Exam (2015)

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This is the exam for the 2015 ensae course.

Exercise 1

Part 1: Efficient frontier, without risk free asset ($\mathcal{E}$)

The context is that of static optimization, i.e. decisions are taken in period 0 and results are observed in period 1. There are N risky assets ($i = 1, \ldots, N$) with stochastic rates of return $(\tilde{r}_i)_{i=1,\ldots,N}$. Expected rates of return are noted $(\mu_i)_{i=1,\ldots,N}$ and the covariance matrix is Σ. Σ is invertible (i.e. there are no redundant assets). The investor chooses portfolio weights $(\pi_i)_{i=1,\ldots,N}$ as a proportion of wealth W_0.

1. Which constraint applies to $(\pi_i)_{i=1,\ldots,N}$?
2. Determine the relationship between wealth in date 1 and wealth in date 0 using the system of weights. Express $\tilde{r}_p$, the rate of return of the portfolio, as a function of the rates of return of the assets $(\tilde{r}_i)_{i=1,\ldots,N}$.
3. Express the mean return and the variance of the portfolio as a function of the system of weights.
4. We are interested in the minimum variance portfolio. Define the corresponding optimization program. Derive the corresponding lagrangian and the solution of the optimization program. Its system of weights is denoted π^1 while μ^1 denotes its expected return. Can a portfolio with expected return lower than μ^1 be mean-variance efficient?
5. Consider an expected return objective $\mu_p \geq \mu^1$. We seek to minimize variance given this return objective. Define the corresponding constrained optimization program. Determine the lagrangian and the corresponding first order condition. Show that the Lagrange multipliers solve a linear system involving the following parameters:
   - the expected return of the portfolio μ_p,
   - the vector μ that consists of the expected return of the risky assets,
   - the covariance matrix of the risky assets.

We will now assume:

$$\frac{\mu'\Sigma^{-1}\mu}{e'\Sigma^{-1}\mu} > \mu^1,$$

as well as $e'\Sigma^{-1}\mu > 0$ where e is the vector of 1s of size N.

6. Show that the linear system can be inverted.
7. Show that there is a value of μ_p, μ^μ, such that the Lagrange multiplier of the budget constraint is zero. Determine the corresponding portfolio, denoted π^μ.
8. Show that in the general case ($\mu_p \geq \mu^1$), the mean-variance efficient portfolio is a linear combination of π^1 and π^μ. Determine the coefficients of this combination and show that they add to one. Explain why the Lagrange multiplier of the budget constraint decreases as we increase the return objective away from μ^1.
9. The efficient frontier is obtained as we move μ_p from μ^1 to $+\infty$. Give its parametric form in the standard deviation (x axis) expected return (y axis) plane.
10. We have just shown that any efficient portfolio can be written as a combination of π^1 and π^μ. Prove that instead, we could have taken two distinct arbitrary portfolios along the efficient frontier to describe the efficient frontier.

Partie 2: Efficient frontier with a riskless asset ($\mathcal{F}$)

We now add a riskless asset with rate of return r^f, to which we ascribe the index $i = 0$. Each vector of portfolio weights can be decomposed into its cash component π_0 and the subportfolio $\pi = (\pi_i)_{i=1,\cdots,N}$ entirely invested in risky assets (thus $\pi_0 + \pi' e = 1$).

11. Show that two optimal portfolios necessarily have the same Sharpe ratio. What is the shape of the efficient frontier $\mathcal{F}$ in the standard deviation (x axis) expected return (y axis) plane. The intersection between $\mathcal{E}$ and $\mathcal{F}$ will be denoted I.
12. Characterize the geometric position of $\mathcal{E}$ and $\mathcal{F}$ around I. What is the name of the portfolio which corresponds to I.
13. Given a return objective μ_p, spell out the constrained optimization program that defines the minimum variance portfolio. What optimization program is satisfied by the risky sub-portfolio π (this problem will be denoted $\mathcal{P}(\mu_p - r^f)$).
14. Show that two problems $\mathcal{P}(\mu^{p,1} - r^f)$ and $\mathcal{P}(\mu^{p,2} - r^f)$ with $r^f < \mu^{p,1} < \mu^{p,2}$ can be mapped to one another using a simple transformation. Show the the unique underlying problem $\mathcal{P}$ consists in finding the risky asset portfolio with the maximum Sharpe ratio. Compute the maximum Sharpe ratio. Relate this result to question 11.

