In[449]:=
$$h[x\_] := \begin{cases} 1 + c01\,x + c02\,x^2 + c03\,x^3 + c04\,x^4 & 0 \le x \le 1/2 \\ c11\,(x-1) + c12\,(x-1)^2 + c13\,(x-1)^3 + c14\,(x-1)^4 & 1/2 < x \le 3/2 \\ c21\,(x-2) + c22\,(x-2)^2 + c23\,(x-2)^3 + c24\,(x-2)^4 & 3/2 < x \le 5/2 \\ 0 & \text{True} \end{cases};$$

```
f[x_] := h[Abs[x]];
```

In[451]:=
```
(*Continuity*)
C1l = Simplify[h[x], 0 ≤ x ≤ 1/2] /. x → 1/2
C1r = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 1/2
C2l = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 3/2
C2r = Simplify[h[x], 3/2 < x ≤ 5/2] /. x → 3/2
C3l = Simplify[h[x], 3/2 < x ≤ 5/2] /. x → 5/2
```

Out[451]=
$$1 + \frac{c01}{2} + \frac{c02}{4} + \frac{c03}{8} + \frac{c04}{16}$$

Out[452]=
$$\frac{1}{2}\left(-c11 + \frac{1}{2}\left(c12 + \frac{1}{2}\left(-c13 + \frac{c14}{2}\right)\right)\right)$$

Out[453]=
$$\frac{1}{2}\left(c11 + \frac{1}{2}\left(c12 + \frac{1}{2}\left(c13 + \frac{c14}{2}\right)\right)\right)$$

Out[454]=
$$\frac{1}{2}\left(-c21 + \frac{1}{2}\left(c22 + \frac{1}{2}\left(-c23 + \frac{c24}{2}\right)\right)\right)$$

Out[455]=
$$\frac{1}{2}\left(c21 + \frac{1}{2}\left(c22 + \frac{1}{2}\left(c23 + \frac{c24}{2}\right)\right)\right)$$

In[456]:=
```
(*Partition of unity and gradient representation*)
T0 = CoefficientList[FullSimplify[Sum[f[x - i], {i, -6, 6}], x > 0 && x < 1/2], x]
T1 = CoefficientList[FullSimplify[Sum[i f[x - i], {i, -6, 6}], x > 0 && x < 1/2], x]
```

Out[456]=
$$\{1,\ c01,\ c02 + 2\,(c12 + c22),\ c03,\ c04 + 2\,(c14 + c24)\}$$

Out[457]=
$$\{0,\ -2\,c11 - 4\,c21,\ 0,\ -2\,(c13 + 2\,c23)\}$$

In[458]:=
```
(*Smoothness*)
S0 = Simplify[D[h[x], x], 0 < x < 1/2] /. x → 0
S1l = Simplify[D[h[x], x], 0 < x < 1/2] /. x → 1/2
S1r = Simplify[D[h[x], x], 1/2 < x < 3/2] /. x → 1/2
S2l = Simplify[D[h[x], x], 1/2 < x < 3/2] /. x → 3/2
S2r = Simplify[D[h[x], x], 3/2 < x < 5/2] /. x → 3/2
S3l = Simplify[D[h[x], x], 3/2 < x < 5/2] /. x → 5/2
```

Out[458]= $c01$

Out[459]= $c01 + \frac{1}{2}\left(2\,c02 + \frac{1}{2}\left(3\,c03 + 2\,c04\right)\right)$

Out[460]= $c11 + \frac{1}{2}\left(-2\,c12 + \frac{1}{2}\left(3\,c13 - 2\,c14\right)\right)$

Out[461]= $c11 + \frac{1}{2}\left(2\,c12 + \frac{1}{2}\left(3\,c13 + 2\,c14\right)\right)$

Out[462]= $c21 + \frac{1}{2}\left(-2\,c22 + \frac{1}{2}\left(3\,c23 - 2\,c24\right)\right)$

Out[463]= $c21 + \frac{1}{2}\left(2\,c22 + \frac{1}{2}\left(3\,c23 + 2\,c24\right)\right)$

