INVERSE SCATTERING AT FIXED ENERGY ON SURFACES WITH EUCLIDEAN ENDS

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Abstract. On a fixed Riemann surface (M_0, g_0) with N Euclidean ends and genus g, we show that, under a topological condition, the scattering matrix $S_V(\lambda)$ at frequency $\lambda > 0$ for the operator $\Delta + V$ determines the potential V if $V \in C^{1,\alpha}(M_0) \cap e^{-\gamma d(\cdot,z_0)^j} L^\infty(M_0)$ for all $\gamma > 0$ and for some $j \in \{1, 2\}$, where $d(z, z_0)$ denotes the distance from z to a fixed point $z_0 \in M_0$. The topological condition is given by $N \geq \max(2g + 1, 2)$ for $j = 1$ and by $N \geq g+1$ if $j = 2$. In $\mathbb{R}^2$ this implies that the operator $S_V(\lambda)$ determines any $C^{1,\alpha}$ potential V such that $V(z) = O(e^{-\gamma|z|^2})$ for all $\gamma > 0$.

1. Introduction

The purpose of this paper is to prove an inverse scattering result at fixed frequency $\lambda > 0$ in dimension 2. The typical question one can ask is to show that the scattering matrix $S_V(\lambda)$ for the Schrödinger operator $\Delta+V$ determines the potential. This is known to be false if V is only assumed to be Schwartz, by the example of Grinevich-Novikov [6], but it is also known to be true for exponentially decaying potentials (i.e. $V \in e^{-\gamma|z|}L^\infty(\mathbb{R}^2)$ for some $\gamma > 0$) with norm smaller than a constant depending on the frequency λ, see Novikov [15]. For other partial results we refer to [2], [10], [19], [20], [21]. The determinacy of V from $S_V(\lambda)$ when V is compactly supported, without any smallness assumption on the norm, follows from the recent work of Bukhgeim [1] on the inverse boundary problem after a standard reduction to the Dirichlet-to-Neumann operator on a large sphere (see [25] for this reduction).

In dimensions $n \geq 3$, it is proved in Novikov [16] (see also [3] for the case of magnetic Schrödinger operators) that the scattering matrix at a fixed frequency λ determines an exponentially decaying potential. When V is compactly supported this also follows directly from the result by Sylvester-Uhlmann [22] on the inverse boundary problem, by reducing to the Dirichlet-to-Neumann operator on a large sphere. Melrose [14] gave a direct proof of the last result based on the methods of [22], and this proof was extended to exponentially decaying potentials in [26] and to the magnetic case in [17]. In the geometric scattering setting, [11, 12] reconstruct the asymptotic expansion of a potential or metrics from the scattering operator at fixed frequency on asymptotically Euclidean/hyperbolic manifolds. Further results of this type are given in [27, 28].

The method for proving the determinacy of V from $S_V(\lambda)$ in [14, 26] is based on the construction of complex geometric optics solutions $u(z) = e^{\rho\cdot z}(1+r(\rho, z))$ of $(\Delta+V-\lambda^2)u = 0$ with $\rho \in \mathbb{C}^n, z \in \mathbb{R}^n$, and the density of the oscillating scattering solutions $u_{\rm sc}(z) = \int_{S^{n-1}} \Phi_V(\lambda, z, \omega)f(\omega)d\omega$ within those complex geometric optics solutions, where $\Phi_V(\lambda, z, \omega) = e^{i\lambda\omega.z} + e^{-i\lambda\omega.z}|z|^{-\frac{1}{2}(n-1)}a(\lambda, z, \omega)$ are the perturbed plane wave solutions (here $\omega \in S^{n-1}$ and $a \in L^\infty$). Unlike when $n \geq 3$, the problem in dimension 2 is that the set of complex geometrical optics solutions of this type is not large enough to show that the Fourier transform of $V_1 - V_2$ is 0.

The real novelty in the recent work of Bukhgeim [1] in dimension 2 is the construction of new complex geometric optics solutions (at least on a bounded domain $\Omega \subset \mathbb{C}$) of $(\Delta + V_i)u_i = 0$ of the form $u_1 = e^{\Phi/h}(1 + r_1(h))$ and $u_2 = e^{-\Phi/h}(1 + r_2(h))$ with $0 < h \ll 1$ where Φ is a holomorphic function in $\mathbb{C}$ with a unique non-degenerate critical point at a fixed $z_0 \in \mathbb{C}$ (for instance $\Phi(z) = (z - z_0)^2$), and $||r_j(h)||_{L^p}$ is small as $h \to 0$ for $p > 1$. These solutions allow to use stationary phase at z_0 to get

$$\int_\Omega (V_1 - V_2)u_1\overline{u_2} = C(V_1(z_0) - V_2(z_0))h + o(h), \quad C \neq 0$$

as $h \to 0$ and thus, if the Dirichlet-to-Neumann operators on $\partial\Omega$ are the same, then $V_1(z_0) = V_2(z_0)$.

One of the problems to extend this to inverse scattering is that a holomorphic function in $\mathbb{C}$ with a non-degenerate critical point needs to grow at least quadratically at infinity, which would somehow force to consider potentials V having Gaussian decay. On the other hand, if we allow the function to be meromorphic with simple poles, then we can construct such functions, having a single critical point at any given point p, for instance by considering $\Phi(z) = (z-p)^2/z$. Of course, with such Φ we then need to work on $\mathbb{C}\setminus\{0\}$, which is conformal to a surface with no hole but with 2 Euclidean ends, and Φ has linear growth in the ends. In general, on a surface with genus g and N Euclidean ends, we can use the Riemann-Roch theorem to construct holomorphic functions with linear or quadratic growth in the ends, the dimension of the space of such functions depending on g, N.

In the present work, we apply this idea to obtain an inverse scattering result for $\Delta_{g_0} + V$ on a fixed Riemann surface (M_0, g_0) with Euclidean ends, under some topological condition on M_0 and some decay condition on V.

Theorem 1.1. *Let (M_0, g_0) be a non-compact Riemann surface with genus g and N ends isometric to $\mathbb{R}^2 \setminus \{|z| \leq 1\}$ with metric $|dz|^2$. Let V_1 and V_2 be two potentials in $C^{1,\alpha}(M_0)$ with $\alpha > 0$, and such that $S_{V_1}(\lambda) = S_{V_2}(\lambda)$ for some $\lambda > 0$. Let $d(z, z_0)$ denote the distance between z and a fixed point $z_0 \in M_0$.*
(i) If $N \geq \max(2g + 1, 2)$ and $V_i \in e^{-\gamma d(\cdot,z_0)}L^\infty(M_0)$ for all $\gamma > 0$, then $V_1 = V_2$.
(ii) If $N \geq g + 1$ and $V_i \in e^{-\gamma d(\cdot,z_0)^2}L^\infty(M_0)$ for all $\gamma > 0$, then $V_1 = V_2$.

In $\mathbb{R}^2$, where $g = 0$ and $N = 1$, we have an immediate corollary:

Corollary 1.2. *Let $\lambda > 0$ and let $V_1, V_2 \in C^{1,\alpha}(\mathbb{R}^2) \cap e^{-\gamma|z|^2}L^\infty(\mathbb{R}^2)$ for all $\gamma > 0$. If the scattering matrices satisfy $S_{V_1}(\lambda) = S_{V_2}(\lambda)$, then $V_1 = V_2$.*

This is an improvement on the result of Bukhgeim [1] which shows identifiability for compactly supported functions, and in a certain sense on the result of Novikov [15] since it is assumed there that the potential has to be of small L^∞ norm.

The structure of the paper is as follows. In Section 2 we employ the Riemann-Roch theorem and a transversality argument to construct Morse holomorphic functions on (M_0, g_0) with linear or quadratic growth in the ends. Section 3 considers Carleman estimates with harmonic weights on (M_0, g_0), where suitable convexification and weights at the ends are required since the surface is non compact. Complex geometrical optics solutions are constructed in Section 4. Section 5 discusses direct scattering theory on surfaces with Euclidean ends and contains the proof that scattering solutions are dense in the set of suitable solutions, and Section 6 gives the proof of Theorem 1.1. Finally, there is an appendix discussing a Paley-Wiener type result for functions with Gaussian decay which is needed to prove density of scattering solutions.

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2. Holomorphic Morse functions on a surface with Euclidean ends

2.1. **Riemann surfaces with Euclidean ends.** Let (M_0, g_0) be a non-compact connected smooth Riemannian surface with N ends $E_1, \dots, E_N$ which are Euclidean, i.e. isometric to $\mathbb{C} \setminus \{|z| \leq 1\}$ with metric $|dz|^2$. By using a complex inversion $z \to 1/z$, each end is also isometric to a pointed disk

$$E_i \simeq \{|z| \leq 1, z \neq 0\} \text{ with metric } \frac{|dz|^2}{|z|^4}$$

thus conformal to the Euclidean metric on the pointed disk. The surface M_0 can then be compactified by adding the points corresponding to $z = 0$ in each pointed disk corresponding to an end E_i, we obtain a closed Riemann surface M with a natural complex structure induced by that of M_0, or equivalently a smooth conformal class on M induced by that of M_0. Another way of thinking is to say that M_0 is the closed Riemann surface M with N points $e_1, \dots, e_N$ removed. The Riemann surface M has holomorphic charts $z_\alpha : U_\alpha \to \mathbb{C}$ and we will denote by $z_1, \dots z_N$ the complex coordinates corresponding to the ends of M_0, or equivalently to the neighbourhoods of the points e_i. The Hodge star operator $\star$ acts on the cotangent bundle T^*M, its eigenvalues are $\pm i$ and the respective eigenspaces $T^*_{1,0}M := \ker(\star + i\mathrm{Id})$ and $T^*_{0,1}M := \ker(\star - i\mathrm{Id})$ are sub-bundles of the complexified cotangent bundle $\mathbb{C}T^*M$ and the splitting $\mathbb{C}T^*M = T^*_{1,0}M \oplus T^*_{0,1}M$ holds as complex vector spaces. Since $\star$ is conformally invariant on 1-forms on M, the complex structure depends only on the conformal class of g. In holomorphic coordinates $z = x + iy$ in a chart U_α, one has $\star(udx + vdy) = -vdx + udy$ and

$$T^*_{1,0}M|_{U_\alpha} \simeq \mathbb{C}dz, \quad T^*_{0,1}M|_{U_\alpha} \simeq \mathbb{C}d\bar{z}$$

where $dz = dx + idy$ and $d\bar{z} = dx - idy$. We define the natural projections induced by the splitting of $\mathbb{C}T^*M$

$$\pi_{1,0} : \mathbb{C}T^*M \to T^*_{1,0}M, \quad \pi_{0,1} : \mathbb{C}T^*M \to T^*_{0,1}M.$$

The exterior derivative d defines the de Rham complex $0 \to \Lambda^0 \to \Lambda^1 \to \Lambda^2 \to 0$ where $\Lambda^k := \Lambda^k T^*M$ denotes the real bundle of k-forms on M. Let us denote $\mathbb{C}\Lambda^k$ the complexification of Λ^k, then the ∂ and $\bar{\partial}$ operators can be defined as differential operators $\partial : \mathbb{C}\Lambda^0 \to T^*_{1,0}M$ and $\bar{\partial} : \mathbb{C}\Lambda 0 \to T^*_{0,1}M$ by

$$\partial f := \pi_{1,0} df, \quad \bar{\partial} f := \pi_{0,1} df,$$

they satisfy $d = \partial + \bar{\partial}$ and are expressed in holomorphic coordinates by

$$\partial f = \partial_z f \, dz, \quad \bar{\partial} f = \partial_{\bar{z}} f \, d\bar{z},$$

with $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$. Similarly, one can define the ∂ and $\bar{\partial}$ operators from $\mathbb{C}\Lambda^1$ to $\mathbb{C}\Lambda^2$ by setting

$$\partial(\omega_{1,0} + \omega_{0,1}) := d\omega_{0,1}, \quad \bar{\partial}(\omega_{1,0} + \omega_{0,1}) := d\omega_{1,0}$$

if $\omega_{0,1} \in T^*_{0,1}M$ and $\omega_{1,0} \in T^*_{1,0}M$. In coordinates this is simply

$$\partial(udz + vd\bar{z}) = \partial v \wedge d\bar{z}, \quad \bar{\partial}(udz + vd\bar{z}) = \bar{\partial} u \wedge dz.$$

If g is a metric on M whose conformal class induces the complex structure $T^*_{1,0}M$, there is a natural operator, the Laplacian acting on functions and defined by

$$\Delta f := -2i \star \bar{\partial}\partial f = d^* d$$

where d^* is the adjoint of d through the metric g and $\star$ is the Hodge star operator mapping Λ^2 to Λ^0 and induced by g as well.

