Complete Markets, Static Case

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In this post, I introduce the notion of market completeness in the context of a two period model. I also review the notion of a stochastic discount factor (SDF). The absence of arbitrage coupled with market completeness imply the existence of a unique strictly positive SDF. In this context, the investor chooses wealth/consumption in each state of the world so as to equalize marginal rates of substitution across states and the ratio of the values the SDF takes in the corresponding states. In other words, the marginal value of wealth/consumption in each state is proportional to the SDF, the constant of proportionality being the same for all states . For a given level of initial wealth and a given utility function, the SDF determines the end wealth/consumption distribution.

Discrete probability space, static case

- Consider the static portfolio problem, i.e. there are two dates. Investment is carried out in period 0 and consumption takes place in period 1 (see this post).
- I now assume that the probability space is discrete, so that the securities payoff distributions are described by a vector of outcomes with components $\tilde{x}_i(\omega_k)$ for state ω_k.
- There are K possible states, $(\omega_1, \ldots, \omega_K)$ with their respective probabilities $(q_1, \ldots, q_K)$ satisfying:

$$\sum_{k=1}^{K} q_k = 1.$$

- Optimization program:

$$\max_{\theta} E_0[u(\tilde{x}_\theta)] = q' u(\tilde{x}\theta)$$

$$\text{s.t.}$$

$$\theta' p = w_0$$

- Notation:

  - $\tilde{x}_\theta$ is the payoff of portfolio theta
  - θ is a $(N, 1)$ vector of quantities
  - $\tilde{x}$ is a (K, N) matrix of payoffs
  - p is a $(N, 1)$ vector of securities prices
  - $u(\tilde{x}\theta)$ is the vector obtained by applying u component by component to the vector $\tilde{x}\theta$.

- Each column of matrix $\tilde{x}$ represents a security. Its components represent the set of payoffs of the security. The vector θ weighs the different securities.

Complete markets

- The case where there are strictly more securities than states of nature is not interesting. Indeed, this means that some securities are redundant. Since they need to have consistent prices (remember that there must be no arbitrage possibilities for the optimization program to have a solution), one can remove redundant securities. We can thus assume $N \leq K$.

- The function to be maximized is:

$$\sum_{k=1}^{K} q_k u(\sum_{i=1}^{N} \theta_i \tilde{x}_i(\omega_k)).$$

- We have the set of first order conditions ($j = 1, \ldots, N$):

$$\sum_{k=1}^{K} q_k \tilde{x}_j(\omega_k) u'(\sum_{i=1}^{N} \theta_i \tilde{x}_i(\omega_k)) = \gamma p_j,$$

  where γ is the Lagrange mutliplier of the budget constraint.

- From an economic standpoint, the securities allow to transfer wealth across states of nature. The transfer possibilities can turn out to be constrained by the set of securities. This is the case in the following example for $\tilde{x}$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

- There is no way, using the two securities, to rebalance wealth across states 1 and 2 (the first two lines of the matrix).

- There is a situation called ‘complete markets’ where the set of available transfers is unconstrained. This situation corresponds to the case where $\tilde{x}$ has full rank and $K = N$.

- In this case, for each k, I can design a portfolio which delivers one unit of wealth in state k and zero in other states. These securities are called Arrow Debreu securities.

Computing the solution

- In complete markets, we can thus change the basis of securities to use the complete set of Arrow Debreu securities.
- In this new basis of securities, the matrix $\tilde{x}$ becomes equal to the identity matrix. The first order conditions simplify greatly ($k = 1, \ldots, K$):

$$q_k u'(\theta_k) = \gamma p_k.$$

  Obviously, to rule out arbitrage, we want $p_k > 0$ for all k.
- The structure of the budget constraint does not change.
- The optimal quantities are given by:

$$\theta_k^\star = u'^{-1}(\frac{\gamma^\star p_k}{q_k}).$$

- We write $m_k = p_k/q_k$. We therefore have:

$$\theta_k^\star = u'^{-1}(\gamma^\star m_k).$$

- The vector m is called the Stochastic Discount Factor.
- The above condition is a slight modification of the condition which states that at the optimum of a consumption problem, marginal rates of substitutions (here, $u'(\theta_i)/u'(\theta_j)$) across goods/states should be equal to price ratios.
- *Marginal rates of substitutions are given by the SDF.*
- To find the solution we vary the value γ, computing the corresponding quantities implied by the first order conditions, so as to satisfy the budget constraint:

$$\sum_{i=1}^{N} p_i u'^{-1}(\gamma^\star m_i) = w_0.$$

- The assumptions on marginal utility are designed to ensure that there is a unique multiplier for all initial levels of wealth.
- Choosing the allocation to the Arrow Debreu securities is like choosing the wealth function $\tilde{w}(\omega_k)$.
- Amongst all wealth distributions which are consistent with the required marginal rates of substitutions (imposed by the SDF, i.e. imposed by the market prices and the state probabilities), we choose the one which can be financed given the initial level of wealth.
- Note that when all Arrow Debreu securities are present, cash can be synthesized as a set of holdings that deliver 1 in each state of the world.

