

```

In[101]:= h[x_] := 
$$\begin{cases} 1 + c01 x + c02 x^2 + c03 x^3 + c04 x^4 & 0 \leq x \leq 1 \\ c11 (x-1) + c12 (x-1)^2 + c13 (x-1)^3 + c14 (x-1)^4 & 1 < x \leq 2; \\ 0 & \text{True} \end{cases}$$


f[x_] := h[Abs[x]];

In[103]:= (*Interpolant constraints*)
I1 = f[1]
I2 = f[2]

Out[103]= 1 + c01 + c02 + c03 + c04

Out[104]= c11 + c12 + c13 + c14

In[105]:= (*Partition of unity and gradient representation*)
T0 = CoefficientList[FullSimplify[f[x + 1] + f[x] + f[x - 1] + f[x - 2], x > 0 && x < 1], x]
T1 = CoefficientList[FullSimplify[-f[x + 1] + f[x - 1] + 2 f[x - 2], x > 0 && x < 1], x]

Out[105]= {2 + c01 + c02 + c03 + c04 + c11 + c12 + c13 + c14, -2 c02 - 3 c03 - 4 c04 - 2 c12 - 3 c13 - 4 c14,
2 c02 + 3 c03 + 4 c04 + 2 c12 + 3 c13 + 4 c14 + 2 (c04 + c14), -4 (c04 + c14), 2 (c04 + c14)}

Out[106]= {1 + c01 + c02 + c03 + c04 + 2 c11 + 2 c12 + 2 c13 + 2 c14,
-c01 - 2 c02 - 3 c03 - 4 c04 - 3 c11 - 4 c12 - 6 c13 - 8 c14,
c02 + 3 c03 + 6 c04 + c12 + 6 c13 + 12 c14, -c03 - 4 c04 - 3 c13 - 8 c14, c04 + c14}

In[120]:= GenSols = Solve[{
I1 == 0,
I2 == 0,
T0[[1]] == 1,
T0[[2]] == 0,
T0[[3]] == 0,
T1[[1]] == 0,
T1[[2]] == 1,
T1[[3]] == 0,
T1[[4]] == 0,
},
{c01, c02, c03, c04, c11, c12, c13, c14}
]

```

 **Solve:** Equations may not give solutions for all "solve" variables.

```

Out[120]= {{c04 -> -1 - c01 - c02 - c03, c11 -> - $\frac{5}{3} - \frac{5 c01}{3} - \frac{2 c02}{3} - \frac{c03}{3}$ ,
c12 -> 2 + 2 c01 + c02 + c03, c13 -> - $\frac{4}{3} - \frac{4 c01}{3} - \frac{4 c02}{3} - \frac{5 c03}{3}$ , c14 -> 1 + c01 + c02 + c03}}

```

```

In[121]:= GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
φ = 1/2;

W1[k_] := 
$$\begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1 - (1 - \varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases};$$


SumF1 =  $\sum_{i=-3}^5 \sum_{j=-3}^5 W1[i-j] f[x-i, y-j] /. \text{GenSol};$ 

{SumF1a1, SumF1a2, SumF1a3, SumF1a4} = Parallelize[{
  Simplify[SumF1, x > 0 && x < 1 && y > 0 && y < 1],
  Simplify[SumF1, x > 0 && x < 1 && y > 1 && y < 2],
  Simplify[SumF1, x > -1 && x < 0 && y > 1 && y < 2],
  Simplify[SumF1, x > -1 && x < 0 && y > 2 && y < 3] }];

{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4} = Parallelize[{
  FullSimplify[D[SumF1a1, {{x, y}}]],
  FullSimplify[D[SumF1a2, {{x, y}}]],
  FullSimplify[D[SumF1a3, {{x, y}}]],
  FullSimplify[D[SumF1a4, {{x, y}}]] }];

{SumF1b1, SumF1b2, SumF1b3, SumF1b4} = Parallelize[{
  Simplify[SumF1, x > 1 && x < 2 && y > 0 && y < 1],
  Simplify[SumF1, x > 1 && x < 2 && y > -1 && y < 0],
  Simplify[SumF1, x > 2 && x < 3 && y > -1 && y < 0],
  Simplify[SumF1, x > 2 && x < 3 && y > -2 && y < -1] }];

{DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4} = Parallelize[{
  FullSimplify[D[SumF1b1, {{x, y}}]],
  FullSimplify[D[SumF1b2, {{x, y}}]],
  FullSimplify[D[SumF1b3, {{x, y}}]],
  FullSimplify[D[SumF1b4, {{x, y}}]] }];

In[130]:= {Err1a1, Err1a2, Err1a3, Err1a4, Err1b1, Err1b2, Err1b3, Err1b4} = Parallelize[{
  Simplify[ $\int_0^1 \int_0^1 (\text{DSumF1a1}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_1^2 \int_0^1 (\text{DSumF1a2}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_1^2 \int_{-1}^0 (\text{DSumF1a3}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_2^3 \int_{-1}^0 (\text{DSumF1a4}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_0^1 \int_1^2 (\text{DSumF1b1}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_{-1}^0 \int_1^2 (\text{DSumF1b2}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_{-1}^0 \int_2^3 (\text{DSumF1b3}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_{-2}^{-1} \int_2^3 (\text{DSumF1b4}.\{1, 1\})^2 dx dy$ ]
}];

