If S carries a probability measure P and a qualitative probability $\leq\cdot$ such that, for every $B,\ C,\ P(B) \leq P(C)$, if and only if $B \leq\cdot\, C$; then P **(strictly) agrees** with $\leq\cdot$. If $B \leq\cdot\, C$ implies $P(B) \leq P(C)$, then P **almost agrees with** $\leq\cdot$. This terminology is obviously consistent in that, if P agrees, that is, strictly agrees, with $\leq\cdot$, P also almost agrees with $\leq\cdot$. It is also easily seen that, if P agrees with $\leq\cdot$, then knowledge of P implies knowledge of $\leq\cdot$. But if p only almost agrees with $\leq\cdot$, it may happen as examples in §4 show, that $P(B) = P(C)$, though $B <\cdot\, C$, so that knowledge of P may imply only imperfect knowledge of $\leq\cdot$.

The rest of this section is mainly a study of qualitative probabilities generally, with a view to discovering interesting conditions under which there is a probability measure that agrees, either strictly or almost, with a given qualitative probability. These conditions suggest a new postulate governing the special qualitative probability $\leq$. The work is necessarily rather tedious and burdened with detail. it will, therefore, be wise for most readers to slim over the material, omitting the proofs but noticing the more obvious logical connections among the theorems and definitions. Some may then find themselves sufficiently interested in the details to return and read or supply the proofs, as the case may require. Others may safely go forward. Here, as elsewhere, technical terms of interest for the moment only are introduced with italics rather than boldface.

An *n-fold almost uniform partition* of B is and n-fold partition of B such that the union of no r elements of the partition is more probable than that of any $r+1$ elements.

THEOREM 1 If there exist n-fold almost uniform partitions of B for arbitrarily large values of n, then there exist m-fold almost uniform partitions for every positive integer m.

PROOF. Let B_i, $i = 1, \cdots, n$, be an n-fold almost uniform partition (of B) with $n \geq m^2$. Using the euclidean algorithm, let n be written $n = am + b$, where a and b are integers such that $m \leq a$ and $0 \leq b < m$. Now let C_j, $j = 1, \cdots, m$, be any m-fold partition such that each C_j is the union of a or $a+1$ of the B_i's. The union of any r of the C_j's, $r < m$, is the union of from ar to $(a+1)r$ of the B_i's and the union of $r+1$ of the C_j's is that of from $a(r+1)$ to $(a+1)(r+1)$ of the B_i's. Since $r < m \leq a$, $(a+1)r = ar + r < ar + a = a(r+1)$.◆

THEOREM 2 If there exist n-fold almost uniform partitions of S for arbitrarily large values of n, then there is one and only one probability measure P that almost agrees with $\leq\cdot$. Furthermore, for any ρ, $0 \leq$

$\rho \leq 1$, any $B \subset S$, and the unique p just defined there exists $C \subset B$ such that $P(C) = \rho P(B)$.†

PROOF. The proof is broken into a sequence of easy steps, left, for the most part, to the reader. These steps are grouped in blocks, only the last step in each being needed in the proof of later steps.

1. There exist n-fold almost uniform partitions of S for every positive n.

2a. If $p_1, \cdots, p_n$ are real numbers such that $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n$, and $\Sigma p_i = 1$; then

$$\sum_1^r p_i \leq r/n, \quad r = 1, \cdots, n. \tag{1}$$

2b. If further

$$\sum_1^{r+1} p_i \geq \sum_{n-r+1}^{n} p_i \quad \text{for } r = 1, \cdots, n-1;$$

then

$$\sum_1^r p_i \geq (r-1)/n, \text{ and } \sum_{n-r+1}^{n} p_i \leq (r+1)/n. \tag{2}$$

2c. The sum of any r of the p_i's lies between $(r-1)/n$ and $(r+1)/n$.

2d. If P almos agrees with $\leq\cdot$, and $C(r,n)$ denotes here and later in this proof any union of r elements of any n-fold almost uniform partition (not necessarily the same from one context to another), then

$$(r-1)/n \leq P(C(r,n)) \leq (r+1)/n. \tag{3}$$

3. Let $k(B,n)$ denote the largest integer r (possibly zero) such that some $C(r,n)$ is not more probable than B. The function $k(B,n)$ is well-defined, and $0 \leq k(N,b) \leq n$.

4a. For any P that almost agrees with $\leq\cdot$,

$$(k(B,n)-1)/n \leq P(B) \leq (k(B,n)+2)/n. \tag{4}$$

4b. At most one P can almost agree with $\leq\cdot$

5a. If B_i and C_i are n-fold partitions (not necessarily almost uniform) so indexed that $B_1 \leq\cdot B_2 \leq\cdot \cdots \leq\cdot B_n$, and $C_1 \geq\cdot C_2 \geq\cdot \cdots \geq\cdot C_n$; then

$$\bigcup_{n-r}^{n} B_i \geq\cdot \bigcup_{n-r}^{n} C_i, \quad r = 0, \cdots, n-1. \tag{5}$$

†Technical note: The mathematical essence of the terminal conclusion of this theorem, and other conclusions related to ir, are given by Sobczyk and Hammer [S15]. It might be conjectured, in analogy with countably additive measures, that this conclusion means only that P ins non-atomic, by that conjecture is false [N5].‡

‡A key reference for further information on the structure of finitely additive measures is (Dubins 1969). Sustained use of finitely additive probability is illustrated in (Dubins and Savage 1965).