

$$h[x_] := \begin{cases} 1 + c_{01} x + c_{02} x^2 + c_{03} x^3 + c_{04} x^4 & 0 \leq x \leq 1 \\ c_{11} (x-1) + c_{12} (x-1)^2 + c_{13} (x-1)^3 + c_{14} (x-1)^4 & 1 < x \leq 2 \\ c_{21} (x-2) + c_{22} (x-2)^2 + c_{23} (x-2)^3 + c_{24} (x-2)^4 & 2 < x \leq 3 \\ 0 & \text{True} \end{cases};$$

`f[x_] := h[Abs[x]];`

`(*Interpolant constraints*)`

`I1 = f[1]`

`I2 = f[2]`

`I3 = f[3]`

`1 + c01 + c02 + c03 + c04`

`c11 + c12 + c13 + c14`

`c21 + c22 + c23 + c24`

`(*Partition of unity and gradient representation*)`

`T0 = CoefficientList[`

`FullSimplify[f[x + 2] + f[x + 1] + f[x] + f[x - 1] + f[x - 2] + f[x - 3], x > 0 && x < 1], x]`

`T1 = CoefficientList[FullSimplify[`

`- 2 f[x + 2] - f[x + 1] + f[x - 1] + 2 f[x - 2] + 3 f[x - 3], x > 0 && x < 1], x]`

`{ 2 + c01 + c02 + c03 + c04 + c11 + c12 + c13 + c14 + c21 + c22 + c23 + c24,`
`- 2 c02 - 3 c03 - 4 c04 - 2 c12 - 3 c13 - 4 c14 - 2 c22 - 3 c23 - 4 c24,`
`2 c02 + 3 c03 + 4 c04 + 2 c12 + 3 c13 + 4 c14 + 2 c22 + 3 c23 + 4 c24 + 2 (c04 + c14 + c24),`
`- 4 (c04 + c14 + c24), 2 (c04 + c14 + c24) }`

`{ 1 + c01 + c02 + c03 + c04 + 2 c11 + 2 c12 + 2 c13 + 2 c14 + 3 c21 + 3 c22 + 3 c23 + 3 c24,`
`- c01 - 2 c02 - 3 c03 - 4 c04 - 3 c11 - 4 c12 - 6 c13 - 8 c14 - 5 c21 - 6 c22 - 9 c23 - 12 c24,`
`c02 + 3 c03 + 6 c04 + c12 + 6 c13 + 12 c14 + c22 + 9 c23 + 18 c24,`
`- c03 - 4 c04 - 3 c13 - 8 c14 - 5 c23 - 12 c24, c04 + c14 + c24 }`

(*Smoothness*)

Df = Simplify[D[f[x], x], x > 0] /. Abs'[x] → 1

c01 + x (2 c02 + x (3 c03 + 4 c04 x)) /. x → 1

c11 + (2 c12 + (3 c13 + 4 c14 (-1 + x)) (-1 + x)) (-1 + x) /. x → 1

c11 + (2 c12 + (3 c13 + 4 c14 (-1 + x)) (-1 + x)) (-1 + x) /. x → 2

c21 + (2 c22 + (3 c23 + 4 c24 (-2 + x)) (-2 + x)) (-2 + x) /. x → 2

c21 + (2 c22 + (3 c23 + 4 c24 (-2 + x)) (-2 + x)) (-2 + x) /. x → 3

$$\begin{cases} c01 + x (2 c02 + x (3 c03 + 4 c04 x)) & x \leq 1 \\ c11 + (2 c12 + (3 c13 + 4 c14 (-1 + x)) (-1 + x)) (-1 + x) & 1 < x \leq 2 \\ c21 + (2 c22 + (3 c23 + 4 c24 (-2 + x)) (-2 + x)) (-2 + x) & 2 < x \leq 3 \\ 0 & \text{True} \end{cases}$$

c01 + 2 c02 + 3 c03 + 4 c04

c11

c11 + 2 c12 + 3 c13 + 4 c14

c21

c21 + 2 c22 + 3 c23 + 4 c24

GenSols = Solve[{

I1 == 0,

I2 == 0,

I3 == 0,

T0[[1]] == 1,

T0[[2]] == 0,

T0[[3]] == 0,

T1[[1]] == 0,

T1[[2]] == 1,

T1[[3]] == 0,

T1[[4]] == 0,

c01 == 0,

c01 + 2 c02 + 3 c03 + 4 c04 == c11,

c11 + 2 c12 + 3 c13 + 4 c14 == c21,

c21 + 2 c22 + 3 c23 + 4 c24 == 0

},

{c01, c02, c03, c04, c11, c12, c13, c14, c21, c22, c23, c24}

]

