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Numerical Analysis of the Basketball Shot

This paper numerically analyzes the dynamics of the basketball shot. The focus of the paper is on the development of a general formulation for the dynamics of the shot beginning when the ball leaves the shooter's hand and ending when the shot is made or missed. The numerical analysis developed in this paper can be used to conduct a parametric study of the dynamics of the basketball shot, which in turn, can be used to improve individual shooting and team strategy. The individual skill level of the shooter enters the formulation through the statistical accuracy of the release. The paper then shows how to determine the shooter's probability of making a given shot. [DOI: 10.1115/1.1636193]

I Introduction

Millions of people play the game of basketball, yet no comprehensive analysis of the basketball shot has ever been undertaken. When shooting a basketball, the ball leaves the hands at some initial position, with some initial velocity and with some initial angular velocity. Can it be said with certainty how these initial conditions should be selected? At what points on the court are the chances greater of making a shot by banking it rather than by aiming directly at the hoop? Where on the court are there significant advantages gained by placing spin on the ball? Overall, where on the court are the chances greatest of making a shot, and where is it least likely? Precise answers to these questions can be found through a detailed numerical analysis of the basketball shot.

The earliest serious mathematical study of the dynamics of the basketball shot was undertaken over two decades ago by Brancazio [2]. Brancazio describes interesting features of the basketball shot. His formulation was analytical, which limited the generality and scope of the results. In other basketball studies, the optimal trajectory of the free throw was developed by Hamilton and Reinschmidt [3] and by Tan and Miller [4], and player shooting patterns were investigated by Liu and Burton [5] and by Miller and Bartlett [6]. This paper couples a general analysis with a numerical procedure for determining the motion of the basketball. The increased generality of the formulation enables a detailed study of the basketball shot to be performed.

The formulation presented in this paper extends the previous results on the subject as follows:

(1) It incorporates any combination of consecutive bounces off the backboard and the rim.

(2) It handles rolling and sliding contact with the backboard and the rim.

(3) It considers the statistical characteristics of the skill level of the individual shooter, making it possible to predict the probability of making a shot.

This paper begins with the development of the equations governing the motion of the basketball and the numerical procedure for determining the time response. Section III describes how to predict the probability of making a shot and Section IV conducts several interesting case studies. Note that the case studies presented in this section are not meant to be comprehensive; comprehensive studies of the basketball shot can be performed in future works based on the shooting dynamics problem formulated in this paper. Conclusions are presented in Section V.

II Mathematical Formulation of the Basketball Shot

Governing Equations. The equations that govern the motion of a basketball are (see Fig. 1)

$$m\mathbf{g}+\mathbf{f}_B+\mathbf{f}_H=m\mathbf{a}_C, \tag{1a}$$

$$\mathbf{r}_{B/C}\times\mathbf{f}_B+\mathbf{r}_{H/C}\times\mathbf{f}_H=I\alpha \tag{1b}$$

where m is the mass of the basketball, $\mathbf{g}=-g\mathbf{k}$, is the gravity vector, $\mathbf{a}_C$ is the acceleration vector of the mass center C of the ball, I is the mass moment of inertia of the ball assuming that it is a thin spherical shell, α is the angular acceleration vector of the ball, $\mathbf{r}_{B/C}=\mathbf{r}_B-\mathbf{r}_C$ and $\mathbf{r}_{H/C}=\mathbf{r}_H-\mathbf{r}_C$ in which $\mathbf{r}_B$, $\mathbf{r}_H$, and $\mathbf{r}_C$ denote the position vectors of the contact point B on the backboard, the contact point H on the hoop and the mass center C of the ball, respectively, and where $\mathbf{f}_B$ is the contact force vector exerted on the ball by the backboard and $\mathbf{f}_H$ is the contact force vector exerted on the ball by the hoop. The dimensions of the backboard, hoop, and court are shown in Fig. 2. Notice in this formulation that the aerodynamic effects are neglected because of their smallness and because they would not be exploited by the shooter.

Contact With the Backboard. The contact force between the basketball and the backboard is $\mathbf{f}_B=\mathbf{N}_B+\mathbf{F}_B$ (Figs. 1 and 3), where $\mathbf{N}_B$ is the normal component and $\mathbf{F}_B$ is the friction component. The normal component and the friction component of the contact force between the ball and the backboard are given by (see Appendix 1):

$$\mathbf{N}_B=N_B\mathbf{i},\quad N_B=ku_B+c\dot{u}_B \tag{2}$$

$$\mathbf{F}_B=-\mu N_B\left(\frac{\mathbf{v}_B}{v_B}\right), \tag{3}$$

where k is the stiffness of the ball, c is the damping of the ball, $u_B=R-a-x>0$ is the compressive displacement of the ball, and where $\mathbf{F}_B=F_{By}\mathbf{j}+F_{Bz}\mathbf{k}$ (see Fig. 3). The velocity $\mathbf{v}_B$ of the contact point on the backboard is given by:

$$\begin{aligned}\mathbf{v}_B&=\mathbf{v}_C+\boldsymbol{\omega}\times\mathbf{r}_{B/C}+(\mathbf{v}_B)_{bf}\\&=(\dot{x}\mathbf{i}+\dot{y}\mathbf{j}+\dot{z}\mathbf{k})+(\omega_x\mathbf{i}+\omega_y\mathbf{j}+\omega_z\mathbf{k})\times(u_B-R)\mathbf{i}-\dot{x}\mathbf{i}\\&=v_{Bx}\mathbf{i}+v_{By}\mathbf{j}+v_{Bz}\mathbf{k},\end{aligned} \tag{4}$$

in which

$$v_{Bx}=0,\quad v_{By}=\dot{y}-\omega_z(a+x),\quad v_{Bz}=\dot{z}+\omega_y(a+x)$$

The friction force given in Eq. (3) is a standard dry friction model [7] in which μ denotes the associated kinetic coefficient of fric-

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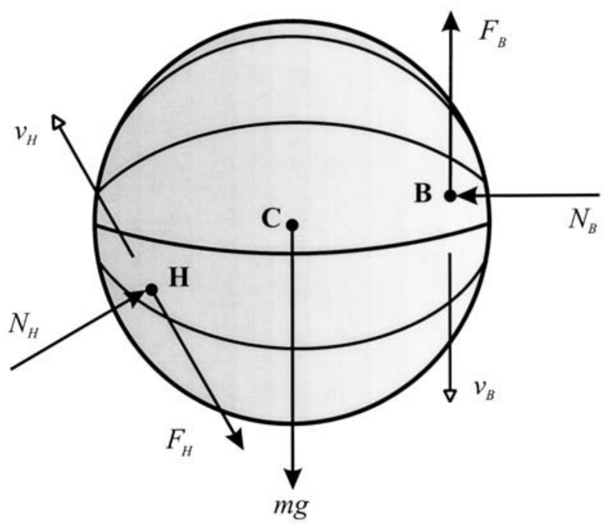


Fig. 1 General free body diagram

tion. The equations governing the motion of the ball will be numerically integrated in time. During contact with the backboard, the basketball will either slide or roll. The dry friction model accurately predicts the motion of the basketball during roll, as well as during sliding (see Appendix 2).

