Foreword

References

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- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

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  - the assumptions on the probability space underlying the models; one should in particular be familiar with Markovianity.
  - the optimization requirements (convexity assumptions needed for first order conditions to be not only necessary but also sufficient, dynamic programming requirements)
- The continuous time setup in particular would require a lot of additional material to become self contained.

The static model

Timing

- Two periods:

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    - x: particular outcome; $\tilde{x}$: random variable

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- I'll assume there are N risky assets ($i = 1, \cdots, N$) and potentially cash (the riskless asset), which will then have index 0
- The set of assets will be denoted by $\mathcal{I}$, with either $\mathcal{I} = (1, \cdots, N)$ (no riskless asset) or $\mathcal{I} = (0, \cdots, N)$ (with a riskless asset)

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- The rate of return of cash is usually denoted r^f; it is known as of date 0

Investment and returns

$t = 0$		$t = 1$
ϕ	$\rightarrow$	$\tilde{R}\phi$
ϕ	$\rightarrow$	$(1 + \tilde{r})\phi$

Table 1:From investment to payoff

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  - wealth shares $(\pi_i)_{i\in\mathcal{I}}$, with $\pi_i = \phi_i / w_0$

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- Wealth shares:

$$\sum_{i \in \mathcal{I}} \pi_i = 1$$

Borrowing

- Borrowing is best understood as a negative position in cash:

$t = 0$		$t = 1$
$\phi = -d$	$\rightarrow$	$-d(1 + r^f)$

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  - portfolio rate of return:

$$\tilde{r}_p = \frac{\tilde{w}}{w_0} = \sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i$$

(since $\sum_{i \in \mathcal{I}} \pi_i = 1$)

The space of excess returns

- In the presence of a riskless asset, it is convenient to introduce excess returns versus the riskless rate:

$$\begin{aligned}\tilde{r}_p &= \sum_{i\in\mathcal{I}} \pi_i \tilde{r}_i \\ &= r^f + \sum_{i=1}^{N} \pi_i (\tilde{r}_i - r^f)\end{aligned}$$

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- The choice variables are initially $(\pi_i)_{i\in\mathcal{I}}$, under the constraint $\sum_{i\in\mathcal{I}} \pi_i = 1$. In the excess return space, the choice variables are $(\pi_i)_{i=1}^{N}$ to which no budget constraint applies since it is enforced by $\pi_0 = 1 - \sum_{i=1}^{N} \pi_i$.

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  - the utility function embodies attitudes towards risk of the decision maker

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- Reminder: an arbitrage is a way to generate a strictly positive pay-off without committing any funds
- The existence of a solution to a portfolio optimization problem thus guarantees the existence of a stochastic discount factor (see below). We will see this principle in action in the case of mean-variance efficiency.

Arbitrage, the law of one price and SDF

- A stochastic discount factor is a random variable $\tilde{m}$ such that for any pay-off $\tilde{x}$, the market price can be recovered:

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- In the presence of a risk free asset, this implies:

$$E[\tilde{R}] - r^f = -r^f \operatorname{cov}(\tilde{m}, \tilde{R}),$$

which describes the structure of risk premia across assets as a result of the covariances with the SDF.

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- Additive certainty equivalent: for a centered distribution $\tilde{\varepsilon}_a$ and an initial level of wealth w, find $\pi_a(w, \tilde{\varepsilon}_a)$ such that:

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- Multiplicative certainty equivalent: for a centered distribution $\tilde{\varepsilon}_m$ and an initial level of wealth w, find $\pi_m(w, \tilde{\varepsilon}_m)$ such that:

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  - range $\mathbb{R}_+^*$
  - relative risk aversion: $\rho(w) = \rho$

CRRA utility functions - fig 1

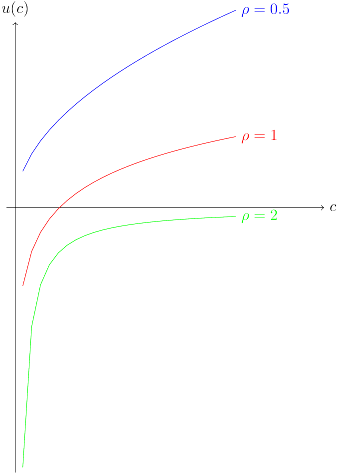

Figure 1:CRRA utility functions

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- For example, CRRA models require $\tilde{R} \geq 0$ i.e. $\tilde{r} \geq 1$. This assumption is sometimes called 'limited liability': the owner of an asset cannot end up having to transfer cash to the issuer.
- This is a problem mainly for discrete time models

