```
h[x_] := { 1 + c01 x + c02 x^2 + c03 x^3          0 ≤ x ≤ 1
         { c11 (x - 1) + c12 (x - 1)^2 + c13 (x - 1)^3   1 < x ≤ 2;
         {              0                          True
```

```
f[x_] := h[Abs[x]];
```

```
(*Interpolant constraints*)
I1 = f[1]
I2 = f[2]
```

$1 + c01 + c02 + c03$

$c11 + c12 + c13$

```
(*Partition of unity and gradient representation*)
T0 = CoefficientList[FullSimplify[f[x + 1] + f[x] + f[x - 1] + f[x - 2], x > 0 && x < 1], x]
T1 = CoefficientList[FullSimplify[-f[x + 1] + f[x - 1] + 2 f[x - 2], x > 0 && x < 1], x]
```

$\{2 + c01 + c02 + c03 + c11 + c12 + c13,\ -2\,c02 - 3\,c03 - 2\,c12 - 3\,c13,\ 2\,c02 + 3\,c03 + 2\,c12 + 3\,c13\}$

$\{1 + c01 + c02 + c03 + 2\,c11 + 2\,c12 + 2\,c13,$
$\ -c01 - 2\,c02 - 3\,c03 - 3\,c11 - 4\,c12 - 6\,c13,\ c02 + 3\,c03 + c12 + 6\,c13,\ -c03 - 3\,c13\}$

```
GenSols = Solve[{
    I1 == 0,
    I2 == 0,
    T0[[1]] == 1,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[1]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0,
    T1[[4]] == 0
    },
    {c01, c02, c03, c11, c12, c13}
]
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\left\{\left\{c03 \to -1 - c01 - c02,\ c11 \to -\frac{4}{3} - \frac{4\,c01}{3} - \frac{c02}{3},\ c12 \to 1 + c01,\ c13 \to \frac{1}{3} + \frac{c01}{3} + \frac{c02}{3}\right\}\right\}$$

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
W1[k_] := Piecewise[{{0, k < 0}, {φ^2/2, k == 0}, {1 - (1 - φ)^2/2, k == 1}, {1, True}}];
SumF1 = Sum[Sum[W1[i - j] f[x - i, y - j] /. GenSol, {j, -3, 5}], {i, -3, 5}];
{SumF1a1, SumF1a2, SumF1a3, SumF1a4} = Parallelize[{
    Simplify[SumF1, x > 0 && x < 1 && y > 0 && y < 1],
    Simplify[SumF1, x > 0 && x < 1 && y > 1 && y < 2],
    Simplify[SumF1, x > -1 && x < 0 && y > 1 && y < 2],
    Simplify[SumF1, x > -1 && x < 0 && y > 2 && y < 3]}];
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4} = Parallelize[{
    FullSimplify[D[SumF1a1, {{x, y}}]],
    FullSimplify[D[SumF1a2, {{x, y}}]],
    FullSimplify[D[SumF1a3, {{x, y}}]],
    FullSimplify[D[SumF1a4, {{x, y}}]]}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4} = Parallelize[{
    Simplify[SumF1, x > 1 && x < 2 && y > 0 && y < 1],
    Simplify[SumF1, x > 1 && x < 2 && y > -1 && y < 0],
    Simplify[SumF1, x > 2 && x < 3 && y > -1 && y < 0],
    Simplify[SumF1, x > 2 && x < 3 && y > -2 && y < -1]}];
{DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4} = Parallelize[{
    FullSimplify[D[SumF1b1, {{x, y}}]],
    FullSimplify[D[SumF1b2, {{x, y}}]],
    FullSimplify[D[SumF1b3, {{x, y}}]],
    FullSimplify[D[SumF1b4, {{x, y}}]]}];
{Err1a1, Err1a2, Err1a3, Err1a4} = Parallelize[{
    Simplify[Integrate[(DSumF1a1.{1, 1})^2 dx dy, {x, 0, 1}, {y, 0, 1}]],
    Simplify[Integrate[(DSumF1a2.{1, 1})^2 dx dy, {x, 1, 2}, {y, 0, 1}]],
    Simplify[Integrate[(DSumF1a3.{1, 1})^2 dx dy, {x, 1, 2}, {y, -1, 0}]],
    Simplify[Integrate[(DSumF1a4.{1, 1})^2 dx dy, {x, 2, 3}, {y, -1, 0}]]}];
{Err1b1, Err1b2, Err1b3, Err1b4} = Parallelize[{
    Simplify[Integrate[(DSumF1b1.{1, 1})^2 dx dy, {x, 0, 1}, {y, 1, 2}]],
    Simplify[Integrate[(DSumF1b2.{1, 1})^2 dx dy, {x, -1, 0}, {y, 1, 2}]],
    Simplify[Integrate[(DSumF1b3.{1, 1})^2 dx dy, {x, -1, 0}, {y, 2, 3}]],
    Simplify[Integrate[(DSumF1b4.{1, 1})^2 dx dy, {x, -2, -1}, {y, 2, 3}]]}];
```

```
Err1 = FullSimplify[Err1a1 + Err1a2 + Err1a3 + Err1a4 + Err1b1 + Err1b2 + Err1b3 + Err1b4];

Err = FullSimplify[Err1 /. φ → 1/2]
DErr = FullSimplify[D[Err, {{c01, c02}}]];
H = FullSimplify[D[Err, {{c01, c02}, 2}]];
Sols = RootReduce[Solve[DErr == 0, {c01, c02}]];
N[Sols]
Sols[[1]]
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}^T]
```