2.2. **Holomorphic functions.** We are going to construct Carleman weights given by holomorphic functions on M_0 which grow at most linearly or quadratically in the ends. We will use the Riemann-Roch theorem, following ideas of [7], however, the difference in the present case is that we have very little freedom to construct these holomorphic functions, simply because there is just a finite dimensional space of such functions by Riemann-Roch. For the convenience of the reader, and to fix notations, we recall the usual Riemann-Roch index theorem (see Farkas-Kra [5] for more details). A divisor D on M is an element

$$D = \big((p_1, n_1), \dots, (p_k, n_k)\big) \in (M \times \mathbb{Z})^k, \text{ where } k \in \mathbb{N}$$

which will also be denoted $D = \prod_{i=1}^k p_i^{n_i}$ or $D = \prod_{p\in M} p^{\alpha(p)}$ where $\alpha(p) = 0$ for all p except $\alpha(p_i) = n_i$. The inverse divisor of D is defined to be $D^{-1} := \prod_{p\in M} p^{-\alpha(p)}$ and the degree of the divisor D is defined by $\deg(D) := \sum_{i=1}^k n_i = \sum_{p\in M} \alpha(p)$. A non-zero meromorphic function on M is said to have divisor D if $(f) := \prod_{p\in M} p^{\mathrm{ord}(p)}$ is equal to D, where $\mathrm{ord}(p)$ denotes the order of p as a pole or zero of f (with positive sign convention for zeros). Notice that in this case we have $\deg(f) = 0$. For divisors $D' = \prod_{p\in M} p^{\alpha'(p)}$ and $D = \prod_{p\in M} p^{\alpha(p)}$, we say that $D' \geq D$ if $\alpha'(p) \geq \alpha(p)$ for all $p \in M$. The same exact notions apply for meromorphic 1-forms on M. Then we define for a divisor D

$$r(D) := \dim(\{f \text{ meromorphic function on } M; (f) \geq D\} \cup \{0\}),$$
$$i(D) := \dim(\{u \text{ meromorphic 1 form on } M; (u) \geq D\} \cup \{0\}).$$

The Riemann-Roch theorem states the following identity: for any divisor D on the closed Riemann surface M of genus g,

$$r(D^{-1}) = i(D) + \deg(D) - g + 1. \tag{1}$$

Notice also that for any divisor D with $\deg(D) > 0$, one has $r(D) = 0$ since $\deg(f) = 0$ for all f meromorphic. By [5, Th. p70], let D be a divisor, then for any non-zero meromorphic 1-form ω on M, one has

$$i(D) = r(D(\omega)^{-1}) \tag{2}$$

which is thus independent of ω. For instance, if $D = 1$, we know that the only holomorphic function on M is 1 and one has $1 = r(1) = r((\omega)^{-1}) - g + 1$ and thus $r((\omega)^{-1}) = g$ if ω is a non-zero meromorphic 1 form. Now if $D = (\omega)$, we obtain again from (1)

$$g = r((\omega)^{-1}) = 2 - g + \deg((\omega))$$

which gives $\deg((\omega)) = 2(g-1)$ for any non-zero meromorphic 1-form ω. In particular, if D is a divisor such that $\deg(D) > 2(g-1)$, then we get $\deg(D(\omega)^{-1}) = \deg(D) - 2(g-1) > 0$ and thus $i(D) = r(D(\omega)^{-1}) = 0$, which implies by (1)

$$\deg(D) > 2(g-1) \Longrightarrow r(D^{-1}) = \deg(D) - g + 1 \geq g. \tag{3}$$

Now we deduce the

Lemma 2.1. *Let $e_1, \dots, e_N$ be distinct points on a closed Riemann surface M with genus g, and let z_0 be another point of $M \setminus \{e_1, \dots, e_N\}$. If $N \geq \max(2g+1, 2)$, the following hold true:*
(i) there exists a meromorphic function f on M with at most simple poles, all contained in $\{e_1, \dots, e_N\}$, such that $\partial f(z_0) \neq 0$,
(ii) there exists a meromorphic function h on M with at most simple poles, all contained in $\{e_1, \dots, e_N\}$, such that z_0 is a zero of order at least 2 of h.

Proof. Let first $g \geq 1$, so that $N \geq 2g+1$. By the discussion before the Lemma, we know that there are at least $g+2$ linearly independent (over $\mathbb{C}$) meromorphic functions $f_0, \dots, f_{g+1}$ on M with at most simple poles, all contained in $\{e_1, \dots, e_{2g+1}\}$. Without loss of generality, one can set $f_0 = 1$ and by linear combinations we can assume that $f_1(z_0) = \dots = f_{g+1}(z_0) = 0$. Now consider the divisor $D_j = e_1 \dots e_{2g+1} z_0^{-j}$ for $j = 1, 2$, with degree $\deg(D_j) = 2g + 1 - j$, then by the Riemann-Roch formula (more precisely (3))

$$r(D_j^{-1}) = g + 2 - j.$$

Thus, since $r(D_1^{-1}) > r(D_2^{-1}) = g$ and using the assumption that $g \geq 1$, we deduce that there is a function in $\mathrm{span}(f_1, \dots, f_{g+1})$ which has a zero of order 2 at z_0 and a function which has a zero of order exactly 1 at z_0. The same method clearly works if $g = 0$ by taking two points e_1, e_2 instead of just e_1. $\square$

If we allow double poles instead of simple poles, the proof of Lemma 2.1 shows the

Lemma 2.2. *Let $e_1, \dots, e_N$ be distinct points on a closed Riemann surface M with genus g, and let z_0 be another point of $M \setminus \{e_1, \dots, e_N\}$. If $N \geq g+1$, then the following hold true:*
(i) there exists a meromorphic function f on M with at most double poles, all contained in $\{e_1, \dots, e_N\}$, such that $\partial f(z_0) \neq 0$,
(ii) there exists a meromorphic function h on M with at most double poles, all contained in $\{e_1, \dots, e_N\}$, such that z_0 is a zero of order at least 2 of h.

2.3. **Morse holomorphic functions with prescribed critical points.** We follow in this section the arguments used in [7] to construct holomorphic functions with non-degenerate critical points (i.e. Morse holomorphic functions) on the surface M_0 with genus g and N ends, such that these functions have at most linear growth (resp. quadratic growth) in the ends if $N \geq \max(2g+1, 2)$ (resp. if $N \geq g+1$). We let $\mathcal{H}$ be the complex vector space spanned by the meromorphic functions on M with divisors larger or equal to $e_1^{-1} \dots e_N^{-1}$ (resp. by $e_1^{-2} \dots e_N^{-2}$) if we work with functions having linear growth (resp. quadratic growth), where $e_1, \dots e_N \in M$ are points corresponding to the ends of M_0 as explained in Section 2. Note that $\mathcal{H}$ is a complex vector space of complex dimension greater or equal to $N - g + 1$ (resp. $2N - g + 1$) for the $e_1^{-1} \dots e_N^{-1}$ divisor (resp. the $e_1^{-2} \dots e_N^{-2}$ divisor). We will also consider the real vector space H spanned by the real parts and imaginary parts of functions in $\mathcal{H}$, this is a real vector space which admits a Lebesgue measure. We now prove the following

Lemma 2.3. *The set of functions $u \in H$ which are not Morse in M_0 has measure 0 in H, in particular its complement is dense in H.*

Proof. We use an argument very similar to that used by Uhlenbeck [24]. We start by defining $m : M_0 \times H \to T^*M_0$ by $(p, u) \mapsto (p, du(p)) \in T_p^*M_0$. This is clearly a smooth map, linear in the second variable, moreover $m_u := m(., u) = (\cdot, du(\cdot))$ is smooth on M_0. The map u is a

Morse function if and only if m_u is transverse to the zero section, denoted $T_0^*M_0$, of T^*M_0, i.e. if

$$\text{Image}(D_p m_u) + T_{m_u(p)}(T_0^*M_0) = T_{m_u(p)}(T^*M_0), \quad \forall p \in M_0 \text{ such that } m_u(p) = (p, 0).$$

This is equivalent to the fact that the Hessian of u at critical points is non-degenerate (see for instance Lemma 2.8 of [24]). We recall the following transversality result, the proof of which is contained in [24, Th.2] by replacing Sard-Smale theorem by the usual finite dimensional Sard theorem:

Theorem 2.4. *Let $m : X \times H \to W$ be a C^k map and X, W be smooth manifolds and H a finite dimensional vector space, if $W' \subset W$ is a submanifold such that $k > \max(1, \dim X - \dim W + \dim W')$, then the transversality of the map m to W' implies that the complement of the set $\{u \in H; m_u \text{ is transverse to } W'\}$ in H has Lebesgue measure 0.*

We want to apply this result with $X := M_0$, $W := T^*M_0$ and $W' := T_0^*M_0$, and with the map m as defined above. We have thus proved our Lemma if one can show that m is transverse to W'. Let (p, u) such that $m(p, u) = (p, 0) \in W'$. Then identifying $T_{(p,0)}(T^*M_0)$ with $T_pM_0 \oplus T_p^*M_0$, one has

$$Dm_{(p,u)}(z, v) = (z, dv(p) + \text{Hess}_p(u)z)$$

where $\text{Hess}_p(u)$ is the Hessian of u at the point p, viewed as a linear map from T_pM_0 to $T_p^*M_0$ (note that this is different from the covariant Hessian defined by the Levi-Civita connection). To prove that m is transverse to W' we need to show that $(z, v) \to (z, dv(p) + \text{Hess}_p(u)z)$ is onto from $T_pM_0 \oplus H$ to $T_pM_0 \oplus T_p^*M_0$, which is realized if the map $v \to dv(p)$ from H to $T_p^*M_0$ is onto. But from Lemma 2.1, we know that there exists a meromorphic function f with real part $v = \text{Re}(f) \in H$ such that $v(p) = 0$ and $dv(p) \neq 0$ as an element of $T_p^*M_0$. We can then take $v_1 := v$ and $v_2 := \text{Im}(f)$, which are functions of H such that $dv_1(p)$ and $dv_2(p)$ are linearly independent in $T_p^*M_0$ by the Cauchy-Riemann equation $\bar{\partial} f = 0$. This shows our claim and ends the proof by using Theorem 2.4. □

In particular, by the Cauchy-Riemann equation, this Lemma implies that the set of Morse functions in $\mathcal{H}$ is dense in $\mathcal{H}$. We deduce

Proposition 2.1. *There exists a dense set of points p in M_0 such that there exists a Morse holomorphic function $f \in \mathcal{H}$ on M_0 which has a critical point at p.*

Proof. Let p be a point of M_0 and let u be a holomorphic function with a zero of order at least 2 at p, the existence is ensured by Lemma 2.1. Let $B(p, \eta)$ be a any small ball of radius $\eta > 0$ near p, then by Lemma 2.3, for any $\epsilon > 0$, we can approach u by a holomorphic Morse function $u_\epsilon \in \mathcal{H}_\epsilon$ which is at distance less than ϵ of u in a fixed norm on the finite dimensional space $\mathcal{H}$. Rouché's theorem for $\partial_z u_\epsilon$ and $\partial_z u$ (which are viewed as functions locally near p) implies that $\partial_z u_\epsilon$ has at least one zero of order exactly 1 in $B(p, \eta)$ if ϵ is chosen small enough. Thus there is a Morse function in $\mathcal{H}$ with a critical point arbitrarily close to p. □

Remark 2.5. *In the case where the surface M has genus 0 and N ends, we have an explicit formula for the function in Proposition 2.1: indeed M_0 is conformal to $\mathbb{C} \setminus \{e_1, \dots, e_{N-1}\}$ for some $e_i \in \mathbb{C}$ - i.e. the Riemann sphere minus N points - then the function $f(z) = (z - z_0)^2/(z - e_1)$ with $z_0 \notin \{e_1, \dots, e_{N-1}\}$ has z_0 for unique critical point in $\mathbb{C} \setminus \{e_1, \dots, e_{N-1}\}$ and it is non-degenerate.*

We end this section by the following Lemmas which will be used for the amplitude of the complex geometric optics solutions but not for the phase.

