MATH1851 Part 2: Ordinary Differential Equations

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1 Integrating Factor

Given an equation of the form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

The integrating factor is given by:

$$I = e^{\int p(x)\,dx}$$

Multiplying the entire equation by I gives $I\frac{dy}{dx} + Ip(x)y = Iq(x)$, which can be used to solve the ODE:

$$\frac{d}{dx}(Iy) = Iq(x) \tag{1}$$

$$Iy = \int Iq(x)\,dx + C \tag{2}$$

$$y = \frac{1}{I}\left(\int Iq(x)\,dx + C\right) \tag{3}$$

2 Homogeneous Linear ODE

Given an equation of the form $\frac{dy}{dx} + p(x)y = 0$

Let $w = \frac{y}{x}$, and substitute $y = wx \implies \frac{dy}{dx} = x\frac{dw}{dx} + w$ into the equation, the equation can then be turned into a solvable separable linear ODE.

3 Bernoulli's equation

A Bernoulli equation is of the form:

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

where $n \neq 0, 1$. Use the substitution:

$$u = y^{1-n} \iff y = u^{\frac{1}{1-n}}$$

The equation will then become a solvable ODE via integrating factor.

4 Exact ODE

An equation of the form:

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the equation is exact, then:

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N \tag{4}$$

$$F = \int M \, dx + g(y) \tag{5}$$

$$F = \int N \, dy + h(x) \tag{6}$$

where $F(x, y)$ is the solution to the ODE. To find $g(y)$ after finding $h(x)$:

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} \left(\int M \, dx + g(y) \right) \tag{7}$$

$$\therefore \frac{\partial F}{\partial y} = N \tag{8}$$

5 Cauchy-Euler equations

A Cauchy-Euler equation is of the form:

$$x^2 \frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$$

where a and b are constants and there are no other constants.

Let $y = x^\lambda$, then solve it as a quadratic equation of sorts.

General solution if:

$$\lambda_1 \neq \lambda_2 \implies y = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$$
$$\lambda_1 = \lambda_2 \implies y = C_1 x^{\lambda_1} + C_2 x^{\lambda_1} \ln(x)$$

If λ_1 and λ_2 are complex, then the general solution is:

$$y = x^\alpha \left(C_1 \cos(\beta \ln(x)) + C_2 \sin(\beta \ln(x)) \right)$$

where $\lambda_{1,2} = \alpha \pm i\beta$.

6 Second order linear equations

A second order linear equation is of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where a, b, and c are constants. Substitute $y = e^{\lambda x}$, and solve it as a normal quadratic equation, since $e^{\lambda x} \neq 0$.

The general solution if:

$$\lambda_1 \neq \lambda_2 \implies y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

$$\lambda_1 = \lambda_2 \implies y = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}$$

If λ_1 and λ_2 are complex, then the general solution is:

$$y = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right)$$

where $\lambda_{1,2} = \alpha \pm i\beta$.

6.1 Non-homogeneous second order linear equations via undetermined coefficients

Guesses list (table):

Form of $g(x)$	Guess
e^{ax}	Ae^{ax}
$\sin(bx)$	$A\sin(bx) + B\cos(bx)$
$\cos(bx)$	$A\sin(bx) + B\cos(bx)$
x^n	$Ax^n + Bx^{n-1} + Cx^{n-2} + ...$

If the guess is a solution to the homogeneous equation, then multiply the guess by x^k where k is the smallest integer such that x^k times the guess is not a solution to the homogeneous equation.

6.2 D-Operator

D-operator is defined as $D = \frac{d}{dx}$, where x is the variable.

If RHS is $f(x)$, then the D-operator to use is:

$$A_0x^n + A_1x^{n-1} + ... + A_n \implies L = D^{n+1}$$

$$A_0x^n e^{ax} + A_1x^{n-1}e^{ax} + ... + A_n e^{ax} \implies L = (D-a)^{n+1}$$

$$A_0x^n e^{ax}\sin(bx) + A_0x^n e^{ax}\cos(bx) + ... + A_n e^{ax}\cos(bx) \implies L = ((D-a)^2 + b^2)^{n+1}$$

7 Riccati's equation

A Riccati equation is of the form:

$$\frac{dy}{dx} = p(x)y^2 + q(x)y + r(x)$$

where $p(x)$, $q(x)$, and $r(x)$ are functions of x.

Do an educated guess to find a solution $u(x)$, and substitute it to:

$$Y(x) = u + \frac{1}{v}$$

where $v(x)$ is a function of x. The equation will then become a linear ODE.

A shortcut to use is:

$$\frac{dv}{dx} + (2p(x)u(x) + q(x))v = -p(x)$$

8 When to use each method

Given a differential equation consisting of the function $f(x) = y$, and one variable x:

If $f(x)$ is a *first order* differential equation:

- Use **Integrating Factor** if the equation is linear (highest order of y is 1).
- Use **Homogeneous Linear ODE** if the equation is homogeneous.
- Use **Bernoulli's equation** if the equation is of the form $\frac{dy}{dx} + p(x)y = q(x)y^n$.
- Use **Riccati's equation** if the equation is of the form $\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2$.
- Use **Exact ODE** if the equation is exact.

If $f(x)$ is a *higher order* differential equation:

- Use **Cauchy-Euler equations** if the equation **contains another variable** x and is of the form $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$.
- Use **Second order linear equations** if the equation is of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where a, b, and c are constants.

9 Variation of Parameters

Given two known solutions y_1 and y_2, the general solutions can be found with: Here is the proof

10 Extras

10.1 Riccati shortcut proof

Given a differential equation $\frac{dy}{dx} = p(x)y^2 + q(x)y + r(x)$, and $u(x)$ is a known solution: Create a substitution $Y(x) = u(x) + \frac{1}{v(x)}$:

$$Y(x) = u + \frac{1}{v} \implies \frac{dY}{dx} = \frac{du}{dx} - \frac{1}{v^2}\frac{dv}{dx} \tag{9}$$

$$\frac{du}{dx} - \frac{1}{v^2}\frac{dv}{dx} = p\left(u^2 + \frac{2u}{v} + \frac{1}{v^2}\right) + q\left(u + \frac{1}{v}\right) + r \tag{10}$$

$$\frac{du}{dx}v^2 - \frac{dv}{dx} = (pu^2 + qu + r)v^2 + (2pu + q)v + p \tag{11}$$

$$\frac{dv}{dx} + (2pu + q)v = (\frac{du}{dx} - (pu^2 + qu + r))v^2 - p \tag{12}$$

$$\therefore \frac{dv}{dx} + (2pu + q)v = -p \tag{13}$$

Applying these principles to our problem: Let $F(s) = \frac{1}{s^2+b^2}$, so $f(t) = \frac{1}{b}\sin(bt)$ Then, using the time-shift property with $a = 2$: