$$\texttt{h[x\_]} := \begin{cases} 1 + c_{0,1}\,x + c_{0,2}\,x^2 + c_{0,3}\,x^3 & 0 \le x \le 1/2 \\ c_{1,1}\,(x-1) + c_{1,2}\,(x-1)^2 + c_{1,3}\,(x-1)^3 & 1/2 < x \le 3/2 \\ c_{2,1}\,(x-2) + c_{2,2}\,(x-2)^2 + c_{2,3}\,(x-2)^3 & 3/2 < x \le 5/2 \\ 0 & \text{True} \end{cases};$$

```
f[x_] := h[Abs[x]];
```

$$\texttt{AllVars} = \{c_{0,1}, c_{0,2}, c_{0,3}, c_{1,1}, c_{1,2}, c_{1,3}, c_{2,1}, c_{2,2}, c_{2,3}\};$$

```
(*Continuity*)
C1 = Limit[h[x], x → 1/2, Direction → 1] == Limit[h[x], x → 1/2, Direction → -1]
C2 = Limit[h[x], x → 3/2, Direction → 1] == Limit[h[x], x → 3/2, Direction → -1]
C3 = Limit[h[x], x → 5/2, Direction → 1] == Limit[h[x], x → 5/2, Direction → -1]
```

$$\frac{1}{8}\left(8 + 4\,c_{0,1} + 2\,c_{0,2} + c_{0,3}\right) == \frac{1}{8}\left(-4\,c_{1,1} + 2\,c_{1,2} - c_{1,3}\right)$$

$$\frac{1}{8}\left(4\,c_{1,1} + 2\,c_{1,2} + c_{1,3}\right) == \frac{1}{8}\left(-4\,c_{2,1} + 2\,c_{2,2} - c_{2,3}\right)$$

$$\frac{1}{8}\left(4\,c_{2,1} + 2\,c_{2,2} + c_{2,3}\right) == 0$$

```
(*Partition of unity and linear term*)
```

$$\texttt{T0} = \texttt{CoefficientList}\left[\texttt{FullSimplify}\left[\sum_{i=-6}^{6} \texttt{f[x - i]},\ x > 0\ \&\&\ x < 1/2\right],\ x\right]$$

$$\texttt{T1} = \texttt{CoefficientList}\left[\texttt{FullSimplify}\left[\sum_{i=-6}^{6} i\ \texttt{f[x - i]},\ x > 0\ \&\&\ x < 1/2\right],\ x\right]$$

$$\{1,\ c_{0,1},\ c_{0,2} + 2\,(c_{1,2} + c_{2,2}),\ c_{0,3}\}$$

$$\{0,\ -2\,c_{1,1} - 4\,c_{2,1},\ 0,\ -2\,(c_{1,3} + 2\,c_{2,3})\}$$

```
(*Smoothness*)
Dh = Simplify[D[h[x], x], x > 0];
S0 = (Dh /. x → 0) == 0
S1 = Limit[Dh, x → 1/2, Direction → 1] == Limit[Dh, x → 1/2, Direction → -1]
S2 = Limit[Dh, x → 3/2, Direction → 1] == Limit[Dh, x → 3/2, Direction → -1]
S3 = Limit[Dh, x → 5/2, Direction → 1] == Limit[Dh, x → 5/2, Direction → -1]
```

$$c_{0,1} == 0$$

$$c_{0,1} + c_{0,2} + \frac{3\,c_{0,3}}{4} == c_{1,1} - c_{1,2} + \frac{3\,c_{1,3}}{4}$$

