```
h[x_] := Piecewise[{{1 + c01 x + c02 x^2 + c03 x^3, 0 ≤ x ≤ 1},
   {c11 (x - 1) + c12 (x - 1)^2 + c13 (x - 1)^3, 1 < x ≤ 2},
   {c21 (x - 2) + c22 (x - 2)^2 + c23 (x - 2)^3, 2 < x ≤ 3}},
  0, True];

f[x_] := h[Abs[x]];

(*Interpolant constraints*)
I1 = f[1]
I2 = f[2]
I3 = f[3]
```

$1 + c01 + c02 + c03$

$c11 + c12 + c13$

$c21 + c22 + c23$

```
(*Partition of unity and gradient representation*)
T0 = CoefficientList[
  FullSimplify[f[x + 2] + f[x + 1] + f[x] + f[x - 1] + f[x - 2] + f[x - 3], x > 0 && x < 1], x]
T1 = CoefficientList[FullSimplify[
   -2 f[x + 2] - f[x + 1] + f[x - 1] + 2 f[x - 2] + 3 f[x - 3], x > 0 && x < 1], x]
```

$\{2 + c01 + c02 + c03 + c11 + c12 + c13 + c21 + c22 + c23,$
$-2\,c02 - 3\,c03 - 2\,c12 - 3\,c13 - 2\,c22 - 3\,c23,\ 2\,c02 + 3\,c03 + 2\,c12 + 3\,c13 + 2\,c22 + 3\,c23\}$

