

$$h[x_] := \begin{cases} 1 + c_{0,1} x + c_{0,2} x^2 + c_{0,3} x^3 + c_{0,4} x^4 & 0 \leq x \leq 1/2 \\ c_{1,1} (x-1) + c_{1,2} (x-1)^2 + c_{1,3} (x-1)^3 + c_{1,4} (x-1)^4 & 1/2 < x \leq 3/2 \\ c_{2,1} (x-2) + c_{2,2} (x-2)^2 + c_{2,3} (x-2)^3 + c_{2,4} (x-2)^4 & 3/2 < x \leq 5/2 \\ 0 & \text{True} \end{cases}$$

f[x\_] := h[Abs[x]];

AllVars = {c<sub>0,1</sub>, c<sub>0,2</sub>, c<sub>0,3</sub>, c<sub>0,4</sub>, c<sub>1,1</sub>, c<sub>1,2</sub>, c<sub>1,3</sub>, c<sub>1,4</sub>, c<sub>2,1</sub>, c<sub>2,2</sub>, c<sub>2,3</sub>, c<sub>2,4</sub>};

(\*Continuity\*)

C1 = Limit[h[x], x → 1/2, Direction → 1] == Limit[h[x], x → 1/2, Direction → -1]

C2 = Limit[h[x], x → 3/2, Direction → 1] == Limit[h[x], x → 3/2, Direction → -1]

C3 = Limit[h[x], x → 5/2, Direction → 1] == Limit[h[x], x → 5/2, Direction → -1]

$$\frac{1}{16} (16 + 8 c_{0,1} + 4 c_{0,2} + 2 c_{0,3} + c_{0,4}) == \frac{1}{16} (-8 c_{1,1} + 4 c_{1,2} - 2 c_{1,3} + c_{1,4})$$

$$\frac{1}{16} (8 c_{1,1} + 4 c_{1,2} + 2 c_{1,3} + c_{1,4}) == \frac{1}{16} (-8 c_{2,1} + 4 c_{2,2} - 2 c_{2,3} + c_{2,4})$$

$$\frac{1}{16} (8 c_{2,1} + 4 c_{2,2} + 2 c_{2,3} + c_{2,4}) == 0$$

(\*Partition of unity and linear term\*)

T0 = CoefficientList[FullSimplify[ $\sum_{i=-6}^6 f[x-i]$ , x > 0 && x < 1/2], x]

T1 = CoefficientList[FullSimplify[ $\sum_{i=-6}^6 i f[x-i]$ , x > 0 && x < 1/2], x]

{1, c<sub>0,1</sub>, c<sub>0,2</sub> + 2 (c<sub>1,2</sub> + c<sub>2,2</sub>), c<sub>0,3</sub>, c<sub>0,4</sub> + 2 (c<sub>1,4</sub> + c<sub>2,4</sub>) }

{0, -2 c<sub>1,1</sub> - 4 c<sub>2,1</sub>, 0, -2 (c<sub>1,3</sub> + 2 c<sub>2,3</sub>) }

(\*Smoothness\*)

Dh = Simplify[D[h[x], x], x > 0];

S0 = (Dh /. x → 0) == 0

S1 = Limit[Dh, x → 1/2, Direction → 1] == Limit[Dh, x → 1/2, Direction → -1]

S2 = Limit[Dh, x → 3/2, Direction → 1] == Limit[Dh, x → 3/2, Direction → -1]

S3 = Limit[Dh, x → 5/2, Direction → 1] == Limit[Dh, x → 5/2, Direction → -1]

c<sub>0,1</sub> == 0

$$c_{0,1} + c_{0,2} + \frac{3 c_{0,3}}{4} + \frac{c_{0,4}}{2} == c_{1,1} - c_{1,2} + \frac{3 c_{1,3}}{4} - \frac{c_{1,4}}{2}$$

$$c_{1,1} + c_{1,2} + \frac{3 c_{1,3}}{4} + \frac{c_{1,4}}{2} == c_{2,1} - c_{2,2} + \frac{3 c_{2,3}}{4} - \frac{c_{2,4}}{2}$$

$$c_{2,1} + c_{2,2} + \frac{3 c_{2,3}}{4} + \frac{c_{2,4}}{2} == 0$$

```

GenSols = Solve[{
  C1, C2, C3,
  T0[[2]] == 0,
  T0[[3]] == 0,
  T0[[4]] == 0,
  T0[[5]] == 0,
  T1[[2]] == 1,
  T1[[4]] == 0,
  S0, S1, S2, S3
},
AllVars
]

```

\*\*\* Solve: Equations may not give solutions for all "solve" variables.

$$\left\{ \left\{ c_{0,1} \rightarrow 0, c_{0,3} \rightarrow 0, c_{1,1} \rightarrow -\frac{13}{6} - \frac{11 c_{0,2}}{12} - \frac{13 c_{0,4}}{48}, c_{1,2} \rightarrow \frac{13}{6} + \frac{2 c_{0,2}}{3} + \frac{c_{0,4}}{3}, \right. \right.$$

$$c_{1,3} \rightarrow \frac{8}{3} + \frac{5 c_{0,2}}{3} + \frac{7 c_{0,4}}{12}, c_{1,4} \rightarrow -\frac{14}{3} - \frac{8 c_{0,2}}{3} - \frac{4 c_{0,4}}{3}, c_{2,1} \rightarrow \frac{5}{6} + \frac{11 c_{0,2}}{24} + \frac{13 c_{0,4}}{96},$$

$$\left. c_{2,2} \rightarrow -\frac{13}{6} - \frac{7 c_{0,2}}{6} - \frac{c_{0,4}}{3}, c_{2,3} \rightarrow -\frac{4}{3} - \frac{5 c_{0,2}}{6} - \frac{7 c_{0,4}}{24}, c_{2,4} \rightarrow \frac{14}{3} + \frac{8 c_{0,2}}{3} + \frac{5 c_{0,4}}{6} \right\}$$

```

RegionXY[k_] := {Quotient[k, 2], 1 + Quotient[-k, 2]};
Regions = Table[RegionXY[k], {k, -4, 7}] - 1/2

