

Ito Calculus

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This post introduces the Ito rule of stochastic calculus. This rule provides the semimartingale decomposition of a nonlinear (this being the non trivial case) function of given semimartingales. Its simplest incarnation is the product rule of stochastic calculus which gives the semimartingale decomposition of the product of two given semimartingales.

Ito calculus for simple polynomial expressions

All semimartingales considered below are assumed continuous.

In this post (I will use the same notation), we established the following algebraic identity :

$$X_t^2 - X_0^2 = \sum_{i=1}^k 2X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) + \sum_{i=1}^k (X_{t_i} - X_{t_{i-1}})^2,$$

along a grid $\pi_{[0,t]}$ satisfying $t_0 = 0, t_k = t$. Applying the polarization identity¹ we can derive :

$$\begin{aligned} X_t Y_t - X_0 Y_0 &= \sum_{i=1}^k Y_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) + \sum_{i=1}^k X_{t_{i-1}}(Y_{t_i} - Y_{t_{i-1}}) + \\ &\quad \sum_{i=1}^k (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}). \end{aligned}$$

We will concentrate on this last relationship since it contains the other one as a special case.

The terms on the right hand side have the following behavior as a function of the nature of $(X_t)_{t \in [0,T]}$ when the grid gets refined:

- If $(X_t)_{t \in [0,T]}$ and $(Y_t)_{t \in [0,T]}$ are continuous local martingales, the last term converges to the cross variation $([X, Y]_t)_{t \in [0,T]}$ and the other terms

¹ $ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]$.

converge to stochastic integrals. We have :

$$X_t Y_t - X_0 Y_0 = \int_0^t X_u dY_u + \int_0^t Y_u dX_u + [X, Y]_t. \quad (1)$$

- If $(X_t)_{t \in [0, T]}$ and $(Y_t)_{t \in [0, T]}$ are continuous bounded variation processes, the last term converges to² 0 and the first two terms converge towards the obvious Lebesgue Stieltjes integrals. Thus :

$$X_t Y_t - X_0 Y_0 = \int_0^t X_u dY_u + \int_0^t Y_u dX_u. \quad (2)$$

We can also study what happens when one of the process is a local continuous martingale, say $(X_t)_{t \in [0, T]}$ while the other one, say $(Y_t)_{t \in [0, T]}$, is continuous bounded variation process. In this case, one can make use of the following identity :

$$X_t Y_t - X_0 Y_0 = \sum_{i=1}^k X_{t_i} (Y_{t_i} - Y_{t_{i-1}}) + \sum_{i=1}^k Y_{t_{i-1}} (X_{t_i} - X_{t_{i-1}}).$$

The dominated convergence theorem of Lebesgue Stieltjes integration shows that the first term converges to the Lebesgue Stieltjes integral³ $(\int_0^t X_u dY_u)_{t \in [0, T]}$ while the second term, as usual, converges to the stochastic integral $(\int_0^t Y_u dX_u)_{t \in [0, T]}$. We thus get :

$$X_t Y_t - X_0 Y_0 = \int_0^t X_u dY_u + \int_0^t Y_u dX_u. \quad (3)$$

It is convenient to summarize these results in differential notation. The logic of the notation, illustrated on the product $X_t Y_t$ of two local martingales, is as follows. We write $X_t Y_t - X_0 Y_0 = \int_0^t d(X_u Y_u)$. Similarly, $[X, Y]_t = [X, Y]_t - [X, Y]_0 = \int_0^t d[X, Y]_u$. We can then write equation (1) as :

$$d(X_u Y_u) = X_u dY_u + Y_u dX_u + d[X, Y]_u.$$

We can wrap up the cases we saw as follows :

- $(X_t)_{t \in [0, T]}$ and $(Y_t)_{t \in [0, T]}$ are local martingales :

$$d(X_u Y_u) = X_u dY_u + Y_u dX_u + d[X, Y]_u \quad (\text{equation 1}).$$

²The cross variation of two bounded variation functions is zero. This can easily be deduced from the fact that the quadratic variation of each process is null and using the Cauchy Schwartz inequality.

³In Lebesgue Stieltjes integration, the sums $\sum_{i=1}^k X_{t_i} (Y_{t_i} - Y_{t_{i-1}})$ and $\sum_{i=1}^k X_{t_{i-1}} (Y_{t_i} - Y_{t_{i-1}})$ converge to the same result, the integral $\int_0^t X_u dY_u$. Within stochastic integration, i.e. when $(Y_t)_{t \in [0, T]}$ is a local martingale, the two sums differ. The second expression converges to the Ito integral $\int_0^t X_u dY_u$ while the first converges to $\int_0^t X_u dY_u + [X, Y]_t$. The Stratonovitch integral is defined as the limit of the average of the two weighted sums, namely $\int_0^t X_u dY_u + \frac{1}{2}[X, Y]_t$.

- $(X_t)_{t \in [0, T]}$ and $(Y_t)_{t \in [0, T]}$ are of bounded variation :

$$d(X_u Y_u) = X_u dY_u + Y_u dX_u \quad (\text{equation 2}).$$

- $(X_t)_{t \in [0, T]}$ is of bounded variation and $(Y_t)_{t \in [0, T]}$ is a local martingale :

$$d(X_u Y_u) = X_u dY_u + Y_u dX_u \quad (\text{equation 3}).$$

In the special case where X is equal to Y , we have :

- $(X_t)_{t \in [0, T]}$ is a local martingale : $dX_u^2 = 2X_u dX_u + d[X]_u$.
- $(X_t)_{t \in [0, T]}$ is of bounded variation : $dX_u^2 = 2X_u dX_u$.

