Solution of the Exam (2018)

Guillaume Rabault

This is the solution of the exam for the 2018 ensae course.

Exercise 1

1. The program is:
$$\min_{\pi} \pi'\Sigma\pi.$$
$$\pi' e = 1,$$
$$\pi'\mu = \bar{\mu}.$$
The Lagrangian is:
$$\frac{1}{2}\pi'\Sigma\pi - \delta\pi' e - \gamma\pi'\mu.$$

2. First order condition:
$$\Sigma\pi = \delta e + \gamma\mu.$$
The condition $\gamma = 0$ corresponds to the minimum variance portfolio, since this corresponds to the soluion of:
$$\min_{\pi} \pi'\Sigma\pi.$$
$$\pi' e = 1,$$
i.e. wihtout the return constraint.

3. If $(\Sigma\bar{\pi})_i$ is the iˆth line of the vector, we have obviously:
$$(\Sigma\bar{\pi})_i = \gamma\mu_i + \gamma.$$
Now $(\Sigma\bar{\pi})_i$ is the covariance of portfolio $\bar{\pi}$ with asset i.

4. Multiplying the first order condition by π', we get:
$$\mathrm{var}(\bar{r}) = \gamma\bar{\mu} + \delta.$$
Beta is defined as:
$$\frac{\mathrm{Cov}(r_i, r_p)}{\mathrm{Var}(r_p)}.$$

We can therefore multiply and divide the left hand side of the equation established in the previous question by $\mathrm{var}(\bar{r})$:

$$\mathrm{var}(\bar{r})\beta_i = \mu_i + \frac{\delta}{\gamma},$$

which then, leads to:

$$\beta_i(\bar{\mu} + \frac{\delta}{\gamma}) = \mu_i + \frac{\delta}{\gamma}.$$

5. If we gather all the above equations using a vectorial notation, we get:

$$\beta(\bar{\mu} + \frac{\delta}{\gamma}) = \mu + \frac{\delta}{\gamma}.$$

We then multiply this by $\tilde{\pi}'$ and then use the fact that $\tilde{\pi}'\beta = 0$ to get:

$$0 = \tilde{\mu} + \frac{\delta}{\gamma},$$

$$\tilde{\mu} = -\frac{\delta}{\gamma}.$$

This provides the required result.

6. We have used the assumption $\gamma \neq 0$. As a consequence, this construct cannot be applied for the minimum variance portfolio.

7. Using the definition of β_i, we can write:

$$\mathrm{Cov}(r_i, \bar{r})(\bar{\mu} - \rho) = \mathrm{Var}(\bar{r})(\mu_i - \rho),$$

which in turn gives:

$$\mathrm{Cov}(r_i, \bar{r}) = \frac{\mathrm{Var}(\bar{r})}{\bar{\mu} - \rho}(\mu_i - \rho),$$

provided $\bar{\mu} - \rho \neq 0$. This has the form:

$$\Sigma\bar{\pi} = \gamma\mu + \delta e.$$

8. The above equation is the first order condition of certain maximization problem as set out in question 1. To identify this program, all that is needed is finding the expected return. It is however just $\bar{\pi}'\mu$. Now, since the first order condition characterizes the solution, $\bar{\pi}$ is, as desired, on the efficient frontier.

Exercice 2

1. $$\frac{dW_t}{W_t} = x_t \frac{dP_t}{P_t} + (1 - x_t)\frac{dD_t}{D_t},$$
$$\frac{dW_t}{W_t} = x_t(\mu dt + \sigma dB_t) + (1 - x_t) r dt,$$
$$\frac{dW_t}{W_t} = r dt + x_t(\mu - r)dt + \pi_t \sigma dB_t,$$
$$E_t\left[dW_t\right] = W_t(r dt + x_t(\mu - r)dt),$$
$$d[W]_t = W_t^2 x_t^2 \sigma^2 dt.$$

2. $$\frac{dW_t}{W_t} = r dt + x_t(\mu - r)dt + x_t \sigma dB_t - c_t dt,$$
where $c_t = C_t/W_t$.

3. The term that depend on C_t in the expression to be maximized is:
$$u(C_t) - J_W C_t,$$
and its maximization with respect to C_t lead to:
$$C_t = u^{'-1}(J_W).$$

4. The term that depends on x_t is:
$$J_W W_t x_t(\mu - r) + \frac{1}{2} J_{WW} W_t^2 x_t^2 \sigma^2.$$
Assuming this is a concave function in x_t, we can assume optimal investment is given by:
$$x_t^* = -\frac{J_W}{W_t J_{WW}} \frac{\mu - r}{\sigma^2}.$$

5. The first fraction on the left hand side is risk tolerance. Investment is proportional to the tagent portfolio.

6. This is standard calculus.

7. These equations are obtained by subsituting the optimal controls (consumption and investment, as function of the partial derivatives of the value function) into the expressions. The algrebra is a bit tedious but without difficulties.

8. Inject the result of 7. into HJB. The algebra is simple.

9. Injecting the above results into HJB, prove that $h(\cdot)$ solves the following differential equation:

$$\beta = \frac{1}{\alpha}\left[-\rho + (1-\alpha)r + \frac{1-\alpha}{2\alpha}\lambda^2\right],$$

the equation is:

$$h' + \beta h + 1 = 0.$$

The solution is of the form:

$$h(t) = a\exp(-\beta t) + b.$$

The differential equation and the terminal constraint determine a and b. We get:

$$h(t) = \frac{1}{\beta}\left(\exp(\beta(T-t)) - 1\right).$$