$\{1 + c01 + c02 + c03 + 2\,c11 + 2\,c12 + 2\,c13 + 3\,c21 + 3\,c22 + 3\,c23,$
$-c01 - 2\,c02 - 3\,c03 - 3\,c11 - 4\,c12 - 6\,c13 - 5\,c21 - 6\,c22 - 9\,c23,$
$c02 + 3\,c03 + c12 + 6\,c13 + c22 + 9\,c23,\ -c03 - 3\,c13 - 5\,c23\}$

```
GenSols = Solve[{
    I1 == 0,
    I2 == 0,
    I3 == 0,
    T0[[1]] == 1,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[1]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0,
    T1[[4]] == 0
    },
    {c01, c02, c03, c11, c12, c13, c21, c22, c23}
]
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\left\{\left\{c03 \to -1 - c01 - c02,\ c13 \to -c11 - c12,\ c21 \to -\frac{7}{5} - \frac{7\,c01}{5} - \frac{2\,c02}{5} - \frac{6\,c11}{5} - \frac{c12}{5},\right.\right.$$
$$\left.\left. c22 \to \frac{6}{5} + \frac{6\,c01}{5} + \frac{c02}{5} + \frac{3\,c11}{5} - \frac{2\,c12}{5},\ c23 \to \frac{1}{5} + \frac{c01}{5} + \frac{c02}{5} + \frac{3\,c11}{5} + \frac{3\,c12}{5}\right\}\right\}$$

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
W1[k_] := Piecewise[{{0, k < 0}, {φ^2/2, k == 0}, {1 - (1 - φ)^2/2, k == 1}}, 1];
SumF1 = Sum[Sum[W1[i - j] f[x - i, y - j] /. GenSol, {j, -5, 6}], {i, -5, 6}];
{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6} = Parallelize[{
    Simplify[SumF1, x > 0 && x < 1 && y > 0 && y < 1],
    Simplify[SumF1, x > 0 && x < 1 && y > 1 && y < 2],
    Simplify[SumF1, x > -1 && x < 0 && y > 1 && y < 2],
    Simplify[SumF1, x > -1 && x < 0 && y > 2 && y < 3],
    Simplify[SumF1, x > -2 && x < -1 && y > 2 && y < 3],
    Simplify[SumF1, x > -2 && x < -1 && y > 3 && y < 4]
}];
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4, DSumF1a5, DSumF1a6} = Parallelize[{
    FullSimplify[D[SumF1a1, {{x, y}}]],
    FullSimplify[D[SumF1a2, {{x, y}}]],
    FullSimplify[D[SumF1a3, {{x, y}}]],
    FullSimplify[D[SumF1a4, {{x, y}}]],
    FullSimplify[D[SumF1a5, {{x, y}}]],
    FullSimplify[D[SumF1a6, {{x, y}}]]
}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6} = Parallelize[{
    Simplify[SumF1, x > 1 && x < 2 && y > 0 && y < 1],
    Simplify[SumF1, x > 1 && x < 2 && y > -1 && y < 0],
    Simplify[SumF1, x > 2 && x < 3 && y > -1 && y < 0],
    Simplify[SumF1, x > 2 && x < 3 && y > -2 && y < -1],
    Simplify[SumF1, x > 3 && x < 4 && y > -2 && y < -1],
    Simplify[SumF1, x > 3 && x < 4 && y > -3 && y < -2]
}];
{DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4, DSumF1b5, DSumF1b6} = Parallelize[{
    FullSimplify[D[SumF1b1, {{x, y}}]],
    FullSimplify[D[SumF1b2, {{x, y}}]],
    FullSimplify[D[SumF1b3, {{x, y}}]],
    FullSimplify[D[SumF1b4, {{x, y}}]],
    FullSimplify[D[SumF1b5, {{x, y}}]],
    FullSimplify[D[SumF1b6, {{x, y}}]]
}];
```

```
DSumF1a1 = Simplify[DSumF1a1 /. φ → 1 / 2];
DSumF1a2 = Simplify[DSumF1a2 /. φ → 1 / 2];
DSumF1a3 = Simplify[DSumF1a3 /. φ → 1 / 2];
DSumF1a4 = Simplify[DSumF1a4 /. φ → 1 / 2];
DSumF1a5 = Simplify[DSumF1a5 /. φ → 1 / 2];
DSumF1a6 = Simplify[DSumF1a6 /. φ → 1 / 2];
DSumF1b1 = Simplify[DSumF1b1 /. φ → 1 / 2];
DSumF1b2 = Simplify[DSumF1b2 /. φ → 1 / 2];
DSumF1b3 = Simplify[DSumF1b3 /. φ → 1 / 2];
DSumF1b4 = Simplify[DSumF1b4 /. φ → 1 / 2];
DSumF1b5 = Simplify[DSumF1b5 /. φ → 1 / 2];
DSumF1b6 = Simplify[DSumF1b6 /. φ → 1 / 2];

{Err1a1, Err1a2, Err1a3, Err1a4, Err1a5, Err1a6} = Parallelize[{
    Simplify[∫_0^1 ∫_0^1 (DSumF1a1.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_1^2 ∫_0^1 (DSumF1a2.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_1^2 ∫_-1^0 (DSumF1a3.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_2^3 ∫_-1^0 (DSumF1a4.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_2^3 ∫_-2^-1 (DSumF1a5.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_3^4 ∫_-2^-1 (DSumF1a6.{1, 1})^2 𝕕x 𝕕y]
}];
{Err1b1, Err1b2, Err1b3, Err1b4, Err1b5, Err1b6} = Parallelize[{
    Simplify[∫_0^1 ∫_1^2 (DSumF1b1.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-1^0 ∫_1^2 (DSumF1b2.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-1^0 ∫_2^3 (DSumF1b3.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-2^-1 ∫_2^3 (DSumF1b4.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-2^-1 ∫_3^4 (DSumF1b5.{1, 1})^2 𝕕x 𝕕y],
    Simplify[∫_-3^-2 ∫_3^4 (DSumF1b6.{1, 1})^2 𝕕x 𝕕y]
}];
Err1 = FullSimplify[Err1a1 + Err1a2 + Err1a3 + Err1a4 +
     Err1a5 + Err1a6 + Err1b1 + Err1b2 + Err1b3 + Err1b4 + Err1b5 + Err1b6];
```

```
Err = Err1
DErr = FullSimplify[D[Err, {{c01, c02, c11, c12}}]];
H = FullSimplify[D[Err, {{c01, c02, c11, c12}, 2}]];
```