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  - intuition suggests people accept greater dollar risk as their wealth rises

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- I assume that there is no labor come
- $\boldsymbol{\pi} = (\pi_i)'_{i \in \mathcal{I}}$

$$\max_{\boldsymbol{\pi}} E[-\exp(-\alpha \tilde{w})]$$

$$\text{s.t. } \tilde{w} = w_0 \sum_{i \in \mathcal{I}} \pi_i \tilde{R}_i$$

$$\sum_{i \in \mathcal{I}} \pi_i = 1$$

CARA normal case (1)

- The random variable $\tilde{w}$ is normally distributed. In this case, we know that:

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- Given that the function $u(\cdot)$ is increasing, the program consists in maximizing the certainty equivalent $E[\tilde{w}] - (\alpha/2)V[\tilde{w}]$, which reads, mean wealth minus the variance of wealth weighted by one half absolute risk aversion.

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- The maximized criterion is thus:

$$E[\sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i] - (\alpha w_0^2/2) V[\sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i].$$

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    - if this was not the case, optimal portfolio composition would be independent on the wealth level; this would imply that the investor take more dollar risk at higher wealth levels; in the CARA case, the appetite for dollar risk is independent of the level of wealth; thus the correction.

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  - in the presence of stochastic labour income, mean variance needs to be amended

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- In the presence of normally distributed stochastic labor income, the optimal programme is:

$$\max_{\boldsymbol{\pi}} E[-\exp(-\alpha\tilde{w})]$$

$$\text{s.t. } \tilde{w} = \tilde{y} + \sum_{i\in\mathcal{I}} \theta_i \tilde{x}_i$$

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- It is this time more convenient to take as control variables the quantities: $(\theta_i)_{i\in\mathcal{I}}$.

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- As before, we need to maximize the certainty equivalent: $E[\tilde{w}] - (\alpha/2)V[\tilde{w}]$. This is equivalent to maximizing:

$$E\left[\sum_{i \in \mathcal{I}} \theta_i \tilde{x}_i\right] - (\alpha/2)V\left[\tilde{y} + \sum_{i \in \mathcal{I}} \theta_i \tilde{x}_i\right].$$

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- We can decompose the variance term as:

$$V\left[\tilde{y}\right] + V\left[\sum_{i\in\mathcal{I}}\theta_i\tilde{x}_i\right] + 2\mathrm{Cov}\left(\sum_{i\in\mathcal{I}}\theta_i\tilde{x}_i, \tilde{y}\right).$$

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  - $\mathrm{Cov}(\tilde{x}, \tilde{y})$ is the $N \times 1$ vector where each entry measures the covariance of a financial instrument with labour income
  - $E[\tilde{x}]$ is the $N \times 1$ vector of the expected pay-offs of the financial instruments.

CARA normal case (7)

- The first order condition leads to, in matrix notation:

$$\theta = V[\tilde{x}]^{-1}\left(-\mathrm{Cov}(\tilde{x},\tilde{y}) + \frac{1}{\alpha}E[\tilde{x}]\right).$$

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- Remember that $1/\alpha$ is risk tolerance.
- The structure of the solution is as follows: the optimal porfolio consists of a hedging portfolio (which tries to replicate income variability using financial assets) and a speculative portfolio which has the same structure as in the case without labour income. The latter portfolio receives a weight equal to risk tolerance.

Mean variance without a riskfree asset (1)

- The program: it consists in minimizing portfolio variance for a given level of expected returns

$$\min_{\boldsymbol{\pi}} V\left[\sum_{i=1}^{N} \pi_i \tilde{r}_i\right] = \boldsymbol{\pi}'\Sigma\boldsymbol{\pi}$$

$$\text{s.t.} \sum_{i=1}^{N} \pi_i = \boldsymbol{\pi}'\boldsymbol{e} = 1$$

$$E\left[\sum_{i=1}^{N} \pi_i \tilde{r}_i\right] = \boldsymbol{\pi}'\boldsymbol{\mu} = \mu_p$$

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  - $\boldsymbol{e}$ is a vector of ones
  - $\tilde{\boldsymbol{r}}$ is the vector of returns
  - $\boldsymbol{\mu}$ is the vector of expected returns
- We assume $\boldsymbol{\mu} \neq \boldsymbol{e}$ to avoid degeneracy

Mean variance without a riskfree asset (3)

- Lagrangian for the optimization problem (a factor 1/2 is convenient):

$$\frac{1}{2}\boldsymbol{\pi}'\Sigma\boldsymbol{\pi} - \delta(\boldsymbol{\pi}'\boldsymbol{\mu} - \mu_p) - \gamma(\boldsymbol{\pi}'\boldsymbol{e} - 1)$$