Lemma 2.6. *For any $p_0, p_1, \dots p_n \in M_0$ some points of M_0 and $L \in \mathbb{N}$, then there exists a function $a(z)$ holomorphic on M_0 which vanishes to order L at all p_j for $j = 1, \dots, n$ and such that $a(p_0) \neq 0$. Moreover $a(z)$ can be chosen to have at most polynomial growth in the ends, i.e. $|a(z)| \leq C|z|^J$ for some $J \in \mathbb{N}$.*

Proof. It suffices to find on M some meromorphic function with divisor greater or equal to $D := e_1^{-J} \dots e_N^{-J} p_1^L \dots p_n^L$ but not greater or equal to Dp_0 and this is insured by Riemann-Roch theorem as long as $JN - nL \geq 2g$ since then $r(D) = -g + 1 + JN - nL$ and $r(Dp_0) = -g + JN - nL$. □

Lemma 2.7. *Let $\{p_0, p_1, .., p_n\} \subset M_0$ be a set of $n+1$ disjoint points. Let $c_0, c_1, \dots, c_K \in \mathbb{C}$, $L \in \mathbb{N}$, and let z be a complex coordinate near p_0 such that $p_0 = \{z = 0\}$. Then there exists a holomorphic function f on M_0 with zeros of order at least L at each p_j, such that $f(z) = c_0 + c_1 z + ... + c_K z^K + O(|z|^{K+1})$ in the coordinate z. Moreover f can be chosen so that there is $J \in \mathbb{N}$ such that, in the ends, $|\partial_z^\ell f(z)| = O(|z|^J)$ for all $\ell \in \mathbb{N}_0$.*

Proof. The proof goes along the same lines as in Lemma 2.6. By induction on K and linear combinations, it suffices to prove it for $c_0 = \cdots = c_{K-1} = 0$. As in the proof of Lemma 2.6, if J is taken large enough, there exists a function with divisor greater or equal to $D := e_1^{-J} \dots e_N^{-J} p_0^{K-1} p_1^L \dots p_n^L$ but not greater or equal to Dp_0. Then it suffices to multiply this function by c_K times the inverse of the coefficient of z^K in its Taylor expansion at $z = 0$. □

2.4. **Laplacian on weighted spaces.** Let x be a smooth positive function on M_0, which is equal to $|z|^{-1}$ for $|z| > r_0$ in the ends $E_i \simeq \{z \in \mathbb{C}; |z| > 1\}$, where r_0 is a large fixed number. We now show that the Laplacian Δ_{g_0} on a surface with Euclidean ends has a right inverse on the weighted spaces $x^{-J} L^2(M_0)$ for $J \notin \mathbb{N}$ positive.

Lemma 2.8. *For any $J > -1$ which is not an integer, there exists a continuous operator G mapping $x^{-J} L^2(M_0)$ to $x^{-J-2} L^2(M_0)$ such that $\Delta_{g_0} G = \mathrm{Id}$.*

Proof. Let $g_b := x^2 g_0$ be a metric conformal to g_0. The metric g_b in the ends can be written $g_b = dx^2/x^2 + d\theta^2_{S^1}$ by using radial coordinates $x = |z|^{-1}, \theta = z/|z| \in S^1$, this is thus a b-metric in the sense of Melrose [13], giving the surface a geometry of surface with cylindrical ends. Let us define for $m \in \mathbb{N}_0$

$$H_b^m(M_0) := \{u \in L^2(M_0; \mathrm{dvol}_{g_b}); (x\partial_x)^j \partial_\theta^k u \in L^2(M_0; \mathrm{dvol}_{g_b}) \text{ for all } j + k \leq m\}.$$

The Laplacian has the form $\Delta_{g_b} = -(x\partial_x)^2 + \Delta_{S^1}$ in the ends, and the indicial roots of Δ_{g_b} in the sense of Section 5.2 of [13] are given by the complex numbers λ such that $x^{-i\lambda} \Delta_{g_b} x^{i\lambda}$ is not invertible as an operator acting on the circle S^1_θ. Thus the indicial roots are the solutions of $\lambda^2 + k^2 = 0$ where k^2 runs over the eigenvalues of Δ_{S^1}, that is, $k \in \mathbb{Z}$. The roots are simple at $\pm ik \in i\mathbb{Z} \setminus \{0\}$ and 0 is a double root. In Theorem 5.60 of [13], Melrose proves that Δ_{g_b} is Fredholm on $x^a H_b^2(M_0)$ if and only if $-a$ is not the imaginary part of some indicial root, that is here $a \notin \mathbb{Z}$. For $J > 0$, the kernel of Δ_{g_b} on the space $x^J H_b^2(M_0)$ is clearly trivial by an energy estimate. Thus $\Delta_{g_b} : x^{-J} H_b^0(M_0) \to x^{-J} H_b^{-2}(M_0)$ is surjective for $J > 0$ and $J \notin \mathbb{Z}$, and the same then holds for $\Delta_{g_b} : x^{-J} H_b^2(M_0) \to x^{-J} H_b^0(M_0)$ by elliptic regularity.

Now we can use Proposition 5.64 of [13], which asserts, for all positive $J \notin \mathbb{Z}$, the existence of a pseudodifferential operator G_b mapping continuously $x^{-J}H_b^0(M_0)$ to $x^{-J}H_b^2(M_0)$ such that $\Delta_{g_b}G_b = \mathrm{Id}$. Thus if we set $G = G_b x^{-2}$, we have $\Delta_{g_0}G = \mathrm{Id}$ and G maps continuously $x^{-J+1}L^2(M_0)$ to $x^{-J-1}L^2(M_0)$ (note that $L^2(M_0) = xH_b^0(M_0)$). $\square$

3. Carleman Estimate for Harmonic Weights with Critical Points

3.1. **The linear weight case.** In this section, we prove a Carleman estimate using harmonic weights with non-degenerate critical points, in a way similar to [7]. Here however we need to work on a non compact surface and with weighted spaces. We first consider a Morse holomorphic function $\Phi \in \mathcal{H}$ obtained from Proposition 2.1 with the condition that Φ has linear growth in the ends, which corresponds to the case where $V \in e^{-\gamma/x}L^\infty(M_0)$ for all $\gamma > 0$. The Carleman weight will be the harmonic function $\varphi := \mathrm{Re}(\Phi)$. We let x be a positive smooth function on M_0 such that $x = |z|^{-1}$ in the complex charts $\{z \in \mathbb{C}; |z| > 1\} \simeq E_i$ covering the end E_i.

Let $\delta \in (0,1)$ be small and let us take $\varphi_0 \in x^{-\alpha}L^2(M_0)$ a solution of $\Delta_{g_0}\varphi_0 = x^{2-\delta}$, a solution exists by Proposition 2.8 if $\alpha > 1 + \delta$. Actually, by using Proposition 5.61 of [13], if we choose $\alpha < 2$, then it is easy to see that φ_0 is smooth on M_0 and has polyhomogeneous expansion as $|z| \to \infty$, with leading asymptotic in the end E_i given by $\varphi_0 = -x^{-\delta}/\delta^2 + c_i \log(x) + d_i + O(x)$ for some c_i, d_i which are smooth functions in S^1. For $\epsilon > 0$ small, we define the convexified weight $\varphi_\epsilon := \varphi - \frac{h}{\epsilon}\varphi_0$.

We recall from the proof of Proposition 3.1 in [7] the following estimate which is valid in any compact set $K \subset M_0$: for all $w \in C_0^\infty(K)$, we have

$$\frac{C}{\epsilon}\Big(\frac{1}{h}\|w\|_{L^2}^2 + \frac{1}{h^2}\|w|d\varphi|\|_{L^2}^2 + \frac{1}{h^2}\|w|d\varphi_\epsilon|\|_{L^2}^2 + \|dw\|^2_{L^2(K)}\Big) \leq \|e^{\varphi_\epsilon/h}\Delta_g e^{-\varphi_\epsilon/h}w\|_{L^2}^2 \tag{4}$$

where C depends on K but not on h and ϵ.

So for functions supported in the end E_i, it clearly suffices to obtain a Carleman estimate in $E_i \simeq \mathbb{R}^2 \setminus \{|z| \leq 1\}$ by using the Euclidean coordinate z of the end.

Proposition 3.1. *Let $\delta \in (0,1)$, and φ_ϵ as above, then there exists $C > 0$ such that for all $\epsilon \gg h > 0$ small enough, and all $u \in C_0^\infty(E_i)$*

$$h^2||e^{\varphi_\epsilon/h}(\Delta - \lambda^2)e^{-\varphi_\epsilon/h}u||_{L^2}^2 \geq \frac{C}{\epsilon}(||x^{1-\frac{\delta}{2}}u||_{L^2}^2 + h^2||x^{1-\frac{\delta}{2}}du||_{L^2}^2).$$

Proof. The metric g_0 can be extended to $\mathbb{R}^2$ to be the Euclidean metric and we shall denote by Δ the flat positive Laplacian on $\mathbb{R}^2$. Let us write $P := \Delta_{g_0} - \lambda^2$, then the operator $P_h := h^2 e^{\varphi_\epsilon/h}Pe^{-\varphi_\epsilon/h}$ is given by

$$P_h = h^2\Delta - |d\varphi_\epsilon|^2 + 2h\nabla\varphi_\epsilon.\nabla - h\Delta\varphi_\epsilon - h^2\lambda^2,$$

following the notation of [4, Chap. 4.3], it is a semiclassical operator in $S^0(\langle\xi\rangle^2)$ with semiclassical full Weyl symbol

$$\sigma(P_h) := |\xi|^2 - |d\varphi_\epsilon|^2 - h^2\lambda^2 + 2i\langle d\varphi_\epsilon, \xi\rangle = a + ib.$$

We can define $A := (P_h + P_h^*)/2 = h^2\Delta - |d\varphi_\epsilon|^2 - h^2\lambda^2$ and $B := (P_h - P_h^*)/2i = -2ih\nabla\varphi_\epsilon.\nabla + ih\Delta\varphi_\epsilon$ which have respective semiclassical full symbols a and b, i.e. $A = \mathrm{Op}_h(a)$ and $B =$

$\mathrm{Op}_h(b)$ for the Weyl quantization. Notice that A, B are symmetric operators, thus for all $u \in C_0^\infty(E_i)$

$$||(A+iB)u||^2 = \langle (A^2+B^2+i[A,B])u,u\rangle. \tag{5}$$

It is easy to check that the operator $ih^{-1}[A,B]$ is a semiclassical differential operator in $S^0(\langle\xi\rangle^2)$ with full semiclassical symbol

$$\{a,b\}(\xi) = 4(D^2\varphi_\epsilon(d\varphi_\epsilon, d\varphi_\epsilon) + D^2\varphi_\epsilon(\xi,\xi)) \tag{6}$$

