In[342]:=
```
h[x_] := Piecewise[{{1 + c01 x + c02 x^2 + c03 x^3, 0 ≤ x ≤ 1/2},
  {c11 (x - 1) + c12 (x - 1)^2 + c13 (x - 1)^3, 1/2 < x ≤ 3/2},
  {c21 (x - 2) + c22 (x - 2)^2 + c23 (x - 2)^3, 3/2 < x ≤ 5/2}}, 0];
f[x_] := h[Abs[x]];
```

In[344]:=
```
(*Continuity*)
C1l = Simplify[h[x], 0 ≤ x ≤ 1/2] /. x → 1/2
C1r = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 1/2
C2l = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 3/2
C2r = Simplify[h[x], 3/2 < x ≤ 5/2] /. x → 3/2
C3l = Simplify[h[x], 3/2 < x ≤ 5/2] /. x → 5/2
```

Out[344]= $1 + \frac{c01}{2} + \frac{c02}{4} + \frac{c03}{8}$

Out[345]= $\frac{1}{2}\left(-c11 + \frac{1}{2}\left(c12 - \frac{c13}{2}\right)\right)$

Out[346]= $\frac{1}{2}\left(c11 + \frac{1}{2}\left(c12 + \frac{c13}{2}\right)\right)$

Out[347]= $\frac{1}{2}\left(-c21 + \frac{1}{2}\left(c22 - \frac{c23}{2}\right)\right)$

Out[348]= $\frac{1}{2}\left(c21 + \frac{1}{2}\left(c22 + \frac{c23}{2}\right)\right)$

In[349]:=
```
(*Partition of unity and gradient representation*)
T0 = CoefficientList[FullSimplify[Sum[f[x - i], {i, -6, 6}], x > 0 && x < 1/2], x]
T1 = CoefficientList[FullSimplify[Sum[i f[x - i], {i, -6, 6}], x > 0 && x < 1/2], x]
```