Contact With the Hoop. Now consider the contact between the basketball and the hoop (see Fig. 4). When the basketball is in contact with the hoop, the use of any one of the popular coordinate systems, i.e., rectangular, polar, tangential-normal, and spherical [7], is cumbersome. It proves advantageous to develop a customized coordinate system for the contact with the hoop. The customized coordinate system is referred to here as the hoop coordinate system. Using the hoop coordinate system, the position vector C of the basketball is expressed as:

$$\mathbf{r}_C = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = r_n\mathbf{u}_n + \mathbf{r}_\phi\mathbf{u}_\phi,$$
$$r_n = R - u_H - R_H\cos\phi, \tag{5}$$
$$r_\phi = R_H\sin\phi,$$

where u_H denotes the compressive displacement of the basketball. The hoop unit vectors are related to the rectangular unit vectors by the orthogonal transformation:

$$\begin{pmatrix}\mathbf{u}_n\\ \mathbf{u}_\phi\\ \mathbf{u}_\theta\end{pmatrix} = \begin{bmatrix} -\cos\phi\cos\theta & -\cos\phi\sin\theta & \sin\phi\\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi\\ -\sin\theta & \cos\theta & 0\end{bmatrix}\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{pmatrix}. \tag{6}$$

Differentiating Eq. (6) with respect to time yields:

$$\begin{pmatrix}\dot{\mathbf{u}}_n\\ \dot{\mathbf{u}}_\phi\\ \dot{\mathbf{u}}_\theta\end{pmatrix} = \begin{bmatrix} 0 & \dot\phi & -\dot\theta\cos\phi\\ -\dot\phi & 0 & \dot\theta\sin\phi\\ \dot\theta\cos\phi & -\dot\theta\sin\phi & 0\end{bmatrix}\begin{pmatrix}\mathbf{u}_n\\ \mathbf{u}_\phi\\ \mathbf{u}_\theta\end{pmatrix}$$
$$= [\boldsymbol{\omega}_H\times]\begin{pmatrix}\mathbf{u}_n\\ \mathbf{u}_\phi\\ \mathbf{u}_\theta\end{pmatrix} \tag{7}$$

in which $\boldsymbol{\omega}_H = \dot\theta\sin\phi\mathbf{u}_n + \dot\theta\cos\phi\mathbf{u}_\phi + \dot\phi\mathbf{u}_\theta$ denotes the angular velocity vector of the hoop coordinates. Differentiating Eq. (5) with respect to time yields the velocity vector $\mathbf{v}_C = v_n\mathbf{u}_n + v_\phi\mathbf{u}_\phi + v_\theta\mathbf{u}_\theta$, and differentiating again yields the acceleration vector $\mathbf{a}_C = a_n\mathbf{u}_n + a_\phi\mathbf{u}_\phi + a_\theta\mathbf{u}_\theta$, in which:

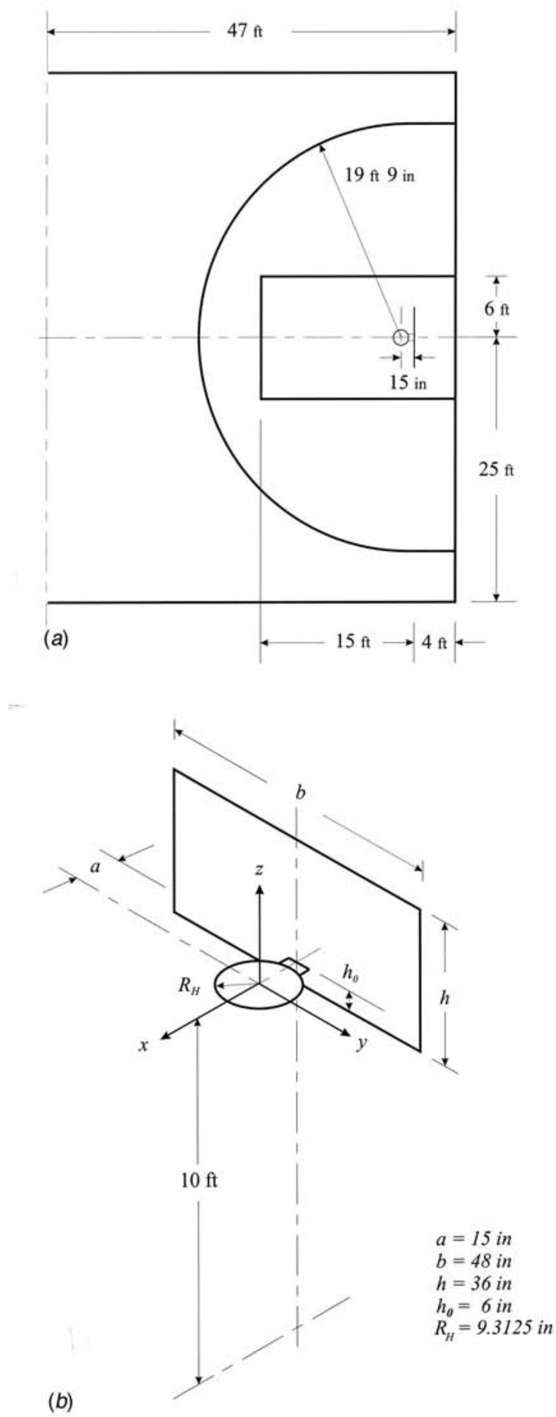


Fig. 2 Dimensions of backboard, hoop and court

$$v_n = -\dot u_H, \quad v_\phi = (R-u_H)\dot\phi, \quad v_\theta = [R_H - (R-u_H)\cos(\phi)]\dot\theta \tag{8}$$

and

$$a_n = -\ddot u_H - (R-u_H)\dot\phi^2 + [R_H - (R-u_H)\cos\phi]\cos\phi\,\dot\theta^2,$$
$$a_\phi = (R-u_H)\ddot\phi - 2\dot u_H\dot\phi - [R_H - (R-u_H)\cos\phi]\sin\phi\,\dot\theta^2, \tag{9}$$
$$a_\theta = [R_H - (R-u_H)\cos\phi]\ddot\theta + 2\dot u_H\dot\theta\cos\phi + 2(R-u_H)\dot\theta\dot\phi\sin\phi.$$

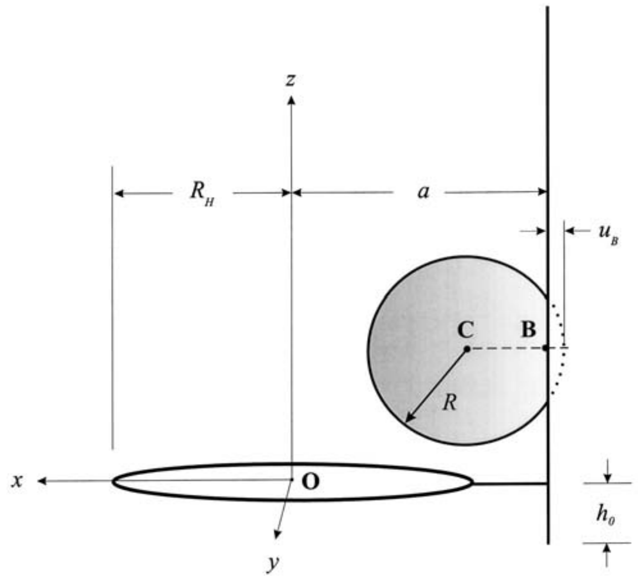


Fig. 3 Backboard

For future reference, the angular acceleration vector, expressed in terms of hoop coordinates, is obtained by differentiating the angular velocity vector with respect to time as follows:

$$\alpha=\alpha_n\mathbf{u}_n+\alpha_\phi\mathbf{u}_\phi+\alpha_\theta\mathbf{u}_\theta=\dot{\omega}+\omega_H\times\omega, \tag{10}$$

in which

$$\begin{aligned}\alpha_n&=\dot{\omega}_n-\dot{\phi}\omega_\phi+\dot{\theta}\cos\phi\,\omega_\theta,\\ \alpha_\phi&=\dot{\omega}_\phi+\dot{\phi}\omega_n-\dot{\theta}\sin\phi\,\omega_\theta,\\ \alpha_\theta&=\dot{\omega}_\theta+\dot{\theta}(-\cos\phi\,\omega_n+\sin\phi\,\omega_\phi)\end{aligned} \tag{11}$$

