Exam (2016)

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This is the exam for the 2016 ensae course.

Exercise 1

We are in a static investment setup. The investment universe is made of cash (asset with index 0) earning a riskless rate r and N risky assets with rates of return $(\tilde{r}_i)_{i=1,\dots,N}$. The vector of risky asset risky is denoted $\tilde{\boldsymbol{r}}$ (size $(N,1)$). Its mean is $\boldsymbol{\mu}$ and its variance Σ is assumed to be positive definite (and therefore invertible).

A portfolio is a vector of proportions. The proportion invested on cash is π_0 and the vector of risky asset positions is $\boldsymbol{\pi}$ (size $(N,1)$). The vector of size $(N,1)$ with all components equal to one is denoted $\boldsymbol{e}$. A portfolio therefore satisfies:

$$\pi_0 + \boldsymbol{e}'\boldsymbol{\pi} = 1.$$

We will assume $\boldsymbol{e}'\Sigma^{-1}(\boldsymbol{\mu} - r\boldsymbol{e}) > 0$.

1. Given an excess return target $\mu_p - r > 0$, find the risky portfolio $\boldsymbol{\pi}_{\mu_p - r}$ with minimum variance amongst portfolios that achieve the return target $\mu_p - r > 0$. Let $\gamma_{\mu_p - r}$ be the lagrange multiplier in the optimization problem. Changing $\mu_{\mu_p - r}$ and therefore $\gamma_{\mu_p - r}$ allows to map the efficient frontier of the complete problem (cash plus risky asset). What is the geometrical shape of this curve in the (x=standard deviation,y=excess-return) space?
2. Show that for a certain value of μ_p, say μ^*, the optimal risky asset portfolio $\boldsymbol{\pi}^*$ is fully invested, i.e. $\boldsymbol{e}'\boldsymbol{\pi}^* = 1$.
3. What is the definition of the risky assets efficient frontier?
4. Why is the risky assets efficient frontier always below the efficient frontier of the complete problem? Describe the overall geometric situation, including the relative position of the different curves at $\boldsymbol{\pi}^*$.
5. Show that:
$$\boldsymbol{\pi}_{\mu_p - r} = \frac{\mu_p - r}{\mu^* - r}\boldsymbol{\pi}^*.$$

6. What is the interpretation of the quantity $\Sigma \boldsymbol{\pi}_{\mu_p - r}$? Consider the first order condition satisfied by the Lagrangian at the optimum of the problem described in question 1. What is its marginalist interpretation?

7. Using the proportionality condition established in question 5, show that the first order condition for an investor with excess return target $\mu_p - r > 0$ is in fact equivalent to that of an investor with excess return target $\mu^* - r$. As a consequence, all investors actually apply the same trade-off?

8. Show that:
$$\mu_i - r = \beta(\tilde{r}_i, \tilde{r}^*)(\mu^* - r),$$
with $\tilde{r}^* = \boldsymbol{\pi}^{*\prime}\tilde{\boldsymbol{r}}$. Give the explicit value of $\beta(\tilde{r}_i, \tilde{r}^*)$ as well as its interpretation.

9. Suppose the economy is composed of mean-variance investors who share the same return and risk expectations but have different levels of risk aversion. The risky asset supply is assumed exogenous. It must be held by the set of investors. The overall portfolio is called the market portfolio. Using the above results, explain how the excess return of individual assets relate to the excess return of the market portfolio.

Exercice 2

The setup is that of the previous exercise. Assumptions on risk and expected returns are specialized further. Tildas are used to denote random variables.

Assume the risky asset returns satisfy:

$$\tilde{r}_i = \mu_i + \sigma_i \tilde{\epsilon}_i,$$

where $(\tilde{\epsilon}_i)_{i=1,\ldots,N}$ are i.i.d., with zero mean and variance equal to 1. As usual, the cash return is denoted r.

Within questions 1 to 4, portfolios are assumed to be fully invested on risky assets (i.e. $\boldsymbol{e}'\boldsymbol{\pi} = 1$).

1. Determine the vector of weights $\boldsymbol{\pi}^{mv}$ of the minimum variance portfolio.

2. For a given portfolio $\boldsymbol{\pi}$, the profile of sensivities to risk is the vector of formal derrivatives of $\boldsymbol{\pi}'\tilde{\boldsymbol{r}}$ with respect to $(\tilde{\epsilon}_i)_{i=1,\ldots,N}$. Compute the profile of sensitivities of $\boldsymbol{\pi}^{mv}$.

3. A portfolio is more diversified if its sensitivity profile constant. I note $\boldsymbol{\pi}^{ew}$ the portfolio with a constant sensitivity profile. Compute $\boldsymbol{\pi}^{ew}$ and give its variance.

4. Under what condition do we have $\boldsymbol{\pi}^{mv} = \boldsymbol{\pi}^{ew}$? Why is $\boldsymbol{\pi}^{mv}$ usually underdiversified?

I now consider portfolios that are invested on all assets, i.e. including cash ($\pi_0 + \boldsymbol{e}'\boldsymbol{\pi} = 1$). In this context, risk can be reduced witout limit by reducing the weight of risky assets ($\boldsymbol{e}'\boldsymbol{\pi}$).

