```
In[266]:= h[x_] := Piecewise[{
    {1 + c01 x + c02 x^2 + c03 x^3 + c04 x^4,  0 ≤ x ≤ 1/2},
    {c11 (x - 1) + c12 (x - 1)^2 + c13 (x - 1)^3 + c14 (x - 1)^4,  1/2 < x ≤ 3/2},
    {0,  True}}];
f[x_] := h[Abs[x]];

In[268]:= (*Continuity*)
C1l = Simplify[h[x], 0 ≤ x ≤ 1/2] /. x → 1/2
C1r = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 1/2
C2l = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 3/2
```

Out[268]= $1 + \frac{c01}{2} + \frac{c02}{4} + \frac{c03}{8} + \frac{c04}{16}$

Out[269]= $\frac{1}{2}\left(-c11 + \frac{1}{2}\left(c12 + \frac{1}{2}\left(-c13 + \frac{c14}{2}\right)\right)\right)$

Out[270]= $\frac{1}{2}\left(c11 + \frac{1}{2}\left(c12 + \frac{1}{2}\left(c13 + \frac{c14}{2}\right)\right)\right)$

```
In[271]:= (*Partition of unity and gradient representation*)
T0 = CoefficientList[FullSimplify[Sum[f[x - i], {i, -3, 3}], x > 0 && x < 1/2], x]
T1 = CoefficientList[FullSimplify[Sum[i f[x - i], {i, -3, 3}], x > 0 && x < 1/2], x]
```