The contact force between the basketball and the hoop is $\mathbf{f}_H=\mathbf{N}_H+\mathbf{F}_H$ in which $\mathbf{N}_H$ denotes the normal component and $\mathbf{F}_H$ denotes the friction component. The normal and friction components of the contact force are given by:

$$\mathbf{N}_H=N_H\mathbf{u}_n,\quad N_H=ku_H+c\dot{u}_H \tag{12}$$

$$\mathbf{F}_H=-\mu N_H\left(\frac{\mathbf{v}_H}{v_H}\right), \tag{13}$$

where the compressive displacement of the ball is $u_H=R-D_\phi>0$ where $D_\phi=\sqrt{(R_H-D_\theta)^2+z^2}$, $D_\theta=\sqrt{x^2+y^2}$ and where $\mathbf{F}_H=F_{H\phi}\mathbf{u}_\phi+F_{H\theta}\mathbf{u}_\theta$. The velocity $\mathbf{v}_H$ of the contact point H between the basketball and the hoop in Eq. (13) is given by

$$\begin{aligned}\mathbf{v}_H&=v_{H\theta}\mathbf{u}_\theta+v_{H\phi}\mathbf{u}_\phi=v_{Hx}\mathbf{i}+v_{Hy}\mathbf{j}+v_{Hz}\mathbf{k},\\ v_{Hx}&=\dot{x}-z\omega_y-(R_H-D_\theta)\omega_z\sin\theta,\\ v_{Hy}&=\dot{y}+z\omega_x+(R_H-D_\theta)\omega_z\cos\theta,\\ v_{Hz}&=\dot{z}+(R_H-D_\theta)(\omega_x\sin\theta-\omega_y\cos\theta).\end{aligned} \tag{14}$$

Numerical Integration. The equations governing the motion of the basketball are numerically integrated using the fourth-order Runga-Kutta method with a variable step size that has been optimized for numerical speed. Notice that the state variables are rectangular, not explicitly requiring Eqs. (9)–(11) during the numerical integration. The governing equations are expressed in the state form:

$$\dot{\mathbf{x}}=\mathbf{g}(\mathbf{x}) \tag{15}$$

in which $\mathbf{x}=[x_1\ldots x_9]^T=[x\,\dot{x}\,y\,\dot{y}\,z\,\dot{z}\,\omega_x\,\omega_y\,\omega_z]^T$ denotes the state vector. The elements $g_1g_2\ldots g_9$ of the state equations are

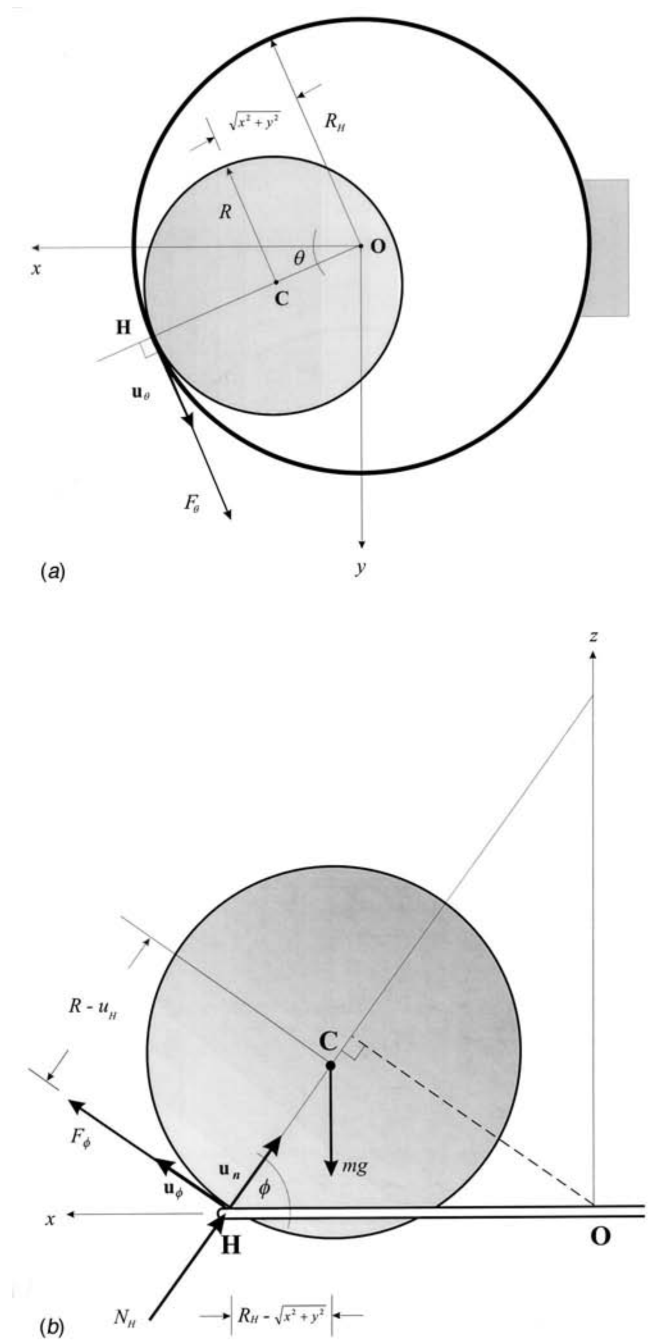


Fig. 4 Hoop

given below along with a listing of the mathematical relationships developed above that are used in the numerical integration.

Backboard equations:

$$\begin{aligned}&N_B=ku_B+c\dot{u}_B,\quad u_B=R-a-x_1,\quad \dot{u}_B=x_2,\\ &v_{By}=x_4-(a+x_1)x_9,\quad v_{Bz}=x_6+(a+x_1)x_8,\\ &v_B=\sqrt{v_{By}^2+v_{Bz}^2},\\ &F_{By}=-\mu N_Bv_{By}/v_B,\quad F_{Bz}=-\mu N_Bv_{Bz}/v_B.\\ &M_{Bx}=0,\quad M_{By}=(a+x_1)F_{Bz},\quad M_{Bz}=-(a+x_1)F_{By}.\end{aligned} \tag{16a}$$

Hoop equations:

$$D_\theta=\sqrt{x_1^2+x_3^2},\quad D_\phi=\sqrt{(R_H-D_\theta)^2+x_5^2},$$

$$u_H = R - D_\phi, \quad \dot{u}_H = \frac{1}{D_\phi}[(R_H - D_\theta)(x_1x_2 + x_3x_4)/D_\theta - x_5x_6],$$

$$\cos\theta = x_1/D_\theta, \quad \sin\theta = x_3/D_\theta,$$

$$\cos\phi = (R_H - D_\theta)/D_\phi, \quad \sin\phi = x_5/D_\phi$$

$$v_{Hx} = x_2 - x_5x_8 - (R_H - D_\theta)x_9 \sin\theta,$$

$$v_{Hy} = x_4 + x_5x_7 + (R_H - D_\theta)x_9 \cos\theta,$$

$$v_{Hz} = x_6 + (R_H - D_\theta)(x_7 \sin\theta - x_8 \cos\theta),$$

$$v_{H\theta} = -v_{Hx}\sin\theta + v_{Hy}\cos\theta,$$

$$v_{H\phi} = v_{Hx}\sin\phi\cos\theta + v_{Hy}\sin\phi\sin\theta + v_{Hz}\cos\phi,$$

$$v_H = \sqrt{v_{H\theta}^2 + v_{H\phi}^2}, \tag{16b}$$

$$N_H = ku_H + c\dot{u}_H,$$

$$F_{H\theta} = -\mu N_H v_{H\theta}/v_H, \quad F_{H\phi} = -\mu N_H v_{H\phi}/v_H,$$

$$F_{\phi x} = \sin\phi\cos\theta F_{H\phi}, \quad F_{\phi y} = \sin\phi\sin\theta F_{H\phi},$$

$$F_{\phi z} = \cos\phi F_{H\phi},$$

$$F_{\theta x} = -\sin\theta F_{H\theta}, \quad F_{\theta y} = \cos\theta F_{H\theta}, \quad F_{\theta z} = 0,$$

$$N_{Hx} = -\cos\phi\cos\theta N_H, \quad N_{Hy} = -\cos\phi\sin\theta N_H,$$

$$N_{Hz} = \sin\phi N_H.$$

$$M_{Hx} = (R_H - D_\theta)\sin\theta F_{\phi z} + x_5(F_{\theta y} + F_{\phi y}),$$

$$M_{Hy} = -x_5(F_{\theta x} + F_{\phi x}) - (R_H - D_\theta)\cos\theta F_{\phi z},$$

$$M_{Hz} = (R_H - D_\theta)[\cos\theta(F_{\phi y} + F_{\theta y}) - \sin\theta(F_{\phi x} + F_{\theta x})]$$

