Inverse problems of quantum and acoustic scattering at fixed frequency

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1.Basic problems

Consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d \ \ (d = 2 \text{ or } d = 3), \ E > 0, \qquad (1)$$

where v is a sufficiently regular function on $\mathbb{R}^d$ with sufficient decay at infinity, for example:

$$|v(x)| \le q(1+|x|)^{-\sigma} \text{ for some } q \ge 0 \text{ and } \sigma > d. \qquad (2)$$

For (1) we consider the scattering eigenfunctions $\psi^+(x,k)$, $k \in \mathbb{R}^d$, $k^2 = E$, specified by

$$\psi^+(x,k) = e^{ikx} + c(d)\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f\Big(k, |k|\frac{x}{|x|}\Big) + o\Big(\frac{1}{|x|^{(d-1)/2}}\Big) \quad (3)$$

as $|x| \to +\infty$, $c(2) = -i\pi\sqrt{2\pi}e^{-i\pi/4}$, $c(3) = -2\pi^2$,
for some a priori unknown f. The function f on

$$\mathcal{M}_E = \{k, l \in \mathbb{R}^d : \quad k^2 = l^2 = E\}$$

arising in (3) is the scattering amplitude for equation (1).

We consider

$$DSP: \quad v \to \psi^+ \to f;$$

$$ISP: \quad f \text{ on } \Gamma_E \subseteq \mathcal{M}_E \to v \text{ on } \mathbb{R}^d.$$

For the case when v is unknown in some open bounded domain D only, our ISP is closely related also with

$$IBVP: \quad \Phi(E) \to v \text{ on } D,$$

where $\Phi(E)$ is the Dirichlet-to-Neumann map on ∂D for the Schrödinger equation (1) in D.

Applications of these studies include:

- Inverse problem of quantum scattering arising in nuclear physics and in tomographies using some elementary particles;
- Acoustic tomography.

As regards to the acoustic tomography, we consider the acoustic equation

$$-\Delta\psi = \Big(\frac{\omega}{c(x)} + i\alpha(x,\omega)\Big)^2\psi, \quad x \in \mathbb{R}^d, \tag{4}$$

with velocity of sound $c(x)$, absorption coefficient $\alpha(x,\omega)$, at fixed frequency ω, under the assumption that

$$c(x) \equiv c_0, \quad \alpha(x,\omega) \equiv 0 \text{ for } |x| \geq r.$$

This equation can be written in the form of the Schrödinger equation (1), where

$$v = \frac{\omega^2}{c_0^2} - \Big(\frac{\omega}{c(x)} + i\alpha(x,\omega)\Big)^2, \quad E = \frac{\omega^2}{c_0^2}, \tag{5}$$

$$v = v(x,\omega) \equiv 0 \text{ for } |x| \geq r.$$

2. Born approximation for small potentials

If $q \to 0$, where q is the upper bound of (2) for v, then

$$f(k,l) \approx \hat{v}(k-l),\ (k,l) \in \mathcal{M}_E, \tag{6}$$

$$\hat{v}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d.$$

In addition,

$$(k,l) \in \mathcal{M}_E \Rightarrow k - l \in \mathcal{B}_{2\sqrt{E}},$$

$$p \in \mathcal{B}_{2\sqrt{E}} \Rightarrow p = k - l \text{ for some } (k,l) \in \mathcal{M}_E,$$

$$\mathcal{B}_r = \{p \in \mathbb{R}^d : \ \ |p| \le r\}.$$

Therefore, in the Born approximation f on $\mathcal{M}_E$ is reduced to $\hat{v}$ on $\mathcal{B}_{2\sqrt{E}}$.

