Static Portfolio Choice

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In this section of the course, I review the static portfolio choice problem. The investor chooses a portfolio structure which is then left alone. The investment criterion is the expected utility of wealth at a terminal date. I briefly review specifications for the utility function together with risk aversion concepts. I look at the case of constant absolute risk aversion and normal returns, with or without labour income. I then introduce mean variance preferences, linking them to expected utility. Mean variance with and without a risk free asset is studied. The link between mean variance preferences and the expected returns/beta relationship is explained (the key ingredient of the CAPM). I then touch on the implementation problem.

Timing

- Two periods:
  - portfolio decisions in t=0
  - outcome observed in t=1
- Outcomes in date 1 are uncertain as of date 0; they are described by random variables which we will identify in the notation using tildas
  - x: particular outcome; $\tilde{x}$: random variable

Instruments

- Instrument i with price p_i in period 0 gives right to pay-off $\tilde{x}_i$ in period 1
- A cash instrument is an instrument with known date 1 pay-off as of date 0
- For risky assets, $\tilde{x}$ is uncertain as of date 0
- I'll assume there are N risky assets ($i = 1, \cdots, N$) and potentially cash (the riskless asset), which will then have index 0
- The set of assets will be denoted by $\mathcal{I}$, with either $\mathcal{I} = (1, \cdots, N)$ (no riskless asset) or $\mathcal{I} = (0, \cdots, N)$ (with a riskless asset)

Returns

- The return of an instrument with price p and pay-off $\tilde{x}$ is:

$$\tilde{R} = \frac{\tilde{x}}{p}$$

- The **rate** of return is $\tilde{r} = \tilde{R} - 1$
- The rate of return of cash is usually denoted r^f; it is known as of date 0

Investment and returns

- From investment to pay-off
- From $t = 0$ to $t = 1$:
  - $\phi \longrightarrow \tilde{R}\phi$
  - $\phi \longrightarrow (1 + \tilde{r})\phi$

Portfolios

- Wealth in period 0 is w_0
- The portfolio is invested in period 0; quantities $(\theta_i)_{i \in \mathcal{I}}$ are purchased
- They need to satisfy:

$$\sum_{i \in \mathcal{I}} \theta_i p_i = w_0$$

- One can choose as control variables:
  - quantities $(\theta_i)_{i \in \mathcal{I}}$
  - dollar amounts invested on instruments $(\phi_i)_{i \in \mathcal{I}}$ with $\phi_i = \theta_i p_i$
  - wealth shares $(\pi_i)_{i \in \mathcal{I}}$, with $\pi_i = \phi_i / w_0$

Budget constraints

- Quantities:

$$\sum_{i \in \mathcal{I}} \theta_i p_i = w_0$$

- Dollar amounts:

$$\sum_{i \in \mathcal{I}} \phi_i = w_0$$

- Wealth shares:

$$\sum_{i \in \mathcal{I}} \pi_i = 1$$

Borrowing

- Borrowing is best understood as a negative position in cash:
  - from $t = 0$ to $t = 1$
  - $\phi = -d \longrightarrow -d(1 + r^f)$

Accounting for future wealth

- for a given initial wealth w_0, a portfolio allocation leads to a random final wealth $\tilde{w}$ with:
  - quantities: $\tilde{w} = \sum_{i \in \mathcal{I}} \theta_i \tilde{x}_i$
  - invested amounts: $\tilde{w} = \sum_{i \in \mathcal{I}} \phi_i \tilde{R}_i$
  - wealth shares: $\tilde{w} = w_0 \sum_{i \in \mathcal{I}} \pi_i \tilde{R}_i$
  - It is sometimes useful to introduce at date 1 an exogenous income (amount to be received) or liability (amount to be paid) $\tilde{y}$
  - $\tilde{w} = \tilde{y} + \sum_{i \in \mathcal{I}} \theta_i \tilde{x}_i$
  - $\tilde{w} = \tilde{y} + \sum_{i \in \mathcal{I}} \phi_i \tilde{R}_i$
  - $\tilde{w} = \tilde{y} + w_0 \sum_{i \in \mathcal{I}} \pi_i \tilde{R}_i$

Some return arithmetic

- Without liability, we have:
  - portfolio return:
  $$\tilde{R}_p = \frac{\tilde{w}}{w_0} = \sum_{i \in \mathcal{I}} \pi_i \tilde{R}_i$$
  - portfolio rate of return:
  $$\tilde{r_p} = \frac{\tilde{w}}{w_0} = \sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i$$
  (since $\sum_{i \in \mathcal{I}} \pi_i = 1$)

The space of excess returns

- In the presence of a riskless asset, it is convenient to introduce excess returns versus the riskless rate:
$$\tilde{r_p} = \sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i$$
$$= r^f + \sum_{i=1}^{N} \pi_i (\tilde{r}_i - r^f).$$

- The choice variables are initially $(\pi_i)_{i\in\mathcal{I}}$, under the constraint $\sum_{i\in\mathcal{I}} \pi_i = 1$.
- In the excess return space, the choice variables are $(\pi_i)_{i=1}^N$ to which no budget constraint applies since it is enforced by $\pi_0 = 1 - \sum_{i=1}^N \pi_i$.