```

$$\left\{ \left\{ -\frac{5}{2}, \frac{5}{2} \right\}, \left\{ -\frac{5}{2}, \frac{3}{2} \right\}, \left\{ -\frac{3}{2}, \frac{3}{2} \right\}, \left\{ -\frac{3}{2}, \frac{1}{2} \right\}, \left\{ -\frac{1}{2}, \frac{1}{2} \right\}, \left\{ -\frac{1}{2}, -\frac{1}{2} \right\}, \right.$$

$$\left. \left\{ \frac{1}{2}, -\frac{1}{2} \right\}, \left\{ \frac{1}{2}, -\frac{3}{2} \right\}, \left\{ \frac{3}{2}, -\frac{3}{2} \right\}, \left\{ \frac{3}{2}, -\frac{5}{2} \right\}, \left\{ \frac{5}{2}, -\frac{5}{2} \right\}, \left\{ \frac{5}{2}, -\frac{7}{2} \right\} \right\}$$

```

GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
φ = 1/2;

```

$$W[k_] := \begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1 - (1 - \varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases}$$

$$\text{SumF} = \sum_{i=-5}^6 \sum_{j=-5}^6 W[i-j] f[x-i, y-j] /. \text{GenSol};$$

```

SimplifySquare[f_, x0_, y0_] := Simplify[f, x > x0 && x < x0 + 1 && y > y0 && y < y0 + 1];
DSimplifySquare[f_, {x0_, y0_}] := Simplify[D[SimplifySquare[f, x0, y0], {{x, y}}]];
DSumF = ParallelMap[DSimplifySquare[SumF, #] &, Regions];

```

```

AnisoInt[df_, {x0_, y0_}] :=
  Simplify[Integrate[Expand[(df.{1, 1})^2], {x, x0, x0 + 1}, {y, y0, y0 + 1}]];
AnisoInts = Parallelize[MapThread[AnisoInt, {DSumF, Regions}]];
Err = Simplify[Total[AnisoInts]]

1
5 618 427 494 400
(30 138 878 443 520 + 589 736 057 600 c0,24 + 13 767 277 232 128 c0,4 + 2 205 392 332 800 c0,42 +
136 508 262 112 c0,43 + 3 241 946 411 c0,44 + 256 c0,23 (27 664 541 552 + 2 442 488 383 c0,4) +
96 c0,22 (315 593 557 760 + 58 816 453 520 c0,4 + 2 632 493 507 c0,42) +
16 c0,2 (3 203 480 279 552 + 1 015 838 792 832 c0,4 + 94 651 470 240 c0,42 + 2 890 448 743 c0,43))

FreeVars = Variables[Err];
DErr = Simplify[D[Err, {FreeVars}]];
H = D[DErr, {FreeVars}];
Sols = Solve[DErr == 0, FreeVars, Reals];
TableForm[
  {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}^T]
1    0.0690686    True

RootReduce[Sols[[1]]]
{c0,2 → Root[1 689 662 005 856 337 976 240 041 904 499 715 134 372 237 013 891 +
1 913 730 991 673 238 426 329 051 917 073 993 219 776 357 823 234 #1 +
875 576 459 165 396 432 047 441 848 633 307 153 441 108 337 957 #12 +
220 353 172 042 287 473 274 228 459 981 792 313 551 037 813 848 #13 +
34 801 416 482 455 760 933 198 507 141 290 225 454 678 189 258 #14 +
3 691 137 740 398 847 524 505 779 769 450 660 705 708 056 556 #15 +
269 781 047 608 097 129 415 603 204 681 660 622 987 864 608 #16 +
13 495 678 805 173 790 703 538 473 545 709 318 352 456 896 #17 +
431 424 428 610 766 564 410 277 417 438 563 815 617 152 #18 +
7 052 647 884 593 722 360 448 514 392 761 188 016 896 #19 &, 1],
c0,4 → Root[-12 052 730 059 666 565 027 942 383 263 597 277 906 151 992 998 912 +
8 724 149 967 135 470 183 666 169 667 339 895 553 162 816 345 600 #1 -
2 307 186 105 966 193 393 075 892 682 083 728 786 314 507 275 328 #12 +
291 580 584 404 141 632 572 557 804 308 371 318 670 966 217 856 #13 -
19 458 521 328 012 163 527 706 445 700 672 395 426 531 724 968 #14 +
806 520 397 583 283 384 374 587 130 856 734 796 383 219 852 #15 -
21 683 052 237 286 257 632 433 005 453 889 060 211 834 536 #16 +
387 555 349 882 436 704 314 878 114 069 617 899 997 788 #17 -
4 358 609 338 880 559 588 643 958 975 730 339 962 778 #18 +
27 549 405 799 194 227 970 502 009 346 723 390 691 #19 &, 1]}

```

```

NSol = N[Sols[[1]]];
FullSol = Join[GenSol /. NSol, NSol]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle -> Black, Background -> White]
{c0,1 -> 0, c0,3 -> 0, c1,1 -> -0.828758, c1,2 -> 1.62969,
c1,3 -> 0.524805, c1,4 -> -2.51876, c2,1 -> 0.164379, c2,2 -> -0.427537,
c2,3 -> -0.262402, c2,4 -> 0.919919, c0,2 -> -2.40431, c0,4 -> 3.19769}

```

