In[413]:=
$$h[x\_] := \begin{cases} 1 + c01\,x + c02\,x^2 + c03\,x^3 & 0 \le x \le 1/2 \\ c11\,(x-1) + c12\,(x-1)^2 + c13\,(x-1)^3 & 1/2 < x \le 3/2 \\ c21\,(x-2) + c22\,(x-2)^2 + c23\,(x-2)^3 & 3/2 < x \le 5/2 \\ 0 & \text{True} \end{cases};$$

```
f[x_] := h[Abs[x]];
```

In[415]:=
```
(*Continuity*)
C1l = Simplify[h[x], 0 ≤ x ≤ 1/2] /. x → 1/2
C1r = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 1/2
C2l = Simplify[h[x], 1/2 < x ≤ 3/2] /. x → 3/2
C2r = Simplify[h[x], 3/2 < x ≤ 5/2] /. x → 3/2
C3l = Simplify[h[x], 3/2 < x ≤ 5/2] /. x → 5/2
```

Out[415]= $1 + \frac{c01}{2} + \frac{c02}{4} + \frac{c03}{8}$

Out[416]= $\frac{1}{2}\left(-c11 + \frac{1}{2}\left(c12 - \frac{c13}{2}\right)\right)$

Out[417]= $\frac{1}{2}\left(c11 + \frac{1}{2}\left(c12 + \frac{c13}{2}\right)\right)$

Out[418]= $\frac{1}{2}\left(-c21 + \frac{1}{2}\left(c22 - \frac{c23}{2}\right)\right)$

Out[419]= $\frac{1}{2}\left(c21 + \frac{1}{2}\left(c22 + \frac{c23}{2}\right)\right)$

In[420]:=
```
(*Partition of unity and gradient representation*)
T0 = CoefficientList[FullSimplify[∑_{i=-6}^{6} f[x - i], x > 0 && x < 1/2], x]
T1 = CoefficientList[FullSimplify[∑_{i=-6}^{6} i f[x - i], x > 0 && x < 1/2], x]
```

Out[420]= $\{1,\ c01,\ c02 + 2\,(c12 + c22),\ c03\}$

Out[421]= $\{0,\ -2\,c11 - 4\,c21,\ 0,\ -2\,(c13 + 2\,c23)\}$

In[422]:=
```
(*Smoothness*)
S0 = Simplify[D[h[x], x], 0 < x < 1/2] /. x → 0
S1l = Simplify[D[h[x], x], 0 < x < 1/2] /. x → 1/2
S1r = Simplify[D[h[x], x], 1/2 < x < 3/2] /. x → 1/2
S2l = Simplify[D[h[x], x], 1/2 < x < 3/2] /. x → 3/2
S2r = Simplify[D[h[x], x], 3/2 < x < 5/2] /. x → 3/2
S3l = Simplify[D[h[x], x], 3/2 < x < 5/2] /. x → 5/2
```

Out[422]= $c01$

Out[423]= $c01 + \frac{1}{2}\left(2\,c02 + \frac{3\,c03}{2}\right)$

Out[424]= $c11 + \frac{1}{2}\left(-2\,c12 + \frac{3\,c13}{2}\right)$

Out[425]= $c11 + \frac{1}{2}\left(2\,c12 + \frac{3\,c13}{2}\right)$

Out[426]= $c21 + \frac{1}{2}\left(-2\,c22 + \frac{3\,c23}{2}\right)$

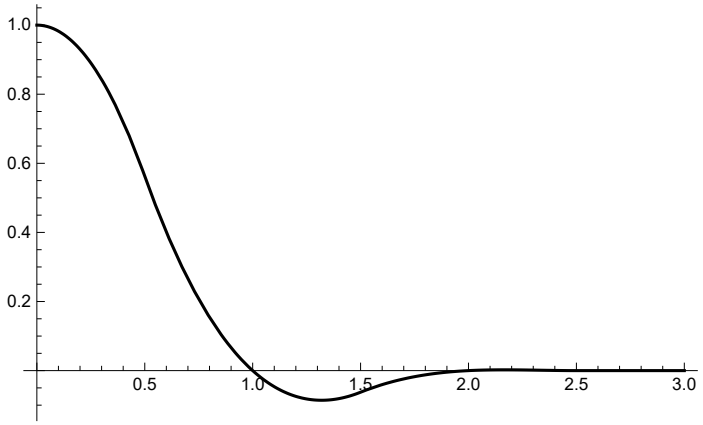
Out[427]= $c21 + \frac{1}{2}\left(2\,c22 + \frac{3\,c23}{2}\right)$

