```
h[x_] := Piecewise[{
   {1 + c0,1 x + c0,2 x^2 + c0,3 x^3, 0 ≤ x ≤ 1},
   {c1,1 (x - 1) + c1,2 (x - 1)^2 + c1,3 (x - 1)^3, 1 < x ≤ 2},
   {c2,1 (x - 2) + c2,2 (x - 2)^2 + c2,3 (x - 2)^3, 2 < x ≤ 3},
   {0, True}}];

f[x_] := h[Abs[x]];
AllVars = {c0,1, c0,2, c0,3, c1,1, c1,2, c1,3, c2,1, c2,2, c2,3};

(*Interpolant constraints*)
I1 = f[1]
I2 = f[2]
I3 = f[3]
```

$$1 + c_{0,1} + c_{0,2} + c_{0,3}$$

$$c_{1,1} + c_{1,2} + c_{1,3}$$

$$c_{2,1} + c_{2,2} + c_{2,3}$$

```
(*Partition of unity and linear term*)
T0 = CoefficientList[FullSimplify[Sum[f[x - k], {k, -2, 3}], x > 0 && x < 1], x]

T1 = CoefficientList[FullSimplify[Sum[k f[x - k], {k, -2, 3}], x > 0 && x < 1], x]
```

$$\{2 + c_{0,1} + c_{0,2} + c_{0,3} + c_{1,1} + c_{1,2} + c_{1,3} + c_{2,1} + c_{2,2} + c_{2,3},\ -2 c_{0,2} - 3 c_{0,3} - 2 c_{1,2} - 3 c_{1,3} - 2 c_{2,2} - 3 c_{2,3},\ 2 c_{0,2} + 3 c_{0,3} + 2 c_{1,2} + 3 c_{1,3} + 2 c_{2,2} + 3 c_{2,3}\}$$

$$\{1 + c_{0,1} + c_{0,2} + c_{0,3} + 2 c_{1,1} + 2 c_{1,2} + 2 c_{1,3} + 3 c_{2,1} + 3 c_{2,2} + 3 c_{2,3},\ -c_{0,1} - 2 c_{0,2} - 3 c_{0,3} - 3 c_{1,1} - 4 c_{1,2} - 6 c_{1,3} - 5 c_{2,1} - 6 c_{2,2} - 9 c_{2,3},\ c_{0,2} + 3 c_{0,3} + c_{1,2} + 6 c_{1,3} + c_{2,2} + 9 c_{2,3},\ -c_{0,3} - 3 c_{1,3} - 5 c_{2,3}\}$$

```
GenSols = Solve[{
    I1 == 0,
    I2 == 0,
    I3 == 0,
    T0[[1]] == 1,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[1]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0,
    T1[[4]] == 0
    },
    AllVars
]
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{c_{0,3} \to -1 - c_{0,1} - c_{0,2},\ c_{1,3} \to -c_{1,1} - c_{1,2},\ c_{2,1} \to -\frac{7}{5} - \frac{7 c_{0,1}}{5} - \frac{2 c_{0,2}}{5} - \frac{6 c_{1,1}}{5} - \frac{c_{1,2}}{5},\ c_{2,2} \to \frac{6}{5} + \frac{6 c_{0,1}}{5} + \frac{c_{0,2}}{5} + \frac{3 c_{1,1}}{5} - \frac{2 c_{1,2}}{5},\ c_{2,3} \to \frac{1}{5} + \frac{c_{0,1}}{5} + \frac{c_{0,2}}{5} + \frac{3 c_{1,1}}{5} + \frac{3 c_{1,2}}{5}\right\}\right\}$$

```
RegionXY[k_] := {Quotient[k, 2], 1 + Quotient[-k, 2]};
Regions = Table[RegionXY[k], {k, -4, 7}]
```

```
{{-2, 3}, {-2, 2}, {-1, 2}, {-1, 1}, {0, 1},
 {0, 0}, {1, 0}, {1, -1}, {2, -1}, {2, -2}, {3, -2}, {3, -3}}
```

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
φ = 1/2;
```

$$\mathtt{W[k\_]} := \begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1 - (1-\varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases};$$

$$\mathtt{SumF} = \sum_{i=-5}^{6}\sum_{j=-5}^{6} \mathtt{W[i-j]\ f[x-i,\ y-j]\ /.\ GenSol;}$$

```
SimplifySquare[f_, x0_, y0_] := Simplify[f, x > x0 && x < x0 + 1 && y > y0 && y < y0 + 1];
DSimplifySquare[f_, {x0_, y0_}] := Simplify[D[SimplifySquare[f, x0, y0], {{x, y}}]];
DSumF = ParallelMap[DSimplifySquare[SumF, #] &, Regions];
```

```
AnisoInt[df_, {x0_, y0_}] :=
  Simplify[Integrate[Expand[(df.{1, 1})^2], {x, x0, x0 + 1}, {y, y0, y0 + 1}]];
AnisoInts = Parallelize[MapThread[AnisoInt, {DSumF, Regions}]];
Err = Simplify[Total[AnisoInts]]
```

