Modeling a Class of Partially Exchangeable Sequences of Markov Chains via Dice Processes

Imanol Nuñez Morales imanol.nunez@cimat.mx
Centro de Investigación en Matemáticas, A.C.

Joint work with Adrián González-Casanova, Noemi Kurt & José-Luis Pérez
XV Symposium on Probability and Stochastic Processes

Motivation

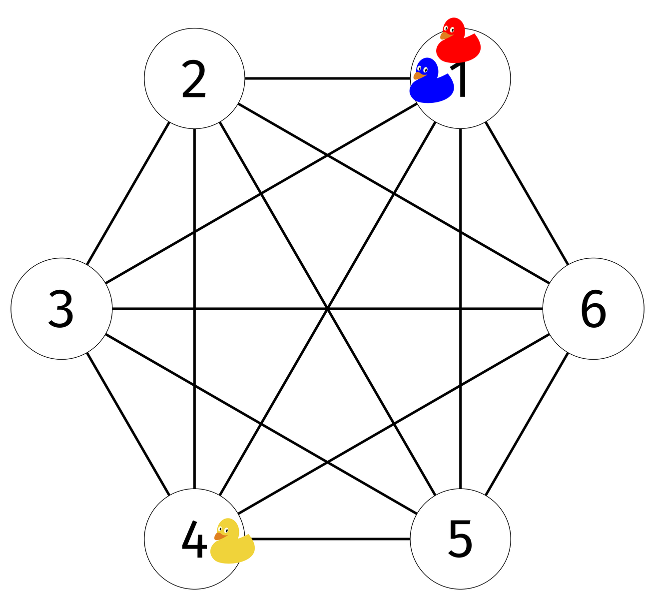
Dice processes provide a natural framework to study systems of Markov chains that are both **partially exchangeable** and **consistent**.

- They serve as a **common ground** for different probabilistic models:
  - In **population genetics**: describe genealogical models with coordinated mutation.
  - In **statistical mechanics**: recover and extend the averaging process and its variants.
- They offer a **constructive and flexible** setting:
  - Explicit rules define the evolution of a countable sequence of processes.
  - Allow for an integral representation of transition mechanisms.

Definition of Dice Processes

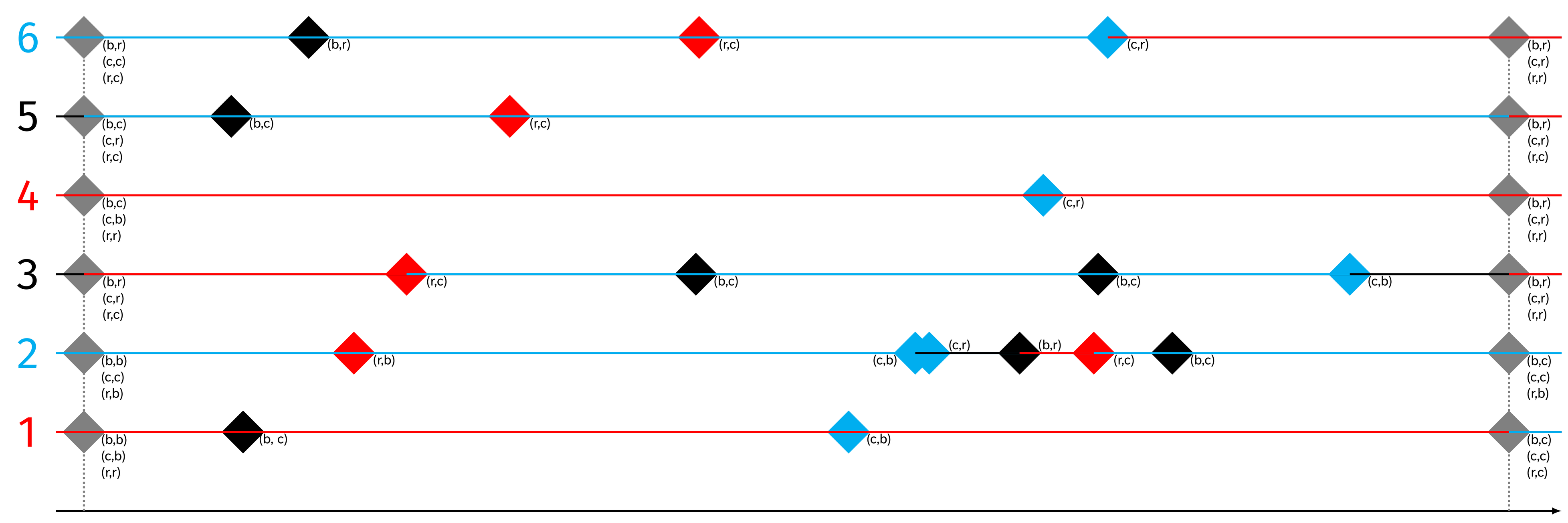
Consider an array $A = (a_{ij})_{i\in[d],j\in[d]\{i\}}$ of non-negative real numbers and a measure ν over Δ^d_{d-1} such that $\int_{\Delta^d_{d-1}} \sum_{i=1}^d (1-u_{ii})\,\nu(dU) < \infty$.
The ***n*-dice process** is defined as a $[d]^n$-valued Markov process that has transition rates