Out[349]= $\{1, c01, c02 + 2\,(c12 + c22), c03\}$

Out[350]= $\{0, -2\,c11 - 4\,c21, 0, -2\,(c13 + 2\,c23)\}$

In[351]:=
```
GenSols = Solve[{
    C1l == C1r,
    C2l == C2r,
    C3l == 0,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T0[[4]] == 0,
    T1[[2]] == 1,
    T1[[4]] == 0
    },
    {c01, c02, c03, c11, c12, c13, c21, c22, c23}
]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[351]=
$$\left\{\left\{c01 \to 0,\ c03 \to 0,\ c12 \to 1,\ c13 \to -6 - 2\,c02 - 4\,c11,\ c21 \to -\frac{1}{4} - \frac{c11}{2},\ c22 \to -1 - \frac{c02}{2},\ c23 \to 3 + c02 + 2\,c11\right\}\right\}$$

In[361]:=
```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
W1[k_] := Piecewise[{{0, k < 0}, {φ^2/2, k == 0}, {1 - (1 - φ)^2/2, k == 1}, {1, True}}];
SumF1 = Sum[Sum[W1[i - j] f[x - i, y - j] /. GenSol, {j, -5, 6}], {i, -5, 6}];
SumF1 = SumF1 /. φ → 1/2;
```

In[366]:=
```
{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6} = Parallelize[{
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 3 - 1/2 && y < 4 - 1/2]
}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6} = Parallelize[{
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -3 - 1/2 && y < -2 - 1/2]
}];
```

In[368]:=
```
TableForm[{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6}]
TableForm[{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6}]
```

Out[368]//TableForm=

$\frac{1}{64}\left(-8\left(-1+\left(2-3\,c11\right)y+\left(1+3\,c02\right)y^2+6\left(3+c02+2\,c11\right)y^3\right)-2\,x^2\left(1+2\,y\right)\left(4-\left(1+14\,c11\right)y+18\right.$

$\frac{1}{32}\left(-77+28\,x-47\,x^2+234\,x^3+221\,y-58\,x\,y+59\,x^2\,y-858\,x^3\,y-204\,y^2+55\,x\,y^2-48\,x^2\,y^2+852\,x^3\,y^2+60\right.$

$\frac{1}{32}\left(-\left(1+x\right)\left(3+2\,x\right)\left(5+6\,x+2\,c02\left(1+x\right)+c11\left(2+4\,x\right)\right)\left(1+c02\left(-1+y\right)^2\right)+4\left(-1-x\right)\left(-1+c11-\right.\right.$

$\frac{1}{128}\left(-2+y\right)\left(-4\left(1+x\right)\left(3+2\,x\right)\left(5+6\,x+2\,c02\left(1+x\right)+c11\left(2+4\,x\right)\right)\left(-2+c11-2\left(3+c02+2\,c11\right)\left(-\right.\right.$

$-\frac{1}{128}\left(2+x\right)\left(5+2\,x\right)\left(11+6\,x+2\,c02\left(2+x\right)+c11\left(6+4\,x\right)\right)\left(-2+y\right)\left(-5+2\,y\right)\left(-11+2\,c02\left(-2+y\right)+\right.$

0

Out[369]//TableForm=

$\frac{1}{64}\left(-96+432\,x-424\,x^2+144\,x^3-9\,y-41\,x\,y+58\,x^2\,y-24\,x^3\,y-94\,y^2+358\,x\,y^2-424\,x^2\,y^2+168\,x^3\,y^2-21\right.$

$\frac{1}{32}\left(-106+636\,x-655\,x^2+234\,x^3-636\,y+2514\,x\,y-2515\,x^2\,y+858\,x^3\,y-655\,y^2+2515\,x\,y^2-2508\,x^2\,y^2+8\right.$

$\frac{1}{32}\left(6251-9032\,x+4369\,x^2-702\,x^3+17\,475\,y-25\,382\,x\,y+12\,281\,x^2\,y-1974\,x^3\,y+16\,478\,y^2-23\,931\,x\,y^2+\right.$

$\frac{1}{128}\left(-38\,812+55\,918\,x-26\,432\,x^2+4104\,x^3-55\,918\,y+80\,295\,x\,y-37\,952\,x^2\,y+5892\,x^3\,y-26\,432\,y^2+37\,9\right.$

$\frac{1}{128}\left(-39\,142+38\,170\,x-12\,320\,x^2+1320\,x^3-56\,763\,y+55\,173\,x\,y-17\,808\,x^2\,y+1908\,x^3\,y-27\,132\,y^2+26\,3\right.$

1

In[370]:=
```
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4, DSumF1a5,
   DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4, DSumF1b5} = Parallelize[{
    Simplify[D[SumF1a1, {{x, y}}]],
    Simplify[D[SumF1a2, {{x, y}}]],
    Simplify[D[SumF1a3, {{x, y}}]],
    Simplify[D[SumF1a4, {{x, y}}]],
    Simplify[D[SumF1a5, {{x, y}}]],
