Martingale Representation

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The martingale representation problem in its simplest form is the following. Given a filtration generated by a martingale M and given another martingale N adapted to the filtration, can we express N as a stochastic integral with M as the integrator? The martingale N is generally closed, i.e. it can be expressed as the conditional expectation of a terminal variable N_T. In this case, the integrand H_t of the stochastic integral representation is heuristically the sensitivity of N_T to the shock dM_t.The Brownian filtration is the most important example where a Martingale Representation Theorem holds.

The theory of martingale representation is concerned with the following problem.

Consider a filtered probability space $(\Omega, \mathcal{F}, P)$ with index space $\mathbb{T} = [0, T]$ where T is finite. Such a space supports a set of martingales $\mathcal{M}$ against which we can compute stochastic integrals for predictable integrands.

We are given an $\mathcal{F}_T$-measurable random variable X_T. It induces a martingale $(E_t[X_T])_{t\in\mathbb{T}}$. This process represents, within the model, the anticipation of X_T at any point t. The changes in $E_t[X_T]$ as a function of t reflect the real time acquisition of information on X_T. New information comes as surprises as modeled in martingale differences (see this post). Heuristically, martingale representation asks the following question: can we represent the surprises in $(E_t[X_T])_{t\in\mathbb{T}}$ for any X_T as a linear function of the (contemporaneous) surprises embedded in our set $\mathcal{M}$ of martingales. More precisely, can we represent the martingale $(E_t[X_T])_{t\in\mathbb{T}}$ as a sum of stochastic integrals against some martingales in $\mathcal{M}$.

A striking incarnation of this issue is found when the filtered probability space is generated by a Brownian motion[1].

Theorem (Martingale Representation for the Brownian Filtration):*Let $\mathcal{F}$ be the smallest right continuous and complete filtration generated by a univariate Brownian motion $(B_t)_{t\in\mathbb{T}}$. Let X_T be an $\mathcal{F}_T$-measurable random variable with finite second moment $E_0[X_T^2] < \infty$. Then there is a predictable process*

[1] The following results can be found in Bass[2011], p. 80.

$(H_t)_{t\in\mathbb{T}}$ with $\int_0^T H_s^2 ds < \infty$ such that:

$$X_T = E[X_T] + \int_0^T H_s dB_s.$$

■

In the same context as above, we have a simple yet important corollary:

Corollary: *For any square integrable[2] right continuous martingale $(M_t)_{t\in\mathbb{T}}$ with $M_0 = 0$, there exists a predictable process $(H_t)_{t\in\mathbb{T}}$ with $\int_0^T H_s^2 ds < \infty$ such that:*

$$M_t = \int_0^t H_s dB_s.$$

■

In other words, all square integrable right continuous martingales with initial value zero are Brownian stochastic integrals. Actually, in our context, all square integrable martingales have a version which is still a martingale and is right continuous with left limits. They can therefore be represented as Brownian integrals. Since Brownian integrals have continuous trajectories, all square integrable martingales in this setup have a continuous version. Finally, one can extend the above result to show that all local martingales can be represented as a Brownian stochastic integral.

It is quite easy to generate setups where the filtration is the minimal filtration generated by a given martingale $(M_t)_{t\in\mathbb{T}}$, and yet, the filtration supports other martingales which cannot be written as sotchastic integrals of $(M_t)_{t\in\mathbb{T}}$. In this post, an example is given where $\mathbb{T}$ is discrete and $(M_t)_{t\in\mathbb{T}}$ has standardized gaussian increments. If, on the other hand, $(M_t)_{t\in\mathbb{T}}$ has binomial increments, the martingale representation holds with the set $\mathcal{M}$ consisting of $(M_t)_{t\in\mathbb{T}}$. A solution to recover a martingale representation result when it does not hold for $\mathcal{M} = \{(M_t)_{t\in\mathbb{T}}\}$ is to add other martingales in $\mathcal{M}$, based on higher order moments of $(M_t)_{t\in\mathbb{T}}$ for instance. Indeed, the problems generally come from the difficulty of generating non linear functions of $(M_t)_{t\in\mathbb{T}}$ through the stochastic integral which, in the end, is just a linear reweighting of the increments of $(M_t)_{t\in\mathbb{T}}$.

