In[504]:=
$$h[x\_] := \begin{cases} 1 + c01\,x + c02\,x^2 + c03\,x^3 + c04\,x^4 & 0 \le x \le 1/2 \\ c11\,(x-1) + c12\,(x-1)^2 + c13\,(x-1)^3 + c14\,(x-1)^4 & 1/2 < x \le 3/2 \\ c21\,(x-2) + c22\,(x-2)^2 + c23\,(x-2)^3 + c24\,(x-2)^4 & 3/2 < x \le 5/2 \\ 0 & \text{True} \end{cases};$$

```
f[x_] := h[Abs[x]];
```

In[506]:=
```
(*Continuity*)
C1l = Simplify[h[x], 0 ≤ x ≤ 1/2] /. x → 1/2
C1r = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 1/2
C2l = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 3/2
C2r = Simplify[h[x], 3/2 < x ≤ 5/2] /. x → 3/2
C3l = Simplify[h[x], 3/2 < x ≤ 5/2] /. x → 5/2
```

Out[506]=
$$1 + \frac{c01}{2} + \frac{c02}{4} + \frac{c03}{8} + \frac{c04}{16}$$

Out[507]=
$$\frac{1}{2}\left(-c11 + \frac{1}{2}\left(c12 + \frac{1}{2}\left(-c13 + \frac{c14}{2}\right)\right)\right)$$

Out[508]=
$$\frac{1}{2}\left(c11 + \frac{1}{2}\left(c12 + \frac{1}{2}\left(c13 + \frac{c14}{2}\right)\right)\right)$$

Out[509]=
$$\frac{1}{2}\left(-c21 + \frac{1}{2}\left(c22 + \frac{1}{2}\left(-c23 + \frac{c24}{2}\right)\right)\right)$$

Out[510]=
$$\frac{1}{2}\left(c21 + \frac{1}{2}\left(c22 + \frac{1}{2}\left(c23 + \frac{c24}{2}\right)\right)\right)$$

In[511]:=
```
(*Partition of unity and gradient representation*)
T0 = CoefficientList[FullSimplify[∑_{i=-6}^{6} f[x - i], x > 0 && x < 1/2], x]
T1 = CoefficientList[FullSimplify[∑_{i=-6}^{6} i f[x - i], x > 0 && x < 1/2], x]
```

