```
h[x_] := Piecewise[{
   {1 + c[0,1] x + c[0,2] x^2 + c[0,3] x^3, 0 ≤ x ≤ 1/2},
   {c[1,1] (x - 1) + c[1,2] (x - 1)^2 + c[1,3] (x - 1)^3, 1/2 < x ≤ 3/2},
   {c[2,1] (x - 2) + c[2,2] (x - 2)^2 + c[2,3] (x - 2)^3, 3/2 < x ≤ 5/2},
   {0, True}}];
```

$$h[x\_] := \begin{cases} 1 + c_{0,1}\,x + c_{0,2}\,x^2 + c_{0,3}\,x^3 & 0 \le x \le 1/2 \\ c_{1,1}\,(x-1) + c_{1,2}\,(x-1)^2 + c_{1,3}\,(x-1)^3 & 1/2 < x \le 3/2 \\ c_{2,1}\,(x-2) + c_{2,2}\,(x-2)^2 + c_{2,3}\,(x-2)^3 & 3/2 < x \le 5/2 \\ 0 & \text{True} \end{cases};$$

```
f[x_] := h[Abs[x]];
AllVars = {c0,1, c0,2, c0,3, c1,1, c1,2, c1,3, c2,1, c2,2, c2,3};

(*Continuity*)
C1 = Limit[h[x], x → 1/2, Direction → 1] == Limit[h[x], x → 1/2, Direction → -1]
C2 = Limit[h[x], x → 3/2, Direction → 1] == Limit[h[x], x → 3/2, Direction → -1]
C3 = Limit[h[x], x → 5/2, Direction → 1] == Limit[h[x], x → 5/2, Direction → -1]
```

$$\frac{1}{8}\left(8 + 4\,c_{0,1} + 2\,c_{0,2} + c_{0,3}\right) == \frac{1}{8}\left(-4\,c_{1,1} + 2\,c_{1,2} - c_{1,3}\right)$$

$$\frac{1}{8}\left(4\,c_{1,1} + 2\,c_{1,2} + c_{1,3}\right) == \frac{1}{8}\left(-4\,c_{2,1} + 2\,c_{2,2} - c_{2,3}\right)$$

$$\frac{1}{8}\left(4\,c_{2,1} + 2\,c_{2,2} + c_{2,3}\right) == 0$$

```
(*Partition of unity and linear term*)
```

$$\text{T0} = \text{CoefficientList}\left[\text{FullSimplify}\left[\sum_{i=-6}^{6} f[x-i],\ x > 0\ \&\&\ x < 1/2\right],\ x\right]$$

$$\text{T1} = \text{CoefficientList}\left[\text{FullSimplify}\left[\sum_{i=-6}^{6} i\, f[x-i],\ x > 0\ \&\&\ x < 1/2\right],\ x\right]$$

$$\{1,\ c_{0,1},\ c_{0,2} + 2\,(c_{1,2} + c_{2,2}),\ c_{0,3}\}$$

$$\{0,\ -2\,c_{1,1} - 4\,c_{2,1},\ 0,\ -2\,(c_{1,3} + 2\,c_{2,3})\}$$

```
GenSols = Solve[{
    C1, C2, C3,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T0[[4]] == 0,
    T1[[2]] == 1,
    T1[[4]] == 0
    },
    AllVars
]
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{c_{0,1} \to 0,\ c_{0,3} \to 0,\ c_{1,2} \to 1,\ c_{1,3} \to -6 - 2\,c_{0,2} - 4\,c_{1,1},\ c_{2,1} \to -\frac{1}{4} - \frac{c_{1,1}}{2},\ c_{2,2} \to -1 - \frac{c_{0,2}}{2},\ c_{2,3} \to 3 + c_{0,2} + 2\,c_{1,1}\right\}\right\}$$

```
RegionXY[k_] := {Quotient[k, 2], 1 + Quotient[-k, 2]};
Regions = Table[RegionXY[k], {k, -4, 7}] - 1/2
```

$$\left\{\left\{-\frac{5}{2}, \frac{5}{2}\right\}, \left\{-\frac{5}{2}, \frac{3}{2}\right\}, \left\{-\frac{3}{2}, \frac{3}{2}\right\}, \left\{-\frac{3}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{3}{2}\right\}, \left\{\frac{3}{2}, -\frac{3}{2}\right\}, \left\{\frac{3}{2}, -\frac{5}{2}\right\}, \left\{\frac{5}{2}, -\frac{5}{2}\right\}, \left\{\frac{5}{2}, -\frac{7}{2}\right\}\right\}$$

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
φ = 1/2;
```

