In[237]:=
```
h[x_] := Piecewise[{{1 + c01 x + c02 x^2, 0 ≤ x ≤ 1/2}, {c11 (x - 1) + c12 (x - 1)^2, 1/2 < x ≤ 3/2}, {0, True}}];
f[x_] := h[Abs[x]];
```

In[239]:=
```
(*Continuity*)
C1l = Simplify[h[x], 0 ≤ x ≤ 1/2] /. x → 1/2
C1r = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 1/2
C2l = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 3/2
```

Out[239]= $1 + \frac{c01}{2} + \frac{c02}{4}$

Out[240]= $\frac{1}{2}\left(-c11 + \frac{c12}{2}\right)$

Out[241]= $\frac{1}{2}\left(c11 + \frac{c12}{2}\right)$

In[242]:=
```
(*Partition of unity and gradient representation*)
T0 = CoefficientList[FullSimplify[Sum[f[x - i], {i, -3, 3}], x > 0 && x < 1/2], x]
T1 = CoefficientList[FullSimplify[Sum[i f[x - i], {i, -3, 3}], x > 0 && x < 1/2], x]
```

Out[242]= $\{1, c01, c02 + 2\, c12\}$

Out[243]= $\{0, -2\, c11\}$

In[244]:=
```
GenSols = Solve[{
    C1l == C1r,
    C2l == 0,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[2]] == 1
    },
    {c01, c02, c11, c12}
]
```

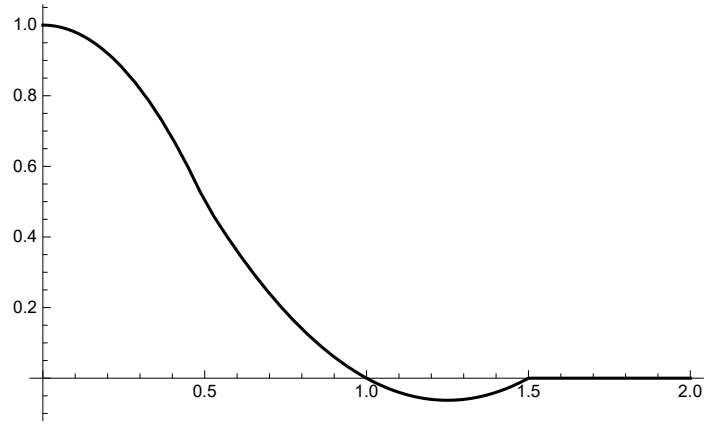
Out[244]= $\left\{\left\{c01 \to 0,\ c02 \to -2,\ c11 \to -\frac{1}{2},\ c12 \to 1\right\}\right\}$

In[245]:=
```
h[x] /. GenSols[[1]]
```

Out[245]=
$$\begin{cases} 1 - 2x^2 & 0 \le x \le \frac{1}{2} \\ \frac{1-x}{2} + (-1+x)^2 & \frac{1}{2} < x \le \frac{3}{2} \\ 0 & \text{True} \end{cases}$$

In[246]:=
```
Plot[h[x] /. GenSols[[1]], {x, 0, 2}, PlotStyle → Black]
```

Out[246]=



In[247]:=
```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
W1[k_] := Piecewise[{{0, k < 0}, {φ^2/2, k == 0}, {1 - (1 - φ)^2/2, k == 1}, {1, True}}];
SumF1 = Sum[Sum[W1[i - j] f[x - i, y - j] /. GenSol, {j, -5, 6}], {i, -5, 6}];
```

In[251]:=
```
{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6} = Parallelize[{
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 3 - 1/2 && y < 4 - 1/2]
}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6} = Parallelize[{
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -3 - 1/2 && y < -2 - 1/2]
}];
```

In[253]:=
```
TableForm[{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6}]
TableForm[{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6}]
```

Out[253]//TableForm=

$\frac{1}{4}\left(2\varphi^2+y^2\left(2+4\varphi-6\varphi^2\right)+y\left(-1-2\varphi+\varphi^2\right)+x\left(1+2\varphi+2y^2\left(-2+\varphi\right)\varphi-\varphi^2+y\left(-1+\varphi^2\right)\right)+2x^2\left(1+2\right.\right.$