A natural way for solving our ISP at fixed E in the Born approximation:

$$v(x) = v^{lin}_{appr}(x, E) + v^{lin}_{err}(x, E), \tag{7}$$

$$v^{lin}_{appr}(x, E) = \int\limits_{|p| \le 2\sqrt{E}} e^{-ipx} \hat{v}(p) dp,$$

$$v^{lin}_{err}(x, E) = \int\limits_{|p| \ge 2\sqrt{E}} e^{-ipx} \hat{v}(p) dp.$$

In addition, if $v \in W^{m,1}(\mathbb{R}^d)$, then

$$\|v(x)^{lin}_{err}(\cdot, E)\|_{L^\infty(\mathbb{R}^d)} = O(E^{-(m-d)/2}), \quad E \to +\infty. \tag{8}$$

Efficient generalization to the non-linearized case: [Novikov 1999, 2005, 2014].

3. Old general result

If v satisfies (2), then

$$f(k,l) = \hat{v}(k-l) + O(E^{-1/2}),\ E \to +\infty,\ (k,l) \in \mathcal{M}_E. \qquad (9)$$

As a mathematical theorem (9) goes back to [Faddeev 1956].
But this gives no method to reconstruct v from $f\big|_{\mathcal{M}_E}$ with the error smaller than $O(E^{-1/2})$ even if $v \in S(\mathbb{R}^d)$.
Applying the inverse Fourrier transform F^{-1} to both sides of (9), one can obtain an explicit linear formula for $u_1 = u_1(x,E)$ in terms of f on $\mathcal{M}_E$, where

$$u_1(x,E) = v(x) + O(E^{-\alpha_1}),\ E \to +\infty, \qquad (10)$$

$$\alpha_1 = \frac{m-d}{2m} \ \text{ if } \ v \in W^{m,1}(\mathbb{R}^d).$$

One can see that

$$\alpha_1 \le 1/2 \ \text{ even } \ \text{if } m \to +\infty.$$

4. Results of [Novikov 1999, 2005]

Consider

$$W_s^{m,1}(\mathbb{R}^d) = \{u : \ (1+|x|)^s \partial^J v(x) \in L^1(\mathbb{R}^d), \ |J| \le m\}.$$

[R.Novikov 1999]: $v \in W_s^{m,1}(\mathbb{R}^2), \ m > 2, \ s > 0,$

$$f\big|_{\mathcal{M}_E} \to v_{appr}(\cdot, E) \ \text{ on } \ \mathbb{R}^2 \tag{11}$$

- stable nonlinear reconstruction such that -

$$\|v - v_{appr}(\cdot, E)\|_{L^\infty(\mathbb{R}^2)} = O(E^{-(m-2)/2}), \ \text{as } E \to +\infty.$$

Reconstruction (11) is based on Fredholm linear integral equations of the second type. Among these linear integral equations, the most important ones arise from a non-local Riemann-Hilbert problem. Riemann-Hilbert problems of such type go back to [Manakov 1981].

Reconstruction (11) together with its multifrequency generalization was implemented numerically in [Burov, Alekseenko, Rumyantseva 2009].

[R.Novikov 2005]: $v \in W_s^{m,1}(\mathbb{R}^3),\ m > 3,\ s > 0,$

$$f\big|_{\mathcal{M}_E} \to v_{appr}(\cdot, E) \ \text{ on } \ \mathbb{R}^3 \tag{12}$$

- stable nonlinear reconstruction such that -

$$\|v - v_{appr}(\cdot, E)\|_{L^\infty(\mathbb{R}^3)} = O(E^{-(m-3)/2} \ln E), \ \text{as } E \to +\infty.$$

Reconstruction (12) is based on linear and nonlinear integral equations. Among these integral equations, the most important are nonlinear ones arising from $\bar{\partial}$-approach to 3D inverse scattering at fixed energy. This $\bar{\partial}$-approach goes back to [Beals, Coifman 1985], [Henkin, R.Novikov 1987].
Reconstruction (12) was implemented numerically in [Alekseenko, Burov, Rumyantseva 2008].
Main disadvantage: overdetermination of $f\big|_{\mathcal{M}_E}$ for $d = 3$,
$dim\, \mathcal{M}_E = 2d - 2 = 4,\ d = 3.$

5. 2d multi-channel approach to 3d inverse problems at fixed energy

Consider the following 3d equation

$$-\Delta_{x,z}\psi + v(x,z)\psi = E\psi, \quad (x,z) \in D \times L, \; E > 0, \tag{13}$$

D is an open bounded domain in $\mathbb{R}^2$ with a C^2 boundary, $L = [a, b]$, $a, b \in \mathbb{R}$, v is a sufficiently regular function on $D \times L$, and $\psi\big|_{D\times\partial L} = 0$ (for example).