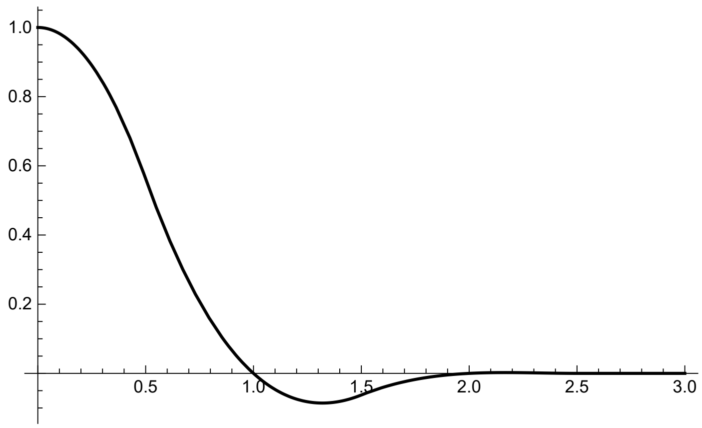
$$c_{1,1} + c_{1,2} + \frac{3\,c_{1,3}}{4} == c_{2,1} - c_{2,2} + \frac{3\,c_{2,3}}{4}$$

$$c_{2,1} + c_{2,2} + \frac{3\,c_{2,3}}{4} == 0$$

```
GenSols = Solve[{
    C1, C2, C3,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T0[[4]] == 0,
    T1[[2]] == 1,
    T1[[4]] == 0,
    S0, S1, S2, S3
    },
    AllVars
]
```

$$\left\{\left\{c_{0,1}\to 0,\ c_{0,2}\to -\frac{7}{4},\ c_{0,3}\to 0,\ c_{1,1}\to -\frac{9}{16},\ c_{1,2}\to 1,\ c_{1,3}\to -\frac{1}{4},\ c_{2,1}\to \frac{1}{32},\ c_{2,2}\to -\frac{1}{8},\ c_{2,3}\to \frac{1}{8}\right\}\right\}$$

```
Plot[h[x] /. GenSols[[1]], {x, 0, 3}, PlotStyle → Black, Background → White]
```




```
RegionXY[k_] := {Quotient[k, 2], 1 + Quotient[-k, 2]};
Regions = Table[RegionXY[k], {k, -4, 7}] - 1/2
```

$$\left\{\left\{-\frac{5}{2},\frac{5}{2}\right\},\left\{-\frac{5}{2},\frac{3}{2}\right\},\left\{-\frac{3}{2},\frac{3}{2}\right\},\left\{-\frac{3}{2},\frac{1}{2}\right\},\left\{-\frac{1}{2},\frac{1}{2}\right\},\left\{-\frac{1}{2},-\frac{1}{2}\right\},\left\{\frac{1}{2},-\frac{1}{2}\right\},\left\{\frac{1}{2},-\frac{3}{2}\right\},\left\{\frac{3}{2},-\frac{3}{2}\right\},\left\{\frac{3}{2},-\frac{5}{2}\right\},\left\{\frac{5}{2},-\frac{5}{2}\right\},\left\{\frac{5}{2},-\frac{7}{2}\right\}\right\}$$

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
φ = 1/2;
```

$$\texttt{W[k\_]} := \begin{cases} 0 & k<0 \\ \varphi^2/2 & k==0 \\ 1-(1-\varphi)^2/2 & k==1 \\ 1 & \text{True} \end{cases};$$

$$\texttt{SumF} = \sum_{i=-5}^{6}\sum_{j=-5}^{6} \texttt{W[i - j] f[x - i, y - j] /. GenSol;}$$

```
SimplifySquare[f_, x0_, y0_] := Simplify[f, x > x0 && x < x0 + 1 && y > y0 && y < y0 + 1];
DSimplifySquare[f_, {x0_, y0_}] := Simplify[D[SimplifySquare[f, x0, y0], {{x, y}}]];
DSumF = ParallelMap[DSimplifySquare[SumF, #] &, Regions];
```

```
AnisoInt[df_, {x0_, y0_}] :=
  Simplify[Integrate[Expand[(df.{1, 1})^2], {x, x0, x0 + 1}, {y, y0, y0 + 1}]];
AnisoInts = Parallelize[MapThread[AnisoInt, {DSumF, Regions}]];
Err = Simplify[Total[AnisoInts]]
```

$$\frac{94\,211\,027}{660\,602\,880}$$