```
h[x_] := Piecewise[{{1 + c0,1 x + c0,2 x^2 + c0,3 x^3 + c0,4 x^4, 0 ≤ x ≤ 1},
                    {c1,1 (x - 1) + c1,2 (x - 1)^2 + c1,3 (x - 1)^3 + c1,4 (x - 1)^4, 1 < x ≤ 2}},
                   0, True];
f[x_] := h[Abs[x]];
AllVars = {c0,1, c0,2, c0,3, c0,4, c1,1, c1,2, c1,3, c1,4};

(*Interpolant constraints*)
I1 = f[1]
I2 = f[2]
```

$$1 + c_{0,1} + c_{0,2} + c_{0,3} + c_{0,4}$$

$$c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4}$$

```
(*Partition of unity and linear term*)
T0 = CoefficientList[FullSimplify[Sum[f[x - k], {k, -1, 2}], x > 0 && x < 1], x]
T1 = CoefficientList[FullSimplify[Sum[k f[x - k], {k, -1, 2}], x > 0 && x < 1], x]
```

$$\{2 + c_{0,1} + c_{0,2} + c_{0,3} + c_{0,4} + c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4},\ -2\,c_{0,2} - 3\,c_{0,3} - 4\,c_{0,4} - 2\,c_{1,2} - 3\,c_{1,3} - 4\,c_{1,4},\ 2\,c_{0,2} + 3\,c_{0,3} + 6\,c_{0,4} + 2\,c_{1,2} + 3\,c_{1,3} + 6\,c_{1,4},\ -4\,c_{0,4} - 4\,c_{1,4},\ 2\,c_{0,4} + 2\,c_{1,4}\}$$

$$\{1 + c_{0,1} + c_{0,2} + c_{0,3} + c_{0,4} + 2\,c_{1,1} + 2\,c_{1,2} + 2\,c_{1,3} + 2\,c_{1,4},\ -c_{0,1} - 2\,c_{0,2} - 3\,c_{0,3} - 4\,c_{0,4} - 3\,c_{1,1} - 4\,c_{1,2} - 6\,c_{1,3} - 8\,c_{1,4},\ c_{0,2} + 3\,c_{0,3} + 6\,c_{0,4} + c_{1,2} + 6\,c_{1,3} + 12\,c_{1,4},\ -c_{0,3} - 4\,c_{0,4} - 3\,c_{1,3} - 8\,c_{1,4},\ c_{0,4} + c_{1,4}\}$$

```
(*Smoothness*)
Dh = Simplify[D[h[x], x], x > 0];
S0 = (Dh /. x → 0) == 0
S1 = Limit[Dh, x → 1, Direction → 1] == Limit[Dh, x → 1, Direction → -1]
S2 = Limit[Dh, x → 2, Direction → 1] == Limit[Dh, x → 2, Direction → -1]
```

$$c_{0,1} == 0$$

$$c_{0,1} + 2\,c_{0,2} + 3\,c_{0,3} + 4\,c_{0,4} == c_{1,1}$$

$$c_{1,1} + 2\,c_{1,2} + 3\,c_{1,3} + 4\,c_{1,4} == 0$$

```
GenSols = Solve[{
    I1 == 0,
    I2 == 0,
    T0[[1]] == 1,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[1]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0,
    T1[[4]] == 0,
    S0, S1, S2
    },
    AllVars
]
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{c_{0,1} \to 0,\ c_{0,3} \to -\frac{7}{2} - 2\,c_{0,2},\ c_{0,4} \to \frac{5}{2} + c_{0,2},\right.\right.$$
$$\left.\left. c_{1,1} \to -\frac{1}{2},\ c_{1,2} \to -\frac{3}{2} - c_{0,2},\ c_{1,3} \to \frac{9}{2} + 2\,c_{0,2},\ c_{1,4} \to -\frac{5}{2} - c_{0,2}\right\}\right\}$$

```
RegionXY[k_] := {Quotient[k, 2], 1 + Quotient[-k, 2]};
Regions = Table[RegionXY[k], {k, -2, 5}]
```

```
{{-1, 2}, {-1, 1}, {0, 1}, {0, 0}, {1, 0}, {1, -1}, {2, -1}, {2, -2}}
```

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
φ = 1 / 2;
```

$$W[k\_] := \begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1 - (1-\varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases};$$

$$\text{SumF} = \sum_{i=-3}^{5} \sum_{j=-3}^{5} W[i-j]\, f[x-i, y-j] \;/.\; \text{GenSol};$$

```
SimplifySquare[f_, x0_, y0_] := Simplify[f, x > x0 && x < x0 + 1 && y > y0 && y < y0 + 1];
DSimplifySquare[f_, {x0_, y0_}] := Simplify[D[SimplifySquare[f, x0, y0], {{x, y}}]];
DSumF = ParallelMap[DSimplifySquare[SumF, #] &, Regions];
```

```
AnisoInt[df_, {x0_, y0_}] :=
  Simplify[Integrate[Expand[(df.{1, 1})^2], {x, x0, x0 + 1}, {y, y0, y0 + 1}]];
AnisoInts = Parallelize[MapThread[AnisoInt, {DSumF, Regions}]];
Err = Simplify[Total[AnisoInts]]
```

$$\left(9\,318\,135 + 7\,949\,688\,c_{0,2} + 3\,041\,872\,c_{0,2}^2 + 323\,456\,c_{0,2}^3 + 12\,544\,c_{0,2}^4\right) / 33\,868\,800$$

```
FreeVars = Variables[Err];
DErr = Simplify[D[Err, {FreeVars}]];
H = D[DErr, {FreeVars}];
Sols = Solve[DErr == 0, FreeVars, Reals];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#] & /@ Sols}^T]
```

```
1    0.0917079    True
```

```
RootReduce[Sols[[1]]]
```

$$\left\{c_{0,2} \to \text{Root}\left[993\,711 + 760\,468\,\#1 + 121\,296\,\#1^2 + 6272\,\#1^3 \&,\ 1\right]\right\}$$

```
NSol = N[Sols[[1]]];
FullSol = Join[GenSol /. NSol, NSol]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

$$\left\{c_{0,1} \to 0,\ c_{0,3} \to 0.00379778,\ c_{0,4} \to 0.748101,\ c_{1,1} \to -\frac{1}{2},\right.$$
$$\left. c_{1,2} \to 0.251899,\ c_{1,3} \to 0.996202,\ c_{1,4} \to -0.748101,\ c_{0,2} \to -1.7519\right\}$$