In[428]:=
```
GenSols = Solve[{
    C1l == C1r,
    C2l == C2r,
    C3l == 0,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T0[[4]] == 0,
    T1[[2]] == 1,
    T1[[4]] == 0,
    S0 == 0,
    S1l == S1r,
    S2l == S2r,
    S3l == 0
    },
    {c01, c02, c03, c11, c12, c13, c21, c22, c23}
]
```

Out[428]= $\left\{\left\{c01 \to 0,\ c02 \to -\frac{7}{4},\ c03 \to 0,\ c11 \to -\frac{9}{16},\ c12 \to 1,\ c13 \to -\frac{1}{4},\ c21 \to \frac{1}{32},\ c22 \to -\frac{1}{8},\ c23 \to \frac{1}{8}\right\}\right\}$

In[429]:=
```
Plot[h[x] /. GenSols[[1]], {x, 0, 3}, PlotStyle → Black]
```

Out[429]=



In[430]:=
```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
W1[k_] := Piecewise[{{0, k < 0}, {φ^2/2, k == 0}, {1 - (1 - φ)^2/2, k == 1}}, 1];
SumF1 = Sum[Sum[W1[i - j] f[x - i, y - j] /. GenSol, {j, -5, 6}], {i, -5, 6}];
SumF1 = SumF1 /. φ → 1/2;
```

In[435]:=
```
{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6} = Parallelize[{
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 0 - 1/2 && x < 1 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 1 - 1/2 && y < 2 - 1/2],
    Simplify[SumF1, x > -1 - 1/2 && x < 0 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 2 - 1/2 && y < 3 - 1/2],
    Simplify[SumF1, x > -2 - 1/2 && x < -1 - 1/2 && y > 3 - 1/2 && y < 4 - 1/2]
}];
{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6} = Parallelize[{
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > 0 - 1/2 && y < 1 - 1/2],
    Simplify[SumF1, x > 1 - 1/2 && x < 2 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -1 - 1/2 && y < 0 - 1/2],
    Simplify[SumF1, x > 2 - 1/2 && x < 3 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -2 - 1/2 && y < -1 - 1/2],
    Simplify[SumF1, x > 3 - 1/2 && x < 4 - 1/2 && y > -3 - 1/2 && y < -2 - 1/2]
}];
```

In[437]:=
```
TableForm[{SumF1a1, SumF1a2, SumF1a3, SumF1a4, SumF1a5, SumF1a6}]
TableForm[{SumF1b1, SumF1b2, SumF1b3, SumF1b4, SumF1b5, SumF1b6}]
```

Out[437]//TableForm=

$$\frac{-32\left(-16+59\,y-68\,y^2+12\,y^3\right)-4\,x^3\left(-96+101\,y+140\,y^2+52\,y^3\right)+4\,x^2\left(544+315\,y-1476\,y^2+140\,y^3\right)-x\left(-1888+957\,y+1260\,y^2+404\,y^3\right)}{4096}$$

$$\frac{x\left(689-1097\,y+512\,y^2-60\,y^3\right)-4\,x^3\left(-51+131\,y-64\,y^2+4\,y^3\right)+4\left(-5+41\,y-56\,y^2+20\,y^3\right)-4\,x^2\left(-281+523\,y-298\,y^2+52\,y^3\right)}{1024}$$

$$\frac{1}{512}\left(-179+377\,y-296\,y^2+80\,y^3+8\,x^3\left(-10+17\,y-10\,y^2+2\,y^3\right)+4\,x^2\left(-74+133\,y-87\,y^2+20\,y^3\right)+x\left(-\right.\right.$$

$$-\frac{(1+x)(-2+y)\left(-901+476\,y-52\,y^2+12\,x\left(-39+4\,y+4\,y^2\right)+4\,x^2\left(11-36\,y+12\,y^2\right)\right)}{8192}$$

$$-\frac{(2+x)(5+2\,x)^2(5-2\,y)^2(-2+y)}{8192}$$

0

Out[438]//TableForm=

$$\frac{-864-1989\,y+5548\,y^2-436\,y^3-16\,x^2\left(208-3\,y-474\,y^2+4\,y^3\right)+4\,x^3\left(96+101\,y-140\,y^2+52\,y^3\right)-x\left(-7392+351\,y+14\,748\,y^2+92\,y^3\right)}{4096}$$