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  where I have introduced the Lagrange multipliers δ and γ.
- The necessary and sufficient first order condition (positive definite quadratic problem) is:

$$\Sigma\boldsymbol{\pi} = \delta\boldsymbol{\mu} + \gamma\boldsymbol{e},$$

  or, assuming the covariance matrix is invertible:

$$\boldsymbol{\pi} = \delta\Sigma^{-1}\boldsymbol{\mu} + \gamma\Sigma^{-1}\boldsymbol{e}.$$

Mean variance without a riskfree asset (4)

- Injecting this into the constraints leads to a system for the Lagrange multipliers:

$$\delta \boldsymbol{\mu}' \Sigma^{-1} \boldsymbol{\mu} + \gamma \boldsymbol{\mu}' \Sigma^{-1} \boldsymbol{e} = \mu_p,$$
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- reminder:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

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- It is useful to introduce two specific portfolios:

$$\boldsymbol{\pi}_1 = \frac{1}{\mathbf{e}' \Sigma^{-1} \mathbf{e}} \Sigma^{-1} \mathbf{e},$$
$$\boldsymbol{\pi}_\mu = \frac{1}{\mathbf{e}' \Sigma^{-1} \boldsymbol{\mu}} \Sigma^{-1} \boldsymbol{\mu}.$$

Mean variance without a riskfree asset (5)

- We can write :

$$
\begin{aligned}
\boldsymbol{\pi} &= (\delta \mathbf{e}' \Sigma^{-1} \boldsymbol{\mu}) \boldsymbol{\pi}_{\mu} + (\gamma \mathbf{e}' \Sigma^{-1} \mathbf{e}) \boldsymbol{\pi}_{1} \\
&= \lambda \boldsymbol{\pi}_{\mu} + (1 - \lambda) \boldsymbol{\pi}_{1}
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- Thus, any optimal portfolio is a combination of the two portfolios we singled out:
  - $\boldsymbol{\pi}_1$ is the minimum variance portfolio
  - $\boldsymbol{\pi}_{\mu}$ is another portfolio as soon as $\boldsymbol{\mu} \neq \boldsymbol{e}$

Mean variance without a riskfree asset (6)

- $A = \boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu},\ B = \boldsymbol{\mu}'\Sigma^{-1}\mathbf{e},\ C = \mathbf{e}'\Sigma^{-1}\mathbf{e}.$

$$\lambda = \frac{BC\mu_p - B^2}{AC - B^2},$$

$$\sigma_p^2 = \frac{A - 2B\mu_p + C\mu_p^2}{AC - B^2}.$$

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- Check this
- The efficient frontier (in the standard deviation mean space) is the subset of non dominated portfolios in the set:

$$\{(\sigma_p, \mu_p),\ \mu_p \geq \mu_1\}$$

where $\mu_1 = \boldsymbol{\pi}_1'\boldsymbol{\mu}$.

Mean variance without a riskfree asset (7) - fig 2

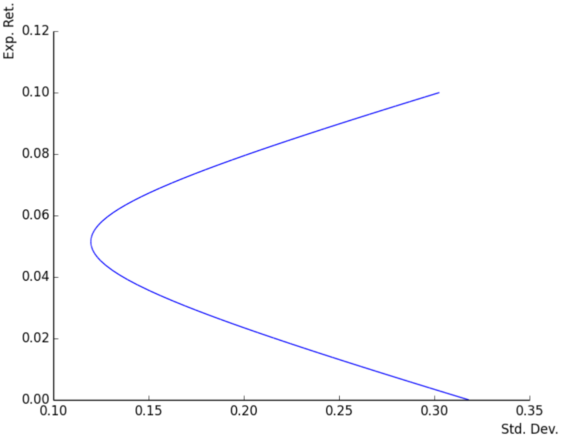

Figure 2:Efficient Frontier (without a risk free asset)

Mean variance with a riskfree asset (1)

- It is convenient in this case to use the notation $\boldsymbol{\pi}$ to denote the vector of positions on the risky assets (see the slide on the space of excess returns). The cash position is thus:

$$\pi_0 = 1 - \boldsymbol{e}'\boldsymbol{\pi}.$$

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- The vector $\boldsymbol{\pi}$ is unconstrained. The optimization problem can be written:

$$\min_{\boldsymbol{\pi}} \boldsymbol{\pi}'\Sigma\boldsymbol{\pi}$$
$$\text{s.t. } \boldsymbol{\pi}'(\boldsymbol{\mu} - r^f \boldsymbol{e}) = \mu_p - r^f$$

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- The standard deviation of the portfolio is:

$$\frac{|\mu_p - r^f|}{\sqrt{(\boldsymbol{\mu} - r^f\boldsymbol{e})'\Sigma^{-1}(\boldsymbol{\mu} - r^f\boldsymbol{e})}}.$$

Mean variance with a riskfree asset (3)

- The tangency portfolio is:

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- It is a portfolio fully invested in risky assets which is on the overall efficient frontier. It is thus also on the risky asset efficient frontier.

