```
h[x_] := {
  1 + c01 x + c02 x^2 + c03 x^3 + c04 x^4                 0 ≤ x ≤ 1
  c11 (x - 1) + c12 (x - 1)^2 + c13 (x - 1)^3 + c14 (x - 1)^4   1 < x ≤ 2
  c21 (x - 2) + c22 (x - 2)^2 + c23 (x - 2)^3 + c24 (x - 2)^4   2 < x ≤ 3
  0                                                       True
};

f[x_] := h[Abs[x]];

(*Interpolant constraints*)
I1 = f[1]
I2 = f[2]
I3 = f[3]
```

$$1 + c01 + c02 + c03 + c04$$

$$c11 + c12 + c13 + c14$$

$$c21 + c22 + c23 + c24$$

```
(*Partition of unity and gradient representation*)
T0 = CoefficientList[
  FullSimplify[f[x + 2] + f[x + 1] + f[x] + f[x - 1] + f[x - 2] + f[x - 3], x > 0 && x < 1], x]
T1 = CoefficientList[FullSimplify[
    -2 f[x + 2] - f[x + 1] + f[x - 1] + 2 f[x - 2] + 3 f[x - 3], x > 0 && x < 1], x]
```

$$\{2 + c01 + c02 + c03 + c04 + c11 + c12 + c13 + c14 + c21 + c22 + c23 + c24,$$
$$-2\,c02 - 3\,c03 - 4\,c04 - 2\,c12 - 3\,c13 - 4\,c14 - 2\,c22 - 3\,c23 - 4\,c24,$$
$$2\,c02 + 3\,c03 + 4\,c04 + 2\,c12 + 3\,c13 + 4\,c14 + 2\,c22 + 3\,c23 + 4\,c24 + 2\,(c04 + c14 + c24),$$
$$-4\,(c04 + c14 + c24),\ 2\,(c04 + c14 + c24)\}$$

$$\{1 + c01 + c02 + c03 + c04 + 2\,c11 + 2\,c12 + 2\,c13 + 2\,c14 + 3\,c21 + 3\,c22 + 3\,c23 + 3\,c24,$$
$$-c01 - 2\,c02 - 3\,c03 - 4\,c04 - 3\,c11 - 4\,c12 - 6\,c13 - 8\,c14 - 5\,c21 - 6\,c22 - 9\,c23 - 12\,c24,$$
$$c02 + 3\,c03 + 6\,c04 + c12 + 6\,c13 + 12\,c14 + c22 + 9\,c23 + 18\,c24,$$
$$-c03 - 4\,c04 - 3\,c13 - 8\,c14 - 5\,c23 - 12\,c24,\ c04 + c14 + c24\}$$

```
GenSols = Solve[{
    I1 == 0,
    I2 == 0,
    I3 == 0,
    T0[[1]] == 1,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[1]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0,
    T1[[4]] == 0
    },
    {c01, c02, c03, c04, c11, c12, c13, c14, c21, c22, c23, c24}
]
```

Solve::svars : Equations may not give solutions for all "solve" variables. >>

$$\left\{\left\{c04 \to -1 - c01 - c02 - c03,\ c14 \to -c11 - c12 - c13,\right.\right.$$
$$c21 \to -\frac{9}{5} - \frac{9\,c01}{5} - \frac{4\,c02}{5} - \frac{2\,c03}{5} - \frac{7\,c11}{5} - \frac{2\,c12}{5} - \frac{c13}{5},$$
$$c22 \to \frac{12}{5} + \frac{12\,c01}{5} + \frac{7\,c02}{5} + \frac{6\,c03}{5} + \frac{6\,c11}{5} + \frac{c12}{5} + \frac{3\,c13}{5},$$
$$c23 \to -\frac{8}{5} - \frac{8\,c01}{5} - \frac{8\,c02}{5} - \frac{9\,c03}{5} - \frac{4\,c11}{5} - \frac{4\,c12}{5} - \frac{7\,c13}{5},$$
$$\left.\left.c24 \to 1 + c01 + c02 + c03 + c11 + c12 + c13\right\}\right\}$$

