In[188]:=
$$h[x\_] := \begin{cases} 1 + c01\,x + c02\,x^2 & 0 \le x \le 1/2 \\ c11\,(x-1) + c12\,(x-1)^2 & 1/2 < x \le 3/2 \\ c21\,(x-2) + c22\,(x-2)^2 & 3/2 < x \le 5/2 \\ 0 & \text{True} \end{cases};$$

```
f[x_] := h[Abs[x]];
```

In[195]:=
```
(*Continuity*)
C1l = Simplify[h[x], 0 ≤ x ≤ 1/2] /. x → 1/2
C1r = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 1/2
C2l = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 3/2
C2r = Simplify[h[x], 3/2 < x ≤ 5/2] /. x → 3/2
C3l = Simplify[h[x], 3/2 < x ≤ 5/2] /. x → 5/2
```

Out[195]= $1 + \frac{c01}{2} + \frac{c02}{4}$

Out[196]= $\frac{1}{2}\left(-c11 + \frac{c12}{2}\right)$

Out[197]= $\frac{1}{2}\left(c11 + \frac{c12}{2}\right)$

Out[198]= $\frac{1}{2}\left(-c21 + \frac{c22}{2}\right)$

Out[199]= $\frac{1}{2}\left(c21 + \frac{c22}{2}\right)$

In[200]:=
```
(*Partition of unity and gradient representation*)
```
$$T0 = \text{CoefficientList}\left[\text{FullSimplify}\left[\sum_{i=-6}^{6} f[x-i],\ x > 0\ \&\&\ x < 1/2\right],\ x\right]$$
$$T1 = \text{CoefficientList}\left[\text{FullSimplify}\left[\sum_{i=-6}^{6} i\, f[x-i],\ x > 0\ \&\&\ x < 1/2\right],\ x\right]$$

