How the damping term ensures the uniqueness in the final data inverse source problems related to vibration of the Euler-Bernoulli beam ?

Alemdar Hasanov Hasanoglu

Department of Mathematics, Kocaeli University, Turkey

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Quasilinear Equations,
Inverse Problems and Their Applications

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COLLOBORATIONS & SUPPORTS

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3. A thermodynamic process is reversible if the process can return back in such a that both the system and the surroundings return to their original states.
4. The inverse problems based on final time measured output are closely related to the notions of reversibility and irreversibility.

0.2. The Backward Parabolic Problem

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2. We formulate this problem as follows:

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$$u_0(x) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} e^{n^2} u_{T,n} \sin(nx)$$

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4. No mathematical trick can recover information about the higher Fourier coefficients, due to the factor e^{n^2}.
5. Details are given in Alemdar Hasanov, Jennifer L. Mueller, *Applied Numerical Mathematics* 37 (2001) 55–78.

0.3 Identification of Spacewise and Time Dependent Source Terms in Heat Equation

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5. Similar results are obtained if the output $u_T(x) := u(x,T)$, $x \in (0,\ell)$ is replaced by the final velocity $v_T(x) := u_t(x,T)$.

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1.1a. Is an inverse source problem with final time measured output feasible? A counterexample

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3. **Proposition 1.** For unique determination of the unknown source $F(x)$ in (1), the final time $T > 0$ must satisfy the following condition

$$T \neq \frac{2m}{n},\ \text{for all } m, n = 1, 2, 3,\ \ldots .$$

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1.1b. A counterexample (continued)

1. Otherwise, i.e. when $T = 2m/n$, an infinite number of singular values σ_n defined as

$$\boxed{\sigma_n = \frac{1}{\lambda_n}\left[1 - \cos\left(\sqrt{\lambda_n}\,T\right)\right], \text{ for all } n = 1, 2, 3, \ \dots,}$$

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2. in the *singular value expansion* (SVE)

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} u_{T,n}\psi_n(x), \; x \in (0, \ell) \tag{1}$$

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3. Here and below, $u_{T,n} := (u_T, \psi_m)_{L^2(0,\ell)}$ is the nth Fourier coefficient of the output $u_T(x)$ and $\{\lambda_n, \psi_n(x)\}_{n=1}^{\infty}$ is the eigensystem of the operator $-u''(x)$ subject to the boundary conditions in $u(0,t) = u(\ell,t) = 0$.

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1.1c. A counterexample (continued)

1. As a consequence, the Picard criterion

$$\sum_{n=1}^{\infty} \frac{u_{T,n}^2}{\sigma_n^2} < \infty$$

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3. Therefore, if $\sigma_n = 0$ for some n, then the nth Fourier coefficient $F_n := (F, \psi_n)_{L^2(0,\ell)}$ of the unknown function $F(x)$ can not be determined uniquely.

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1.1d. A counterexample: **Conclusion**

1. From the condition

$$\boxed{T \neq \frac{2m}{n}, \text{ for all } m, n = 1, 2, 3, \ \dots}$$

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   it follows that for unique determination of the unknown source $F(x)$ the final time $T > 0$ can not be a rational number.
2. Evidently, fulfilment of this necessary condition is impossible in practice.
3. For this reason in [AH & VGR, *Introduction to Inverse Problems for Differential Equations* (New York: Springer, 2017)] the above final time output inverse problems for *undamped wave equation* were defined as infeasible inverse problems.

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3. Moreover, the situation does not change, if the above equations with pure spatial load is replaced with the equations $u_{tt} = u_{xx} + F(x)G(t)$ and $\rho(x)u_{tt} + (r(x)u_{xx})_{xx} = F(x)G(t)$ containing the temporal component $G(t)$ of the load.

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4. Namely, for an arbitrary function $G(t)$ the uniqueness of the solutions to the above inverse problems can not be guaranteed.
5. For an inverse problem related to parabolic equations with a final time output, this important issue was studied in [V.L. Kamynin, *Mathematical Notes* 73 ((2003) 2002-2011].

2. The inverse source problem with final time output

2.1. Formulation of inverse source problem with final time output

1. **The ISP:** Find $F(x)$ in

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2. Here, $\rho(x) > 0$ is the mass density, $r(x) = E(x)I(x) > 0$ is the spatial varying flexural rigidity, while $E(x) > 0$ and $I(x) > 0$ are the elasticity modulus and moment of inertia of the cross-section.

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3. For convenience of the analysis, we assume that the damping coefficient $\mu > 0$ is constant. $F \not\equiv 0$ and $G(t) > 0$ are the spatial and temporal components of the acting load, and $\Omega_T = (0,\ell) \times (0,T)$.

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2. Simply supported damped Euler-Bernoulli beam bridge subjected to spatial and temporal loads

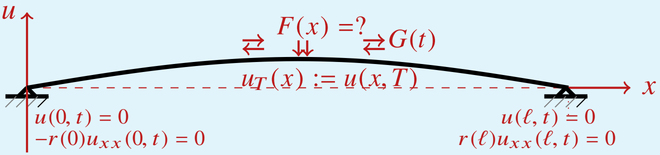

Figure: 1. Geometry of the inverse problem with final time output

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2.3a. The role of damping in vibration theory

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3. It provides an explanation for the fact that the response of a vibratory system excited at resonance does not grow without limit.
4. In most dynamic systems which are of interest from the point of view of vibrations the damping is small.
5. The values for loss factor that are encountered in practice range from about $\mu = 10^{-5}$ to $\mu = 2 \times 10^{-1}$ [S. H. Crandall, The role of damping in vibration theory, *Journal of Sound and Vibration*, **11**(1), 3–18, 1970]; although larger values of $\mu > 0$ are found in instrument mechanisms, transducers and vehicle suspensions.

2. The inverse source problem with final time output

2.3b. Effect of the damping coefficient μ

Fig. 2 shows that in the undamped case, a mass on a beam, displaced out of its equilibrium position will oscillate about this position for all time, while in the damped case, it will relax towards that equilibrium.

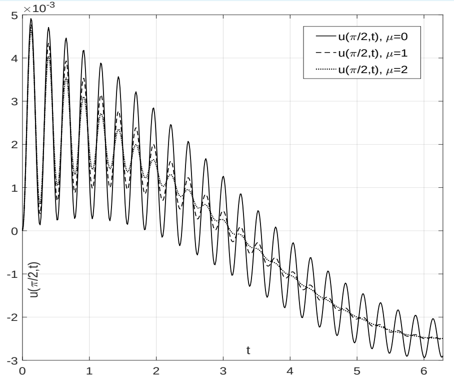

Figure: 2. The case: $\Omega_T = (0, \pi) \times (0, 2\pi)$, $F(x) = x \sin x$, $G(t) = \cos(\omega t)$, $\rho = r = 1$

2. The inverse source problem with final time output

2.4a. Can the damping coefficient play a positive role in unique determination of unknown source?

1. The second reason motivating this research is that damping, as the physical phenomenon responsible for the dissipation of energy, drastically changes the nature of the solution to the vibration problem, controlling also the response of the beam.

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3. Furthermore, the damped natural frequency $\omega_n = \sqrt{4\lambda_n - \mu^2}/2$, $\mu > 0$ of a beam, which is of the order $\mathcal{O}(\sqrt{\lambda_n})$, is always less than the undamped natural frequency $\omega_n = \lambda_n$, $\mu = 0$.

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4. Consequently, the singular values corresponding to the damped problem must be greater than the singular values corresponding to the undamped problem, since σ_n and λ_n are inversely proportional as formula $\sigma_n = \left[1 - \cos\left(\sqrt{\lambda_n}\,T\right)\right]/\lambda_n$, $n = 1, 2, 3, \ldots$ suggests.

2. The inverse source problem with final time output

2.4b. Can the damping coefficient play a positive role in unique determination of unknown source? (Continued)

1. The above considerations suggest that the damping parameter can naturally play a positive role in the unique determination of a spatial load $F(x)$ from the final state displacement or velocity, since due to this parameter the system decays more slowly towards its equilibrium configuration.

2. The inverse source problem with final time output

2.4b. Can the damping coefficient play a positive role in unique determination of unknown source? (Continued)

1. The above considerations suggest that the damping parameter can naturally play a positive role in the unique determination of a spatial load $F(x)$ from the final state displacement or velocity, since due to this parameter the system decays more slowly towards its equilibrium configuration.
2. The behavior of solutions of the direct problem corresponding to the different values of the damping coefficient μ shown in the previous Figure 2 also clearly illustrates this typical situation.

