Ratio process of multi-type branching processes with immigration

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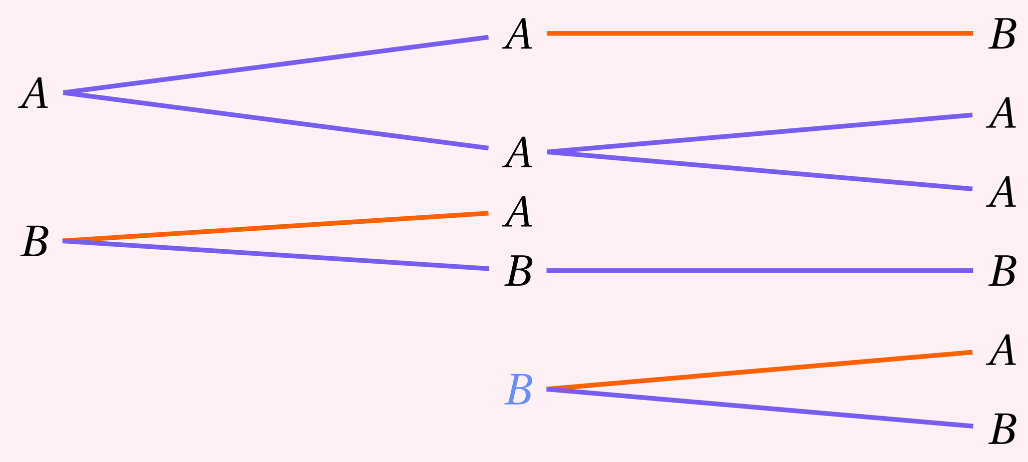
Two-type branching processes

Discrete case



Continuous case

Let $X^{(1)}$ and $X^{(2)}$ be the solution to the SDE:

$$X_t^{(i)} = x^{(i)} + \int_0^t \left(\eta_i + b_{i1}(X_s^{(1)}) + b_{i2}(X_s^{(2)}) + X_s^{(i)} \int_{U_2} (z_i - 1)^+ \mu_i(\mathrm{d}z) \right) \mathrm{d}s + \int_0^t \sqrt{2c^i X_{s-}^{(i)}}\, \mathrm{d}B_s^{(i)}$$
$$+ \int_0^t \int_{U_2} \int_0^\infty z_i \mathbb{1}_{\{u \leqslant X_{s-}^{(i)}\}} \widetilde{N}_R^i(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{U_2} \int_0^\infty z_i \mathbb{1}_{\{u \leqslant X_{s-}^{(j)}\}} N_R^j(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{U_2} z_i\, N_I(\mathrm{d}s, \mathrm{d}z),$$

where $i \neq j$, $i, j \in \{1, 2\}$.

Continuous branching — Same-type discontinous branching

Inter-type discontinous branching — External immigration

Frequency and total population

Objective: to study the frequency process R associated to individuals of type 1, i.e.

$$R_t = \frac{X_t^{(1)}}{X_t^{(1)} + X_t^{(2)}}, \quad t \geqslant 0.$$

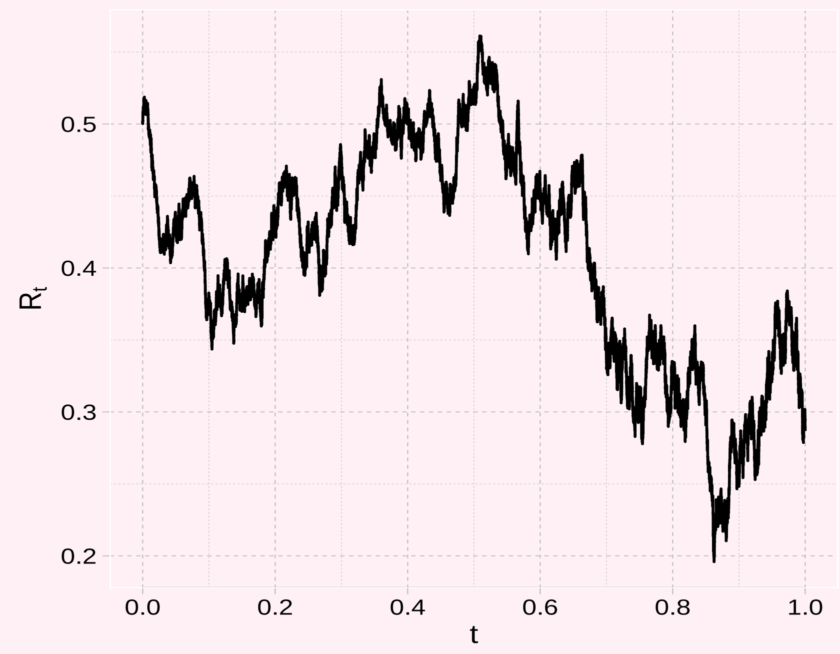
Problem: This process is NOT Markovian.