Out[200]= $\{1,\ c01,\ c02 + 2\,(c12 + c22)\}$

Out[201]= $\{0,\ -2\,(c11 + 2\,c21)\}$

In[202]:=
```
GenSols = Solve[{
    C1l == C1r,
    C2l == C2r,
    C3l == 0,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[2]] == 1
    },
    {c01, c02, c11, c12, c21, c22}
]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[202]= $\left\{\left\{c01 \to 0,\ c11 \to -\frac{3}{2} - \frac{c02}{2},\ c12 \to 1,\ c21 \to \frac{1}{2} + \frac{c02}{4},\ c22 \to -1 - \frac{c02}{2}\right\}\right\}$

In[203]:=
```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
W1[k_] := Piecewise[{{0, k < 0}, {φ^2/2, k == 0}, {1 - (1 - φ)^2/2, k == 1}}, 1];
```

$$\text{SumF1} = \sum_{i=-5}^{6} \sum_{j=-5}^{6} \text{W1}[i-j]\, f[x-i, y-j] \text{ /. GenSol;}$$

In[209]:=
```
{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6} = Parallelize[{
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 3 - 1/2 && y < 4 - 1/2]
}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6} = Parallelize[{
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -3 - 1/2 && y < -2 - 1/2]
}];
```

In[211]:=
```
TableForm[{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6}]
TableForm[{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6}]
```

Out[211]//TableForm=

$\frac{1}{16}\left(8\varphi^2+8y^2(-1+\varphi)(1+c02-\varphi+c02\,\varphi)+4y\left(-1-2(3+c02)\varphi+(3+c02)\varphi^2\right)+x\left(4+8(3+c02)\varphi\right.\right.$

$\frac{1}{16}\left(2(2+c02)\,x(1+2x)(5+c02-2y)(-1+y)+(2+c02)^2x(1+2x)(-1+y)(-3+2y)-2(2+c02)\right.$

$\frac{1}{16}\left(\left(2+c02\right)^2(3+5x+2x^2)(-1+y)(-3+2y)-2(2+c02)(3+5x+2x^2)\left(1+c02(-1+y)^2\right)\varphi^2-2\right.$

$\frac{1}{32}(2+c02)(-2+y)\left(2(3+5x+2x^2)(7+c02-2y)\varphi^2-2(1+x)(5+c02+2x)(-5+2y)\varphi^2-(2+c0\right.$

$\frac{1}{32}(2+c02)^2(10+9x+2x^2)(-2+y)(-5+2y)\varphi^2$

0

Out[212]//TableForm=

$\frac{1}{16}\left(-c02^2(-1+x)\,y\left(-6-10\varphi+13\varphi^2+6x\left(1+2y(-2+\varphi)\varphi-\varphi^2\right)-6y\left(1-4\varphi+\varphi^2\right)\right)-4\left(2-4\varphi-3\varphi^2\right.\right.$

$\frac{1}{16}\left(-c02^2(-1+x)(1+y)\left(9+28\varphi-24\varphi^2-4y\left(-3-3\varphi+4\varphi^2\right)+4x\left(-3-3\varphi+4\varphi^2+y\left(-3+2\varphi^2\right)\right)\right)-\right.$

$\frac{1}{16}\left(c02^2(-2+x)(1+y)\left(-21+40\varphi-12\varphi^2-4y\left(3-9\varphi+4\varphi^2\right)+4x\left(2+y-5\varphi-4y\varphi+2\varphi^2+2y\varphi^2\right)\right)+\right.$

$\frac{1}{32}\left(c02^2(-2+x)(2+y)\left(y\left(16+8\varphi-14\varphi^2\right)+5\left(6+8\varphi-9\varphi^2\right)+2x\left(-8-4\varphi+7\varphi^2+2y\left(-2+\varphi^2\right)\right)\right)+4\right.$

$\frac{1}{32}\left(-4\,c02(21-13x+2x^2)(10+9y+2y^2)(-1+\varphi)^2-c02^2(21-13x+2x^2)(10+9y+2y^2)(-1+\varphi)^2\right.$

1

In[213]:=
```
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4, DSumF1a5,
   DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4, DSumF1b5} = Parallelize[{
    FullSimplify[D[SumF1a1, {{x, y}}]],
    FullSimplify[D[SumF1a2, {{x, y}}]],
    FullSimplify[D[SumF1a3, {{x, y}}]],
    FullSimplify[D[SumF1a4, {{x, y}}]],
    FullSimplify[D[SumF1a5, {{x, y}}]],
    FullSimplify[D[SumF1b1, {{x, y}}]],
    FullSimplify[D[SumF1b2, {{x, y}}]],
    FullSimplify[D[SumF1b3, {{x, y}}]],
    FullSimplify[D[SumF1b4, {{x, y}}]],
    FullSimplify[D[SumF1b5, {{x, y}}]]
}];
```

In[214]:=
```
DSumF1a1 = Simplify[DSumF1a1 /. φ → 1/2];
DSumF1a2 = Simplify[DSumF1a2 /. φ → 1/2];
DSumF1a3 = Simplify[DSumF1a3 /. φ → 1/2];
DSumF1a4 = Simplify[DSumF1a4 /. φ → 1/2];
DSumF1a5 = Simplify[DSumF1a5 /. φ → 1/2];
DSumF1b1 = Simplify[DSumF1b1 /. φ → 1/2];
DSumF1b2 = Simplify[DSumF1b2 /. φ → 1/2];
DSumF1b3 = Simplify[DSumF1b3 /. φ → 1/2];
DSumF1b4 = Simplify[DSumF1b4 /. φ → 1/2];
DSumF1b5 = Simplify[DSumF1b5 /. φ → 1/2];
```

In[224]:=
```
{Err1a1, Err1a2, Err1a3, Err1a4, Err1a5} = Parallelize[{
    Simplify[∫_{0-1/2}^{1-1/2} ∫_{0-1/2}^{1-1/2} (DSumF1a1.{1, 1})^2 dx dy],
    Simplify[∫_{1-1/2}^{2-1/2} ∫_{0-1/2}^{1-1/2} (DSumF1a2.{1, 1})^2 dx dy],
    Simplify[∫_{1-1/2}^{2-1/2} ∫_{-1-1/2}^{0-1/2} (DSumF1a3.{1, 1})^2 dx dy],
    Simplify[∫_{2-1/2}^{3-1/2} ∫_{-1-1/2}^{0-1/2} (DSumF1a4.{1, 1})^2 dx dy],
    Simplify[∫_{2-1/2}^{3-1/2} ∫_{-2-1/2}^{-1-1/2} (DSumF1a5.{1, 1})^2 dx dy]
  }];
{Err1b1, Err1b2, Err1b3, Err1b4, Err1b5} = Parallelize[{
    Simplify[∫_{0-1/2}^{1-1/2} ∫_{1-1/2}^{2-1/2} (DSumF1b1.{1, 1})^2 dx dy],
    Simplify[∫_{-1-1/2}^{0-1/2} ∫_{1-1/2}^{2-1/2} (DSumF1b2.{1, 1})^2 dx dy],
    Simplify[∫_{-1-1/2}^{0-1/2} ∫_{2-1/2}^{3-1/2} (DSumF1b3.{1, 1})^2 dx dy],
    Simplify[∫_{-2-1/2}^{-1-1/2} ∫_{2-1/2}^{3-1/2} (DSumF1b4.{1, 1})^2 dx dy],
    Simplify[∫_{-2-1/2}^{-1-1/2} ∫_{3-1/2}^{4-1/2} (DSumF1b5.{1, 1})^2 dx dy]
  }];
```

In[226]:=
```
Err1 = FullSimplify[
   Err1a1 + Err1a2 + Err1a3 + Err1a4 + Err1a5 + Err1b1 + Err1b2 + Err1b3 + Err1b4 + Err1b5];
```

In[227]:=
```
Err = Err1
DErr = FullSimplify[D[Err, c02]];
H = FullSimplify[D[Err, {{c02}, 2}]];
Sols = Solve[DErr == 0, c02];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}^T]
```

Out[227]=

$$\frac{1}{368\,640}\left(1\,360\,512+c02\,\left(2\,161\,584+c02\,\left(1\,188\,000+c02\,\left(253\,708+20\,325\,c02\right)\right)\right)\right)$$

Out[231]//TableForm=

1	0.0996855	True
2	1.58408 − 0.861657 i	False
3	1.58408 + 0.861657 i	False

In[232]:=
```
RootReduce[Sols[[1]]]
```

Out[232]=
```
{c02 → Root[180132 + 198000 #1 + 63427 #1^2 + 6775 #1^3 &, 1]}
```

In[233]:=
```
Sol = Sols[[1]];
FullSol = N[Join[GenSol /. Sol, Sol]]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

Out[234]= {c01 → 0., c11 → -0.721136, c12 → 1., c21 → 0.110568, c22 → -0.221136, c02 → -1.55773}

Out[236]=