Solution of the Exam (2015)

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This is the solution of the exam for the 2015 ensae course.

Exercise 1

Part 1

1. $\sum_{i=1}^{N} \pi_i = 1$.
2. $W_1 = W_0 R_p$ with $R_p = \sum_{i=1}^{N} \pi_i R_i$ or $r_p = \sum_{i=1}^{N} \pi_i r_i$.
3. $\mu_p = \sum_{i=1}^{N} \pi_i \mu_i = \pi' \mu$ and $\sigma_p^2 = \pi' \Sigma \pi$.
4. $$\min_{\pi} \pi' \Sigma \pi$$
   $$\text{s.t.}: \quad \pi' e = 1.$$
   Lagrangian: $\frac{1}{2}\pi'\Sigma\pi - \gamma(\pi' e - 1)$. First order condition: $\Sigma\pi^1 = \gamma e$. Value of the Lagrange multiplier and solution:
   $$\gamma = \frac{1}{e'\Sigma e}, \; \pi^1 = \frac{\Sigma^{-1} e}{e'\Sigma e}.$$
   All risky asset portfolios which are efficient must have a rate of return in excess of:
   $$\mu^1 = \frac{\mu'\Sigma^{-1} e}{e'\Sigma e} = \frac{e'\Sigma^{-1}\mu}{e'\Sigma e}.$$
5. $$\min_{\pi} \pi' \Sigma \pi$$
   $$\text{s.t.}: \quad \pi' e = 1, \; \pi'\mu = \mu_p.$$
   Lagrangian: $\frac{1}{2}\pi'\Sigma\pi - \gamma(\pi' e - 1) - \delta(\pi'\mu - \mu_p)$. First order condition: $\Sigma\pi^p = \gamma e + \delta\mu$ or $\pi^p = \gamma\Sigma^{-1} e + \delta\Sigma^{-1}\mu$. The linear system which determines the multipliers comprises two equations which correspond to the constraints in which the value of π^p has been injected:
   $$\delta\mu'\Sigma^{-1}\mu + \gamma\mu'\Sigma^{-1} e = \mu_p,$$
   $$\delta e'\Sigma^{-1}\mu + \gamma e'\Sigma^{-1} e = 1.$$

6. The determinant of the linear system is: $(\mu'\Sigma^{-1}\mu)(e'\Sigma^{-1}e) - (\mu'\Sigma^{-1}e)^2$. Under the conditions quoted in the text, it is strictly positive.
7. The linear system has a unique solution for each μ_p. If γ is zero, we must have $\delta = \frac{1}{e'\Sigma^{-1}\mu}$. As a consequence:
$$\mu^\mu = \frac{\mu'\Sigma^{-1}\mu}{e'\Sigma^{-1}\mu}.$$
We have:
$$\pi^\mu = \frac{\Sigma^{-1}\mu}{e'\Sigma^{-1}\mu}.$$
8. The formula for π^p can be written:
$$\gamma(e'\Sigma^{-1}e)\frac{\Sigma^{-1}e}{e'\Sigma^{-1}e} + \delta(\mu'\Sigma^{-1}\mu)\frac{\Sigma^{-1}\mu}{\mu'\Sigma^{-1}\mu}.$$
Thus:
$$\gamma(e'\Sigma^{-1}e)\pi^1 + \delta(\mu'\Sigma^{-1}\mu)\pi^\mu.$$
We have:
$$\gamma(e'\Sigma^{-1}e) + \delta(\mu'\Sigma^{-1}\mu) = \lambda + (1-\lambda) = 1,$$
from the linear system, using $\lambda = \gamma(e'\Sigma^{-1}e)$. When the required rate of return increases from μ^1 to μ^μ, λ decreases from 1 to 0. The Lagrange multiplier of the budget constraint decreases until it reaches 0. When the required rate of return exceeds μ^μ, the solution consists in selling the minimum variance portfolio so as to leverage π^μ. The Lagrange mutliplier of the budget constraint measures the price of the constraint. The price is high for the minimum variance portfolio because lowering the sum of weights below one would obviously allow to lower the variance further. When a rate of return constraint is imposed ($\mu_p > \mu^1$), the latter constraint prevents from lowering the variance. The price of the budget constraint comes down.
9. The linear system delivers the Lagrange multipliers as a function of $A = \mu'\Sigma^{-1}\mu, B = \mu'\Sigma^{-1}e, C = e'\Sigma^{-1}e$:
$$\sigma_p^2 = \frac{A - 2B\mu_p + C\mu_p^2}{AC - B^2}.$$
The efficient frontier is the trace in the half plane $\sigma_p > 0$ of the geometric locus of the quadratic equation:
$$C\mu_p^2 - (AC - B^2)\sigma_p^2 - 2B\mu_p + A = 0.$$
This defines a hyperbola (see conic sections).
10. Two distinct efficient portfolios:
$$\pi_{p_1} = \lambda_{p_1}\pi^1 + (1-\lambda_{p_1})\pi^\mu,$$
$$\pi_{p_2} = \lambda_{p_2}\pi^1 + (1-\lambda_{p_2})\pi^\mu.$$
The matrix that maps π^1 and π^μ to the new efficient portfolios is invertible as soon as $\lambda_{p_1} \neq \lambda_{p_2}$.