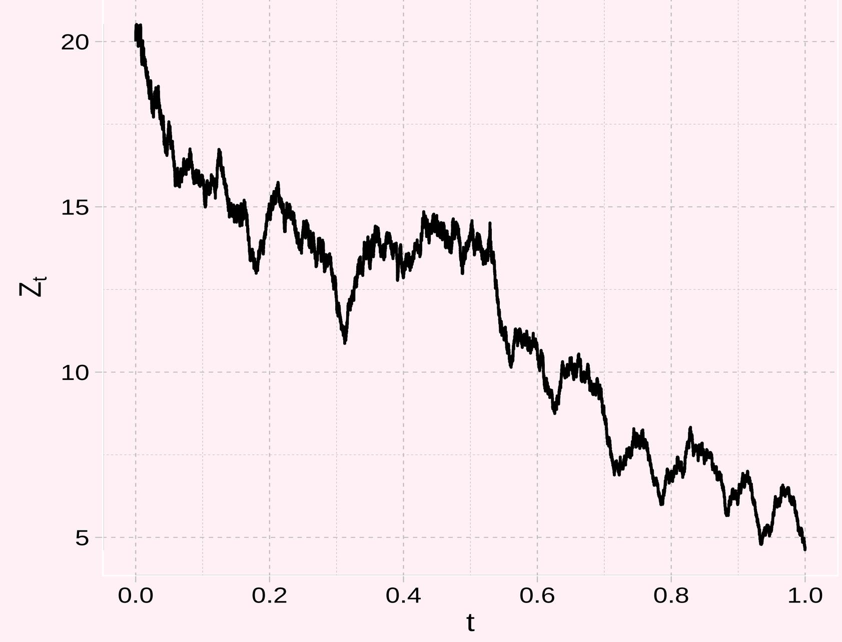
Question: How do we recover the dynamic of R with a Markov process under the assumption that the total population is constant?

Answer: We use the bi-dimensional process (R, Z) where

$$Z_t = X_t^{(1)} + X_t^{(2)}, \quad t \geqslant 0,$$

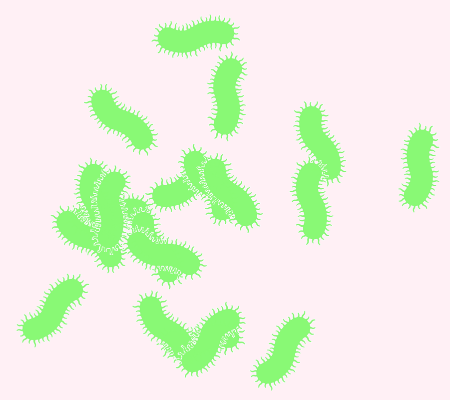
and the *culling* procedure.



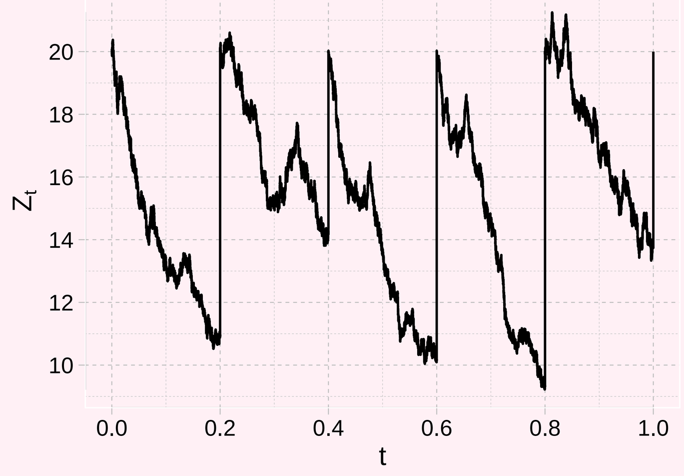


Culling procedure

Lenski's experiment



- Set a population level $z > 0$ and let (R, Z) evolve over a deterministic period of time t_n, $n \in \mathbb{N}$.
- At time t_n set $Z_{t_n} = z$ and log the value R_{t_n}. "Restart" the process (R, Z) at (R_{t_n}, z).
- Logged frequencies will be the values of a pure jump Markov process that jumps at rate n.
- If $t_n \downarrow 0$ as $n \to \infty$, the aforementioned Markov processes will converge to another Markov process $R^{(z,r)}$, which is also Feller, and satisfies certain SDE.