In[464]:=
```
GenSols = Solve[{
    C1l == C1r,
    C2l == C2r,
    C3l == 0,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T0[[4]] == 0,
    T0[[5]] == 0,
    T1[[2]] == 1,
    T1[[4]] == 0,
    S0 == 0,
    S1l == S1r,
    S2l == S2r,
    S3l == 0
    },
    {c01, c02, c03, c04, c11, c12, c13, c14, c21, c22, c23, c24}
]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[464]=
$$\left\{\left\{c01 \to 0,\ c03 \to 0,\ c11 \to -\frac{13}{6} - \frac{11\,c02}{12} - \frac{13\,c04}{48},\ c12 \to \frac{13}{6} + \frac{2\,c02}{3} + \frac{c04}{3},\right.\right.$$
$$c13 \to \frac{8}{3} + \frac{5\,c02}{3} + \frac{7\,c04}{12},\ c14 \to -\frac{14}{3} - \frac{8\,c02}{3} - \frac{4\,c04}{3},\ c21 \to \frac{5}{6} + \frac{11\,c02}{24} + \frac{13\,c04}{96},$$
$$\left.\left.c22 \to -\frac{13}{6} - \frac{7\,c02}{6} - \frac{c04}{3},\ c23 \to -\frac{4}{3} - \frac{5\,c02}{6} - \frac{7\,c04}{24},\ c24 \to \frac{14}{3} + \frac{8\,c02}{3} + \frac{5\,c04}{6}\right\}\right\}$$

In[465]:=
```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
W1[k_] := Piecewise[{{0, k < 0}, {φ^2/2, k == 0}, {1 - (1 - φ)^2/2, k == 1}}, 1];
SumF1 = Sum[Sum[W1[i - j] f[x - i, y - j] /. GenSol, {j, -5, 6}], {i, -5, 6}];
SumF1 = SumF1 /. φ → 1/2;
```

In[470]:=
```
{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6} = Parallelize[{
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 3 - 1/2 && y < 4 - 1/2]
}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6} = Parallelize[{
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -3 - 1/2 && y < -2 - 1/2]
}];
```

In[472]:=
```
TableForm[{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6}]
TableForm[{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6}]
```

Out[472]//TableForm=

$$\frac{96\,(48-3\,(136+44\,c02+13\,c04)\,y-8\,(13+22\,c02+2\,c04)\,y^2+12\,(32+20\,c02+7\,c04)\,y^3+16\,(14+8\,c02-5\,c04)\,y^4)-16\,x^4\,(1+2\,y)\,(32\,c02^2\,y\,(-77-90\,y+326}{73\,728}$$

$$-16\,c02^2\,x\,(-1+y)\,(x\,(786-4856\,y+5416\,y^2-1632\,y^3)+11\,(-27+42\,y+4\,y^2-8\,y^3)+20\,x^2\,(27-42\,y-4\,y^2+8\,y^3)+8\,x^3\,(3+412\,y-596\,y^2+192\,y^3))+c04^2\,x\,(-1$$

$$48\,(1+x)\,(3+2\,x)^2\,(32+7\,c04+112\,x+20\,c04\,x+4\,c02\,(5+16\,x))\,(1+c02\,(-1+y)^2+c04\,(-1+y)^4)+48\,(1+c02\,(1+x)^2+c04\,(1+x)^4)\,(3-2\,y)^2\,(-1+y)\,(-32$$

$$(1+x)\,(-2+y)\,(7\,(3+2\,x)^2\,(32+7\,c04+112\,x+20\,c04\,x+4\,c02\,(5+16\,x))\,(5-2\,y)^2\,(16\,(-9+7\,y)+c04\,(-27+20\,y)+c02\,(-84+64\,y))-2\,(8\,(18+103\,x+100$$

$$(2+x)\,(5+2\,x)^2\,(16\,(9+7\,x)+c04\,(27+20\,x)+c02\,(84+64\,x))\,(5-2\,y)^2\,(-2+y)\,(16\,(-9+7\,y)+c04\,(-27+20\,y)+c02\,(-84+64\,y))$$

0

Out[473]//TableForm=

$$16\,c02^2\,(-1+x)\,y\,(3\,(-649+804\,y+1180\,y^2-3552\,y^3)+32\,x^3\,(77-244\,y-140\,y^2+640\,y^3)+4\,x\,(2827-3236\,y-5140\,y^2+11\,168\,y^3)-4\,x^2\,(2563-5036\,y-4660\,y$$