Out[511]=
$$\{1,\ c01,\ c02 + 2\,(c12 + c22),\ c03,\ c04 + 2\,(c14 + c24)\}$$

Out[512]=
$$\{0,\ -2\,c11 - 4\,c21,\ 0,\ -2\,(c13 + 2\,c23)\}$$

In[513]:=
```
GenSols = Solve[{
    C1l == C1r,
    C2l == C2r,
    C3l == 0,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T0[[4]] == 0,
    T0[[5]] == 0,
    T1[[2]] == 1,
    T1[[4]] == 0
    },
    {c01, c02, c03, c04, c11, c12, c13, c14, c21, c22, c23, c24}
]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[513]=
$$\left\{\left\{c01 \to 0,\ c03 \to 0,\ c13 \to -6-2\,c02-\frac{c04}{2}-4\,c11,\ c14 \to 4-4\,c12,\right.\right.$$
$$\left.\left.c21 \to -\frac{1}{4}-\frac{c11}{2},\ c22 \to -\frac{c02}{2}-c12,\ c23 \to 3+c02+\frac{c04}{4}+2\,c11,\ c24 \to -4-\frac{c04}{2}+4\,c12\right\}\right\}$$

In[521]:=
```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
W1[k_] := Piecewise[{{0, k < 0}, {φ^2/2, k == 0}, {1 - (1 - φ)^2/2, k == 1}}, 1];
SumF1 = Sum[Sum[W1[i - j] f[x - i, y - j] /. GenSol, {j, -5, 6}], {i, -5, 6}];
SumF1 = SumF1 /. φ → 1/2;
```

In[526]:=
```
{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6} = Parallelize[{
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 3 - 1/2 && y < 4 - 1/2]
}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6} = Parallelize[{
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -3 - 1/2 && y < -2 - 1/2]
}];
```

In[528]:=
```
TableForm[{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6}]
TableForm[{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6}]
```

Out[528]//TableForm=

$\frac{1}{64}\left(-\left(12+4\,c02+c04+8\,c11\right)x^3\left(-12-2\left(1+13\,c11\right)y-2\left(9\,c02+7\,c12\right)y^2+13\left(12+4\,c02+c04+8\,c1\right.\right.$

$\frac{1}{32}\left(-7\,x\left(1+2\,x\right)\left(-1+2\left(1+c02+2\,c12\right)x+\left(8+c04-8\,c12\right)x^2+c11\left(-2+4\,x\right)\right)\left(1+c02\left(-1+y\right)^2+c\right.\right.$

$\frac{1}{32}\left(-\left(1+x\right)\left(3+2\,x\right)\left(9+2\,c11-4\,c12+18\,x+4\,c11\,x-12\,c12\,x+8\,x^2-8\,c12\,x^2+2\,c02\left(1+x\right)+c04\left(1+\right.\right.\right.$

$\frac{1}{128}\left(-7\left(1+x\right)\left(3+2\,x\right)\left(9+2\,c11-4\,c12+18\,x+4\,c11\,x-12\,c12\,x+8\,x^2-8\,c12\,x^2+2\,c02\left(1+x\right)+c04\left(\right.\right.\right.$

$-\frac{1}{128}\left(2+x\right)\left(5+2\,x\right)\left(35+6\,c11-24\,c12+34\,x+4\,c11\,x-28\,c12\,x+8\,x^2-8\,c12\,x^2+2\,c02\left(2+x\right)+c04\left(2\right.\right.\right.$

0

Out[529]//TableForm=

$\frac{1}{128}\left(128\left(1+c02\left(-1+x\right)^2+c04\left(-1+x\right)^4\right)y\left(c11+c12\,y-\frac{1}{2}\left(12+4\,c02+c04+8\,c11\right)y^2-4\left(-1+c12\right)\right.\right.$

$\frac{1}{64}\left(8\,c02^2\left(-1+x\right)^2\left(1+y\right)^2\left(13+8\,x\,y\right)+4\,c04^2\left(-1+x\right)^3\left(1+y\right)^3\left(-17-14\,y+2\,x\left(7+5\,y\right)\right)+2\,c02\left(29\right.\right.$

$\frac{1}{64}\left(-8\,c04^2\left(-2+x\right)^3\left(1+y\right)^3\left(2+\left(-1+x\right)y\right)-8\,c02^2\left(-2+x\right)^2\left(1+y\right)^2\left(-43-36\,y+4\,x\left(5+4\,y\right)\right)+2\,c\right.\right.$

$-\frac{1}{4}\left(1+c02\left(-2+x\right)^2+c04\left(-2+x\right)^4\right)\left(2+y\right)\left(5+2\,y\right)\left(35+6\,c11-24\,c12+34\,y+4\,c11\,y-28\,c12\,y+8\,y\right.\right.$