Mean variance with a riskfree asset (4) - fig 3

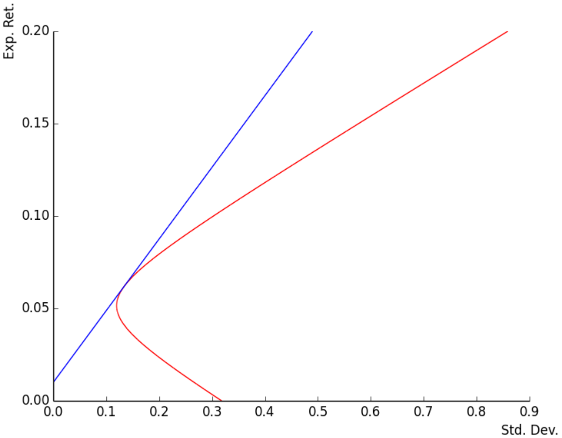

Figure 3:Efficient Frontier (with risk free asset)

Mean variance with a riskfree asset (5) - fig4

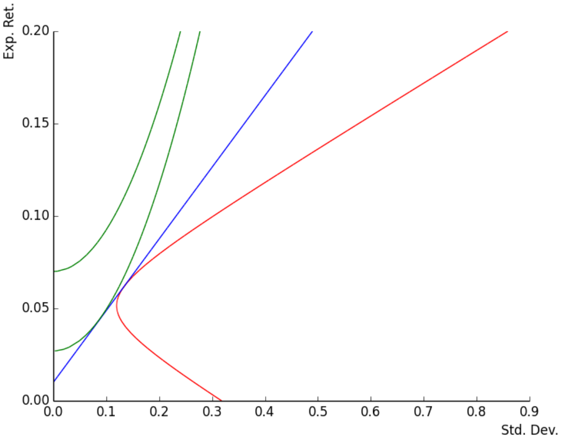

Figure 4:Efficient Frontier (with risk free asset): iso-utility curve

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  - $\mathrm{sharpe}(\boldsymbol{\pi_*}) = 0.39$

Data for the graphs (2)

- The graphs shown assume positive Sharpe ratios for the underlying assets. This is the ‘normal’ situation. It ensures that the efficient frontier (with a riskfree asset!) is upward sloping.

A different description of the efficient frontier (1)

- Maximize the expected return penalized for portfolio variance ($\rho > 0$):

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- Exercise: recover the lagrange multipliers of the traditional approach
- The criteria are given by quadratic utility functions, indexed by ρ

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- We will remember that:

$$\rho\hat{\pi} = \frac{\mu_* - r^f}{\text{var}(\pi_*)},$$

which is therefore independent of the risk aversion level of the investor. This will play a role in the derivation of the CAPM.

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- Consider that the optimal portfolio of a mean-variance investor (p with weights $\boldsymbol{\pi}$) is tilted by adding a long-short portfolio $\boldsymbol{\delta\pi}$. How does that affect quadratic utility?

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- The utility level changes by (first order approximation):

$$
\begin{aligned}
&\mu_\delta - \rho\,\text{cov}(r_\delta, r_p)\\
&= \mu_\delta - \rho\frac{\text{cov}(r_\delta, r_p)}{\text{var}(r_p)}\text{var}(r_p),\\
&= \mu_\delta - \rho\beta(\delta, p)\text{var}(r_p),\\
&= \mu_\delta - \rho\hat{\pi}\beta(\delta, r_*)\text{var}(r_*).
\end{aligned}
$$

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- Given the optimality of the tangency portfolio, the above quantity should be zero for all long short deviations to the tangency portfolio:

$$\mu_\delta = (\mu_* - r^f)\beta(\delta, r_*).$$

Interpretation of the first order condition (2)

- Because the quantity $\rho\hat{\pi}$ is independent of ρ, the trade off between return and beta is a well defined consequence of the mean and variance assumptions.
- Injecting the value of $\rho\hat{\pi}$ into the first order condition delivers the quantity:

$$= \mu_\delta - (\mu_* - r^f)\beta(\delta, r_*).$$

- Given the optimality of the tangency portfolio, the above quantity should be zero for all long short deviations to the tangency portfolio:

$$\mu_\delta = (\mu_* - r^f)\beta(\delta, r_*).$$

- For long short portfolios which borrow to buy a stock, the condition reads:

$$(r_i - r^f) = (\mu_* - r^f)\beta(r_i, r_*).$$

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- At this stage, no equilibrium assumption has been made. We are looking at the implications of a portfolio being mean-variance optimal.
- Note that the tangency portfolio can be replaced by any other efficient portfolio in the relationship.

