A frequency process associated to the multi-type Λ-coalescent

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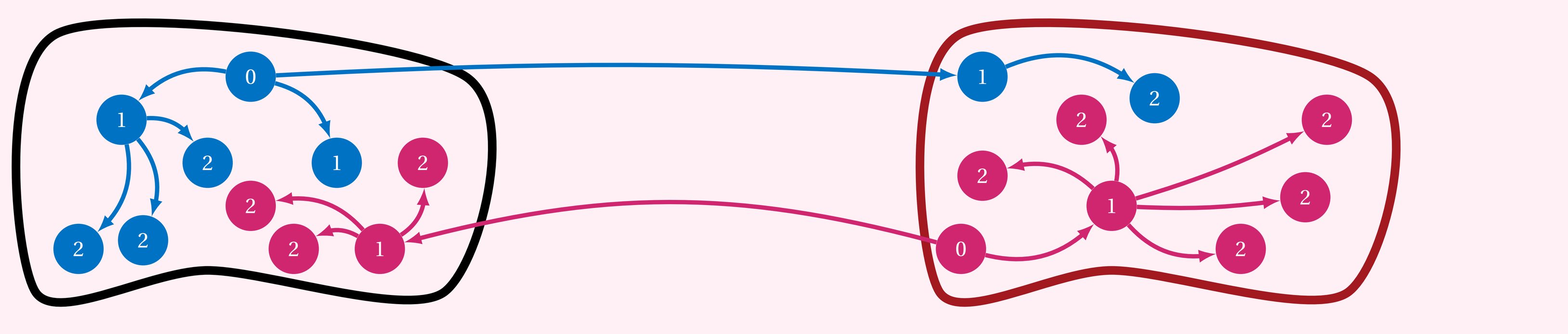
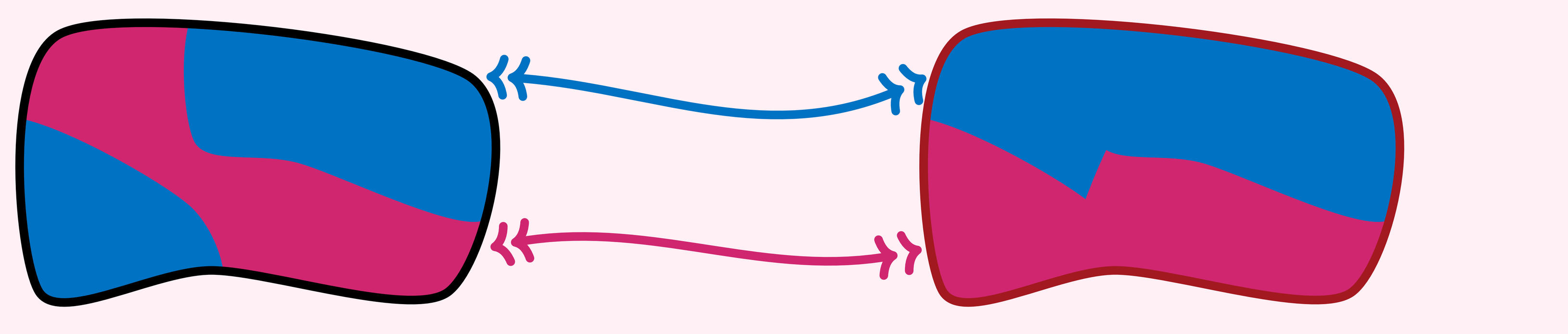
A branching model on islands

Microscopic scale



Macroscopic scale



The dynamics of the branching processes

Formally, we consider two independent and identically distributed d-type branching processes X and Y with infinitesimal generator

$$\mathcal{L}f(x) = \left\langle \beta + Bx, \nabla f(x) \right\rangle + \sum_{i=1}^{d} c_i x_i \partial_{ii} f(x) + \sum_{i=1}^{d} x_i \int_{\mathbb{R}_+^d \setminus \{0\}} \Big(f(x+w) - f(x) - \xi_i(w) \partial_i f(x) \Big) \mu_i(\mathrm{d}w) + \int_{\mathbb{R}_+^d \setminus \{0\}} \Big(f(x+w) - f(x) \Big) \nu(\mathrm{d}w).$$

Frequency process

Goal

Objective: to study the frequency process $R = (R^{(1)}, \dots, R^{(d)})$ associated to individuals of species X, i.e.

$$R_t^{(i)} = \frac{X_t^{(i)}}{X_t^{(i)} + Y_t^{(i)}}, \quad t \geqslant 0, \quad i \in [d].$$

Problem: This process is NOT Markovian.