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
φ = 1 / 2;
```

$$W1[k\_] := \begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1 - (1-\varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases};$$

```
SumF1 = Sum[Sum[W1[i - j] f[x - i, y - j] /. GenSol, {j, -5, 6}], {i, -5, 6}];
{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6} = Parallelize[{
    Simplify[SumF1, x > 0 && x < 1 && y > 0 && y < 1],
    Simplify[SumF1, x > 0 && x < 1 && y > 1 && y < 2],
    Simplify[SumF1, x > -1 && x < 0 && y > 1 && y < 2],
    Simplify[SumF1, x > -1 && x < 0 && y > 2 && y < 3],
    Simplify[SumF1, x > -2 && x < -1 && y > 2 && y < 3],
    Simplify[SumF1, x > -2 && x < -1 && y > 3 && y < 4]
}];
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4, DSumF1a5, DSumF1a6} = Parallelize[{
    FullSimplify[D[SumF1a1, {{x, y}}]],
    FullSimplify[D[SumF1a2, {{x, y}}]],
    FullSimplify[D[SumF1a3, {{x, y}}]],
    FullSimplify[D[SumF1a4, {{x, y}}]],
    FullSimplify[D[SumF1a5, {{x, y}}]],
    FullSimplify[D[SumF1a6, {{x, y}}]]
}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6} = Parallelize[{
    Simplify[SumF1, x > 1 && x < 2 && y > 0 && y < 1],
    Simplify[SumF1, x > 1 && x < 2 && y > -1 && y < 0],
    Simplify[SumF1, x > 2 && x < 3 && y > -1 && y < 0],
    Simplify[SumF1, x > 2 && x < 3 && y > -2 && y < -1],
    Simplify[SumF1, x > 3 && x < 4 && y > -2 && y < -1],
    Simplify[SumF1, x > 3 && x < 4 && y > -3 && y < -2]
}];
{DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4, DSumF1b5, DSumF1b6} = Parallelize[{
    FullSimplify[D[SumF1b1, {{x, y}}]],
    FullSimplify[D[SumF1b2, {{x, y}}]],
    FullSimplify[D[SumF1b3, {{x, y}}]],
    FullSimplify[D[SumF1b4, {{x, y}}]],
    FullSimplify[D[SumF1b5, {{x, y}}]],
    FullSimplify[D[SumF1b6, {{x, y}}]]
}];
```

```
DSumF1a1 = Simplify[DSumF1a1 /. φ → 1 / 2];
DSumF1a2 = Simplify[DSumF1a2 /. φ → 1 / 2];
DSumF1a3 = Simplify[DSumF1a3 /. φ → 1 / 2];
DSumF1a4 = Simplify[DSumF1a4 /. φ → 1 / 2];
DSumF1a5 = Simplify[DSumF1a5 /. φ → 1 / 2];
DSumF1a6 = Simplify[DSumF1a6 /. φ → 1 / 2];
DSumF1b1 = Simplify[DSumF1b1 /. φ → 1 / 2];
DSumF1b2 = Simplify[DSumF1b2 /. φ → 1 / 2];
DSumF1b3 = Simplify[DSumF1b3 /. φ → 1 / 2];
DSumF1b4 = Simplify[DSumF1b4 /. φ → 1 / 2];
DSumF1b5 = Simplify[DSumF1b5 /. φ → 1 / 2];
DSumF1b6 = Simplify[DSumF1b6 /. φ → 1 / 2];