```

```
In[131]:= Err1 = FullSimplify[Err1a1 + Err1a2 + Err1a3 + Err1a4 + Err1b1 + Err1b2 + Err1b3 + Err1b4];
```

```
In[143]:= Err = FullSimplify[Err1 /.  $\varphi \rightarrow 1/2$ ]
DErr = FullSimplify[D[Err, {{c01, c02, c03}}]];
H = FullSimplify[D[Err, {{c01, c02, c03}, 2}]];
NSols = NSolve[DErr == 0, {c01, c02, c03}, Reals]
TableForm[
  {Range[Length[NSols]], Err /. NSols, PositiveDefiniteMatrixQ[H /. N[#]] & /@ NSols}^T]
```

```
Out[143]= 
$$\frac{1}{6350400} \left( 2 \left( 14121001 + 4601736 c01^4 + \right. \right. \\ \left. 4 c01^3 \left( 6646912 + 2075877 c02 \right) + 6 c01^2 \left( 9235274 + c02 \left( 6080303 + 945524 c02 \right) \right) + \right. \\ \left. 6 c01 \left( 7852793 + c02 \left( 8567389 + 16 c02 \left( 174999 + 18160 c02 \right) \right) \right) + \right. \\ \left. c02 \left( 22099269 + 4 c02 \left( 3006474 + c02 \left( 648409 + 51042 c02 \right) \right) \right) \right) + \\ 2 \left( 8555243 + c01 \left( 19760445 + 2 c01 \left( 6949914 + 1567813 c01 \right) \right) + 9321723 c02 + \right. \\ \left. 6 c01 \left( 2148514 + 721303 c01 \right) c02 + 6 \left( 500663 + 336310 c01 \right) c02^2 + 319400 c02^3 \right) c03 + \\ 2 \left( 1840904 + 838616 c01^2 + c02 \left( 1175017 + 190430 c02 \right) + c01 \left( 2508664 + 791074 c02 \right) \right) c03^2 + \\ \left. 4 \left( 77435 + 52438 c01 + 25600 c02 \right) c03^3 + 10461 c03^4 \right)$$

```

```
Out[146]= {{c01 → -0.617904, c02 → -0.432006, c03 → 0.1175}}
```

```
Out[147]//TableForm=
```

```
1      0.0493978      True
```

```
(*Bugged without the Reals dmain restriction*)
```

```
Err = FullSimplify[Err1 /.  $\varphi \rightarrow 1/2$ ]
DErr = FullSimplify[D[Err, {{c01, c02, c03}}]];
H = FullSimplify[D[Err, {{c01, c02, c03}, 2}]];
Sols = Solve[DErr == 0, {c01, c02, c03}, Reals];
TableForm[
  {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}^T]
```

```
Out[191]= 
$$\frac{1}{6350400} \left( 2 \left( 14121001 + 4601736 c01^4 + \right. \right. \\ \left. 4 c01^3 \left( 6646912 + 2075877 c02 \right) + 6 c01^2 \left( 9235274 + c02 \left( 6080303 + 945524 c02 \right) \right) + \right. \\ \left. 6 c01 \left( 7852793 + c02 \left( 8567389 + 16 c02 \left( 174999 + 18160 c02 \right) \right) \right) + \right. \\ \left. c02 \left( 22099269 + 4 c02 \left( 3006474 + c02 \left( 648409 + 51042 c02 \right) \right) \right) \right) + \\ 2 \left( 8555243 + c01 \left( 19760445 + 2 c01 \left( 6949914 + 1567813 c01 \right) \right) + 9321723 c02 + \right. \\ \left. 6 c01 \left( 2148514 + 721303 c01 \right) c02 + 6 \left( 500663 + 336310 c01 \right) c02^2 + 319400 c02^3 \right) c03 + \\ 2 \left( 1840904 + 838616 c01^2 + c02 \left( 1175017 + 190430 c02 \right) + c01 \left( 2508664 + 791074 c02 \right) \right) c03^2 + \\ \left. 4 \left( 77435 + 52438 c01 + 25600 c02 \right) c03^3 + 10461 c03^4 \right)$$

```

```
Out[195]//TableForm=
```

```
1      0.0493978      True
```

```
(*Huge in symbolic form*)
```

```
N[Sols[[1]]]
```

```
Out[196]= {c01 → -0.617904, c02 → -0.432006, c03 → 0.1175}
```

```
In[203]:= RootReduce[Sols[[1]]]
```

```
Out[203]= $Aborted
```

```
In[198]:= Sol = N[Sols[[1]]];  
FullSol = N[Join[GenSol /. Sol, Sol]]  
fo[x_] := f[x] /. FullSol;  
Plot[fo[x], {x, -3, 3}, PlotStyle -> Black, Background -> White]  
Out[199]= {c04 -> -0.0675902, c11 -> -0.38799, c12 -> 0.449686, c13 -> -0.129287,  
c14 -> 0.0675902, c01 -> -0.617904, c02 -> -0.432006, c03 -> 0.1175}
```

Out[201]=