State equations:

$$g_1 = x_2,$$

$$g_2 = CONDB[N_B/m] + CONDH[(N_{Hx} + F_{\theta x} + F_{\phi x})/m],$$

$$g_3 = x_4,$$

$$g_4 = CONDB[F_{By}/m] + CONDH[(N_{Hy} + F_{\theta y} + F_{\phi y})/m],$$

$$g_5 = x_6, \tag{17}$$

$$g_6 = -g + CONDB[F_{Bz}/m] + CONDH[N_{Hz} + F_{\theta z} + F_{\phi z})/m],$$

$$g_7 = CONDH[M_{Hx}/I],$$

$$g_8 = CONDB[(a + x_1)F_{Bz}/I] + CONDH[M_{Hy}/I],$$

$$g_9 = CONDB[-(a + x_1)F_{By}/I] + CONDH[M_{Hz}/I].$$

The terms *CONDB* and *CONDH* in Eqs. (17) are contact conditions. When there is contact with the backboard $CONDB = 1$, during which $u_B > 0$, $|x_3| < b/2$, and $|x_5 - h_0| < h/2$. Otherwise $CONDB = 0$. When there is contact with the hoop $CONDH = 1$, during which $u_H > 0$. Otherwise $CONDH = 0$. In free flight (no contact), the governing equations are quadratic, from which it follows that the step size in the Runga-Kutta method can be arbitrarily large and still admit no numerical error (recall that the Runga-Kutta method is a second-order expansion). On the other hand, the step size during contact, as a rule of thumb, needs to be 10 times smaller than the contact time, in which the contact time is bound from below by one-half the damped fundamental period of the basketball (see Appendix 1). In view of these characteristics, the optimal step size needs to be as large as possible in free flight and equal to ten times less than the damped fundamental period during contact. The complexity in the optimization of the step size enters during the transition between free space and contact. Toward this end, a relatively simple tuned bi-section method can be employed. The tuned bi-section method incrementally reduces the step size as the ball approaches the contact surface from free space and it incrementally increases the step size as the ball enters free space from the contact surface. A flow chart of the tuned bi-section is shown in Table 1, in which $dt_{\max} = 0.015$ and $dt_{\min} = 0.0005$.

III Basketball Shot Probability

Initial Conditions. A shooter throws a ball from a particular point $\mathbf{r}_0$ on the court with a particular initial velocity $\mathbf{v}_0$ and with a particular initial angular velocity $\boldsymbol{\omega}_0$, expressed as (Fig. 5)

$$\mathbf{v}_0 = v_0[-\cos\alpha_0[\cos\beta_0\mathbf{i} + \sin\beta_0\mathbf{j}] + \sin\alpha_0\mathbf{k}],$$

$$\boldsymbol{\omega}_0 = \omega_{r0}[\cos\beta_0\mathbf{i} + \sin\beta_0\mathbf{j}] + \omega_{\beta 0}[-\sin\beta_0\mathbf{i} + \cos\beta_0\mathbf{j}] + \omega_{z0}\mathbf{k}, \tag{18}$$

$$\mathbf{r}_0 = r_0(\cos\theta_0\mathbf{i} + \sin\theta_0\mathbf{j}) + z_0\mathbf{k},$$

where v_0 is speed, α_0 is pitch angle, β_0 is polar angle, ω_{r0} is radial angular velocity, $\omega_{\beta 0}$ is polar angular velocity, ω_{z0} is vertical angular velocity, r_0 is radial distance, θ_0 is radial angle, and z_0 is vertical distance. The initial conditions represent nine statistical quantities, expressed either in terms of rectangular coordinates as v_{x0}, v_{y0}, v_{z0}, ω_{x0}, ω_{y0}, ω_{z0}, x_0, y_0, and z_0 or expressed above in terms of v_0, α_0, β_0, ω_{r0}, $\omega_{\beta 0}$, $\omega_{z0} r_0$, θ_0, and z_0.

Statistical Distributions. In practice, the shooter can prescribe these initial conditions only approximately, so the initial conditions must be regarded as statistical quantities. Let's denote them by $a_1 = v_0$, $a_2 = \alpha_0$, $a_3 = \beta_0$, $a_4 = \omega_{r0}$, $a_5 = \omega_{\beta 0}$, $a_6 = \omega_{z0}$, $a_7 = r_0$, $a_8 = \theta_0$, and $a_9 = z_0$. The statistical quantities can also be written as

$$a_i = a_{i0} + a_{i1} \quad (i = 1,2,\ldots,9), \tag{19}$$

where the subscript 0 denotes nominal values and the subscript 1 denotes statistical errors. The nominal values are the values that are desired by the shooter. The statistical errors are associated with the physical and mental limitations of the shooter. It is further assumed that the statistical quantities a_i $(i = 1,2,\ldots,9)$ are perceived coordinates, that is the coordinates that are chosen by the shooter to perceive and for which corrections are made. Mathematically, this is tantamount to assuming that the errors a_{i1} $(i = 1,2,\ldots,9)$ are statistically independent. Furthermore assume that the errors are normally distributed, in which case their normal probability density functions are given by:

$$p_i(a_i, a_{i0}) = \frac{1}{\sigma_i\sqrt{2\pi}} e^{-1/2(a_i - a_{i0}/\sigma_i)^2}, \quad (i = 1,2,\ldots,9) \tag{20}$$

where σ_i is the standard deviation of the i-th error. The probability density functions satisfy the conditions [8]:

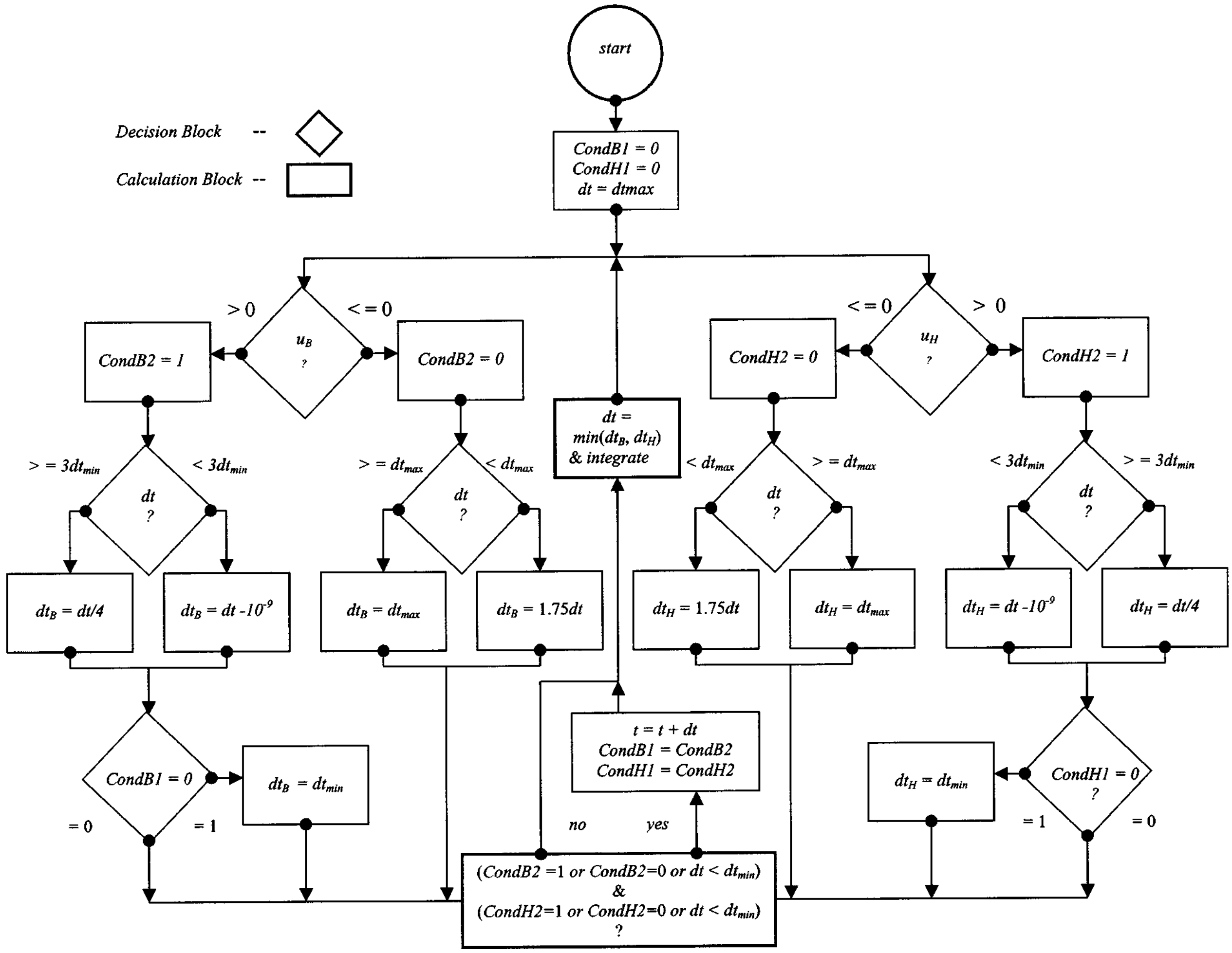
$$1 = \int_{-\infty}^{\infty} p_i(a_i, a_{i0})\,da_i, \quad (i = 1,2,\ldots,9) \tag{21}$$