Out[271]= {1, c01, c02 + 2 c12, c03, c04 + 2 c14}

Out[272]= {0, -2 c11, 0, -2 c13}

```
In[273]:= GenSols = Solve[{
    C1l == C1r,
    C2l == 0,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T0[[4]] == 0,
    T0[[5]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0,
    T1[[4]] == 0
    },
    {c01, c02, c03, c04, c11, c12, c13, c14}
]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[273]= $\left\{\left\{c01 \to 0,\ c03 \to 0,\ c04 \to -8 - 4\,c02,\ c11 \to -\frac{1}{2},\ c12 \to -\frac{c02}{2},\ c13 \to 0,\ c14 \to 4 + 2\,c02\right\}\right\}$

In[274]:=
```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
W1[k_] := Piecewise[{{0, k < 0}, {φ^2/2, k == 0}, {1 - (1 - φ)^2/2, k == 1}, {1, True}}];
SumF1 = Sum[W1[i - j] f[x - i, y - j], {i, -5, 6}, {j, -5, 6}] /. GenSol;
```

$$W1[k\_] := \begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1 - (1-\varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases}$$

$$\text{SumF1} = \sum_{i=-5}^{6}\sum_{j=-5}^{6} W1[i-j]\, f[x-i, y-j] \;/.\; \text{GenSol};$$

In[278]:=
```
{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6} = Parallelize[{
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 3 - 1/2 && y < 4 - 1/2]
}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6} = Parallelize[{
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -3 - 1/2 && y < -2 - 1/2]
}];
```

In[280]:=
```
TableForm[{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6}]
TableForm[{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6}]
```

Out[280]//TableForm=

$\frac{1}{8}\left(2\,x\left(1 - c02\,x + 4\left(2 + c02\right)x^3\right)y\left(-1 - c02\,y + 4\left(2 + c02\right)y^3\right) + x\left(-1 - c02\,x + 4\left(2 + c02\right)x^3\right)y\left(-1 - c\right.\right.$

$\frac{1}{8}\left(2\,x\left(1 - c02\,x + 4\left(2 + c02\right)x^3\right)\left(1 + c02\left(-1 + y\right)^2 - 4\left(2 + c02\right)\left(-1 + y\right)^4\right)\varphi^2 + 2\left(1 + c02\,x^2 - 4\left(2 + c02\right.\right.\right.$

$\frac{1}{8}\left(3 + 5\,x + 2\,x^2\right)\left(3 + c02 + 6\,x + 3\,c02\,x + 4\,x^2 + 2\,c02\,x^2\right)\left(1 - c02\left(-1 + y\right)^2 + 4\left(2 + c02\right)\left(-1 + y\right)^4 - y\right)\varphi^2$

0

0

0

Out[281]//TableForm=

$\frac{1}{8}\left(4\left(1 + c02\left(-1 + x\right)^2 - 4\left(2 + c02\right)\left(-1 + x\right)^4\right)y\left(-1 - c02\,y + 4\left(2 + c02\right)y^3\right) + 2\left(-1 - c02\left(-1 + x\right)^2 + 4\right.\right.$

$\frac{1}{8}\left(2\left(-1 - c02\left(-1 + x\right)^2 + 4\left(2 + c02\right)\left(-1 + x\right)^4 + x\right)\left(3 + 5\,y + 2\,y^2\right)\left(3 + c02 + 6\,y + 3\,c02\,y + 4\,y^2 + 2\,c02\,y^2\right.\right.$

$\frac{1}{4}\left(2\left(1 + c02\left(-2 + x\right)^2 - 4\left(2 + c02\right)\left(-2 + x\right)^4\right)\left(3 + 5\,y + 2\,y^2\right)\left(3 + c02 + 6\,y + 3\,c02\,y + 4\,y^2 + 2\,c02\,y^2\right) + \left(\right.\right.$

1

1

1

In[282]:=
```
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1b1, DSumF1b2, DSumF1b3} = Parallelize[{
    FullSimplify[D[SumF1a1, {{x, y}}]],
    FullSimplify[D[SumF1a2, {{x, y}}]],
    FullSimplify[D[SumF1a3, {{x, y}}]],
    FullSimplify[D[SumF1b1, {{x, y}}]],
    FullSimplify[D[SumF1b2, {{x, y}}]],
    FullSimplify[D[SumF1b3, {{x, y}}]]
}];
```

In[283]:=
```
DSumF1a1 = Simplify[DSumF1a1 /. φ → 1/2];
DSumF1a2 = Simplify[DSumF1a2 /. φ → 1/2];
DSumF1a3 = Simplify[DSumF1a3 /. φ → 1/2];
DSumF1b1 = Simplify[DSumF1b1 /. φ → 1/2];
DSumF1b2 = Simplify[DSumF1b2 /. φ → 1/2];
DSumF1b3 = Simplify[DSumF1b3 /. φ → 1/2];
```

In[289]:=
```
{Err1a1, Err1a2, Err1a3} = Parallelize[{
    Simplify[∫_{0-1/2}^{1-1/2} ∫_{0-1/2}^{1-1/2} (DSumF1a1.{1, 1})^2 dx dy],
    Simplify[∫_{1-1/2}^{2-1/2} ∫_{0-1/2}^{1-1/2} (DSumF1a2.{1, 1})^2 dx dy],
    Simplify[∫_{1-1/2}^{2-1/2} ∫_{-1-1/2}^{0-1/2} (DSumF1a3.{1, 1})^2 dx dy]
}];
{Err1b1, Err1b2, Err1b3} = Parallelize[{
    Simplify[∫_{0-1/2}^{1-1/2} ∫_{1-1/2}^{2-1/2} (DSumF1b1.{1, 1})^2 dx dy],
    Simplify[∫_{-1-1/2}^{0-1/2} ∫_{1-1/2}^{2-1/2} (DSumF1b2.{1, 1})^2 dx dy],
    Simplify[∫_{-1-1/2}^{0-1/2} ∫_{2-1/2}^{3-1/2} (DSumF1b3.{1, 1})^2 dx dy]
}];
```

In[291]:=
```
Err1 = FullSimplify[Err1a1 + Err1a2 + Err1a3 + Err1b1 + Err1b2 + Err1b3];
```

In[292]:=
```
Err = Err1
DErr = FullSimplify[D[Err, {c02}]];
H = FullSimplify[D[Err, {{c02}, 2}]];
Sols = Solve[DErr == 0, c02];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}^T]
```

Out[292]= $\left(9\,621\,843 + 2\,c02\,\left(2\,288\,145 + c02\,\left(489\,617 + c02\,\left(32\,705 + 1133\,c02\right)\right)\right)\right) / 16\,934\,400$

Out[296]//TableForm=

1	0.183005	True
2	0.240425 + 1.30089 i	False
3	0.240425 − 1.30089 i	False

In[297]:=
```
RootReduce[Sols[[1]]]
```

Out[297]= $\{c02 \to \text{Root}[2\,288\,145 + 979\,234\,\#1 + 98\,115\,\#1^2 + 4532\,\#1^3\,\&,\,1]\}$

In[298]:=
```
Sol = Sols[[1]];
FullSol = N[Join[GenSol /. Sol, Sol]]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

Out[299]=
```
{c01 → 0., c03 → 0., c04 → 4.8882, c11 → -0.5,
 c12 → 1.61102, c13 → 0., c14 → -2.4441, c02 → -3.22205}
```

Out[301]=