```
h[x_] := Piecewise[{
    {1 + c[0,1] x + c[0,2] x^2 + c[0,3] x^3,  0 ≤ x ≤ 1},
    {c[1,1] (x - 1) + c[1,2] (x - 1)^2 + c[1,3] (x - 1)^3,  1 < x ≤ 2},
    {0, True}}];
f[x_] := h[Abs[x]];
AllVars = {c[0,1], c[0,2], c[0,3], c[1,1], c[1,2], c[1,3]};
```

$$h[x\_] := \begin{cases} 1 + c_{0,1}\, x + c_{0,2}\, x^2 + c_{0,3}\, x^3 & 0 \le x \le 1 \\ c_{1,1}\,(x-1) + c_{1,2}\,(x-1)^2 + c_{1,3}\,(x-1)^3 & 1 < x \le 2 \\ 0 & \text{True} \end{cases};$$

f[x_] := h[Abs[x]];

AllVars = $\{c_{0,1}, c_{0,2}, c_{0,3}, c_{1,1}, c_{1,2}, c_{1,3}\}$;

```
(*Interpolant constraints*)
I1 = f[1]
I2 = f[2]
```

$1 + c_{0,1} + c_{0,2} + c_{0,3}$

$c_{1,1} + c_{1,2} + c_{1,3}$

```
(*Partition of unity and linear term*)
```

$$\text{T0} = \text{CoefficientList}\left[\text{FullSimplify}\left[\sum_{k=-1}^{2} f[x-k],\ x > 0 \,\&\&\, x < 1\right],\ x\right]$$

$$\text{T1} = \text{CoefficientList}\left[\text{FullSimplify}\left[\sum_{k=-1}^{2} k\, f[x-k],\ x > 0 \,\&\&\, x < 1\right],\ x\right]$$

$\{2 + c_{0,1} + c_{0,2} + c_{0,3} + c_{1,1} + c_{1,2} + c_{1,3},\ -2\, c_{0,2} - 3\, c_{0,3} - 2\, c_{1,2} - 3\, c_{1,3},\ 2\, c_{0,2} + 3\, c_{0,3} + 2\, c_{1,2} + 3\, c_{1,3}\}$

$\{1 + c_{0,1} + c_{0,2} + c_{0,3} + 2\, c_{1,1} + 2\, c_{1,2} + 2\, c_{1,3},$
$-c_{0,1} - 2\, c_{0,2} - 3\, c_{0,3} - 3\, c_{1,1} - 4\, c_{1,2} - 6\, c_{1,3},\ c_{0,2} + 3\, c_{0,3} + c_{1,2} + 6\, c_{1,3},\ -c_{0,3} - 3\, c_{1,3}\}$

```
GenSols = Solve[{
    I1 == 0,
    I2 == 0,
    T0[[1]] == 1,
    T0[[2]] == 0,
    T0[[3]] == 0,
    T1[[1]] == 0,
    T1[[2]] == 1,
    T1[[3]] == 0,
    T1[[4]] == 0
    },
    AllVars
]
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{c_{0,3} \to -1 - c_{0,1} - c_{0,2},\ c_{1,1} \to -\frac{4}{3} - \frac{4\, c_{0,1}}{3} - \frac{c_{0,2}}{3},\ c_{1,2} \to 1 + c_{0,1},\ c_{1,3} \to \frac{1}{3} + \frac{c_{0,1}}{3} + \frac{c_{0,2}}{3}\right\}\right\}$$

```
RegionXY[k_] := {Quotient[k, 2], 1 + Quotient[-k, 2]};
Regions = Table[RegionXY[k], {k, -2, 5}]
```

{{-1, 2}, {-1, 1}, {0, 1}, {0, 0}, {1, 0}, {1, -1}, {2, -1}, {2, -2}}

```
GenSol = GenSols[[1]];
f[x_, y_] := f[x] f[y];
```

$$W[k\_] := \begin{cases} 0 & k < 0 \\ \varphi^2/2 & k == 0 \\ 1-(1-\varphi)^2/2 & k == 1 \\ 1 & \text{True} \end{cases};$$

$$\text{SumF} = \sum_{i=-3}^{5}\sum_{j=-3}^{5} W[i-j]\ f[x-i,\ y-j]\ /.\ \text{GenSol};$$