{Err1a1, Err1a2, Err1a3, Err1a4, Err1a5, Err1a6} = Parallelize[{
    Simplify[∫_0^1 ∫_0^1 (DSumF1a1.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_1^2 ∫_0^1 (DSumF1a2.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_1^2 ∫_-1^0 (DSumF1a3.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_2^3 ∫_-1^0 (DSumF1a4.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_2^3 ∫_-2^-1 (DSumF1a5.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_3^4 ∫_-2^-1 (DSumF1a6.{1, 1})^2 𝕕x 𝕕y]
}];
{Err1b1, Err1b2, Err1b3, Err1b4, Err1b5, Err1b6} = Parallelize[{
    Simplify[∫_0^1 ∫_1^2 (DSumF1b1.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-1^0 ∫_1^2 (DSumF1b2.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-1^0 ∫_2^3 (DSumF1b3.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-2^-1 ∫_2^3 (DSumF1b4.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-2^-1 ∫_3^4 (DSumF1b5.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-3^-2 ∫_3^4 (DSumF1b6.{1, 1})^2 𝕕x 𝕕y]
}];
Err1 = FullSimplify[Err1a1 + Err1a2 + Err1a3 + Err1a4 +
     Err1a5 + Err1a6 + Err1b1 + Err1b2 + Err1b3 + Err1b4 + Err1b5 + Err1b6];
```

```
Err = FullSimplify[Err1]
DErr = FullSimplify[D[Err, {{c01, c02, c03, c11, c12, c13}}]];
H = FullSimplify[D[Err, {{c01, c02, c03, c11, c12, c13}, 2}]];
NSols = NSolve[DErr == 0, {c01, c02, c03, c11, c12, c13}];
TableForm[
 {Range[Length[NSols]], Err /. NSols, PositiveDefiniteMatrixQ[H /. N[#]] & /@ NSols}^T]

FindMinimum[Err,
 {{c01, -1}, {c02, 0}, {c03, 0}, {c11, 0}, {c12, 0}, {c13, 0}}, Method → "Newton"]
FindMinimum[Err, {{c01, -1}, {c02, 0}, {c03, 0}, {c11, 0}, {c12, 0}, {c13, 0}},
 Method → "QuasiNewton"]
FindMinimum[Err, {{c01, -1}, {c02, 0}, {c03, 0}, {c11, 0}, {c12, 0}, {c13, 0}},
 Method → "ConjugateGradient"]
FindMinimum[Err, {{c01, -1}, {c02, 0}, {c03, 0}, {c11, 0}, {c12, 0}, {c13, 0}},
 Method → "PrincipalAxis"]
```

```
{0.029495, {c01 → -0.443271, c02 → -0.70886,
  c03 → 0.118277, c11 → -0.54828, c12 → 0.389883, c13 → 0.141874}}

{0.029495, {c01 → -0.443271, c02 → -0.70886,
  c03 → 0.118277, c11 → -0.54828, c12 → 0.389883, c13 → 0.141874}}
```

FindMinimum::cvmit : Failed to converge to the requested accuracy or precision within 100 iterations. >>

```
{0.029495, {c01 → -0.443261, c02 → -0.708896,
  c03 → 0.118313, c11 → -0.54829, c12 → 0.389918, c13 → 0.141836}}

{0.029495, {c01 → -0.443272, c02 → -0.708859,
  c03 → 0.118276, c11 → -0.54828, c12 → 0.389882, c13 → 0.141875}}
```

```
FindMinimum[Err, {{c01, -0.435330221908065559679420525119965044 47`10.226366502496674},
  {c02, -0.7533366962200895`}, {c03, 0.18866691812815506`},
  {c11, -0.54806244893545961068469588662520138975`9.012895622311857},
  {c12, 0.37946785619540035693120056280664868902`8.451373514162633},
  {c13, 0.16859459274005925375349532381855270073`7.954032520769908}}, Method → "Newton"]
```

```
{0.029495, {c01 → -0.443271, c02 → -0.70886,
  c03 → 0.118277, c11 → -0.54828, c12 → 0.389883, c13 → 0.141874}}
```

```
Sol = {c01 → -0.4432714716264854`, c02 → -0.708859891578325`, c03 → 0.11827663982134162`,
   c11 → -0.548279764036396`, c12 → 0.3898828766198676`, c13 → 0.1418740156335347`};
FullSol = N[Join[GenSol /. Sol, Sol]]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

```
{c04 → 0.0338547, c14 → 0.0165229, c21 → 0.10093, c22 → -0.00915814,
 c23 → -0.0413939, c24 → -0.0503776, c01 → -0.443271, c02 → -0.70886,
 c03 → 0.118277, c11 → -0.54828, c12 → 0.389883, c13 → 0.141874}
```