$\frac{1}{128}\left(-565\,822-170\,520\,c11-12\,600\,c11^2+829\,080\,c12+126\,000\,c11\,c12-302\,400\,c12^2+717\,850\,x+197\,96\right.\right.$

1

In[530]:=
```
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4, DSumF1a5,
   DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4, DSumF1b5} = Parallelize[{
    Simplify[D[SumF1a1, {{x, y}}]],
    Simplify[D[SumF1a2, {{x, y}}]],
    Simplify[D[SumF1a3, {{x, y}}]],
    Simplify[D[SumF1a4, {{x, y}}]],
    Simplify[D[SumF1a5, {{x, y}}]],
    Simplify[D[SumF1b1, {{x, y}}]],
    Simplify[D[SumF1b2, {{x, y}}]],
    Simplify[D[SumF1b3, {{x, y}}]],
    Simplify[D[SumF1b4, {{x, y}}]],
    Simplify[D[SumF1b5, {{x, y}}]]
}];
```

In[531]:=
```
{Err1a1, Err1a2, Err1a3, Err1a4, Err1a5,
   Err1b1, Err1b2, Err1b3, Err1b4, Err1b5} = Parallelize[{
    Simplify[∫_{0-1/2}^{1-1/2} ∫_{0-1/2}^{1-1/2} (DSumF1a1.{1, 1})^2 dx dy],
    Simplify[∫_{1-1/2}^{2-1/2} ∫_{0-1/2}^{1-1/2} (DSumF1a2.{1, 1})^2 dx dy],
    Simplify[∫_{1-1/2}^{2-1/2} ∫_{-1-1/2}^{0-1/2} (DSumF1a3.{1, 1})^2 dx dy],
    Simplify[∫_{2-1/2}^{3-1/2} ∫_{-1-1/2}^{0-1/2} (DSumF1a4.{1, 1})^2 dx dy],
    Simplify[∫_{2-1/2}^{3-1/2} ∫_{-2-1/2}^{-1-1/2} (DSumF1a5.{1, 1})^2 dx dy],
        Simplify[∫_{0-1/2}^{1-1/2} ∫_{1-1/2}^{2-1/2} (DSumF1b1.{1, 1})^2 dx dy],
    Simplify[∫_{-1-1/2}^{0-1/2} ∫_{1-1/2}^{2-1/2} (DSumF1b2.{1, 1})^2 dx dy],
    Simplify[∫_{-1-1/2}^{0-1/2} ∫_{2-1/2}^{3-1/2} (DSumF1b3.{1, 1})^2 dx dy],
    Simplify[∫_{-2-1/2}^{-1-1/2} ∫_{2-1/2}^{3-1/2} (DSumF1b4.{1, 1})^2 dx dy],
    Simplify[∫_{-2-1/2}^{-1-1/2} ∫_{3-1/2}^{4-1/2} (DSumF1b5.{1, 1})^2 dx dy]
}];
```

In[532]:=
```
Err1 = FullSimplify[
   Err1a1 + Err1a2 + Err1a3 + Err1a4 + Err1a5 + Err1b1 + Err1b2 + Err1b3 + Err1b4 + Err1b5];
```

In[533]:=
```
Err1
```

Out[533]=
$$\frac{1}{69\,363\,302\,400}\Big(3\,435\,022\,080\,c02^4 + 13\,203\,915\,c04^4 + 48\,c04^3\,(13\,451\,975 + 150\,478\,c11 + 2795\,c12) + 10\,752\,c02^3\,(3\,711\,112 + 316\,425\,c04 - 6012\,c11 + 13\,794\,c12) + 64\,c04^2\,(222\,183\,873 + 1\,739\,784\,c11^2 + 28\,c12\,(-1413 + 1322\,c12) - 6\,c11\,(100\,757 + 23\,092\,c12)) + 2048\,c04\,(58\,355\,292 + 126\,756\,c11^3 + 6\,c11^2\,(331\,526 - 21\,335\,c12) + c11\,(5\,645\,217 + 6\,(37\,259 - 7708\,c12)\,c12) - c12\,(5\,002\,167 + 10\,c12\,(28\,613 + 493\,c12))) + 64\,c02\,(3\,306\,360\,c04^3 + c04^2\,(120\,471\,210 + 1\,169\,382\,c11 + 98\,707\,c12) + 128\,(52\,225\,182 + 3\,c11\,(2\,311\,657 + 42\,c11\,(20\,259 + 1006\,c11)) - (2\,843\,219 + 6\,c11\,(2749 + 5271\,c11))\,c12 + 8\,(21\,983 - 5781\,c11)\,c12^2 + 20\,590\,c12^3) + 8\,c04\,(203\,126\,499 + 1\,056\,384\,c11^2 + c11\,(879\,834 - 385\,536\,c12) - 8\,c12\,(221\,317 + 1901\,c12))) + 32\,c02^2\,(39\,789\,115\,c04^2 + 8\,c04\,(118\,209\,037 + 654\,822\,c11 + 85\,559\,c12) + 32\,(191\,723\,345 + 887\,544\,c11^2 + 630\,c11\,(2063 - 1004\,c12) + 4\,c12\,(-158\,593 + 50\,726\,c12))) + 8192\,(46\,358\,295 + 528\,444\,c11^3 + 507\,024\,c11^4 + 36\,c11\,(531\,177 + (17\,935 - 4449\,c12)\,c12) + 6\,c11^2\,(1\,429\,427 + 8\,c12\,(4531 + 1504\,c12)) + 2\,c12\,(-4\,225\,587 + c12\,(948\,209 + 8\,c12\,(6943 + 880\,c12))))\Big)$$