Let us now decompose the Hessian of φ_ϵ in the basis $(d\varphi_\epsilon, \theta)$ where θ is a covector orthogonal to $d\varphi_\epsilon$ and of norm $|d\varphi_\epsilon|$. This yields coordinates $\xi = \xi_0 d\varphi_\epsilon + \xi_1\theta$ and there exist smooth functions M, N, K so that

$$D^2\varphi_\epsilon(\xi,\xi) = |d\varphi_\epsilon|^2(M\xi_0^2 + N\xi_1^2 + 2K\xi_0\xi_1).$$

Notice that φ_ϵ has a polyhomogeneous expansion at infinity of the form

$$\varphi_\epsilon(z) = \gamma.z + \frac{h}{\epsilon}\frac{r^\delta}{\delta^2} + c_1\log(r) + c_2 + c_3 r^{-1} + O(r^{-2})$$

where $r = |z|, \omega = z/r, \gamma = (\gamma_1,\gamma_2) \in \mathbb{R}^2$ and c_i are some smooth functions on S^1 depending on h; in particular we have

$$d\varphi_\epsilon = \gamma_1 dz_1 + \gamma_2 dz_2 + O(r^{-1+\delta}), \quad \partial_z^\alpha\partial_{\bar z}^\beta\varphi_\epsilon(z) = O(r^{-2+\delta}) \text{ for all } \alpha+\beta \geq 2$$

which implies that $M, N, K \in r^{-2+\delta}L^\infty(E_i)$. Then one can write

$$\begin{aligned}\{a,b\} =& 4|d\varphi_\epsilon|^2(M + M\xi_0^2 + N\xi_1^2 + 2K\xi_0\xi_1)\\ =& 4(N(a+h^2\lambda^2) + ((M-N)\xi_0 + 2K\xi_1)b/2 + (N+M)|d\varphi_\epsilon|^2)\end{aligned}$$

and since $M + N = \mathrm{Tr}(D^2\varphi_\epsilon) = -\Delta\varphi_\epsilon = h\Delta\varphi_0/\epsilon$ we obtain

$$\begin{gathered}\{a,b\} = 4|d\varphi_\epsilon|^2(c(z)(a+h^2\lambda^2) + \ell(z,\xi)b + \frac{h}{\epsilon}r^{-2+\delta}),\\ c(z) = \frac{N}{|d\varphi_\epsilon|^2}, \;\; \ell(z,\xi) = \frac{(M-N)\xi_0 + 2K\xi_1}{2|d\varphi_\epsilon|^2}.\end{gathered} \tag{7}$$

Now, we take a smooth extension of $|d\varphi_\epsilon|^2, a(z,\xi), \ell(z,\xi)$ and r to $z \in \mathbb{R}^2$, this can done for instance by extending r as a smooth positive function on $\mathbb{R}^2$ and then extending $d\varphi$ and $d\varphi_0$ to smooth non vanishing 1-forms on $\mathbb{R}^2$ (not necessarily exact) so that $|d\varphi_\epsilon|^2$ is smooth positive (for small h) and polynomial in h and a, ℓ are of the same form as in $\{|z| > 1\}$. Let us define the symbol and quantized differential operator on $\mathbb{R}^2$

$$e := 4|d\varphi_\epsilon|^2(c(z)(a+h^2\lambda^2) + \ell(z,\xi)b), \quad E := \mathrm{Op}_h(e)$$

and write

$$\begin{aligned}&ih^{-1}r^{1-\frac{\delta}{2}}[A,B]r^{1-\frac{\delta}{2}} = hF + r^{1-\frac{\delta}{2}}Er^{1-\frac{\delta}{2}} - \frac{h}{\epsilon}(A^2+B^2),\\ &\text{with } F := h^{-1}r^{1-\frac{\delta}{2}}(ih^{-1}[A,B] - E)r^{1-\frac{\delta}{2}} + \frac{1}{\epsilon}(A^2+B^2).\end{aligned} \tag{8}$$

We deduce from (6) and (7) the following

Lemma 3.2. *The operator F is a semiclassical differential operator in the class $S^0(\langle\xi\rangle^4)$ with semiclassical principal symbol*

$$\sigma(F)(\xi) = \frac{4|d\varphi|^2}{\epsilon} + \frac{1}{\epsilon}(|\xi|^2 - |d\varphi|^2)^2 + \frac{4}{\epsilon}(\langle\xi, d\varphi\rangle)^2.$$

By the semiclassical Gårding estimate, we obtain the

Corollary 3.3. *The operator F of Lemma 3.2 is such that there is a constant C so that*

$$\langle Fu, u\rangle \geq \frac{C}{\epsilon}(||u||^2_{L^2} + h^2||du||^2_{L^2}).$$

Proof. It suffices to use that $\sigma(F)(\xi) \geq \frac{C'}{\epsilon}(1+|\xi|^4)$ for some $C' > 0$ and use the semiclassical Gårding estimate. $\square$

So by writing $\langle i[A,B]u,u\rangle = \langle ir^{1-\frac{\delta}{2}}[A,B]r^{1-\frac{\delta}{2}}r^{-1+\frac{\delta}{2}}u, r^{-1+\frac{\delta}{2}}u\rangle$ in (5) and using (8) and Corollary 3.3, we obtain that there exists $C > 0$ such that for all $u \in C_0^\infty(E_i)$

$$\begin{aligned} ||P_h u||^2_{L^2} \geq & \langle (A^2 + B^2)u, u\rangle + \frac{Ch^2}{\epsilon}(||r^{-1+\frac{\delta}{2}}u||^2_{L^2} + h^2||r^{-1+\frac{\delta}{2}}du||^2_{L^2}) + h\langle Eu, u\rangle \\ & - \frac{h^2}{\epsilon}(||A(r^{-1+\frac{\delta}{2}}u)||^2_{L^2} + ||B(r^{-1+\frac{\delta}{2}}u)||^2_{L^2}). \end{aligned} \tag{9}$$

We observe that $h^{-1}[A, r^{-1+\frac{\delta}{2}}]r^{1+\frac{\delta}{2}} \in S^0(\langle\xi\rangle)$ and $h^{-1}[B, r^{-1+\frac{\delta}{2}}]r^{1+\frac{\delta}{2}} \in hS^0(1)$, and thus

$$||A(r^{-1+\frac{\delta}{2}}u)||^2_{L^2} + ||B(r^{-1+\frac{\delta}{2}}u)||^2_{L^2}) \leq C'(||Au||^2_{L^2} + ||Bu||^2_{L^2} + h^2||r^{-1+\frac{\delta}{2}}u||^2_{L^2} + h^4||r^{-1+\frac{\delta}{2}}du||^2_{L^2})$$

for some $C' > 0$. Taking h small, this implies with (9) that there exists a new constant $C > 0$ such that

$$||P_h u||^2_{L^2} \geq \frac{1}{2}\langle (A^2 + B^2)u, u\rangle + \frac{Ch^2}{\epsilon}(||r^{-1+\frac{\delta}{2}}u||^2_{L^2} + h^2||r^{-1+\frac{\delta}{2}}du||^2_{L^2}) + h\langle Eu, u\rangle. \tag{10}$$

It remains to deal with $h\langle Eu, u\rangle$: we first write $E = 4|d\varphi_\epsilon|^2(c(z)(A + h^2\lambda^2) + \mathrm{Op}_h(\ell)B) + hr^{-1+\frac{\delta}{2}}Sr^{-1+\frac{\delta}{2}}$ where S is a semiclassical differential operator in the class $S^0(\langle\xi\rangle)$ by the decay estimates on $c(z), \ell(z,\xi)$ as $z \to \infty$, then by Cauchy-Schwartz (and with $L := \mathrm{Op}_h(\ell)$)

$$\begin{aligned} |\langle hEu, u\rangle| \leq & Ch(||Au||_{L^2} + h^2||r^{-1+\frac{\delta}{2}}u||_{L^2} + h||Sr^{-1+\frac{\delta}{2}}u||_{L^2})||r^{-1+\frac{\delta}{2}}u||_{L^2} + Ch||Bu||_{L^2}||Lu||_{L^2} \\ \leq & \frac{1}{4}||Au||^2_{L^2} + h^2||Sr^{-1+\frac{\delta}{2}}u||^2_{L^2} + Ch^2||r^{-1+\frac{\delta}{2}}u||^2_{L^2} + \frac{1}{4}||Bu||^2_{L^2} + Ch^2||Lu||^2_{L^2} \end{aligned}$$

where C is a constant independent of h, ϵ but may change from line to line. Now we observe that $Lr^{1-\frac{\delta}{2}}$ and S are in $S^0(\langle\xi\rangle)$ and thus

$$||Sr^{-1+\frac{\delta}{2}}u||^2_{L^2} + ||Lu||^2_{L^2} \leq C(||r^{-1+\frac{\delta}{2}}u||^2_{L^2} + h^2||r^{-1+\frac{\delta}{2}}du||^2_{L^2}),$$

which by (10) implies that there exists $C > 0$ such that for all $\epsilon \gg h > 0$ with ϵ small enough

$$||P_h u||^2_{L^2} \geq \frac{Ch^2}{\epsilon}(||r^{-1+\frac{\delta}{2}}u||^2_{L^2} + h^2||r^{-1+\frac{\delta}{2}}du||^2_{L^2})$$

for all $u \in C_0^\infty(E_i)$. The proof is complete. $\square$

Combining now Proposition 3.1 and (4), we obtain

Proposition 3.4. *Let (M_0, g_0) be a Riemann surface with Euclidean ends with x a boundary defining function of the radial compactification $\overline{M}_0$ and let $\varphi_\epsilon = \varphi - \frac{h}{\epsilon}\varphi_0$ where φ is a harmonic function with non-degenerate critical points and linear growth on M_0 and φ_0 satisfies $\Delta_{g_0}\varphi_0 = x^{2-\delta}$ as above. Then for all $V \in x^{1-\frac{\delta}{2}}L^\infty(M_0)$ there exists an $h_0 > 0$, ϵ_0 and $C > 0$ such that for all $0 < h < h_0$, $h \ll \epsilon < \epsilon_0$ and $u \in C_0^\infty(M_0)$, we have*

$$\frac{1}{h}\|x^{1-\frac{\delta}{2}}u\|_{L^2}^2 + \frac{1}{h^2}\|x^{1-\frac{\delta}{2}}u|d\varphi|\|_{L^2}^2 + \|x^{1-\frac{\delta}{2}}du\|_{L^2}^2 \leq C\epsilon\|e^{\varphi_\epsilon/h}(\Delta_g + V - \lambda^2)e^{-\varphi_\epsilon/h}u\|_{L^2}^2 \tag{11}$$

Proof. As in the proof of Proposition 3.1 in [7], by taking ϵ small enough, we see that the combination of (4) and Proposition 3.1 shows that for any $w \in C_0^\infty(M_0)$,

$$\begin{aligned}\frac{C}{\epsilon}\Big(\frac{1}{h}\|x^{1-\frac{\delta}{2}}w\|_{L^2}^2 + \frac{1}{h^2}\|x^{1-\frac{\delta}{2}}w|d\varphi|\|_{L^2}^2 + \frac{1}{h^2}\|x^{1-\frac{\delta}{2}}w|d\varphi_\epsilon|\|_{L^2}^2 + \|x^{1-\frac{\delta}{2}}dw\|_{L^2}^2\Big)\\ \leq \|e^{\frac{\varphi_\epsilon}{h}}(\Delta - \lambda^2)e^{-\frac{\varphi_\epsilon}{h}}w\|_{L^2}^2\end{aligned}$$

which ends the proof. □

3.2. **The quadratic weight case for surfaces.** In this section, φ has quadratic growth at infinity, which corresponds to the case where $V \in e^{-\gamma/x^2}L^\infty$ for all $\gamma > 0$. The proof when φ has quadratic growth at infinity is even simpler than the linear growth case. We define $\varphi_0 \in x^{-2}L^\infty$ to be a solution of $\Delta_{g_0}\varphi_0 = 1$, this is possible by Lemma 2.8 and one easily obtains from Proposition 5.61 of [13] that $\varphi_0 = -x^{-2}/4 + O(x^{-1})$ as $x \to 0$. We let $\varphi_\epsilon := \varphi - \frac{h}{\epsilon}\varphi_0$ which satisfies $\Delta_{g_0}\varphi_\epsilon/h = -1/\epsilon$.