The portfolio problem

- Future wealth is a random variable, with a specific distribution
- The portfolio problem:
  - choose quantities (amounts, wealth shares) so as to obtain the best wealth distribution possible
- How do we compare random outcomes?
  - expected utility (Von Neumann Morgenstern - VNM) of outcome: $E[u(\tilde{w})]$
  - the utility function embodies attitudes towards risk of the decision maker

Some remarks

- **The optimization problem cannot have a solution if there are arbitrage opportunities**
- Reminder: an arbitrage is a way to generate a strictly positive pay-off without committing any funds
- The existence of a solution to a portfolio optimization problem thus guarantees the existence of a strictly positive stochastic discount factor (see below). We will see this principle in action

Arbitrage, the law of one price and SDFs

- A stochastic discount factor is a random variable $\tilde{m}$ such that for any pay-off $\tilde{x}$, the market price can be recovered:

$$p = E[\tilde{m}\tilde{x}].$$

- The law of one price is equivalent to the existence of a stochastic discount factor. The absence of arbitrage is equivalent to the existence of an almost everywhere strictly positive discount factor. Broadly speaking, strict positivity ensures that a (possibly synthetic) asset with strictly positive payoff cannot have a strictly negative price (this would be an arbitrage).
- In the return space, the above relationship reads:

$$E[\tilde{m}\tilde{R}] = 1.$$

- The expectation of the discount factor is linked to the risk free rate:

$$E[\tilde{m}](1+r^f) = 1.$$

- In the excess return space, this reads:

$$E[\tilde{m}(\tilde{r} - r^f)] = 0.$$

- We thus have, in the presence of a risk free asset[1]:

$$E[\tilde{r}] - r^f = -R^f \operatorname{cov}(\tilde{m}, \tilde{R}),$$

  which describes the structure of risk premia across assets as a result of the covariances with the SDF.

Reminder on utility functions (1)

- VNM utility functions are determined up to a linear transformation
- Absolute risk aversion: $\alpha(w) = -u''(w)/u'(w)$
- Relative risk aversion: $\rho(w) = w\alpha(w)$
- Risk tolerance: $\tau(w) = 1/\alpha(w)$
- Additive certainty equivalent: for a centered distribution $\tilde{\varepsilon}_a$ and an initial level of wealth w, find $\pi_a(w, \tilde{\varepsilon}_a)$ such that:

$$u(w - \pi_a) = E[u(w + \tilde{\varepsilon}_a)].$$

- Multiplicative certainty equivalent: for a centered distribution $\tilde{\varepsilon}_m$ and an initial level of wealth w, find $\pi_m(w, \tilde{\varepsilon}_m)$ such that:

$$u(w(1 - \pi_m)) = E[u(w(1 + \tilde{\varepsilon}_m))].$$

Reminder on utility functions (2)

- For small (centered) additive risks of variance σ^2: $\pi_a \approx \frac{1}{2}\sigma_a^2 \alpha(w)$
- For small (centered) multiplicative risks of variance σ^2: $\pi_m \approx \frac{1}{2}\sigma_m^2 \rho(w)$

[1] Write the discount factor condition as:

$$E[\tilde{m}\tilde{R}] = E[(\tilde{m} - E[\tilde{m}] + E[\tilde{m}])\tilde{R}] = 1,$$

and use the fact:

$$E[(\tilde{m} - E[\tilde{m}])\tilde{R}] = \operatorname{Cov}(\tilde{m}, \tilde{R}).$$

Some important utility functions

- CARA: $u(w) = -\exp(-\alpha w)$
  - range: $\mathbb{R}$
  - absolute risk aversion: $\alpha(w) = \alpha$
- CRRA:

$$u_\rho(w) = \frac{c^{1-\rho}}{1-\rho},\ \rho \geq 0,\ \rho \neq 1,$$

$$u_\rho(w) = \log(w),\ \rho = 1,$$

  - range $\mathbb{R}_+^*$
  - relative risk aversion: $\rho(w) = \rho$

CRRA utility functions - fig 1

Utility functions and return distributions

- Utility functions often have a restricted domain (frequently: positive consumption)
- Assumptions on return distributions have to be consistent
- For example, CRRA models require $\tilde{R} \geq 0$ i.e. $\tilde{r} \geq -1$. This assumption is sometimes called 'limited liability': the owner of an asset cannot end up having to transfer cash to the issuer.
- This is a problem mainly for discrete time models (or continuous times models where prices can jump)

Absolute or relative?

- The key consideration is the dependence of risk attitudes vis-à-vis the level of wealth
  - intuition suggests people accept greater dollar risk as their wealth rises

An important benchmark: CARA & normally distributed returns

- Note that with normal returns, returns can be arbitrarily negative (no limited liability). Accordingly, the range of the utility function is $\mathbb{R}$.
- I assume that there is no labor income
- $\boldsymbol{\pi} = (\pi_i)'_{i \in \mathcal{I}}$

$$\max_{\boldsymbol{\pi}} E[-\exp(-\alpha \tilde{w})]$$

s.t. :