Exercice 2

We consider a terminal utility of wealth problem, with term T (no consumption before T), in the continuous time setup. Terminal utility of wealth is CRRA:

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma},\ \gamma > 0,\ \gamma <> 1$$

$$u(w) = \log(w),\ \gamma = 1.$$

The market comprises a riskless asset with rate of return r^f and N risky assets. Risky asset processes are geometric diffusions ($i = 1, \ldots, N$):

$$\frac{dP_i}{P_i} = \mu_i dt + \sigma_i dB_t,$$

where B_t is a multivariate Brownian motion of dimension $(N, 1)$ and covariance matrix equal to the identity. The matrix σ_i is of size $(1, N)$. We denote by μ the vector with components μ_i i.e. the expected returns of the risky assets, and by σ the matrix obtained by stacking the 'lines' σ_i ($i = 1, \ldots, N$). We assume σ is invertible, and put $\Sigma = \sigma\sigma'$. The vector of size $(N, 1)$ with all components equal to 1 is denoted e. As a simplification, we assume all diffusion coefficients are constant, i.e. independent of time.

The value function $V(t, W_t)$ is the solution to the problem:

$$V(t, W_t) = \max_{\pi_{[t,T]}} E_t\Big[u(W_T)\Big]$$

$$\text{s.t.}: \quad \frac{dW_u}{W_u} = r^f du + \pi'_u(\mu - r^f e)du + \pi'_u \sigma dB_u.$$

The variable $\pi_{[t,T]}$ refers to the collection of risky assets portfolios π_τ for $\tau \in [t, T]$ (π_t is the vector of the portfolio positions on risky assets at time t). The principle of dynamic programming says that ($\Delta > 0,\ t + \Delta \leq T$):

$$V(t, W_t) = \max_{\pi_{[t,t+\Delta]}} E_t[V(t + \Delta, W_{t+\Delta})].$$

1. Show that the value function is homogenous of degree $1 - \gamma$ in W for $\gamma \neq 1$ and additive in $\log(W)$ for $\gamma = 1$.
2. Using Ito's lemma, decompose $E_t[V(t + \Delta, W_{t+\Delta}) - V(t, W_t)]$.
3. Derive the infinitesimal version of the dynamic programming equation, taking the limit $\Delta \to 0$ (HJB).
4. Using the first order condition of the infinitesimal optimization problem (HJB), derive the expression of the optimal portfolio $\pi_t^\star$ as a function of the partial derivatives of the value function (the use of the first order condition will be justified). Question 1 will be used to then get rid of the partial derivatives of V. Does the optimal portfolio depend on t or T?

5. Prove that:
$$\pi^{\star\prime}\Sigma\pi^\star = \frac{1}{\gamma}\pi^{\star\prime}(\mu - r^f e),$$
$$\pi^{\star\prime}(\mu - r_f e) = \frac{1}{\gamma}\kappa^2,$$
with $\lambda = \sigma^{-1}(\mu - r_f e)$ and $\kappa^2 = \lambda\lambda'$.
6. For each γ, determine the integral formula for $W_T^{\gamma,\star}$ as a fonction of initial wealth W_0 when the optimal policy is used.
7. Determine and then rank (as a function of γ) the limits $(\log(W_T^{\gamma,*}) - \log(W_1))/T$ (log growth). For which value of γ is this quantity maximized? This quantity will be denoted $\gamma^\star$. The corresponding optimal policy will be called the maximum growth investment policy.
8. We get back to a finite investment horizon T. Is the maximum growth investment policy determined in the previous question optimal for an investor with $\gamma \neq \gamma^*$? Explain.