Discrete Time Lognormal Dynamics

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This post introduces discrete time price dynamics in a log-normal setup. This framework allows an easy translation of some of the continuous time results discussed on this site.

Reminder on log normality The random variable X has a log-normal distribution if $x = \log(X)$ has a normal distribution. If the distribution of x is $\mathcal{N}(\mu, \sigma)$, then we will note the distribution of X $\mathcal{L}(\mu, \sigma)$. Its mean is:

$$\exp(\mu + \frac{1}{2}\sigma^2).$$

This is consistent with the convexity of the exponential function and the Jensen effect[1]. Whereas x has a distribution which is centered around its mean μ, the distribution of X is positively skewed. Its median is $\exp(\mu)$ which is lower than its mean $\exp(\mu + \frac{1}{2}\sigma^2)$ by the factor $\exp(-\frac{1}{2}\sigma^2)$. Log normal price dynamics Suppose now that we have a value process $(P_t)_{t \in \mathbb{N}}$ such that as seen from date t, P_{t+1}/P_t is log-normal $\mathcal{L}(\mu_{\sqcup+\infty}, \sigma_{\sqcup+\infty})$. More explicitly, we assume that the variables μ_{t+1} and σ_{t+1} are $\mathcal{F}_{\sqcup}$ measurable. Then:

$$E_t[\frac{P_{t+1}}{P_t}] = \exp(\mu_{t+1} + \frac{1}{2}\sigma_{t+1}^2).$$

I will note $r_{t+1} = \mu_{t+1} + \frac{1}{2}\sigma_{t+1}^2$ the log of the sequential expected return as of date t. With this notation, we can now write:

$$\frac{P_{t+1}}{P_t} = \exp(r_{t+1} + \sigma_{t+1}\varepsilon_{t+1} - \frac{1}{2}\sigma_{t+1}^2),$$

where by construction, ε_{t+1} is $\mathcal{N}(\prime, \infty)$. We can also write:

$$\frac{P_{t+1}}{P_t} = \exp(r_{t+1}) \exp(\sigma_{t+1}\varepsilon_{t+1} - \frac{1}{2}\sigma_{t+1}^2),$$

where:

$$E_t[\exp(\sigma_{t+1}\varepsilon_{t+1} - \frac{1}{2}\sigma_{t+1}^2)] = 1,$$

[1] If $f(\cdot)$ is a conxex function ($\mathbb{R} \to \mathbb{R}$) and z is a random variable, $E[f(z)] \geq f(E[z])$.

so that $\exp(\sigma_{t+1}\varepsilon_{t+1} - \frac{1}{2}\sigma_{t+1}^2)$ can be interpreted as a multiplicative surprise. It is indeed the martingale ratio of a multiplicative martingale (see the notion of martingale difference in this post.). This multiplicative surprise has mean 1 but median $\exp(-\frac{1}{2}\sigma_{t+1}^2) < 1$. We have thus decomposed the return into its expected component and a surprise. This is really a discrete time version of the usual geometric Brownian diffusion:

$$\frac{dP_t}{P_t} = r_t dt + \sigma_t dB_t.$$

Intuition is sometimes easier to develop working on the discrete time model. As a final remark, we need to emphasize that although the shocks in the above discrete time model are conditionally log-normal, it does not imply that prices have log-normal marginal distributions. This is the case if $(\mu_t)_{t\in\mathbb{N}}$ and $(\sigma_t)_{t\in\mathbb{N}}$ are deterministic processes, but it fails to be the case in general.