Variational approach to non-local curvature flows

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Introduction

- Perimeters, and Geometric “gradient” flows
- Study of a non-local, singular perimeter
- A word on existence and uniqueness
- Construction of a flow by an implicit variational scheme: a general framework

Classical mean curvature flow

The classical mean curvature flow is typically defined as the "gradient flow" of the perimeter. If ϕ is a smooth vector field, E a smooth set, and $E_\varepsilon = \{x + \varepsilon\phi(x), x \in E\}$ ("inner variations"):

$$\lim_{\varepsilon \to 0} \frac{P(E_\varepsilon) - P(E)}{\varepsilon} = \int_{\partial E} \kappa(\phi \cdot \nu) \, d\mathcal{H}^{d-1}$$

Then one looks for sets $E(t)$ whose normal velocity is $V = -\kappa$
This can be generalized in many ways, replacing the perimeter with "similar" set functions.

General Perimeters

A non-negative function $P(E)$ of measurable sets of $\Omega \subset \mathbb{R}^N$ is a Perimeter if

- $P(\emptyset) = P(\Omega) = 0$;
- P is l.s.c. with respect to L^1 convergence;
- $P(E \cup F) + P(E \cap F) \ \leq \ P(E) + P(F)$

This last property is called "submodularity" in discrete combinatorial optimization, and is related to convexity (Lovasz, 82).

If P is a "perimeter", we can associate a "total variation" through the *generalized coarea formula* (Visintin, C-Giacomini-Lussardi)

$$J(u) \;=\; \int_{-\infty}^{+\infty} P(\{u > s\})\, ds$$

Then: the convexity of J is equivalent to the "submodularity" of P (cf Lovasz's extension in the discrete setting).
It implies a comparison principle for the minimizers of energies of the form

$$P(E) \;+\; \int_E g(x)\, dx$$

How to define the associated "curvature flow"?

Natural ways to define a curvature flow associated to the perimeter P are

- a "classical" way: one defines a curvature by computing the first variation

$$\lim_{\varepsilon\to 0}\frac{P(E_\varepsilon)-P(E)}{\varepsilon} = \int_{\partial E}\kappa_P(E,x)(\phi\cdot\nu)\,d\mathcal{H}^{d-1}$$

  and one then tries to find sets $E(t)$ with $V=-\kappa_P$ (the normal velocity), at least in a weak sense. The natural setting is the *level set* approach, where one looks for u solving in an appropriate weak sense

$$\frac{\partial u}{\partial t} = |Du|\kappa_p(\partial\{u\geq u(x)\},x)$$

How to define the associated "curvature flow"?

- a variational way (cf Almgren-Taylor-Wang, Luckhaus-Sturzenhecker, 90-95): one fixes a time-step $h > 0$, and given E_0 a (compact) set, one defines E^{n+1} from E^n, $n \geq 0$ by solving

$$\min_E P(E) \; + \; \frac{1}{h} \int_E d_{E^n}$$

where $d_F(x) = \operatorname{dist}(x, F) - \operatorname{dist}(x, F^c)$ is the signed distance to the boundary of F. The Euler-Lagrange to this problem is (formally)

$$\frac{d_{E^n}(x)}{h} \; = \; -\kappa_P(E^{n+1}, x)$$

Then one defines $E_h(t) := E_{[t/h]}$ and tries to send $h \to 0$.

Of course, one would like (when κ_P can be defined) to show that the second approach yields a solution to the first point of view.

$\Rightarrow$ We will see that this is true under quite natural assumptions.

Similar works: C. (04), Caselles-C. (07),
Bellettini-Caselles-C.-Novaga (06), C.-Novaga (06/08) for convex crystals, C.-Novaga (Non-regular forcing terms, 2009, Obstacle, 2012), Ciomaga-Thouroude (mobilities, 20xx?),
T. Eto-Y. Giga-K Ishii (Unbounded, 2012).