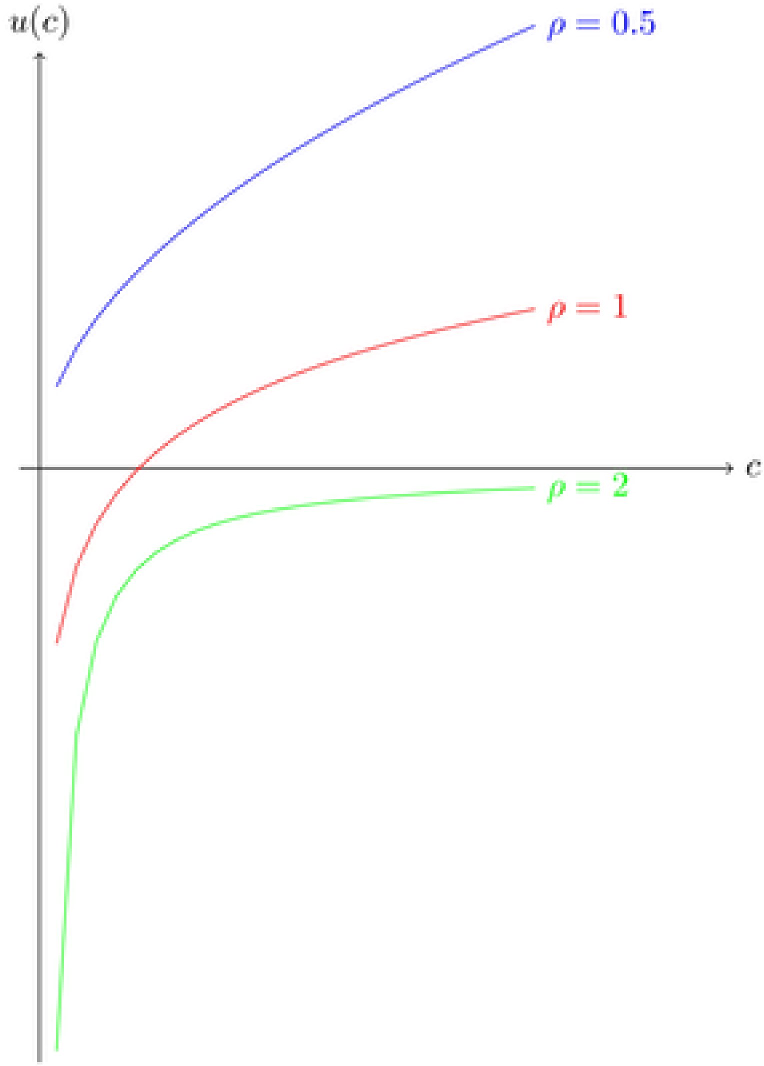

Figure 1: Figure 1: CRRA utility functions

$$\tilde{w} = w_0 \sum_{i \in \mathcal{I}} \pi_i \tilde{R}_i$$
$$\sum_{i \in \mathcal{I}} \pi_i = 1.$$

CARA normal case (1)

- The random variable $\tilde{w}$ is normally distributed. In this case, we know that:

$$\begin{aligned} E[-\exp(-\alpha\tilde{w})] = & -\exp(-\alpha E[\tilde{w}] + (\alpha^2/2)V[\tilde{w}]) \\ = & \, u(E[\tilde{w}] - (\alpha/2)V[\tilde{w}]). \end{aligned}$$

- Given that the function $u(\cdot)$ is increasing, the program consists in maximizing the certainty equivalent $E[\tilde{w}] - (\alpha/2)V[\tilde{w}]$, which reads, mean wealth minus the variance of wealth weighted by one half absolute risk aversion.

CARA normal case (2)

- Preferences over the distribution of final wealth are thus entirely determined by the mean and the variance of the wealth distribution. This is an example of mean variance preferences.
- We have:

$$\tilde{w} = w_0 \sum_{i \in \mathcal{I}} \pi_i \tilde{R}_i = w_0 + w_0 \sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i$$

- The maximized criterion is thus (dividing by $w_0 > 0$):

$$E[\sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i] - (\alpha w_0/2)V[\sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i].$$

CARA normal case (3)

- This is a standard mean-variance criterion, up to the fact that the risk aversion parameter depends on the level of wealth.
  - if this was not the case, optimal portfolio composition would be independent of the wealth level; this would imply that the investor take more dollar risk at higher wealth levels; in the CARA case, the appetite for dollar risk is independent of the level of wealth; thus the correction.

When do we get mean variance preferences?

- How general is mean variance ?
  - preferences induced by utility functions will not, in general, correspond to mean-variance; additional assumptions are needed.

- when the distribution of portfolio returns is characterized by mean and variance, all utility functions naturally lead to mean variance preferences (see elliptic distributions).
- in the presence of stochastic labour income, mean variance needs to be amended

CARA normal case (4)

- In the presence of normally distributed stochastic labor income, the optimal programme is:

$$\max_{\boldsymbol{\pi}} E[-\exp(-\alpha\tilde{w})]$$

$$\text{s.t.}$$

$$\tilde{w} = \tilde{y} + \sum_{i\in\mathcal{I}} \theta_i \tilde{x}_i$$

$$\sum_{i\in\mathcal{I}} \theta_i p_i = w_0.$$

- It is this time more convenient to take as control variables the quantities: $(\theta_i)_{i\in\mathcal{I}}$.

CARA normal case (5)

- As before, we need to maximize the certainty equivalent: $E[\tilde{w}] - (\alpha/2)V[\tilde{w}]$. This is equivalent to maximizing:

$$E\left[\sum_{i\in\mathcal{I}} \theta_i \tilde{x}_i\right] - (\alpha/2)V\left[\tilde{y} + \sum_{i\in\mathcal{I}} \theta_i \tilde{x}_i\right].$$

- We can decompose the variance term as:

$$V[\tilde{y}] + V\left[\sum_{i\in\mathcal{I}} \theta_i \tilde{x}_i\right] + 2\mathrm{Cov}\left(\sum_{i\in\mathcal{I}} \theta_i \tilde{x}_i, \tilde{y}\right).$$

CARA normal case (6)

- I give the result assuming there is a riskless asset.
- We assume the price of cash is $p_0 = 1$, and the payoff $\tilde{x}_0 = 1 + r^f$.
- Using the budget constraint $\theta_0 = w_0 - \sum_{i=1}^{N} \theta_i p_i$, we can rewrite the criterion as:

$$E\left[w_0(1+r^f) + \sum_{i=1}^{N} \theta_i(\tilde{x}_i - p_i(1+r^f))\right] - (\alpha/2)V\left[\tilde{y} + \sum_{i=1}^{N} \theta_i \tilde{x}_i\right].$$

- Notation:
  - θ is the $N \times 1$ vector of quantities invested on each risky asset
  - $V[\tilde{x}]$ is the $N \times N$ matrix where each (i, j) is the covariance of the pay-offs of asset i and j. It is assumed to have full rank, so that no financial asset is riskless or redundant.
  - $\mathrm{Cov}(\tilde{x}, \tilde{y})$ is the $N \times 1$ vector where each entry measures the covariance of a financial instrument with labour income
  - $E[\tilde{\tilde{x}}]$ is the $N \times 1$ vector of the expected excess pay-offs $(\tilde{x}_i - p_i(1+r^f))$ of the risky instruments instruments.