5. Assuming $\mu_i - r = \lambda\sigma_i > 0$ (the risk structure is unchanged), find the efficient portfolio with target excess return $\mu_p - r > 0$. Give its sensitivity profile and comment.

We now consider a more general condition where $\text{Cov}(\tilde{r}_i, \tilde{r}_j) = \rho\sigma_i\sigma_j,\ i \neq j$ with ρ neither 0 nor 1 (homogenous correlations). The assumption $\mu_i - r = \lambda\sigma_i > 0$ is maintained.

6. Find the efficient portfolio with target excess return $\mu_p - r > 0$. Give its sensitivity profile and comment. Compare with question 5.

For a given portfolio π, I note $\sigma_p(\pi)$ its standard deviation. The derivative $\partial\sigma_p/\partial\pi_i$ is called the marginal contribution to risk of position π_i. The total contribution to risk of position π_i is defined as $\pi_i\partial\sigma_p/\partial\pi_i$.

7. In the context of mean-variance optimization (with cash and risky assets) and a target excess return, what relation is satisfied by the marginal contribution to risk (use the first order condition).

8. Show that the total contributions to risk sum to the standard error:
$$\sum_{i=1}^{N} \pi_i \frac{\partial\sigma_p}{\partial\pi_i} = \sigma_p.$$

9. Show that in the context of question 6, total contributions to risk are equal.

10. Some people propose to build portfolios by equalizing total contributions to risk. What do you think of this principle? Is this a credible diversification principle?

Exercice 3

We have a multivariate Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ of size N. Its covariance matrix is the identity.

There are N risky assets with dynamics given by (for $i = 1, \ldots, N$):
$$d\log(P_{i,t}) = \gamma_{i,t}dt + \xi_{i,t}dB_t,$$
where $\xi_{i,t}$ is of size $(1, N)$. The covariance matrix of the risky assets is assumed invertible.

All portfolios are assumed fully invested on risky assets, with all weights strictly positive. It is given by a set of weights $\pi_t = (\pi_{i,t})_{i=1,\ldots,N}$. The value process is $V_{\pi,t}$. Its dynamics is:
$$d\log(V_{\pi,t}) = \gamma_{\pi,t}dt + \xi_{\pi,t}dB_t.$$

The market is composed of one unit of each asset i. The total value of the market is thus:

$$V_{\mu,t} = \sum_{i=1}^{N} P_{i,t}.$$

It is the value of a portfolio characterized by weights $\mu_t = (\mu_{i,t})_{i=1,\ldots,N}$ with $\mu_{i,t} = P_{i,t}/V_{\mu,t}$. We will assume that all these weights are strictly positive for almost all price trajectories (i.e. don't shy away from dividing by the weights).

1. Give the diffusions followed by $dP_{i,t}/P_{i,t}$ and $dV_{\pi,t}/V_{\pi,t}$. Their respective drifts will be noted $r_{i,t}$ et $r_{\pi,t}$.

2. Show that:

$$d\log(V_{\pi,t}) = \sum_{i=1}^{N} \pi_{i,t} d\log(P_{i,t}) + \gamma^*_{\pi,t} dt,$$

   and give the expression for $\gamma^*_{\pi,t}$.

3. Suppose we have another portfolio $\eta_t = (\eta_{i,t})_{i=1,\ldots,N}$, with value $V_{\eta,t}$, show that:

$$d\log(V_{\pi,t}/V_{\eta,t}) = \sum_{i=1}^{N} \pi_{i,t} d\log(P_{i,t}/V_{\eta,t}) + \gamma^*_{\pi,t} dt.$$

4. Show that:

$$\gamma^*_{\pi,t} dt = -\sum_{i=1}^{N} \pi_{i,t} d\log(P_{i,t}/V_{\pi,t}).$$

5. Show that:

$$\gamma^*_{\pi,t} dt \geq 0.$$

6. Show that:

$$d\log(V_{\pi,t}/V_{\mu,t}) = \sum_{i=1}^{N} \pi_{i,t} d\log(\mu_{i,t}) + \gamma^*_{\pi,t} dt.$$

7. We now choose for π the equally weighted portfolio $\pi_{i,t} = 1/N$.Given a vector $x = (x_i)_{i=1,\ldots,N}$ with strictly positive components, I note:

$$S(x) = \frac{1}{N} \sum_{i=1}^{N} \log(x_i).$$

   Show that $S(\cdot)$ is concave, has a maximum is π, and tends to $-\infty$ towards the edges of the simplex ($\{x \in \mathbb{R}^N_+ | \forall i, x_i > 0, \sum_{i=1}^{N} x_i = 1\}$).

8. Supposing $\int_0^t \gamma^*_{\pi,u} du$ is negligible, comment on the performance of π relative to μ as a fonction of the evolution of $S(\cdot)$ between 0 et t. If portfolio μ is equally weighted at date 0, can portfolio π outperform the market portfolio?

9. Assuming that we can ensure that $S(\mu_t) \geq B$ (B est une constante) and $\gamma^*_{\pi,t} \geq \epsilon > 0$ on almost all price trajectories, show that the equally weighted portfolio outperforms the market portfolio almost surely in finite time.