Given the above remarks, the Brownian martingale representation theorem looks like a nice accident. I now sketch the proof. An $\mathcal{F}_T$-measurable random variable is, roughly speaking, a function of the increments of the Brownian motion. A simple example would be a function $f(B_{t_1} - B_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}})$ where the time intervals $[t_i, t_{i-1}]$ do not overlap. Such functions can however be recovered

[2] $E_0[M_T^2] < \infty$.

through Fourier transform from products of complex exponentials[3]:

$$\exp(iu_1(B_{t_1} - B_{t_0})) \cdots \exp(iu_n(B_{t_n} - B_{t_{n-1}})).$$

It is conceivable that if a martingale representation were to hold for such a function, the representation could be extended by limiting arguments to all $\mathcal{F}_T$-measurable random variables. However, Ito calculus implies that:

$$\exp(iu_k(B_t - B_{t_{k-1}}) + \frac{1}{2}u_k^2(t - t_{k-1})) = 1+$$

$$\int_{t_{k-1}}^{t_k} iu_k \exp(iu_k(B_s - B_{t_{k-1}}) + \frac{1}{2}u_k^2(s - t_{k-1}))dB_s,$$

i.e.

$$d\left(\exp(iu_k(B_t - B_{t_{k-1}}) + \frac{1}{2}u_k^2(t - t_{k-1}))\right) = \exp(iu_k(B_t - B_{t_{k-1}}) + \frac{1}{2}u_k^2(t - t_{k-1}))dB_t.$$

This complex exponential is a geometric martingale with initial value 1 at $t = t_{k-1}$.

From this, we get (taking $t = t_k$ and rearranging terms):

$$Z_{k-1} = \exp(iu_k(B_{t_k} - B_{t_{k-1}})) = \exp(-\frac{1}{2}u_k^2(t_k - t_{k-1}))+$$

$$\int_{t_{k-1}}^{t_k} iu_k \exp(iu_k(B_s - B_{t_{k-1}}) + \frac{1}{2}u_k^2(s - t_k))dB_s$$

$$= F_{k-1} + \int_{t_{k-1}}^{t_k} H_{k-1}(s)dB_s,$$

where Z_{k-1} is the random variable of interest, F_{k-1} is a function of non random parameters only and H_{k-1} is the integrand within the stochastic integral. We thus have the right representation for a single exponential of a Brownian increment.

When multiplying two such terms attached to non overlapping intervals, say $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$, the product rule entails no covariation terms because the stochastic integrals refer to non overlapping time intervals:

$$[\int_{t_{k-1}}^{t_k} H_{k-1}(s)dB_s, \int_{t_k}^{t_{k+1}} H_k(s)dB_s] = 0.$$

We thus have the following representation for the product:

$$Z_{k-1}Z_k = F_{k-1}F_k + \int_{t_{k-1}}^{t_k} F_k H_{k-1}(s)dB_s + \int_{t_k}^{t_{k+1}} Z_{k-1}H_k(s)dB_s,$$

[3]In our context, the Fourier transform amounts to mixing functions indexed by $(u_1, \ldots, u_n)$ using a weighting scheme $\hat{f}(u_1, \ldots, u_n)$.

which still has the right structure. It is now clear that any product involving a finite number of such exponentials involving non overlapping intervals has a martingale representation. The rest of the proof is a matter of spelling out the limiting arguments that allow to extend[4] the representation to any function $f(B_{t_1} - B_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}})$ and then to any $\mathcal{F}_T$-measurable random variable (through a density argument).

In the Brownian context thus, Brownian integrals allow to generate all the local martingales supported by the filtration[5]. Amongst them are all the martingales generated by moments B_t^α, for instance $X_t = B_t^2 - t = 2\int_0^t B_s dB_s$.

A striking illustration of this involves Hermite polynomial functions. If $H_n(x,y) = \left(\frac{y}{2}\right)^{\frac{n}{2}} h_n(\frac{x}{\sqrt{2y}})$ $(n \geq 0)$ where $h_n(\cdot)$ are Hermite polynomials[6], then $H_n(B_t, t)$ are martingales and we have the following integral representation:

$$H_n(B_t, t) = \int_0^t nH_{n-1}(B_u, u)dB_u = n! \int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_1} dB_s dB_{t_1} \cdots dB_{t_{n-1}}.$$

This result can be found for instance in Chung[1990], chapter 6.

Reference: Chung K.L and R.J. Williams, 1990 : *An introduction to Stochastic Integration*, Birkhauser.

Bass R.F., 2011, *Stochastic Processes*, Cambridge University Press

[4]Through the Fourier transform, which amounts to integrating the integral representations attached to different parameters $(u_1, \ldots, u_n)$, using a weighting scheme $\hat{f}(u_1, \ldots, u_n)$.

[5]It is important that the filtration be the minimal filtration generated by the Brownian motion, i.e. the smallest right continuous and complete filtration generated by the Brownian motion.

[6]$H_0(x,y) = 1, H_1(x,y) = x, H_2(x,y) = x^2 - y, \ldots$