and the total probability of shooting the ball within the complete range is then:

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{i=1}^{9} p_i(a_i, a_{i0})\prod_{i=1}^{9} da_i \tag{22}$$

Now let $\mathrm{cond}(a_1, \ldots, a_9)$ indicate whether a shot is made or missed. This function is determined by simulating the dynamics of a basketball shot per the development presented in the previous section of the paper. Let $\mathrm{cond}(a_1, \ldots, a_9) = 1$ when the shot is made and let $\mathrm{cond}(a_1, \ldots, a_9) = 0$ when the shot is missed. From Eq. (22), the total probability of making the shot is:

Table 1 Flow Chart



$$P(a_{10},\ldots,a_{90})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{cond}(a_1,\ldots,a_9)\times\prod_{i=1}^{9}p_i(a_i,a_{i0})\prod_{i=1}^{9}da_i \tag{23}$$

The best shot is then:

$$P^*(a_{10}^*,\ldots,a_{90}^*)=\max\{P(a_{10},\ldots,a_{90})\} \tag{24}$$

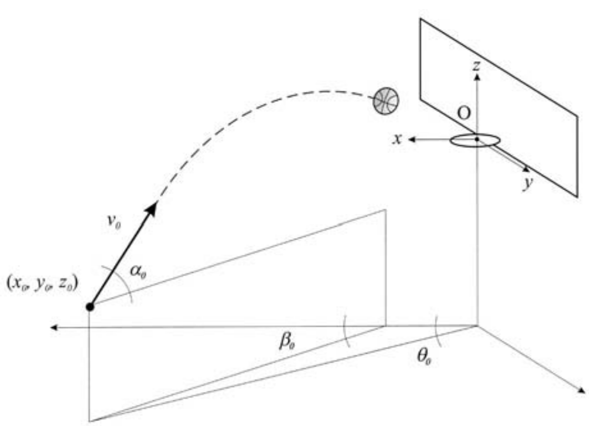


Fig. 5 Initial conditions

Numerical Considerations. For numerical purposes, divide the statistical parameters a_i into $n-1$ uniformly spaced increments a_{i1}, a_{i2}, $\ldots$, a_{in}, where $\Delta a_i=(a_{in}-a_{i1})/(n-1)$. Equations (23) and (24) become:

$$P(a_{10},\ldots,a_{90})=\left\{\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\sum_{k_3=1}^{n}\cdots\sum_{k_9=1}^{n}\mathrm{cond}(a_{1k_1},\ldots,a_{9k_9})\times\prod_{i=1}^{9}p_i(a_{ik_i},a_{i0})\right\}\left\{\prod_{i=1}^{9}\Delta a_i\right\} \tag{25}$$

and

$$P^*(a_{10}^*,\ldots,a_{90}^*)=\max_{(k_1k_2\ldots k_9)}\{P(a_{10k_1},\ldots,a_{90k_9})\} \tag{26}$$

The number of simulations that is required in Eq. (25) is n^9. To gain an appreciation for the extent of the computational effort, note that the simulation of a single basketball shot required an average of 150 numerical integration steps that were completed in about 1 second on a personal computer. Furthermore, the nature of the optimization space dictated the need for $n=40$ increments per parameter. Therefore, the total computational time was estimated at 40^9 seconds, which is about 8.3 million years.

Since the computational time associated with calculating the total probability of making a shot is prohibitive, the conditional probabilities associated with making a shot will be considered. The conditional probabilities refer to the probabilities obtained by varying some of the initial conditions, regarding them as statistical, while not varying others, regarding them as deterministic. The conditional probability associated with making a shot from a given location on the court (regarding the three locations as deterministic) reduces the estimated number of computations from 40^9 to 40^6. By neglecting the effect of off-axis spin (letting $\omega_{n0} = \omega_{z0} = 0$), the estimated number of calculations is further reduced to 40^4 seconds which is about 30 days. By bracketing the optimization space associated with the shots that have potential of being made, an additional factor of savings of ten is achieved. This reduces the computational time to the order of several hours.

IV. Testing the Individual Shooter

Each shooter possesses a unique set of standard deviations associated with the statistical errors. Even so, there exist general tendencies associated with these errors that are common to each shooter. Specifically, the following tendencies are assumed.

1. When far from the hoop, the error of the initial speed a_{11} increases with increasing initial speed a_{10}, and when near the hoop the error of the initial speed increases with decreasing initial speed.
2. The error a_{21} of the pitch angle a_{20} does not change significantly over the range of values for which the shooter makes the shot.
3. The error a_{31} of the polar angle a_{30} does not change significantly over the range of values for which the shooter makes the shot. The polar angle is nominally directed toward the center of the hoop for a direct shot and nominally directed toward the backboard reflection of the center of the hoop for a bank shot. The initial polar angle β_0 for the bank shot is then given by:

$$\tan(\beta_0) = \frac{\sin(\theta_0)}{\frac{a-R}{D_\theta}\left(1+\frac{3}{5\gamma}\right)+\cos(\theta_0)}, \tag{27}$$

   where $\gamma=0.8457$ is the coefficient of restitution[1]. The coefficient of restitution is determined from the stiffness k and damping c of the basketball (see Appendix 1).
4. The error a_{41} of the initial angular velocity increases with increasing angular velocity a_{40}. The off-axis components of the angular velocity are zero ($a_{50} = a_{60} = a_{51} = a_{61} = 0$).
5. The initial position is deterministic ($a_{71} = a_{81} = a_{91} = 0$).

Based on these tendencies, the standard deviations of the errors can be expressed in the form:

$$\sigma_1 = c_{10} + c_{11}a_{10} + c_{12}a_{10}^2, \quad \sigma_2 = c_{20}$$
$$\sigma_3 = c_{30}, \quad \sigma_4 = c_{40} + c_{41}a_{40} + c_{42}a_{40}^2, \tag{28}$$
$$\sigma_5 = \sigma_6 = \sigma_7 = \sigma_8 = \sigma_9 = 0,$$

where c_{10}, c_{11}, c_{12}, c_{20}, c_{30}, c_{40}, c_{41}, c_{42} are coefficients that are determined by testing the individual shooter. The determination of the coefficients requires a testing procedure, which is beyond the scope of this paper. Nevertheless, such a procedure would have the individual shooter take a sufficient number of shots from selected points on the court during which the shots are photographed and the errors are determined, after which the coefficients are determined through a curve-fitting procedure.