```
Err = Err1;
DErr = FullSimplify[D[Err, {{c02, c04, c11, c12}}]];
H = FullSimplify[D[Err, {{c02, c04, c11, c12}, 2}]];
```

In[25]:=
```
NSols = NSolve[DErr == 0, {c02, c04, c11, c12}];
TableForm[
 {Range[Length[NSols]], Err /. NSols, PositiveDefiniteMatrixQ[H /. N[#]] & /@ NSols}^T]
```

Out[26]//TableForm=

1	$-371.818-962.248\,i$	False
2	$-371.818+962.248\,i$	False
3	$41.277+5.50383\,i$	False
4	$41.277-5.50383\,i$	False
5	$35.2303-2.26307\,i$	False
6	$35.2303+2.26307\,i$	False
7	$11.4979-39.2821\,i$	False
8	$11.4979+39.2821\,i$	False
9	$-1.60967-7.2478\,i$	False
10	$-1.60967+7.2478\,i$	False
11	$1.65238-4.12597\,i$	False
12	$1.65238+4.12597\,i$	False
13	$1.16618-3.32555\,i$	False
14	$1.16618+3.32555\,i$	False
15	$-0.748242-6.9968\,i$	False
16	$-0.748242+6.9968\,i$	False
17	$-1.13705-1.95979\,i$	False
18	$-1.13705+1.95979\,i$	False
19	$0.492741+0.315781\,i$	False
20	$0.492741-0.315781\,i$	False
21	$1.88457-1.1297\,i$	False
22	$1.88457+1.1297\,i$	False
23	$7.00001-3.75823\,i$	False
24	$7.00001+3.75823\,i$	False
25	$-1.7295-7.55784\,i$	False
26	$-1.7295+7.55784\,i$	False
27	$1.36523+0.489093\,i$	False
28	$1.36523-0.489093\,i$	False
29	$1.64857-1.6851\,i$	False
30	$1.64857+1.6851\,i$	False
31	$2.28973-3.00686\,i$	False
32	$2.28973+3.00686\,i$	False
33	$-0.204138-0.692522\,i$	False
34	$-0.204138+0.692522\,i$	False
35	$-68.8987+37.4208\,i$	False
36	$-68.8987-37.4208\,i$	False
37	$1.87087-0.790439\,i$	False
38	$1.87087+0.790439\,i$	False
39	$-5.02256+6.48535\,i$	False
40	$-5.02256-6.48535\,i$	False
41	$1.99203-2.33353\,i$	False
42	$1.99203+2.33353\,i$	False
43	$-0.206467+2.37946\,i$	False
44	$-0.206467-2.37946\,i$	False
45	$1.77111+3.98441\,i$	False
46	$1.77111-3.98441\,i$	False
47	$-1.80967-7.08748\,i$	False
48	$-1.80967+7.08748\,i$	False
49	$2.84915-0.894647\,i$	False
50	$2.84915+0.894647\,i$	False
51	$0.637082-0.848384\,i$	False
52	$0.637082+0.848384\,i$	False
53	$0.990693+3.06665\,i$	False
54	$0.990693-3.06665\,i$	False
55	$0.933043-0.201305\,i$	False

56	0.933043 + 0.201305 i	False
57	1.98039 − 2.83645 i	False
58	1.98039 + 2.83645 i	False
59	0.725031 − 0.647079 i	False
60	0.725031 + 0.647079 i	False
61	0.629028 − 0.176407 i	False
62	0.629028 + 0.176407 i	False
63	0.0857839 − 0.370587 i	False
64	0.0857839 + 0.370587 i	False
65	−28.3771 − 7.93182 i	False
66	−28.3771 + 7.93182 i	False
67	1.91734 − 0.309608 i	False
68	1.91734 + 0.309608 i	False
69	0.011362 + 1.92331 i	False
70	0.011362 − 1.92331 i	False
71	0.994576 + 1.40524 i	False
72	0.994576 − 1.40524 i	False
73	−1.2097 + 2.32849 i	False
74	−1.2097 − 2.32849 i	False
75	0.617988 + 1.21712 i	False
76	0.617988 − 1.21712 i	False
77	−0.0604634 − 1.59919 i	False
78	−0.0604634 + 1.59919 i	False
79	0.963861 + 0.0862702 i	False
80	0.963861 − 0.0862702 i	False
81	0.0688145	True

In[562]:=
```
Sol = NSols[[81]];
FullSol = N[Join[GenSol /. Sol, Sol]]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

Out[563]=
```
{c01 → 0., c03 → 0., c13 → 0.519807, c14 → −2.68505, c21 → 0.163347, c22 → −0.460913,
 c23 → −0.259904, c24 → 1.05668, c02 → −2.4207, c04 → 3.25674, c11 → −0.826694, c12 → 1.67126}
```

Out[565]=