Equation (13) can be approximated by the 2D multi-channel equation

$$-\Delta\psi + V(x)\psi = E\psi, \quad x \in D, \; E > 0, \tag{14}$$

where ψ, V are $M_n(\mathbb{C})$- valued functions on D,

$$V_{ij}(x) = \lambda_j\delta_{ij} + \int_L \bar{\varphi}_i(z)v(x,z)\varphi_j(z)dz, \quad x \in D, \tag{15}$$

for $1 \leq i,j \leq n$, where $n \in \mathbb{N}$, $\{\varphi_i\}_{j\in\mathbb{N}}$ is the orthonormal basis of $L^2(L)$ given by eigenfunctions of $-d^2/dz^2$ such that $\varphi_j\big|_{\partial L} = 0$, $-\frac{d^2\varphi_j}{dz^2} = \lambda_j\varphi_j$ for $j \in \mathbb{N}$, $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

For (14) we consider the Dirichlet-to-Neumann map $\Phi(E)$ such that

$$\Phi(E)(\psi|_{\partial D}) = \frac{\partial \psi}{\partial \nu}|_{\partial D} \tag{16}$$

for all sufficiently regular solutions ψ of (14) in $\bar{D} = D \cup \partial D$.
[R.Novikov, Santacesaria 2013]: $V \in W^{m,1}(\mathbb{R}^2, M_n(\mathbb{C}))$, $m \geq 3$, $suppV \subset D$,

$$\Phi(E) \rightarrow V_{appr}(\cdot, E) \;\; \text{on} \;\; \mathbb{R}^2 \tag{17}$$

- stable nonlinear reconstruction such that -

$$\|V - V_{appr}(\cdot, E)\|_{L^\infty(\mathbb{R}^2)} = O(E^{-(m-2)/2}) \;\; \text{as} \;\; E \rightarrow +\infty.$$

Reconstruction (17) is based on Fredholm linear integral equations of the second type. Among these equations the most important ones arise from a non-local matrix Riemann-Hilbert problem. Numerical implementations are started in [Burov, Shurup, Rumyantseva, Zotov 2012].

The aforementioned results of [R.Novikov 1999, 2005], [R.Novikov, M.Santacesaria 2013] were obtained on the basis of synthesis of results going back to [Faddeev 1965, 1974] and results going back to the theory of solitons.
In addition, some related results of [R.Novikov 2014], obtained via some pure iterative approach will be discussed in the next section.

6. Iterative approach of [R.Novikov 2014]

$$ISP: \quad f \ \text{ on } \ \Gamma_E^\delta \subset \mathcal{M}_E \to v \text{ on } \mathbb{R}^d, \tag{18}$$

$$\Gamma_E^\delta = \{k = k_E(p),\ l = l_E(p): \quad p \in \mathcal{B}_{2\delta\sqrt{E}}\},\ 0 < \delta \le 1,$$

$$k_E(p) = \frac{p}{2} + \eta_E(p), \quad l_E(p) = -\frac{p}{2} + \eta_E(p),$$

$$\mathcal{B}_r = \{p \in \mathbb{R}^d: \quad |p| \le r\},$$

where η_E is a piecewise continuous vector-function on $\mathcal{B}_{2\delta\sqrt{E}}$ such that

$$\eta_E(p)p = 0,\ \frac{p^2}{4} + (\eta_E(p))^2 = E, \quad p \in \mathcal{B}_{2\delta\sqrt{E}}.$$

One can see that

$$dim\,\mathcal{M}_E = 2d - 2, \quad dim\,\Gamma_E^\delta = d \ \text{ for } \ d \ge 2,$$

$$dim\,\mathcal{M}_E > d \ \text{ for } \ d \ge 3.$$

Therefore, the problem of finding v from f on $\mathcal{M}_E$ is overdetermined for $d \ge 3$, whereas the problem of finding v from f on Γ_E^δ is non-overdetermined.