$$\frac{1}{63\,000\,000}\left(168\,920\,343\,c01^4 + 2\,150\,743\,c02^4 + c02^3\left(39\,247\,427 + 11\,652\,126\,c11 + 4\,020\,801\,c12\right) + 3\,c01^3\left(338\,365\,884 + 75\,342\,324\,c02 + 102\,576\,067\,c11 + 34\,569\,742\,c12\right) + 3\,c02^2\left(86\,246\,666 + 9\,448\,279\,c11^2 + c12\left(19\,003\,966 + 1\,129\,579\,c12\right) + c11\left(53\,849\,841 + 6\,446\,208\,c12\right)\right) + 3\,c01^2\left(743\,772\,216 + 37\,897\,886\,c02^2 + 472\,541\,241\,c11 + 83\,257\,779\,c11^2 + 163\,357\,766\,c12 + 56\,304\,408\,c11\,c12 + 9\,728\,279\,c12^2 + c02\left(342\,228\,127 + 103\,130\,826\,c11 + 35\,011\,901\,c12\right)\right) + 3\,c01\left(681\,031\,404 + 8\,495\,924\,c02^3 + 9\,c11\left(79\,014\,809 + 6\,c11\left(4\,732\,732 + 630\,747\,c11\right)\right) + 251\,022\,806\,c12 + 2\,c11\left(87\,948\,503 + 17\,271\,132\,c11\right)c12 + 2\left(15\,597\,639 + 5\,912\,207\,c11\right)c12^2 + 1\,367\,488\,c12^3 + c02^2\left(115\,746\,802 + 34\,626\,301\,c11 + 11\,844\,926\,c12\right) + 2\,c02\left(252\,539\,891 + 4\,c11\left(39\,875\,404 + 6\,985\,551\,c11\right) + 55\,736\,941\,c12 + 18\,966\,158\,c11\,c12 + 3\,293\,104\,c12^2\right)\right) + 3\,c02\left(232\,270\,229 + 11\,409\,288\,c11^3 + c11^2\left(87\,381\,503 + 11\,614\,814\,c12\right) + c11\left(241\,298\,906 + 44\,c12\left(1\,370\,699 + 90\,856\,c12\right)\right) + c12\left(86\,575\,981 + c12\left(10\,688\,803 + 467\,138\,c12\right)\right)\right) + 3\left(222\,162\,041 + 11\,650\,551\,c11^4 + 8\,c11^3\left(5\,500\,051 + 1\,964\,988\,c12\right) + c11^2\left(191\,553\,249 + 45\,408\,224\,c12 + 8\,025\,506\,c12^2\right) + c11\left(337\,003\,107 + 2\,c12\left(67\,314\,799 + 7\,940\,562\,c12 + 921\,752\,c12^2\right)\right) + 3\,c12\left(40\,231\,594 + c12\left(8\,261\,833 + c12\left(624\,436 + 54\,117\,c12\right)\right)\right)\right)\right)$$

```
NSols = NSolve[DErr == 0, {c01, c02, c11, c12}];
TableForm[
 {Range[Length[NSols]], Err /. N[NSols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ NSols}^T]
```

1	0.528298 + 7.33229 i	False
2	0.528298 − 7.33229 i	False
3	1.91013 + 4.95848 i	False
4	1.91013 − 4.95848 i	False
5	27.777 − 0.673146 i	False
6	27.777 + 0.673146 i	False
7	34.7981 − 9.52804 i	False
8	34.7981 + 9.52804 i	False
9	2.39764 + 1.73273 i	False
10	2.39764 − 1.73273 i	False
11	1.06742 + 1.6692 i	False
12	1.06742 − 1.6692 i	False
13	0.657 + 0.417678 i	False
14	0.657 − 0.417678 i	False
15	1.72704 + 0.121618 i	False
16	1.72704 − 0.121618 i	False
17	1.6985 + 0.11274 i	False
18	1.6985 − 0.11274 i	False
19	1.60671 + 0.174859 i	False
20	1.60671 − 0.174859 i	False
21	−2.75266 + 2.23591 i	False
22	−2.75266 − 2.23591 i	False
23	0.976594 − 0.041467 i	False
24	0.976594 + 0.041467 i	False
25	0.81858 − 0.432571 i	False
26	0.81858 + 0.432571 i	False
27	0.0999374 − 0.0418618 i	False
28	0.0999374 + 0.0418618 i	False
29	1.0239 − 0.14209 i	False