Solve::svars: Equations may not give solutions for all "solve" variables. >>

$$\left\{ \left\{ c01 \rightarrow 0, c04 \rightarrow -1 - c02 - c03, c11 \rightarrow -4 - 2 c02 - c03, c13 \rightarrow \frac{41}{4} + 5 c02 + \frac{5 c03}{2} - 2 c12, \right. \right.$$

$$c14 \rightarrow -\frac{25}{4} - 3 c02 - \frac{3 c03}{2} + c12, c21 \rightarrow \frac{7}{4} + c02 + \frac{c03}{2}, c22 \rightarrow \frac{15}{4} + 2 c02 + \frac{3 c03}{2} - c12,$$

$$\left. c23 \rightarrow -\frac{51}{4} - 7 c02 - \frac{9 c03}{2} + 2 c12, c24 \rightarrow \frac{29}{4} + 4 c02 + \frac{5 c03}{2} - c12 \right\}$$

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GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];

W1[k_] := 
$$\begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1 - (1 - \varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases};$$


SumF1 = 
$$\sum_{i=-5}^6 \sum_{j=-5}^6 W1[i-j] f[x-i, y-j] /. \text{GenSol};$$


{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6} = Parallelize[{
  Simplify[SumF1, x > 0 && x < 1 && y > 0 && y < 1],
  Simplify[SumF1, x > 0 && x < 1 && y > 1 && y < 2],
  Simplify[SumF1, x > -1 && x < 0 && y > 1 && y < 2],
  Simplify[SumF1, x > -1 && x < 0 && y > 2 && y < 3],
  Simplify[SumF1, x > -2 && x < -1 && y > 2 && y < 3],
  Simplify[SumF1, x > -2 && x < -1 && y > 3 && y < 4]
}];

{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4, DSumF1a5, DSumF1a6} = Parallelize[{
  FullSimplify[D[SumF1a1, {{x, y}}]],
  FullSimplify[D[SumF1a2, {{x, y}}]],
  FullSimplify[D[SumF1a3, {{x, y}}]],
  FullSimplify[D[SumF1a4, {{x, y}}]],
  FullSimplify[D[SumF1a5, {{x, y}}]],
  FullSimplify[D[SumF1a6, {{x, y}}]]
}];

{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6} = Parallelize[{
  Simplify[SumF1, x > 1 && x < 2 && y > 0 && y < 1],
  Simplify[SumF1, x > 1 && x < 2 && y > -1 && y < 0],
  Simplify[SumF1, x > 2 && x < 3 && y > -1 && y < 0],
  Simplify[SumF1, x > 2 && x < 3 && y > -2 && y < -1],
  Simplify[SumF1, x > 3 && x < 4 && y > -2 && y < -1],
  Simplify[SumF1, x > 3 && x < 4 && y > -3 && y < -2]
}];

{DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4, DSumF1b5, DSumF1b6} = Parallelize[{
  FullSimplify[D[SumF1b1, {{x, y}}]],
  FullSimplify[D[SumF1b2, {{x, y}}]],
  FullSimplify[D[SumF1b3, {{x, y}}]],
  FullSimplify[D[SumF1b4, {{x, y}}]],
  FullSimplify[D[SumF1b5, {{x, y}}]],
  FullSimplify[D[SumF1b6, {{x, y}}]]
}];

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DSumF1a1 = Simplify[DSumF1a1 /.  $\varphi \rightarrow 1/2$ ];
DSumF1a2 = Simplify[DSumF1a2 /.  $\varphi \rightarrow 1/2$ ];
DSumF1a3 = Simplify[DSumF1a3 /.  $\varphi \rightarrow 1/2$ ];
DSumF1a4 = Simplify[DSumF1a4 /.  $\varphi \rightarrow 1/2$ ];
DSumF1a5 = Simplify[DSumF1a5 /.  $\varphi \rightarrow 1/2$ ];
DSumF1a6 = Simplify[DSumF1a6 /.  $\varphi \rightarrow 1/2$ ];
DSumF1b1 = Simplify[DSumF1b1 /.  $\varphi \rightarrow 1/2$ ];
DSumF1b2 = Simplify[DSumF1b2 /.  $\varphi \rightarrow 1/2$ ];
DSumF1b3 = Simplify[DSumF1b3 /.  $\varphi \rightarrow 1/2$ ];
DSumF1b4 = Simplify[DSumF1b4 /.  $\varphi \rightarrow 1/2$ ];
DSumF1b5 = Simplify[DSumF1b5 /.  $\varphi \rightarrow 1/2$ ];
DSumF1b6 = Simplify[DSumF1b6 /.  $\varphi \rightarrow 1/2$ ];

