Introducing Diffusions

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The purpose of this note is to introduce diffusions which are made up of a drift and a martingale component. I start from the elementary discrete time setup where the drift of the process is most easily understood. I then explain how the specialized decomposition of a diffusion into a drift and a Brownian integral can arise as the limit of the decompositions obtained on the discretized process.

Doob's decomposition

We introduced the martingale concept in the post on martingales. In particular, we explained (see the post on martingales for notation) that in discrete time ($\mathbb{T} = \mathbb{N}$), a martingale $(M_t)_{t\in\mathbb{T}}$ has martingale differences $\Delta M_t = M_t - M_{t-1}$ which are conditionally centered:

$$E_t[\Delta M_{t+1}] = 0.$$

We now wish to relax this constraint and consider more general processes.

Starting from an adapted process $(X_t)_{t\in\mathbb{T}}$, we can consider its differences $\Delta X_t = X_t - X_{t-1}$ and define their one step ahead conditional expectation:

$$\Delta A_{t+1} = E_t[\Delta X_{t+1}].$$

The random variables ΔA_{t+1} are by construction $\mathcal{F}_t$ measurable. In the language introduced to describe stochastic integrals, this process is predictable. Removing ΔA_{t+1} from ΔX_{t+1} is a centering operation. Indeed, setting $\Delta M_{t+1} = \Delta X_{t+1} - \Delta A_{t+1}$, we get:

$$E_t[\Delta M_{t+1}] = 0.$$

The variables ΔM_{t+1} are martingale differences. Cumulating differences, we can recover the level X_t through:

$$X_t = X_0 + \sum_{k=1}^{t} \Delta A_k + \sum_{k=1}^{t} \Delta M_k.$$

Setting $M_0 = 0$ and $A_0 = 0$, we can now define two 'level' processes $(A_t)_{t\in\mathbb{T}}$ and $(M_t)_{t\in\mathbb{T}}$ through:

$$A_t = A_0 + \sum_{k=1}^{t} \Delta A_k,\ t \geq 1,$$

$$M_t = M_0 + \sum_{k=1}^{t} \Delta M_k,\ t \geq 1.$$

The first process $(A_t)_{t\in\mathbb{T}}$ is predictable in the sense that each A_t is $\mathcal{F}_{t-1}$ measurable (its value at date t is known at date $t-1$), and the second process $(M_t)_{t\in\mathbb{T}}$ is a martingale. Both processes are of course adapted to the filtration. We can finally write:

$$X_t = X_0 + A_t + M_t.$$

This decomposition is called Doob's decomposition.

It should be stressed that ΔA_t are one step ahead predictions. Two step ahead predictions for instance involve predicting ΔA_t one step ahead:

$$E_t[\Delta A_{t+1} + \Delta A_{t+2}] = \Delta A_{t+1} + E_t[\Delta A_{t+2}].$$

By constraining the sign of $(\Delta A_t)_{t\in\mathbb{T},\, t\geq 1}$, we obtain sub and supermartingales. A supermartingale is a process which is expected to decrease or remain stable. It is obtained by forcing $(\Delta A_t)_{t\in\mathbb{T},\, t\geq 1}$ to be negative. A submartingale is a process which is expected to increase or remain stable, that is $(\Delta A_t)_{t\in\mathbb{T},\, t\geq 1}$ is forced to be positive (a martingale is thus both a super and a submartingale). Accordingly, the process $(A_t)_{t\in\mathbb{T}}$ is monotonic in both cases.

Quasimartingales

We now look at the case of a continuous time process $(X_t)_{t\in[0,T]}$ where $T < \infty$. To carry out the above decomposition, we can introduce a discretization scheme and apply the previous calculations to the process obtained thereby. For each discretization grid π_n of $[0, 1]$ indexed by n ($t_0 = 0, \ldots, t_n = T$), we can thus split the discretized version $(X_{t_i})_{t_i\in\pi_n}$ of the original process into a discrete time martingale and the cumulated expected changes along the discretization intervals:

$$X^n_{t_i} = X_0 + A^n_{t_i} + M^n_{t_i}.$$

As the grid is refined (n tending to infinity), one can hope to recover:

$$X_t = X_0 + A_t + M_t,$$

where $(M_t)_{t\in\mathbb{R}_+}$ is a continuous time martingale and $(A_t)_{t\in[0,T]}$ is the limit of the discretized processes $(A_{t_i})_{t_i\in\pi_n}$, i.e. the cumulated infinitesimal expected changes of $(X_t)_{t\in[0,T]}$.