A nonstandard perimeter

We have first been interested in a particular perimeter/total variation which has been proposed for image regularization in "*A variational model for infinite perimeter segmentations based on Lipschitz level set functions: denoising while keeping finely oscillatory boundaries*" (SIAM MMS, 2010) by Marco Barchiesi, Sung Ha Kang, Triet M. Le, Massimiliano Morini, Marcello Ponsiglione.

The goal in that paper was to smooth shapes, removing isolated details but keeping the small details of the boundaries.

A strange "perimeter"

Idea: replace the measure of the perimeter with

$$\mathcal{M}_\delta(E) = \frac{1}{2\delta}|\{\mathrm{dist}\,(x, \partial E) \le \delta\}|$$

Heuristics

• "Large objects" (w.r. δ) are measured as with the classical perimeter

• Small details are ignored, if they are close to each other or larger parts

• Small isolated details are very expensive

Remark well known that $\mathcal{M}_\delta(E)$ ($\Gamma-$)converges to the classical perimeter as $\delta \downarrow 0$ ("Minkowski contents").

It is not hard to check that this satisfies the submodularity

$$\mathcal{M}_\delta(E \cup F) + \mathcal{M}_\delta(E \cap F) \ \leq \ \mathcal{M}_\delta(E) + \mathcal{M}_\delta(F)$$

This implies

- comparison principle for the minimizers of $\mathcal{M}_\delta(E) + \int_E g$;
- ellipticity of the associated "curvature";
- convexity of the extension by co-area formula.

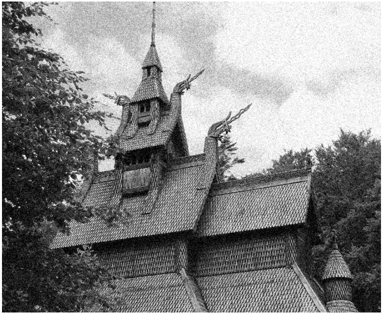
Non-local total variation

The associated "total variation" is

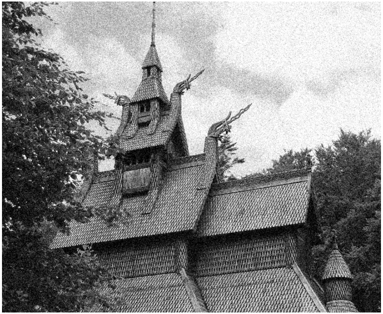
$$J_\delta(u) \;=\; \frac{1}{2\delta}\int \mathrm{osc}(u; B_\delta(x))\,dx \;=\; \int_{-\infty}^{\infty} \mathcal{M}_\delta(\{u \geq s\})\,ds$$

To give an idea of the behaviour, one can solve the associated "image denoising" ("ROF") problem

$$\min_u J_\delta(u) \;+\; \frac{1}{2}\|u-g\|^2$$



Resolution of “ROF” with the functional based on oscillation: large details are smoothed out as with the total variation while small oscillations are kept. [The method is based on graph-cuts]



The same with the standard Total Variation

The “curvature” of the oscillation

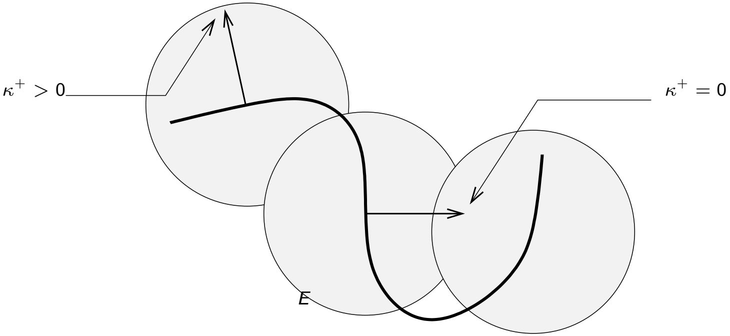
We can associate to the first variation of $\mathcal{M}_\delta(E)$ a “curvature”:

$$\kappa_\delta(x) \ = \ \kappa_\delta^+(x) + \kappa_\delta^-(x) \qquad ,$$

$$\kappa_\delta^+(x) \ = \begin{cases} \dfrac{1}{2\delta}\det(I + \delta\nabla\nu_E(x)) & \text{if there exists } t > \delta : \\ & \operatorname{dist}(x + t\nu_E(x), E) = t\,, \\ 0 & \text{otherwise,} \end{cases}$$

and κ_δ^- in a similar way, replacing E with E^c and the '$+$' with '$-$'.

The nonlocal curvature



There are several issues: it is

- non continuous
- nonlocal
- singular for small sets, in dimension ≥ 3 (indeed $\kappa_\delta^\pm$ are polynomials of degree $d-1$ of the curvatures).

For the first issue, we cannot do much and we regularize in an appropriate way both $\mathcal{M}_\delta$ and its curvature: we define

$$\mathcal{M}^f(E) \;=\; \int_0^\delta \alpha(s)\mathcal{M}_s(E)\,ds$$

where $\alpha \geq 0$ is a continuous function with support in $(0,\delta)$.
In fact, it corresponds to defining

$$\mathcal{M}^f(E) \;=\; \int_\Omega f(d_E(x))\,dx$$

where $f(t) = \int_{|t|}^\delta \frac{\alpha(s)}{2s}\,ds$ is even, C^1, nonincreasing on $[0,+\infty)$, constant near 0, with support in $[-\delta,\delta]$. Then $\mathcal{M}_\delta$ corresponds to $f = \chi_{[-\delta,\delta]}$, that is $\alpha = (2\delta)\delta_\delta$.
The corresponding curvature is $\kappa^f = \int_0^\delta \alpha(s)\kappa_s\,ds$.

Still, there remain the two last issues: κ^f is

- nonlocal
- singular for small sets, in dimension ≥ 3.

A first result: we adapt the works of [Ishii-Souganidis95, Goto94, Slepčev03], for nonlocal, singular evolutions.

Theorem Existence and uniqueness for the BUC viscosity solution of the geometric equation

$$\frac{\partial u}{\partial t} = F(x, Du, D^2u, \{u > u(x)\}), \qquad u(t=0) = u_0 \in BUC(\mathbb{R}^d)$$

if F satisfies:

i) Translational invariance: $F(x+r,p,X,K+r) = F(x,p,X,K)$ for every $r \in \mathbb{R}^d$;

ii) Degenerate ellipticity: $F(x,p,X,E) \geq F(x,p,Y,E)$ if $X \leq Y$;

iii) Monotonicity wr the set: $F(x,p,X,E) \geq F(x,p,X,G)$ if $E \subseteq G$;

iv) Geometric: $F(x,\lambda p,\lambda X + \mu p \otimes p, E) = \lambda F_f(x,p,X,E)$ for all $\lambda \geq 0$, $\mu \in \mathbb{R}$.

v) Continuity: F is continuous with respect to its first variable (x), moreover, the following properties hold:

v.1) If $x_n \to x$, $p_n \to p \neq 0$, $X_n \to X$ and $\{K_n\}$ is a sequence of closed sets converging to K in the Kuratowski sense, then

$$F(x,p,X,K) \leq \liminf_n F(x_n,p_n,X_n,K_n).$$

v.2) If $x_n \to x$, $p_n \to p \neq 0$, $X_n \to X$ and $\{A_n\}$ is a sequence of open sets such that A_n^c converges to A^c in the Kuratowski sense, then

$$F(x,p,X,A) \geq \limsup_n F(x_n,p_n,X_n,A_n).$$

vi) There exists a continuous function $c : (0,+\infty) \mapsto (0,+\infty)$ such that, for all x,p,E,

$$-c(|p|) \leq F(x,p,\pm I,E) \leq c(|p|).$$

The proof is a copy-paste of Ishii-Souganidis' proof with ideas taken from Slepčev's paper.

