Efficient Bayesian Response Surface Maximization

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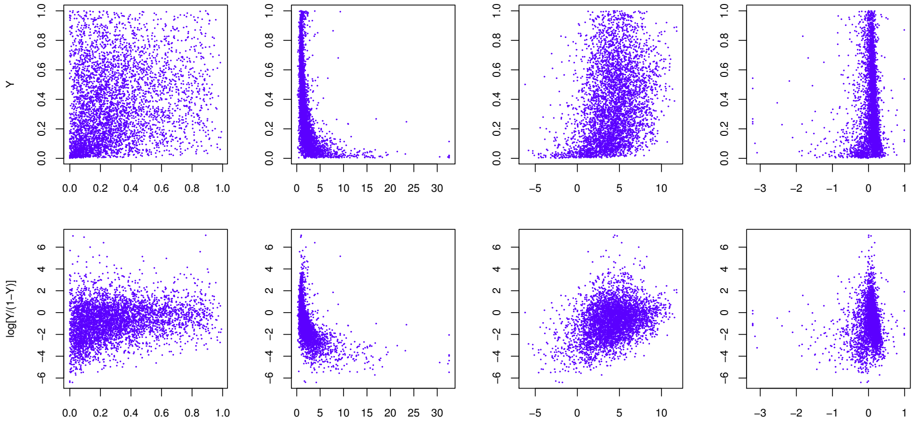
Outline

The firm leverage data

leverage (Y):	total debt/(total debt+book value of equity), 4405 observations;
tang:	tangible assets/book value of total assets;
market2book:	(book value of total assets - book value of equity + market value of equity) / book value of total assets;
logSales:	logarithm of sales;
profit:	(earnings before interest, taxes, depreciation, and amortization) / book value of total assets.



Our interests

- Find an optimal combination of **tang**, **market2book**, **logSales**, and **profis** so that **leverage** reaches the maximum.
- We may write it down in mathematics

$$\arg\max_{x \in R^d} \; f(x)$$

  where x is defined as sample and $y = f(x) + \epsilon$ with ϵ as the noisy observation of the objective function at x.
- But note that
    - The function $f(x)$ is **unknown**, **not noise-free** and **hard to evaluate**.
    - We do not know its derivatives. Common optimization methods usually fail here.
    - The covariates space can be very sparse.

Bayesian Optimization I

1. Assume $D_{1:t} = \boldsymbol{x}_{1:t}, \boldsymbol{y}_{1:t}$ are observations, a prior distribution $P(f)$ over function $f(\cdot)$ is combined with the likelihood function $P(D_{1:t}|f)$ to produce the posterior distribution

$$P(f|D_{1:t}) \propto P(D_{1:t}|f)P(f).$$

2. Bayesian optimization it to find $\boldsymbol{x}_{t+1}$

$$\boldsymbol{x}_{t+1} = \arg\max_{\boldsymbol{x} \in R^d} \int a(\boldsymbol{x}) P(f_{t+1}|D_{1:t}) df_{t+1} \tag{1}$$

where $a(x)$ is called **acquisition function** that guides the search for the optimum that high acquisition corresponds to potentially high values of the objective function.

Bayesian Optimization II

- Common acquisition functions include **probability of improvement**, **expected improvement** and **entropy** (Kushner, 1964; Mockus et al., 1978; Jones, 2001; Cox and John, 1997; Brochu et al., 2010; Villemonteix et al., 2009)
- Recent work in Bayesian Optimization like Jones et al. (1998) Jones (2001), Bergstra and Bengio (2012) can trace back to Cox and John (1997).
- Bayesian Optimization is a popular approach in engineering but not well known in statistics.

Bayesian Optimization III

- We are interested in finding the maximum for the predictive surface

$$p(\tilde{y}_b|\tilde{y}_{-b}, x) = \int \prod_{i \in \mathcal{T}_b} p(y_i|\theta, x_i)p(\theta|\tilde{y}_{-b})d\theta,$$

- However in econometric time series this is much more complicated due to the decomposition

$$\begin{aligned} & p(Y_{(T+1):(T+p)}|Y_{1:T}, X) \\ & = \prod_{i=1}^{p} \int p(Y_{T+i}|\theta, Y_{1:(T+i-1)}, X_{T+i})p(\theta|Y_{1:(T+i-1)}, X_{1:(T+i-1)})d\theta. \end{aligned}$$

Bayesian modeling of $P(f|D_{1:t})$ I

- The function $f(x)$ is usually approximated by a Gaussian Process (Mockus, 1994; Sasena, 2002)

$$f(\mathbf{x}) = \mathcal{GP}(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x}'))$$

  which is restrictive by $\mathcal{GP}$ itself because choosing the covariance function for the $\mathcal{GP}$ is crucial.
- We consider the **multivariate surface model** (Li and Villani, 2013) to model $f(x)$
  - The surface consists of three different components, *linear*, *surface* and *additive* as

$$\mathbf{Y} = \mathbf{X}_o \mathbf{B}_o + \mathbf{X}_s(\xi_s)\mathbf{B}_s + \mathbf{X}_a(\xi_a)\mathbf{B}_a + \mathbf{E}.$$

  - We treat the knots ξ_i as unknown parameters and let them move freely.

