Stochastic Integrals as Martingale Transforms

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Building on the portfolio return post, this article describes the stochastic inregral $\int_0^{\cdot} H_u dM_u$ of an integrand H against a martingale M as a martingale transform. The integral re-weights the increments of M using a system of 'predetermined' weights in such a way that the resulting process remains a martingale. The preservation of the martingale property is a key requirement of the standard stochastic integration theory. The sensitivity of the resulting martingale $\int_0^{\cdot} H_u dM_u$ with respect to the infinitesimal increments of M is an instanciation of the concept of Malliavin derivative. The post ends by considering the martingale representation property which plays a key role in financial theory.

Martingale Transforms

The portfolio return post took the perspective that a portfolio is a weighting scheme applied to the returns of the available instruments. The weights are chosen as a function of the information set of the portfolio manager. This leads to a staggered information structure: weights depend on past information and cannot anticipate future surprises in returns.

Martingale transforms can be understood from this perspective. Assume that we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, P)$ together with a martingale $(M_t)_{t\in\mathbb{T}}$. Let's assume a discrete time setting $\mathbb{T} = \mathbb{N}$. The martingale differences $\Delta M_t = M_t - M_{t-1}$ are our surprises. Let's weigh them using a $\mathcal{F}_{t-1}$ random variable H_{t-1}. We get the reweighted martingale difference:

$$H_{t-1}\Delta M_t = H_{t-1}(M_t - M_{t-1}),$$

which is still a martingale difference in the sense that:

$$E_{t-1}[H_{t-1}\Delta M_t] = H_{t-1}E_{t-1}[\Delta M_t] = 0.$$

Cumulating these differences leads to a new martingale $(\tilde{M}_t)_{t\in\mathbb{T}}$:

$$\tilde{M}_t = \sum_{k=1}^{t} H_{k-1}\Delta M_k$$

which is centered $E_0[\tilde{M}_t] = 0$. This is the discrete time stochastic integral.

In continuous time ($\mathbb{T} = \mathbb{R}_+$), the staggered information structure is more subtle. The key to the construction of the stochastic integral is that we want it to preserve the martingale property. The construction starts with the definition of simple weights, i.e. piecewise constant weighting schemes, where discontinuities take place at stopping times $(T_i)_{i\in\mathbb{N}}$, $T_0 = 0$:

$$H_t = \sum_{i=0}^{N-1} H^i 1_{(T_i,T_{i+1}]}(t),$$

where H^i is $\mathcal{F}_{T_i}$ measurable. The key point is that the weight chosen at date T_i is in place right after T_i, and up until T_{i+1}. In particular, at T_i, H^{i-1} prevails. When applied to the martingale differences, the simple weighting scheme delivers for each ω:

$$\tilde{M}_t = \int_0^t H_u dM_u := \sum_{i=0}^{J(t)-1} H^i (M_{T_{i+1}} - M_{T_i}) + H^{J(t)}(M_t - M_{T_{J(t)}}),$$

where $J(t)(\omega)$ is set such that $T_{J(t)}(\omega) \leq t < T_{J(t)+1}(\omega)$. The staggered information structure implies that each increment is conditionally centered, and $(\tilde{M}_t)$ is a centered ($E_0[\tilde{M}_t] = 0$) martingale as in the discrete time case.

The theory of stochastic integrals consists in extending this construction to more general weighting schemes while preserving the staggered information structure. We will be evasive about the technical conditions needed for stochastic integration to work. We will however keep the information structure in mind. Integrands[1] cannot anticipate on information. Adapted continuous processes (i.e. each H_t is $\mathcal{F}_t$ measurable and trajectories are continuous) are suitable integrands. When the underlying martingale is allowed to jump (while being continuous on the right with limits on the left - cadlag), the integrands can also be allowed to jump provided they are caglad (continuous on the left with limits on the right - see the post on portfolio returns). These requirements are needed to ensure that the stochastic integral of a valid integrand against a martingale remains a martingale.

Sensitivity of a martingale to the underlying shocks

A martingale $(M_t)_{t\in\mathbb{T}}$ is the additive accumulation of the associated shocks, whether these are true differences (ΔM_t) as in the discrete time case or infinitesimal idealizations (dM_t). I use the discrete time case in what follows. Any

[1]The weighting function is called the integrand while the underlying martingale is called the integrator.