The first results generalizing the discrete time decomposition to the continuous time setup and ensuring that the above discretization scheme converges concerned submartingales and supermartingales, for which the process $(A_t)_{t\in[0,T]}$ is monotonic (these assumptions lead to the Doob-Meyer decomposition, the continuous time version of Doob's decomposition). A natural generalization was sought among processes for which $(A_t)_{t\in[0,T]}$ would be of bounded variation, since this class of functions is the simplest extension of monotonic functions (any bounded variation function is the difference of two bounded monotonic functions). This led to the concept of quasimartingales, where the main ingredient consists in bounding the following sums :

$$E[V(\pi)] = E\left[\sum_{i=1}^{n} E_{t_{i-1}}[\left|X_{t_i} - X_{t_{i-1}}\right| |\mathcal{F}_{t_{i-1}}]\right] \leq K < \infty,$$

uniformly with respect to partitions $\pi = (t_0 = 0, \ldots, t_n = T)$ of $[0, T]$ of any size n. The decomposition of a quasimartingale involves a process $(A_t)_{t\in[0,T]}$ which has finite expected variation and *a fortiori* almost everyhere bounded variation.

Stochastic integration can be easily extended from martingales to quasimartingales since bounded variation functions can be used as integrators within a well understood integration theory (cf. Lebesgue-Stieltjes integration). What this means for mathematical finance is that most continuous time calculus can be carried out assuming prices follow quasimartingales[1].

In the case of processes with continuous paths, *necessary and sufficient conditions* have been identified (see Fisk, Quasi-Martingales, *Transactions of the American Mathematical Society*, 1965) for $(X_t)_{t\in[0,T]}$ to be a quasimartingale, in which case the discretization process described above delivers a martingale with continuous paths $(M_t)_{t\in[0,T]}$ and a bounded variation process $(A_t)_{t\in[0,T]}$ with continuous paths as well. This decomposition is unique.

Ito diffusions

We now wish to specialize the above setup so as to get tractable and flexible specifications for a process $(X_t)_{t\in[0,T]}$. We will assume the filtration is generated by a one dimensional Brownian motion $(B_t)_{t\in[0,T]}$. We assume the process $(A_t)_{t\in[0,T]}$ is of the form:

$$\int_0^t r(X_u)du.$$

We thereby select a specific class of bounded variation processes, those that can be written as the integral of a function. In addition, we constrain that function to be a function of the state variable X_u. Similarly the martingale process

[1] Actually, a continuous time process for which stochastic integration can be defined is called a semimartingale and quasimartingales are a strict subset of semimartingales, but the gap is very small from a modeling perspective.

$(M_t)_{t\in[0,T]}$ is assumed to be a Brownian stochastic integral:

$$\int_0^t \sigma(X_u)dB_u,$$

where again $\sigma(\cdot)$ is a function of the state variable X_u only. The functions $r(\cdot)$ and $\sigma(\cdot)$ are real measurable functions. We thus get:

$$X_t = X_0 + \int_0^t r(X_u)du + \int_0^t \sigma(X_u)dB_u.$$

The function $r(\cdot)$ is called the drift of the diffusion and the function $\sigma(\cdot)$ is the volatility of the diffusion. The interpretation of these two terms is now obvious. The drift measures the infinitesimal expected change of the process, while volatility measures the infinitesimal surprises.

The technical conditions usually applied to the drift and volatility coefficients are (almost everywhere):

$$\int_0^T |r(X_u)|\, du < \infty,$$
$$\int_0^T \sigma(X_u)^2 du < \infty.$$

The first condition ensures that $(A_t)_{t\in[0,T]}$ is almost everywhere of bounded variation[2], while the second is needed to define the Brownian integral.

We finally note that it is customary to summarize the integral equations through the differential notation:

$$dX_t = r(X_t)dt + \sigma(X_t)dB_t,$$

with an initial condition X_0. This should however merely be seen as a shorthand for the integral equation.

[2]To be precise, one would have to impose stronger condition in the quasimartingale context. In spirit, the variation has to be integrable:

$$E[\int_0^T |r(X_u)|\, du] < \infty.$$

The condition spelled out in the text corresponds to the requirement that the process $(X_t)_{t\in[0,T]}$ be a semimartingale, rather than a quasimartingale.