The scattering transform for vector fields and dispersionless integrable PDE's:the Cauchy problem for the Pavlov equation

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Integrable hierarchies with non-zero and zero dispersion

Two important classes of intregrable hierarchies:

- Equations with non-zero dispersion
- Dispersionless systems (hydrodynamical-type equations).

Example: Korteweg-de Vries (KdV) equation

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x, \quad u = u(x,t).$$

Dispersionless KdV = Hopf equation (1-D collisionless hydrodynamics)

$$u_t = -\frac{3}{2}uu_x.$$

Small-dispersion KdV

$$u_t = \varepsilon u_{xxx} - \frac{3}{2}uu_x.$$

Linearization of KdV through scattering transform

Lax representation for KdV:

$$L = -\partial_x^2 + u(x,t), \quad A = \partial_X^3 - \frac{3}{2} u \partial_x - \frac{3}{4} u_x.$$

KdV equation is equivalent to the operator relation:

$$\partial_t L + [L, A] = 0, \quad \text{where} \quad [L, A] = LA - AL.$$

Scattering transform: for every fixed t

$$u(x,t) \Rightarrow \left\{ \mathcal{T}(k,t), \mathcal{R}(k,t), E_j(t), \mathcal{B}_j(t), j = 1, \ldots, N \right\}$$

Here:

- $\mathcal{T}(k)$ – is the transition coefficient,
- $\mathcal{R}(k)$ – is the reflection coefficient,
- N is the number of points in discrete spectrum,
- $E_j = -\varkappa_j^2$ are the points of discrete spectrum,
- $\mathcal{B}_j$ connects the asymptotics of the discrete spectrum eigenfunctions at $+\infty$ and $-\infty$.

Theorem (Gardner, Greene, Kruskal, Miura, "Method for Solving the Korteweg-de Vries Equation", Phys. Rev. Lett. 19, 1095 (1967)):

$$\mathcal{T}(k,t) = \mathcal{T}(k,0), \;\; \mathcal{R}(k,t) = e^{2ik^3t}\mathcal{R}(k,0),$$

$$E_j(t) = E_j(0), \;\; \mathcal{B}_j(t) = e^{2\varkappa_j^3 t}\mathcal{B}_j(0).$$

Dispersionless systems

Dispersionless integrable systems:

- They have no soliton solutions
- They may have wave breaking.
- They may have **arbitrary many** spatial variables.

The study of 1+1 dispersionless systems as completely integrable sytems was started in the middle of 1980's.

Dubrovin B.A., Novikov S.P., "Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogolyubov-Whitham averaging method", Soviet Math. Dokl. **27**,(1983), 665-669.

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Different integration methods - generalized Hodograph transformation.

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Different integration methods - generalized Hodograph transformation.

Example 2: Kadomtsev-Petviashvili (KP) equation

$$(u_t + \frac{3}{2}uu_x - \frac{1}{4}u_{xxx})_x = \frac{3}{4}\alpha^2 u_{yy}, \;\; \alpha^2 = \pm 1, \;\; u = u(x,y,t).$$

The case $\alpha = i$ is called KP1, the case $\alpha = 1$ is called KP2.
Lax representation:

$$[L, A] = 0, \;\; L = \alpha\partial_y - \partial_x^2 + u,$$

$$A = \partial_t - \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x + \frac{3}{4}\alpha w, \;\; w_x = u_y.$$

Integration is based on the spectral transform for L:

- $\alpha = -i \;\; \Rightarrow \;\; L = i\partial_y - \partial_x^2 + u$ – nonstationary Schrödinger operator,
- $\alpha = -i \;\; \Rightarrow \;\; L = \partial_y - \partial_x^2 + u$ – heat conductivity operator.

Dispersionless KP – nonlinear Lax pair

Dispersionless Kadomtsev-Petviashvili (dKP) equation = Khokhlov-Zabolotskaya equation.

$$(u_t + \frac{3}{2} u u_x)_x = \frac{3}{4} u_{yy}, \ \ u = u(x, y, t).$$

Nonlinear Lax representation = quasiclassical limit:

$$\partial_t \mathcal{B}_2 - \partial_y \mathcal{B}_3 + \{\mathcal{B}_2, \mathcal{B}_3\} = 0,$$

$$\mathcal{B}_2 = \xi^2 - u, \ \ \mathcal{B}_3 = \xi^3 - \frac{3}{2} u \xi - \frac{3}{4} w, \ \ w_x = u_y,$$