{Err1a1, Err1a2, Err1a3, Err1a4, Err1a5, Err1a6} = Parallelize[{
  Simplify[ $\int_0^1 \int_0^1 (\text{DSumF1a1}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_1^2 \int_0^1 (\text{DSumF1a2}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_1^2 \int_{-1}^0 (\text{DSumF1a3}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_2^3 \int_{-1}^0 (\text{DSumF1a4}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_2^3 \int_{-2}^{-1} (\text{DSumF1a5}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_3^4 \int_{-2}^{-1} (\text{DSumF1a6}.\{1, 1\})^2 dx dy$ ]
}];

{Err1b1, Err1b2, Err1b3, Err1b4, Err1b5, Err1b6} = Parallelize[{
  Simplify[ $\int_0^1 \int_1^2 (\text{DSumF1b1}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_{-1}^0 \int_1^2 (\text{DSumF1b2}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_{-1}^0 \int_2^3 (\text{DSumF1b3}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_{-2}^{-1} \int_2^3 (\text{DSumF1b4}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_{-2}^{-1} \int_3^4 (\text{DSumF1b5}.\{1, 1\})^2 dx dy$ ],
  Simplify[ $\int_{-3}^{-2} \int_3^4 (\text{DSumF1b6}.\{1, 1\})^2 dx dy$ ]
}];

Err1 = FullSimplify[Err1a1 + Err1a2 + Err1a3 + Err1a4 +
  Err1a5 + Err1a6 + Err1b1 + Err1b2 + Err1b3 + Err1b4 + Err1b5 + Err1b6];

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Err = FullSimplify[Err1]
DErr = FullSimplify[D[Err, {{c02, c03, c12}}]];
H = FullSimplify[D[DErr, {{c02, c03, c12}, 2}]];
Sols = Solve[DErr == 0, {c02, c03, c12}];
TableForm[
  {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}^T]

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$$\begin{aligned}
& \frac{1}{541900800} \left(710720512 c02^4 + 64 c02^3 (96603491 + 22528790 c03 - 4600728 c12) + \right. \\
& 16 c02^2 (1265098369 + 68874844 c03^2 + c03 (581444444 - 28506576 c12) + \\
& 96 c12 (-1193613 + 33406 c12)) + 3 (5110913111 - 916196400 c12 + \\
& 8 (c03 (604723137 + c03 (211536105 + 32705908 c03 + 2023278 c03^2)) - \\
& 6 c03 (13769049 + 4 c03 (818366 + 72175 c03)) c12 + \\
& 8 (1121269 + 525692 c03 + 75788 c03^2) c12^2 - 1024 (297 + 98 c03) c12^3 + 6656 c12^4) \left. \right) + \\
& 12 c02 (2419537509 + 31366520 c03^3 + 4 c03^2 (97444621 - 4942360 c12) - \\
& 8 c12 (40513703 + 16 c12 (-130313 + 2832 c12)) + \\
& 2 c03 (843003883 + 64 c12 (-1211445 + 35278 c12))) \left. \right)
\end{aligned}$$

1	0.0497664	True
2	-43542. + 77568.6 i	False
3	-43542. - 77568.6 i	False
4	157568. + 6242.88 i	False
5	157568. - 6242.88 i	False
6	-18159.3 + 14818.5 i	False
7	-18159.3 - 14818.5 i	False
8	-212.426 + 699.396 i	False
9	-212.426 - 699.396 i	False
10	0.850573 + 0.379191 i	False
11	0.850573 - 0.379191 i	False
12	3875.75 + 1429.82 i	False
13	3875.75 - 1429.82 i	False
14	1.703 - 0.711495 i	False
15	1.703 + 0.711495 i	False
16	0.378762 + 0.0717344 i	False
17	0.378762 - 0.0717344 i	False
18	-15600.3 - 70685.4 i	False
19	-15600.3 + 70685.4 i	False
20	0.3265 + 0.0360115 i	False
21	0.3265 - 0.0360115 i	False
22	2.85639 + 0.891125 i	False
23	2.85639 - 0.891125 i	False
24	111126. - 56909.7 i	False
25	111126. + 56909.7 i	False
26	20209.6 + 6622.78 i	False
27	20209.6 - 6622.78 i	False

```
Sol = Sols[[1]]; (*Huge*)  
FullSol = N[Join[GenSol /. Sol, Sol]]  
fo[x_] := f[x] /. FullSol;  
Plot[fo[x], {x, -3, 3}, PlotStyle -> Black, Background -> White]  
{c01 -> 0., c04 -> 0.309774, c11 -> -0.838313, c13 -> 0.958096,  
  c14 -> -0.813626, c21 -> 0.169156, c22 -> 0.165539, c23 -> -0.838547,  
  c24 -> 0.503852, c02 -> -1.85191, c03 -> 0.542139, c12 -> 0.69384}
```