2. The inverse source problem with final time output

2.5. The damping term is a kind of regularization

1. More detailed analysis of the inverse problem shows that the role of the damping coefficient $\mu > 0$ in the Euler-Bernoulli equation $\rho u_{tt} + \mu u_t + (r(x)u_{xx})_{xx} = F(x)G(t)$ is very similar to the role of the regularization parameter $\alpha > 0$ in the regularized Tikhonov functional $J_\alpha(u) = (1/2)\|\Phi F - u_T\|^2 + \alpha\|F\|^2$.

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4. Hence, taking the value of this sum away from zero, the regularization term α/σ_n ensures the sufficient condition for $\sigma_n > 0$, thereby for the uniqueness of the solution.

2. The inverse source problem with final time output

2.5. The damping term is a kind of regularization (Continued)

1. Adding the damping term μu_t to the undamped dynamic E-B operator $\mathcal{E}^0 u := \rho u_{tt} + (r(x)u_{xx})_{xx}$ leads to the fact that the factor $e^{-\mu(T-t)/2}$ appears in the formula

$$\sigma_n = \frac{1}{\omega_n} \int_0^T \sin(\omega_n (T-t))\, G(t) dt, \ \omega_n = \sqrt{\lambda_n}, \ \mu = 0$$

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$$\sigma_n = \frac{1}{\omega_n}\int_0^T \sin(\omega_n(T-t))\,G(t)dt,\ \omega_n = \sqrt{\lambda_n},\ \mu = 0$$

2. And the formula for the *damped* dynamic Euler-Bernoulli operator $\mathcal{E}^{\mu} u := \rho u_{tt} + \mu u_t + (r(x)u_{xx})_{xx}$

$$\sigma_n = \frac{1}{\omega_n}\int_0^T e^{-\mu(T-t)/2}\sin(\omega_n(T-t))\,G(t)dt,\ 0 < \mu < 2\sqrt{\lambda_1},$$

contains the "regularization term" $e^{-\mu(T-t)/2}$.

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contains the "regularization term" $e^{-\mu(T-t)/2}$.

3. This ensures, under certain natural conditions, the positiveness of the singular values σ_n, and thus the uniqueness of the solution to the inverse problem.

3. The purpose and methodology of this study

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3. *If yes, then what is the role of the damping term μu_t in a positive solution to this problem?*
4. *And finally, what is the relationship between the basic parameters, that is, the final time $T > 0$, the damping coefficient $\mu > 0$, and also the temporal load $G(t)$, in order for the solution to be unique?*

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4. *And finally, what is the relationship between the basic parameters, that is, the final time $T > 0$, the damping coefficient $\mu > 0$, and also the temporal load $G(t)$, in order for the solution to be unique?*
5. Despite the popularity of the final time inverse source problems, these important questions from the point of view of the theory of inverse problems, as well as its recent applications, have not yet been investigated so far.

3. The purpose and methodology of this study

3.2a. The methodology

1. As the above analysis shows, a distinctive feature of the considered inverse problem is that it is impossible to generate the required final time measured input $u_T(x) := u(x,T)$ for each value of the final time $T > 0$.

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2. That is, one needs initially to find the acceptable interval $[T_*, T^*]$ of these values $T > 0$, at which the measured input $u_T(x)$ can be generated.

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2. That is, one needs initially to find the acceptable interval $[T_*, T^*]$ of these values $T > 0$, at which the measured input $u_T(x)$ can be generated.
3. In other similar inverse problems, the measured output is given in some way, and there is no discussion as to whether this is acceptable. This is the feature that needs to be emphasized.

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1. As the above analysis shows, a distinctive feature of the considered inverse problem is that it is impossible to generate the required final time measured input $u_T(x) := u(x, T)$ for each value of the final time $T > 0$.
2. That is, one needs initially to find the acceptable interval $[T_*, T^*]$ of these values $T > 0$, at which the measured input $u_T(x)$ can be generated.
3. In other similar inverse problems, the measured output is given in some way, and there is no discussion as to whether this is acceptable. This is the feature that needs to be emphasized.
4. We have developed an approach based on the SVE and Picard's theory, that allows us to find that admissible values of the final time in order to generate the acceptable final time measured input $u_T(x) := u(x, T)$.

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3.2a. The methodology

1. As the above analysis shows, a distinctive feature of the considered inverse problem is that it is impossible to generate the required final time measured input $u_T(x) := u(x,T)$ for each value of the final time $T > 0$.
2. That is, one needs initially to find the acceptable interval $[T_*, T^*]$ of these values $T > 0$, at which the measured input $u_T(x)$ can be generated.
3. In other similar inverse problems, the measured output is given in some way, and there is no discussion as to whether this is acceptable. This is the feature that needs to be emphasized.
4. We have developed an approach based on the SVE and Picard's theory, that allows us to find that admissible values of the final time in order to generate the acceptable final time measured input $u_T(x) := u(x,T)$.
5. Moreover, the proposed methodology allows us to find the relationship between the basic parameters. the final time $T > 0$, the damping coefficient $\mu > 0$, and also the temporal load $G(t)$, for which the solution of the inverse problem exists and is unique.

3. The purpose and methodology of this study

3.2b. The methodology (Continued)

1. In addition to the important role of the approach based on the SVE listed above, the TSVD algorithm constructed as a consequence of this approach, can be implemented as a simple computational tool, when the coefficients of the Euler-Bernoulli equation are constant and other inputs are smooth enough.

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3.2b. The methodology (Continued)

1. In addition to the important role of the approach based on the SVE listed above, the TSVD algorithm constructed as a consequence of this approach, can be implemented as a simple computational tool, when the coefficients of the Euler-Bernoulli equation are constant and other inputs are smooth enough.
2. However, in the case of variable coefficients, additional computational work is required to find the eigensystem of the Euler-Bernoulli operator numerically, which increases the overall cost of the entire computational algorithm.

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1. In addition to the important role of the approach based on the SVE listed above, the TSVD algorithm constructed as a consequence of this approach, can be implemented as a simple computational tool, when the coefficients of the Euler-Bernoulli equation are constant and other inputs are smooth enough.
2. However, in the case of variable coefficients, additional computational work is required to find the eigensystem of the Euler-Bernoulli operator numerically, which increases the overall cost of the entire computational algorithm.
3. Furthermore, the numerically found eigenvalues λ_n, $n = 1, 2, 3, \ldots$ introduce additional error in the TSVD formula

$$F_{\alpha,N}(x) = \sum_{n=1}^{N} \frac{q(\alpha;\sigma_n)}{\sigma_n} u_{T,n}\, \psi_n(x),\ \ x \in (0,\pi),$$

since σ_n depends on λ_n.

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1. For this reason, the adjoint problem approach combined with the Tikhonov functional, is proposed as an alternative method.
2. This method allows to solve the inverse problem for non-homogeneous beam /variable coefficients), and also with non-smooth measured output, unlike the TSVD algorithm.

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1. For this reason, the adjoint problem approach combined with the Tikhonov functional, is proposed as an alternative method.
2. This method allows to solve the inverse problem for non-homogeneous beam /variable coefficients), and also with non-smooth measured output, unlike the TSVD algorithm.
3. Thus, the combination of these two approaches is the ideal methodology for solving this important classes of inverse problems arising in wave and vibration phenomena, as the obtained theoretical and numerical results show.

4. The weak and regular weak solutions of the IBVP

4.1. The IBVP and basic conditions

1. Consider the DP:

$$\begin{cases} \rho(x)u_{tt} + \mu u_t + (r(x)u_{xx})_{xx} = F(x)G(t), \ (x,t) \in \Omega_T, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in (0,\ell), \\ u(0,t) = u_{xx}(0,t) = 0, \ u(\ell,t) = u_{xx}(\ell,t) = 0, \ t \in [0,T], \end{cases} \qquad \text{(IBVP)}$$

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2. We always assume that the inputs in satisfy the following *basic conditions* (for simplicity, in what follows we will refer to these conditions as BC):

$$\begin{cases} \rho, r \in L^\infty(0,\ell), \ 0 < \mu \le \mu^*, \\ 0 < r_0 \le r(x) \le r_1, \ 0 < \rho_0 \le \rho(x) \le \rho_1, \ x \in (0,\ell), \\ u_0 \in H^2(0,\ell), \ u_0(0) = u_0(\ell) = 0, \ u_1 \in L^2(0,\ell), \\ F \in L^2(0,\ell), \ F(x) \not\equiv 0, \ G \in L^2(0,T), \ G(t) > 0. \end{cases}$$

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3. **Remark 4.1.** Indeed in the DP $u_0(x) = u_1(x) = 0$. The presence of these initial data in the IBVP is necessary for the estimates given below.

