Zero Coupon Contracts

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This post introduces zero coupon contracts, contracts which promise a single payment at a fixed date T. Truthful to this introductory post, the price process is derived backwards. The terminal pay-off is modeled as a random variable usually not known before maturity. The price process is then defined as adapted to the filtration, with a drift which is the instantaneous expected return and converging to the terminal pay-off at maturity. This is an example of a very simple backward stochastic differential equation. The data of the zero coupon contract is the terminal pay-off and the drift (expected return) process. A key component of the solution is the volatility of the price process, which can be related to the data of the problem.

Introduction

This note introduces a stylized financial contract, the zero coupon contract. The simple structure of this contract allows to discuss price dynamics without being bogged down in unnecessary complexity. The important features of price dynamics stand out clearly. We will see that price dynamics should be conceived as the solution to a terminal value problem. This will lead us to introduce the concept of a backward stochastic differential equation. This approach should be contrasted with the usual practice of specifying ad-hoc forward dynamics.

A financial contract is the promise of a stream of payments, usually spread across several periods. This latter feature creates analytical challenges. We will instead assume that our contract specifies a single payment at a given date T. This is what we call a **zero coupon contract**.

In the day to day practice of plain vanilla finance, the range of questions raised by the potential buyer of the contract is, 'what is this payment likely to be? How uncertain is it? What is the worst case? What is the best case? How likely is the worst case? How likely is the best case?'. Once these questions have been answered (qualitatively at least), the buyer can judge whether the current price is attractive or not, i.e. whether the contract has the right return to risk characteristics. He will be able to form expectations on the distribution of the return to maturity. If the contract is liquid, the buyer can raise the same questions tomorrow and the day after, given the observed behavior of the market price. He can form short term expectations of the return. The realized short term return will be made up of its expected value and a surprise. This surprise will perhaps correspond to a change in the likely terminal payment. If not, the buyer will be able to conclude that the expected return to maturity has changed.

Zero coupon dynamics: the fundamental solution

To be relevant, modeling should closely mimic the above logic. To be specific, I'll assume that uncertainty in the model is driven by a Brownian motion $B = (B_t)_{t\in[0,T]}$ and that the filtration $\mathcal{F}$ is the minimal filtration generated by B and satisfying the usual conditions. I now proceed to introduce the relevant mathematical objects. The terminal payment is an $\mathcal{F}_T-$ measurable random variable X_T. Technically, we impose the square integrability and strict positivity (almost surely) of X_T. The technical conditions should make sure that we can apply the martingale representation theorem. We also need all processes to be almost surely strictly positive to formulate dynamics in a geometric way.

At a given date $0 \leq t \leq T$, the expectation of the terminal payment is $E_t[X_T]$ which we will denote by X_t. What is the dynamics of $(X_t)_{t\in[0,T]}$? This process is a martingale and the martingale representation theorem tells us that there is an adapted process $\zeta = (\zeta_t)_{t\in[0,T]}$ such that:

$$X_t = X_0 + \int_0^t \zeta_u dB_u.$$

This can be written in differential form:

$$dX_t = \zeta_t dB_t.$$

This makes it clear that ζ_t measures the change in the expectation $E_t[X_T]$ that results from the shock dB_t. To stress the backward angle to expectations dynamics, we note that expectations satisfy the equation:

$$X_t = X_T - \int_t^T \zeta_u dB_u.$$

This emphasizes the fact that expectations converge to the terminal variable X_T which is the key piece of data in this problem. Finally, X is strictly positive and we can write $\eta_t = \zeta_t/X_t$. The process η is called volatility process and the dynamics is determined by a geometric stochastic differential equation:

$$\frac{dX_t}{X_t} = \eta_t dB_t.$$

A historically important model of asset price dynamics is obtained when $P_t = X_t = E_t[X_T]$. In this case, the price dynamics are determined by a stochastic integral. At any point in time, the price is expected to remain constant and all its fluctuations come from revisions in the expected value of the terminal payment. Periods where η_t is low (respectively high) correspond to periods where the newsflow on the terminal payment is light (resp. heavy). We will call $(X_t)_{t\in[0,T]}$ **{the fundamental solution} of the zero coupon contract.**