CARA normal case (7)

- The first order condition leads to, in matrix notation:

$$\theta = V[\tilde{x}]^{-1} \left(-\mathrm{Cov}(\tilde{x}, \tilde{y}) + \frac{1}{\alpha} E[\tilde{\tilde{x}}] \right).$$

- Remember that $1/\alpha$ is risk tolerance.
- The structure of the solution is as follows: the optimal porfolio consists of a hedging portfolio (which tries to replicate income variability using financial assets) and a speculative portfolio which has the same structure as in the case without labour income. The latter portfolio receives a weight equal to risk tolerance.

Optimization and SDF

- I assume there is a solution $\boldsymbol{\pi}_*$ to the following problem:

$$\max_{\boldsymbol{\pi}} E[u(\boldsymbol{\pi}'\tilde{\boldsymbol{R}})]$$

$$\text{s.t.}$$

$$\boldsymbol{\pi}'\boldsymbol{e} = 1,$$

  where $\boldsymbol{e}$ is a vector where all components are equal to 1, and $\boldsymbol{\pi}$ is the vector of asset proportions.

- The Lagrangian reads:

$$\mathcal{L} = E[u(\boldsymbol{\pi}'\tilde{\boldsymbol{R}})] - \gamma\boldsymbol{\pi}'\boldsymbol{e},$$

  and the first order condition reads:

$$E[u'(\boldsymbol{\pi}'\tilde{\boldsymbol{R}})\tilde{\boldsymbol{R}}] = \gamma\boldsymbol{e}.$$

- Let:

$$\tilde{m} = \frac{u'(\boldsymbol{\pi}_*'\tilde{\boldsymbol{R}})}{\gamma}.$$

We then have:

$$E[\tilde{m}\tilde{\boldsymbol{R}}] = \boldsymbol{e},$$

i.e. for any asset i:

$$E[\tilde{m}\tilde{R}_i] = 1.$$

In other words, we have built an SDF from the solution of the optimization problem.

Mean variance efficiency

- A portfolio p with mean and variance (μ_p, σ_p) is dominated by a portfolio q with mean and variance (μ_q, σ_q) if $\mu_q \geq \mu_p$ and $\sigma_q \leq \sigma_p$ with at least one inequality being strict.
- A portfolio is efficient in the mean variance sense if it is not dominated by any other portfolio.
- Domination is a preorder. An efficient portfolio is a maximal element for the preorder. In particular, it is not a total order (all portfolio pairs cannot necessarily be ordered).

Mean variance without a riskfree asset (1)

- The program: it consists in minimizing portfolio variance for a given level of expected returns

$$\min_{\boldsymbol{\pi}} V\left[\sum_{i=1}^{N} \pi_i \tilde{r}_i\right] = \boldsymbol{\pi}'\Sigma\boldsymbol{\pi}$$

$$\text{s.t.}$$

$$\sum_{i=1}^{N} \pi_i = \boldsymbol{\pi}'\boldsymbol{e} = 1$$

$$E\left[\sum_{i=1}^{N} \pi_i \tilde{r}_i\right] = \boldsymbol{\pi}'\boldsymbol{\mu} = \mu_p.$$

Mean variance without a riskfree asset (2)

- Bold notations denote vectors
  - Σ is the covariance matrix of returns, which we assume invertible
  - $\boldsymbol{e}$ is a vector of ones
  - $\tilde{\boldsymbol{r}}$ is the vector of returns
  - $\boldsymbol{\mu}$ is the vector of expected returns
- We assume $\boldsymbol{\mu} \neq \boldsymbol{e}$ to avoid degeneracy

Mean variance without a riskfree asset (3)

- Lagrangian for the optimization problem (a factor 1/2 is convenient):

$$\frac{1}{2}\boldsymbol{\pi}'\Sigma\boldsymbol{\pi} - \delta(\boldsymbol{\pi}'\boldsymbol{\mu} - \mu_p) - \gamma(\boldsymbol{\pi}'\boldsymbol{e} - 1)$$

  where I have introduced the Lagrange multipliers δ and γ.

- The necessary and sufficient first order condition (positive definite quadratic problem) is:

$$\Sigma\boldsymbol{\pi} = \delta\boldsymbol{\mu} + \gamma\boldsymbol{e},$$

  or, assuming the covariance matrix is invertible:

$$\boldsymbol{\pi} = \delta\Sigma^{-1}\boldsymbol{\mu} + \gamma\Sigma^{-1}\boldsymbol{e}.$$

Mean variance without a riskfree asset (4)

- Injecting this into the constraints leads to a system for the Lagrange multipliers:

$$\delta\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu} + \gamma\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{e} = \mu_p,$$
$$\delta\boldsymbol{e}'\Sigma^{-1}\boldsymbol{\mu} + \gamma\boldsymbol{e}'\Sigma^{-1}\boldsymbol{e} = 1.$$

- Reminder:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- It is useful to introduce two specific portfolios:

$$\boldsymbol{\pi}_1 = \frac{1}{\boldsymbol{e}'\Sigma^{-1}\boldsymbol{e}}\Sigma^{-1}\boldsymbol{e},$$

$$\boldsymbol{\pi}_\mu = \frac{1}{\boldsymbol{e}'\Sigma^{-1}\boldsymbol{\mu}}\Sigma^{-1}\boldsymbol{\mu}.$$

