$$h[x\_] := \begin{cases} 1 + c_{0,1}\, x + c_{0,2}\, x^2 + c_{0,3}\, x^3 & 0 \le x \le 1 \\ c_{1,1}\, (x-1) + c_{1,2}\, (x-1)^2 + c_{1,3}\, (x-1)^3 & 1 < x \le 2 \\ c_{2,1}\, (x-2) + c_{2,2}\, (x-2)^2 + c_{2,3}\, (x-2)^3 & 2 < x \le 3 \\ 0 & \text{True} \end{cases};$$

$f[x\_] := h[\text{Abs}[x]];$

$\text{AllVars} = \{c_{0,1}, c_{0,2}, c_{0,3}, c_{1,1}, c_{1,2}, c_{1,3}, c_{2,1}, c_{2,2}, c_{2,3}\};$

(*Interpolant constraints*)

$\text{I1} = f[1]$

$\text{I2} = f[2]$

$\text{I3} = f[3]$

$1 + c_{0,1} + c_{0,2} + c_{0,3}$

$c_{1,1} + c_{1,2} + c_{1,3}$

$c_{2,1} + c_{2,2} + c_{2,3}$

(*Partition of unity and linear term*)

$$\text{T0} = \text{CoefficientList}\left[\text{FullSimplify}\left[\sum_{k=-2}^{3} f[x-k],\ x > 0 \,\&\&\, x < 1\right],\ x\right]$$

$$\text{T1} = \text{CoefficientList}\left[\text{FullSimplify}\left[\sum_{k=-2}^{3} k\, f[x-k],\ x > 0 \,\&\&\, x < 1\right],\ x\right]$$

$$\{2 + c_{0,1} + c_{0,2} + c_{0,3} + c_{1,1} + c_{1,2} + c_{1,3} + c_{2,1} + c_{2,2} + c_{2,3},\\ -2\, c_{0,2} - 3\, c_{0,3} - 2\, c_{1,2} - 3\, c_{1,3} - 2\, c_{2,2} - 3\, c_{2,3},\ 2\, c_{0,2} + 3\, c_{0,3} + 2\, c_{1,2} + 3\, c_{1,3} + 2\, c_{2,2} + 3\, c_{2,3}\}$$

$$\{1 + c_{0,1} + c_{0,2} + c_{0,3} + 2\, c_{1,1} + 2\, c_{1,2} + 2\, c_{1,3} + 3\, c_{2,1} + 3\, c_{2,2} + 3\, c_{2,3},\\ -c_{0,1} - 2\, c_{0,2} - 3\, c_{0,3} - 3\, c_{1,1} - 4\, c_{1,2} - 6\, c_{1,3} - 5\, c_{2,1} - 6\, c_{2,2} - 9\, c_{2,3},\\ c_{0,2} + 3\, c_{0,3} + c_{1,2} + 6\, c_{1,3} + c_{2,2} + 9\, c_{2,3},\ -c_{0,3} - 3\, c_{1,3} - 5\, c_{2,3}\}$$

(*Smoothness*)

$\text{Dh} = \text{Simplify}[D[h[x], x], x > 0];$

$\text{S0} = (\text{Dh} \,/.\, x \to 0) == 0$

$\text{S1} = \text{Limit}[\text{Dh}, x \to 1, \text{Direction} \to 1] == \text{Limit}[\text{Dh}, x \to 1, \text{Direction} \to -1]$

$\text{S2} = \text{Limit}[\text{Dh}, x \to 2, \text{Direction} \to 1] == \text{Limit}[\text{Dh}, x \to 2, \text{Direction} \to -1]$

$\text{S3} = \text{Limit}[\text{Dh}, x \to 3, \text{Direction} \to 1] == \text{Limit}[\text{Dh}, x \to 3, \text{Direction} \to -1]$

$c_{0,1} == 0$

$c_{0,1} + 2\, c_{0,2} + 3\, c_{0,3} == c_{1,1}$

$c_{1,1} + 2\, c_{1,2} + 3\, c_{1,3} == c_{2,1}$

$c_{2,1} + 2\, c_{2,2} + 3\, c_{2,3} == 0$

```
GenSols = Solve[{
    I1 == 0,
    I2 == 0,
    I3 == 0,
    T0[[1]] == 1,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[1]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0,
    T1[[4]] == 0,
    S0, S1, S2, S3
    },
    AllVars
]
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{c_{0,1}\to 0,\ c_{0,3}\to -1-c_{0,2},\ c_{1,1}\to -3-c_{0,2},\ c_{1,2}\to \frac{19}{4}+\frac{3\,c_{0,2}}{2},\ c_{1,3}\to -\frac{7}{4}-\frac{c_{0,2}}{2},\ c_{2,1}\to \frac{5}{4}+\frac{c_{0,2}}{2},\ c_{2,2}\to -\frac{5}{2}-c_{0,2},\ c_{2,3}\to \frac{5}{4}+\frac{c_{0,2}}{2}\right\}\right\}$$

```
RegionXY[k_] := {Quotient[k, 2], 1 + Quotient[-k, 2]};
Regions = Table[RegionXY[k], {k, -4, 7}]
```

{{-2, 3}, {-2, 2}, {-1, 2}, {-1, 1}, {0, 1},
 {0, 0}, {1, 0}, {1, -1}, {2, -1}, {2, -2}, {3, -2}, {3, -3}}

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
φ = 1/2;
```

$$W[k\_] := \begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1-(1-\varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases};$$

$$\text{SumF} = \sum_{i=-5}^{6}\sum_{j=-5}^{6} W[i-j]\, f[x-i,\ y-j]\ /.\ \text{GenSol};$$

```
SimplifySquare[f_, x0_, y0_] := Simplify[f, x > x0 && x < x0 + 1 && y > y0 && y < y0 + 1];
DSimplifySquare[f_, {x0_, y0_}] := Simplify[D[SimplifySquare[f, x0, y0], {{x, y}}]];
DSumF = ParallelMap[DSimplifySquare[SumF, #] &, Regions];

AnisoInt[df_, {x0_, y0_}] :=
  Simplify[Integrate[Expand[(df.{1, 1})^2], {x, x0, x0 + 1}, {y, y0, y0 + 1}]];
AnisoInts = Parallelize[MapThread[AnisoInt, {DSumF, Regions}]];
Err = Simplify[Total[AnisoInts]]
```

$$\left(92\,669\,325+117\,493\,344\,c_{0,2}+52\,220\,952\,c_{0,2}^2+9\,325\,760\,c_{0,2}^3+598\,096\,c_{0,2}^4\right)/\,25\,804\,800$$

```
FreeVars = Variables[Err];
DErr = Simplify[D[Err, {FreeVars}]];
H = D[DErr, {FreeVars}];
Sols = Solve[DErr == 0, FreeVars, Reals];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}ᵀ]
```

1	0.0575003	True

```
RootReduce[Sols[[1]]]
```

$\{c_{0,2} \to \text{Root}[7\,343\,334 + 6\,527\,619\, \#1 + 1\,748\,580\, \#1^2 + 149\,524\, \#1^3 \,\&,\, 1]\}$

```
NSol = N[Sols[[1]]];
FullSol = Join[GenSol /. NSol, NSol]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

$\{c_{0,1} \to 0,\ c_{0,3} \to 1.06787,\ c_{1,1} \to -0.932133,\ c_{1,2} \to 1.6482,\ c_{1,3} \to -0.716067,$
$c_{2,1} \to 0.216067,\ c_{2,2} \to -0.432133,\ c_{2,3} \to 0.216067,\ c_{0,2} \to -2.06787\}$