```
h[x_] := Piecewise[{
  {1 + c0,1 x + c0,2 x^2,              0 ≤ x ≤ 1/2},
  {c1,1 (x - 1) + c1,2 (x - 1)^2,     1/2 < x ≤ 3/2},
  {c2,1 (x - 2) + c2,2 (x - 2)^2,     3/2 < x ≤ 5/2},
  {0,                                   True}}];
f[x_] := h[Abs[x]];
AllVars = {c0,1, c0,2, c1,1, c1,2, c2,1, c2,2};

(*Continuity*)
C1 = Limit[h[x], x → 1/2, Direction → 1] == Limit[h[x], x → 1/2, Direction → -1]
C2 = Limit[h[x], x → 3/2, Direction → 1] == Limit[h[x], x → 3/2, Direction → -1]
C3 = Limit[h[x], x → 5/2, Direction → 1] == Limit[h[x], x → 5/2, Direction → -1]
```

$$\frac{1}{4}\left(4+2\,c_{0,1}+c_{0,2}\right)=\frac{1}{4}\left(-2\,c_{1,1}+c_{1,2}\right)$$

$$\frac{1}{4}\left(2\,c_{1,1}+c_{1,2}\right)=\frac{1}{4}\left(-2\,c_{2,1}+c_{2,2}\right)$$

$$\frac{1}{4}\left(2\,c_{2,1}+c_{2,2}\right)=0$$

```
(*Partition of unity and linear term*)
T0 = CoefficientList[FullSimplify[Sum[f[x - i], {i, -6, 6}], x > 0 && x < 1/2], x]
T1 = CoefficientList[FullSimplify[Sum[i f[x - i], {i, -6, 6}], x > 0 && x < 1/2], x]
```

$$\{1,\ c_{0,1},\ c_{0,2}+2\left(c_{1,2}+c_{2,2}\right)\}$$

$$\{0,\ -2\left(c_{1,1}+2\,c_{2,1}\right)\}$$

```
GenSols = Solve[{
    C1, C2, C3,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[2]] == 1
    },
    AllVars
]
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{c_{0,1}\to 0,\ c_{1,1}\to -\frac{3}{2}-\frac{c_{0,2}}{2},\ c_{1,2}\to 1,\ c_{2,1}\to \frac{1}{2}+\frac{c_{0,2}}{4},\ c_{2,2}\to -1-\frac{c_{0,2}}{2}\right\}\right\}$$

```
RegionXY[k_] := {Quotient[k, 2], 1 + Quotient[-k, 2]};
Regions = Table[RegionXY[k], {k, -4, 7}] - 1/2
```

$$\left\{\left\{-\frac{5}{2},\frac{5}{2}\right\},\left\{-\frac{5}{2},\frac{3}{2}\right\},\left\{-\frac{3}{2},\frac{3}{2}\right\},\left\{-\frac{3}{2},\frac{1}{2}\right\},\left\{-\frac{1}{2},\frac{1}{2}\right\},\left\{-\frac{1}{2},-\frac{1}{2}\right\},\left\{\frac{1}{2},-\frac{1}{2}\right\},\left\{\frac{1}{2},-\frac{3}{2}\right\},\left\{\frac{3}{2},-\frac{3}{2}\right\},\left\{\frac{3}{2},-\frac{5}{2}\right\},\left\{\frac{5}{2},-\frac{5}{2}\right\},\left\{\frac{5}{2},-\frac{7}{2}\right\}\right\}$$

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
φ = 1/2;
W[k_] := Piecewise[{
    {0, k < 0},
    {φ^2/2, k == 0},
    {1 - (1 - φ)^2/2, k == 1},
    {1, True}}];
```

$$\text{SumF} = \sum_{i=-5}^{6}\sum_{j=-5}^{6} \text{W[i - j] f[x - i, y - j] /. GenSol;}$$

```
SimplifySquare[f_, x0_, y0_] := Simplify[f, x > x0 && x < x0 + 1 && y > y0 && y < y0 + 1];
DSimplifySquare[f_, {x0_, y0_}] := Simplify[D[SimplifySquare[f, x0, y0], {{x, y}}]];
DSumF = ParallelMap[DSimplifySquare[SumF, #] &, Regions];

AnisoInt[df_, {x0_, y0_}] :=
  Simplify[Integrate[Expand[(df.{1, 1})^2], {x, x0, x0 + 1}, {y, y0, y0 + 1}]];
AnisoInts = Parallelize[MapThread[AnisoInt, {DSumF, Regions}]];
Err = Simplify[Total[AnisoInts]]
```

$$\frac{1}{368\,640}\left(1\,360\,512 + 2\,161\,584\, c_{0,2} + 1\,188\,000\, c_{0,2}^2 + 253\,708\, c_{0,2}^3 + 20\,325\, c_{0,2}^4\right)$$

```
FreeVars = Variables[Err];
DErr = Simplify[D[Err, {FreeVars}]];
H = D[DErr, {FreeVars}];
Sols = Solve[DErr == 0, FreeVars, Reals];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}ᵀ]
```

1	0.0996855	True

```
RootReduce[Sols[[1]]]
```

$$\left\{c_{0,2} \to \text{Root}\left[180\,132 + 198\,000\, \#1 + 63\,427\, \#1^2 + 6775\, \#1^3\ \&,\ 1\right]\right\}$$

```
NSol = N[Sols[[1]]];
FullSol = Join[GenSol /. NSol, NSol]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

$\{c_{0,1} \to 0,\ c_{1,1} \to -0.721136,\ c_{1,2} \to 1,\ c_{2,1} \to 0.110568,\ c_{2,2} \to -0.221136,\ c_{0,2} \to -1.55773\}$