Mean variance without a riskfree asset (5)

- We can write :

$$\boldsymbol{\pi} = (\delta\boldsymbol{e}'\Sigma^{-1}\boldsymbol{\mu})\boldsymbol{\pi}_\mu + (\gamma\boldsymbol{e}'\Sigma^{-1}\boldsymbol{e})\boldsymbol{\pi}_1 =$$
$$\lambda\boldsymbol{\pi}_\mu + (1-\lambda)\boldsymbol{\pi}_1.$$

- Thus, any optimal portfolio is a combination of the two portfolios we singled out:
  - $\boldsymbol{\pi}_1$ is the minimum variance portfolio
  - $\boldsymbol{\pi}_\mu$ is another portfolio as soon as $\boldsymbol{\mu} \neq \boldsymbol{e}$

Mean variance without a riskfree asset (6)

- $A = \boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}$, $B = \boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{e}$, $C = \boldsymbol{e}'\Sigma^{-1}\boldsymbol{e}$.

$$\lambda = \frac{BC\mu_p - B^2}{AC - B^2},$$

$$\sigma_p^2 = \frac{A - 2B\mu_p + C\mu_p^2}{AC - B^2}.$$

- Check this.
- The efficient frontier (in the standard deviation mean space) is the subset of non dominated portfolios in the set:

$$\{(\sigma_p, \mu_p),\ \mu_p \geq \mu_1\}$$

where $\mu_1 = \boldsymbol{\pi}_1'\boldsymbol{\mu}$.

Mean variance without a riskfree asset (7) - fig 2

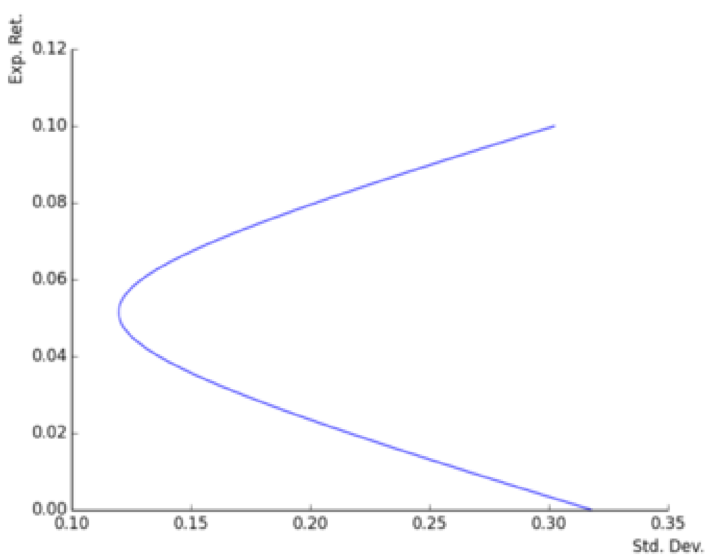

Figure 2: Figure 2: Efficient Frontier (without a risk free asset)

Mean variance without a riskfree asset (8)

- I list the technical conditions below:
  - we assume that $\boldsymbol{\mu}$ and $\boldsymbol{e}$ are not colinear

- we assume $\boldsymbol{e}'\Sigma^{-1}\boldsymbol{\mu} > \mathbf{0}$
- we have $\boldsymbol{e}'\Sigma^{-1}\boldsymbol{e} > 0$ as Σ^{-1} defines a positive definite quadratic form
- we have $\left(\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{e}\right)^2 < \left(\boldsymbol{e}'\Sigma^{-1}\boldsymbol{e}\right)\left(\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}\right)$ from the Cauchy-Schwartz inequality and $\boldsymbol{e}'\Sigma^{-1}\boldsymbol{e} > 0$.

Mean variance with a riskfree asset (1)

- It is convenient in this case to use the notation $\boldsymbol{\pi}$ to denote the vector of positions on the risky assets (see the slide on the space of excess returns). The cash position is thus:

$$\pi_0 = 1 - \boldsymbol{e}'\boldsymbol{\pi}.$$

- The vector $\boldsymbol{\pi}$ is unconstrained. The optimization problem can be written:

$$\min_{\boldsymbol{\pi}} \boldsymbol{\pi}'\Sigma\boldsymbol{\pi}$$

$$\text{s.t.}$$

$$\boldsymbol{\pi}'(\boldsymbol{\mu} - r^f\boldsymbol{e}) = \mu_p - r^f.$$

- For reasons that will be clear below, I assume $\boldsymbol{e}'\Sigma^{-1}(\boldsymbol{\mu} - r^f\boldsymbol{e}) > 0$.

Mean variance with a riskfree asset (2)

- First order condition for the Lagrangian: $\boldsymbol{\pi} = \delta\Sigma^{-1}(\boldsymbol{\mu} - r^f\boldsymbol{e})$
- From $(\boldsymbol{\mu} - r^f\boldsymbol{e})'\boldsymbol{\pi} = \mu_p - r^f$, we get the value of δ and then the value of $\boldsymbol{\pi}$:

$$\boldsymbol{\pi} = \frac{\mu_p - r^f}{(\boldsymbol{\mu} - r^f\boldsymbol{e})'\Sigma^{-1}(\boldsymbol{\mu} - r^f\boldsymbol{e})}\Sigma^{-1}(\boldsymbol{\mu} - r^f\boldsymbol{e}).$$

- The standard deviation of the portfolio is:

$$\frac{|\mu_p - r^f|}{\sqrt{(\boldsymbol{\mu} - r^f\boldsymbol{e})'\Sigma^{-1}(\boldsymbol{\mu} - r^f\boldsymbol{e})}}.$$

Mean variance with a riskfree asset (3)

- The tangency portfolio is:

$$\boldsymbol{\pi}_* = \frac{1}{\boldsymbol{e}'\Sigma^{-1}(\boldsymbol{\mu} - r^f\boldsymbol{e})}\Sigma^{-1}(\boldsymbol{\mu} - r^f\boldsymbol{e}).$$

- It is a portfolio fully invested in risky assets which is on the overall efficient frontier. It is thus also on the risky asset efficient frontier.