$\{,\}$ are the standard Poisson brackets $\{f, g\} = f_\xi g_x - g_\xi f_x$. Let $\mathcal{L} = \xi + v_1 \xi^{1} + v_2 \xi^{-2} + \ldots$, $v_1 = -u/2$, $v_2 = -w/4$.
Dispersionless analog of the linear problem:

$$\partial_y \mathcal{L} = \{\mathcal{B}_2, \mathcal{L}\}, \ \ \partial_t \mathcal{L} = \{\mathcal{B}_3, \mathcal{L}\}.$$

Dispersionless KP – linear Lax pair

Dispersionless Kadomtsev-Petviashvili (dKP) equation (in a different normalization)

$$(u_t + uu_x)_x + u_{yy} = 0, \quad u = u(x, y, t).$$

Linear Lax representation = Hamilton-Jacobi equation:

$$[\hat{L}_1, \hat{L}_2] = 0$$

$$\hat{L}_1 \equiv \partial_y + \lambda\partial_x - u_x\partial_\lambda,$$
$$\hat{L}_2 \equiv \partial_t + (\lambda^2 + u)\partial_x + (-\lambda u_x + u_y)\partial_\lambda$$

where $\lambda \in \mathbb{C}$ – is the spectral parameter.

Dispersionless Kadomtsev-Petviashvili equation

dKP arose in many physical models. May be the first time in:
C. C. Lin, E. Reissner, and H.S. Tsien, “On two-dimensional non-steady motion of a slender body in a compressible fluid”. Journal of Mathematical Physics, 27, (1948). 220-231.

Exact solutions of dKP using algebro-geometrical methods:
I. M. Krichever, “Method of averaging for two-dimensional “integrable”equations”, Funkts. Anal. Prilozh., 22:3 (1988), 3752

The nonlinear Lax representation for DKP.
V. E. Zakharov “Dispersionless limit of integrable systems in 2+1 dimensions”, in *Singular Limits of Dispersive Waves*, edited by N.M.Ercolani et al., Plenum Press, New York, 1994.
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Semiclassical version of the non-local $\bar\partial$ problem:
Konopelchenko, B.; Martnez Alonso, L.; Ragnisco, O. The $\bar\partial$-approach to the dispersionless KP hierarchy. J. Phys. A **34** (2001), no. 47, 1020910217.
Bogdanov, L.V., Konopel'chenko, B. G.; Martines Alonso, L. The quasiclassical $\bar\partial$-method: generating equations for dispersionless integrable hierarchies. Theoret. and Math. Phys. **134** (2003), no. 1, 3946,
Lax pairs for multidimensional dispersionless systems based on vector fields:
V. E. Zakharov and A. B. Shabat, "Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II", Functional Anal. Appl. **13**, (1979), 166-174.

A series of papers by S.V. Manakov, P.M. Santini – how to develop an analog of the dressing method for dispersionless systems with more than one spatial variables?

The approach by Manakov and Santini in based on linear operators.

S.V. Manakov and P. M. Santini “Inverse scattering problem for vector fields and the Cauchy problem for the heavenly equation”, Physics Letters A **359** (2006) 613-619. http://arXiv:nlin.SI/0604017.

S. V. Manakov and P. M. Santini “On the solutions of the second heavenly and Pavlov equations”, J. Phys. A: Math. Theor. **42** (2009) 404013 (11pp). doi: 10.1088/1751-8113/42/40/404013.

S.V. Manakov asked Santini and me to study, how to make this approach mathematically rigorous. It turns out, that development of a proper analog of spectral transform for zero dispersion case is a very non-trivial mathematical problem.

Open problem: How to construct inverse scattering transform for $\hat{L}_1$ from the dKP Lax pair?

Some partial results about holomorphic eigenfunctions: Grinevich P.G., Santini P.M. “Holomorphic eigenfunctions of the vector field associated with the dispersionless Kadomtsev-Petviashvili equation” - Journal of Differential Equations, 2013, v. 255, No. 7, pp. 1469-1491.

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In our paper we solve the Cauchy problem for the mathematically simplest equation of such type – the so-called Pavlov equation.

$$v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0, \quad v = v(x, y, t),$$

assuming, that the Cauchy data

$$v(x, y, 0)$$

is sufficiently small.