$$\mathtt{W[k\_]} := \begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1 - (1-\varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases};$$

$$\mathtt{SumF} = \sum_{i=-5}^{6} \sum_{j=-5}^{6} \mathtt{W[i - j]\ f[x - i, y - j]\ /.\ GenSol};$$

```
SimplifySquare[f_, x0_, y0_] := Simplify[f, x > x0 && x < x0 + 1 && y > y0 && y < y0 + 1];
DSimplifySquare[f_, {x0_, y0_}] := Simplify[D[SimplifySquare[f, x0, y0], {{x, y}}]];
DSumF = ParallelMap[DSimplifySquare[SumF, #] &, Regions];
```

```
AnisoInt[df_, {x0_, y0_}] :=
  Simplify[Integrate[Expand[(df.{1, 1})^2], {x, x0, x0 + 1}, {y, y0, y0 + 1}]];
AnisoInts = Parallelize[MapThread[AnisoInt, {DSumF, Regions}]];
Err = Simplify[Total[AnisoInts]]
```

$$\frac{1}{12\,902\,400}\left(638\,955\, c_{0,2}^4 + c_{0,2}^3\left(7\,449\,812 - 12\,024\, c_{1,1}\right) + 12\, c_{0,2}^2\left(3\,036\,379 + 10\,590\, c_{1,1} + 14\,088\, c_{1,1}^2\right) + 96\, c_{0,2}\left(786\,959 + 109\,083\, c_{1,1} + 40\,016\, c_{1,1}^2 + 2012\, c_{1,1}^3\right) + 96\left(633\,789 + 311\,236\, c_{1,1} + 140\,734\, c_{1,1}^2 + 8388\, c_{1,1}^3 + 8048\, c_{1,1}^4\right)\right)$$

```
FreeVars = Variables[Err];
DErr = Simplify[D[Err, {FreeVars}]];
H = D[DErr, {FreeVars}];
Sols = Solve[DErr == 0, FreeVars, Reals];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}ᵀ]
```

1	0.0902019	True

```
RootReduce[Sols[[1]]]
```

$\{c_{0,2} \to \text{Root}[95\,847\,501\,175\,547\,613\,564\,801\,600 + 290\,787\,026\,673\,489\,172\,616\,069\,184\,\#1 + 427\,693\,530\,277\,085\,972\,126\,756\,216\,\#1^2 + 374\,454\,369\,419\,578\,668\,527\,648\,440\,\#1^3 + 206\,526\,493\,409\,660\,806\,096\,287\,626\,\#1^4 + 73\,985\,811\,686\,932\,641\,040\,952\,616\,\#1^5 + 17\,333\,088\,289\,108\,701\,496\,879\,625\,\#1^6 + 2\,585\,430\,622\,222\,421\,745\,766\,995\,\#1^7 + 224\,999\,970\,811\,293\,497\,663\,025\,\#1^8 + 8\,794\,043\,350\,198\,430\,409\,225\,\#1^9\ \&, 1],$

$c_{1,1} \to \text{Root}[186\,034\,729\,797\,937\,411\,538\,022\,907\,705 + 266\,195\,094\,834\,401\,033\,071\,061\,383\,245\,\#1 + 23\,814\,746\,789\,073\,430\,925\,129\,080\,750\,\#1^2 - 74\,579\,241\,765\,856\,896\,300\,539\,908\,358\,\#1^3 - 31\,992\,383\,923\,550\,463\,074\,170\,477\,624\,\#1^4 + 27\,118\,665\,196\,708\,626\,294\,219\,280\,904\,\#1^5 + 12\,658\,942\,797\,006\,126\,318\,184\,281\,760\,\#1^6 + 12\,703\,022\,498\,519\,592\,874\,591\,024\,320\,\#1^7 + 1\,336\,110\,826\,973\,695\,570\,314\,316\,800\,\#1^8 + 1\,509\,855\,165\,491\,135\,315\,913\,446\,400\,\#1^9\ \&, 1]\}$

```
NSol = N[Sols[[1]]];
FullSol = Join[GenSol /. NSol, NSol]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

$\{c_{0,1} \to 0,\ c_{0,3} \to 0,\ c_{1,2} \to 1,\ c_{1,3} \to 0.463315,\ c_{2,1} \to 0.162576,\ c_{2,2} \to -0.209324,\ c_{2,3} \to -0.231657,\ c_{0,2} \to -1.58135,\ c_{1,1} \to -0.825153\}$