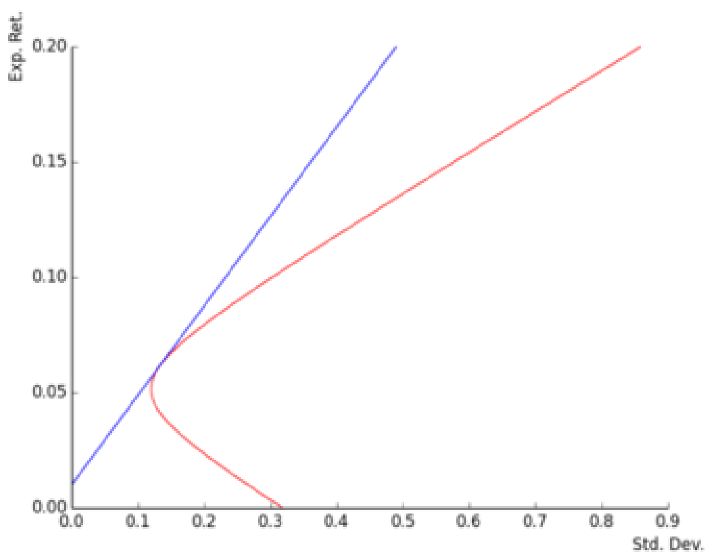

Figure 3: Figure 3: Efficient Frontier (with risk free asset)

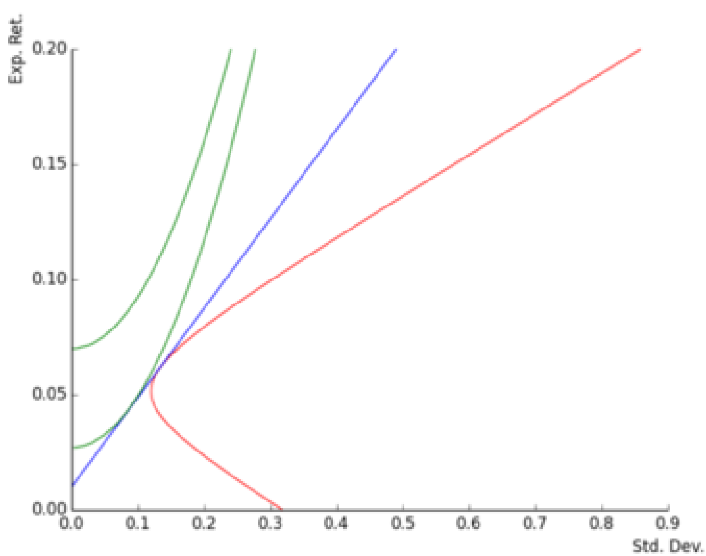

Figure 4: Figure 4: Efficient Frontier (with risk free asset): iso-utility curves

Mean variance with a riskfree asset (4) - fig 3

Mean variance with a riskfree asset (4) - fig4

Data for the graphs (1)

- Two risky assets:
  - $\mu_1 = 0.05$, $\sigma_1 = 0.12$
  - $\mu_2 = 0.07$, $\sigma_2 = 0.16$
  - $\rho = 0.7$
  - $(\mu_1 - r)/\sigma_1 = 0.33$
  - $(\mu_2 - r)/\sigma_2 = 0.375$
  - $\boldsymbol{\pi_1} = (0.93, 0.07)$
  - $\mathrm{vol}(\boldsymbol{\pi_1}) = 0.12$
  - $\boldsymbol{\pi_*} = (0.4, 0.6)$
  - $\mathrm{vol}(\boldsymbol{\pi_*}) = 0.13$
  - $\mathrm{sharpe}(\boldsymbol{\pi_*}) = 0.39$

Data for the graphs (2)

- The graphs shown assume positive Sharpe ratios for the underlying assets. This is the 'normal' situation. It ensures that the efficient frontier (with a riskfree asset!) is upward sloping.

A different description of the efficient frontier (1)

- Maximize the expected return penalized for portfolio variance ($\rho > 0$):

$$\max_{\boldsymbol{\pi}} r^f + \boldsymbol{\pi}'(\boldsymbol{\mu} - r^f \boldsymbol{e}) - \frac{\rho}{2}\boldsymbol{\pi}'\Sigma\boldsymbol{\pi}.$$

- Exercise: recover the lagrange multiplier of the traditional approach
- The criteria are given by quadratic utility functions, indexed by ρ

A different description of the efficient frontier (2)

- The first order condition reads:

$$(\boldsymbol{\mu} - r^f \boldsymbol{e}) = \rho\Sigma\boldsymbol{\pi},$$

  and this implies that the optimal portfolio is proportional to the tangency portfolio.
- How much of the tangency portfolio $\boldsymbol{\pi_*}$ does an investor with the above preferences and beliefs buy?