Some other examples from the paper:
S. V. Manakov, P. M. Santini "Solvable vector nonlinear Riemann problems, exact implicit solutions of dispersionless PDEs and wave breaking". arXiv:1011.2619 [nlin.SI]

2. The vector nonlinear PDE in $N+4$ dimensions:

$$\vec{U}_{t_1 z_2} - \vec{U}_{t_2 z_1} + \left(\vec{U}_{z_1} \cdot \nabla_{\vec{x}}\right) \vec{U}_{z_2} - \left(\vec{U}_{z_2} \cdot \nabla_{\vec{x}}\right) \vec{U}_{z_1} = \vec{0},$$

where $\vec{U}(t_1, t_2, z_1, z_2, \vec{x}) \in \mathbb{R}^N$, $\vec{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $\nabla_{\vec{x}} = (\partial_{x_1}, .., \partial_{x_N})$,
The Lax operators are $(N+1)$ dimensional vector fields

$$\hat{L}_i = \partial_{t_i} + \lambda \partial_{z_i} + \vec{U}_{z_i} \cdot \nabla_{\vec{x}}, \quad i = 1, 2.$$

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3. Its dimensional reduction, for $N = 2$:

$$\vec{U}_{tx} - \vec{U}_{zy} + \left(\vec{U}_y \cdot \nabla_{\vec{x}}\right)\vec{U}_x - \left(\vec{U}_x \cdot \nabla_{\vec{x}}\right)\vec{U}_y = \vec{0},$$
$$\vec{U} \in \mathbb{R}^2, \quad \vec{x} = (x, y), \quad \nabla_{\vec{x}} = (\partial_x, \partial_y),$$

where: $t_1 = z,\ t_2 = t,\ \ x_1 = x,\ \ x_2 = y$ and

$$\hat{L}_1 = \partial_z + \lambda\partial_x + \vec{U}_x \cdot \nabla_{\vec{x}},$$
$$\hat{L}_2 = \partial_t + \lambda\partial_y + \vec{U}_y \cdot \nabla_{\vec{x}}.$$

4. Its Hamiltonian reduction $\nabla_{\vec{x}} \cdot \vec{U} = 0$, $U_1 = \theta_y$, $U_2 = -\theta_x$ gives the celebrated second heavenly equation of Plebanski:

$$\theta_{tx} - \theta_{zy} + \theta_{xx}\theta_{yy} - \theta_{xy}^2 = 0, \quad \theta = \theta(x, y, z, t) \in \mathbb{R}, \qquad x, y, z, t \in \mathbb{R},$$

The Lax operators:

$$\hat{L}_1 \equiv \partial_z + \lambda\partial_x + \theta_{xy}\partial_x - \theta_{xx}\partial_y,$$
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5. The two-dimensional dispersionless Toda (2ddT) equation
J. D. Finley and J. F. Plebanski "The classification of all $\mathcal{K}$ spaces admitting a Killing vector", J. Math. Phys. **20**, 1938 (1979).
V. E. Zakharov "Integrable systems in multidimensional spaces", Lecture Notes in Physics, Springer-Verlag, Berlin **153** (1982), 190-216.

$$\phi_{\zeta_1\zeta_2} = \left(e^{\phi_t}\right)_t, \quad \phi = \phi(\zeta_1, \zeta_2, t)$$

(or $\varphi_{\zeta_1\zeta_2} = (e^{\varphi})_{tt}$, $\varphi = \phi_t$),
The Lax operators:
K. Takasaki and T. Takebe "SDIFF(2) hierarchy", Proceedings of the RIMS Research Project 91 "Infinite Analysis". RIMS-814, 1991.

$$\hat{L}_1 = \partial_{\zeta_1} + \lambda e^{\frac{\phi_t}{2}} \partial_t + \left(-\lambda (e^{\frac{\phi_t}{2}})_t + \frac{\phi_{\zeta_1 t}}{2}\right) \lambda \partial_\lambda,$$
$$\hat{L}_2 = \partial_{\zeta_2} + \lambda^{-1} e^{\frac{\phi_t}{2}} \partial_t + \left(\lambda^{-1} (e^{\frac{\phi_t}{2}})_t - \frac{\phi_{\zeta_2 t}}{2}\right) \lambda \partial_\lambda,$$

6. A system of two nonlinear PDEs in $2+1$ dimensions:

$$\begin{aligned} u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} &= 0, \\ v_{xt} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} &= 0, \end{aligned}$$

with Lax operators:

$$\begin{aligned} \tilde{L}_1 &\equiv \partial_y + (\lambda + v_x)\partial_x - u_x \partial_\lambda, \\ \tilde{L}_2 &\equiv \partial_t + (\lambda^2 + \lambda v_x + u - v_y)\partial_x + (-\lambda u_x + u_y)\partial_\lambda, \end{aligned}$$

describing a general integrable Einstein-Weyl metric
M. Dunajski, “The nonlinear graviton as an integrable system”, PhD Thesis, Oxford University, 1998.
M. Dunajski “An interpolating dispersionless integrable system”; J. Phys. A 41 (2008), no. 31, 315202, 9 pp. arXiv:0804.1234.
M. Dunajski, E. Ferapontov, B. Kruglikov “On the Einstein-Weyl and conformal self-duality equations”, arXiv:1406.0018 [nlin.SI].