30	1.0239 + 0.14209 i	False
31	0.750415 + 0.396665 i	False
32	0.750415 − 0.396665 i	False
33	1.1604 − 0.139167 i	False
34	1.1604 + 0.139167 i	False
35	0.757358 + 0.5026 i	False
36	0.757358 − 0.5026 i	False
37	1.04765 − 0.378955 i	False
38	1.04765 + 0.378955 i	False
39	0.105569 − 0.0443054 i	False
40	0.105569 + 0.0443054 i	False
41	1.13148 + 0.0676579 i	False
42	1.13148 − 0.0676579 i	False
43	0.715452 − 0.262095 i	False
44	0.715452 + 0.262095 i	False
45	0.849866 − 0.351532 i	False
46	0.849866 + 0.351532 i	False
47	0.768947 + 0.381592 i	False
48	0.768947 − 0.381592 i	False
49	0.13038 − 0.0285524 i	False
50	0.13038 + 0.0285524 i	False
51	1.31338 + 0.124248 i	False
52	1.31338 − 0.124248 i	False
53	0.90527 − 0.0317635 i	False
54	0.90527 + 0.0317635 i	False
55	0.828893 + 0.144463 i	False
56	0.828893 − 0.144463 i	False
57	1.21393 − 0.161239 i	False
58	1.21393 + 0.161239 i	False
59	0.584939 + 0.0269389 i	False
60	0.584939 − 0.0269389 i	False
61	1.42522 − 0.269396 i	False
62	1.42522 + 0.269396 i	False
63	1.22402 + 0.213994 i	False
64	1.22402 − 0.213994 i	False
65	1.65848 + 0.552046 i	False
66	1.65848 − 0.552046 i	False
67	0.87112 − 0.232836 i	False
68	0.87112 + 0.232836 i	False
69	26.2029 + 8.75896 i	False
70	26.2029 − 8.75896 i	False
71	1.00509 − 0.246774 i	False
72	1.00509 + 0.246774 i	False
73	0.858078 − 0.750702 i	False
74	0.858078 + 0.750702 i	False
75	0.937562 + 0.371323 i	False
76	0.937562 − 0.371323 i	False
77	0.934118 + 0.349618 i	False
78	0.934118 − 0.349618 i	False
79	0.188007 + 0.00855673 i	False
80	0.188007 − 0.00855673 i	False
81	0.0295579	True

```
Sols = Solve[DErr == 0, {c01, c02, c11, c12}];
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}^T]
```