If $K \subset M_0$ is a compact set, the Carleman estimate (4) in K is satisfied by Proposition 3.1 of [7], it then remains to get the estimate in the ends $E_1, \dots, E_N$. But the exact same proof as in Lemma 3.1 and Lemma 3.2 of [7] gives directly that for any $w \in C_0^\infty(E_i)$

$$\frac{C}{\epsilon}\Big(\frac{1}{h}\|w\|_{L^2}^2 + \frac{1}{h^2}\|w|d\varphi|\|_{L^2}^2 + \frac{1}{h^2}\|w|d\varphi_\epsilon|\|_{L^2}^2 + \|dw\|_{L^2}^2\Big) \leq \|e^{\varphi_\epsilon/h}\Delta_{g_0}e^{-\varphi_\epsilon/h}w\|_{L^2}^2 \tag{12}$$

for some $C > 0$ independent of ϵ, h and it suffices to glue the estimates in K and in the ends E_i as in Proposition 3.1 of [7], to obtain (12) for any $w \in C_0^\infty(M_0)$. Then by using triangle inequality

$$||e^{\varphi_\epsilon/h}(\Delta_{g_0} + V - \lambda^2)e^{-\varphi_\epsilon/h}u||_{L^2} \leq ||e^{\varphi_\epsilon/h}\Delta_{g_0}e^{-\varphi_\epsilon/h}u||_{L^2} + C||u||_{L^2}$$

for some C depending on $\lambda, ||V||_{L^\infty}$, we see that the $V - \lambda^2$ term can be absorbed by the left hand side of (12) and we finally deduce

Proposition 3.5. *Let (M_0, g_0) be a Riemann surface with Euclidean ends and let $\varphi_\epsilon = \varphi - \frac{h}{\epsilon}\varphi_0$ where φ is a harmonic function with non-degenerate critical points and quadratic growth on M_0 and φ_0 satisfies $\Delta_{g_0}\varphi_0 = 1$ with $\varphi_0 \in x^{-2}L^\infty(M_0)$. Then for all $V \in L^\infty$ there exists an $h_0 > 0$, ϵ_0 and $C > 0$ such that for all $0 < h < h_0$, $h \ll \epsilon < \epsilon_0$ and $u \in C_0^\infty(M_0)$*

$$\frac{C}{\epsilon}\Big(\frac{1}{h}\|u\|_{L^2}^2 + \frac{1}{h^2}||u|d\varphi||_{L^2}^2 + \|du\|_{L^2}^2\Big) \leq \|e^{\varphi_\epsilon/h}(\Delta_{g_0} + V - \lambda^2)e^{-\varphi_\epsilon/h}u\|_{L^2}^2.$$

The main difference with the linear weight case is that one can use a convexification which has quadratic growth at infinity which allows to absorb the λ^2 term, while it was not the case for the linearly growing weights.

4. Complex Geometric Optics on a Riemann Surface with Euclidean ends

As in [1, 9, 7], the method for identifying the potential at a point p is to construct complex geometric optic solutions depending on a small parameter $h > 0$, with phase a Morse holomorphic function with a non-degenerate critical point at p, and then to apply the stationary phase method. Here, in addition, we need the phase to be of linear growth at infinity if $V \in e^{-\gamma/x}L^\infty$ for all $\gamma > 0$ while the phase has to be of quadratic growth at infinity if $V \in e^{-\gamma/x^2}L^\infty$ for all $\gamma > 0$.

We shall now assume that M_0 is a non-compact surface with genus g with N ends equipped with a metric g_0 which is Euclidean in the ends, and V is a $C^{1,\alpha}$ function in M_0. Moreover, if $V \in e^{-\gamma/x}L^\infty$ for all $\gamma > 0$, we ask that $N \geq \max(2g+1, 2)$ while if $V \in e^{-\gamma/x^2}L^\infty$ for all $\gamma > 0$, we assume that $N \geq g+1$. As above, let us use a smooth positive function x which is equal to 1 in a large compact set of M_0 and is equal to $x = |z|^{-1}$ in the regions $|z| > r_0$ of the ends $E_i \simeq \{z \in \mathbb{C}; |z| > 1\}$, where r_0 is a fixed large number. This function is a boundary defining function of the radial compactification of M_0 in the sense of Melrose [13]. To construct the complex geometric optics solutions, we will need to work with the weighted spaces $x^{-\alpha}L^2(M_0)$ where $\alpha \in \mathbb{R}_+$.

Let $\mathcal{H}$ be the finite dimensional complex vector space defined in the beginning of Section 2.3. Choose $p \in M_0$ such that there exists a Morse holomorphic function $\Phi = \varphi + i\psi \in \mathcal{H}$ on M_0, with a critical point at p; there is a dense set of such points by Proposition 2.1. The purpose of this section is to construct solutions u on M_0 of $(\Delta - \lambda^2 + V)u = 0$ of the form

$$u = e^{\Phi/h}(a + r_1 + r_2) \tag{13}$$

for $h > 0$ small, where $a \in x^{-J+1}L^2$ with $J \in \mathbb{R}_+ \setminus \mathbb{N}$ is a holomorphic function on M_0, obtained by Lemma 2.6, such that $a(p) \neq 0$ and a vanishing to order L (for some fixed large L) at all other critical points of Φ, and finally r_1, r_2 will be remainder terms which are small as $h \to 0$ and have particular properties near the critical points of Φ. More precisely, $e^{\varphi_0/\epsilon}r_2$ will be a $o_{L^2}(h)$ and r_1 will be a $O_{x^{-J}L^2}(h)$ but with an explicit expression, which can be used to obtain sufficient information in order to apply the stationary phase method.

4.0.1. *Construction of* r_1. We want to construct $r_1 = O_{x^{-J}L^2}(h)$ which satisfies

$$e^{-\Phi/h}(\Delta_{g_0} - \lambda^2 + V)e^{\Phi/h}(a + r_1) = O_{x^{-J}L^2}(h)$$

for some large $J \in \mathbb{R}_+ \setminus \mathbb{N}$ so that $a \in x^{-J+1}L^2$.

Let G be the operator of Lemma 2.8, mapping continuously $x^{-J+1}L^2(M_0)$ to $x^{-J-1}L^2(M_0)$. Then clearly $\bar\partial\partial G = \frac{i}{2}\star^{-1}$ when acting on $x^{-J+1}L^2$, here $\star^{-1}$ is the inverse of $\star$ mapping functions to 2-forms. First, we will search for r_1 satisfying

$$e^{-2i\psi/h}\partial e^{2i\psi/h}r_1 = -\partial G(a(V - \lambda^2)) + \omega + O_{x^{-J}H^1}(h) \tag{14}$$

with $\omega \in x^{-J}L^2(M_0)$ a holomorphic 1-form on M_0 and $\|r_1\|_{x^{-J}L^2} = O(h)$. Indeed, using the fact that Φ is holomorphic we have

$$e^{-\Phi/h}\Delta_{g_0}e^{\Phi/h} = -2i \star \bar\partial e^{-\Phi/h}\partial e^{\Phi/h} = -2i \star \bar\partial e^{-\frac{1}{h}(\Phi-\bar\Phi)}\partial e^{\frac{1}{h}(\Phi-\bar\Phi)} = -2i \star \bar\partial e^{-2i\psi/h}\partial e^{2i\psi/h}$$

and applying $-2i \star \bar\partial$ to (14), this gives

$$e^{-\Phi/h}(\Delta_{g_0} + V)e^{\Phi/h}r_1 = -a(V - \lambda^2) + O_{x^{-J}L^2}(h).$$

Writing $-\partial G(a(V-\lambda^2)) =: c(z)dz$ in local complex coordinates, $c(z)$ is $C^{2,\alpha}$ by elliptic regularity and we have $2i\partial_{\bar z}c(z) = a(V-\lambda^2)$, therefore $\partial_z\partial_{\bar z}c(p') = \partial_{\bar z}^2 c(p') = 0$ at each critical point $p' \neq p$ by construction of the function a. Therefore, we deduce that at each critical point $p' \neq p$, $c(z)$ has Taylor series expansion $\sum_{j=0}^2 c_j z^j + O(|z|^{2+\alpha})$. That is, all the lower order terms of the Taylor expansion of $c(z)$ around p' are polynomials of z only. By Lemma 2.7, and possibly by taking J larger, there exists a holomorphic function $f \in x^{-J}L^2$ such that $\omega := \partial f$ has Taylor expansion equal to that of $\partial G(a(V-\lambda^2))$ at all critical points $p' \neq p$ of Φ. We deduce that, if $b := -\partial G(a(V-\lambda^2)) + \omega = b(z)dz$, we have

$$\begin{aligned} |\partial_{\bar z}^m \partial_z^\ell b(z)| &= O(|z|^{2+\alpha-\ell-m}), \quad \text{for } \ell+m \leq 2, \text{ at critical points } p' \neq p \\ |b(z)| &= O(|z|), \qquad\qquad\qquad\qquad \text{if } p' = p. \end{aligned} \tag{15}$$

Now, we let $\chi_1 \in C_0^\infty(M_0)$ be a cutoff function supported in a small neighbourhood U_p of the critical point p and identically 1 near p, and $\chi \in C_0^\infty(M_0)$ is defined similarly with $\chi = 1$ on the support of χ_1. We will construct r_1 to be a sum $r_1 = r_{11} + hr_{12}$ where r_{11} is a compactly supported approximate solution of (14) near the critical point p of Φ and r_{12} is correction term supported away from p. We define locally in complex coordinates centered at p and containing the support of χ

$$r_{11} := \chi e^{-2i\psi/h} R(e^{2i\psi/h}\chi_1 b) \tag{16}$$

where $Rf(z) := -(2\pi i)^{-1}\int_{\mathbb{R}^2} \frac{1}{\bar z - \bar\xi} f d\bar\xi \wedge d\xi$ for $f \in L^\infty$ compactly supported is the classical Cauchy operator inverting locally ∂_z (r_{11} is extended by 0 outside the neighbourhood of p). The function r_{11} is in $C^{3,\alpha}(M_0)$ and we have

$$\begin{aligned} e^{-2i\psi/h}\partial(e^{2i\psi/h} r_{11}) &= \chi_1(-\partial G(a(V-\lambda^2)) + \omega) + \eta \\ \text{with } \eta &:= e^{-2i\psi/h} R(e^{2i\psi/h}\chi_1 b)\partial\chi. \end{aligned} \tag{17}$$

We then construct r_{12} by observing that b vanishes to order $2+\alpha$ at critical points of Φ other than p (from (15)), and $\partial\chi = 0$ in a neighbourhood of any critical point of ψ, so we can find r_{12} satisfying

$$2ir_{12}\partial\psi = (1-\chi_1)b. \tag{18}$$

This is possible since both $\partial\psi$ and the right hand side are valued in $T^*_{1,0}M_0$ and $\partial\psi$ has finitely many isolated zeroes on M_0: r_{12} is then a function which is in $C^{2,\alpha}(M_0 \setminus P)$ where $P := \{p_1, \dots, p_n\}$ is the set of critical points other than p, it extends to a function in $C^{1,\alpha}(M_0)$ and it satisfies in local complex coordinates z at each p_j

$$|\partial_{\bar z}^\beta \partial_z^\gamma r_{12}(z)| \leq C|z|^{1+\alpha-\beta-\gamma}, \quad \beta+\gamma \leq 2$$

by using also the fact that $\partial\psi$ can be locally be considered as a smooth function with a zero of order 1 at each p_j. Moreover $b \in x^{-J}H^2(M_0)$ thus $r_1 \in x^{-J}H^2(M_0)$ and we have

$$e^{-2i\psi/h}\partial(e^{2i\psi/h} r_1) = b + h\partial r_{12} + \eta = -\partial G(a(V-\lambda^2)) + \omega + h\partial r_{12} + \eta.$$

Lemma 4.1. *The following estimates hold true*

$$||\eta||_{H^2(M_0)} = O(|\log h|), \quad ||\eta||_{H^1(M_0)} \leq O(h|\log h|), \quad ||x^J\partial r_{12}||_{H^1(M_0)} = O(1),$$
$$||x^J r_1||_{L^2} = O(h), \quad ||x^J(r_1 - h\widetilde{r}_{12})||_{L^2} = o(h)$$

where $\widetilde{r}_{12}$ *solves* $2i\widetilde{r}_{12}\partial\psi = b$.

