Exam (2018)

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This is the exam for the 2018 ensae course.

Exercise 1

The investment universe is composed of N risky assets (indices $i = 1, \dots, N$) with expected returns $\mu = (\mu_1, \dots, \mu_N)'$ and covariance matrix Σ (dimension (N, N)) which is assumed non singular. A portfolio is a vector π of size N which records the proportions invested on the assets. The proportions sum to 1, i.e. $\pi' e = 1$ where e is the vector of size N where each component is equal to 1.

1. Given a return objective $\bar{\mu}$, we are looking for the portfolio with the lowest possible level of risk. Give the corresponding optimization problem. Derive the Lagrangian, noting γ the multiplier of the return constraint and δ the multiplier of the full investment constraint.

2. What is the first order condition corresponding to the optimum ? For which portfolio do we get $\gamma = 0$?

3. We consider an optimal portfolio $\bar{\pi}$ with $\gamma \neq 0$ and return $\bar{r}$. Show that the first order condition for this portfolio can be written as:

$$\mathrm{cov}(r_i, \bar{r}) = \gamma \mu_i + \delta.$$

4. Show that:

$$\mathrm{var}(\bar{r}) = \gamma \bar{\mu} + \delta,$$

   and:

$$\beta_i (\bar{\mu} + \frac{\delta}{\gamma}) = \mu_i + \frac{\delta}{\gamma},$$

   where β_i is the beta of asset i against portfolio $\bar{\pi}$.

5. We now consider portfolio $\tilde{\pi}$ with return $\tilde{r}$ satisfying: $\mathrm{cov}(\tilde{r}, \bar{r}) = 0$. Let $\tilde{\mu}$ be its expected return. Show that for asset i:

$$\mu_i - \tilde{\mu} = \beta_i (\bar{\mu} - \tilde{\mu}).$$

   This is called a one factor representation of expected returns.

6. Only one efficient portfolio does not give rise to a one factor representation of expected returns. Which one ?

7. Inversely, assume that portfolio $\bar{\pi}$ with return $\bar{r}$ gives rise to a one factor representation:
$$\beta_i(\bar{\mu} - \rho) = \mu_i - \rho,$$
for all i, where β_i is the beta of asset i with respect to portfolio $\bar{\pi}$. Show that we have the following vectorial relationship:
$$\Sigma\bar{\pi} = \gamma\mu + \delta e,$$
where γ and δ are constants associated to the one factor representation.

8. Conclude that portfolio $\bar{\pi}$ (the portfolio which generates the given one factor representation) is a solution to the problem defined in the first question. As a reminder, in the context of convex objective function and linear constraints, the first order condition attached to the Lagrangian is both necessary and sufficient to define a solution.

Exercice 2

We consider the following continuous time investment problem. The investment universe is composed of two assets, cash with constant rate of return:
$$\frac{dD_t}{D_t} = rdt,$$
and a risky asset that follows a geometric diffusion process:
$$\frac{dP_t}{P_t} = \mu dt + \sigma dB_t = rdt + (\mu - r)dt + \sigma dB_t,$$
where B_t is a scalar Brownian motion. The price of risk is defined as $\lambda = (\mu - r)/\sigma$.

At each point in time, wealth W_t is invested to finance a consumption flow $C_t dt$ over the time interval $[t, t + dt]$. The fraction of wealth invested on the risky asset at time t is noted x_t.

1. Give the stochastic differential equation followed by wealth assuming consumption is zero. As a reminder, this is the infinitesimal version of the discrete time equation. Give $E_t[dW_t]$ (the drift of wealth) and $d[W]_t$ (the quadratic variation of wealth).

2. Same question without assuming $C^t = 0$.

At each time t, the investor maximizes:
$$E_t\left[\int_t^T e^{-\rho(u-t)} u(C_u) du\right],$$

by making consumption - C_t - and investment - x_t - choices. The associated value function is noted $J(t, W_t)$. It is admitted that the dynamic programming principle implies the following partial differential equation (HJB):

$$0 = \max_{(C_t, x_t)} \left[u(C_t) - \rho J + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial W} E[dW_t] + \frac{1}{2}\frac{\partial^2 J}{\partial W^2} d[W]_t \right].$$

We will use the following notations:

$$\frac{\partial J}{\partial t} = J_t,$$

$$\frac{\partial J}{\partial W} = J_W,$$

$$\frac{\partial^2 J}{\partial W^2} = J_{WW}.$$

3. Give the optimal consumption rate C_t^* as a function of J_W and the utility function.
4. Give the optimal risky asset weight x_t^*, outlining the relevant property of the value function underlying your reasoning.
5. Describe the structure of this solution.

We now assume the utility function is:

$$u(C) = \frac{C^{1-\alpha}}{1-\alpha},$$

with $\alpha > 1$ and admit that the value function has a similar structure:

$$J(t, W_t) = h(t)^\alpha \frac{W_t^{1-\alpha}}{1-\alpha}.$$

6. Show that:

$$\frac{C_t^*}{W_t^*} = h(t)^{-1},$$

   and:

$$x_t^* = \frac{1}{\alpha}\frac{\lambda}{\sigma}.$$

7. Show that at each date t:

$$u(C^*) - J_W C^* = \frac{\alpha}{1-\alpha} J_W^{(\alpha-1)/\alpha},$$

$$J_W W x^* (\mu - r) + \frac{1}{2} J_{WW} W^2 \sigma^2 x^{*2} = -\frac{1}{2}\frac{J_W^2}{J_{WW}} \lambda^2.$$

8. Injecting the above results into HJB, prove that $h(\cdot)$ solves the following differential equation:

$$h' + \frac{1}{\alpha}\left[-\rho + (1-\alpha)r + \frac{1-\alpha}{2\alpha}\lambda^2\right] h + 1 = 0,$$

   with $h(T) = 0$.

9. Find the solution $h(\cdot)$.

10. Give the stochastic differential equation followed by log wealth and log consumption.