Question: How do we recover the dynamic of R with a Markov process under the assumption that the total population is constant?

Answer: We use the Markov process (R, Z) where

$$Z_t^{(i)} = X_t^{(i)} + Y_t^{(i)}, \quad t \geqslant 0,$$

and the *culling* procedure.

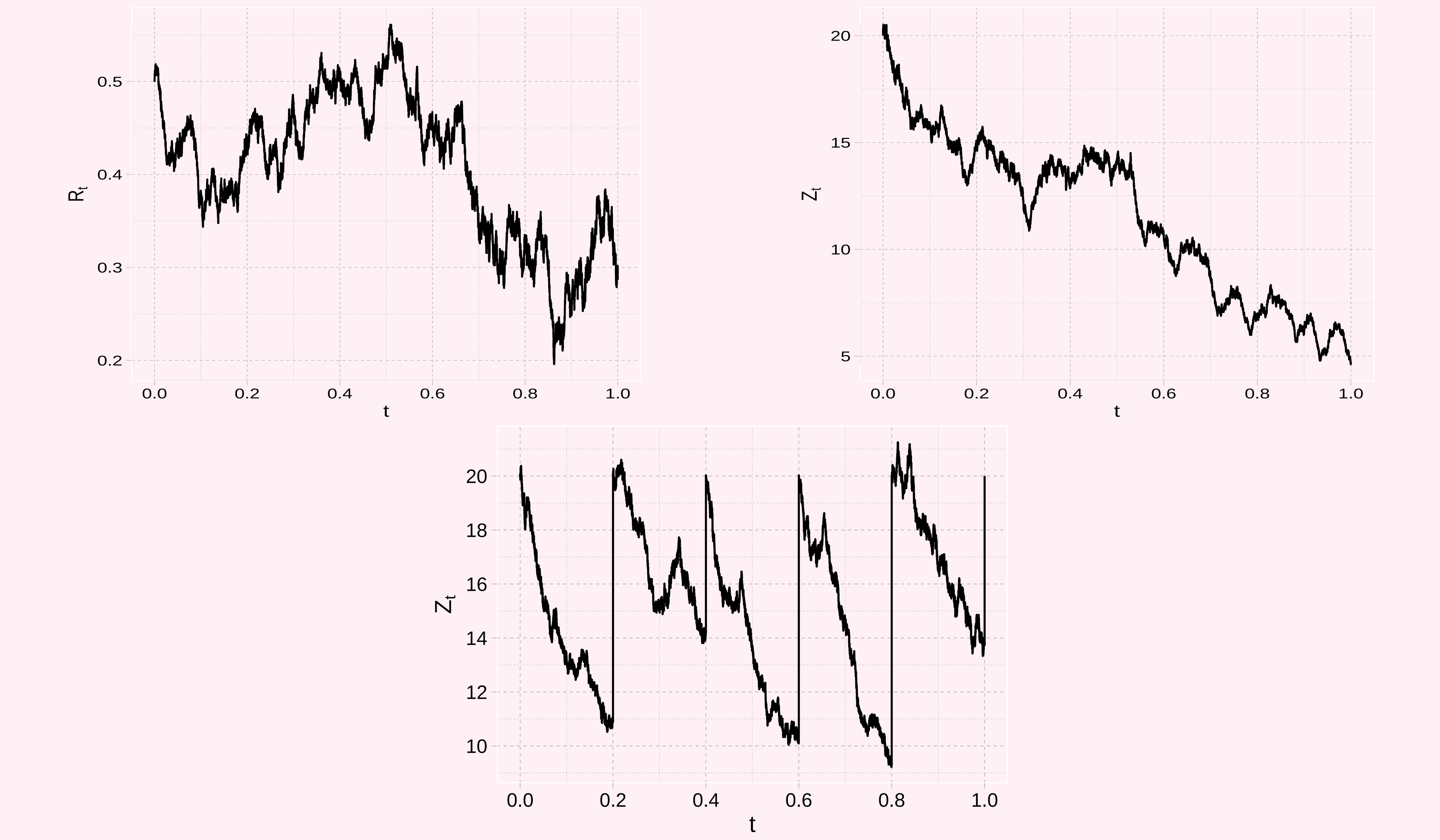
The target infinitesimal generator

By Itô's formula we have, for any $f \in C^2([0,1]^d)$,

$$\begin{aligned} \mathcal{A}^{(z)} f(r) = {} & \sum_{i=1}^{d} \left(\frac{\beta_i}{z_i}(1 - 2r_i) + \sum_{j \in [d] \setminus \{i\}} \left(b_{ij} \frac{z_j}{z_i} + z_j \int_{[0,1]^d} u_i \mathbf{T}_z \mu_j(\mathrm{d}u) \right)(r_j - r_i) \right) \partial_i f(r) + \sum_{i=1}^{d} \frac{c_i}{z_i} r_i (1 - r_i) \partial_{ii} f(r) \\ & + \sum_{i=1}^{d} z_i r_i \int_{[0,1]^d} \Big(f(r + (1-r) \odot u) - f(r) - \left\langle (1-r) \odot u, \nabla f(r) \right\rangle \Big) \mathbf{T}_z \mu_i(\mathrm{d}u) \\ & + \sum_{i=1}^{d} z_i (1 - r_i) \int_{[0,1]^d} \Big(f(r - r \odot u) - f(r) + \left\langle r \odot u, \nabla f(r) \right\rangle \Big) \mathbf{T}_z \mu_i(\mathrm{d}u) \\ & + \int_{[0,1]^d} \Big(f(r + (1-r) \odot u) + f(r - r \odot u) - 2f(r) \Big) \mathbf{T}_z \nu(\mathrm{d}u). \end{aligned} \tag{$\star$}$$

The culling method

Evolution of R and Z



Theorem 1