7. Its $v = 0$ reduction is dKP (Khokhlov-Zabolotskaya).

Pavlov equation

The so-called Pavlov equation:

$$v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0, \quad v = v(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R},$$

commutativity condition for the following pair of vector fields:

$$\begin{aligned} L &= \partial_y + (\lambda + v_x)\partial_x, \\ M &= \partial_t + (\lambda^2 + \lambda v_x - v_y)\partial_x, \end{aligned}$$

where $\lambda \in \mathbb{C}$ – spectral parameter.

M. V. Pavlov "Integrable hydrodynamic chains", J. Math. Phys. **44** (2003) 4134-4156.

M. Dunajski "A class of Einstein-Weyl spaces associated to an integrable system of hydrodinamic type", J. Geom. Phys. **51** (2004), 126-137.

Prandtl equation and Pavlov equation

The zero pressure Prandtl equation for the potential Φ

$$\Phi_{xt} - \Phi_{xxx} + \Phi_x \Phi_{xy} - \Phi_y \Phi_{xx} = 0.$$

has the same nonlinear terms as the Pavlov equation

$$v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0,$$

but the dissipative term $-\Phi_{xxx}$ instead of the diffractive v_{yy}.

While the zero-pressure Prandtl equation with suitable boundary conditions gives rise to blow-up at finite time:
W. E, and B. Engquist "Blowup of solutions of the unsteady Prandtl's equation", Communications on Pure and Applied Mathematics, **50**, Issue 12 (1997), 1287-1293,
we prove in this paper that localized and sufficiently small initial data for Pavlov equation remain smooth at all times.

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Prandtl equation and Pavlov equation

The inviscid Prandtl equation

$$\Phi_{yt} + \Phi_y \Phi_{xy} - \Phi_x \Phi_{yy} = 0$$

can be linearized using some partial Legendre transformation, and it also shows formation of singularities at finite time (private communication by E.A. Kuznetsov).

This equation can be obtained as the zero-diffraction limit of the Pavlov equation. In this limit the constants in our estimates goes to infinity, therefore there is no contradiction.

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We show, that for “sufficiently good” Cauchy data, satisfying, in particular, the “small norm condition”, the spectral transform for the Pavlov equation provides us a regular solution for all $t > 0$.

Remark. Manakov and Santini used two different formulations for the inverse spectral problem:

- The approach based on a singular integrable equation for the wave function.
- The approach based on the nonlinear Riemann-Hilbert problem.

They are not equivalent.

We us the first one.

To avoid extra technicalities, we assume that $v_0(x,y) = v(x,y,0) \in \mathbb{R}$ is smooth and has compact support:

$$v_0(x,y) = 0 \text{ outside the area } -D_x \le x \le D_x, -D_y \le x \le D_y.$$

Step 1: We construct the Jost functions and the classical scattering data. By definition, the Jost functions are solutions of:

$$L\varphi_{\pm}(x,y,\lambda) = 0, \quad L = \partial_y + (\lambda + v_x)\partial_x,$$

such that

$$\varphi_{\pm}(x,y,\lambda) \to x - \lambda y \text{ as } y \to \pm\infty.$$

The zero eigenfunctions of L – are exactly the functions, which are constant on the characteristics, i.e. are constant on the solutions of the corresponding ODE:

$$\frac{dx}{dy} = \lambda + v_x(x,y).$$

Direct spectral transform

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Direct spectral transform

Consider the solutions of the following Cauchy problem:

$$x(y) = x_0 \quad y = y_0.$$

We have the following asymptotic:

$$x(y) \to \lambda y + x_\pm(x_0, y_0, \lambda), \;\; y \to \pm\infty.$$

It is easy to see that

$$x_\pm(x_0, y_0, \lambda) \to x_0 - \lambda y_0 \;\text{ as }\; y_0 \to \pm\infty;$$

therefore

$$\varphi_\pm(x_0, y_0, \lambda) = x_\pm(x_0, y_0, \lambda).$$

Direct spectral transform

The classical scattering amplitude $\sigma(\xi,\lambda)$ is defined $\xi \in \mathbb{R}$, $\lambda \in \mathbb{R}$ as the function connecting the asymptotic at $y \to +\infty$ and $y \to -\infty$

$$x_+(x_0, y_0, \lambda) = x_-(x_0, y_0, \lambda) + \sigma(x_-(x_0, y_0, \lambda), \lambda).$$

Therefore

$$\varphi_+(x, y, \lambda) \to x - \lambda y + \sigma(x - \lambda y, \lambda) \ \text{ as } \ y \to -\infty.$$

It is easy to prove the analytic properties of $\sigma(\xi,\lambda)$ using the standard ODE theory.

