$$
h[x\_] := \begin{cases} 1 + c_{0,1}\, x + c_{0,2}\, x^2 + c_{0,3}\, x^3 + c_{0,4}\, x^4 & 0 \le x \le 1/2 \\ c_{1,1}\,(x-1) + c_{1,2}\,(x-1)^2 + c_{1,3}\,(x-1)^3 + c_{1,4}\,(x-1)^4 & 1/2 < x \le 3/2 \\ c_{2,1}\,(x-2) + c_{2,2}\,(x-2)^2 + c_{2,3}\,(x-2)^3 + c_{2,4}\,(x-2)^4 & 3/2 < x \le 5/2 \\ 0 & \text{True} \end{cases};
$$

```
f[x_] := h[Abs[x]];
AllVars = {c0,1, c0,2, c0,3, c0,4, c1,1, c1,2, c1,3, c1,4, c2,1, c2,2, c2,3, c2,4};

(*Continuity*)
C1 = Limit[h[x], x → 1/2, Direction → 1] == Limit[h[x], x → 1/2, Direction → -1]
C2 = Limit[h[x], x → 3/2, Direction → 1] == Limit[h[x], x → 3/2, Direction → -1]
C3 = Limit[h[x], x → 5/2, Direction → 1] == Limit[h[x], x → 5/2, Direction → -1]
```

$$\frac{1}{16}\left(16 + 8\, c_{0,1} + 4\, c_{0,2} + 2\, c_{0,3} + c_{0,4}\right) == \frac{1}{16}\left(-8\, c_{1,1} + 4\, c_{1,2} - 2\, c_{1,3} + c_{1,4}\right)$$

$$\frac{1}{16}\left(8\, c_{1,1} + 4\, c_{1,2} + 2\, c_{1,3} + c_{1,4}\right) == \frac{1}{16}\left(-8\, c_{2,1} + 4\, c_{2,2} - 2\, c_{2,3} + c_{2,4}\right)$$

$$\frac{1}{16}\left(8\, c_{2,1} + 4\, c_{2,2} + 2\, c_{2,3} + c_{2,4}\right) == 0$$

```
(*Partition of unity and linear term*)
```

$$T0 = \text{CoefficientList}\left[\text{FullSimplify}\left[\sum_{i=-6}^{6} f[x-i],\ x > 0\ \&\&\ x < 1/2\right],\ x\right]$$

$$T1 = \text{CoefficientList}\left[\text{FullSimplify}\left[\sum_{i=-6}^{6} i\, f[x-i],\ x > 0\ \&\&\ x < 1/2\right],\ x\right]$$

$$\{1,\ c_{0,1},\ c_{0,2} + 2\,(c_{1,2} + c_{2,2}),\ c_{0,3},\ c_{0,4} + 2\,(c_{1,4} + c_{2,4})\}$$

$$\{0,\ -2\, c_{1,1} - 4\, c_{2,1},\ 0,\ -2\,(c_{1,3} + 2\, c_{2,3})\}$$

```
GenSols = Solve[{
    C1, C2, C3,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T0[[4]] == 0,
    T0[[5]] == 0,
    T1[[2]] == 1,
    T1[[4]] == 0
    },
    AllVars
]
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{c_{0,1} \to 0,\ c_{0,3} \to 0,\ c_{1,3} \to -6 - 2\, c_{0,2} - \frac{c_{0,4}}{2} - 4\, c_{1,1},\ c_{1,4} \to 4 - 4\, c_{1,2},\ c_{2,1} \to -\frac{1}{4} - \frac{c_{1,1}}{2},\ c_{2,2} \to -\frac{c_{0,2}}{2} - c_{1,2},\ c_{2,3} \to 3 + c_{0,2} + \frac{c_{0,4}}{4} + 2\, c_{1,1},\ c_{2,4} \to -4 - \frac{c_{0,4}}{2} + 4\, c_{1,2}\right\}\right\}$$

```
RegionXY[k_] := {Quotient[k, 2], 1 + Quotient[-k, 2]};
Regions = Table[RegionXY[k], {k, -4, 7}] - 1/2
```

$$\left\{\left\{-\frac{5}{2}, \frac{5}{2}\right\}, \left\{-\frac{5}{2}, \frac{3}{2}\right\}, \left\{-\frac{3}{2}, \frac{3}{2}\right\}, \left\{-\frac{3}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{3}{2}\right\}, \left\{\frac{3}{2}, -\frac{3}{2}\right\}, \left\{\frac{3}{2}, -\frac{5}{2}\right\}, \left\{\frac{5}{2}, -\frac{5}{2}\right\}, \left\{\frac{5}{2}, -\frac{7}{2}\right\}\right\}$$

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
φ = 1/2;
```

$$\texttt{W[k\_] :=} \begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1 - (1-\varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases};$$