For any $z \in (0, \infty)^d$ there exists a Markov process $\overline{R}$ that is the unique strong solution of the SDE

$$\begin{aligned} \mathrm{d}\overline{R}_t^{(i)} = {} & \left(\frac{\beta_i}{z_i}(1 - 2\overline{R}_t^{(i)}) + \sum_{j \in [d] \setminus \{i\}} \left(b_{ij} \frac{z_j}{z_i} + z_j \int_{[0,1]^d} u_i \mathbf{T}_z \mu_j(du) \right) (\overline{R}_t^{(j)} - \overline{R}_t^{(i)}) \right) \mathrm{d}t \\ & + \sqrt{\frac{2c_i}{z_i} \overline{R}_t^{(i)} (1 - \overline{R}_t^{(i)})} \, \mathrm{d}B_t^{(i)} \\ & + \sum_{j=1}^{d} \int_{[0,1]^d} \int_0^\infty u_i (1 - \overline{R}_{t-}^{(i)}) 1_{\{v \leqslant z_i \overline{R}_{t-}^{(j)}\}} \widetilde{N}_1^j(\mathrm{d}t, \mathrm{d}u, \mathrm{d}v) \\ & + \sum_{j=1}^{d} \int_{[0,1]^d} \int_0^\infty (-u_i \overline{R}_{t-}^{(i)}) 1_{\{v \leqslant z_i (1 - \overline{R}_{t-}^{(j)})\}} \widetilde{N}_2^j(\mathrm{d}t, \mathrm{d}u, \mathrm{d}v) \\ & + \int_{[0,1]^d} u_i (1 - \overline{R}_{t-}^{(i)}) N_1^I(\mathrm{d}t, \mathrm{d}u) - \int_{[0,1]^d} u_i \overline{R}_{t-}^{(i)} N_2^I(\mathrm{d}t, \mathrm{d}u), \end{aligned}$$

for $i \in [d]$, and whose infinitesimal generator is given by ($\star$). Moreover, $\overline{R}$ is the weak limit in $D([0,T],[0,1]^d)$ of a sequence of processes generated by culling.