Step 2: We construct the eigenfunction, analytic in the spectral parameter.

For complex λ let us introduce the following complex notations:

$$z = x - \lambda y, \quad \bar{z} = x - \bar{\lambda} y$$

Equation on the wave function takes the form:

$$L\Phi^{\pm}(x, y, \lambda) = 0, \quad L = \partial_y + (\lambda + v_x)\partial_x.$$

and can be written as Beltrami equation:

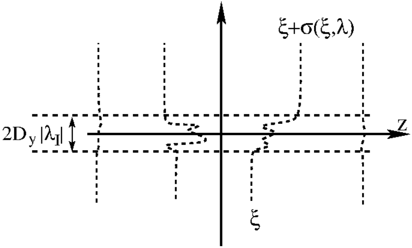
$$\left[\partial_{\bar{z}} + b(z, \bar{z}, \lambda)\partial_z\right] \Phi(z, \bar{z}, \lambda) = 0,$$

where

$$b(z, \bar{z}, \lambda) = \frac{v_x(z, \bar{z})}{2i\lambda_I + v_x(z, \bar{z})}.$$

It is uniquely solvable without the small norm assumption, and is holomorphic in λ for $\mathrm{Im}\,\lambda \neq 0$.

What happens if $\mathrm{Im}\,\lambda \ll 1$, $\mathrm{Im}\,\lambda < 0$?



Outside a small neighborhood of the real line in the z-plane thef function $\Phi^{-}(x, y, \lambda)$ is holomorphic in z and almost constant on the characteristics. We show that the limit $\hat{\Phi}^{\pm}(z, \lambda) = \Phi^{-}(x, y, \lambda)$ and $\mathrm{Im}\,\lambda \to -0$ is well-defined and satisfy the shifted Riemann problem:

$$\hat{\Phi}(\xi - i\epsilon, \lambda) \sim \hat{\Phi}(\xi + \tilde{\sigma}(\xi, \lambda) + i\epsilon, \lambda).$$

Direct spectral transform

Thertefore

$$\Phi^-(x,y,\lambda) = \varphi_-(x,y,\lambda) + \chi_-(\varphi_-(x,y,\lambda),\lambda) =$$
$$= \varphi_+(x,y,\lambda) + \chi_+(\varphi_+(x,y,\lambda),\lambda)$$
$$\Phi^+(x,y,\lambda) = \overline{\Phi^-(x,y,\lambda)},$$

and the spectral data $\chi_\pm(\xi,\lambda)$ satisfy the shifted Riemann problem:

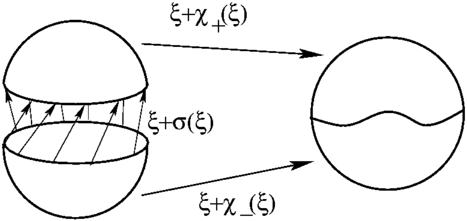
$$\sigma(\xi,\lambda) + \chi_+(\xi+\sigma(\xi,\lambda),\lambda) - \chi_-(\xi,\lambda) = 0, \quad \xi \in \mathbb{R},$$

where the functions $\chi_\pm(\xi,\lambda)$ are analytic in ξ at the upper half-plane and in the lower half-plane respectively,

$$\chi(\xi,\lambda) = \chi_+(\xi,\lambda), \ \ \mathrm{Im}\,\xi > 0$$
$$\chi(\xi,\lambda) = \chi_-(\xi,\lambda), \ \ \mathrm{Im}\,\xi < 0$$
$$\partial_{\bar{\xi}}\chi(\xi,\lambda) = 0 \ \text{ for } \ \xi \in \mathbb{C}^{\pm}, \ \mathrm{Im}\,\chi \neq 0,$$
$$\chi(\xi,\lambda) \to 0 \ \text{ as } \ |\xi| \to \infty.$$

Direct spectral transform

Equivalently, we define a new complex structure on S^2 by gluing the lower semi-sphere to the upper semi-sphere along the map $\xi + \sigma(\xi, \lambda)$, $\lambda =$ const, and introducing the uniformizing coordinate $\xi + \chi(\xi, \lambda)$.



