```
h[x_] := ⎡ 1 + c_{0,1} x + c_{0,2} x^2 + c_{0,3} x^3 + c_{0,4} x^4                        0 ≤ x ≤ 1
         ⎢ c_{1,1} (x - 1) + c_{1,2} (x - 1)^2 + c_{1,3} (x - 1)^3 + c_{1,4} (x - 1)^4  1 < x ≤ 2
         ⎢ c_{2,1} (x - 2) + c_{2,2} (x - 2)^2 + c_{2,3} (x - 2)^3 + c_{2,4} (x - 2)^4  2 < x ≤ 3 ;
         ⎣ 0                                                                           True
f[x_] := h[Abs[x]];
AllVars = {c_{0,1}, c_{0,2}, c_{0,3}, c_{0,4}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{2,1}, c_{2,2}, c_{2,3}, c_{2,4}};

(*Interpolant constraints*)
I1 = f[1]
I2 = f[2]
I3 = f[3]
```

$1 + c_{0,1} + c_{0,2} + c_{0,3} + c_{0,4}$

$c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4}$

$c_{2,1} + c_{2,2} + c_{2,3} + c_{2,4}$

```
(*Partition of unity and linear term*)
T0 = CoefficientList[FullSimplify[Sum_{k=-2}^{3} f[x - k], x > 0 && x < 1], x]
T1 = CoefficientList[FullSimplify[Sum_{k=-2}^{3} k f[x - k], x > 0 && x < 1], x]
```

$$\{2 + c_{0,1} + c_{0,2} + c_{0,3} + c_{0,4} + c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4} + c_{2,1} + c_{2,2} + c_{2,3} + c_{2,4},\ -2 c_{0,2} - 3 c_{0,3} - 4 c_{0,4} - 2 c_{1,2} - 3 c_{1,3} - 4 c_{1,4} - 2 c_{2,2} - 3 c_{2,3} - 4 c_{2,4},\ 2 c_{0,2} + 3 c_{0,3} + 6 c_{0,4} + 2 c_{1,2} + 3 c_{1,3} + 4 c_{1,4} + 2 c_{2,2} + 3 c_{2,3} + 4 c_{2,4} + 2 (c_{1,4} + c_{2,4}),\ -4 c_{0,4} - 4 (c_{1,4} + c_{2,4}),\ 2 c_{0,4} + 2 (c_{1,4} + c_{2,4})\}$$

$$\{1 + c_{0,1} + c_{0,2} + c_{0,3} + c_{0,4} + 2 c_{1,1} + 2 c_{1,2} + 2 c_{1,3} + 2 c_{1,4} + 3 c_{2,1} + 3 c_{2,2} + 3 c_{2,3} + 3 c_{2,4},\ -c_{0,1} - 2 c_{0,2} - 3 c_{0,3} - 4 c_{0,4} - 3 c_{1,1} - 4 c_{1,2} - 6 c_{1,3} - 8 c_{1,4} - 5 c_{2,1} - 6 c_{2,2} - 9 c_{2,3} - 12 c_{2,4},\ c_{0,2} + 3 c_{0,3} + 6 c_{0,4} + c_{1,2} + 6 c_{1,3} + 12 c_{1,4} + c_{2,2} + 9 c_{2,3} + 18 c_{2,4},\ -c_{0,3} - 4 c_{0,4} - 3 c_{1,3} - 8 c_{1,4} - 5 c_{2,3} - 12 c_{2,4},\ c_{0,4} + c_{1,4} + c_{2,4}\}$$

```
(*Smoothness*)
Dh = Simplify[D[h[x], x], x > 0];
S0 = (Dh /. x → 0) == 0
S1 = Limit[Dh, x → 1, Direction → 1] == Limit[Dh, x → 1, Direction → -1]
S2 = Limit[Dh, x → 2, Direction → 1] == Limit[Dh, x → 2, Direction → -1]
S3 = Limit[Dh, x → 3, Direction → 1] == Limit[Dh, x → 3, Direction → -1]
```