4. The weak and regular weak solutions of the IBVP

4.1. The weak solution: Existence, uniqueness and estimates

1. **Theorem 4.1.** Assume that the BC hold. Then there exists a unique weak solution $u \in L^2(0,T;\mathcal{V}^2(0,\ell))$ with $u_t \in L^2(0,T;L^2(0,\ell))$ and $u_{tt} \in L^2(0,T;H^{-2}(0,\ell))$ of the IBVP,

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2. where $\mathcal{V}^2(0,\ell) := \{v \in H^2(0,\ell) : v(0) = v(\ell) = 0\}$.
3. Moreover, the following estimates hold:

$$\begin{aligned}\|u_t\|^2_{L^2(0,T;L^2(0,\ell))} \le C_1^2 \Big[&\|F\|^2_{L^2(0,\ell)}\|G\|^2_{L^2(0,T)} \\ &+C_u^2\left(\|u_0''\|^2_{L^2(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)}\right)\Big],\end{aligned}$$

$$\begin{aligned}\|u_{xx}\|^2_{L^2(0,T;L^2(0,\ell))} \le C_2^2 \Big[&\|F\|^2_{L^2(0,\ell)}\|G\|^2_{L^2(0,T)} \\ &+C_u^2\left(\|u_0''\|^2_{L^2(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)}\right)\Big],\end{aligned}$$

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\|u_{xx}\|^2_{L^2(0,T;L^2(0,\ell))} \le C_2^2 \Big[&\|F\|^2_{L^2(0,\ell)} \|G\|^2_{L^2(0,T)} \\
&+ C_u^2 \left(\|u_0''\|^2_{L^2(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)} \right) \Big],
\end{aligned}$$

4. where $C_1^2 = \exp(T/\rho_0) - 1$, $C_2^2 = \rho_0$, C_1^2/r_0, $C_u^2 = \max(r_1, \; \rho_1)$, and $\rho_0, r_0 > 0$ are the constants introduced in the BC.

4. The weak and regular weak solutions of the DP

4.2. The regular weak solution: Existence, uniqueness and estimates

1. **Theorem 4.2.** Assume that in addition to the BC the following regularity conditions hold:

$$r \in H^3(0,\ell),\ G \in H^1(0,T),\ u_0 \in H^4(0,\ell),\ u_1 \in H^2(0,\ell). \qquad \text{(RC)}$$

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2. Then there exists a unique regular weak solution $u \in L^2(0,T;H^4(0,\ell))$ with $u_t \in L^2(0,T;\mathcal{V}^2(0,\ell))$, $u_{tt} \in L^2(0,T;L^2(0,\ell))$ and $u_{ttt} \in L^2(0,T;H^{-2}(0,\ell))$ of the IBVP.

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3. Moreover, the following estimates hold:

$$\begin{aligned}
\|u_{tt}\|^2_{L^2(0,T;L^2(0,\ell))} &\le C_1^2 \Big[C_3^2\|F\|^2_{L^2(0,\ell)}\|G\|^2_{H^1(0,T)} \\
&\quad + C_4^2\left(\|u_0\|^2_{H^4(0,\ell)} + \|u_1\|^2_{H^2(0,\ell)}\right)\Big], \\
\|u_{xxt}\|^2_{L^2(0,T;L^2(0,\ell))} &\le C_2^2 \Big[C_3^2\|F\|^2_{L^2(0,\ell)}\|G\|^2_{H^1(0,T)} \\
&\quad + C_4^2\left(\|u_0\|^2_{H^4(0,\ell)} + \|u_1\|^2_{H^2(0,\ell)}\right)\Big].
\end{aligned}$$

4. The weak and regular weak solutions of the DP

4.3. Estimates for the final time outputs

1. **Corollary 4.1.** Assume that conditions of Theorem 4.1 hold. Then final time output $u(x,T)$ (displacement) the following estimate holds:

$$\|u(\cdot,T)\|^2_{L^2(0,\ell)} \leq \widetilde{C}_1^2 \|F\|^2_{L^2(0,\ell)} \|G\|^2_{L^2(0,T)} + \widetilde{C}_0^2 \left[\|u_0\|^2_{H^2(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)} \right],$$

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$\square$

2. **Corollary 4.2.** Assume that conditions of Theorem 4.1 hold. Then the following estimate holds:

$$\|u_x(\cdot,T)\|^2_{L^2(0,\ell)} \leq \tilde{C}_1^2 \|F\|^2_{L^2(0,\ell)} \|G\|^2_{H^1(0,T)} + \tilde{C}_0^2 \left[\|u_0\|^2_{H^4(0,\ell)} + \|u_1\|^2_{H^2(0,\ell)} \right]$$

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3. **Corollary 4.3.** Assume that conditions of Theorem 4.1 hold. Then final time output (velocity) $u_t(x,T)$ the following estimate holds:

$$\|u_t(\cdot,T)\|^2_{L^2(0,\ell)} \le \widehat{C}_1^2 \|F\|^2_{L^2(0,\ell)} \|G\|^2_{H^1(0,T)} + \widehat{C}_0^2 \left[\|u_0\|^2_{H^4(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)} \right]$$

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$\square$

4. All the constants above are defined through the physical parameters

5. Input-output operator and singular values

5.1. The input-output operator

1. We define the *set of admissible spatial loads*:

$$\mathcal{F} = \{F \in L^2(0,\ell) : \|F\|_{L^2(0,\ell)} \leq \gamma_F,\ \gamma_F > 0\}.$$

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2. And then, the input-output operator

$$(\Phi F)(x) := u(x,T;F),\ \Phi : \mathcal{F} \subset L^2(0,\ell) \mapsto L^2(0,\ell),$$

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$$(\Phi F)(x) := u(x,T;F), \; \Phi : \mathcal{F} \subset L^2(0,\ell) \mapsto L^2(0,\ell),$$

3. where $u(x,t;F)$ is the solution of the *direct problem* (DP)

$$\begin{cases} \rho(x)u_{tt} + \mu u_t + (r(x)u_{xx})_{xx} = F(x)G(t), \; (x,t) \in \Omega_T, \\ u(x,0) = 0, \; u_t(x,0) = 0, \; x \in (0,\ell), \\ u(0,t) = u_{xx}(0,t) = 0, \; u(\ell,t) = u_{xx}(\ell,t) = 0, \; t \in [0,T], \end{cases} \qquad \text{(DP)}$$

corresponding to a given $F \in \mathcal{F}$.

5. Input-output operator and singular values

5.2. Properties of the input-output operator: compactness

1. **Lemma 5.1.** *Assume that the BCs hold.*

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3. Compactness of the input-output operator implies that the considered inverse problem is *ill-posed*.
4. **Remark 5.1.** We can prove that if $u_T \subset H^1(0,\ell)$ (more regular measured output), the input-output map defined as $\Phi : \mathcal{F} \subset L^2(0,\ell) \mapsto H^1(0,\ell)$ is still compact operator.

5. Input-output operator and singular values

5.3a. Properties of the input-output operator: singular values

1. **Lemma 5.2.** *Let the BC are satisfied. Then the following hold true:*

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2. **(i)** *The Φ is a positive defined self-adjoint operator.*

5. Input-output operator and singular values

5.3a. Properties of the input-output operator: singular values

1. **Lemma 5.2.** *Let the BC are satisfied. Then the following hold true:*
2. **(i)** *The Φ is a positive defined self-adjoint operator.*
3. **(ii)** *Furthermore, $(\Phi\psi_n)(x) = \sigma_n\psi_n(x)$, $n = 1, 2, 3, \ldots,$, where $\{\sigma_n, \psi_n\}_{n=1}^{\infty}$ is the eigensystem of the input-output operator Φ, and*

$$\sigma_n = \frac{1}{\omega_n}\int_0^T e^{-\mu(T-t)/2}\sin(\omega_n(T-t))\,G(t)dt,$$

$$\omega_n = \tfrac{1}{2}\sqrt{4\lambda_n - \mu^2},\ \mu < 2\sqrt{\lambda_n},$$

$$\sigma_n = \int_0^T (T-t)e^{-\mu(T-t)/2}G(t)dt,\ \mu = 2\sqrt{\lambda_n},$$

$$\sigma_n = \frac{1}{2\widehat{\omega}_n}\int_0^T e^{-\mu(T-t)/2}\left[e^{\widehat{\omega}_n(T-t)} - e^{-\widehat{\omega}_n(T-t)}\right]G(t)dt,$$

$$\mu > 2\sqrt{\lambda_n},\ \widehat{\omega}_n = \tfrac{1}{2}\sqrt{\mu^2 - 4\lambda_n}.$$