This procedure is analogous to the construction of non-local Riemann problem data in the classical paper by Manakov dedicated to KP-1.

By analogy with dispersive systems, there are two ways of defining the time dynamics: By introducing the time-dependence in the spectral data:

$$\sigma(\xi,\lambda,t) = \sigma(\xi-\lambda^2 t,\lambda,0),$$
$$\chi_{\pm}(\xi,\lambda,t) = \chi_{\pm}(\xi-\lambda^2 t,\lambda,0),$$

or by introducing the t-dependence in the asymptotic of the wave function. We use the second approach.

The inverse spectral problem equation has the form:

$$\psi_-(x,y,t,\lambda)-H_\lambda\chi_{-I}\Big(\psi_-(x,y,t,\lambda),\lambda\Big)+\chi_{-R}\Big(\psi_-(x,y,t,\lambda),\lambda\Big)=x-\lambda y-\lambda^2 t,$$

where χ_{-R} and χ_{-I} denote the real and imaginary parts of χ_- respectively, H_λ – denotes the Hilbert transform in λ

$$H_\lambda f(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(\lambda')}{\lambda-\lambda'}d\lambda'.$$

Inverse spectral transform

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Inverse spectral transform

In terms of the Hilbert transform analyticity of $\chi_-(\xi,\lambda)$ ξ in the lower half-plane is equivalent to: $\chi_{-R} - H_\xi \chi_{-I} = 0$.

Theorem

Let the spectral data $\chi_-(\xi,\lambda)$ satisfy the following constraints:

1. $\chi_-(\xi,\lambda)$, $\partial_\xi \chi_-(\xi,\lambda)$ *are differentiable*
2. $$|\partial_\xi \chi_{-R}(\xi,\lambda)| \le \frac{1}{4}, \quad |\partial_\xi \chi_{-I}(\xi,\lambda)| \le \frac{1}{4}.$$
3. *For some $C > 0$*
$$|\chi_-(\xi,\lambda)| \le \frac{C}{1+|\lambda|}$$

Then for all $x, y, t \in \mathbb{R}$, $t \ge 0$ inverse problem equations are uniquely solvable and $\psi(x,y,t,\lambda) = x - \lambda y - \lambda^2 t + \omega(x,y,t,\lambda)$, where $\omega(x,y,t,\lambda) \in L^2(d\lambda) \cap L^\infty(d\lambda)$.

Theorem

Assume, that we have the following constraints on the inverse spectral data: Let the spectral data $\chi_-(\xi,\lambda)$ satisfy the following constraints:

1. $|\partial_\xi \chi_{-R}(\xi,\lambda)| \leq \frac{1}{4}\tan\left(\frac{\pi}{8}\right), \quad |\partial_\xi \chi_{-I}(\xi,\lambda)| \leq \frac{1}{4}\tan\left(\frac{\pi}{8}\right).$
2. $|\partial_\xi^n \chi_-(\xi,\lambda)| \leq \frac{C}{1+|\lambda|^{2+n}}$, $n = 0,1,2,3.$
3. $|\partial_\xi^n \partial_\lambda \chi_-(\xi,\lambda)| \leq \frac{C}{1+|\lambda|^{3+n}}$, $n = 0,1.$

Then

- *The regularized wave functions $\omega_x, \omega_y, \omega_t \in L^2(d\lambda) \cap L^4(d\lambda)$, $\omega_{xx}, \omega_{xy}, \omega_{xt}, \omega_{yy} \in L^2(d\lambda)$, and $\psi(x,y,t,\lambda)$ satisfy the Lax pair for the Pavlov equation.*
- *The functions $v_x, v_y, v_{xx}, v_{x,y}, v_{xt}, v_{yy}$ are well-defined and satisfy the Pavlov equation.*

Inverse spectral transform: the small norm condition

Let us associate the following constants with the Cauchy data

$$B_0 = \int_{-\infty}^{+\infty} \left[\max_{x\in\mathbb{R}} |v_x(x,y)| \right] dy,$$

$$B_1 = \exp\left[\int_{-\infty}^{+\infty} \left[\max_{x\in\mathbb{R}} |v_{xx}(x,y)| \right] dy \right] - 1,$$

$$B_2 = \left[\int_{-\infty}^{+\infty} \left[\max_{x\in\mathbb{R}} |v_{xxx}(x,y)| \right] dy \right] (1+B_1)^3,$$

$$B_3 = \left[\int_{-\infty}^{+\infty} \left[\max_{x\in\mathbb{R}} |v_{xxx}(x,y)| \right] dy \right] 3(1+B_1)^2 B_2 +$$

$$+ \left[\int_{-\infty}^{+\infty} \left[\max_{x\in\mathbb{R}} |v_{xxxx}(x,y)| \right] dy \right] (1+B_1)^4,$$