$$(1)\quad \tilde{\gamma}^{(n)}_{x,y} = \sum_{i=1}^{d} \sum_{j\in[d]\{i\}} a_{ij} 1_{\{\exists l\in[n]: y=x+(j-i)e_l\}} + \int_{\Delta^d_{d-1}} \prod_{i=1}^{d} \prod_{j=1}^{d} u_{ij}^{\#\{m\in[n]: x_m=i, y_m=j\}}\, \nu(dU).$$



The **dice process** is the **projective limit** of n-dice processes.

Graphical Construction



Characterization Theorem

Let $Y^{(\infty)} = (Y^{(\infty)}_i)_{i\in\mathbb{N}}$ be a $[d]^\infty$-valued process such that $(Y^{(\infty)}_i)_{i\in[n]}$ is a $[d]^n$-valued Markov chain over $[d]^n$ for each $n \in \mathbb{N}$.
Then $Y^{(\infty)}$ is partially exchangeable **if and only if** it is a dice process.
Moreover, $Y^{(\infty)}$ is **exchangeable** if and only if $Y^{(\infty)}(0)$ is exchangeable.

Main ideas of proof:

- Rates of projections of $Y^{(\infty)}$ are of the form (1).
- Use exchangeability and the Markov property to extend exchangeability beyond time 0.

The de Finetti Process

For an **exchangeable** dice process X, define the empirical distribution

$$R_i(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{\{X_j(t)=i\}}, \quad \text{almost surely.}$$

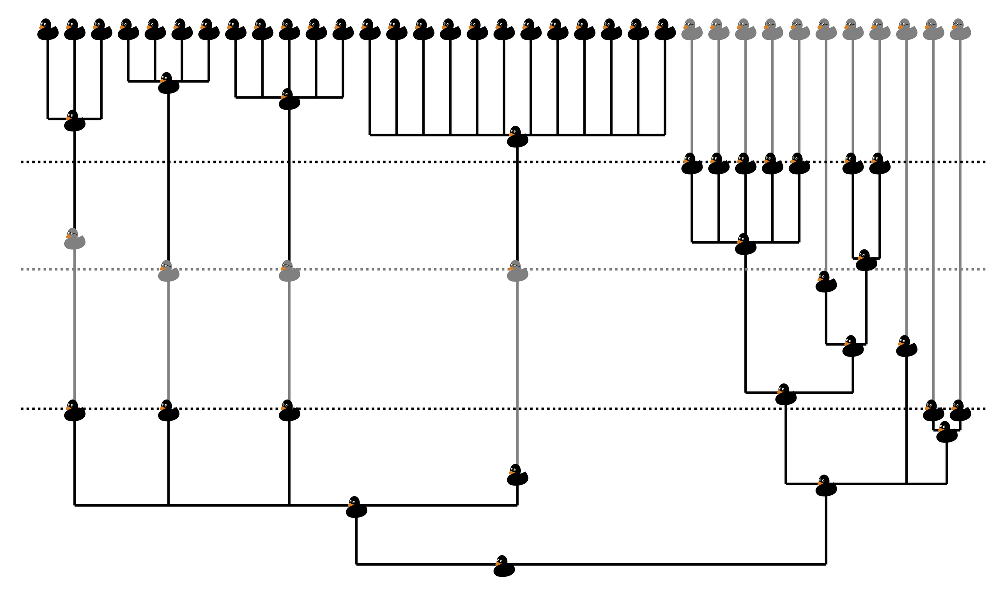
Then $R = (R_i)_{i\in[d]}$ defines a Markov process with values on Δ_{d-1} known as the **de Finetti process** of X.
Its infinitesimal generator is formally given by