From product rules to Ito calculus

We now consider two continuous semimartingales $(X_t)_{t \in [0, T]}$ and $(Y_t)_{t \in [0, T]}$ with their decompositions :

$$X_t = M_t + A_t,$$

$$Y_t = N_t + B_t,$$

into their local continuous martingale component $((M_t)_{t \in [0, T]}$ and $(N_t)_{t \in [0, T]}$) and their continuous bounded variation process $((A_t)_{t \in [0, T]}$ and $(B_t)_{t \in [0, T]}$). We can write :

$$\begin{aligned} d(X_u Y_u) &= X_u dY_u + Y_u dX_u + d[X, Y]_u \\ &= X_u dB_u + Y_u dA_u + X_u dN_u + Y_u dM_u + d[X, Y]_u, \end{aligned}$$

with $[X, Y]_u = [M, N]_u$. We note that if X or Y is actually a bounded variation process, then $[X, Y]_u = 0$.

This decomposition of $d(X_u Y_u)$ is actually the semimartingale decomposition of $(X_t Y_t)_{t \in [0, T]}$ into its local martingale component :

$$\int_0^t X_u dN_u + \int_0^t Y_u dM_u,$$

and its finite variation component :

$$\int_0^t X_u dB_u + \int_0^t Y_u dA_u + [X, Y]_t.$$

We can thus compute $d(X_u Y_u Z_u)$ for a third semimartingale $(Z_t)_{t \in [0, T]}$ by considering the product of the three semimartingales $X_u Y_u Z_u$ as a product of two semimartingales $(X_u Y_u) Z_u$. This remark leads to consider the differential of polynomial functions of a set of underlying semimartingales.

One can show by recurrence along the above lines that for any polynomial expression $F(X_{1,u}, X_{2,u}, \dots, X_{n,u})$ in n semimartingales, we have :

$$dF(X_{1,u}, \dots, X_{n,u}) = \sum_i F'_i(X_{1,u}, \dots, X_{n,u})dX_{i,u} + \frac{1}{2} \sum_{i,j} F''_{i,j}(X_{1,u}, \dots, X_{n,u})d[X_i, X_j]_u.$$

The reader can check that this works for all simple polynomials considered in the first section, and that if it works for two polynomial expressions, it works for their product.

What then remains to be seen is whether this can be applied to general twice continuously differentiable functions. This is indeed the case. The key step in the proof of this result consists in restricting the semimartingales to compact subspaces through localization, and then approximate the function with polynomials. In a compact domain, a twice continuously differentiable function together with its first and second derivatives can be arbitrarily closely approximated (in the sense of uniform convergence) by a polynomial.

To conclude, let's state Ito's formula precisely :

Theorem (Ito's Formula): *Consider n continuous semimartingales $(X_{1,t}, \dots, X_{n,t})$ with t in $[0, T]$ and a twice continuously differentiable function of n variables $F : \mathbb{R}^n \rightarrow \mathbb{R}$. Then we have with probability one for all u in $[0, T]$:*

$$dF(X_{1,u}, \dots, X_{n,u}) = \sum_i F'_i(X_{1,u}, \dots, X_{n,u})dX_{i,u} + \frac{1}{2} \sum_{i,j} F''_{i,j}(X_{1,u}, \dots, X_{n,u})d[X_i, X_j]_u,$$

or in integral form, for all t in $[0, T]$:

$$F(X_{1,t}, \dots, X_{n,t}) - F(X_{1,0}, \dots, X_{n,0}) = \sum_i \int_0^t F'_i(X_{1,u}, \dots, X_{n,u})dX_{i,u} + \frac{1}{2} \sum_{i,j} \int_0^t F''_{i,j}(X_{1,u}, \dots, X_{n,u})d[X_i, X_j]_u.$$

■

This theorem singles out the semimartingale decomposition of the process $(F(X_{1,t}, \dots, X_{n,t}))_{t \in [0, T]}$. If we decompose each semimartingale $(X_{i,t})_{t \in [0, T]}$ into its bounded variation component $(A_{i,t})_{t \in [0, T]}$ and its local martingale component

$(M_{i,t})_{t \in [0,T]}$, then the bounded variation component of $(F(X_{1,t}, \dots, X_{n,t}))_{t \in [0,T]}$ is :

$$\sum_i \int_0^t F'_i(X_{1,u}, \dots, X_{n,u}) dA_{i,u} + \frac{1}{2} \sum_{i,j} \int_0^t F''_{i,j}(X_{1,u}, \dots, X_{n,u}) d[X_i, X_j]_u,$$

while the local martingale component is:

$$\sum_i \int_0^t F'_i(X_{1,u}, \dots, X_{n,u}) dM_{i,u}.$$

Note : This post takes its inspiration from section IV.32 of Rogers[2000].

Reference: Rogers L.C.G. and D. Williams[2000] : *Diffusions, Markov Processes and Martingales, Vol 2, Ito Calculus*, Cambridge University Press.