Inverse spectral transform: the small norm condition

$$\hat{B}_0 = \left[\int\limits_{-\infty}^{+\infty} \left(\sqrt{\int\limits_{-\infty}^{+\infty} |v_x(x,y)|^2 dx} \right) dy \right] \cdot \frac{1}{\sqrt{1 - B_1}},$$

$$\hat{B}_1 = \left[\int\limits_{-\infty}^{+\infty} \left(\sqrt{\int\limits_{-\infty}^{+\infty} |v_{xx}(x,y)|^2 dx} \right) dy \right] \cdot \frac{1 + B_1}{\sqrt{1 - B_1}}.$$

Theorem

Assume that

1. $v(x,y)=0$ *outside the area* $-D_x \leq x \leq D_x,\ -D_y \leq x \leq D_y$.
2. $B_0 \leq \frac{1}{4}$,
3. $B_1 \leq \frac{1}{2}$,
4. $8B_0 + 8B_2 + 2\sqrt{2}\hat{B}_0 < \pi$,
5. $2B_1 + \frac{\sqrt{2}}{\pi}(32B_1 + 16\hat{B}_0) + \frac{1}{\pi}(8B_3 + 16B_2^2 + 56B_1 + 16B_1^2)\left(B_0 + \frac{2}{\pi}[2B_0 + \hat{B}_0]\right) < \tan\left(\frac{\pi}{8}\right)$.

Then the unique solubility conditions for the inverse problem are fulfilled.

Inverse spectral transform

By analogy with the standard KP equation the behavior of v_t at $t = 0$ requires an extra investigation.

Open question: how to characterize analogs of Manakov conditions for KP? = How to select well-localized at all times solutions?

Another question. What happens, if we consider the inverse problem

$$\psi_-(x,y,t,\lambda) - H_\lambda \chi_{-I}\Big(\psi_-(x,y,t,\lambda),\lambda\Big) + \chi_{-R}\Big(\psi_-(x,y,t,\lambda),\lambda\Big) = x - \lambda y - \lambda^2 t,$$

with inverse data such that:

$$\chi_{-R} - H_\xi \chi_{-I} \neq 0?$$

It can be shown, that we obtain the same solutions of the Pavlov equation, but the normalization of the wave function will be different from the Jost one at $y \to -\infty$.

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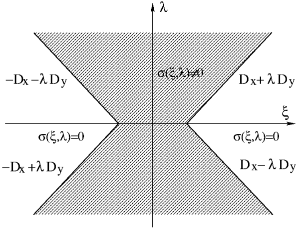
Using the standard methods of ordinary differential equations theory it is easy to prove the following properties of the function $\sigma(\xi,\lambda)$

- $\sigma(\xi,\lambda)$ is a smooth regular function of 2 real variables.
- $\partial_\xi(\xi+\sigma(\xi,\lambda))>0$.
- The support of $\sigma(\xi,\lambda)$ lies in the area $|\xi|\leq|D_x+|\lambda|D_y|$, $\lambda,\xi\in\mathbb{R}$.



- For sufficiently large λ one can write $\sigma(\xi,\lambda)=\frac{1}{\lambda^2}\mathring{\sigma}\left(\frac{\xi}{\lambda},\frac{1}{\lambda}\right)$, where $\mathring{\sigma}\left(\mathring{\xi},\mathring{\lambda}\right)$ is a regular function for sufficiently small $\mathring{\lambda}$.

Direct spectral transform - technical details

From these properties it is easy to check, that

$$\|\partial_\xi^n \chi(\xi,\lambda)\|_{L^\infty(d\xi)} = O\left(\frac{1}{\lambda^{2+n}}\right), \quad n = 0,1,2,3,$$

$$\|\partial_\xi^n \partial_\lambda \chi(\xi,\lambda)\|_{L^\infty(d\xi)} = O\left(\frac{1}{\lambda^{3+n}}\right), \quad n = 0,1.$$

Moreover, if the small norm conditions are fulfilled, then

$$|\chi_\xi(\xi,\lambda)| \leq \frac{1}{4}\tan\left(\frac{\pi}{8}\right).$$

To estimate $\chi(\xi,\lambda)$ for large ξ we use the formula:

$$\chi(\xi,\lambda) = \frac{1}{2\pi i}\int_{-D_x-|\lambda|D_y}^{D_x+|\lambda|D_y} \frac{(\chi_+(\tau,\lambda)-\chi_-(\tau,\lambda))}{\tau-\xi}d\tau.$$