5. Input-output operator and singular values

5.3b. **Lemma 5.2.** Continued

1. **(iii)** *The above* λ_n *is the eigenvalue of the Bernoulli operator* $(\mathcal{B}w)(x) := (r(x)w''(x))''$ *defined on* $\mathcal{D}(B) := \{w \in \mathcal{V}^2(0,\ell) \cap H^4(0,\ell) :\ w''(0) = w''(\ell) = 0\}$, *provided that there exists a function* ψ_n, *not identically equal to zero, solving the (S-L) problem*

$$\begin{cases} (r(x)\psi_n''(x))'' = \lambda_n \psi_n(x),\ x \in (0,\ell), \\ \psi_n(0) = \psi_n''(0) = \psi_n(\ell) = \psi_n''(\ell) = 0. \end{cases} \qquad (S-L)$$

5. Input-output operator and singular values

5.3b. **Lemma 5.2.** Continued

1. **(iii)** *The above λ_n is the eigenvalue of the Bernoulli operator $(\mathcal{B}w)(x) := (r(x)w''(x))''$ defined on $\mathcal{D}(B) := \{w \in \mathcal{V}^2(0,\ell) \cap H^4(0,\ell) : \ w''(0) = w''(\ell) = 0\}$, provided that there exists a function ψ_n, not identically equal to zero, solving the (S-L) problem*

$$\begin{cases} (r(x)\psi_n''(x))'' = \lambda_n \psi_n(x), \ x \in (0,\ell), \\ \psi_n(0) = \psi_n''(0) = \psi_n(\ell) = \psi_n''(\ell) = 0. \end{cases} \qquad (S-L)$$

2. **(iv)** *The input-output operator possesses the following SVE:*

$$(\Phi F)(x) = \sum_{n=1}^{\infty} \sigma_n F_n \psi_n(x),$$

where $F_n := (F, \psi_n)_{L^2(0,\ell)}$ is the nth Fourier coefficient of $F(x)$.

5. Input-output operator and singular values

5.4. Bernoulli operator $(\mathcal{B}w)(x) := (r(x)w''(x))''$

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2. **(i)** *Then the Bernoulli operator* $(\mathcal{B}w)(x) := (r(x)w''(x))''$ *defined on* $\mathcal{D}(B)$ *is a self-adjoint and positive defined operator.*

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2. **(i)** *Then the Bernoulli operator $(\mathcal{B}w)(x) := (r(x)w''(x))''$ defined on $\mathcal{D}(B)$ is a self-adjoint and positive defined operator.*
3. **(ii)** *The system $\{\psi_n\}_{n=1}^{\infty}$, forms an orthonormal basis for $L^2(0,\ell)$, and hence the Fourier series expansion*

$$w(x) = \sum_{n=1}^{\infty} w_n \psi_n(x), \; w_n := (w, \psi_n)_{L^2(0,\ell)} \qquad (FSE)$$

for the weak solution of the eigenvalue problem (S-L) converges in $L^2(0,\ell)$. □

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for the weak solution of the eigenvalue problem (S-L) converges in $L^2(0,\ell)$. $\square$

4. **Remark 5.1.** *This "weak" convergence* L^2*-norm is insufficient for our purpose since the unique weak solution of the DP is defined in* $L^2(0,T;\mathcal{V}^2(0,\ell))$, *where* $\mathcal{V}^2(0,\ell) := \{v \in H^2(0,\ell) : \; v(0) = v(\ell) = 0\}$.

5. Input-output operator and singular values

5.5a. The strong convergence H^2-norm of the FSE

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5.5a. The strong convergence H^2-norm of the FSE

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5. Input-output operator and singular values

5.5a. The strong convergence H^2-norm of the FSE

1. **Lemma 5.4.** *Let the BC are satisfied.*
2. *Then the FSE converges in $\mathcal{V}^2(0,\ell) \subset H^2(0,\ell)$.* $\square$
3. **Proof.** Introduce the symmetric bilinear form associated by the Bernoulli operator:

$$B[w,v] := \int_0^\ell r(x)w''(x)v''(x)dx,\ \ w,v \in \mathcal{V}^2(0,\ell).$$

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4. Evidently, the energy norm $(B[w,w])^{1/2}$ is equivalent to the norm $\|w\|_{\mathcal{V}^2(0,\ell)}$. Furthermore, from the eq. in (S-L) it follows that

$$B[\psi_n,\psi_m] = \lambda_n(\psi_n,\psi_m)_{L^2(0,\ell)} = \lambda_n\delta_{n,m},\ \ n,m = \overline{1,\infty},$$

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5. or

$$B\left[\frac{\psi_n}{\sqrt{\lambda_n}}, \frac{\psi_m}{\sqrt{\lambda_n}}\right] = \delta_{n,m}, \;\; n, m = \overline{1,\infty},$$

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$$B\left[\frac{\psi_n}{\sqrt{\lambda_n}}, \frac{\psi_m}{\sqrt{\lambda_m}}\right] = \delta_{n,m}, \;\; n,m = \overline{1,\infty},$$

6. where $\delta_{n,m}$ is the Kronecker symbol.

5. Input-output operator and singular values

5.5b. The strong convergence H^2-norm of the FSE (continued)

1. The equality $B\left[\frac{\psi_n}{\sqrt{\lambda_n}}, \frac{\psi_m}{\sqrt{\lambda_n}}\right] = \delta_{n,m},\ n, m = \overline{1, \infty}$ implies that the system $\{\psi_n/\sqrt{\lambda_n}\}_{n=0}^{\infty}$ is an orthonormal subset of $\mathcal{V}^2(0, \ell)$ endowed with the new inner product $B[w, v]$.

5. Input-output operator and singular values

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2. We prove that $\{\psi_n/\sqrt{\lambda_n}\,\}_{n=0}^{\infty}$ is in fact an orthonormal basis of $\mathcal{V}^2(0,\ell)$.

5. Input-output operator and singular values

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1. The equality $B\left[\frac{\psi_n}{\sqrt{\lambda_n}}, \frac{\psi_m}{\sqrt{\lambda_n}}\right] = \delta_{n,m},\ n, m = \overline{1, \infty}$ implies that the system $\{\psi_n/\sqrt{\lambda_n}\}_{n=0}^{\infty}$ is an orthonormal subset of $\mathcal{V}^2(0,\ell)$ endowed with the new inner product $B[w, v]$.
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3. To this end, we need to show that $B[\psi_n/\sqrt{\lambda_n}, w] = 0$, for all $n = 1, 2, 3, \ldots,$ implies $w \equiv 0$.

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4. But this assertion is evidently holds since $B[\psi_n/\sqrt{\lambda_n}, w] = \sqrt{\lambda_n}(\psi_n, w)_{L^2(0,\ell)}$, and the conditions

$$(\psi_n, w)_{L^2(0,\ell)} = 0, \text{ for all } n = 1, 2, 3, \ldots$$

imply $w(x) \equiv 0$, as $\{\psi_n\}_{n=0}^{\infty}$ is a basis for $L^2(0, \ell)$.

5. Input-output operator and singular values

5.5b. The strong convergence H^2-norm of the FSE (continued)

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2. We prove that $\{\psi_n/\sqrt{\lambda_n}\,\}_{n=0}^{\infty}$ is in fact an orthonormal basis of $\mathcal{V}^2(0, \ell)$.
3. To this end, we need to show that $B[\psi_n/\sqrt{\lambda_n}, w] = 0$, for all $n = 1, 2, 3, \ldots$, implies $w \equiv 0$.
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imply $w(x) \equiv 0$, as $\{\psi_n\}_{n=0}^{\infty}$ is a basis for $L^2(0, \ell)$.

5. Thus, $\{\psi_n/\sqrt{\lambda_n}\,\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathcal{V}^2(0, \ell)$.

5. Input-output operator and singular values

5.5c. The strong convergence H^2-norm of the FSE (continued)

1. Since $\{\psi_n/\sqrt{\lambda_n}\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathcal{V}^2(0,\ell)$, the series

$$\sum_{n=1}^{\infty} \widehat{w}_n \frac{\psi_n}{\sqrt{\lambda_n}}, \ \widehat{w}_n := B\left[w, \frac{\psi_n}{\sqrt{\lambda_n}}\right]$$

converges in $\mathcal{V}^2(0,\ell)$.

5. Input-output operator and singular values

5.5c. The strong convergence H^2-norm of the FSE (continued)

1. Since $\{\psi_n/\sqrt{\lambda_n}\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathcal{V}^2(0,\ell)$, the series

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converges in $\mathcal{V}^2(0,\ell)$.