Issue

How do we define F?

$$F_f(x,p,X,K) := F_f^+(x,p,X,K) \, + \, F_f^-(x,p,X,K)$$

where

$$F_f^\pm(x,p,X,K) \;=\; \int_0^\delta \alpha(s) F_s^\pm(x,p,X,K)\, ds,$$

and

$$F_s^+(x,p,X,K) \;=\; \begin{cases} -\frac{|p|}{2s}\,\mathrm{det}\left(I + \frac{s}{|p|}\mathcal{P}_{\hat{p}} X \mathcal{P}_{\hat{p}}\right) \\ \qquad\qquad\qquad \text{if} \begin{cases} p \neq 0,\; I + \frac{s}{|p|}\mathcal{P}_{\hat{p}} X \mathcal{P}_{\hat{p}} \geq 0, \\ \mathrm{dist}\left(x + s\hat{p}, K^c\right) \geq s, \end{cases} \\ 0 \text{ otherwise.} \end{cases}$$

$F_s^-(x,p,X,K)$ is defined in a symmetric way. Here $\mathcal{P}_{\hat{p}} := (I - \hat{p} \otimes \hat{p})$, where, for $p \neq 0$, $\hat{p} = p/|p|$.

Issue...

- This F_f does not satisfy the continuity assumptions (v,v.1,v.2)!
- Possible solution: smooth the Hamiltonian, letting for instance:

$$F_\varepsilon(x,p,X,E) \ := \ \frac{|p|}{2s}\det\left[I-\frac{s}{|p|}\mathcal{P}_{\hat{p}}X\mathcal{P}_{\hat{p}}\right]^{+} H_\varepsilon(\mathrm{dist}\,(x-s\hat{p},E)-s)$$
$$- \ \frac{|p|}{2s}\det\left[I+\frac{s}{|p|}\mathcal{P}_{\hat{p}}X\mathcal{P}_{\hat{p}}\right]^{+} H_\varepsilon(\mathrm{dist}\,(x+s\hat{p},E^c)-s)$$

$H_\varepsilon(t)$ is a continuous approximation of the Heavyside function, which is 1 for $t \geq 0$, 0 for $t \leq -\varepsilon$, and nondecreasing. Here, for a symmetric matrix X, $[X]^+$ denotes the matrix X with all eigenvalues replaced with their positive part, in particular, its determinant is zero if X is not positive definite.

But this F_ε is not variational (different from a smoothing F_f).

- Work in progress: Uniqueness for our Hamiltonian, for compactly supported evolutions (which we are currently focussed on — still problematic).

The requirements

What we need is a proof which uses only the Hamiltonian on its "natural" domain, which would be $F(x, p, X, K)$ for $p = D\phi(x), X = D^2\phi(x)$ and $K = \{\phi \geq \phi(x)\}$ (or $>$), for ϕ smooth and $D\phi(x) \neq 0$. (By extension, on appropriately defined sub/superjets.)

On this class, "true" Hamiltonian has good semicontinuity properties (wr C^2 convergence of ϕ).
However there are still difficulties for showing uniqueness without additional assumptions (Reminiscent of similar results of Cardaliaguet-Ley (2008) using "tubes" — viscosity theory for characteristic functions — see also Cardaliaguet (2000,2001), Cardaliaguet-Rouy (2006)).

Existence: a general framework for variational curvature flows

General assumptions: we consider a set function J whith

- $J \geq 0$, $J(\emptyset) = J(\mathbb{R}^N) = 0$;
- $J(E) = J(E')$ if $|E \triangle E'| = 0$;
- J is L^1-l.s.c.: if $|E_n \triangle E| \to 0$, $J(E) \leq \liminf_n J(E_n)$;
- J is "submodular": for any E, F,

$$J(E \cup F) + J(E \cap F) \ \leq \ J(E) + J(F) \ ; \qquad (1)$$

- J is translational invariant:

$$J(x + E) \ = \ J(E) \quad \forall E \text{ measurable and } x \in \mathbb{R}^N . \qquad (2)$$

and...