$$16\,c02^2\,(-1+x)\,(1+y)\,(3\,(351+646\,y+204\,y^2+8\,y^3)+8\,x^3\,(-3+412\,y+596\,y^2+192\,y^3)-4\,x^2\,(-153+2262\,y+3596\,y^2+1192\,y^3)+2\,x\,(-969+1676\,y+4524\,y^2+1$$

$$16\,c02^2\,(-2+x)\,(1+y)\,(4\,x^3\,(327+1308\,y+1180\,y^2+320\,y^3)-4\,x^2\,(2142+8623\,y+7840\,y^2+2140\,y^3)-3\,(4014+16\,373\,y+15\,120\,y^2+4180\,y^3)+x\,(18\,123+73\,4$$

$$-64\,c04\,(-2+x)\,(2+y)\,(4\,x^3\,(9891+15\,546\,y+7724\,y^2+1232\,y^3)-9\,(35\,250+55\,425\,y+27\,550\,y^2+4396\,y^3)-2\,x^2\,(123\,975+194\,882\,y+96\,840\,y^2+15\,448\,y^3)+$$

$$-16\,c02^2\,(7-2\,x)^2\,(111-85\,x+16\,x^2)\,(5+2\,y)^2\,(42+53\,y+16\,y^2)-c04^2\,(7-2\,x)^2\,(141-107\,x+20\,x^2)\,(5+2\,y)^2\,(54+67\,y+20\,y^2)-16\,c04\,(7-2\,x)^2\,(-3+x)$$

1

In[475]:=
```
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4, DSumF1a5,
   DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4, DSumF1b5} = Parallelize[{
    Simplify[D[SumF1a1, {{x, y}}]],
    Simplify[D[SumF1a2, {{x, y}}]],
    Simplify[D[SumF1a3, {{x, y}}]],
    Simplify[D[SumF1a4, {{x, y}}]],
    Simplify[D[SumF1a5, {{x, y}}]],
    Simplify[D[SumF1b1, {{x, y}}]],
    Simplify[D[SumF1b2, {{x, y}}]],
    Simplify[D[SumF1b3, {{x, y}}]],
    Simplify[D[SumF1b4, {{x, y}}]],
    Simplify[D[SumF1b5, {{x, y}}]]
}];
```

In[476]:=
```
{Err1a1, Err1a2, Err1a3, Err1a4, Err1a5,
   Err1b1, Err1b2, Err1b3, Err1b4, Err1b5} = Parallelize[{
    Simplify[∫_(0-1/2)^(1-1/2) ∫_(0-1/2)^(1-1/2) (DSumF1a1.{1, 1})^2 dx dy],
    Simplify[∫_(1-1/2)^(2-1/2) ∫_(0-1/2)^(1-1/2) (DSumF1a2.{1, 1})^2 dx dy],
    Simplify[∫_(1-1/2)^(2-1/2) ∫_(-1-1/2)^(0-1/2) (DSumF1a3.{1, 1})^2 dx dy],
    Simplify[∫_(2-1/2)^(3-1/2) ∫_(-1-1/2)^(0-1/2) (DSumF1a4.{1, 1})^2 dx dy],
    Simplify[∫_(2-1/2)^(3-1/2) ∫_(-2-1/2)^(-1-1/2) (DSumF1a5.{1, 1})^2 dx dy],
        Simplify[∫_(0-1/2)^(1-1/2) ∫_(1-1/2)^(2-1/2) (DSumF1b1.{1, 1})^2 dx dy],
    Simplify[∫_(-1-1/2)^(0-1/2) ∫_(1-1/2)^(2-1/2) (DSumF1b2.{1, 1})^2 dx dy],
    Simplify[∫_(-1-1/2)^(0-1/2) ∫_(2-1/2)^(3-1/2) (DSumF1b3.{1, 1})^2 dx dy],
    Simplify[∫_(-2-1/2)^(-1-1/2) ∫_(2-1/2)^(3-1/2) (DSumF1b4.{1, 1})^2 dx dy],
    Simplify[∫_(-2-1/2)^(-1-1/2) ∫_(3-1/2)^(4-1/2) (DSumF1b5.{1, 1})^2 dx dy]
}];
```

In[484]:=
```
Err1 = FullSimplify[
   Err1a1 + Err1a2 + Err1a3 + Err1a4 + Err1a5 + Err1b1 + Err1b2 + Err1b3 + Err1b4 + Err1b5];
```

In[485]:=
```
Err = Err1
DErr = FullSimplify[D[Err, {{c02, c04}}]];
H = FullSimplify[D[Err, {{c02, c04}, 2}]];
Sols = Solve[DErr == 0, {c02, c04}];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}^T]
```

Out[485]=