- From the first order condition of the utility maximization problem[2], we get that the weight $\hat{\pi} = 1 - \pi_0$ invested in the tangency portfolio is:

$$\hat{\pi} = \frac{1}{\rho} \frac{\mu_* - r^f}{\mathrm{var}(\tilde{r}_*)}.$$

- We will remember that:

$$\rho\hat{\pi} = \frac{\mu_* - r^f}{\mathrm{var}(\tilde{r}_*)},$$

  which is therefore independent of the risk aversion level of the investor. This will play a role in the derivation of the CAPM.

Interpretation of the first order condition (1)

- Consider that the optimal portfolio of a mean-variance investor (p with weights $\boldsymbol{\pi}$) is tilted by adding a long-short portfolio $\boldsymbol{\pi}_\delta$. How does that affect quadratic utility?
- The utility level changes by (first order approximation):

$$\begin{aligned}
&\mu_\delta - \rho \mathrm{cov}(\tilde{r}_\delta, \tilde{r}_p) \\
&= \mu_\delta - \rho \frac{\mathrm{cov}(\tilde{r}_\delta, \tilde{r}_p)}{\mathrm{var}(\tilde{r}_p)} \mathrm{var}(\tilde{r}_p), \\
&= \mu_\delta - \rho\beta(\tilde{r}_\delta, \tilde{r}_p)\mathrm{var}(\tilde{r}_p), \\
&= \mu_\delta - \rho\hat{\pi}\beta(\tilde{r}_\delta, \tilde{r}_*)\mathrm{var}(\tilde{r}_*).
\end{aligned}$$

Interpretation of the first order condition (2)

- Because the quantity $\rho\hat{\pi}$ is independent of ρ, the trade off between return and beta is a well defined consequence of the mean and variance assumptions.
- Injecting the value of $\rho\hat{\pi}$ into the first order condition delivers the quantity:

$$\mu_\delta - (\mu_* - r^f)\beta(\tilde{r}_\delta, \tilde{r}_*).$$

- Given the optimality of the tangency portfolio, the above quantity should be zero for all long short deviations to the tangency portfolio:

$$\mu_\delta = (\mu_* - r^f)\beta(\tilde{r}_\delta, \tilde{r}_*).$$

- For long short portfolios which borrow to buy a stock, the condition reads:

$$(\mu_i - r^f) = (\mu_* - r^f)\beta(\tilde{r}_i, \tilde{r}_*).$$

[2] The first order condition reads:

$$(\boldsymbol{\mu} - r^f \boldsymbol{e}) = \rho\Sigma\boldsymbol{\pi}.$$

Multiply both sides on the left by $\boldsymbol{\pi}'$. Then use $\boldsymbol{\pi} = \hat{\pi}\boldsymbol{\pi}_*$.

Interpretation of the first order condition (3)

- The above relationship embodies the return beta trade off embedded in the mean variance assumptions.
- At this stage, no equilibrium assumption has been made. We are looking at the implications of a portfolio being mean-variance optimal.
- Note that the tangency portfolio can be replaced by any other efficient portfolio in the relationship.

The excess return-beta relationship (1) - fig 5

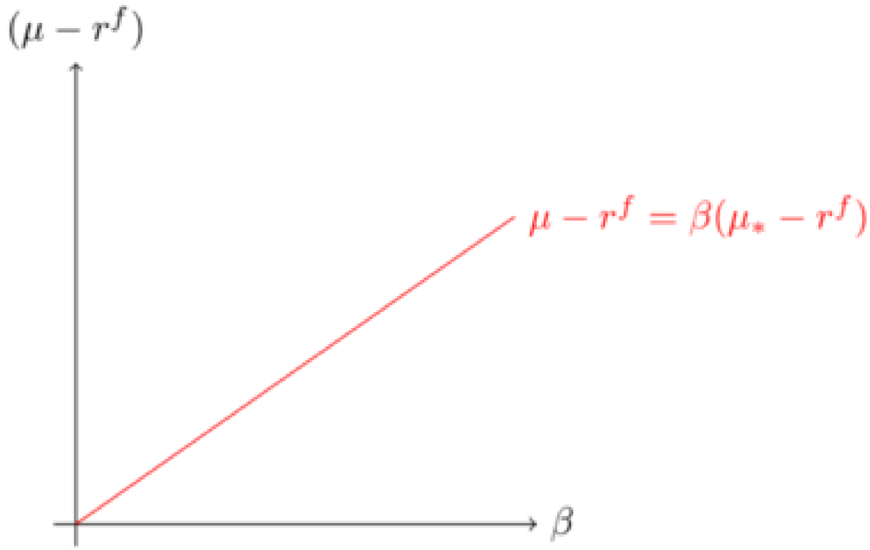

Figure 5: Figure 5: Return/beta relationship

The excess return-beta relationship (2) - fig 6

The two fund theorem and the CAPM

- We now move to equilibrium considerations. We assume all investors share the same beliefs on expected returns and risk, and all choose mean variance efficient portfolios.
- As a result, they all hold a mixture of the risk free asset and a unique portfolio of risky asset, the tangency portfolio.
- This is an instance of the two fund theorem, which also holds in more general contexts