$$\frac{1}{907\,200}\left(2\,073\,435 + 630\,023\,c01^4 + 4\,c01^3\left(949\,251 + 211\,676\,c02\right) + 6\,c01^2\left(1\,366\,684 + c02\left(649\,053 + 71\,515\,c02\right)\right) + 4\,c01\left(1\,745\,927 + 2\,c02\left(712\,023 + 2\,c02\left(83\,661 + 6080\,c02\right)\right)\right) + c02\left(2\,452\,640 + c02\left(998\,496 + c02\left(153\,882 + 8351\,c02\right)\right)\right)\right)$$

```
{{c01 → -0.624491, c02 → -0.370817}, {c01 → -6.05904 - 2.68977 i, c02 → 14.3617 + 5.60167 i},
 {c01 → -6.05904 + 2.68977 i, c02 → 14.3617 - 5.60167 i},
 {c01 → -4.80989 - 2.81766 i, c02 → 11.053 + 6.88527 i},
 {c01 → -4.80989 + 2.81766 i, c02 → 11.053 - 6.88527 i},
 {c01 → -3.27799 - 2.54381 i, c02 → 4.91809 + 6.37681 i},
 {c01 → -3.27799 + 2.54381 i, c02 → 4.91809 - 6.37681 i},
 {c01 → -1.70423 - 0.515319 i, c02 → -0.860432 + 2.5564 i},
 {c01 → -1.70423 + 0.515319 i, c02 → -0.860432 - 2.5564 i}}
```

$$\left\{c01 \to \text{Root}\left[11\,136\,212\,089\,427\,417\,799\,940 + 40\,565\,503\,933\,508\,466\,119\,319\,\#1 + 56\,175\,410\,802\,925\,780\,786\,716\,\#1^2 + 41\,251\,698\,626\,094\,358\,074\,618\,\#1^3 + 18\,239\,143\,335\,129\,521\,212\,776\,\#1^4 + 5\,173\,229\,177\,780\,774\,994\,150\,\#1^5 + 956\,712\,737\,540\,209\,331\,904\,\#1^6 + 112\,632\,888\,717\,491\,520\,636\,\#1^7 + 7\,734\,928\,656\,195\,327\,168\,\#1^8 + 239\,273\,075\,197\,171\,456\,\#1^9 \&, 1\right],\ c02 \to \text{Root}\left[3\,374\,023\,697\,485\,280\,698\,110\,075 + 8\,537\,532\,930\,936\,839\,854\,857\,150\,\#1 - 1\,103\,781\,936\,366\,867\,775\,279\,410\,\#1^2 + 951\,787\,233\,357\,756\,794\,329\,620\,\#1^3 - 383\,780\,785\,728\,198\,277\,951\,170\,\#1^4 + 81\,595\,240\,042\,195\,026\,782\,580\,\#1^5 - 9\,947\,775\,659\,359\,580\,484\,972\,\#1^6 + 712\,083\,875\,694\,813\,372\,936\,\#1^7 - 28\,030\,349\,234\,173\,638\,144\,\#1^8 + 478\,546\,150\,394\,342\,912\,\#1^9 \&, 1\right]\right\}$$

```
1    0.0494529               True
2    0.290471 + 0.22153 i    False
3    0.290471 - 0.22153 i    False
4    0.303282 + 0.206304 i   False
5    0.303282 - 0.206304 i   False
6    0.43778 + 0.168334 i    False
7    0.43778 - 0.168334 i    False
8    0.658936 - 0.0980122 i  False
9    0.658936 + 0.0980122 i  False
```

```
Sol = Sols[[1]];
FullSol = N[Join[GenSol /. Sol, Sol]]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

```
{c03 → -0.00469238, c11 → -0.377073, c12 → 0.375509,
 c13 → 0.00156413, c01 → -0.624491, c02 → -0.370817}
```