$$
\texttt{h[x\_] :=} \begin{cases} 1 + c01\,x + c02\,x^2 + c03\,x^3 + c04\,x^4 & 0 \le x \le 1 \\ c11\,(x-1) + c12\,(x-1)^2 + c13\,(x-1)^3 + c14\,(x-1)^4 & 1 < x \le 2 \\ 0 & \text{True} \end{cases};
$$

```
f[x_] := h[Abs[x]];
```

```
(*Interpolant constraints*)
I1 = f[1]
I2 = f[2]
```

```
1 + c01 + c02 + c03 + c04
```

```
c11 + c12 + c13 + c14
```

```
(*Partition of unity and gradient representation*)
T0 = CoefficientList[FullSimplify[f[x + 1] + f[x] + f[x - 1] + f[x - 2], x > 0 && x < 1], x]
T1 = CoefficientList[FullSimplify[-f[x + 1] + f[x - 1] + 2 f[x - 2], x > 0 && x < 1], x]
```

```
{2 + c01 + c02 + c03 + c04 + c11 + c12 + c13 + c14, -2 c02 - 3 c03 - 4 c04 - 2 c12 - 3 c13 - 4 c14,
 2 c02 + 3 c03 + 4 c04 + 2 c12 + 3 c13 + 4 c14 + 2 (c04 + c14), -4 (c04 + c14), 2 (c04 + c14)}
```

```
{1 + c01 + c02 + c03 + c04 + 2 c11 + 2 c12 + 2 c13 + 2 c14,
 -c01 - 2 c02 - 3 c03 - 4 c04 - 3 c11 - 4 c12 - 6 c13 - 8 c14,
 c02 + 3 c03 + 6 c04 + c12 + 6 c13 + 12 c14, -c03 - 4 c04 - 3 c13 - 8 c14, c04 + c14}
```

```
(*Smoothness*)
Df = Simplify[D[f[x], x], x > 0] /. Abs′[x] → 1
c01 + x (2 c02 + x (3 c03 + 4 c04 x)) /. x → 1
c11 + (2 c12 + (3 c13 + 4 c14 (-1 + x)) (-1 + x)) (-1 + x) /. x → 1
c11 + (2 c12 + (3 c13 + 4 c14 (-1 + x)) (-1 + x)) (-1 + x) /. x → 2
```

$$
\begin{cases} c01 + x\,(2\,c02 + x\,(3\,c03 + 4\,c04\,x)) & x \le 1 \\ c11 + (2\,c12 + (3\,c13 + 4\,c14\,(-1+x))\,(-1+x))\,(-1+x) & 1 < x \le 2 \\ 0 & \text{True} \end{cases}
$$

```
c01 + 2 c02 + 3 c03 + 4 c04
```

```
c11
```

```
c11 + 2 c12 + 3 c13 + 4 c14
```

```
GenSols = Solve[{
    I1 == 0,
    I2 == 0,
    T0[[1]] == 1,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[1]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0,
    T1[[4]] == 0,
    c01 == 0,
    c01 + 2 c02 + 3 c03 + 4 c04 == c11,
    c11 + 2 c12 + 3 c13 + 4 c14 == 0
    },
    {c01, c02, c03, c04, c11, c12, c13, c14}
]
```

Solve::svars : Equations may not give solutions for all "solve" variables. >>

$$\left\{\left\{c01 \to 0,\ c03 \to -\frac{7}{2} - 2\,c02,\ c04 \to \frac{5}{2} + c02,\ c11 \to -\frac{1}{2},\ c12 \to -\frac{3}{2} - c02,\ c13 \to \frac{9}{2} + 2\,c02,\ c14 \to -\frac{5}{2} - c02\right\}\right\}$$

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
W1[k_] := Piecewise[{{0, k < 0}, {φ^2/2, k == 0}, {1 - (1 - φ)^2/2, k == 1}}, 1];
```

$$\mathrm{W1}[k\_] := \begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1 - (1-\varphi)^2/2 & k == 1 \\ 1 & \mathrm{True} \end{cases};$$

$$\mathrm{SumF1} = \sum_{i=-3}^{5}\sum_{j=-3}^{5} \mathrm{W1}[i-j]\, f[x-i, y-j] \text{ /. GenSol};$$