2. Comparing this series with the series in (FSE), i.e. with

$$w(x) = \sum_{n=1}^{\infty} w_n \psi_n(x), \ w_n := (w, \psi_n)_{L^2(0,\ell)}, \qquad (FSE)$$

we find: $\widehat{w}_n = \sqrt{\lambda_n}\, w_n$.

5. Input-output operator and singular values

5.5c. The strong convergence H^2-norm of the FSE (continued)

1. Since $\{\psi_n/\sqrt{\lambda_n}\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathcal{V}^2(0,\ell)$, the series

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we find: $\widehat{w}_n = \sqrt{\lambda_n}\, w_n$.

3. This means that the series in (FSE) in fact converges also in $\mathcal{V}^2(0,\ell)$.

5. Input-output operator and singular values

5.5c. The strong convergence H^2-norm of the FSE (continued)

1. Since $\{\psi_n/\sqrt{\lambda_n}\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathcal{V}^2(0,\ell)$, the series

$$\sum_{n=1}^{\infty} \widehat{w}_n \frac{\psi_n}{\sqrt{\lambda_n}}, \ \widehat{w}_n := B\left[w, \frac{\psi_n}{\sqrt{\lambda_n}}\right]$$

   converges in $\mathcal{V}^2(0,\ell)$.

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   we find: $\widehat{w}_n = \sqrt{\lambda_n}\, w_n$.

3. This means that the series in (FSE) in fact converges also in $\mathcal{V}^2(0,\ell)$.

4. *It is this convergence that agrees with the corresponding norm of the weak solution space $\mathcal{V}^2(0,\ell)$.*

5. Input-output operator and singular values

5.6a. Classification of the damped cases

1. In Lemma 5.2, we have derived formulas for the singular values σ_n, corresponding to the following cases:

$$\mu < 2\sqrt{\lambda_n},\ \mu = 2\sqrt{\lambda_n},\ \mu > 2\sqrt{\lambda_n}$$

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2. Although these cases appeared as a result of the sign of the discriminant of the characteristic equation associated with the Cauchy problem

$$\begin{cases} u_n''(t) + \mu u_n'(t) + \lambda_n u_n(t) = F_n G(t), \\ u_n(0) = u_n'(0) = 0, \end{cases}$$

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3. each of these cases has a clear physical meaning and unambiguously correspond to certain commonly accepted vibrating systems.
4. Namely, the above cases correspond to underdamped, critically damped and overdamped vibrating systems, respectively, according to the commonly accepted classification.

5. Input-output operator and singular values

5.6b. Classification of the damped cases (Continued)

1. By Lemma 5.2, the input-output operator is positive defined self-adjoint, and $0 < \lambda_1 < \lambda_2 \ldots,$.

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5.6b. Classification of the damped cases (Continued)

1. By Lemma 5.2, the input-output operator is positive defined self-adjoint, and $0 < \lambda_1 < \lambda_2 \ldots,$.
2. It should be emphasized that only one term σ_{n_*} associated with the case $\mu = 2\sqrt{\lambda_{n_*}}$, corresponding to the critically damped case, can appear in the SVE $(\Phi F)(x) = \sum_{n=1}^{\infty} \sigma_n F_n \psi_n(x)$.

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3. If $\mu = 2\sqrt{\lambda_{n_*}}$ and $n_* > 1$, then the terms σ_n, $n = 1, 2, , .. n_* - 1$ associated with the case $\mu > 2\sqrt{\lambda_n}$, defined as overdamped case, appear in the above SVE, due to the fact that the sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ increases monotonically as $n \to \infty$.

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4. Finally, the case $\mu \in (2\sqrt{\lambda_m}, 2\sqrt{\lambda_{m+1}})$ means that the terms $\sigma_1, \ldots, \sigma_m$ in the above SVE correspond to the overdamped case.

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4. Finally, the case $\mu \in (2\sqrt{\lambda_m}, 2\sqrt{\lambda_{m+1}})$ means that the terms $\sigma_1, \ldots, \sigma_m$ in the above SVE correspond to the overdamped case.
5. These three cases can occur simultaneously in the same inverse problem.

6. Sufficient condition for uniqueness. Convergence of SVE

6.1. Picard criterion and positivity of the singular values

1. Consider the underdamped case $\mu < 2\sqrt{\lambda_n}$ as the most common one. Then

$$\sigma_n = \frac{1}{\omega_n} \int_0^T e^{-\mu(T-t)/2} \sin(\omega_n(T-t))\, G(t)dt,\ \omega_n = \frac{1}{2}\sqrt{4\lambda_n - \mu^2}.$$

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2. This formula shows that even the positivity $G(t) > 0$ of the temporal load can not guarantee the positivity $\sigma_n > 0$ of the singular values for all $n = 1, 2, , \ldots,$ which means that the SVE

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} u_{T,n}\psi_n(x),\ x \in (0, \ell)$$

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is *only formal*

3. Note that the Picard criterion $\sum_{n=1}^{\infty}\frac{u_{T,n}^2}{\sigma_n^2} < \infty$ also requires the positivity of the singular values.

6. Sufficient condition for uniqueness. Convergence of SVE

6.2. The uniqueness theorem

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3. Suppose that the damping coefficient, final time and the temporal load satisfy the following inequality:

$$G(T) > \left(G(0)e^{-\mu T/2} + \left((1 - e^{-\mu T})/\mu\right)^{1/2} \|G'\|_{L^2(0,T)}\right) \times \left(1 - \left(\mu/(2\sqrt{\lambda_1})\right)^2\right)^{-1/2}. \qquad \text{(MI)}$$

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4. Then the SVE of the solution of the final time inverse source problem is unique.

6. Sufficient condition for uniqueness. Convergence of SVE

6.3. Special cases encountered in applications

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3. Then

$$\left[1 + \frac{\mu^2}{4\lambda_1 - \mu^2}\right]^{1/2} = \sqrt{2},\ e^{-\mu T/2} = e^{-T\sqrt{\lambda_1/2}},\ \left(\frac{1 - e^{-\mu T}}{\mu}\right)^{1/2} < \frac{1}{\sqrt{\mu}}.$$

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4. In this case the inequality (MI) is valid for all large enough values of $T > 0$, and $G(T) > \|G'\|_{L^2(0,T)}(2/\lambda_1)^{1/4}$.

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4. In this case the inequality (MI) is valid for all large enough values of $T > 0$, and $G(T) > \|G'\|_{L^2(0,T)}(2/\lambda_1)^{1/4}$.
5. The latter inequality holds if, for example, $G(t) = \exp(\alpha t)$ with large enough $\alpha > 0$.

6. Sufficient condition for uniqueness. Convergence of SVE

6.4a. Convergence of SVE

1. The positivity condition does not guarantee the convergence of SVE as the Picard criterion $\sum_{n=1}^{\infty} \frac{u_{T,n}^2}{\sigma_n^2} < \infty$ shows.

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2. Indeed, from the above formula for σ_n it follows that

$$0 < \sigma_n < \frac{1}{\omega_n} \left(\int_0^T \sin^2(\omega_n t) dt \right)^{1/2} \left(\int_0^T G^2(T-t)\, dt \right)^{1/2}$$

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4. As a consequence of this and the Picard criterion we deduce that the SVE converges if and only if $\sum_{n=1}^{\infty} n^4 u_{T,n}^2 < \infty$.

6. Sufficient condition for uniqueness. Convergence of SVE

6.4b. Convergence of SVE (Continued)

1. Based on characterization of Sobolev spaces by Fourier transform we conclude that $\sum_{n=1}^{\infty} n^4 u_{T,n}^2 < \infty$ holds, if the measured output $u_T(x)$ satisfies the following regularity and consistency conditions:

$$u_T \in H^2(0,\ell),\ u_T(0) = u_T(\ell) = 0. \qquad \text{(RC)}$$

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2. **Theorem 6.2.** Assume that conditions of Theorem 6.1 hold and the measured output $u_T(x)$ defined satisfies the conditions (RC).
3. Then the inverse problem has a unique solution.
4. Furthermore, this solution possesses the convergent SVE

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} u_{T,n}\psi_n(x),\ x \in (0,\ell).$$

7. Some conclusions related to SVE based approach

7.1a. Positive and negative aspects of the SVE based approach

1. From the above analysis it follows that the main merit of the approach based on the singular value decomposition of the input-output operator, is that it allows us to find the relationship between the main inputs, namely, the final time, the damping coefficient and the temporal load, in which the inverse problem has a unique solution.