[1]Equation (27) is obtained by first showing that the y component of velocity just after the collision with the backboard is given by $v_2 = 3/5v \sin\beta + 2/5R\omega_1$, in which v denotes the y component of velocity of the ball just before the collision with the backboard, and ω_1 denotes the angular velocity about the z axis just before the collision (see Fig. 6). This equation is found by summing forces in the y direction, by summing moments about the z axis, and by letting the velocity of the contact point of the ball with the backboard be zero at the end of the collision. Equation (27) then assumes that $\omega_1 = 0$.

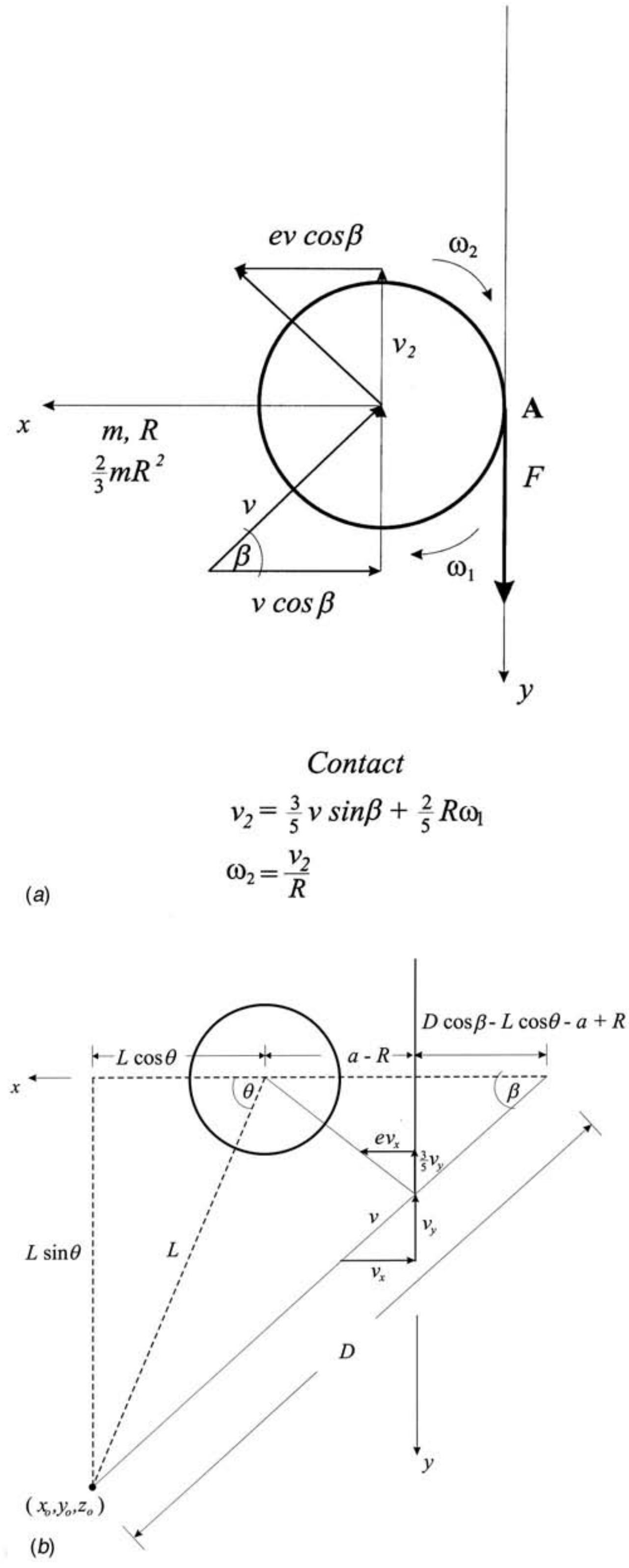


Fig. 6 Nominal bank shot angle

IV Numerical Examples

First consider several cases in which the ball is thrown from the side ($r_0 = 13.75$ ft, $\theta_0 = 67.5$ deg). A direct swish is shown in Fig. 7 and a bank shot is shown in Fig. 8. The direct shot leads to several bounces and rims in Fig. 9 and a bank shot leads to a

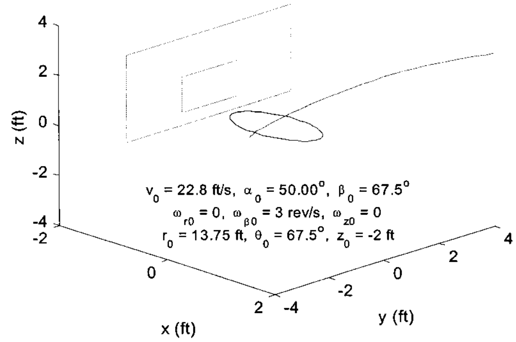


Fig. 7 Swish shot

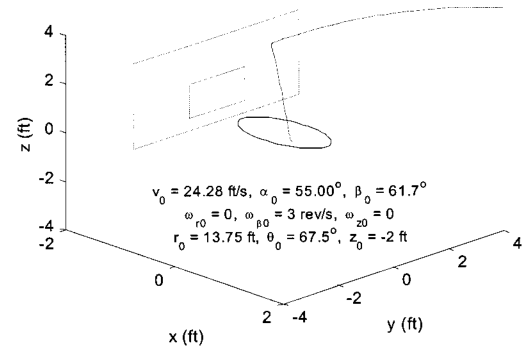


Fig. 10 Shot that bounces off of backboard and rim

bounce and a rim in Fig. 10. Next, the initial conditions are set to cause the ball to swirl in Fig. 11. In order to achieve the swirl, the steady-state swirl conditions were calculated. At steady-state, the initial conditions are given by

$$x=0,\quad y=R_H-(R-u_H)\sin\delta,\quad z=(R-u_H)\cos\delta,$$

$$\dot{x}=-\sqrt{g\tan\delta(R_H-(R-u_H)\sin\delta)},\quad \dot{y}=0,\quad \dot{z}=0,$$

$$\omega_x=0,\quad \omega_y=0,\quad \omega_z=-\frac{\dot{x}R_H}{[R_H-(R-u_H)\sin\delta](R-u_H)},$$

$$u_H=\frac{mg}{k\cos\delta},$$

in which $\delta=87.5$ deg is the angle between the vertical and the line of action of the normal force. Figures 12 through 15 show the interdependences between initial velocity and initial pitch angle for shots thrown from the foul line ($r_0=13.75$ ft, $\theta_0=0$ deg) and from the side ($r_0=13.75$ ft, $\theta_0=67.5$ deg).

Figure 12 considers direct foul shots thrown with no spin. Notice that the region in which the basket is made consists of six rings: from the left, a thin ring, a wide ring, 2 thin rings, a wide ring, and a thin ring. The first ring is associated with the lowest initial velocities for which the ball hits the front of the rim and then enters the hoop. As the velocity is increased, the interior of the second ring is associated with swish shots. The third and forth rings are associated with shots for which the ball hits near the back rim, the fifth is associated with bank shots and the sixth is associated with bank shots that hit the front of the rim. The large ring associated with swish shots is wider than the large ring associated with bank shots indicating that the probability of making a swish shot is higher than of a bank shot.