[R.Novikov 2014]: Suppose that v is a perturbation of some known background v_0 satisfying (2), where $v - v_0 \in W^{m,1}(\mathbb{R}^d)$, $m > d$, $supp\,(v - v_0) \subset D$, where D is an open bounded domain (fixed a priori). Then from f on Γ_E^δ we iteratively construct (by stable explicit formulas) approximations $u_j(x, E)$, $j \geq 1$, to the unknown $v(x)$, $x \in D$, such that

$$\|u_j(\cdot, E) - v\|_{L^\infty(D)} = O(E^{-\alpha_j}) \text{ as } E \to +\infty, \tag{19}$$

$$\alpha_j = \left(1 - \left(\frac{m-d}{m}\right)^j\right)\frac{m-d}{2d}, \quad j \geq 1.$$

One can see that:

$$\alpha_1 = \frac{m-d}{m} \text{ is the number of (10)},$$

$$\alpha_j \to \alpha_\infty = \frac{m-d}{2d} \text{ if } j \to +\infty,$$

$$\alpha_\infty \to +\infty \text{ if } m \to +\infty.$$

Besides, in fact, f on $\Gamma_E^{\delta(E)}$ only is used in this iterative approximate reconstruction, where

$$\delta(E) = \tau E^{-(d-1)/(2d)}, \quad \tau \in]0,1],$$

$$\delta(E) \to 0 \ \text{ as } \ E \to +\infty.$$

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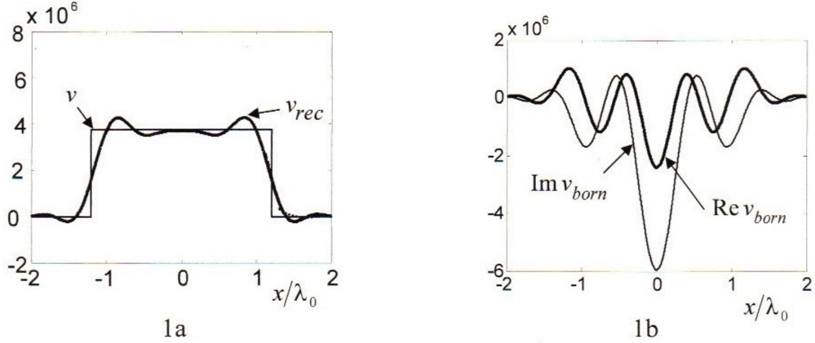

Fig 1. [N.V. Alexeenko, V.A. Burov, O.D. Rumyantseva 2008]:
3D monochromatic reconstruction of a ball-type scatterer via the $\bar{\partial}$-approach of [R. Novikov 2005].

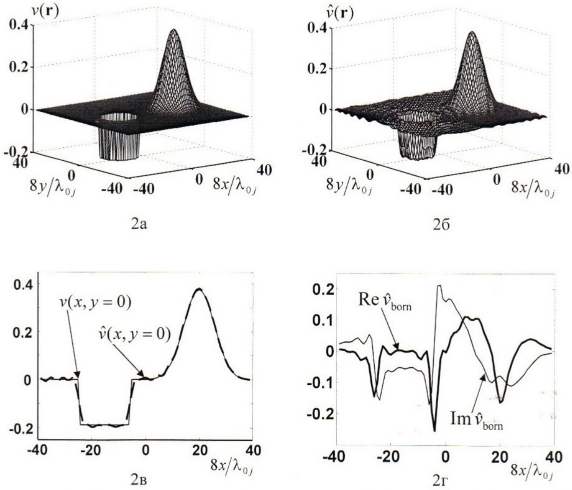

Fig 2. [V.A. Burov, A.S. Shurup, O.D. Rumyantseva, D.I. Zotov 2012]:
Example of 2D monochromatic reconstruction from near-field scattering data via the Riemann-Hilbert problem approach of [R. Novikov 1999], [R. Novikov, M. Santacesaria 2013].