Proof. The proof is exactly the same as the proof of Lemma 4.2 in [8], except that one needs to add the weight x^J to have bounded integrals. $\square$

As a direct consequence, we have

Corollary 4.2. *With $r_1 = r_{11} + hr_{12}$, there exists $J > 0$ such that*

$$||e^{-\Phi/h}(\Delta_{g_0} - \lambda^2 + V)e^{\Phi/h}(a + r_1)||_{x^{-J}L^2(M_0)} = O(h|\log h|).$$

4.0.2. *Construction of r_2.* In this section, we complete the construction of the complex geometric optic solutions. We deal with the general case of surfaces and we shall show the following

Proposition 4.1. *If φ_0 is the subharmonic function constructed in Section 3, then for ϵ small enough there exist solutions to $(\Delta_{g_0} - \lambda^2 + V)u = 0$ of the form $u = e^{\Phi/h}(a + r_1 + r_2)$ with $r_1 = r_{11} + hr_{12}$ constructed in the previous section and $r_2 \in e^{-\varphi_0/\epsilon}L^2$ satisfying $\|e^{\varphi_0/\epsilon}r_2\|_{L^2} \leq Ch^{3/2}|\log h|$.*

This is a consequence of the following Lemma (which follows from the Carleman estimate obtained in Section 3 above)

Lemma 4.3. *Let $\delta \in (0, 1)$, $V \in x^{1-\frac{\delta}{2}}L^\infty(M_0)$, and $\varphi_\epsilon = \varphi - \frac{h}{\epsilon}\varphi_0$ a weight with linear growth at infinity as in Proposition 3.4. For all $f \in L^2(M_0)$ and all $h > 0$ small enough, there exists a solution $v \in L^2(M_0)$ to the equation*

$$e^{-\varphi_\epsilon/h}(\Delta_g - \lambda^2 + V)e^{\varphi_\epsilon/h}v = x^{1-\frac{\delta}{2}}f \tag{19}$$

satisfying

$$\|v\|_{L^2(M_0)} \leq Ch^{\frac{1}{2}}\|f\|_{L^2(M_0)}.$$

If φ_ϵ has quadratic growth at infinity, the same result is true when $V \in L^\infty(M_0)$ but $x^{1-\frac{\delta}{2}}f$ can be replaced by $f \in L^2$ in (19).

Proof. The proof is based on a duality argument. Let $P_h := e^{\varphi_\epsilon/h}(\Delta_g - \lambda^2 + V)e^{-\varphi_\epsilon/h}$ and for all $h > 0$ the real vector space $\mathcal{A} := \{u \in x^{-1+\frac{\delta}{2}}H^1(M_0); P_h u \in L^2(M_0)\}$ equipped with the real scalar product

$$(u, w)_{\mathcal{A}} := \langle P_h u, P_h w\rangle_{L^2}.$$

By the Carleman estimate of Proposition 3.4, the space $\mathcal{A}$ is a Hilbert space equipped with the scalar product above if $h < h_0$, and thus the linear functional $L : w \to \int_{M_0} x^{1-\frac{\delta}{2}}fw\,\mathrm{dvol}_{g_0}$ on $\mathcal{A}$ is continuous with norm bounded by $Ch^{\frac{1}{2}}||f||_{L^2}$ by Proposition 3.4, and by Riesz theorem there is an element $u \in \mathcal{A}$ such that $(., u)_{\mathcal{A}} = L$ and with norm bounded by the norm of L. It remains to take $v := P_h u$ which solves $P_h^* v = x^{1-\frac{\delta}{2}}f$ where $P_h^* = e^{-\varphi_\epsilon/h}(\Delta_g - \lambda^2 + V)e^{\varphi_\epsilon/h}$ is the adjoint of P_h and v satisfies the desired norm estimate. The proof when the weight φ_ϵ has quadratic growth at infinity is the same, but improves slightly due to the Carleman estimate of Proposition 3.5. $\square$

Proof of Proposition 4.1. We first solve the equation

$$(\Delta + V - \lambda^2)e^{\varphi_\epsilon/h}\widetilde{r}_2 = x^{1-\frac{\delta}{2}}e^{\varphi_\epsilon/h}\Big(x^{-1+\frac{\delta}{2}}e^{-\varphi_\epsilon/h}(\Delta + V - \lambda^2)e^{\Phi/h}(a + r_1)\Big)$$

by using Lemma 4.3 and the fact that for J large, there is $C>0$ such that for all $h<h_0$

$$||x^{-1+\frac{\delta}{2}}e^{-\varphi_\epsilon/h}(\Delta+V-\lambda^2)e^{\Phi/h}(a+r_1)||_{L^2}\leq C||x^Je^{-\Phi/h}(\Delta-\lambda^2+V)e^{\Phi/h}(a+r_1)||_{L^2}$$

since $x^{-J-1}e^{\varphi_0/\epsilon}\in L^\infty(M_0)$ for all J (recall that $\varphi_0\sim -x^{-\delta}/\delta^2$ as $x\to 0$). But now the right hand side is bounded by $O(h|\log h|)$ according to Corollary 4.2, therefore we set $r_2:=-e^{-i\psi/h-\varphi_0/\epsilon}\widetilde{r}_2$ which satisfies $(\Delta_{g_0}-\lambda^2+V)e^{\Phi/h}(a+r_1+r_2)=0$ and, by Lemma 4.3, the norm estimate $||e^{\varphi_0/\epsilon}r_2||_{L^2}\leq O(h^{3/2}|\log h|)$. □

5. Scattering on surface with Euclidean ends

Let (M_0,g_0) be a surface with Euclidean ends and $V\in e^{-\gamma/x}L^\infty(M_0)$ for some γ. The scattering theory in this setting is described for instance in Melrose [14], here we will follow this presentation (see also Section 3 in Uhlmann-Vasy [26] for the $\mathbb{R}^n$ case). First, using standard methods in scattering theory, we define the resolvent on the continuous spectrum as follows

Lemma 5.1. *The resolvent $R_V(\lambda):=(\Delta_{g_0}+V-\lambda^2)^{-1}$ admits a meromorphic extension from $\{\mathrm{Im}(\lambda)<0\}$ to $\{\mathrm{Im}(\lambda)\leq A,\mathrm{Re}(\lambda)\neq 0\}$, as a family of operators mapping $e^{-\gamma/x}L^2(M_0)$ to $e^{\gamma/x}L^2(M_0)$ for any $\gamma>A$. Moreover, for $\lambda\in\mathbb{R}\setminus\{0\}$ not a pole, $R_V(\lambda)$ maps continuously $x^\alpha L^2$ to $x^{-\alpha}L^2$ for any $\alpha>1/2$.*

Proof. The statement is known for $V=0$ and $M_0=\mathbb{R}^2$ by using the explicit formula of the resolvent convolution kernel on $\mathbb{R}^2$ in terms of Hankel functions (see for instance [14]), we shall denote $R_0(\lambda)$ this continued resolvent. More precisely, for all $A>0$, the operator $R_0(\lambda)$ continues analytically from $\{\mathrm{Im}(\lambda)<0\}$ to $\{\mathrm{Im}(\lambda)\leq A,\mathrm{Re}(\lambda)\neq 0\}$ as a family of bounded operators mapping $e^{-\gamma/x}L^2$ to $e^{\gamma/x}L^2$ for any $\gamma>A$. Now we can set $\chi\in C_0^\infty(M_0)$ such that $1-\chi$ is supported in the ends E_i, and let $\chi_0,\chi_1\in C_0^\infty(M_0)$ such that $(1-\chi_0)=1$ on the support of $(1-\chi)$ and $\chi_1=1$ on the support of χ. Let $\lambda_0\in -i\mathbb{R}_+$ with $i\lambda_0\gg 0$, then the resolvent $R_0(\lambda_0)$ is well defined from $L^2(M_0)$ to $H^2(M_0)$ since the Laplacian is essentially self-adjoint [23, Proposition 8.2.4], and we have a parametrix

$$E(\lambda):=(1-\chi_0)R_0(\lambda)(1-\chi)+\chi_1R_0(\lambda_0)\chi$$

which satisfies

$$(\Delta_{g_0}-\lambda^2+V)E(\lambda)=1+K(\lambda),$$
$$K(\lambda):=([\Delta_{g_0},\chi_1]-(\lambda^2-\lambda_0^2)\chi_1)R_0(\lambda_0)\chi-[\Delta_{g_0},\chi_0]R_0(\lambda)(1-\chi)+VE(\lambda),$$

where here we use the notation $R_0(\lambda)$ for an integral kernel on M_0, which in the charts $\{z\in\mathbb{R}^2;|z|>1\}$ corresponding the ends $E_1,\dots E_N$, is given by the integral kernel of $(\Delta_{\mathbb{R}^2}-\lambda^2)^{-1}$. Using the explicit expression of the convolution kernel of $R_0(\lambda)$ in the ends (see for instance Section 1.5 of [14]) and the decay assumption on V, it is direct to see that for $\mathrm{Im}(\lambda)<A,\mathrm{Re}(\lambda)\neq 0$, the map $\lambda\mapsto K(\lambda)$ a is compact analytic family of bounded operators from $e^{-\gamma/x}L^2$ to $e^{-\gamma/x}L^2$ for any $\gamma>A$. Moreover $1+K(\lambda_0)$ is invertible since $||K(\lambda_0)||_{L^2\to L^2}\leq 1/2$ if $i\lambda_0$ is large enough. Then by analytic Fredholm theory, the resolvent $R_V(\lambda)$ has an meromorphic extension to $\mathrm{Im}(\lambda)<A,\mathrm{Re}(\lambda)\neq 0$ as a bounded operator from $e^{-\gamma/x}L^2$ to $e^{\gamma/x}L^2$ if $\gamma>A$, given by

$$R_V(\lambda)=E(\lambda)(1+K(\lambda))^{-1}.$$

Now $(1+K(\lambda))^{-1} = 1 + Q(\lambda)$ for some $Q(\lambda) = -K(\lambda)(1+K(\lambda))^{-1}$ mapping $e^{-\gamma/x}L^2$ to itself for any $\gamma > A$, which proves the mapping properties of $R_V(\lambda)$ on exponential weighted spaces. For the mapping properties on $\{\mathrm{Re}(\lambda) = 0\}$, a similar argument works. □

A corollary of this Lemma is the mapping property

Corollary 5.2. *For $\lambda \in \mathbb{R} \setminus \{0\}$ not a pole of $R_V(\lambda)$, and $f \in e^{-\gamma/x}L^\infty$ for some $\gamma > 0$, then there exists $v \in C^\infty(\partial \overline{M}_0)$ such that*

$$R_V(\lambda)f - x^{\frac{1}{2}}e^{-i\lambda/x}v \in L^2.$$

Proof. Using the expression $R_V(\lambda) = E(\lambda)(1+Q(\lambda))$ of the proof of Lemma 5.1, it suffices to know the mapping property of $E(\lambda)$ on $e^{-\gamma/x}L^2$, but since outside a compact set (i.e. in the ends) $E(\lambda)$ is given by the free resolvent on $\mathbb{R}^2$, this amounts to proving the statement in $\mathbb{R}^2$, which is well-known: for instance, this is proved for $f \in C_0^\infty(\mathbb{R}^2)$ in Section 1.7 [14] but the proof extends easily to $f \in e^{-\gamma/x}L^\infty(\mathbb{R}^2)$ since the only used assumption on f for applying a stationary phase argument is actually that the Fourier transform $\hat{f}(z)$ has a holomorphic extension in a complex neighbourhood of $\mathbb{R}^2$. □

We also have a boundary pairing, the proof of which is exactly the same as [14, Lemma 2.2] (see also Proposition 3.1 of [26]).