```
SimplifySquare[f_, x0_, y0_] := Simplify[f, x > x0 && x < x0 + 1 && y > y0 && y < y0 + 1];
DSimplifySquare[f_, {x0_, y0_}] := Simplify[D[SimplifySquare[f, x0, y0], {{x, y}}]];
DSumF = ParallelMap[DSimplifySquare[SumF, #] &, Regions];
```

```
AnisoInt[df_, {x0_, y0_}] :=
  Simplify[Integrate[Expand[(df.{1, 1})^2], {x, x0, x0 + 1}, {y, y0, y0 + 1}]];
AnisoInts = Parallelize[MapThread[AnisoInt, {DSumF, Regions}]];
Err = Simplify[Total[AnisoInts] /. φ → 1/2]
```

$$\frac{1}{907\,200}\left(2\,073\,435 + 630\,023\, c_{0,1}^4 + 2\,452\,640\, c_{0,2} + 998\,496\, c_{0,2}^2 + 153\,882\, c_{0,2}^3 + 8351\, c_{0,2}^4 + 4\, c_{0,1}^3 \left(949\,251 + 211\,676\, c_{0,2}\right) + 6\, c_{0,1}^2 \left(1\,366\,684 + 649\,053\, c_{0,2} + 71\,515\, c_{0,2}^2\right) + 4\, c_{0,1} \left(1\,745\,927 + 1\,424\,046\, c_{0,2} + 334\,644\, c_{0,2}^2 + 24\,320\, c_{0,2}^3\right)\right)$$

```
FreeVars = Variables[Err];
DErr = Simplify[D[Err, {FreeVars}]];
H = D[DErr, {FreeVars}];
Sols = RootReduce[Solve[DErr == 0, FreeVars, Reals]];
RootReduce[Sols[[1]]]
TableForm[
 {Range[Length[Sols]], Err /. N[Sols], PositiveDefiniteMatrixQ[H /. N[#]] & /@ Sols}ᵀ]
```

$$\{c_{0,1} \to \text{Root}[11\,136\,212\,089\,427\,417\,799\,940 + 40\,565\,503\,933\,508\,466\,119\,319\,\#1 + 56\,175\,410\,802\,925\,780\,786\,716\,\#1^2 + 41\,251\,698\,626\,094\,358\,074\,618\,\#1^3 + 18\,239\,143\,335\,129\,521\,212\,776\,\#1^4 + 5\,173\,229\,177\,780\,774\,994\,150\,\#1^5 + 956\,712\,737\,540\,209\,331\,904\,\#1^6 + 112\,632\,888\,717\,491\,520\,636\,\#1^7 + 7\,734\,928\,656\,195\,327\,168\,\#1^8 + 239\,273\,075\,197\,171\,456\,\#1^9\ \&,\ 1],$$
$$c_{0,2} \to \text{Root}[3\,374\,023\,697\,485\,280\,698\,110\,075 + 8\,537\,532\,930\,936\,839\,854\,857\,150\,\#1 - 1\,103\,781\,936\,366\,867\,775\,279\,410\,\#1^2 + 951\,787\,233\,357\,756\,794\,329\,620\,\#1^3 - 383\,780\,785\,728\,198\,277\,951\,170\,\#1^4 + 81\,595\,240\,042\,195\,026\,782\,580\,\#1^5 - 9\,947\,775\,659\,359\,580\,484\,972\,\#1^6 + 712\,083\,875\,694\,813\,372\,936\,\#1^7 - 28\,030\,349\,234\,173\,638\,144\,\#1^8 + 478\,546\,150\,394\,342\,912\,\#1^9\ \&,\ 1]\}$$

1	0.0494529	True

```
NSol = N[Sols[[1]]];
FullSol = Join[GenSol /. NSol, NSol]
fo[x_] := f[x] /. FullSol;
Plot[fo[x], {x, -3, 3}, PlotStyle → Black, Background → White]
```

$\{c_{0,3} \to -0.00469238,\ c_{1,1} \to -0.377073,\ c_{1,2} \to 0.375509,\ c_{1,3} \to 0.00156413,\ c_{0,1} \to -0.624491,\ c_{0,2} \to -0.370817\}$