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4. Moreover, if the output $u_T(x)$ contains a low random noise of, say, 10^{-2}, in the 3th or 4th terms of the SVE $F(x) = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} u_{T,n}\psi_n(x),\ x \in (0,\ell)$ this error is amplified by the factor of the order 10!

7. Some conclusions related to SVE based approach

7.1b. Positive and negative aspects of the SVE based approach (Continued)

1. The second drawback of this method is that it requires very smooth final time output as the conditions $u_T \in H^2(0,\ell),\ u_T(0) = u_T(\ell) = 0$ show, which is not always the case in applications.

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2. In the case of non-smooth data $u_T \in L^2(0, T)$, of course, the convergence of the SVE cannot be guaranteed.

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2. In the case of non-smooth data $u_T \in L^2(0, T)$, of course, the convergence of the SVE cannot be guaranteed.
3. The advantage of the adjoint problem approach based on minimization the Tikhonov functional is, in particular, that this method does not require such a restriction.

8. Adjoint method combined with Tikhonov RM

8.1. Tikhonov functional and quasi-solution

1. The measured output $u_T(x)$ always contains a random noise and, as a result, exact equality in the equation $(\Phi F)(x) = u_T(x)$ is not possible in practice.

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$$J(F) = \frac{1}{2} \|\Phi F - u_T\|^2_{L^2(0,\ell)}, \; F \in \mathcal{F}$$

and reformulate the inverse problem as the minimization problem

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3. A solution of this problem is defined as a quasi-solution of the inverse problem.

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8.2. Lipschitz continuity of the the input-output operator

1. **Lemma 8.1.** *Let the basic conditions (BC) hold. Then the input-output operator is Lipschitz continuous, that is,*

$$\|\Phi F_1 - \Phi F_2\|_{L^2(0,\ell)} \le \widetilde{C}_1 \|G\|_{L^2(0,T)} \|\delta F\|_{L^2(0,\ell)}$$

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$$\begin{cases} \rho(x)\delta u_{tt} + \mu \delta u_t + (r(x)\delta u_{xx})_{xx} = \delta F(x) G(t), \ (x,t) \in \Omega_T, \\ \delta u(x,0) = \delta u_t(x,0) = 0, \ x \in (0,\ell), \\ \delta u(0,t) = \delta u_{xx}(0,t) = 0, \ \delta u(\ell,t) = \delta u_{xx}(\ell,t) = 0, \ t \in [0,T], \end{cases}$$

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4. Apply Corollary 1 to the weak solution of the above problem:

$$\|\delta u(\cdot, T)\|_{L^2(0,\ell)} \leq \widetilde{C}_1 \|G\|_{L^2(0,T)} \|\delta F\|_{L^2(0,\ell)}.$$

8. Adjoint method combined with Tikhonov RM

8.3. Lipschitz continuity of the Tikhonov functional and existence of a quasi-solution

1. **Lemma 8.2.** *Let the basic conditions (BC) hold and $u_T \in L^2(0,\ell)$. Then the Tikhonov functional is Lipschitz continuous, that is*

$$|J(F_1) - J(F_2)| \leq L_J \|F_1 - F_2\|_{L^2(0,\ell)}, \ \forall F_1, F_2 \in \mathcal{F},$$

where $L_J = \widetilde{C}_1 \left[\widetilde{C}_1 \gamma_F + \|u_T\|_{L^2(0,\ell)} \right] \|G\|_{L^2(0,T)}$

8. Adjoint method combined with Tikhonov RM

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2. **Proof.** Use the inequality

$$|J(F_1) - J(F_2)| \leq \frac{1}{2} \left[\|\Phi F_1\|_{L^2(0,\ell)} + \|\Phi F_2\|_{L^2(0,\ell)} + 2\|u_T\|_{L^2(0,\ell)} \right] \times \|\Phi F_1 - \Phi F_2\|_{L^2(0,\ell)}$$

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4. **Theorem 8.1.** *Assume that conditions of Lemma 8.2 hold. Then there exists a quasi-solution of the final time output inverse source problem.*

8. Adjoint method combined with Tikhonov RM

8.4. The integral relationship between inputs and outputs

1. **Lemma 8.3.** *Let the basic conditions (BC) hold and* $u_T \in L^2(0,\ell)$. *Then between the inputs and outputs the following integral relationship holds:*

$$-\int_0^\ell \rho(x)q(x)\delta u(x,T)dx = \int_0^\ell \left(\int_0^T G(t)\phi(x,t;q)dt\right)\delta F(x)dx,$$

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$$\begin{cases} \rho(x)\phi_{tt} - \mu(x)\phi_{tt} + (r(x)\phi_{xx})_{xx} = 0, \ (x,t) \in \Omega_T, \\ \phi(x,T) = 0, \ \phi_t(x,T) = q(x), \ x \in (0,\ell), \\ \phi(0,t) = \phi_{xx}(0,t) = 0, \ \phi(\ell,t) = \phi_{xx}(\ell,t) = 0, \ t \in (0,T), \end{cases}$$

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3. *with the arbitrary input (final velocity) $q(x)$, where $\delta u(x,t)$ is the weak solution of the DP with $F(x)$ replaced by $\delta F(x)$.*

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3. *with the arbitrary input (final velocity)* $q(x)$, *where* $\delta u(x,t)$ *is the weak solution of the DP with* $F(x)$ *replaced by* $\delta F(x)$.
4. Note that the backward problem is a well-posed problem as the change of the variable t with $\tau = T - t$ shows. Hence all estimates derived above can also be applied to the solution of this problem.

8. Adjoint method combined with Tikhonov RM

8.5. The input and output relationship

1. Choose the arbitrary final time input $q(x)$ in the BP as follows:

$$q(x) = -\frac{1}{\rho(x)}\,[u(x,T;F) - u_T(x)],\ \ x \in (0,\ell).$$

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2. Then the integral relationship turns to the input-output relationship

$$\int_0^\ell [u(x,T;F) - u_T(x)]\delta u(x,T)dx = \int_0^T \int_0^\ell \delta F(x)G(t)\phi(x,t;F)dxdt,$$

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3. through the weak solution $\phi \in L^2(0,T;\mathcal{V}^2(0,\ell))$ is of the following adjoint problem

$$\begin{cases} \rho(x)\phi_{tt} - \mu(x)\phi_{tt} + (r(x)\phi_{xx})_{xx} = 0, \ (x,t) \in \Omega_T, \\ \phi(x,T) = 0, \ \phi_t(x,T) = -\frac{1}{\rho(x)} [u(x,T;F) - u_T(x)], \ x \in (0,\ell), \\ \phi(0,t) = \phi_{xx}(0,t) = 0, \ \phi(\ell,t) = \phi_{xx}(\ell,t) = 0, \ t \in (0,T), \end{cases}$$

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8.6. The first variation of the Tikhonov functional

1. Assume that $F, F + \delta F \in \mathcal{F}$ and $u(x,t;F)$ is the solution of the DP, corresponding to $F \in \mathcal{F}$, and $\delta u(x,t;\delta F) := u(x,t;F+\delta F) - u(x,t;F)$ solves the DP with $F(x)$ replaced by $\delta F(x)$.

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2. Then *the first variation* $\delta J(F) := J(F+\delta F) - J(F)$ of the Tikhonov functional $J(F)$, $\delta F \in \mathcal{F}$ is

$$\delta J(F) = \int_0^\ell \underline{[u(x,T,F) - u_T(x)]\delta u(x,T)}dx + \frac{1}{2}\int_0^l [\delta u(x,T)]^2 dx.$$

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3. With the input-output relationship $\int_0^{\ell} \underline{[u(x,T;F) - u_T(x)]\delta u(x,T)}dx = \int_0^T \int_0^{\ell} \delta F(x) G(t)\phi(x,t;F)dxdt$, this yields:

$$\delta J(F) = \int_0^{\ell} \left(\underbrace{\int_0^T G(t)\phi(x,t;q)dt} \right) \delta F(x)dx + \frac{1}{2}\int_0^{l} [\delta u(x,T)]^2 dx.$$

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8.7. Fréchet differentiability of Tikhonov functional

1. **Theorem 8.2.** *Assume that the basic conditions hold and* $u_T \in L^2(0,\ell)$.