Lemma 5.3. *For $\lambda > 0$ and $V \in e^{-\gamma/x}L^\infty(M_0)$, if $u_\pm \in x^{-\alpha}L^2(M_0)$ for some $\alpha > 1/2$ and $(\Delta_{g_0} - \lambda^2 + V)u_\pm \in x^\alpha L^2(M_0)$ with*

$$u_+ - x^{\frac{1}{2}}e^{i\lambda/x}f_{++} - x^{\frac{1}{2}}e^{-i\lambda/x}f_{+-} \in L^2, \quad u_- - x^{\frac{1}{2}}e^{i\lambda/x}f_{-+} - x^{\frac{1}{2}}e^{-i\lambda/x}f_{--} \in L^2$$

for some $f_{\pm\pm} \in C^\infty(\partial\overline{M}_0)$, then

$$\langle u_+, (\Delta_{g_0} + V - \lambda^2)u_-\rangle - \langle(\Delta_{g_0} + V - \lambda^2)u_+, u_-\rangle = 2i\lambda \int_{\partial\overline{M}_0} (f_{++}\overline{f_{-+}} - f_{+-}\overline{f_{--}})$$

where the volume form on $\partial\overline{M}_0 \simeq \sqcup_{i=1}^N S^1$ is induced by the metric $x^2 g|_{T\partial\overline{M}_0}$.

As a corollary, the same exact arguments as in Sections 2.2 to 2.5 in [14] show [1]

Corollary 5.4. *The operator $R_V(\lambda)$ is analytic on $\lambda \in \mathbb{R} \setminus \{0\}$ as a bounded operator from $x^\alpha L^2$ to $x^{-\alpha}L^2$ if $\alpha > 1/2$.*

In $\mathbb{R}^2$ there is a Poisson operator $P_0(\lambda)$ mapping $C^\infty(S^1)$ to $x^{-\alpha}L^2(\mathbb{R}^2)$ for $\alpha > 1/2$, which satisfies that for any $f_+ \in C^\infty(S^1)$ there exists $f_- \in C^\infty(S^1)$ such that

$$P_0(\lambda)f_+ - x^{\frac{1}{2}}e^{i\lambda/x}f_+ - x^{\frac{1}{2}}e^{-i\lambda/x}f_- \in L^2, \quad (\Delta - \lambda^2)P_0(\lambda)f_+ = 0.$$

We can therefore define in our case a similar Poisson operator $P_V(\lambda)$ mapping $C^\infty(\partial\overline{M}_0)$ to $x^{-\alpha}L^2$ for $\alpha > 1/2$, by

$$P_V(\lambda)f_+ := (1-\chi)P_0(\lambda)f_+ - R_V(\lambda)(\Delta_{g_0} + V - \lambda^2)(1-\chi)P_0(\lambda)f_+ \tag{20}$$

where $1 - \chi \in C^\infty(M_0)$ equals 1 in the ends E_i and $P_0(\lambda)$ denotes here the Schwartz kernel of the Poisson operator on $\mathbb{R}^2$ pulled back to each of the Euclidean ends E_i of M_0 in the

[1] In [14], a unique continuation is used for Schwartz solutions of $(\Delta + V - \lambda^2)u = 0$ when V is a compactly supported potential on $\mathbb{R}^n$ but the same result is also true in our setting, this is a consequence of a standard Carleman estimate.

obvious way. Then, since $(\Delta_{g_0}+V-\lambda^2)(1-\chi)P_0(\lambda)f_+\in e^{-\gamma/x}L^2$ for all $\gamma>0$, it suffices to use Corollaries 5.2 and 5.4 to see that it defines an analytic Poisson operator $P_V(\lambda)$ on $\lambda\in\mathbb{R}\setminus\{0\}$ satisfying that for all $f_+\in C^\infty(\partial\overline{M}_0)$, there exists $f_-\in C^\infty(\partial\overline{M}_0)$ such that

$$P_V(\lambda)f_+-x^{\frac12}e^{i\lambda/x}f_+-x^{\frac12}e^{-i\lambda/x}f_-\in L^2,\quad (\Delta+V-\lambda^2)P_V(\lambda)f_+=0. \tag{21}$$

Moreover, it is easily seen to be the unique solution of (21): indeed, if two such solutions exist then the difference is a solution u with asymptotic $x^{\frac12}e^{-i\lambda/x}f_-+L^2$ for some $f_-\in C^\infty(\partial\overline{M}_0)$, but applying Lemma 5.3 with $u_-=u_+=u$ shows that $f_-=0$, thus $u\in L^2$, which implies $u=0$ by Corollary 5.4.

Definition 5.5. *The scattering matrix $S_V(\lambda):C^\infty(\partial\overline{M}_0)\to C^\infty(\partial\overline{M}_0)$ for $\lambda\in\mathbb{R}\setminus\{0\}$ is defined to be the map $S_V(\lambda)f_+:=f_-$ where f_- is given by the asymptotic*

$$P_V(\lambda)f_+=x^{\frac12}e^{i\lambda/x}f_++x^{\frac12}e^{-i\lambda/x}f_-+g,\quad \textit{with}\ \ g\in L^2.$$

We remark that, using Lemma 5.3 and the uniqueness of the Poisson operator, one easily deduces for $\lambda\in\mathbb{R}\setminus\{0\}$

$$S_V(\lambda)^*=S_V(-\lambda)=S_V(\lambda)^{-1} \tag{22}$$

where the scalar product on $L^2(\partial\overline{M}_0)$ is induced by the metric $x^2g_0|_{T\partial\overline{M}_0}$.

We can now state a density result similar to Proposition 3.3 of [26]:

Proposition 5.6. *If $V\in e^{-\gamma_0/x}L^\infty(M_0)$ (resp. $V\in e^{-\gamma_0/x^2}L^\infty(M_0)$) for some $\gamma_0>0$, and $\lambda\in\mathbb{R}\setminus\{0\}$, then for any $0<\gamma<\gamma'<\gamma_0$ the set*

$$\mathcal{F}:=\{P_V(\lambda)f_+;f_+\in C^\infty(\partial\overline{M}_0)\}$$

is dense in the null space of $\Delta_{g_0}+V-\lambda^2$ in $e^{\gamma/x}L^2(M_0)$ for the topology of $e^{\gamma'/x}L^2(M_0)$ (resp. in $e^{\gamma/x^2}L^2(M_0)$ for the topology of $e^{\gamma'/x^2}L^2(M_0)$).

Proof. First assume $V\in e^{-\gamma_0/x}L^\infty(M_0)$. Let $w\in e^{-\gamma'/x}L^2$ be orthogonal to $\mathcal{F}$, and set $u_-:=R_V(\lambda)w$ and $u_+=P_V(\lambda)f_{++}$ for some $f_{++}\in C^\infty(\partial\overline{M}_0)$. Then, define $f_{--}\in C^\infty(\partial\overline{M}_0)$ by $R_V(\lambda)w-x^{\frac12}e^{-i\lambda/x}f_{--}\in L^2$, and from Lemma 5.3 we obtain $\langle f_{+-},f_{--}\rangle=0$ since $\langle w,P_V(\lambda)f_{++}\rangle=0$ by assumption. Since $f_{+-}=S_V(\lambda)f_{++}$ is arbitrary, then $f_{--}=0$ and $u_-\in L^2$. In particular, from the parametrix constructed in the proof of Lemma 5.1

$$R_V(\lambda)w-(1-\chi_0)R_0(\lambda)(1-\chi)(1+Q(\lambda))w\in L^2$$

with $(1+Q(\lambda))w\in e^{-\gamma'/x}L^2$. Since in each end, $R_0(\lambda)$ is the integral kernel of the free resolvent of the Euclidean Laplacian on $\mathbb{R}^2$ and $(1-\chi_0)$ and $(1-\chi)$ are supported in the ends, we can view the term $(1-\chi_0)R_0(\lambda)(1-\chi)(1+Q(\lambda))w$ as a disjoint sum (over the ends) of functions on $\mathbb{R}^2$ of the form

$$(1-\chi_0(z))\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{iz\xi}(\xi^2-\lambda^2-i0)^{-1}\hat{f}(\xi)d\xi \tag{23}$$

where in each end E_i, $f=(1-\chi)(1+Q(\lambda))w\in e^{-\gamma'/x}L^2(E_i)$ can be considered as a function in $e^{-\gamma'|z|}L^2(\mathbb{R}^2)$. By the Paley-Wiener theorem, $\hat{f}$ is holomorphic in a strip $U=\{|\mathrm{Im}(\xi)|<\gamma'\}$ with bound $\sup_{\eta\leq\gamma}||\hat{f}(\cdot+i\eta)||_{L^2(\mathbb{R}^2)}<\infty$ for all $\gamma<\gamma'$, so the fact that (23) is in L^2 implies that $\hat{f}$ vanishes at the real sphere $\{\xi\in\mathbb{R}^2;\xi^2=\lambda^2\}$, and thus there exists h holomorphic in U such that $\hat{f}(\xi)=(\xi^2-\lambda^2)h(\xi)$ (see e.g. the proof of Lemma 2.5 in [17]), and satisfying the same types of L^2 estimates as $\hat{f}$ in U on lines $\mathrm{Im}(\xi)=\mathrm{cst}$. By the Paley-Wiener theorem again, we deduce that (23) is in $e^{-\gamma|z|}L^2$ and thus $R_V(\lambda)w \in e^{-\gamma/x}L^2(M_0)$ for any $\gamma < \gamma'$. Then if $v \in e^{\gamma/x}L^2(M_0)$ and $(\Delta_{g_0} + V - \lambda^2)v = 0$, one has by integration by parts

$$0 = \langle R_V(\lambda)w, (\Delta_{g_0} + V - \lambda^2)v\rangle = \langle w, v\rangle$$

which ends the proof in the case $V \in e^{-\gamma_0/x}L^\infty(M_0)$. The quadratic decay case $V \in e^{-\gamma_0/x^2}L^\infty(M_0)$ is exactly similar but instead of Paley-Wiener theorem, we use Corollary 7.3 and the inclusions $e^{-\gamma'/x^2}L^2 \subset e^{-\gamma''/x^2}L^1 \cap e^{-\gamma''/x^2}L^2$ and $e^{-\gamma'/x^2}L^\infty \subset e^{-\gamma/x^2}L^2$ for all $\gamma < \gamma'' < \gamma'$. □

6. Identifying the potential

6.1. **The case of a surface.** On a Riemann surface (M_0, g_0) with N Euclidean ends and genus g, we assume that $V_1, V_2 \in C^{1,\alpha}(M_0)$ are two real valued potentials such that the respective scattering operators $S_{V_1}(\lambda)$ and $S_{V_2}(\lambda)$ agree for a fixed $\lambda > 0$. We also assume that for all $\gamma > 0$

$$V_1, V_2 \in \begin{cases} e^{-\gamma/x}L^\infty(M_0) & \text{if } N \geq \max(2g+1, 2) \\ e^{-\gamma/x^2}L^\infty(M_0) & \text{if } N \geq g+1. \end{cases}$$