$\frac{1}{8}\left(-2\left(-1+2x^2\right)\left(3-5y+2y^2\right)\varphi^2-2x\left(1+2x\right)\left(1-4y+2y^2\right)\varphi^2-x\left(1+2x\right)\left(3-5y+2y^2\right)\left(-1-2\varphi\right.\right.$

$\frac{1}{8}\left(3+5x+2x^2\right)\left(3-5y+2y^2\right)\varphi^2$

0

0

0

Out[254]//TableForm=

$\frac{1}{4}\left(-2y\varphi+\varphi\left(-4+5\varphi\right)+y^2\left(2+16\varphi-16\varphi^2\right)+x^2\left(2\varphi\left(-4+3\varphi\right)-4y^2\varphi\left(-6+5\varphi\right)-2y\left(-1+\varphi^2\right)\right)+x\left(\right.\right.$

$\frac{1}{8}\left(2-24\varphi+21\varphi^2+y^2\left(4-32\varphi+22\varphi^2\right)+y\left(4-68\varphi+49\varphi^2\right)+2x^2\left(2-16\varphi+11\varphi^2+2y^2\left(2-8\varphi+5\varphi^2\right)\right.\right.$

$\frac{1}{8}\left(9x\left(3+5y+2y^2\right)\left(-1+\varphi\right)^2-2x^2\left(3+5y+2y^2\right)\left(-1+\varphi\right)^2-2\left(11+25y\left(-1+\varphi\right)^2+10y^2\left(-1+\varphi\right)^2-\right.\right.$

1

1

1

In[255]:=
```
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1b1, DSumF1b2, DSumF1b3} = Parallelize[{
    FullSimplify[D[SumF1a1, {{x, y}}]],
    FullSimplify[D[SumF1a2, {{x, y}}]],
    FullSimplify[D[SumF1a3, {{x, y}}]],
    FullSimplify[D[SumF1b1, {{x, y}}]],
    FullSimplify[D[SumF1b2, {{x, y}}]],
    FullSimplify[D[SumF1b3, {{x, y}}]]
}];
DSumF1a1 = Simplify[DSumF1a1 /. φ → 1/2];
DSumF1a2 = Simplify[DSumF1a2 /. φ → 1/2];
DSumF1a3 = Simplify[DSumF1a3 /. φ → 1/2];
DSumF1b1 = Simplify[DSumF1b1 /. φ → 1/2];
DSumF1b2 = Simplify[DSumF1b2 /. φ → 1/2];
DSumF1b3 = Simplify[DSumF1b3 /. φ → 1/2];
```

In[262]:=
```
{Err1a1, Err1a2, Err1a3} = Parallelize[{
    Simplify[∫_(0-1/2)^(1-1/2) ∫_(0-1/2)^(1-1/2) (DSumF1a1.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_(1-1/2)^(2-1/2) ∫_(0-1/2)^(1-1/2) (DSumF1a2.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_(1-1/2)^(2-1/2) ∫_(-1-1/2)^(0-1/2) (DSumF1a3.{1, 1})^2 𝕕x 𝕕y]
}];
{Err1b1, Err1b2, Err1b3} = Parallelize[{
    Simplify[∫_(0-1/2)^(1-1/2) ∫_(1-1/2)^(2-1/2) (DSumF1b1.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_(-1-1/2)^(0-1/2) ∫_(1-1/2)^(2-1/2) (DSumF1b2.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_(-1-1/2)^(0-1/2) ∫_(2-1/2)^(3-1/2) (DSumF1b3.{1, 1})^2 𝕕x 𝕕y]
}];
```

In[264]:=
```
Err1 = FullSimplify[Err1a1 + Err1a2 + Err1a3 + Err1b1 + Err1b2 + Err1b3]
N[Err1]
```

Out[264]= $\frac{1061}{4608}$

Out[265]= 0.230252