We solve the nonlinear integral equation of the inverse problem:

$$\psi(x, y, t, \lambda) = x - \lambda y - \lambda^2 t - \chi_{-R}(\psi(x, y, t, \lambda), \lambda) - \frac{1}{\pi} \fint_{\mathbb{R}} \frac{\chi_{-I}(\psi(x, y, t, \zeta), \zeta)}{\zeta - \lambda} d\zeta$$

How to select the proper functional space for this equation?
We know, that

$$\|H_\lambda\|_{L^2(d\lambda)} = 1, \quad \|H_\lambda\|_{L^4(d\lambda)} = \cot\left(\frac{\pi}{8}\right)$$

therefore we have convergence in L^2, L^4 spaces. But to show that the x, y, t-derivatives of ψ satisfy the linearized equation, we require L^∞ convergence.
How to prove the L^∞ convergence?

We use the following unequality:

$$|f(\lambda)| \leq \sqrt{||f||_{L^2(d\lambda)} \cdot ||f_\lambda||_{L^2(d\lambda)}}$$

Our iteration procedure has the following form:

$$\omega^{(n+1)} = -\chi_{-R}(x-\lambda y-\lambda^2 t+\omega^{(n)},\lambda) + H_\lambda\left[\chi_{-I}(x-\lambda y-\lambda^2 t+\omega^{(n)},\lambda)\right]$$

$$\begin{aligned}\omega_\lambda^{(n+1)} =& g_\lambda^{(n)} - \partial_\xi\chi_{-R}(x-\lambda y-\lambda^2 t+\omega^{(n)},\lambda)\,\omega_\lambda^{(n)} + \\ & + H_\lambda\left[\partial_\xi\chi_{-I}(x-\lambda y-\lambda^2 t+\omega^{(n)},\lambda)\,\omega_\lambda^{(n)}\right],\end{aligned}$$

$$\begin{aligned}g_\lambda^{(n)} =& -\partial_\lambda\chi_{-R}(x-\lambda y-\lambda^2 t+\omega^{(n)},\lambda) + \\ & + H_\lambda\left[\partial_\lambda\chi_{-I}(x-\lambda y-\lambda^2 t+\omega^{(n)},\lambda)\right] + \\ & + (\partial_\xi\chi_{-R}(x-\lambda y-\lambda^2 t+\omega^{(n)},\lambda))\cdot(y+2\lambda t) - \\ & - H_\lambda\left[(\partial_\xi\chi_{-I}(x-\lambda y-\lambda^2 t+\omega^{(n)},\lambda))\cdot(y+2\lambda t)\right]\end{aligned}$$

Inverse spectral transform - technical details

For any fixed x, y, t the functions $g_\lambda^{(n)}$ are uniformly bounded in $L^2(d\lambda)$.
Therefore $\omega_\lambda^{(n+1)}$ are also uniformly bounded in $L^2(d\lambda)$, and L^2 convergence implies L^∞ convergence.
The iteration procedure for the first derivatives of ψ also converges in L^4. Why do we need L^4 convergence?
For the second derivatives of the wave function we have:

$$\begin{aligned}\psi_{\alpha\beta} = &-\partial_\xi^2 \chi_{-R}(\psi,\lambda)\,\psi_\alpha\psi_\beta - \partial_\xi\chi_{-R}(\psi,\lambda)\,\psi_{\alpha\beta} + \\ &+ H_\lambda\left[\partial_\xi^2\chi_{-I}(\psi,\lambda)\,\psi_\alpha\psi_\beta\right] + H_\lambda\left[\partial_\xi\chi_{-I}(\psi,\lambda)\,\psi_{\alpha\beta}\right] = \\ &= g_{\alpha\beta} - \partial_\xi\chi_{-R}(\psi,\lambda)\,\psi_{\alpha\beta} + H_\lambda\left[\partial_\xi\chi_{-I}(\psi,\lambda)\,\psi_{\alpha\beta}\right],\end{aligned}$$

where

$$g_{\alpha\beta} = -\partial_\xi^2\chi_{-R}(\psi,\lambda)\,\psi_\alpha\psi_\beta + H_\lambda\left[\partial_\xi^2\chi_{-I}(\psi,\lambda)\,\psi_\alpha\psi_\beta\right].$$

$\psi_{alpha} \in L^4(d\lambda)$, therefore $g_{\alpha\beta} \in L^2(d\lambda)$.