By considering the asymptotics of $u_1 := P_{V_1}(\lambda)f_1$ and $P_{V_2}(-\lambda)f_2$ for $f_i \in C^\infty(\partial\overline{M}_0)$ we easily have by integration by parts that

$$\begin{aligned} \int_{M_0} (V_1 - V_2)u_1\overline{u_2}\,\mathrm{dvol}_{g_0} &= -2i\lambda \int_{\partial\overline{M}_0} S_{V_1}(\lambda)f_1.\overline{f_2} - f_1.\overline{S_{V_2}(-\lambda)f_2} \\ &= -2i\lambda \int_{\partial\overline{M}_0} (S_{V_1}(\lambda) - S_{V_2}(\lambda))f_1.\overline{f_2} = 0 \end{aligned} \tag{24}$$

by using (22). From Proposition 5.6, this implies by density that, if $V \in e^{-\gamma/x}L^\infty$ (resp. $V \in e^{-\gamma/x^2}L^\infty$ for all $\gamma > 0$), then for all solutions u_i of $(\Delta_{g_0} + V_i - \lambda^2)u_i = 0$ in $e^{\gamma'/x}L^2(M_0)$ (resp. $u_i \in e^{\gamma'/x^2}L^2(M_0)$) for some $\gamma' > 0$, we have

$$\int_{M_0} (V_1 - V_2)u_1\overline{u_2}\,\mathrm{dvol}_{g_0} = 0. \tag{25}$$

We shall now use our complex geometric optics solutions as special solutions in the weighted space $e^{-\gamma'/hx}L^2(M_0)$ (resp. $e^{-\gamma'/hx^2}L^2(M_0)$) for some $\gamma' > 0$ if $V \in e^{-\gamma/x}L^\infty$ (resp. $V \in e^{-\gamma/x^2}L^\infty$) for all $\gamma > 0$.

Let $p \in M_0$ be such that, using Proposition 2.1, we can choose a holomorphic Morse function $\Phi = \varphi + i\psi$ with linear or quadratic growth on M_0 (depending on the topological assumption), with a critical point at p. Then for the complex geometric optics solutions u_1, u_2 with phase Φ constructed in Section 4, the identity (25) holds true. We will then deduce the

Proposition 6.1. *Let $\lambda \in (0, \infty)$ and assume that $S_{V_1}(\lambda) = S_{V_2}(\lambda)$, then $V_1(p) = V_2(p)$.*

Proof. Let u_1 and u_2 be solutions on M_0 to

$$(\Delta_g + V_j - \lambda^2)u_j = 0$$

constructed in Section 4 with phase Φ for u_1 and $-\Phi$ for u_2, thus of the form

$$u_1 = e^{\Phi/h}(a + r_1^1 + r_2^1), \quad u_2 = e^{-\Phi/h}(a + r_1^2 + r_2^2).$$

We have the identity

$$\int_{M_0} u_1(V_1 - V_2)\overline{u_2}\,\mathrm{dvol}_{g_0} = 0$$

Then by using the estimates in Lemma 4.1 and Proposition 4.1 we have, as $h \to 0$,

$$\int_{M_0} e^{2i\psi/h}|a|^2(V_1 - V_2)\,\mathrm{dvol}_{g_0} + h\int_{M_0} e^{2i\psi/h}(\overline{a}\widetilde{r}^1_{12} + a\overline{\widetilde{r}^2_{12}})(V_1 - V_2)\,\mathrm{dvol}_{g_0} + o(h) = 0$$

where $\widetilde{r}^j_{12} \in L^\infty(M_0)$ are defined in Lemma 4.1, with the superscript j refering to the solution for the potential V_j; in particular these functions $\widetilde{r}^j_{12}$ are independent of h.

By splitting $V_i(\cdot) = (V_i(\cdot) - V_i(p)) + V_i(p)$ and using the $C^{1,\alpha}$ regularity assumption on V_i, one can use stationary phase for the $V_i(p)$ term and integration by parts to gain a power of h for the $V_i(\cdot) - V_i(p)$ term (see the proof of Lemma 5.4 in [8] for details) to deduce

$$\int_{M_0} e^{2i\psi/h}|a|^2(V_1 - V_2)\,\mathrm{dvol}_{g_0} = Ch(V_1(p) - V_2(p)) + o(h)$$

for some $C \neq 0$. Therefore,

$$Ch(V_1(p) - V_2(p)) + h\int_{M_0} e^{2i\psi/h}(\overline{a}\widetilde{r}^1_{12} + a\overline{\widetilde{r}^2_{12}})(V_1 - V_2)\,\mathrm{dvol}_{g_0} = o(h).$$

Now to deal with the middle terms, it suffices to apply a Riemann-Lebesgue type argument like Lemma 5.3 of [8] to deduce that it is a $o(h)$. The argument is simply to approximate the amplitude in the $L^1(M_0)$ norm by a smooth compactly supported function and then use stationary phase to deal with the smooth function. We have thus proved that $V_1(p) = V_2(p)$ by taking $h \to 0$. □

7. Appendix

To obtain mapping properties of the resolvent of $\Delta_{\mathbb{R}^2}$ acting on functions with Gaussian decay, we shall give two Lemmas on Fourier transforms of functions with Gaussian decay.

Lemma 7.1. *Let $f(z) \in e^{-\gamma|z|^2}L^2(\mathbb{R}^2)$ for some $\gamma > 0$. Then the Fourier transform $\hat{f}(\xi)$ extends analytically to $\mathbb{C}^2$ and for all $\xi, \eta \in \mathbb{R}^2$,*

$$||\hat{f}(\xi + i\eta)||_{L^2(\mathbb{R}^2, d\xi)} \leq 2\pi e^{\frac{|\eta|^2}{4\gamma}}||e^{\gamma|z|^2}f||_{L^2(\mathbb{R}^2)}.$$

If $f(z) \in e^{-\gamma|z|^2}L^1(\mathbb{R}^2)$ for some $\gamma > 0$ then

$$\sup_{\xi\in\mathbb{R}^2}|\hat{f}(\xi + i\eta)| \leq e^{\frac{|\eta|^2}{4\gamma}}||e^{\gamma|z|^2}f||_{L^1(\mathbb{R}^2)}.$$

Proof. The first statement is clear. For the bound, we write

$$\hat{f}(\xi + i\eta) = e^{\frac{|\eta|^2}{4\gamma}}\int_{\mathbb{R}^2} e^{-i\xi.z}e^{-\gamma|z-\frac{\eta}{2\gamma}|^2}e^{\gamma|z|^2}f(z)dz = e^{\frac{|\eta|^2}{4\gamma}}\mathcal{F}_{z\to\xi}(e^{-\gamma|z-\frac{\eta}{2\gamma}|^2}e^{\gamma|z|^2}f(z)).$$

But the function $e^{-\gamma|z-\frac{\eta}{2\gamma}|^2}e^{\gamma|z|^2}f(z)$ is in $L^2(\mathbb{R}^2, dz)$ and its norm is bounded uniformly by $||e^{\gamma|z|^2}f||_{L^2}$, thus it suffices to use the Plancherel theorem to obtain the desired bound. The L^∞ bound is similar. □

Lemma 7.2. *Let $F(\xi+i\eta)$ be a complex analytic function on $\mathbb{R}^2+i\mathbb{R}^2=\mathbb{C}^2$ such that there is $C>0$ and $\gamma>0$ with*

$$||F(\xi+i\eta)||_{L^2(\mathbb{R}^2,d\xi)}\leq Ce^{\frac{|\eta|^2}{4\gamma}} \text{ and } \sup_{\xi\in\mathbb{R}^2}|F(\xi+i\eta)|\leq Ce^{\frac{|\eta|^2}{4\gamma}}.$$

If F vanishes on the real submanifold $\{|\xi|^2=\lambda^2\}$, then $\mathcal{F}^{-1}_{\xi\to z}(\frac{F(\xi)}{|\xi|^2-\lambda^2})\in e^{-\gamma|z|^2}L^\infty(\mathbb{R}^2)$.

Proof. First by analyticity of F, one has that F vanishes on the complex hypersurface $M_\lambda:=\{\zeta\in\mathbb{C}^2;\zeta.\zeta=\lambda^2\}$ (see for instance the proof of Lemma 2.5 of [17]), and in particular $G(\zeta)=F(\zeta)/(\zeta.\zeta-\lambda^2)$ is an analytic function on $\mathbb{C}^2$. We will first prove that for each $\eta\in\mathbb{R}^2$, $G(\xi+i\eta)\in L^1(\mathbb{R}^2,d\xi)\cap L^\infty(\mathbb{R}^2,d\xi)$ and

$$||G(\xi+i\eta)||_{L^1(\mathbb{R}^2,d\xi)}\leq Ce^{\frac{|\eta|^2}{4\gamma}}. \tag{26}$$

If $|\eta|\leq 2$ we choose the disc $B:=\{\xi\in\mathbb{R}^2;|\xi|^2<2(4+\lambda^2)\}$ and let $\zeta:=\xi+i\eta$. Then $||G(\xi+i\eta)||_{L^1(B,d\xi)}$ and $||(\zeta.\zeta-\lambda^2)^{-1}||_{L^2(\mathbb{R}^2\setminus B,d\xi)}$ are uniformly bounded for $|\eta|\leq 2$, and we obtain by Cauchy-Schwarz that (26) holds for $|\eta|\leq 2$. For the case $|\eta|>2$ we define $U_\eta:=\{\xi\in\mathbb{R}^2;|\zeta.\zeta-\lambda^2|>|\eta|\}$ and note that

$$\sup_{|\eta|>2}||(\zeta.\zeta-\lambda^2)^{-1}||_{L^1(\mathbb{R}^2\setminus U_\eta,d\xi)}<\infty,$$
$$\sup_{|\eta|>2}||(\zeta.\zeta-\lambda^2)^{-1}||_{L^2(U_\eta,d\xi)}<\infty.$$

These results follow by decomposing the integration sets to parts where one can change coordinates $\xi_1+i\xi_2$ to $\tilde{\xi_1}+i\tilde{\xi_2}:=\zeta.\zeta-\lambda^2$, and by evaluating simple integrals. Then (26) follows from Cauchy-Schwarz and the estimates for F.

Let $\eta=2\gamma z$, we use a contour deformation from $\mathbb{R}^2$ to $2i\gamma z+\mathbb{R}^2$ in $\mathbb{C}^2$,

$$\int_{\mathbb{R}^2}e^{iz.\xi}G(\xi)d\xi=\int_{\mathbb{R}^2}e^{iz.(\xi+2i\gamma z)}G(\xi+2i\gamma z)d\xi,$$

which is justified by the fact that $G(\xi+i\eta)\in L^1(\mathbb{R}^2\times K,d\xi\,d\eta)$ for any compact set K in $\mathbb{R}^2$ by the uniform bound (26). Now using (26) again shows that

$$\Big|\int_{\mathbb{R}^2}e^{iz.\xi}G(\xi)d\xi\Big|\leq Ce^{-\gamma|z|^2}$$

which ends the proof. □

Corollary 7.3. *Let $f(z)\in e^{-\gamma|z|^2}L^2(\mathbb{R}^2)\cap e^{-\gamma|z|^2}L^1(\mathbb{R}^2)$ for some $\gamma>0$. Assume that its Fourier transform $\hat{f}(\xi)$ vanishes on the sphere $\{|\xi|=|\lambda|\}$, then one has*

$$\mathcal{F}^{-1}_{\xi\to z}\Big(\frac{\hat{f}(\xi)}{|\xi|^2-\lambda^2}\Big)\in e^{-\gamma|z|^2}L^\infty(\mathbb{R}^2).$$

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