```
h[x_] := Piecewise[{
    {1 + c01 x + c02 x^2,                0 ≤ x ≤ 1},
    {c11 (x - 1) + c12 (x - 1)^2,        1 < x ≤ 2},
    {0,                                  True}
}];
f[x_] := h[Abs[x]];

(*Interpolant constraints*)
I1 = f[1]
I2 = f[2]
```

```
1 + c01 + c02

c11 + c12
```

```
(*Partition of unity and gradient representation*)
T0 = CoefficientList[FullSimplify[f[x + 1] + f[x] + f[x - 1] + f[x - 2], x > 0 && x < 1], x]
T1 = CoefficientList[FullSimplify[-f[x + 1] + f[x - 1] + 2 f[x - 2], x > 0 && x < 1], x]
```

```
{2 + c01 + c02 + c11 + c12, -2 (c02 + c12), 2 (c02 + c12)}

{1 + c01 + c02 + 2 c11 + 2 c12, -c01 - 2 c02 - 3 c11 - 4 c12, c02 + c12}
```

```
GenSols = Solve[{
    I1 == 0,
    I2 == 0,
    T0[[1]] == 1,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[1]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0
    },
    {c01, c02, c11, c12}
]
```

```
Solve::svars : Equations may not give solutions for all "solve" variables. >>

{{c02 -> -1 - c01, c11 -> -1 - c01, c12 -> 1 + c01}}
```

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
W1[k_] := Piecewise[{{0, k < 0}, {φ^2/2, k == 0}, {1 - (1 - φ)^2/2, k == 1}, {1, True}}];
```

$$\text{W1[k\_]} := \begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1 - (1-\varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases};$$

$$\text{SumF1} = \sum_{i=-3}^{5}\sum_{j=-3}^{5} \text{W1[i - j] f[x - i, y - j] /. GenSol;}$$

```
{SumF1a1, SumF1a2, SumF1a3, SumF1a4} = Parallelize[{
    Simplify[SumF1, x > 0 && x < 1 && y > 0 && y < 1],
    Simplify[SumF1, x > 0 && x < 1 && y > 1 && y < 2],
    Simplify[SumF1, x > -1 && x < 0 && y > 1 && y < 2],
    Simplify[SumF1, x > -1 && x < 0 && y > 2 && y < 3]}];
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4} = Parallelize[{
    FullSimplify[D[SumF1a1, {{x, y}}]],
    FullSimplify[D[SumF1a2, {{x, y}}]],
    FullSimplify[D[SumF1a3, {{x, y}}]],
    FullSimplify[D[SumF1a4, {{x, y}}]]}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4} = Parallelize[{
    Simplify[SumF1, x > 1 && x < 2 && y > 0 && y < 1],
    Simplify[SumF1, x > 1 && x < 2 && y > -1 && y < 0],
    Simplify[SumF1, x > 2 && x < 3 && y > -1 && y < 0],
    Simplify[SumF1, x > 2 && x < 3 && y > -2 && y < -1]}];
{DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4} = Parallelize[{
    FullSimplify[D[SumF1b1, {{x, y}}]],
    FullSimplify[D[SumF1b2, {{x, y}}]],
    FullSimplify[D[SumF1b3, {{x, y}}]],
    FullSimplify[D[SumF1b4, {{x, y}}]]}];
```

$$\{\text{Err1a1, Err1a2, Err1a3, Err1a4}\} = \text{Parallelize}\Big[\Big\{$$
$$\quad \text{Simplify}\Big[\int_0^1\int_0^1 (\text{DSumF1a1}.\{1,1\})^2\,dx\,dy\Big],$$
$$\quad \text{Simplify}\Big[\int_1^2\int_0^1 (\text{DSumF1a2}.\{1,1\})^2\,dx\,dy\Big],$$
$$\quad \text{Simplify}\Big[\int_1^2\int_{-1}^0 (\text{DSumF1a3}.\{1,1\})^2\,dx\,dy\Big],$$
$$\quad \text{Simplify}\Big[\int_2^3\int_{-1}^0 (\text{DSumF1a4}.\{1,1\})^2\,dx\,dy\Big]\Big\}\Big];$$

$$\{\text{Err1b1, Err1b2, Err1b3, Err1b4}\} = \text{Parallelize}\Big[\Big\{$$
$$\quad \text{Simplify}\Big[\int_0^1\int_1^2 (\text{DSumF1b1}.\{1,1\})^2\,dx\,dy\Big],$$
$$\quad \text{Simplify}\Big[\int_{-1}^0\int_1^2 (\text{DSumF1b2}.\{1,1\})^2\,dx\,dy\Big],$$
$$\quad \text{Simplify}\Big[\int_{-1}^0\int_2^3 (\text{DSumF1b3}.\{1,1\})^2\,dx\,dy\Big],$$
$$\quad \text{Simplify}\Big[\int_{-2}^{-1}\int_2^3 (\text{DSumF1b4}.\{1,1\})^2\,dx\,dy\Big]\Big\}\Big];$$

```
Err1 = FullSimplify[Err1a1 + Err1a2 + Err1a3 + Err1a4 + Err1b1 + Err1b2 + Err1b3 + Err1b4];

Err = FullSimplify[Err1 /. φ → 1/2]
DErr = FullSimplify[D[Err, c01]];
H = D[DErr, c01];
Sols = RootReduce[Solve[DErr == 0, c01]];
N[Sols]
RootReduce[Sols[[1]]]
TableForm[{Range[Length[Sols]], Err /. N[Sols], Sign[H /. N[Sols]]}ᵀ]
```

$$\frac{1}{1440}\left(752 + c01\left(2611 + 2\,c01\left(1596 + c01\left(667 + 98\,c01\right)\right)\right)\right)$$

{{c01 → -0.621913}, {c01 → -2.24134 + 0.57569 i}, {c01 → -2.24134 - 0.57569 i}}

$\{c01 \to \mathrm{Root}[2611 + 6384\,\#1 + 4002\,\#1^2 + 784\,\#1^3\,\&,\ 1]\}$

1	0.0494532	1
2	0.583152 - 0.112147 i	-0.334955 - 0.942234 i
3	0.583152 + 0.112147 i	-0.334955 + 0.942234 i

```
Sol = Sols[[1]];
FullSol = N[Join[GenSol /. Sol, Sol]]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

{c02 → -0.378087, c11 → -0.378087, c12 → 0.378087, c01 → -0.621913}