particular shock ΔM_u has a fully persistent impact on the subsequent level $(M_t)_{t \geq u}$ of the martingale. We could write this as:

$$\frac{\partial M_t}{\partial \Delta M_u} = 1,\, t \geq u.$$

We can see the stochastic integral as a method to create a new martingale with a different sensitivity to the underlying shocks. The new sensitivity is *a priori* given by H_{u-1}. This is right when the integrand is a deterministic function, in which case we thus have:

$$\frac{\partial \tilde{M}_t}{\partial \Delta M_u} = H_{u-1},\, t \geq u.$$

When integrands are stochastic however, H_{u-1} cannot measure the overall impact of the shock on the future level of the new martingale. Indeed, the shock might alter the integrand H_t at a future date. We would like to write:

$$\frac{\partial \tilde{M}_t}{\partial \Delta M_u} = H_{u-1} + \sum_{u+1}^{t} \frac{\partial H_{v-1}}{\partial \Delta M_u} \Delta M_v,\, t \geq u.$$

The meaning of $\frac{\partial H_{v-1}}{\partial \Delta M_u}$ is however quite unclear unless H_{v-1} can itself be described as a stochastic integral with deterministic integrand! The martingale $(\tilde{M}_t)_{t \in \mathbb{T}}$ would then be a double stochastic integral.

What is sketched above is precisely the program of defining Malliavin derivatives, initially developed for Brownian filtrations and integrals. The program can be carried out because in the Brownian filtration, all square integrable random variables can be approximated as a multiple stochastic integral[2]. As such, their derivatives with respect to the underlying Brownian shocks can be computed. We will not need the full force of this theory, but we will keep in mind the idea. We will often use deterministic integrands for which Malliavin derivatives are trivial. Stochastic Brownian integrals with deterministic integrands are Gaussian variables. They are extremely handy because they allow to express a wide range of phenomena while permitting analytical computation.

Martingales and martingale representation

Readers with a mathematical inclination could raise the following problem. Stochastic integrals allow to produce new martingales from an initial one. Can we produce all martingales in this way?

Let's then start from a filtered probability space as above, with the filtration being the filtration generated by a given martingale $(M_t)_{t \in \mathbb{T}}$. We take $\mathbb{T}$ to be an interval within $\mathbb{N}$ or $\mathbb{R}$. We therefore start with a minimal setup: a single martingale described in the most parsimonious filtered probability space which

[2]See for instance D. Nualart, *The Malliavin Calculus and Related Topics*, Springer.

can support it. We can now define new martingales using stochastic integration based on suitably adapted integrands. We thereby get a wealth of new (centered) martingales as stochastic integrals. On the other hand, any $\mathcal{F}_T$ integrable variable X_T with mean $E_0[X_T] = 0$ defines a centered martingale $(E_t[X_T])$. The latter set of centered martingales is *a priori* larger than the previous set of centered martingales defined through stochastic integrals against the initial martingale. Indeed a stochastic integral is a closed centered martingale:

$$\int_0^t H_u dM_u = E_t[\int_0^T H_u dM_u].$$

The question is then whether the inclusion is strict[3]. It turns out again that in the Brownian filtration, the two sets of martingales coincide. Martingales can be represented as stochastic integrals: the martingale representation theorem holds. A similar result holds when the underlying martingale is the compensated Poisson process. There are however plenty of cases where the inclusion is strict. I give below two trivial examples that shed some light on what is going on.

I assume an extreme discrete time case, with only two dates $T = 0, 1$. The sigma algebra $\mathcal{F}_0$ is trivial and $\mathcal{F}_1$ is the sigma algebra generated by a variable ϵ. Stochastic integrals are just proportional functions of ϵ since $\mathcal{F}_0$ random variables (integrands) are constant. Assume ϵ is a standardized Gaussian variable. Consider $X_1 = \epsilon^2 - 1$. This defines a centered martingale which is not linear in ϵ. The martingale representation theorem does not hold.

If instead, ϵ equals a binomial variable taking values $+1$ with probability $1/2$ and -1 with probability $1/2$, the martingale representation theorem holds. Indeed, any centered $\mathcal{F}_1$ measurable variable is defined by two values $X_1(1)$ and $X_1(-1)$ such that:

$$\frac{1}{2}X_1(-1) + \frac{1}{2}X_1(1) = 0.$$

Thus:

$$X_1(-1) = -X_1(1),$$

and:

$$X_1 = X_1(1)\epsilon,$$

which is a stochastic integral since $X_1(1)$ is a constant.

This gives a good sense of why the compensated Poisson model has a martingale representation theorem. In the Brownian case, it is clear that continuous time plays an important role. One can perhaps close this post by noting that in this case:

$$M_1^2 - 1 = \int_0^1 M_t dM_t.$$

[3]This question has an important interpretation in finance, in a broader but related context. In a market model, if all terminal pay-offs can be derived as a porfolio value (i.e. a stochastic integral) where the portfolio trades underlying financial instruments, the market is said to be complete. Completeness therefore means, loosely speaking, that a representation theorem holds with respect to the tradable instruments.

Moments shifted so as to be centered and more generally centered non linear functions of M_1 can be obtained as stochastic integrals. Continuous time does achieve a few miracles!