$$Lf(r) = \sum_{i\in[d]} \sum_{j\in[d]\{i\}} (a_{ij} r_j - a_{ji} r_i)\partial_i f(r) + \int_{\Delta^d_{d-1}} \left(f(U^T r) - f(r)\right) \nu(dU).$$

Notation

- For $d \in \mathbb{N}$, $[d] := \{1, \dots, d\}$
- $\Delta_{d-1} := \{x \in \mathbb{R}^d_+ : |x| = 1\}$ is the d-dimensional simplex
- Δ^d_{d-1} is the set of $d \times d$ stochastic matrices
- A permutation $\sigma : \mathbb{N} \to \mathbb{N}$ is finite if there exists $j_\sigma \in \mathbb{N}$ such that $\sigma(i) = i$ for all $i \geq j_\sigma$

Coalescents with Multiple Switches



Example of a seedbank coalescent with multiple switches

Partial Exchangeability

A sequence $X = (X_i)_{i\in\mathbb{N}}$ of stochastic processes with values in $[d]$ is **partially exchangeable**, with respect to the initial condition, if for any given $x \in [d]^\infty$,

$$X \stackrel{d}{=} X_\sigma \text{ for every } \sigma \in \mathrm{Perm}_x.$$

- $X_\sigma = (X_{\sigma(i)})_{i\in\mathbb{N}}$
- Let $B^x_i = \{n \in \mathbb{N} : x_n = i\}$ for each $i \in [d]$
- $\sigma \in \mathrm{Perm}_x$ if $\sigma : \mathbb{N} \to \mathbb{N}$ is a finite permutation such that $\sigma[B^x_i] = B^x_i$ for all $i \in [d]$

Consistency

A collection of stochastic processes $(X^{(n)})_{n\in\mathbb{N}}$, where $X^{(n)} = (X^{(n)}_i)_{i\in[n]}$ is $[d]^n$-valued, is **consistent** if

$$X^{(m)} \stackrel{d}{=} (X^{(n)}_i)_{i\in[m]} \text{ for all } m \leq n.$$

Stochastic Exchange Models

A Markov Process $R = (R_i)_{i\in[d]}$ with values in Δ_{d-1} is a stochastic exchange model if at "rate" $\beta_{ij}(du, dv)$, with $i \neq j$,

$$R_i(t) \text{ and } R_j(t)$$

are replaced, respectively, by

$$uR_i(t) + (1-v)R_j(t) \text{ and } (1-u)R_i(t) + vR_j(t).$$

- β_{ij} is a measure over $[0,1]^2$ such that $\int_{[0,1]^2} ((1-u)+(1-v))\beta_{ij}(du, dv) < \infty$.

References & Simulations

Ciencia y Tecnología
Secretaría de Ciencia, Humanidades, Tecnología e Innovación

CIMAT
Centro de Investigación en Matemáticas, A.C.