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8.7. Fréchet differentiability of Tikhonov functional

1. **Theorem 8.2.** *Assume that the basic conditions hold and* $u_T \in L^2(0,\ell)$.
2. *Then the Tikhonov functional is Fréchet differentiable.*

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1. **Theorem 8.2.** *Assume that the basic conditions hold and* $u_T \in L^2(0,\ell)$.
2. *Then the Tikhonov functional is Fréchet differentiable.*
3. *Furthermore, for the Fréchet gradient* $\nabla J(F)$ *the following formula holds:*

$$\nabla J(F)(x) = \int_0^T \phi(x,t;F)G(t)dt, \;\; F \in \mathcal{F},$$

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4. *where* $\phi \in L^2(0,T;\mathcal{V}^2(0,\ell))$ *is the weak solution of the adjoint problem.*
5. **Proof.** The last right-hand-side integral in the modified increment formula is of the order $\mathcal{O}\left(\|\delta F\|^2_{L^2(0,\ell)}\right)$, as the trace estimate shows. Then the proof follows from the definition of the gradient.

8. Adjoint method combined with Tikhonov RM

8.8. Lipschitz continuity of the Fréchet gradient

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$$\|\nabla J(F_1) - \nabla J(F_2)\|_{L^2(0,\ell)} \le L_\nabla \, \|F_1 - F_2\|_{L^2(0,\ell)}, \;\; F_1, F_2 \in \mathcal{F},$$

where $L_\nabla = T^{3/2}\, C \|G\|_{L^2(0,T)}$ *is the Lipschitz constant.*

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where $L_\nabla = T^{3/2}\, C \|G\|_{L^2(0,T)}$ *is the Lipschitz constant.*

3. Proof follows from the inequality $\|J'(F_1) - J'(F_2)\|^2_{L^2(0,\ell)} \le \|G\|^2_{L^2(0,T)} \|\delta\phi\|^2_{L^2(0,T;L^2(0,\ell))}$ and the estimate $\|\delta\phi\|^2_{L^2(0,T;L^2(0,\ell))} \le \frac{T^2}{2}\, C_1^2 C_u^2 \, \|\delta u(\cdot,T)\|^2_{L^2(0,\ell)}$ for the weak solution of the problem

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where $L_\nabla = T^{3/2}\, C\|G\|_{L^2(0,T)}$ is the Lipschitz constant.

3. Proof follows from the inequality $\|J'(F_1) - J'(F_2)\|^2_{L^2(0,\ell)} \le \|G\|^2_{L^2(0,T)} \|\delta\phi\|^2_{L^2(0,T;L^2(0,\ell))}$ and the estimate $\|\delta\phi\|^2_{L^2(0,T;L^2(0,\ell))} \le \frac{T^2}{2}\, C_1^2 C_u^2 \, \|\delta u(\cdot,T)\|^2_{L^2(0,\ell)}$ for the weak solution of the problem

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4. An important consequence of this lemma is the monotonicity of the iterations $\{J(F^{(n)})\}$ in the gradient algorithm.

9. Application to vibration under harmonic temporal load

9.1a. Forced vibration under harmonic temporal load

1. The case, when $G(t) = \cos(\omega t)$, is called harmonic loading with the frequency of the applied temporal load $\omega > 0$.

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2. The ISPhl of recovering the unknown spatial load $F(x)$ is

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \rho u_{tt} + \mu u_t + (r(x) u_{xx})_{xx} = F(x)\cos(\omega t),\ (x,t) \in \Omega_T, \\ u(x,0) = u_t(x,0) = 0,\ x \in (0,\ell), \\ u(0,t) = u_{xx}(0,t) = u(\ell,t) = u_{xx}(\ell,t) = 0,\ t \in [0,T], \end{array} \right. \\ \quad u_T(x) := u(x,T),\ x \in (0,\ell). \end{array} \right.$$

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3. Harmonic excitation, in which the magnitude of the external load varies within a harmonic envelope, is one of the most encountered loading type in applications.

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3. Harmonic excitation, in which the magnitude of the external load varies within a harmonic envelope, is one of the most encountered loading type in applications.
4. Determination of the harmonic response of engineering structures in which the beam-like elements are involved is of great importance especially at the design stage.

9. Application to vibration under harmonic temporal load

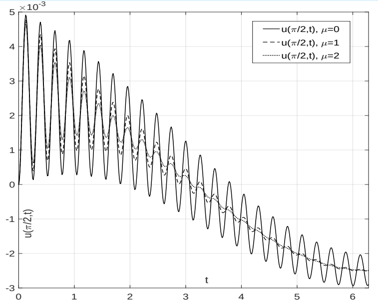
9.1b. Forced vibration under harmonic temporal load

1. An important problem here is the determination of $F(x)$ from the final time displacement or velocity, at *different admissible values of the frequency* $\omega > 0$ of the applied harmonic load $G(t) = \cos(\omega t)$.

9. Application to vibration under harmonic temporal load

9.1b. Forced vibration under harmonic temporal load

1. An important problem here is the determination of $F(x)$ from the final time displacement or velocity, at *different admissible values of the frequency* $\omega > 0$ of the applied harmonic load $G(t) = \cos(\omega t)$.
2. Positions in Fig.3, as function of $t \in [0, T]$, for the undamped and two damped vibrations, shows that in the undamped case, a mass on a beam, displaced out of its equilibrium position, will oscillate about that equilibrium for all time, while in the damped case, it will relax towards that equilibrium.



9. Application to vibration under harmonic temporal load

9.2. The sufficient condition for positivity of the singular values

1. **Lemma 9.1.** *Let* $G(t) = \cos(\omega t)$ *and* $\mu < 2\sqrt{\lambda_1}$, $0 < \omega < \sqrt{\lambda_1}$, $\omega T < \pi/4$, *where* $\lambda_1 > 0$ *is the principal eigenvalue of the Euler-Bernoulli operator.*

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2. *Denote by* $\omega^* = \omega^*(\lambda_1, \mu)$ *the root of the equation*

$$e^{\mu\pi/(4\omega)} = 2\left[1 + \frac{\mu^2(\lambda_1 + \omega^2)^2}{(4\lambda_1 - \mu^2)(\lambda_1 - \omega^2)^2}\right]$$

and let $T_* = \frac{\pi}{4\omega^*(\lambda_1,\mu)}$, $T^* = \frac{\pi}{4\omega}$.

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$$e^{\mu\pi/(4\omega)} = 2\left[1 + \frac{\mu^2(\lambda_1 + \omega^2)^2}{(4\lambda_1 - \mu^2)(\lambda_1 - \omega^2)^2}\right]$$

and let $T_* = \frac{\pi}{4\omega^*(\lambda_1,\mu)}$, $T^* = \frac{\pi}{4\omega}$.

3. *Then for all of* $\omega \in (0, \omega^*(\lambda_1, \mu))$ *and* $T \in (T_*, T^*)$, *the singular values*

$$\sigma_n = \frac{1}{(\lambda_n - \omega^2)^2 + \mu^2\omega^2}\left\{\left[2\mu\omega\sin(\omega T) + (\lambda_n - \omega^2)\cos(\omega T)\right]\right.$$
$$\left. - \left[\frac{\mu(\lambda_n + \omega^2)}{2\omega_n}\sin(\omega_n T) + (\lambda_n - \omega^2)\cos(\omega_n T)\right]e^{-\mu T/2}\right\},$$

are positive.

9. Application to vibration under harmonic temporal load

9.3. The unique SVE for the ISPhl

1. This lemma not only provides the sufficient condition for positivity, specific formula, for the singular values σ_n, and thus a SVE for the solution to the ISPhl.

9. Application to vibration under harmonic temporal load

9.3. The unique SVE for the ISPhl

1. This lemma not only provides the sufficient condition for positivity, specific formula, for the singular values σ_n, and thus a SVE for the solution to the ISPhl.
2. The lemma gives a constructive method for selecting the final time, depending on a given value of the frequency of the applied temporal load $\omega > 0$, for which the final data inverse problem makes sense, i.e. feasible.

9. Application to vibration under harmonic temporal load

9.3. The unique SVE for the ISPhl

1. This lemma not only provides the sufficient condition for positivity, specific formula, for the singular values σ_n, and thus a SVE for the solution to the ISPhl.
2. The lemma gives a constructive method for selecting the final time, depending on a given value of the frequency of the applied temporal load $\omega > 0$, for which the final data inverse problem makes sense, i.e. feasible.
3. **Theorem 9.1.** *Assume that conditions of Lemma 9.1 are satisfied and $u_T \in H^2(0, \ell)$. Then the ISPhl has a unique solution. Furthermore, this solution possesses the convergent singular value expansion given, with the singular values defined in Lemma 9.1.*

9. Application to vibration under harmonic temporal load

9.4. The typical example: the admissible intervals for the values of the frequency of the applied temporal load $\omega > 0$ and final time $T > 0$.

