```
In[302]:= h[x_] := Piecewise[{
    {1 + c01 x + c02 x^2 + c03 x^3 + c04 x^4, 0 ≤ x ≤ 1/2},
    {c11 (x - 1) + c12 (x - 1)^2 + c13 (x - 1)^3 + c14 (x - 1)^4, 1/2 < x ≤ 3/2},
    {0, True}}];
          f[x_] := h[Abs[x]];

In[304]:= (*Continuity*)
          C1l = Simplify[h[x], 0 ≤ x ≤ 1/2] /. x → 1/2
          C1r = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 1/2
          C2l = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 3/2
```

Out[304]= $1+\frac{c01}{2}+\frac{c02}{4}+\frac{c03}{8}+\frac{c04}{16}$

Out[305]= $\frac{1}{2}\left(-c11+\frac{1}{2}\left(c12+\frac{1}{2}\left(-c13+\frac{c14}{2}\right)\right)\right)$

Out[306]= $\frac{1}{2}\left(c11+\frac{1}{2}\left(c12+\frac{1}{2}\left(c13+\frac{c14}{2}\right)\right)\right)$

```
In[307]:= (*Partition of unity and gradient representation*)
          T0 = CoefficientList[FullSimplify[Sum[f[x - i], {i, -3, 3}], x > 0 && x < 1/2], x]
          T1 = CoefficientList[FullSimplify[Sum[i f[x - i], {i, -3, 3}], x > 0 && x < 1/2], x]
```

Out[307]= $\{1, c01, c02+2\,c12, c03, c04+2\,c14\}$

Out[308]= $\{0, -2\,c11, 0, -2\,c13\}$

```
In[309]:= (*Smoothness*)
          S0 = Simplify[D[h[x], x], 0 < x < 1/2] /. x → 0
          S1l = Simplify[D[h[x], x], 0 < x < 1/2] /. x → 1/2
          S1r = Simplify[D[h[x], x], 1/2 < x < 3/2] /. x → 1/2
          S2l = Simplify[D[h[x], x], 1/2 < x < 3/2] /. x → 3/2
```

Out[309]= $c01$

Out[310]= $c01+\frac{1}{2}\left(2\,c02+\frac{1}{2}\left(3\,c03+2\,c04\right)\right)$

Out[311]= $c11+\frac{1}{2}\left(-2\,c12+\frac{1}{2}\left(3\,c13-2\,c14\right)\right)$

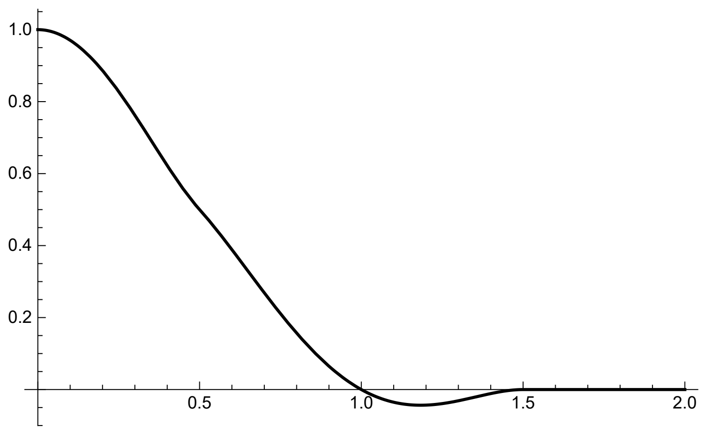
Out[312]= $c11+\frac{1}{2}\left(2\,c12+\frac{1}{2}\left(3\,c13+2\,c14\right)\right)$

In[313]:=
```
GenSols = Solve[{
    C1l == C1r,
    C2l == 0,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T0[[4]] == 0,
    T0[[5]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0,
    T1[[4]] == 0,
    S0 == 0,
    S1l == S1r,
    S2l == 0
    },
    {c01, c02, c03, c04, c11, c12, c13, c14}
]
```

Out[313]= $\left\{\left\{c01 \to 0,\ c02 \to -3,\ c03 \to 0,\ c04 \to 4,\ c11 \to -\frac{1}{2},\ c12 \to \frac{3}{2},\ c13 \to 0,\ c14 \to -2\right\}\right\}$

In[314]:=
```
Plot[f[x] /. GenSols[[1]], {x, 0, 2}, PlotStyle → Black]
```

Out[314]=



In[315]:=
```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
```

$$W1[k\_] := \begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1 - (1-\varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases};$$

$$SumF1 = \sum_{i=-5}^{6} \sum_{j=-5}^{6} W1[i-j]\, f[x-i, y-j] \ /.\ GenSol;$$

In[319]:=
```
{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6} = Parallelize[{
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 3 - 1/2 && y < 4 - 1/2]
}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6} = Parallelize[{
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -3 - 1/2 && y < -2 - 1/2]
}];
```

In[321]:=
```
TableForm[{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6}]
TableForm[{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6}]
```