Curvature

A "curvature" defined for smooth sets: for E with compact C^2 boundary, we have $J(E) < +\infty$, and at any $x \in \partial E$ there exists a "curvature" $\kappa(x, E)$ with:

(A) If E_n, E have compact C^2 boundaries and $\partial E_n \to \partial E$ in C^2, $x_n \to x$, then $\kappa(x_n, E_n) \to \kappa(x, E)$;

(B) For $\varphi \in C^2(\mathbb{R}^N)$ constant out of a compact set and $\bar{x} \in \mathbb{R}^N$ such that $\varphi(\bar{x})$ not critical there exists δ s.t. if $|x - \bar{x}| \leq \delta$ and $W \subset B(\bar{x}, \delta)$,

$$
\begin{aligned}
J(\{\varphi \geq \varphi(x)\} \cup W) \; - \; J(\{\varphi \geq \varphi(x)\}) & \\
\geq \; |W \cap \{\varphi < \varphi(x)\}| \, (\kappa(\bar{x}, \{\varphi \geq \varphi(\bar{x})\}) - \varepsilon) \; , &
\end{aligned}
$$

$$
\begin{aligned}
J(\{\varphi > \varphi(x)\}) \; - \; J(\{\varphi > \varphi(x)\} \setminus W) & \\
\leq \; |W \cap \{\varphi > \varphi(x)\}| \, (\kappa(\bar{x}, \{\varphi > \varphi(\bar{x})\}) + \varepsilon) \; . &
\end{aligned}
$$

It includes

- Standard curvature flow (cf Ciomaga-Thouroude, Eto-Giga-Ishii)
- The non-local curvature flow studied here
- The fractional curvature flow of [Imbert 2009, Caffarelli-Souganidis 2010]

It does not include the crystalline curvature flow which is not continuous under C^2 convergence.

The Hamiltonian

We define a geometric Hamiltonian $H(x,p,X,K) = |p|\kappa(x,p,X,K)$ as follows: if K is a closed set and (p,X) is a superjet of χ_K at x, then

$$\kappa_*(x,p,X,K) = \sup\left\{\kappa(x,E) \,:\, E \in C^2\,, E \supseteq K\,, (p,X) \in J^{2,-}\chi_{\mathring{E}}(x)\right\}$$

and if A is open and (p,X) is a subjet of χ_A at x,

$$\kappa^*(x,p,X,A) = \inf\left\{\kappa(x,E) \,:\, E \in C^2\,, E \subseteq A\,, (p,X) \in J^{2,+}\chi_{\overline{E}}(x)\right\}$$

Remarks

- $\kappa(x, E) = \kappa_*(x, p, X, E)$ if E is C^2 with p, X corresponding to the 2nd form at $x \in \partial E$;
- Coincides with the l.s.c. envelope of κ for the convergence: $E_n \to K$ Hausdorff, with $(p, X_n) \in J^{2,+}\chi_{E_n}(x)$, and for any $\delta > 0$, in a neighborhood of x then for all n large E_n is in $\{p \cdot (y-x) + \frac{1}{2}((X_n + \delta)(y-x), (y-x)) \geq 0\}$
- May not depend on X! (except that we require that p, X is a superjet of χ_E), cf Imbert 2009, Caffarelli-Souganidis 2010.

General result

Theorem under these assumptions, we have convergence of the time-discrete variational scheme to a viscosity solution, when the initial data is compactly supported.

Existence through the time-implicit discretization

As we mentioned, we can pick a time step, and for E_0 a compact set, we define E^{n+1} from E^n, $n \geq 0$ by solving

$$\min_E J(E) \; + \; \frac{1}{h} \int_E d_{E^n}$$

We can do a level set approach: if $u_0 \in BUC(\mathbb{R}^d)$, we define u_{n+1} from u_n by performing the scheme independently for each level set $\{u_n \geq s\}$ (we assume u_n has compact support).