$$\frac{-973-3549\,y-1736\,y^2-204\,y^3+4\,x^3\left(51+131\,y+64\,y^2+4\,y^3\right)-8\,x^2\left(217+458\,y+245\,y^2+32\,y^3\right)+x\left(3549+6853\,y+3664\,y^2+524\,y^3\right)}{1024}$$

$$\frac{1}{512}\left(-8\,x^3\left(10+17\,y+10\,y^2+2\,y^3\right)+4\,x^2\left(134+235\,y+147\,y^2+32\,y^3\right)+4\left(361+444\,y+314\,y^2+78\,y^3\right)-x\right.$$

$$\frac{13\,492+4170\,y+584\,y^2-88\,y^3+4\,x^3\left(22+83\,y+60\,y^2+12\,y^3\right)-4\,x^2\left(-146+191\,y+252\,y^2+60\,y^3\right)+x\left(-4170-1885\,y+764\,y^2+332\,y^3\right)}{8192}$$

$$\frac{842-9555\,y-4116\,y^2-588\,y^3+133\,x(2+y)(5+2\,y)^2-40\,x^2(2+y)(5+2\,y)^2+4\,x^3(2+y)(5+2\,y)^2}{8192}$$

1

In[439]:=
```
{DSumF1a1, DSumF1a2, DSumF1a3, DSumF1a4, DSumF1a5,
   DSumF1b1, DSumF1b2, DSumF1b3, DSumF1b4, DSumF1b5} = Parallelize[{
    Simplify[D[SumF1a1, {{x, y}}]],
    Simplify[D[SumF1a2, {{x, y}}]],
    Simplify[D[SumF1a3, {{x, y}}]],
    Simplify[D[SumF1a4, {{x, y}}]],
    Simplify[D[SumF1a5, {{x, y}}]],
    Simplify[D[SumF1b1, {{x, y}}]],
    Simplify[D[SumF1b2, {{x, y}}]],
    Simplify[D[SumF1b3, {{x, y}}]],
    Simplify[D[SumF1b4, {{x, y}}]],
    Simplify[D[SumF1b5, {{x, y}}]]
}];
```

In[440]:=
```
{Err1a1, Err1a2, Err1a3, Err1a4, Err1a5,
   Err1b1, Err1b2, Err1b3, Err1b4, Err1b5} = Parallelize[{
    Simplify[∫_(0-1/2)^(1-1/2) ∫_(0-1/2)^(1-1/2) (DSumF1a1.{1, 1})^2 dx dy],
    Simplify[∫_(1-1/2)^(2-1/2) ∫_(0-1/2)^(1-1/2) (DSumF1a2.{1, 1})^2 dx dy],
    Simplify[∫_(1-1/2)^(2-1/2) ∫_(-1-1/2)^(0-1/2) (DSumF1a3.{1, 1})^2 dx dy],
    Simplify[∫_(2-1/2)^(3-1/2) ∫_(-1-1/2)^(0-1/2) (DSumF1a4.{1, 1})^2 dx dy],
    Simplify[∫_(2-1/2)^(3-1/2) ∫_(-2-1/2)^(-1-1/2) (DSumF1a5.{1, 1})^2 dx dy],
        Simplify[∫_(0-1/2)^(1-1/2) ∫_(1-1/2)^(2-1/2) (DSumF1b1.{1, 1})^2 dx dy],
    Simplify[∫_(-1-1/2)^(0-1/2) ∫_(1-1/2)^(2-1/2) (DSumF1b2.{1, 1})^2 dx dy],
    Simplify[∫_(-1-1/2)^(0-1/2) ∫_(2-1/2)^(3-1/2) (DSumF1b3.{1, 1})^2 dx dy],
    Simplify[∫_(-2-1/2)^(-1-1/2) ∫_(2-1/2)^(3-1/2) (DSumF1b4.{1, 1})^2 dx dy],
    Simplify[∫_(-2-1/2)^(-1-1/2) ∫_(3-1/2)^(4-1/2) (DSumF1b5.{1, 1})^2 dx dy]
}];
```

In[447]:=
```
Err1 = FullSimplify[
  Err1a1 + Err1a2 + Err1a3 + Err1a4 + Err1a5 + Err1b1 + Err1b2 + Err1b3 + Err1b4 + Err1b5]
N[
 Err1]
```

Out[447]= $\frac{94\,211\,027}{660\,602\,880}$

Out[448]= 0.142614