Part 2

10. The Sharpe ratio of a portfolio is inherited from the risky asset subportfolio. Take two portfolios. They have risky asset subportfolios. If the two subportfolios are proportional to one another, the original portfolios have the same Sharpe ratio. Indeed, the Sharpe ratio of a portfolio (π_0, π) satisfies:
$$\frac{\pi'(\mu - r^f)}{Vol(\pi'\tilde{r})} = \frac{\theta\pi'(\mu - r^f)}{Vol(\theta\pi'\tilde{r})},$$
for any real number $\theta > 0$. By adjusting the cash level and changing in a proportionate way the risky asset portfolio (so as to preserve $\pi_0 + e'\pi = 1$), one can reach any level of volatility without changing the Sharpe ratio of the initial portfolio. One can thus 'port' a given Sharpe ratio over the whole volatility range. Thus, starting from two portfolios, P_1 and $P_{2} $with distinct Sharpe ratios, we can choose the one with the highest Sharpe ratio (say P_1), adjust its cash holding and risky asset subportfolio (in a proportionate way) so as to build a new portfolio which matches the volatility of P_2, and has the Sharpe ratio of P_1. Thus P_2 cannot be mean variant efficient if it has a Sharpe ratio that is strictly lower that that of P_1. From a geometric standpoint, the efficient frontier is a half-line in the standard deviation/expected return space. It goes through the point $(0, r^f)$ and through a point on the risky asset efficient frontier.

11. Consider the half-lines which satisfy the criteria established in the previous questions. These half-lines can be ordered because for any pair of such distinct half-lines, one is necessarily above the other. The ranking metric is the Sharpe ratio. The upper enveloppe of the set of half-lines is the efficient frontier $\mathcal{F}$. It is necessarily tangent to the risky asset efficient frontier $\mathcal{E}$. To see this, one has to realize that the efficient frontier $\mathcal{E}$ is vertical at (μ^1, σ^1), that it is concave: its slope in the (σ, μ) space is decreasing in σ; finally this slope converges to zero asymptotically. In this configuration, the half-line which maximizes the Sharpe ratio has to intersect the efficient frontier $\mathcal{E}$ at an 'interior' point, at a point with volatility σ satisfying $\sigma^1 < \sigma < \infty$.

12. The optimization program reads:
$$\min_{\pi} \pi'\Sigma\pi$$
$$\text{s.t.}: \quad \pi_0 + \pi'e = 1, \; \pi'(\mu - r^f) = \mu_p - r^f.$$
This is equivalent to: $\mathcal{P}(\mu_p - r^f)$:
$$\min_{\pi} \pi'\Sigma\pi$$
$$\text{s.t.}: \quad \pi'(\mu - r^f) = \mu_p - r^f,$$
imposing $\pi_0 = 1 - \pi'e$.

13. If π is a portfolio that satisfies the contraint $\pi'(\mu - r^f) = \mu_{p,1} - r^f$, $\hat{\pi} = \theta\pi$ with $\theta = (\mu_{p,2} - r^f)/(\mu_{p,1} - r^f)$ satisfies $\hat{\pi}'(\mu - r^f) = \mu_{p,2} - r^f$. $\mathcal{P}(\mu_{p,1} - r^f)$ can be written:

$$\min_{\pi} \theta^2 \pi' \Sigma \pi$$

$$\text{s.t.}: \quad \pi'(\mu - r^f) = \mu_{p,1} - r^f,$$

since θ is a constant. Thanks to this transformation, this program becomes:

$$\min_{\hat{\pi}} \hat{\pi}' \Sigma \hat{\pi}$$

$$\text{s.t.}: \quad \hat{\pi}'(\mu - r^f) = \mu_{p,2} - r^f,$$

which is just $\mathcal{P}(\mu_{p,2} - r^f)$. Thus if π^* is the solution of $\mathcal{P}(\mu_{p,1} - r^f)$, $\theta\pi^*$ is the solution of $\mathcal{P}(\mu_{p,2} - r^f)$. This works backwards as well. If the two corresponding optimal portfolios are proportional via θ, their excess-returns and their volatility are both multiplied by the same factor. As a consequence, the Sharpe ratio is conserved. It is maximized at the optimum since the program minimizes variance and thus volatility for a given excess return. The programs $\mathcal{P}$ identify the unique portfolio of maximum Sharpe ratio. This portfolio corresponds to point I.