The case of a constant expected return

A more realistic model assumes that the financial contract has a constant positive expected return r. We now have to modify the price dynamics. We assume that the price process is adapted and follows a geometric diffusion with drift r:

$$\frac{dP_t}{P_t} = rdt + Z_t dB_t.$$

We still want to complete this with the requisite that the price process at date T equals the terminal payment X_T, any other value leading to an unrealistic arbitrage. We would therefore hope to be able to specify the complete dynamics with the terminal condition as:

$$\frac{dP_t}{P_t} = rdt + Z_t dB_t,\ P_T = X_T$$

The stochastic differential equation is really the shorthand for:

$$P_t = P_0 + \int_0^t rP_u du + \int_0^t Z_u P_u du,$$

and, to emphasize the terminal value condition, we can write:

$$P_t = X_T - \int_t^T rP_u du - \int_t^T Z_u P_u du.$$

A natural candidate for a solution is:

$$P_t = \exp(-r(T-t))X_t = E_t[\exp(-r(T-t))X_T],$$

and this indeed works. It is easily checked by applying Ito:

$$d(\exp(-r(T-t))X_t) = r\exp(-r(T-t))X_t dt + \exp(-r(T-t))X_t \eta_t dB_t,$$

or:

$$dP_t = rP_t dt + \eta_t P_t dB_t.$$

We have thus found a solution to our problem, with $Z_t = \eta_t$, i.e. the volatility of the fundamental solution. This solution rises in expectations at a rate r and its fluctuations entirely reflect revisions of the terminal payment X_T.

The case of an exogenous expected return

We now assume we have a stochastic expected return process $(r_t)_{t\in[0,T]}$. For the given terminal payment X_T and the given expected return process $(r_t)_{t\in[0,T]}$ (the data), we are looking for an adapted price process $(P_t)_{t\in[0,T]}$ with an adapted volatility function $(Z_t)_{t\in[0,T]}$ such that:

$$\frac{dP_t}{P_t} = r_t dt + Z_t dB_t,\ P_T = X_T.$$

In integral form, we are trying to solve:

$$P_t = X_T - \int_t^T r_u P_u du - \int_t^T Z_u P_u du.$$

This is an instance of a backward stochastic differential equation.

We observe that if there is a solution, we should have the following representation:

$$P_t = E_t[\exp(-\int_t^T r_u du) X_T].$$

Note that this is indeed equivalent to:

$$\exp(-\int_0^t r_u du) P_t = E_t[\exp(-\int_0^T r_u du) X_T] = E_t[\exp(-\int_0^T r_u du) P_T],$$

but then if P and Z are solutions, the left hand side has to be a martingale (apply Ito) which means that the above equation should hold for any solution.

As for existence, it directly follows from the above observation. The variable $Y_t = E_t[\exp(-\int_0^T r_u du) X_T]$ defines a continuous martingale. It can be represented as a Brownian integral:

$$dY_t = Y_t Z_t dB_t.$$

If we define P_t through $P_t = \exp(\int_0^t r_u du) Y_t$ and we apply Ito, we get:

$$dP_t = r_t P_t dt + Z_t P_t dB_t.$$

We have thus found a solution to our simple backward stochastic differential equation.

In all cases we have seen, the volatility of the price is determined by the volatility of the martingale $(E_t[\exp(-\int_0^T r_u du) X_T])_{t\in[0,T]}$. When $(r_t)_{t\in[0,T]}$ is deterministic, this is the same thing as the volatility of the martingale $(E_t[X_T])_{t\in[0,T]}$, i.e. fundamental volatility. But in general, it is not. We will come back to this in a later post.

Summary

We have thus found a description of the price process which sticks to the financial logic. What drives the dynamics is two things:

- **revisions of the terminal pay-off, as reflected in the dynamics of the fundamental solution** $(X_t)_{t\in[0,T]}$**,**
- **changes in the expected return.**

These two items determine the volatility function of the price process. We will see that price volatility obeys an accounting identity that links it to the two items above. There is thus more structure that in a traditional forward stochastic differential equation. As argued above, this structure is ideally suited to raising investment questions.

Note: A short and clear introduction to backward stochastic differential equation, Hu[2013] can be found here.

References: Hu Y., 2013, *Backward Stochastic Differential Equations and Applications in Finance*, Lecture Notes, Rennes University.