```
h[x_] := Piecewise[{{1 + c01 x + c02 x^2 + c03 x^3, 0 ≤ x ≤ 1},
    {c11 (x - 1) + c12 (x - 1)^2 + c13 (x - 1)^3, 1 < x ≤ 2},
    {c21 (x - 2) + c22 (x - 2)^2 + c23 (x - 2)^3, 2 < x ≤ 3},
    {0, True}}];

f[x_] := h[Abs[x]];

(*Interpolant constraints*)
I1 = f[1]
I2 = f[2]
I3 = f[3]
```

$$1 + c01 + c02 + c03$$

$$c11 + c12 + c13$$

$$c21 + c22 + c23$$

```
(*Partition of unity and gradient representation*)
T0 = CoefficientList[
  FullSimplify[f[x + 2] + f[x + 1] + f[x] + f[x - 1] + f[x - 2] + f[x - 3], x > 0 && x < 1], x]
T1 = CoefficientList[FullSimplify[
    -2 f[x + 2] - f[x + 1] + f[x - 1] + 2 f[x - 2] + 3 f[x - 3], x > 0 && x < 1], x]
```

$$\{2 + c01 + c02 + c03 + c11 + c12 + c13 + c21 + c22 + c23,\ -2\,c02 - 3\,c03 - 2\,c12 - 3\,c13 - 2\,c22 - 3\,c23,\ 2\,c02 + 3\,c03 + 2\,c12 + 3\,c13 + 2\,c22 + 3\,c23\}$$

$$\{1 + c01 + c02 + c03 + 2\,c11 + 2\,c12 + 2\,c13 + 3\,c21 + 3\,c22 + 3\,c23,\ -c01 - 2\,c02 - 3\,c03 - 3\,c11 - 4\,c12 - 6\,c13 - 5\,c21 - 6\,c22 - 9\,c23,\ c02 + 3\,c03 + c12 + 6\,c13 + c22 + 9\,c23,\ -c03 - 3\,c13 - 5\,c23\}$$

```
(*Smoothness*)
Df = Simplify[D[f[x], x], x > 0] /. Abs'[x] → 1
c01 + x (2 c02 + 3 c03 x) /. x → 1
c11 + (2 c12 + 3 c13 (-1 + x)) (-1 + x) /. x → 1
c11 + (2 c12 + 3 c13 (-1 + x)) (-1 + x) /. x → 2
c21 + (2 c22 + 3 c23 (-2 + x)) (-2 + x) /. x → 2
c21 + (2 c22 + 3 c23 (-2 + x)) (-2 + x) /. x → 3
```

$$\begin{cases} c01 + x\,(2\,c02 + 3\,c03\,x) & x \le 1 \\ c11 + (2\,c12 + 3\,c13\,(-1 + x))\,(-1 + x) & 1 < x \le 2 \\ c21 + (2\,c22 + 3\,c23\,(-2 + x))\,(-2 + x) & 2 < x \le 3 \\ 0 & \text{True} \end{cases}$$