$$\frac{1}{5\,618\,427\,494\,400}\Big(256\,c02\,\big(200\,217\,517\,472 + c02\,\big(118\,347\,584\,160 + c02\,\big(27\,664\,541\,552 + 2\,303\,656\,475\,c02\big)\big)\big) + 256\,c02\,\big(63\,489\,924\,552 + c02\,\big(22\,056\,170\,070 + 2\,442\,488\,383\,c02\big)\big)\,c04 + 96\,\big(22\,972\,836\,800 + c02\,\big(15\,775\,245\,040 + 2\,632\,493\,507\,c02\big)\big)\,c04^2 + 16\,\big(8\,531\,766\,382 + 2\,890\,448\,743\,c02\big)\,c04^3 + 3\,241\,946\,411\,c04^4 + 103\,333\,888\,\big(291\,665 + 133\,231\,c04\big)\Big)$$

Out[489]//TableForm=

1	0.0690686	True
2	0.485241 + 0.750128 i	False
3	0.485241 − 0.750128 i	False
4	0.475722 + 0.71243 i	False
5	0.475722 − 0.71243 i	False
6	0.606476 + 2.0615 i	False
7	0.606476 − 2.0615 i	False
8	1.07829 + 0.117982 i	False
9	1.07829 − 0.117982 i	False

In[490]:=
```
RootReduce[Sols[[1]]]
```

Out[490]=
$$\Big\{c02 \to \text{Root}\Big[1\,689\,662\,005\,856\,337\,976\,240\,041\,904\,499\,715\,134\,372\,237\,013\,891 + 1\,913\,730\,991\,673\,238\,426\,329\,051\,917\,073\,993\,219\,776\,357\,823\,234\,\#1 + 875\,576\,459\,165\,396\,432\,047\,441\,848\,633\,307\,153\,441\,108\,337\,957\,\#1^2 + 220\,353\,172\,042\,287\,473\,274\,228\,459\,981\,792\,313\,551\,037\,813\,848\,\#1^3 + 34\,801\,416\,482\,455\,760\,933\,198\,507\,141\,290\,225\,454\,678\,189\,258\,\#1^4 + 3\,691\,137\,740\,398\,847\,524\,505\,779\,769\,450\,660\,705\,708\,056\,556\,\#1^5 + 269\,781\,047\,608\,097\,129\,415\,603\,204\,681\,660\,622\,987\,864\,608\,\#1^6 + 13\,495\,678\,805\,173\,790\,703\,538\,473\,545\,709\,318\,352\,456\,896\,\#1^7 + 431\,424\,428\,610\,766\,564\,410\,277\,417\,438\,563\,815\,617\,152\,\#1^8 + 7\,052\,647\,884\,593\,722\,360\,448\,514\,392\,761\,188\,016\,896\,\#1^9\ \&,\ 1\Big],$$
$$c04 \to \text{Root}\Big[-12\,052\,730\,059\,666\,565\,027\,942\,383\,263\,597\,277\,906\,151\,992\,998\,912 + 8\,724\,149\,967\,135\,470\,183\,666\,169\,667\,339\,895\,553\,162\,816\,345\,600\,\#1 - 2\,307\,186\,105\,966\,193\,393\,075\,892\,682\,083\,728\,786\,314\,507\,275\,328\,\#1^2 + 291\,580\,584\,404\,141\,632\,572\,557\,804\,308\,371\,318\,670\,966\,217\,856\,\#1^3 - 19\,458\,521\,328\,012\,163\,527\,706\,445\,700\,672\,395\,426\,531\,724\,968\,\#1^4 + 806\,520\,397\,583\,283\,384\,374\,587\,130\,856\,734\,796\,383\,219\,852\,\#1^5 - 21\,683\,052\,237\,286\,257\,632\,433\,005\,453\,889\,060\,211\,834\,536\,\#1^6 + 387\,555\,349\,882\,436\,704\,314\,878\,114\,069\,617\,899\,997\,788\,\#1^7 - 4\,358\,609\,338\,880\,559\,588\,643\,958\,975\,730\,339\,962\,778\,\#1^8 + 27\,549\,405\,799\,194\,227\,970\,502\,009\,346\,723\,390\,691\,\#1^9\ \&,\ 1\Big]\Big\}$$

In[491]:=
```
Sol = Sols[[1]];
FullSol = N[Join[GenSol /. Sol, Sol]]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

Out[492]=
```
{c01 → 0., c03 → 0., c11 → -0.828758, c12 → 1.62969,
 c13 → 0.524805, c14 → -2.51876, c21 → 0.164379, c22 → -0.427537,
 c23 → -0.262402, c24 → 0.919919, c02 → -2.40431, c04 → 3.19769}
```

Out[494]=