Then, one checks easily that u_n is BUC (with same bound and modulus of continuity as u_0) (translational invariance+comparison);

We let $u_h(x, t) = u(x, [t/h])$: one also checks that u_h is "uniformly continuous" in time
$\longrightarrow$ by Ascoli-Arzelà, we can assume $u_h \to u$ locally uniformly (up to a subsequence).

Sketch of proof of consistency

Consider $(\bar{x}, \bar{t})$ a local maximum of $u - \varphi$, one wants to show that (if $D\varphi(\bar{x}, \bar{t}) \neq 0$)

$$\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}) + |D\varphi(\bar{t}, \bar{x})| \kappa_*(\bar{x}, D\varphi(\bar{t}, \bar{x}), D^2\varphi(\bar{t}, \bar{x}), \{\varphi(\bar{t}, \cdot) \geq \varphi(\bar{t}, \bar{x})\}) \ \leq \ 0.$$

Sketch of proof

As usual we find $(x_k, t_k) \to (\bar{x}, \bar{t})$ a maximum of $u_{h_k} - \varphi$, then we use the minimality of the level set $\{u_{h_k}(\cdot, t_k) \geq u_{h_k}(x_k, t_k)\}$ to obtain information on the tangent level set $\{\varphi(\cdot, t_k) \geq \varphi(x_k, t_k)\}$. Using only the "submodularity" we arrive to:

$$
\begin{aligned}
J(\{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\} \cup W_\varepsilon) - J(\{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\}) \\
+ \frac{1}{h_k} \int_{W_\varepsilon} d_{\{\varphi(t_k - h_k, \cdot) \geq s_\varepsilon - c_k\}}(x)\, dx \ \leq \ 0\,.
\end{aligned}
$$

So that

$$
|W_\varepsilon| \kappa(\bar{x}, \{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\}) + \frac{1}{h_k} \int_{W_\varepsilon} d_{\{\varphi(t_k - h_k, \cdot) \geq s_\varepsilon - c_k\}}(x)\, dx \ \lesssim \ 0\,.
$$

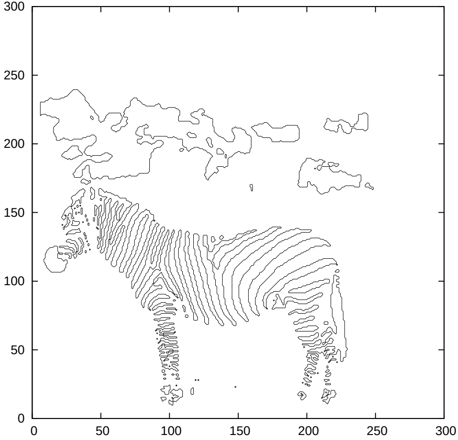
The equation will follow from the lower semicontinuity of κ_*.

Uniqueness

To get uniqueness for compactly supported initial data, we still need strong additional conditions on κ_*, κ^*; we haven't found yet a "natural" condition on the initial $\kappa(x, E)$ (which could be stronger than simple continuity?) for this property to hold.

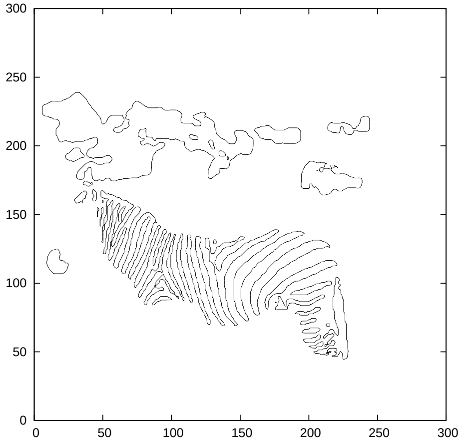
The iterative problem can be solved numerically (with a parametric maximal flow approach).