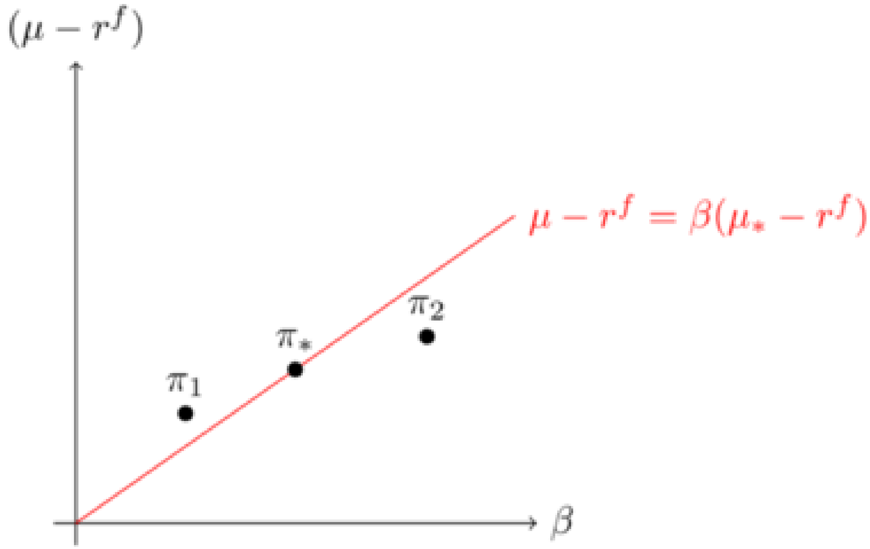

Figure 6: Figure 6: Imperfectly priced portfolios

- The risky asset portfolio should be equal to the market portfolio of risky asset, with return r_m. This gives:

$$(\mu_i - r^f) = (\mu_m - r^f)\beta(\tilde{r}_i, \tilde{r}_m).$$

Illustration of the CAPM - fig 7

The low beta anomaly - fig 8

Equity pricing anomalies

- Take an investment universe (stocks) and an equity index
- Follow the steps:
  - build equity portfolios by sorting stocks according to a financial characteristic
  - compute the beta of the portfolios and graph realized returns against betas
  - is the pricing error significant?
- Examples of characteristics: size, book value, momentum, beta, vol
- This procedure asks whether the index is mean variance efficient in sample
- The pricing errors should be statistically significant

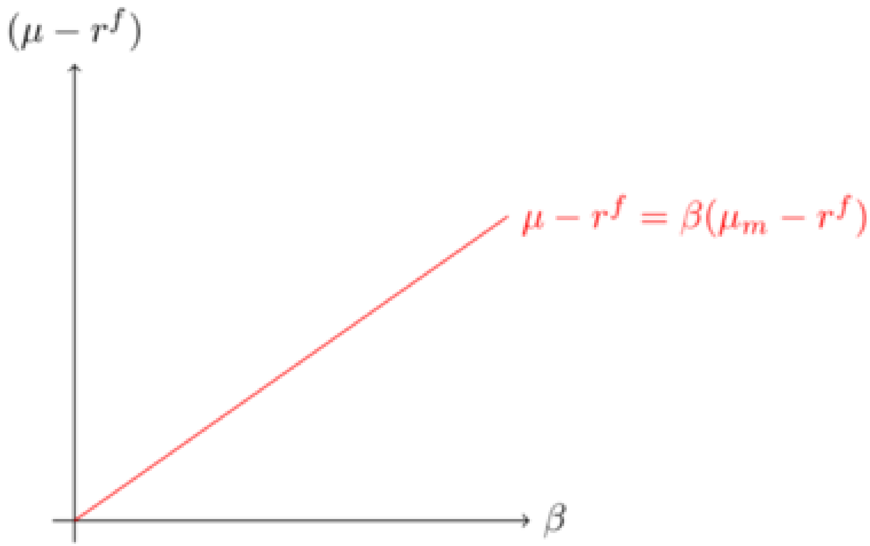

Figure 7: Figure 7: CAPM

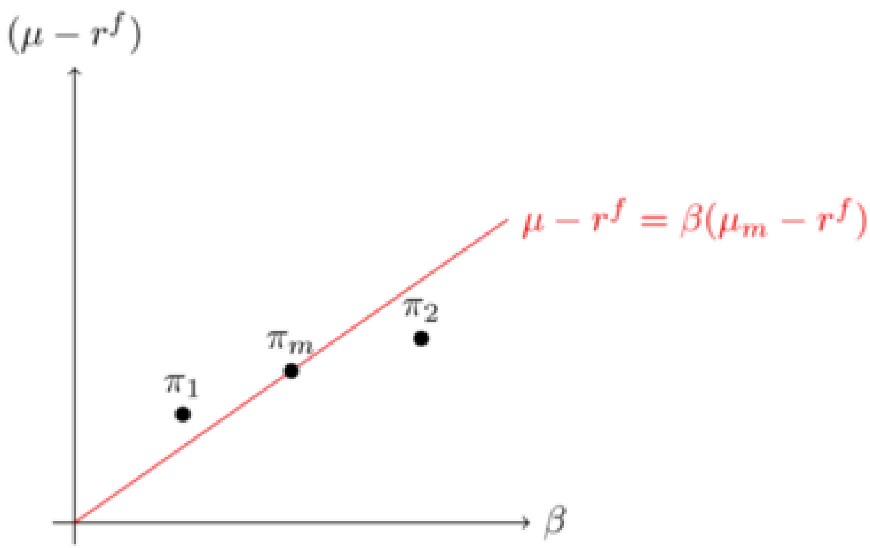

Figure 8: Figure 8: The low beta anomaly

Mean variance in practice: the challenges

- First, there is a question of interpretation: what is the investment horizon?
  - in particular, this conditions the nature of the risky asset (bonds or cash).
- One also needs to be clear on whether real returns or nominal returns are considered
- Once this has been clarified, input data needs to be estimated:
  - getting hold of expected returns
  - getting hold of the covariance matrix
- The optimal portfolio is very sensitive to inputs
  - garbage in, garbage out

Examples of implementations

- Give the same return to all risky assets
  - this delivers the minimum variance portfolio, which is not optimal unless the expected returns are truly equal across assets
- Link the return assumptions to the risk estimates
  - this leads to various solutions…
- For returns: estimate the payoffs of the asset and derive the implies return from the current asset price
- Example: ERC ?

Links

- The beamer presentation is here.