$$c01 + 2\,c02 + 3\,c03$$

$$c11$$

$$c11 + 2\,c12 + 3\,c13$$

$$c21$$

$$c21 + 2\,c22 + 3\,c23$$

```
GenSols = Solve[{
    I1 == 0,
    I2 == 0,
    I3 == 0,
    T0[[1]] == 1,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[1]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0,
    T1[[4]] == 0,
    c01 == 0,
    c01 + 2 c02 + 3 c03 == c11,
    c11 + 2 c12 + 3 c13 == c21,
    c21 + 2 c22 + 3 c23 == 0
    },
    {c01, c02, c03, c11, c12, c13, c21, c22, c23}
]
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\left\{\left\{\text{c01} \to 0,\ \text{c03} \to -1-\text{c02},\ \text{c11} \to -3-\text{c02},\ \text{c12} \to \frac{19}{4}+\frac{3\,\text{c02}}{2},\right.\right.$$
$$\left.\left.\text{c13} \to -\frac{7}{4}-\frac{\text{c02}}{2},\ \text{c21} \to \frac{5}{4}+\frac{\text{c02}}{2},\ \text{c22} \to -\frac{5}{2}-\text{c02},\ \text{c23} \to \frac{5}{4}+\frac{\text{c02}}{2}\right\}\right\}$$

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
W1[k_] := Piecewise[{{0, k < 0}, {φ^2/2, k == 0}, {1 - (1 - φ)^2/2, k == 1}}, 1];
SumF1 = Sum[Sum[W1[i - j] f[x - i, y - j] /. GenSol, {j, -5, 6}], {i, -5, 6}];
{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6} = Parallelize[{
    Simplify[SumF1, x > 0 && x < 1 && y > 0 && y < 1],
    Simplify[SumF1, x > 0 && x < 1 && y > 1 && y < 2],
    Simplify[SumF1, x > -1 && x < 0 && y > 1 && y < 2],
    Simplify[SumF1, x > -1 && x < 0 && y > 2 && y < 3],
    Simplify[SumF1, x > -2 && x < -1 && y > 2 && y < 3],
    Simplify[SumF1, x > -2 && x < -1 && y > 3 && y < 4]
}];
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4, DSumF1a5, DSumF1a6} = Parallelize[{
    FullSimplify[D[SumF1a1, {{x, y}}]],
    FullSimplify[D[SumF1a2, {{x, y}}]],
    FullSimplify[D[SumF1a3, {{x, y}}]],
    FullSimplify[D[SumF1a4, {{x, y}}]],
    FullSimplify[D[SumF1a5, {{x, y}}]],
    FullSimplify[D[SumF1a6, {{x, y}}]]
}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6} = Parallelize[{
    Simplify[SumF1, x > 1 && x < 2 && y > 0 && y < 1],
    Simplify[SumF1, x > 1 && x < 2 && y > -1 && y < 0],
    Simplify[SumF1, x > 2 && x < 3 && y > -1 && y < 0],
    Simplify[SumF1, x > 2 && x < 3 && y > -2 && y < -1],
    Simplify[SumF1, x > 3 && x < 4 && y > -2 && y < -1],
    Simplify[SumF1, x > 3 && x < 4 && y > -3 && y < -2]
}];
{DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4, DSumF1b5, DSumF1b6} = Parallelize[{
    FullSimplify[D[SumF1b1, {{x, y}}]],
    FullSimplify[D[SumF1b2, {{x, y}}]],
    FullSimplify[D[SumF1b3, {{x, y}}]],
    FullSimplify[D[SumF1b4, {{x, y}}]],
    FullSimplify[D[SumF1b5, {{x, y}}]],
    FullSimplify[D[SumF1b6, {{x, y}}]]
}];
```

```
{Err1a1, Err1a2, Err1a3, Err1a4, Err1a5, Err1a6} = Parallelize[{
    Simplify[∫_0^1 ∫_0^1 (DSumF1a1.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_1^2 ∫_0^1 (DSumF1a2.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_1^2 ∫_-1^0 (DSumF1a3.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_2^3 ∫_-1^0 (DSumF1a4.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_2^3 ∫_-2^-1 (DSumF1a5.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_3^4 ∫_-2^-1 (DSumF1a6.{1, 1})^2 𝕕x 𝕕y]
}];
{Err1b1, Err1b2, Err1b3, Err1b4, Err1b5, Err1b6} = Parallelize[{
    Simplify[∫_0^1 ∫_1^2 (DSumF1b1.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-1^0 ∫_1^2 (DSumF1b2.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-1^0 ∫_2^3 (DSumF1b3.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-2^-1 ∫_2^3 (DSumF1b4.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-2^-1 ∫_3^4 (DSumF1b5.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-3^-2 ∫_3^4 (DSumF1b6.{1, 1})^2 𝕕x 𝕕y]
}];
Err1 = FullSimplify[Err1a1 + Err1a2 + Err1a3 + Err1a4 +
     Err1a5 + Err1a6 + Err1b1 + Err1b2 + Err1b3 + Err1b4 + Err1b5 + Err1b6];

Err = FullSimplify[Err1 /. φ → 1/2]
DErr = FullSimplify[D[Err, {{c02}}]];
H = FullSimplify[D[Err, {{c02}, 2}]];
Sols = RootReduce[Solve[DErr == 0, {c02}]];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}^T]
```

$$\left(92\,669\,325 + 8\,c02\left(14\,686\,668 + c02\left(6\,527\,619 + 2\,c02\left(582\,860 + 37\,381\,c02\right)\right)\right)\right) / 25\,804\,800$$

```
1    0.0575003               True
2    0.692089 - 0.0754737 i  False
3    0.692089 + 0.0754737 i  False
```

```
Sols[[1]]
```

$$\left\{c02 \to \text{Root}\left[7\,343\,334 + 6\,527\,619\,\#1 + 1\,748\,580\,\#1^2 + 149\,524\,\#1^3\ \&,\ 1\right]\right\}$$

```
Sol = Sols[[1]];
FullSol = N[Join[GenSol /. Sol, Sol]]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

```
{c01 → 0., c03 → 1.06787, c11 → -0.932133, c12 → 1.6482, c13 → -0.716067,
 c21 → 0.216067, c22 → -0.432133, c23 → 0.216067, c02 → -2.06787}
```