Figure 13 considers direct foul shots thrown with 3 rev/s of back-spin. Notice that the region in which the basket is made again consists of six rings. Notice that the two wide rings are

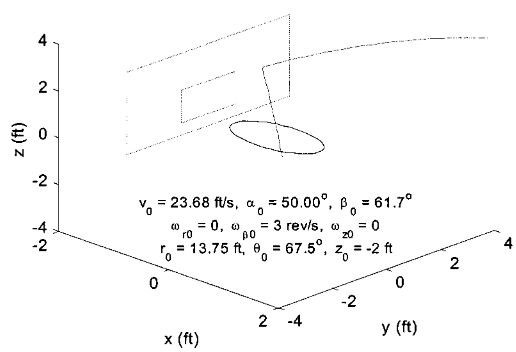


Fig. 8 Bank shot

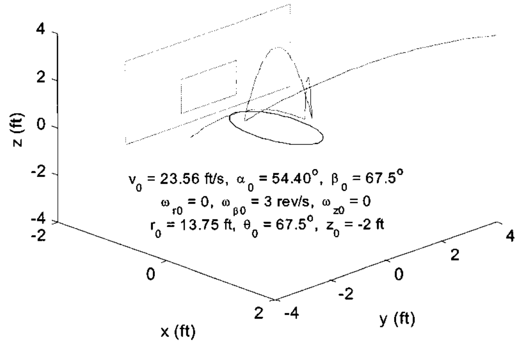


Fig. 9 Shot that bounces repeatedly off of backboard and rim

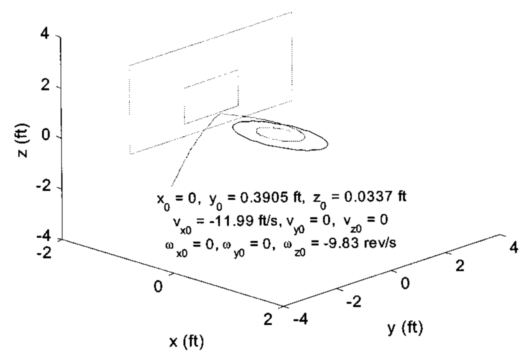


Fig. 11 Swirl shot

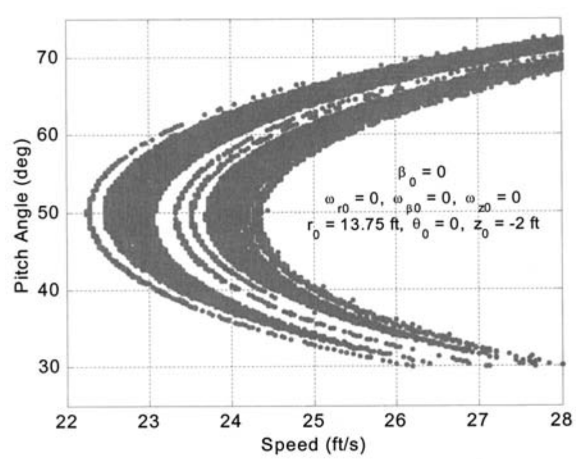


Fig. 12 Initial velocity versus initial pitch angle for direct foul shot with no spin

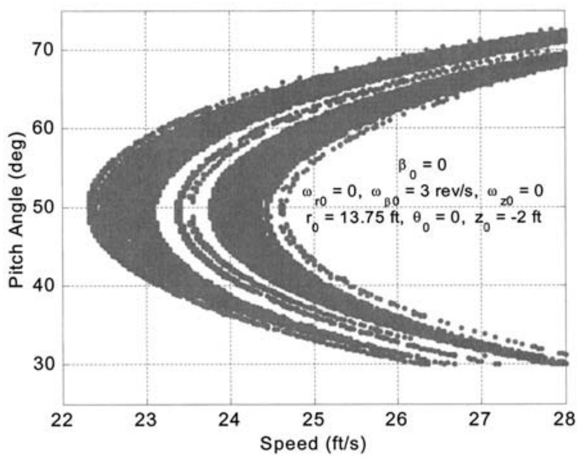


Fig. 13 Initial velocity versus initial pitch angle for direct foul shot with spin

larger than the corresponding rings in Fig. 12 indicating that the spin increases the probability of making the shot.

Figure 14 considers direct shots thrown from the side with 3 rev/s of back-spin. Notice that the region in which the basket is made again consists of three rings all associated with the rim. The first ring is associated with the ball hitting the front rim, the interior of the large middle ring is associated with swish shots, and the last ring is associated with the ball hitting the back of the rim.

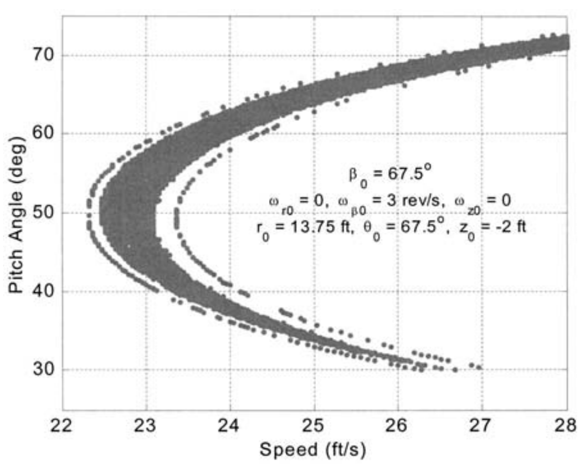


Fig. 14 Initial velocity versus initial pitch angle for direct side shot with spin

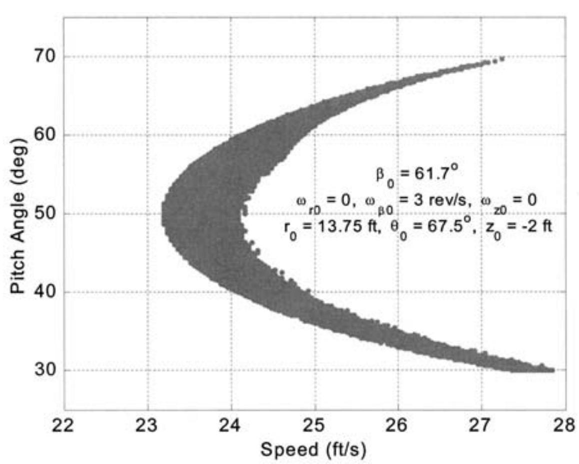


Fig. 15 Initial velocity versus initial pitch angle for bank side shot with spin

Figure 15 considers bank shots thrown from the side with 3 rev/s of back-spin. Notice that the region is dominated by a single large ring. The large size of the ring indicates that the probability of making the bank shot is relatively high.

V Summary

This paper formulated the equations governing the motion of a basketball from the point at which the ball leaves the shooter's hand to the point at which the basket is either made or missed. A numerical approach was adopted because of the high level of generality that it affords and because it enables a comprehensive parametric study of the problem to be undertaken. This paper also formulated the probabilistic problem that can lead to the calculation of the probability of making a shot from a given location on the court. The probability depends on the skill level of the shooter, which would be determined by testing the shooter. Ordinary extensions of the results presented in this paper consist of performing parametric studies of the basketball shot and developing a procedure for testing the skill level of the individual shooter.

Nomenclature

x, y, z	=	rectangular coordinates; ft
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	=	rectangular unit vectors
r_n, r_ϕ	=	hoop coordinates; ft
$\mathbf{u}_n, \mathbf{u}_\phi, \mathbf{u}_\theta$	=	hoop unit vectors
$\mathbf{r}_C, \mathbf{r}_B, \mathbf{r}_H$	=	position vectors of the ball's mass center C, the contact point B with the backboard, and the contact point H with the hoop; ft
$\mathbf{f}_B, \mathbf{f}_H$	=	contact forces with the backboard and the hoop; lb
ω, α	=	angular velocity and angular acceleration vectors; rad/s, rad/s^2
r_0, θ_0, z_0	=	initial position components; ft, rad, ft
v_0, α_0, β_0	=	initial velocity components; ft/s, rad/s, rad/s
$\omega_{r0}, \omega_{\beta 0}, \omega_{z0}$	=	initial angular velocity components; rad/s
a_i, a_{i0}, a_{i1}	=	i-th statistical parameter, nominal value, error
$p(a_i, a_{i0})_-$	=	normal probability density function of i-th error
cond	=	shot condition (=1 if shot is made and=0 if missed)
CONDB	=	contact condition (=1 when in contact with backboard, =0 if not)
CONDH	=	contact condition (=1 when in contact with hoop, =0 if not)
$\mathbf{g} = -g\mathbf{k}$	=	gravity vector; ft/s^2 ($g = 32.2$)
R, R_H	=	ball radius and hoop radius; ft ($R = 29.75/24\pi$, $R_H = (9+5/16)/12$) [NCAA 2001]
$m, I = 2mR^2/3$	=	mass, mass moment of inertia; slugs, slug ft^2 ($m = 21/16/32.2$) [NCAA 2001]
k, c	=	stiffness, damping; 1b/ft, lb s/ft ($k = 2000$, $c = 1.53$)
μ	=	kinetic coefficient of dry friction ($\mu = 1.00$)
a, b, h, h_0	=	backboard parameters shown in Fig. 1; ft ($a = 1.25$, $b = 6.0$, $h = 3.5$, $h_0 = 1.25$) [1]