$c_{0,1} == 0$

$c_{0,1} + 2 c_{0,2} + 3 c_{0,3} + 4 c_{0,4} == c_{1,1}$

$c_{1,1} + 2 c_{1,2} + 3 c_{1,3} + 4 c_{1,4} == c_{2,1}$

$c_{2,1} + 2 c_{2,2} + 3 c_{2,3} + 4 c_{2,4} == 0$

```
GenSols = Solve[{
    I1 == 0,
    I2 == 0,
    I3 == 0,
    T0[[1]] == 1,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[1]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0,
    T1[[4]] == 0,
    S0, S1, S2, S3
    },
    AllVars
]
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{c_{0,1}\to 0,\ c_{0,4}\to -1-c_{0,2}-c_{0,3},\ c_{1,1}\to -4-2\,c_{0,2}-c_{0,3},\ c_{1,3}\to \frac{41}{4}+5\,c_{0,2}+\frac{5\,c_{0,3}}{2}-2\,c_{1,2},\ c_{1,4}\to -\frac{25}{4}-3\,c_{0,2}-\frac{3\,c_{0,3}}{2}+c_{1,2},\ c_{2,1}\to \frac{7}{4}+c_{0,2}+\frac{c_{0,3}}{2},\ c_{2,2}\to \frac{15}{4}+2\,c_{0,2}+\frac{3\,c_{0,3}}{2}-c_{1,2},\ c_{2,3}\to -\frac{51}{4}-7\,c_{0,2}-\frac{9\,c_{0,3}}{2}+2\,c_{1,2},\ c_{2,4}\to \frac{29}{4}+4\,c_{0,2}+\frac{5\,c_{0,3}}{2}-c_{1,2}\right\}\right\}$$

```
RegionXY[k_] := {Quotient[k, 2], 1 + Quotient[-k, 2]};
Regions = Table[RegionXY[k], {k, -4, 7}]
```

```
{{-2, 3}, {-2, 2}, {-1, 2}, {-1, 1}, {0, 1},
 {0, 0}, {1, 0}, {1, -1}, {2, -1}, {2, -2}, {3, -2}, {3, -3}}
```

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
φ = 1/2;
```

$$W[k\_] := \begin{cases} 0 & k<0 \\ \varphi^2/2 & k==0 \\ 1-(1-\varphi)^2/2 & k==1 \\ 1 & \text{True} \end{cases};$$

$$\text{SumF} = \sum_{i=-5}^{6}\sum_{j=-5}^{6} W[i-j]\ f[x-i,\ y-j]\ /.\ \text{GenSol};$$

```
SimplifySquare[f_, x0_, y0_] := Simplify[f, x > x0 && x < x0 + 1 && y > y0 && y < y0 + 1];
DSimplifySquare[f_, {x0_, y0_}] := Simplify[D[SimplifySquare[f, x0, y0], {{x, y}}]];
DSumF = ParallelMap[DSimplifySquare[SumF, #] &, Regions];
```

```
AnisoInt[df_, {x0_, y0_}] :=
  Simplify[Integrate[Expand[(df.{1, 1})^2], {x, x0, x0 + 1}, {y, y0, y0 + 1}]];
AnisoInts = Parallelize[MapThread[AnisoInt, {DSumF, Regions}]];
Err = Simplify[Total[AnisoInts]]
```

$$\frac{1}{541900800}\Big(710720512\,c_{0,2}^4+64\,c_{0,2}^3\left(96603491+22528790\,c_{0,3}-4600728\,c_{1,2}\right)+16\,c_{0,2}^2\left(1265098369+68874844\,c_{0,3}^2+c_{0,3}\left(581444444-28506576\,c_{1,2}\right)-114586848\,c_{1,2}+3206976\,c_{1,2}^2\right)+12\,c_{0,2}\left(2419537509+31366520\,c_{0,3}^3-324109624\,c_{1,2}+16680064\,c_{1,2}^2-362496\,c_{1,2}^3-4\,c_{0,3}^2\left(-97444621+4942360\,c_{1,2}\right)+2\,c_{0,3}\left(843003883-77532480\,c_{1,2}+2257792\,c_{1,2}^2\right)\right)+3\left(5110913111+16186224\,c_{0,3}^4-916196400\,c_{1,2}+71761216\,c_{1,2}^2-2433024\,c_{1,2}^3+53248\,c_{1,2}^4-32\,c_{0,3}^3\left(-8176477+433050\,c_{1,2}\right)+8\,c_{0,3}^2\left(211536105-19640784\,c_{1,2}+606304\,c_{1,2}^2\right)-8\,c_{0,3}\left(-604723137+82614294\,c_{1,2}-4205536\,c_{1,2}^2+100352\,c_{1,2}^3\right)\right)\Big)$$

```
FreeVars = Variables[Err];
DErr = Simplify[D[Err, {FreeVars}]];
H = D[DErr, {FreeVars}];
Sols = Solve[DErr == 0, FreeVars, Reals];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}ᵀ]
```

| | | |
|---|---|---|
| 1 | 0.0497664 | True |

```
NSol = N[Sols[[1]]];
FullSol = Join[GenSol /. NSol, NSol]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

$\{c_{0,1}\to 0,\ c_{0,4}\to 0.309775,\ c_{1,1}\to -0.838311,\ c_{1,3}\to 0.958101,\ c_{1,4}\to -0.813628,\ c_{2,1}\to 0.169156,\ c_{2,2}\to 0.165542,\ c_{2,3}\to -0.83855,\ c_{2,4}\to 0.503853,\ c_{0,2}\to -1.85191,\ c_{0,3}\to 0.542138,\ c_{1,2}\to 0.693839\}$