Out[321]//TableForm=

$\frac{1}{8}\left(2\left(-1+x\right)x\left(1+2x\right)^2 y\left(1-3y+4y^3\right)+\left(-1+x\right)x\left(1+2x\right)^2\left(-1+y\right)y\left(1+2y\right)^2\varphi^2+x\left(1-3x+4x^3\right.\right.$

$\frac{1}{8}\left(2\left(x+3x^2-4x^4\right)\left(1-3\left(-1+y\right)^2+4\left(-1+y\right)^4\right)\varphi^2-2\left(1-3x^2+4x^4\right)\left(3-2y\right)^2\left(-1+y\right)y\varphi^2+\left(x+3\right.\right.$

$\frac{1}{8}x\left(1+x\right)\left(3+2x\right)^2\left(3-2y\right)^2\left(-1+y\right)y\varphi^2$

0

0

0

Out[322]//TableForm=

$\frac{1}{4}\left(-4\left(-2+\varphi\right)\varphi+y\left(-3-2\varphi+3\varphi^2\right)+3y^2\left(1-10\varphi+7\varphi^2\right)-4y^4\left(1-10\varphi+7\varphi^2\right)-21x^2\left(y-y\varphi^2+\varphi\left(-4\right.\right.\right.$

$\frac{1}{8}\left(2\left(1-2x\right)^2\left(2-3x+x^2\right)y\left(1+y\right)\left(3+2y\right)^2-4\left(1-3\left(-1+x\right)^2+4\left(-1+x\right)^4\right)\left(1+2y\right)^2\left(2+3y+y^2\right)\right.$

$\frac{1}{8}\left(8-9\left(5-2x\right)^2\left(2-3x+x^2\right)y\left(-1+\varphi\right)^2-21\left(5-2x\right)^2\left(2-3x+x^2\right)y^2\left(-1+\varphi\right)^2-16\left(5-2x\right)^2\left(2-3\right.\right.$

1

1

1

In[323]:=
```
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1b1, DSumF1b2, DSumF1b3} = Parallelize[{
    FullSimplify[D[SumF1a1, {{x, y}}]],
    FullSimplify[D[SumF1a2, {{x, y}}]],
    FullSimplify[D[SumF1a3, {{x, y}}]],
    FullSimplify[D[SumF1b1, {{x, y}}]],
    FullSimplify[D[SumF1b2, {{x, y}}]],
    FullSimplify[D[SumF1b3, {{x, y}}]]
}];
DSumF1a1 = Simplify[DSumF1a1 /. φ → 1/2];
DSumF1a2 = Simplify[DSumF1a2 /. φ → 1/2];
DSumF1a3 = Simplify[DSumF1a3 /. φ → 1/2];
DSumF1b1 = Simplify[DSumF1b1 /. φ → 1/2];
DSumF1b2 = Simplify[DSumF1b2 /. φ → 1/2];
DSumF1b3 = Simplify[DSumF1b3 /. φ → 1/2];
```

In[330]:=

$$\{\text{Err1a1}, \text{Err1a2}, \text{Err1a3}\} = \text{Parallelize}\Big[\Big\{$$

$$\text{Simplify}\Big[\int_{0-1/2}^{1-1/2}\int_{0-1/2}^{1-1/2} (\text{DSumF1a1}.\{1, 1\})^2\, dx\, dy\Big],$$

$$\text{Simplify}\Big[\int_{1-1/2}^{2-1/2}\int_{0-1/2}^{1-1/2} (\text{DSumF1a2}.\{1, 1\})^2\, dx\, dy\Big],$$

$$\text{Simplify}\Big[\int_{1-1/2}^{2-1/2}\int_{-1-1/2}^{0-1/2} (\text{DSumF1a3}.\{1, 1\})^2\, dx\, dy\Big]$$

$$\Big\}\Big];$$

$$\{\text{Err1b1}, \text{Err1b2}, \text{Err1b3}\} = \text{Parallelize}\Big[\Big\{$$

$$\text{Simplify}\Big[\int_{0-1/2}^{1-1/2}\int_{1-1/2}^{2-1/2} (\text{DSumF1b1}.\{1, 1\})^2\, dx\, dy\Big],$$

$$\text{Simplify}\Big[\int_{-1-1/2}^{0-1/2}\int_{1-1/2}^{2-1/2} (\text{DSumF1b2}.\{1, 1\})^2\, dx\, dy\Big],$$

$$\text{Simplify}\Big[\int_{-1-1/2}^{0-1/2}\int_{2-1/2}^{3-1/2} (\text{DSumF1b3}.\{1, 1\})^2\, dx\, dy\Big]$$

$$\Big\}\Big];$$

In[332]:=
```
Err1 = FullSimplify[Err1a1 + Err1a2 + Err1a3 + Err1b1 + Err1b2 + Err1b3]
N[Err1]
```

Out[332]= $\frac{208\,237}{1\,128\,960}$

Out[333]= 0.18445