Bayesian modeling of $P(f|D_{1:t})$ II

- A model with a minimal number of free knots outperforms model with lots of fixed knots.
- For notational convenience, we sometimes write model in compact form

$$\mathbf{Y} = \mathbf{X}\mathbf{B} + \mathbf{E},$$

where $\mathbf{X} = [\mathbf{X}_o, \mathbf{X}_s, \mathbf{X}_a]$ and $\mathbf{B} = [\mathbf{B}_o{}', \mathbf{B}_s{}', \mathbf{B}_a{}']'$ and $\mathbf{E} \sim \mathbf{N}_p(\mathbf{0},\ \mathbf{\Sigma})$

The Efficient MCMC algorithm I

- The coefficients $(\mathbf{B})$ are directly sampled from normal distribution.
- We update covariance $(\boldsymbol{\Sigma})$, all knots $(\boldsymbol{\xi})$ and shrinkages $(\boldsymbol{\lambda})$ jointly by using Metropolis-Hastings within Gibbs.
- The proposal density for $\boldsymbol{\Sigma}$ is the inverse Wishart density on previous slide.
- The proposal density for $\boldsymbol{\xi}$ and $\boldsymbol{\lambda}$ is a multivariate t-density with $\nu > 2$ df,

$$\theta_p|\theta_c \sim \mathbf{MVT}\left[\hat{\theta},\ -\left(\frac{\partial^2 \ln p(\theta|\mathbf{Y})}{\partial\theta\partial\theta'}\right)^{-1}\Bigg|_{\theta=\hat{\theta}},\ \nu\right],$$

where $\hat{\theta}$ is obtained by R steps $(R \leqslant 3)$ Newton's iterations during the proposal with analytical gradients for matrices.

The Efficient MCMC algorithm II

- The Metropolis-Hastings acceptance probability is

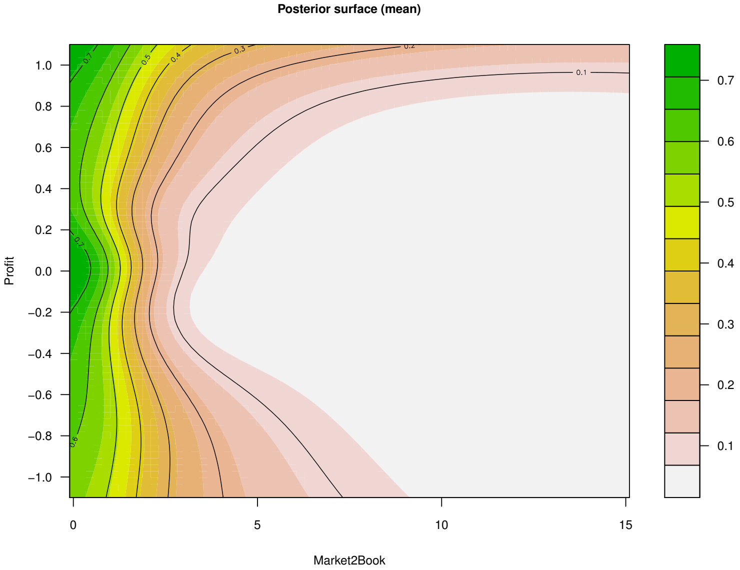
$$a\left(\theta_c \to \theta_p\right) = \min\left[1,\ \frac{p(\mathbf{Y}|\theta_p)p(\theta_p)g(\theta_c|\theta_p)}{p(\mathbf{Y}|\theta_c)p(\theta_c)g(\theta_p|\theta_c)}\right].$$

- The analytical gradients are very complicated and we have implemented it in an efficient way.
- Bayesian variable selection can be naturally applied in MCMC procedure.
- The MCMC implementations are straightforward.
- We allow the parameters to be updated via:
  - parallel mode for small datasets,
  - batched mode for big datasets.
- MCMC method allows us to evaluate the integral in Eq. (1) easily.

Optimizing the acquisition function

- With the posterior, a deterministic, derivative-free optimizer can then be used in optimizing the acquisition function (Jones et al., 1993; Mockus, 1994; Lizotte, 2008).
- By taking the advantage of MCMC, we may integrate the two steps together. *Working in progress*...

The firm leverage data, a revisit



Extensions

- Our approach can be applied experimental design.
- High dimensional response surfaces will be considered.

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Thank you!