Main result: Large population limits

- What happens to the process obtained through culling when $z \to \infty$?
- This a natural concern when we want to approximate $R^{(z,r)}$ whenever z is large.
- Under certain hypotheses over

$$\beta_{ij}(z) = \lim_{z \to \infty} \frac{b_{ij}^z(rz)}{z},$$

it was proved that for any given $T > 0$,

$$\lim_{z \to \infty} \mathbb{E}\left[\sup_{t \leqslant T} |R_t^{(z,r)} - R_t^{(\infty,r)}|^2 \right] = 0,$$ where $R^{(\infty,r)}$ solves the ODE given by $R_0^{(\infty,r)} = r$ and

$$\begin{aligned} \mathrm{d}R_t^{(\infty,r)} = {} & \beta_{11}(R_t^{(\infty,r)})(1 - R_t^{(\infty,r)})\, \mathrm{d}t - \beta_{22}(1 - R_t^{(\infty,r)}) R_t^{(\infty,r)}\, \mathrm{d}t && \text{Same-type continuous branching} \\ & + \beta_{12}(1 - R_t^{(\infty,r)})(1 - R_t^{(\infty,r)})\, \mathrm{d}t - \beta_{21}(R_t^{(\infty,r)}) R_t^{(\infty,r)}\, \mathrm{d}t && \text{Inter-type continuous branching} \\ & + R_t^{(\infty,r)}(1 - R_t^{(\infty,r)}) \left(\int_{U_2} (w_1 - 1)^+ \mu_1(\mathrm{d}w) - \int_{U_2} (w_2 - 1)^+ \mu_2(\mathrm{d}w) \right) \mathrm{d}t && \text{Same-type discontinous branching} \\ & - (R_t^{(\infty,r)})^2 \int_{U_2} w_2\, \mu_1(\mathrm{d}w)\, \mathrm{d}t + (1 - R_t^{(\infty,r)})^2 \int_{U_2} w_1\, \mu_2(\mathrm{d}w)\, \mathrm{d}t. && \text{Inter-type discontinous branching} \end{aligned}$$

There are NO terms due to immigration.

- Interest: to know and classify the equilibria in the system, as they might represent coexistence between different types of individuals.