Shrinking a Zebra



$t = 0$

Shrinking a Zebra



$$t = 2$$

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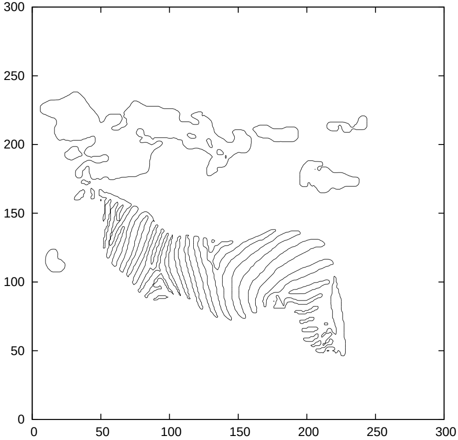

$$t = 4$$

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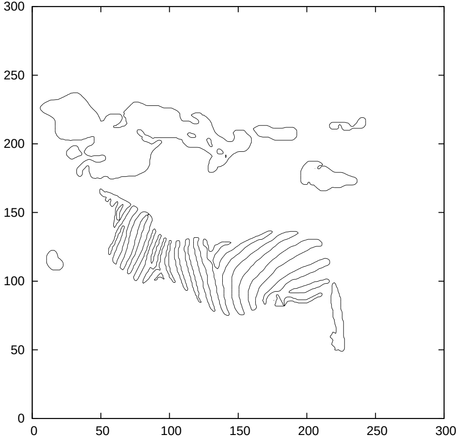

$t = 8$

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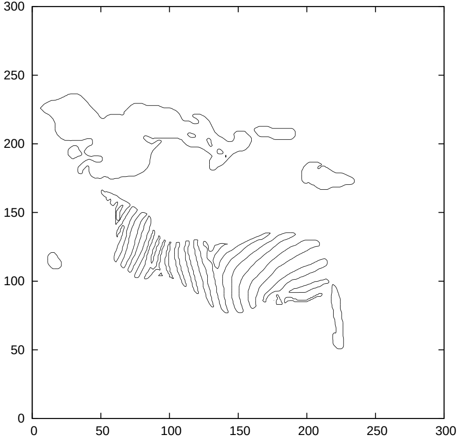

$$t = 12$$

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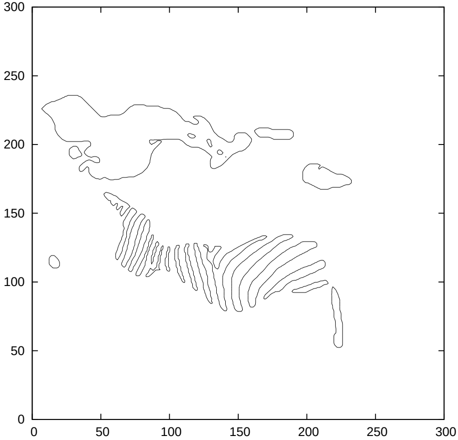

$t = 16$

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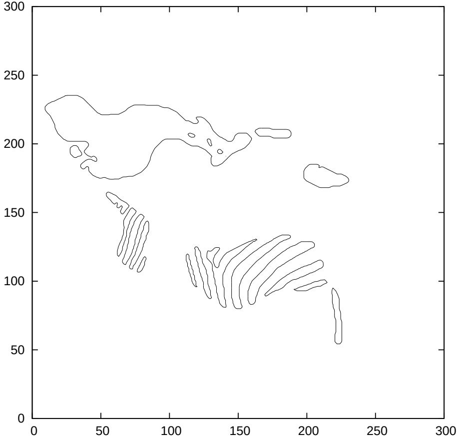