VI Appendices

Appendix 1: Numerical Treatment of Visco-Elasticity. The basketball is a thin, lightly-damped elastic body that undergoes small motions. As such, it's behavior can be accurately characterized by a linear, visco-elastic model in which the normal contact force acting on the ball is of the form given in Eq. (2). An alternative approach, which is a special case of the analysis given below, assumes that the contact is instantaneous. This is the classical elastic-plastic collision that is characterized by a coefficient of restitution γ.

In general, assume that the contact is between two bodies of masses m_1 and m_2, respectively, between which the elasticity is k and the plasticity is c. The relative displacement of the two bodies is $x=x_2-x_1$. The relative motion of the two masses is governed by the second-order differential equation $\ddot{x}+2\alpha\dot{x}+(\alpha^2+\omega^2)x=0$, in which $\alpha=c/2[1/m_1+1/m_2]$, and $\alpha^2+\omega^2=k[1/m_1+1/m_2]$. At the initial time ($t=0$) and at the final time ($t=T$) the relative displacements and velocities are:

$$0=x(0)=x_2(0)-x_1(0), \quad \dot{x}(0)=\dot{x}_2(0)-\dot{x}_1(0),$$

$$0=x(T)=x_2(T)-x_1(T), \quad \dot{x}(T)=\dot{x}_2(T)-\dot{x}_1(T)$$

The general solution is:

$$x(t)=\frac{\dot{x}(0)}{\omega}e^{-\alpha t}\sin\omega t$$

The final contact time is then $T=\pi/\omega$, which is one-half of the natural period of oscillation of the masses. This also yields the classical coefficient of restitution:

$$\gamma=e^{-\alpha\pi/\omega}=-\frac{\dot{x}(T)}{\dot{x}(0)}$$

The damping c, in terms of the coefficient of restitution, is then:

$$c=-\frac{2m\omega}{\pi}\ln(\gamma),$$

in which we have let m_1 approach infinity and m_2 equal m. The visco-elastic model is used because it handles those situations in which the contact time should not be regarded as instantaneous. In basketball experiments, the average stiffness was measured to be $k=2000$ lb/ft and the coefficient of restitution was measured to be $\gamma=0.8457$, which yields a damping of $c=1.53$ lb-s/ft. The tests show that the ball normally rolls after every bounce. During exceptions to this behavior, the bounce makes a high-pitched screech. Both experimental results and numerical results confirm that the bounces are insensitive to the values of stiffness and damping provided the coefficient of restitution does not change.

Appendix 2: Numerical Treatment of Roll and Sliding. When the basketball slides on a surface, the friction force acts to decrease the basketball's velocity until the velocity of the contact point changes direction. Then, the friction force will change direction, so as to oppose the velocity of the contact point. Eventually, the object may begin to roll. Analytically, this situation is handled by drawing two free body diagrams, one for the case of roll and the other for the case of sliding. Numerically, however, it is sufficient to consider the sliding forces; roll occurs as a limiting case.

To demonstrate this, consider Fig. 16. During sliding, the friction force is:

$$F=\begin{cases}\mu N, & v_A>0\\ -\mu N, & v_A<0\end{cases}$$

During roll, the friction force is different. Summing forces in the x and y directions and summing moments (positive counterclockwise) yields:

$$ma=mg\sin(\beta)-F, \quad N=mg\cos(\beta)$$

$$I\alpha=-FR, \quad v_A=v+R\omega$$

where I is the mass moment of inertia of the sphere, v is the velocity of the mass center C, v_A is the velocity of the contact point A, a is the acceleration of C, ω is the angular velocity and α

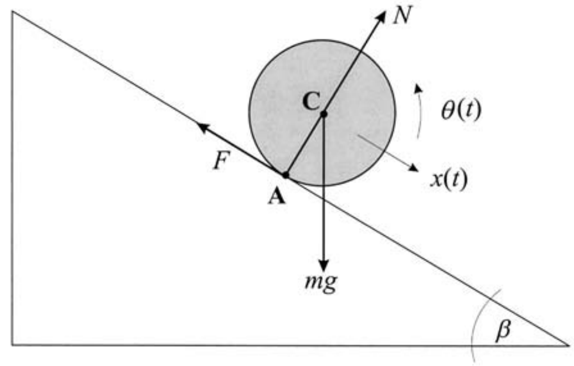


Fig. 16 Rolling and sliding down an incline

is the angular acceleration. During roll, the velocity of the contact point is zero. The friction force F and the acceleration a are given by:

$$F=\frac{mg\sin(\beta)}{1+\frac{mR^2}{I}}, \quad a=g\sin(\beta)\frac{\frac{mR^2}{I}}{1+\frac{mR^2}{I}}$$

Now determine the acceleration of the sphere during roll in a different way. Assume that the sphere is sliding and that the friction force is given by the expressions determined earlier. Assume that:

$$v_A>0 \quad (v_A<0).$$

The acceleration a and the angular acceleration α are then:

$$a_{1,2}=g(\sin(\beta)\mp\mu\cos(\beta)),$$

$$\alpha_{1,2}=\mp\frac{R}{I}\mu mg\cos(\beta)$$

The acceleration of the sphere alternates with the alternating sign of the velocity of the contact point. The friction force alternates when the magnitude of the velocity reaches some $\varepsilon>0$. Start when $v_A=\varepsilon$. The first interval of time is t_1, which occurs from the instant $t=0$ when $v_A=\varepsilon$ until the instant when $v_A=-\varepsilon$. The second interval of time is t_2-t_1, which occurs from the instant $t=t_1$ when $v_A=-\varepsilon$ until the instant when again $v_A=\varepsilon$. Over each of the time intervals:

$$\begin{aligned}v_A(t_1)&=-\varepsilon=(v(0)+a_1t_1)+R(\omega(0)+\alpha_1t_1)\\&=(v(0)+R\omega(0))+(a_1+R\alpha_1)t_1=\varepsilon+(a_1+R\alpha_1)t_1,\\v_A(t_2)&=\varepsilon=(v(t_1)+a_1(t_2-t_1))+R(\omega(t_1)+\alpha_1(t_2-t_1))\\&=(v(t_1)+R\omega(t_1))+(a_1+R\alpha_1)(t_2-t_1)\\&=-\varepsilon+(a_1+R\alpha_1)(t_2-t_1),\end{aligned}$$

so

$$t_1=-\frac{2\varepsilon}{a_1+R\alpha_1},$$

$$t_2-t_1=\frac{2\varepsilon}{a_2+R\alpha_2}.$$

The average acceleration is then:

$$a_{ave}=\frac{a_1t_1+a_2(t_2-t_1)}{t_2}=\ldots=g\sin(\beta)\frac{\frac{mR^2}{I}}{1+\frac{mR^2}{I}}$$

which is precisely the same expression for acceleration that was obtained during roll. The average friction force is:

$$F_{ave}=\frac{\mu Nt_1-\mu N(t_2-t_1)}{t_2}=\ldots=\frac{mg\sin(\beta)}{1+\frac{mR^2}{I}}$$

which again is exactly the same as the friction force on the sphere during roll. These results imply that the velocity and angular velocity of the basketball using separate equations for roll and for sliding yield the same average velocity and the same average angular velocity of the basketball using the sliding condition alone.

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