Particular case: linear coefficients, classic selection and mutation

Taking

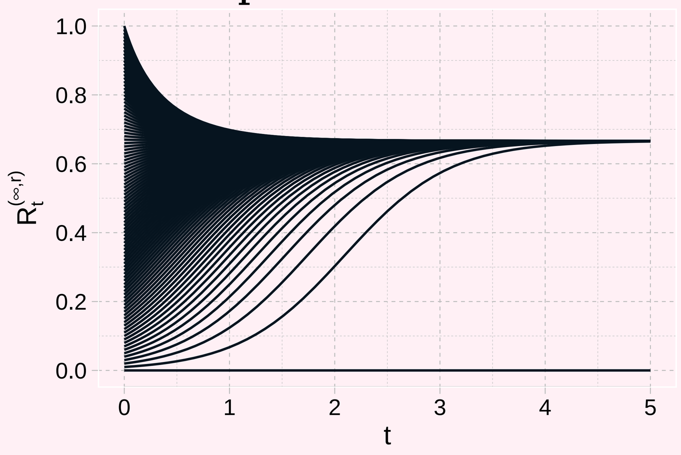
$$\beta_{ij}(z) = a_{ij} z,$$

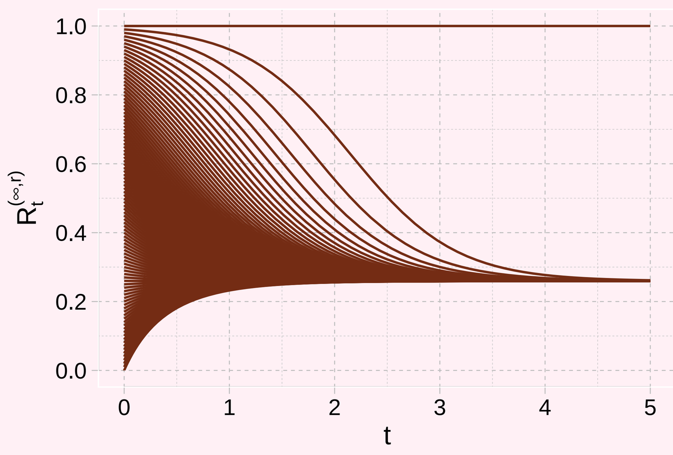
we get

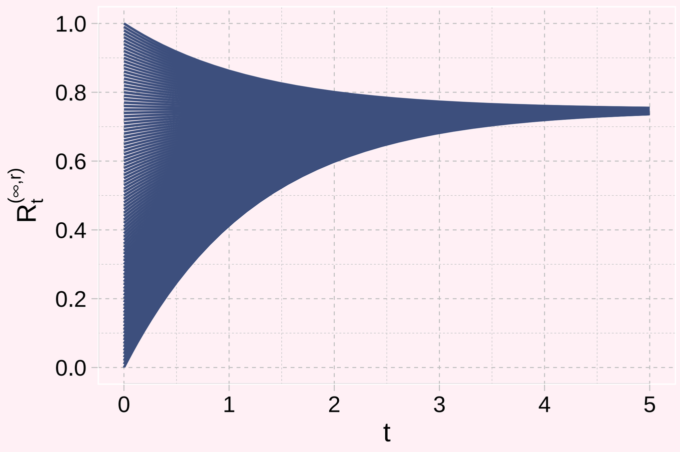
$$\mathrm{d}R_t^{(\infty,r)} = (d_1 - d_2 + d_3) R_t^{(\infty,r)} (1 - R_t^{(\infty,r)})\, \mathrm{d}t + d_2 (1 - R_t^{(\infty,r)})\, \mathrm{d}t - d_3 R_t^{(\infty,r)}\, \mathrm{d}t$$

for some constants d_1, d_2, and d_3.

Some examples:







Particular case: quadratic coefficients, classic and balancing selection

Taking

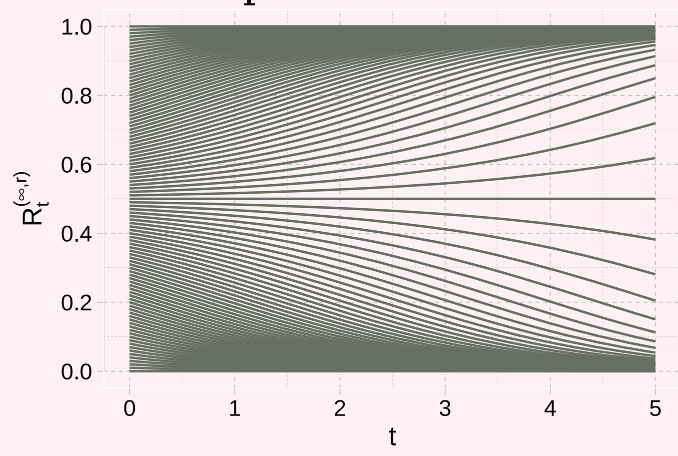
$$\beta_{ii}(z) = c_{ii} z^2 + a_{ii} z \quad \text{and, if } i \neq j, \quad \beta_{ij} = 0, \quad \text{and} \quad \int_{U_2} w_i\, \mu_j(\mathrm{d}w) = 0,$$

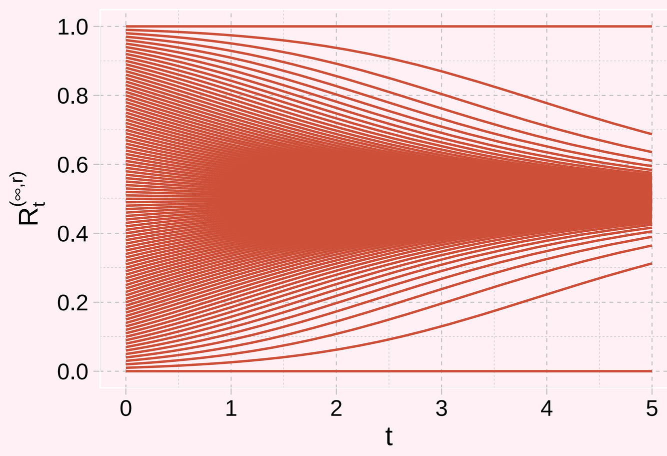
we get

$$\mathrm{d}R_t^{(\infty,r)} = \gamma_1 R_t^{(\infty,r)} (1 - R_t^{(\infty,r)}) \left(R_t^{(\infty,r)} - \frac{\gamma_2}{\gamma_1} \right) \mathrm{d}t,$$

for certain constants γ_1 and γ_2.

Some examples:





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