1	0.0295579	True
2	0.904847 − 10.8784 i	False
3	0.904847 + 10.8784 i	False

4	$1.42522 - 0.269396\,i$	False
5	$1.42522 + 0.269396\,i$	False
6	$1.21393 - 0.161239\,i$	False
7	$1.21393 + 0.161239\,i$	False
8	$0.188007 + 0.00855673\,i$	False
9	$0.188007 - 0.00855673\,i$	False
10	$1.22402 + 0.213994\,i$	False
11	$1.22402 - 0.213994\,i$	False
12	$4.72904 + 5.11114\,i$	False
13	$4.72904 - 5.11114\,i$	False
14	$0.899583 - 0.735862\,i$	False
15	$0.899583 + 0.735862\,i$	False
16	$30.5054 + 94.4651\,i$	False
17	$30.5054 - 94.4651\,i$	False
18	$-2.75266 - 2.23591\,i$	False
19	$-2.75266 + 2.23591\,i$	False
20	$-22.9962 + 12.9345\,i$	False
21	$-22.9962 - 12.9345\,i$	False
22	$174.667 - 52.7064\,i$	False
23	$174.667 + 52.7064\,i$	False
24	$-192.099 + 70.049\,i$	False
25	$-192.099 - 70.049\,i$	False
26	$-8.45113 - 17.1864\,i$	False
27	$-8.45113 + 17.1864\,i$	False
28	$-83.4379 + 50.3345\,i$	False
29	$-83.4379 - 50.3345\,i$	False
30	$-7.71688 - 20.003\,i$	False
31	$-7.71688 + 20.003\,i$	False
32	$-5.31714 \times 10^{14} + 3.04748 \times 10^{14}\,i$	False
33	$-5.31714 \times 10^{14} - 3.04748 \times 10^{14}\,i$	False
34	$-202.856 + 84.5498\,i$	False
35	$-202.856 - 84.5498\,i$	False
36	$290.694 + 2422.67\,i$	False
37	$290.694 - 2422.67\,i$	False
38	$-5.88089 \times 10^{8} + 5.91726 \times 10^{9}\,i$	False
39	$-5.88089 \times 10^{8} - 5.91726 \times 10^{9}\,i$	False
40	$-470.259 + 339.438\,i$	False
41	$-470.259 - 339.438\,i$	False
42	$-5.99938 \times 10^{10} + 5.04895 \times 10^{12}\,i$	False
43	$-5.99938 \times 10^{10} - 5.04895 \times 10^{12}\,i$	False
44	$-372.009 + 3525.32\,i$	False
45	$-372.009 - 3525.32\,i$	False
46	$-122.439 - 245.574\,i$	False
47	$-122.439 + 245.574\,i$	False
48	$-544.074 - 98.1515\,i$	False
49	$-544.074 + 98.1515\,i$	False
50	$108.599 - 159.312\,i$	False
51	$108.599 + 159.312\,i$	False
52	$-854.48 + 108.168\,i$	False
53	$-854.48 - 108.168\,i$	False
54	$-694.859 - 70.2148\,i$	False
55	$-694.859 + 70.2148\,i$	False
56	$-59.5842 - 390.921\,i$	False
57	$-59.5842 + 390.921\,i$	False
58	$-6.71817 \times 10^{15} + 1.38041 \times 10^{15}\,i$	False
59	$-6.71817 \times 10^{15} - 1.38041 \times 10^{15}\,i$	False
60	$-2.92834 \times 10^{48} + 2.53342 \times 10^{48}\,i$	False
61	$-2.92834 \times 10^{48} - 2.53342 \times 10^{48}\,i$	False

62	$-294.521-486.566\,i$	False
63	$-294.521+486.566\,i$	False
64	$-2817.03+1406.07\,i$	False
65	$-2817.03-1406.07\,i$	False
66	$167.009-566.413\,i$	False
67	$167.009+566.413\,i$	False
68	$978.143-428.3\,i$	False
69	$978.143+428.3\,i$	False
70	$233.846-1362.21\,i$	False
71	$233.846+1362.21\,i$	False
72	$-6227.23-5696.33\,i$	False
73	$-6227.23+5696.33\,i$	False
74	$598.76-8702.26\,i$	False
75	$598.76+8702.26\,i$	False
76	$-6084.41-34\,503.9\,i$	False
77	$-6084.41+34\,503.9\,i$	False
78	$-1.13217\times10^{55}+5.58864\times10^{54}\,i$	False
79	$-1.13217\times10^{55}-5.58864\times10^{54}\,i$	False
80	$-5.47206\times10^{50}+1.02241\times10^{51}\,i$	False
81	$-5.47206\times10^{50}-1.02241\times10^{51}\,i$	False

```
Sol = Sols[[1]];
FullSol = N[Join[GenSol /. Sol, Sol]]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

```
{c03 → 0.188667, c13 → 0.16859459, c21 → 0.0925784, c22 → 0.0463118, c23 → -0.13889,
 c01 → -0.4353302219, c02 → -0.753337, c11 → -0.548062449, c12 → 0.37946786}
```