```
{SumF1a1, SumF1a2, SumF1a3, SumF1a4} = Parallelize[{
    Simplify[SumF1, x > 0 && x < 1 && y > 0 && y < 1],
    Simplify[SumF1, x > 0 && x < 1 && y > 1 && y < 2],
    Simplify[SumF1, x > -1 && x < 0 && y > 1 && y < 2],
    Simplify[SumF1, x > -1 && x < 0 && y > 2 && y < 3]}];
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4} = Parallelize[{
    FullSimplify[D[SumF1a1, {{x, y}}]],
    FullSimplify[D[SumF1a2, {{x, y}}]],
    FullSimplify[D[SumF1a3, {{x, y}}]],
    FullSimplify[D[SumF1a4, {{x, y}}]]}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4} = Parallelize[{
    Simplify[SumF1, x > 1 && x < 2 && y > 0 && y < 1],
    Simplify[SumF1, x > 1 && x < 2 && y > -1 && y < 0],
    Simplify[SumF1, x > 2 && x < 3 && y > -1 && y < 0],
    Simplify[SumF1, x > 2 && x < 3 && y > -2 && y < -1]}];
{DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4} = Parallelize[{
    FullSimplify[D[SumF1b1, {{x, y}}]],
    FullSimplify[D[SumF1b2, {{x, y}}]],
    FullSimplify[D[SumF1b3, {{x, y}}]],
    FullSimplify[D[SumF1b4, {{x, y}}]]}];
```

{Err1a1, Err1a2, Err1a3, Err1a4} = Parallelize[{

$$\mathrm{Simplify}\left[\int_0^1\int_0^1 (\mathrm{DSumF1a1}.\{1, 1\})^2\, dx\, dy\right],$$

$$\mathrm{Simplify}\left[\int_1^2\int_0^1 (\mathrm{DSumF1a2}.\{1, 1\})^2\, dx\, dy\right],$$

$$\mathrm{Simplify}\left[\int_1^2\int_{-1}^0 (\mathrm{DSumF1a3}.\{1, 1\})^2\, dx\, dy\right],$$

$$\mathrm{Simplify}\left[\int_2^3\int_{-1}^0 (\mathrm{DSumF1a4}.\{1, 1\})^2\, dx\, dy\right]\}];$$

{Err1b1, Err1b2, Err1b3, Err1b4} = Parallelize[{

$$\mathrm{Simplify}\left[\int_0^1\int_1^2 (\mathrm{DSumF1b1}.\{1, 1\})^2\, dx\, dy\right],$$

$$\mathrm{Simplify}\left[\int_{-1}^0\int_1^2 (\mathrm{DSumF1b2}.\{1, 1\})^2\, dx\, dy\right],$$

$$\mathrm{Simplify}\left[\int_{-1}^0\int_2^3 (\mathrm{DSumF1b3}.\{1, 1\})^2\, dx\, dy\right],$$

$$\mathrm{Simplify}\left[\int_{-2}^{-1}\int_2^3 (\mathrm{DSumF1b4}.\{1, 1\})^2\, dx\, dy\right]\}];$$

```
Err1 = FullSimplify[Err1a1 + Err1a2 + Err1a3 + Err1a4 + Err1b1 + Err1b2 + Err1b3 + Err1b4];
```

```
Err = FullSimplify[Err1 /. φ → 1/2]
DErr = FullSimplify[D[Err, {{c02}}]];
H = FullSimplify[D[Err, {{c02}, 2}]];
Sols = RootReduce[Solve[DErr == 0, c02]];
N[Sols]
Sols[[3]]
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}ᵀ]
```

$(9318135 + 8\,c02\,(993711 + 2\,c02\,(190117 + 56\,c02\,(361 + 14\,c02)))) / 33868800$

{{c02 → -8.79369 + 3.62046 i}, {c02 → -8.79369 - 3.62046 i}, {c02 → -1.7519}}

{c02 → Root[993711 + 760468 #1 + 121296 #1^2 + 6272 #1^3 &, 1]}

1	0.813096 - 0.330049 i	False
2	0.813096 + 0.330049 i	False
3	0.0917079	True

```
Sol = Sols[[3]];
FullSol = N[Join[GenSol /. Sol, Sol]]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

{c01 → 0., c03 → 0.00379778, c04 → 0.748101, c11 → -0.5,
 c12 → 0.251899, c13 → 0.996202, c14 → -0.748101, c02 → -1.7519}