The excess return-beta relationship (1) - fig 5

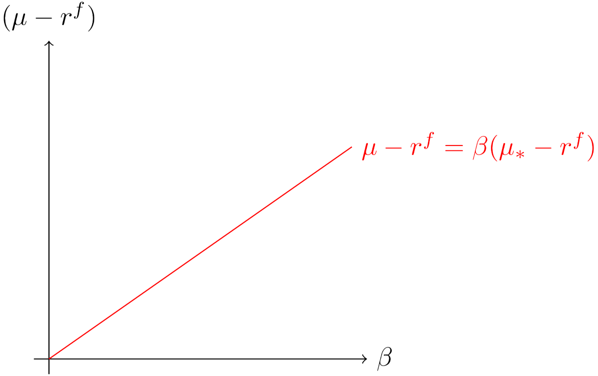

Figure 5:Return/beta relationship

The excess return-beta relationship (2) - fig 6

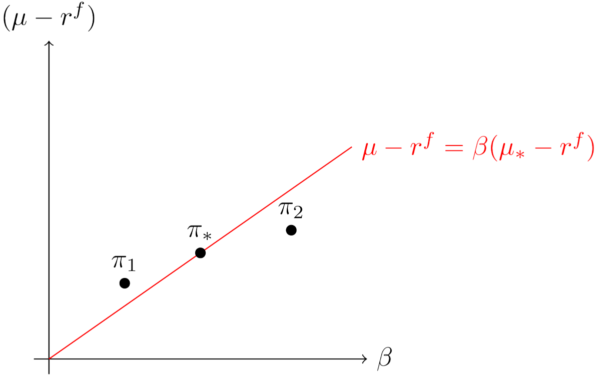

Figure 6:Imperfectly priced portfolios

The two fund theorem and the CAPM

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- As a result, they all hold a mixture of the risk free asset and a unique portfolio of risky asset, the tangency portfolio.
- This is an instance of the two fund theorem, which also holds in more general contexts
- The risky asset portfolio should be equal to the market portfolio of risky asset, with return r_m. This gives:

$$(r_i - r^f) = (\mu_m - r^f)\beta(r_i, r_m).$$

Conditions for the CAPM to hold

Illustration of the CAPM - fig 7

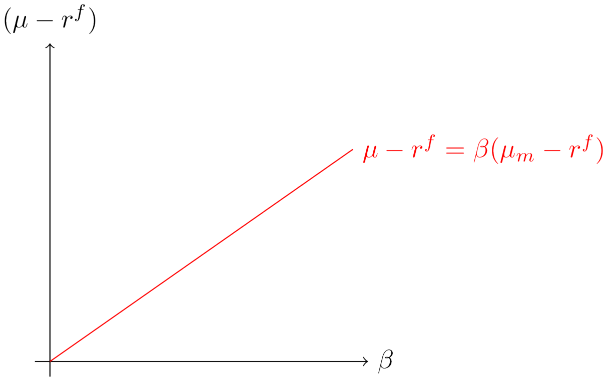

Figure 7:CAPM

The low beta anomaly - fig 8

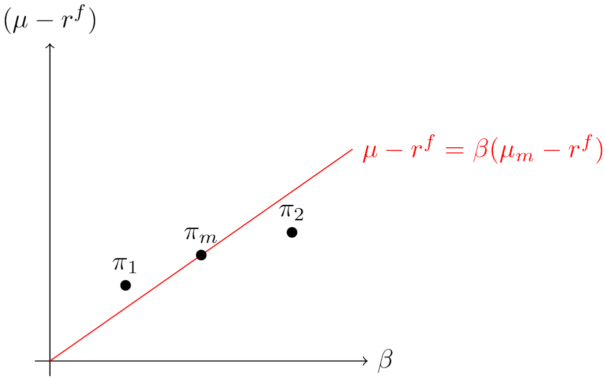

Figure 8:The low beta anomaly

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- Example: ERC ?