    Simplify[D[SumF1b1, {{x, y}}]],
    Simplify[D[SumF1b2, {{x, y}}]],
    Simplify[D[SumF1b3, {{x, y}}]],
    Simplify[D[SumF1b4, {{x, y}}]],
    Simplify[D[SumF1b5, {{x, y}}]]
}];
```

In[373]:=
```
{Err1a1, Err1a2, Err1a3, Err1a4, Err1a5,
   Err1b1, Err1b2, Err1b3, Err1b4, Err1b5} = Parallelize[{
    Simplify[∫_(0-1/2)^(1-1/2) ∫_(0-1/2)^(1-1/2) (DSumF1a1.{1, 1})^2 dx dy],
    Simplify[∫_(1-1/2)^(2-1/2) ∫_(0-1/2)^(1-1/2) (DSumF1a2.{1, 1})^2 dx dy],
    Simplify[∫_(1-1/2)^(2-1/2) ∫_(-1-1/2)^(0-1/2) (DSumF1a3.{1, 1})^2 dx dy],
    Simplify[∫_(2-1/2)^(3-1/2) ∫_(-1-1/2)^(0-1/2) (DSumF1a4.{1, 1})^2 dx dy],
    Simplify[∫_(2-1/2)^(3-1/2) ∫_(-2-1/2)^(-1-1/2) (DSumF1a5.{1, 1})^2 dx dy],
        Simplify[∫_(0-1/2)^(1-1/2) ∫_(1-1/2)^(2-1/2) (DSumF1b1.{1, 1})^2 dx dy],
    Simplify[∫_(-1-1/2)^(0-1/2) ∫_(1-1/2)^(2-1/2) (DSumF1b2.{1, 1})^2 dx dy],
    Simplify[∫_(-1-1/2)^(0-1/2) ∫_(2-1/2)^(3-1/2) (DSumF1b3.{1, 1})^2 dx dy],
    Simplify[∫_(-2-1/2)^(-1-1/2) ∫_(2-1/2)^(3-1/2) (DSumF1b4.{1, 1})^2 dx dy],
    Simplify[∫_(-2-1/2)^(-1-1/2) ∫_(3-1/2)^(4-1/2) (DSumF1b5.{1, 1})^2 dx dy]
  }];
```

In[374]:=
```
Err1 = FullSimplify[
   Err1a1 + Err1a2 + Err1a3 + Err1a4 + Err1a5 + Err1b1 + Err1b2 + Err1b3 + Err1b4 + Err1b5];
```

In[384]:=
```
Err = Err1
DErr = FullSimplify[D[Err, {{c02, c11}}]];
H = FullSimplify[D[Err, {{c02, c11}, 2}]];
Sols = Solve[DErr == 0, {c02, c11}];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}^T]
```

Out[384]=
$$\frac{1}{12\,902\,400}\left(638\,955\,c02^4 + c02^3\left(7\,449\,812 - 12\,024\,c11\right) + 12\,c02^2\left(3\,036\,379 + 6\,c11\left(1765 + 2348\,c11\right)\right) + 96\,c02\left(786\,959 + c11\left(109\,083 + 4\,c11\left(10\,004 + 503\,c11\right)\right)\right) + 96\left(633\,789 + 2\,c11\left(155\,618 + c11\left(70\,367 + 4194\,c11 + 4024\,c11^2\right)\right)\right)\right)$$

Out[388]//TableForm=

1	0.0902019	True
2	1.51531 + 0.202274 i	False
3	1.51531 − 0.202274 i	False
4	1.62536 + 2.08458 i	False
5	1.62536 − 2.08458 i	False
6	−0.624601 + 3.89872 i	False
7	−0.624601 − 3.89872 i	False
8	−0.218441 + 3.24005 i	False
9	−0.218441 − 3.24005 i	False

In[389]:=
```
RootReduce[Sols[[1]]]
```

Out[389]=
```
{c02 → Root[95 847 501 175 547 613 564 801 600 + 290 787 026 673 489 172 616 069 184 #1 +
      427 693 530 277 085 972 126 756 216 #1^2 + 374 454 369 419 578 668 527 648 440 #1^3 +
      206 526 493 409 660 806 096 287 626 #1^4 + 73 985 811 686 932 641 040 952 616 #1^5 +
      17 333 088 289 108 701 496 879 625 #1^6 + 2 585 430 622 222 421 745 766 995 #1^7 +
      224 999 970 811 293 497 663 025 #1^8 + 8 794 043 350 198 430 409 225 #1^9 &, 1],
 c11 → Root[186 034 729 797 937 411 538 022 907 705 + 266 195 094 834 401 033 071 061 383 245 #1 +
      23 814 746 789 073 430 925 129 080 750 #1^2 - 74 579 241 765 856 896 300 539 908 358 #1^3 -
      31 992 383 923 550 463 074 170 477 624 #1^4 + 27 118 665 196 708 626 294 219 280 904 #1^5 +
      12 658 942 797 006 126 318 184 281 760 #1^6 + 12 703 022 498 519 592 874 591 024 320 #1^7 +
      1 336 110 826 973 695 570 314 316 800 #1^8 + 1 509 855 165 491 135 315 913 446 400 #1^9 &, 1]}
```

In[390]:=
```
Sol = Sols[[1]];
FullSol = N[Join[GenSol /. Sol, Sol]]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

Out[391]=
```
{c01 → 0., c03 → 0., c12 → 1., c13 → 0.463315, c21 → 0.162576,
 c22 → -0.209324, c23 → -0.231657, c02 → -1.58135, c11 → -0.825153}
```

Out[393]=