$t = 20$

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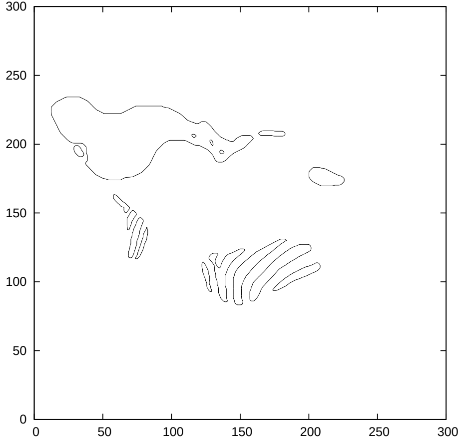
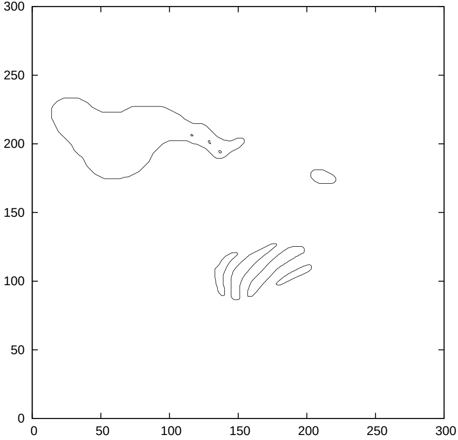


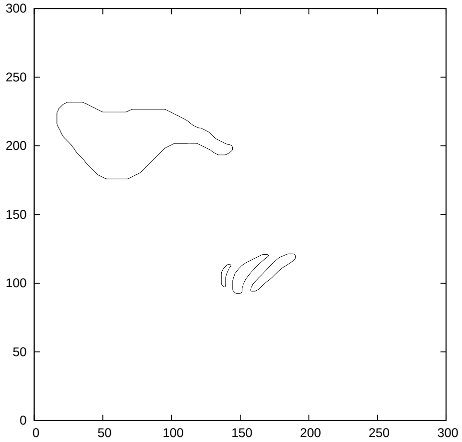
$$t = 30$$

Shrinking a Zebra



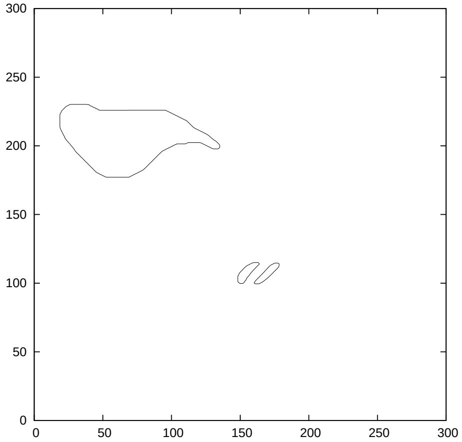
$$t = 40$$

Shrinking a Zebra

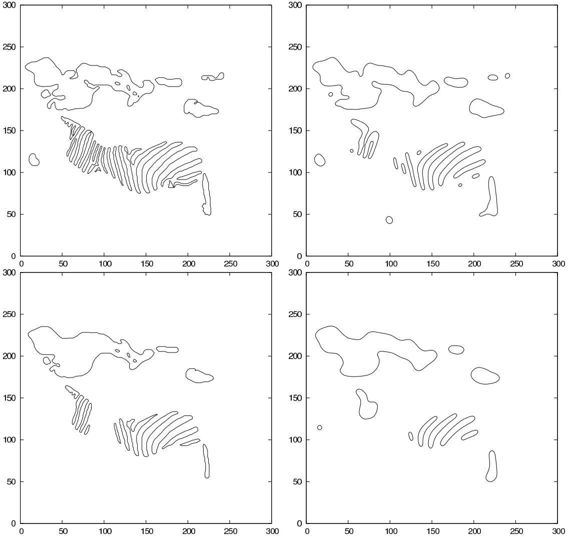


$t = 60$

Shrinking a Zebra



$t = 80$



left: the non-local motion, *right:* the standard curvature flow

Thank you for your attention