- When cash is present and there are K states of the world, $K-1$ non redundant risky securities are needed to complete the market.

Reminder on SDFs

- We do not assume complete markets at this stage and we do not assume that we are dealing with Arrow Debreu securities. We might also have less securities than states of nature.
- The law of one price says that if two portfolios deliver the same payoff, then they must have the same price:
  - if $\tilde{x}\theta = \tilde{x}\bar{\theta}$ then $p'\theta = p'\bar{\theta}$.
- It can be restated as: if $\tilde{x}\theta = \nvdash$ (vector of zeros), then $p'\theta = 0$. This just says that if θ is orthogonal to all rows of $\tilde{x}$, then it should be orthogonal to p as well. The vector p should thus be in the linear span of the rows of $\tilde{x}$, i.e. there should be a $(K,1)$ vector λ such that $p = \tilde{x}'\lambda$.
- This means that for any achievable payoff $\tilde{x}_\theta$ we can write the corresponding price as $p_\theta = \tilde{x}_\theta'\lambda$.
- If we can build from the securities the Arrow Debreu security θ_k (it's a portfolio of the original securities) that delivers one unit in state ω_k, we get $\lambda_k = p_{\theta_k}$. Thus $\lambda_k = \lambda(\omega_k)$ can be interpreted as a state price.
- Taking $m(\omega_k) = \lambda(\omega_k)/q(\omega_k)$ (assuming all states have strictly positive probability!), we can write:

  $$p_\theta = \sum_{k=1}^{K} \lambda(\omega_k) x_\theta(\omega_k) = E_0[\tilde{m}\tilde{x}_\theta].$$

  The stochastic variable $\tilde{m}$ is the Stochastic Discount Factor.
- When the market is complete, the state prices are pinned down and the SDF is unique. Indeed, if we had two such SDFs, we could build two vectors λ and $\bar{\lambda}$ such that $\tilde{x}_\theta'\lambda = \tilde{x}_\theta'\bar{\lambda}$ or $\lambda'\tilde{x}_\theta = \bar{\lambda}'\tilde{x}_\theta$ for all portfolios θ. We would thus have $\lambda'\tilde{x} = \bar{\lambda}'\tilde{x}$ with $\lambda \neq \bar{\lambda}$ but this is however not possible since $\tilde{x}$ is invertible (we have removed redundant securities, $K = N$).
- State prices and the SDF are not uniquely determined if the market is incomplete.
- Remark: the law of one price implies that we can define a linear pricing function (the function which to portfolios associates their price) through a parameter λ. This parameter is necessarily unique when the market is complete. These results do not use the probability distribution. The probability structure gets into play when we derive the SDF.

- The value of wealth at date 0 is linked to the wealth outcome in date 1 through:
$$w_0 = \theta' p = E_0[\tilde{m}\tilde{x}_\theta] = E_0[\tilde{m}\tilde{w}].$$

- The law of one price is a weaker condition than the absence of arbitrage. The absence of arbitrage implies the strict positivity of the discount factor.

- This is intuitive: any asset with a positive pay-off in all states of the world and a strictly positive pay-off in at least one state of the world needs to have a strictly positive price. This forces the strict positivity of the SDF.

- The absence of arbitrage coupled with the assumption of complete markets implies the existence of a unique strictly positive SDF.

- In what follows, we assume the absence of arbitrage since otherwise, the optimization would not have a solution.

The risk neutral measure

- Consider the cash security. Its price is p_0 and it delivers one unit of good in all states of the world:
$$p_0 = E_0[\tilde{m}],$$
and thus:
$$1 = E_0[\tilde{m}/p_0].$$

- Consider the quantites $\bar{q}_k = \tilde{m}(\omega_k)q_k/p_0$. They obey:
$$\sum_{k=1}^{N} \bar{q}_k = 1,$$
and can thus be considered as fictitious probabilities.

- If we define $\bar{E}_0$ as the expectation with respect to the new probabilities, we have:
$$p_\theta = \bar{E}_0[\tilde{x}_\theta p_0].$$

- Since $p_0 = 1/(1 + r^f)$, under this new probability measure, we have:
$$p_\theta = \bar{E}_0[\tilde{x}_\theta/(1 + r^f)],$$
and similarly for wealth:
$$w_0 = \bar{E}_0[\tilde{w}/(1 + r^f)].$$

- This equation says that the value of assets and portfolios equal their expected payoff discounted by the risk free rate, evaluated under the modified probability measure.

- It's as if, under the modified probability measure, assets were priced by risk neutral investors. Accordingly, this new probability measure is called the *risk neutral probability measure*.
- If one redefines prices and wealth in period one by discounting them back to period 0 using the risk free rate, we find that the corresponding processes are martingales. The risk neutral measure is therefore also called the *martingale measure*.