1. Formula $\lambda_n = \pi^4 r n^4/(\ell^4 \rho)$ for simply supported Euler-Bernoulli beam suggests that if $r = \rho = 1$ then $\lambda_1 = 1$, and the underdamped case $\mu < 2\sqrt{\lambda_1}$ means $\mu \in (0, 2)$.

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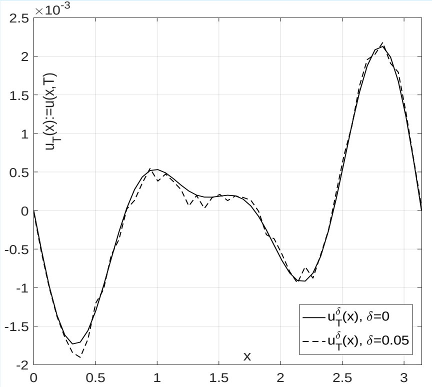
2. Lower and upper limits T_* and T^* of admissible values $T \in (T_*, T^*]$ of the final time corresponding to the root $\omega^* \in (0, \sqrt{\lambda_1}\,)$ of the equation introduced in Lemma 9.1.

Table 1. $G(t) = \cos(\omega t)$, $\rho = r(x) = 1$, $\ell = \pi$ and $\lambda_1 = 1$.

μ	0.1	0.5	1.0	1.2	1.5
$\omega^* = \omega^*(\lambda_1, \mu)$	0.113	0.465	0.543	0.539	0.517
$T_* = \pi/(4\omega^*)$	6.950	1.689	1.446	1.457	1.519
$\omega_* = \omega^*/2$	0.057	0.233	0.271	0.268	0.208
$T^* = \pi/(4\omega_*)$	13.780	3.371	2.910	3.931	3.776

10. Numerical algorithms and computational experiments

10.1. The TSVE reconstruction



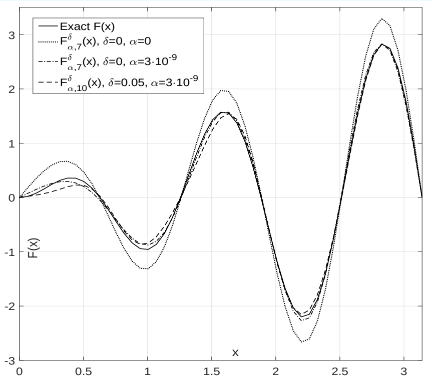


Figure: 4. Synthetic noise free and noisy output data (left), reconstruction of the spatial component of the load by SVD (right).

10. Numerical algorithms and computational experiments

10.2. Some conclusions related to the TSVE reconstruction

1. The worst numerical result is obtained when the regularization parameter is zero: $\alpha = 0$ (thin dotted line), as expected.

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3. Furthermore, the numerically found eigenvalues introduce additional error in the TSVE.
4. Finally, there is no defined criterion for finding the optimal value of the truncation parameter N, which ensures obtaining accurate TSVE solution.
5. The CG-algorithm does not contain any of the disadvantages listed above.

10. Numerical algorithms and computational experiments

10.2. Reconstruction by the CG-algorihtm-1

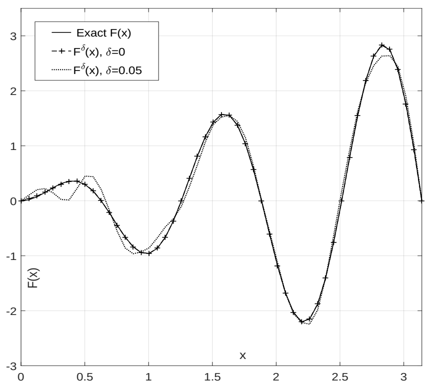


Figure: 5. Reconstruction of the spatial component of the load by the CG-algorihtm (The previous example solved by the TSVD).

10. Numerical algorithms and computational experiments

10.3. Reconstruction by the CG-algorihtm-2

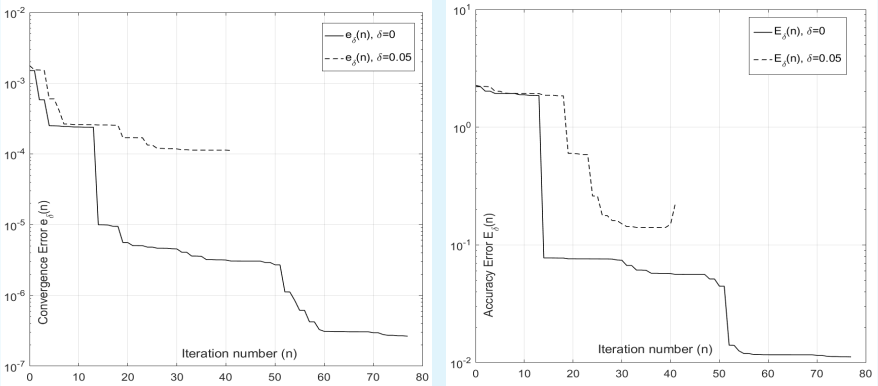
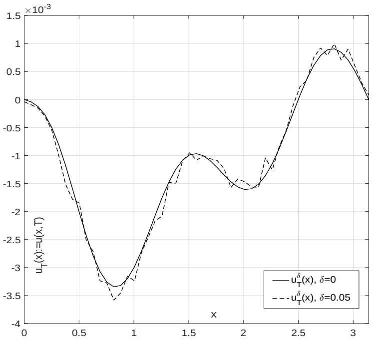


Figure: 6. Convergence error $e(n; F; \delta) = \|u(\cdot, T; F^{(n)}) - u_T^\delta\|_{L^2(0,\ell)}$ (left) and accuracy error $E(n; F; \delta) = \|F - F^{(n)}\|_{L^2(0,\ell)}$ (right) of CGA for the case $r(x) \equiv 1$.

10. Numerical algorithms and computational experiments

10.3. Reconstruction by the CG-algorihtm-3



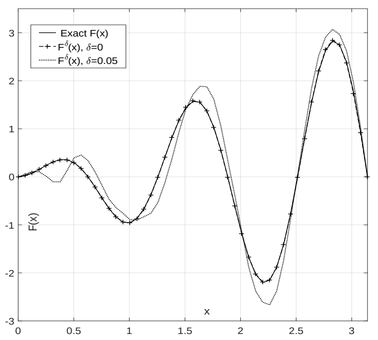
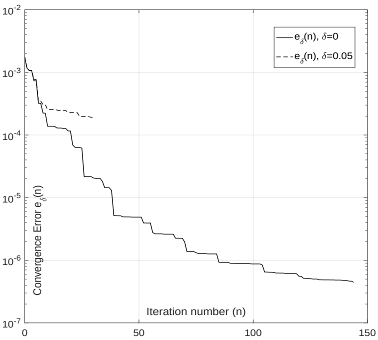


Figure: 7. The goal of this experiment is to find out how the variability of the coefficient $r(x)$ affects the accuracy of the reconstruction: synthetic noise free and noisy final time output (left), reconstruction of spatial load (right) for the case $r(x) = \exp(-\sqrt{x})$.

10. Numerical algorithms and computational experiments

10.3. Convergence error (left) and accuracy error (right)



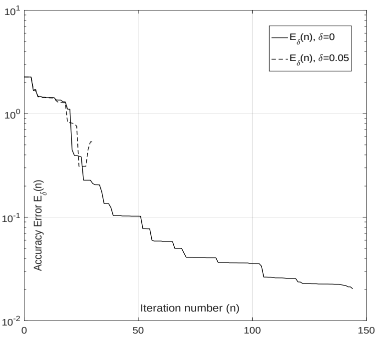


Figure: 8. The right figure show that the accuracy error decreases at first, and after a certain number of iterations n_*, it starts to increase. This means that stopping the CG-algorithm a few iterations earlier, say at $n_* - 2$, will give a better reconstruction of $F(x)$, especially for the case of noisy output. But since this function is unknown in real applications, such an artificial interference into the algorithm is impossible.

11. Conclusions

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3. Moreover, the same sufficient condition expresses the relationship, in the form of an inequality, between the main parameters: the damping coefficient, final time and temporal load.
4. Thus, we have revealed a new ability of the SVE approach.
5. At the same time, we propose the adjoint problem approach combined with the Tikhonov functional, as an alternative method, which allows to solve this class of inverse problems with non-smooth measured output, unlike the TSVD algorithm.

12. Some references

1. A. Hasanov and O. Baysal, *Automatica*, 71(2016) 106-117. DOI: 10.1016/j.automatica.2016.04.034

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4. A. Hasanov, V. Romanov and O. Baysal, Unique recovery of unknown spatial load in damped Euler-Bernoulli beam equation from final time measured output (submitted)

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2. Thank You For Your Attention.