$$\texttt{SumF} = \sum_{i=-5}^{6} \sum_{j=-5}^{6} \texttt{W[i - j] f[x - i, y - j] /. GenSol;}$$

```
SimplifySquare[f_, x0_, y0_] := Simplify[f, x > x0 && x < x0 + 1 && y > y0 && y < y0 + 1];
DSimplifySquare[f_, {x0_, y0_}] := Simplify[D[SimplifySquare[f, x0, y0], {{x, y}}]];
DSumF = ParallelMap[DSimplifySquare[SumF, #] &, Regions];

AnisoInt[df_, {x0_, y0_}] :=
  Simplify[Integrate[Expand[(df.{1, 1})^2], {x, x0, x0 + 1}, {y, y0, y0 + 1}]];
AnisoInts = Parallelize[MapThread[AnisoInt, {DSumF, Regions}]];
Err = Simplify[Total[AnisoInts]]
```

$$\frac{1}{69\,363\,302\,400}\Big(3\,435\,022\,080\, c_{0,2}^4 + 13\,203\,915\, c_{0,4}^4 + 48\, c_{0,4}^3 \left(13\,451\,975 + 150\,478\, c_{1,1} + 2795\, c_{1,2}\right) + 10\,752\, c_{0,2}^3 \left(3\,711\,112 + 316\,425\, c_{0,4} - 6012\, c_{1,1} + 13\,794\, c_{1,2}\right) + 64\, c_{0,4}^2 \left(222\,183\,873 + 1\,739\,784\, c_{1,1}^2 - 39\,564\, c_{1,2} + 37\,016\, c_{1,2}^2 - 6\, c_{1,1} \left(100\,757 + 23\,092\, c_{1,2}\right)\right) + 2048\, c_{0,4} \left(58\,355\,292 + 126\,756\, c_{1,1}^3 - 5\,002\,167\, c_{1,2} - 286\,130\, c_{1,2}^2 - 4930\, c_{1,2}^3 - 6\, c_{1,1}^2 \left(-331\,526 + 21\,335\, c_{1,2}\right) + c_{1,1} \left(5\,645\,217 + 223\,554\, c_{1,2} - 46\,248\, c_{1,2}^2\right)\right) + 8192 \left(46\,358\,295 + 528\,444\, c_{1,1}^3 + 507\,024\, c_{1,1}^4 - 8\,451\,174\, c_{1,2} + 1\,896\,418\, c_{1,2}^2 + 111\,088\, c_{1,2}^3 + 14\,080\, c_{1,2}^4 - 36\, c_{1,1} \left(-531\,177 - 17\,935\, c_{1,2} + 4449\, c_{1,2}^2\right) + 6\, c_{1,1}^2 \left(1\,429\,427 + 36\,248\, c_{1,2} + 12\,032\, c_{1,2}^2\right)\right) + 32\, c_{0,2}^2 \left(39\,789\,115\, c_{0,4}^2 + 8\, c_{0,4} \left(118\,209\,037 + 654\,822\, c_{1,1} + 85\,559\, c_{1,2}\right) + 32 \left(191\,723\,345 + 887\,544\, c_{1,1}^2 - 634\,372\, c_{1,2} + 202\,904\, c_{1,2}^2 - 630\, c_{1,1} \left(-2063 + 1004\, c_{1,2}\right)\right)\right) + 64\, c_{0,2} \left(3\,306\,360\, c_{0,4}^3 + c_{0,4}^2 \left(120\,471\,210 + 1\,169\,382\, c_{1,1} + 98\,707\, c_{1,2}\right) + 8\, c_{0,4} \left(203\,126\,499 + 1\,056\,384\, c_{1,1}^2 + c_{1,1} \left(879\,834 - 385\,536\, c_{1,2}\right) - 1\,770\,536\, c_{1,2} - 15\,208\, c_{1,2}^2\right) + 128 \left(52\,225\,182 + 126\,756\, c_{1,1}^3 - 2\,843\,219\, c_{1,2} + 175\,864\, c_{1,2}^2 + 20\,590\, c_{1,2}^3 - 126\, c_{1,1}^2 \left(-20\,259 + 251\, c_{1,2}\right) - 3\, c_{1,1} \left(-2\,311\,657 + 5498\, c_{1,2} + 15\,416\, c_{1,2}^2\right)\right)\right)\Big)$$

```
FreeVars = Variables[Err];
DErr = Simplify[D[Err, {FreeVars}]];
H = D[DErr, {FreeVars}];
Sols = Solve[DErr == 0, FreeVars, Reals];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols, 32], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}ᵀ]
```

1	0.06881454653131627777477218405	True

```
NSol = N[Sols[[1]], 32];
FullSol = Join[GenSol /. NSol, NSol]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

$\{c_{0,1} \to 0,\ c_{0,3} \to 0,\ c_{1,3} \to 0.51980718987666977283111451604 03,$
$c_{1,4} \to -2.6850530621058698539505862102221,$
$c_{2,1} \to 0.16334716743234760510660644494250,\ c_{2,2} \to -0.46091346966448189934824423790 65,$
$c_{2,3} \to -0.25990359493833488641555725802 01,\ c_{2,4} \to 1.0566837290758165293712399081263,$
$c_{0,2} \to -2.4206995917239711282788046292981,\ c_{0,4} \to 3.2567386660601066491586926041917,$
$c_{1,1} \to -0.82669433486469521021321288988500,\ c_{1,2} \to 1.6712632655264674634876465525555\}$