$$\frac{1}{63\,000\,000}\Big(168\,920\,343\,c_{0,1}^4 + 2\,150\,743\,c_{0,2}^4 + c_{0,2}^3\left(39\,247\,427 + 11\,652\,126\,c_{1,1} + 4\,020\,801\,c_{1,2}\right) + 3\,c_{0,1}^3\left(338\,365\,884 + 75\,342\,324\,c_{0,2} + 102\,576\,067\,c_{1,1} + 34\,569\,742\,c_{1,2}\right) + 3\,c_{0,2}^2\left(86\,246\,666 + 9\,448\,279\,c_{1,1}^2 + 19\,003\,966\,c_{1,2} + 1\,129\,579\,c_{1,2}^2 + c_{1,1}\left(53\,849\,841 + 6\,446\,208\,c_{1,2}\right)\right) + 3\,c_{0,1}^2\Big(37\,897\,886\,c_{0,2}^2 + 83\,257\,779\,c_{1,1}^2 + c_{0,2}\left(342\,228\,127 + 103\,130\,826\,c_{1,1} + 35\,011\,901\,c_{1,2}\right) + c_{1,1}\left(472\,541\,241 + 56\,304\,408\,c_{1,2}\right) + 11\left(67\,615\,656 + 14\,850\,706\,c_{1,2} + 884\,389\,c_{1,2}^2\right)\Big) + 3\,c_{0,2}\Big(232\,270\,229 + 11\,409\,288\,c_{1,1}^3 + 86\,575\,981\,c_{1,2} + 10\,688\,803\,c_{1,2}^2 + 467\,138\,c_{1,2}^3 + c_{1,1}^2\left(87\,381\,503 + 11\,614\,814\,c_{1,2}\right) + c_{1,1}\left(241\,298\,906 + 60\,310\,756\,c_{1,2} + 3\,997\,664\,c_{1,2}^2\right)\Big) + 3\Big(222\,162\,041 + 11\,650\,551\,c_{1,1}^4 + 120\,694\,782\,c_{1,2} + 24\,785\,499\,c_{1,2}^2 + 1\,873\,308\,c_{1,2}^3 + 162\,351\,c_{1,2}^4 + 8\,c_{1,1}^3\left(5\,500\,051 + 1\,964\,988\,c_{1,2}\right) + c_{1,1}^2\left(191\,553\,249 + 45\,408\,224\,c_{1,2} + 8\,025\,506\,c_{1,2}^2\right) + c_{1,1}\left(337\,003\,107 + 134\,629\,598\,c_{1,2} + 15\,881\,124\,c_{1,2}^2 + 1\,843\,504\,c_{1,2}^3\right)\Big) + 3\,c_{0,1}\Big(681\,031\,404 + 8\,495\,924\,c_{0,2}^3 + 34\,060\,338\,c_{1,1}^3 + 251\,022\,806\,c_{1,2} + 31\,195\,278\,c_{1,2}^2 + 1\,367\,488\,c_{1,2}^3 + 24\,c_{1,1}^2\left(10\,648\,647 + 1\,439\,261\,c_{1,2}\right) + c_{0,2}^2\left(115\,746\,802 + 34\,626\,301\,c_{1,1} + 11\,844\,926\,c_{1,2}\right) + c_{1,1}\left(711\,133\,281 + 175\,897\,006\,c_{1,2} + 11\,824\,414\,c_{1,2}^2\right) + 2\,c_{0,2}\left(252\,539\,891 + 27\,942\,204\,c_{1,1}^2 + 55\,736\,941\,c_{1,2} + 3\,293\,104\,c_{1,2}^2 + 2\,c_{1,1}\left(79\,750\,808 + 9\,483\,079\,c_{1,2}\right)\right)\Big)\Big)$$

```
FreeVars = Variables[Err];
DErr = Simplify[D[Err, {FreeVars}]];
H = D[DErr, {FreeVars}];
Sols = Solve[DErr == 0, FreeVars, Reals];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols, 32], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}ᵀ]
```

```
1    0.029557870106865588473407724386    True
```

```
NSol = N[Sols[[1]], 32];
FullSol = Join[GenSol /. NSol, NSol]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

$\{c_{0,3} \to 0.18866691811874665403809911616 16,$
$c_{1,3} \to 0.16859459244859876135301681109694,\ c_{2,1} \to 0.09257835676271371355310814434 33,$
$c_{2,2} \to 0.04631178233019487406632176554 72,\ c_{2,3} \to -0.13889013909290858761942990989047,$
$c_{0,1} \to -0.43533022189865716985707333758223,\ c_{0,2} \to -0.75333669622008948418102577857932,$
$c_{1,1} \to -0.54806244912683812981019179807705,\ c_{1,2} \to 0.37946785667823936845717498698011\}$