Exercise 2

1. I assume $\gamma <> 1$, the log case being very similar. The dynamics of the wealth process is:

$$\frac{dW_u}{W_u} = r^f du + \pi_u'(\mu - r^f e)du + \pi_u' \sigma dB_u.$$

It is a geometric process. If W_t is multiplied by θ, the value of W_T is multiplied by θ as well **for any trajectory of the Brownian motion and all portfolio choices** $\pi_{[t,T]}$. The criterion to be maximized for the starting point $\hat{W}_t = \theta W_t$ is thus $E_t[u(\theta W_T)] = \theta^{1-\gamma} E_t[u(W_T)]$ for $\gamma \neq 1$. The investment process that is optimal for W_t still is for the initial condition $\hat{W}_t$ and $V(t, \hat{W}_t) = \theta^{1-\gamma} V(t, W_t)$.

2. We have:

$$V(t+\Delta, W_{t+\Delta}) - V(t, W_t) =$$

$$\int_t^{t+\Delta} [V_1(u, W_u)du + V_2(u, W_u)dB_u] + \frac{1}{2}\int_t^{t+\Delta} V_{2,2}(u, W_u)d[B]_t.$$

Taking the expectation:

$$\int_t^{t+\Delta} [V_1(u, W_u)du + V_2(u, W_u)W_u(r^f du + \pi_u'(\mu - r^f e)du)] +$$

$$\frac{1}{2}\int_t^{t+\Delta} V_{2,2}(u, W_u)W_u^2 \pi_u' \Sigma \pi_u du.$$

3. We have:

$$\max_{\pi_t} \left\{ V_1(t,W_t)+V_2(t,W_t)W_t(r^f+\pi_t'(\mu-r^f e))+\frac{1}{2}V_{2,2}(t,W_t)W_t^2\pi_t'\Sigma\pi_t \right\} = 0.$$

4. The function to be maximized:

$$V_2(t,W_t)W_t\pi_t'(\mu-r^f e)+\frac{1}{2}V_{2,2}(t,W_t)W_t^2\pi_t'\Sigma\pi_t.$$

   Since V is concave in W_t, the second derivative is negative and the maximimum of the quadratic form is characterized by the first order condition:

$$\pi_t^* = -\frac{V_2(t,W_t)}{W_t V_{2,2}(t,W_t)}\Sigma^{-1}(\mu-r^f e).$$

   Given question 1, on have:

$$-\frac{V_2(t,W_t)}{W_t V_{2,2}(t,W_t)} = \frac{1}{\gamma}.$$

   The solution depends neither on t nor on T.

5. We have:

$$\Sigma\pi^\star = \frac{1}{\gamma}(\mu-r^f e),$$

   which gives the first equation. We have as well:

$$\sigma\sigma'\pi^\star = \frac{1}{\gamma}(\mu-r^f e),$$

$$\sigma'\pi^\star = \frac{1}{\gamma}\lambda,$$

$$\pi^{\star\prime}\sigma = \frac{1}{\gamma}\lambda',$$

$$\pi^{\star\prime} = \frac{1}{\gamma}\lambda'\sigma^{-1},$$

   and this gives the second equation by multiplying by $(\mu-r^f e)$ on the right.

6.
$$d\log(W_u^{\gamma,\star}) = r^f du + (1-\frac{1}{2\gamma})\frac{1}{\gamma}\kappa^2 du + \frac{1}{\gamma}\lambda' dB_u,$$

$$W_T^{\gamma,\star} = W_0 \exp\left(r^f T + (1-\frac{1}{2\gamma})\frac{1}{\gamma}\kappa^2 T + \frac{1}{\gamma}\lambda' B_T \right).$$

7. $$\frac{\log(W_T^{\gamma,*}) - \log(W_1)}{T} =$$
$$r^f + (1 - \frac{1}{2\gamma})\frac{1}{\gamma}\kappa^2 + \frac{1}{\gamma}\frac{\lambda' B_T}{T}.$$

   The last term converges almost surely to 0 when T goes to $+\infty$ (law of large numbers). The asymptotic rate of log growth reads:

$$r^f + (1 - \frac{1}{2\gamma})\frac{1}{\gamma}\kappa^2.$$

   It increases as γ rises up to 1 and then decreases beyond that point.

8. For a finite T, the level of risk taken enters the picture whereas it does not enter into account in the asymptotic ranking of growth. Preference for risk matters for finite T. The maximum growth portfolio is not optimal for $\gamma <> 1$. For $\gamma < 1$, the investor likes risk so much that he finds the max growth portfolio too safe. The investor wants to gamble more. For $\gamma > 1$, the investor finds the max growth portfolio too risky.