Main result: Multi-type Λ-coalescent block counting process

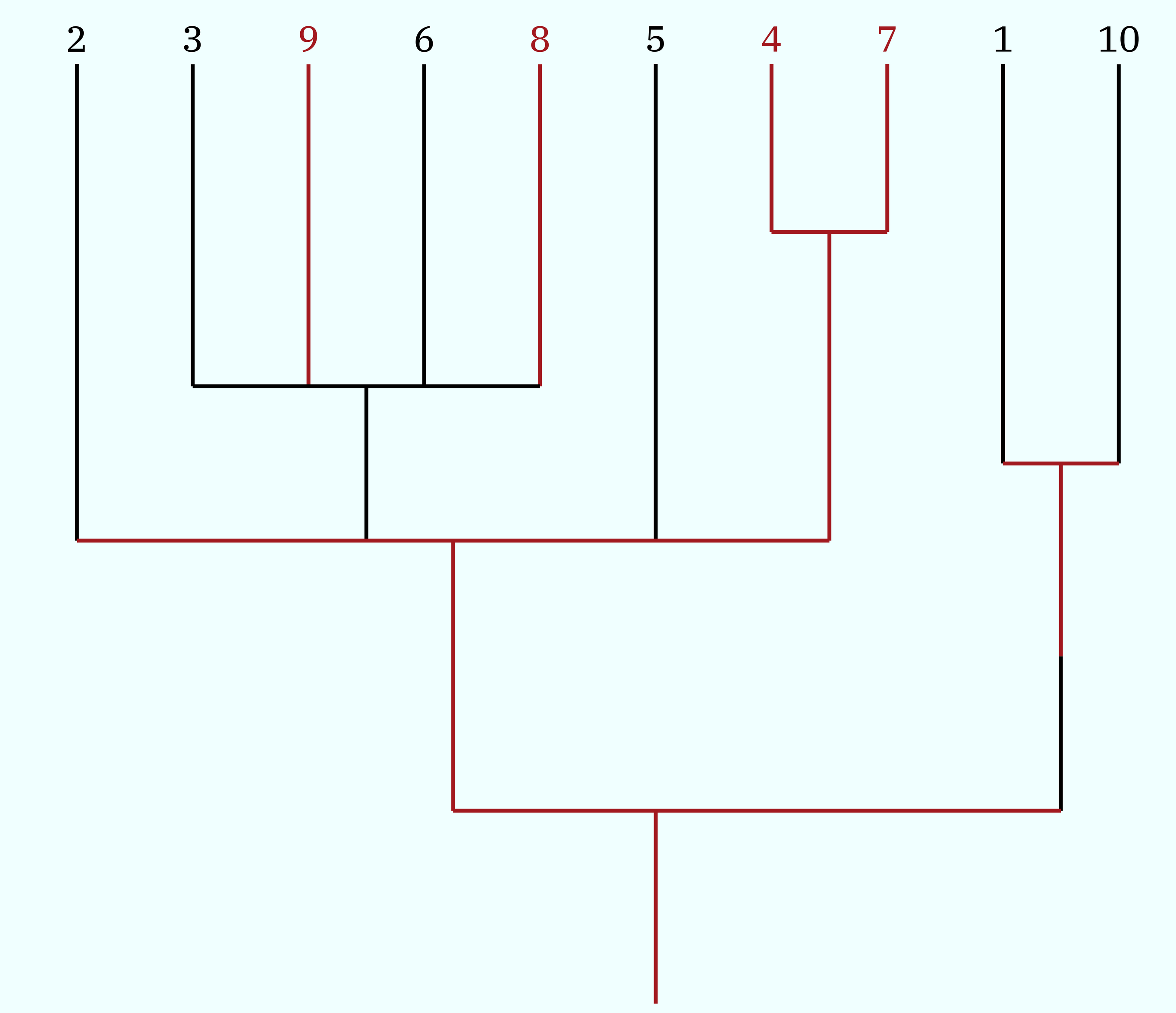
Theorem 2: Moment duality for $\overline{\mathrm{R}}$

For any $r \in [0,1]^d$, every $n \in \mathbb{N}_0^d \cup \{\Delta\}$ and any $t > 0$ we have

$$\mathbb{E}_r\Big[\prod_{i=1}^{d} (\overline{R}_t^{(i)})^{n_i} \Big] = \mathbb{E}_n\Big[\prod_{i=1}^{d} r_i^{N_t^{(i)}} \Big],$$

where N is a Markov process on $\mathbb{N}_0^d \cup \{\Delta\}$ and transition rates

$$q_{nm} = \begin{cases} n_i \frac{\beta_i}{z_i} + 2\frac{c_i}{z_i} \binom{n_i}{2} & \text{if } n \in \mathbb{N}_0^d \text{ and } m = n - e_i, \\ n_i b_{ij} \frac{z_j}{z_i} & \text{if } n \in \mathbb{N}_0^d \text{ and } m = n - e_i + e_j, \\ z_i \int_{[0,1]^d} \prod_{j=1}^{d} \binom{n_j}{k_j} u_j^{k_j} (1 - u_j)^{n_j - k_j} \mathbf{T}_z \mu_i(\mathrm{d}u) & \text{if } n \in \mathbb{N}_0^d \text{ and } m = n - k + e_i, \\ \int_{[0,1]^d} \prod_{j=1}^{d} \binom{n_j}{k_j} u_j^{k_j} (1 - u_j)^{n_j - k_j} \mathbf{T}_z \nu(\mathrm{d}u) & \text{if } n \in \mathbb{N}_0^d \text{ and } m = n - k, \\ \sum_{i=1}^{d} n_i \frac{\beta_i}{z_i} + \int_{[0,1]^d} \Big(1 - \prod_{j=1}^{d} (1 - u_j)^{n_j} \Big) \mathbf{T}_z \nu(\mathrm{d}u) & \text{if } n \in \mathbb{N}_0^d \text{ and } m = \Delta, \\ 0 & \text{otherwise.} \end{cases}$$



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