Generalized boundary conditions and inverse problems

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Joint work with

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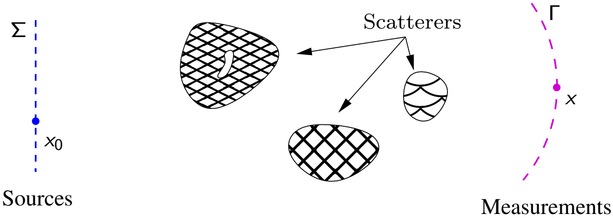
General applicative context

Radar, Sonar, Medical Imaging, Non destructive testing, $\cdots$



Inverse problem: Determine the geometry (imaging) and some physical properties (identification) of inclusions from the knowledge of diffracted fields (associated with several incident waves).

- nonlinear problem
- unstable with respect to measurement error (ill-posed problem)

None of the existing numerical methods can efficiently treat the problem in its general setting.

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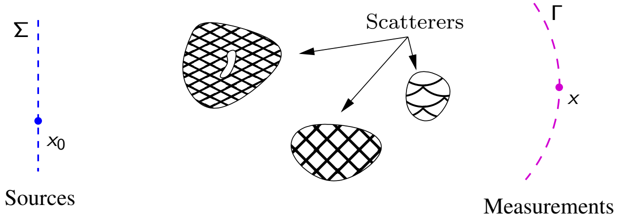

Main interests: We consider problems for which the linearization is not possible (strongly non linear problems)

- inclusions with high contrasts, frequency in the resonant regime
- multiple scales, complex topology

$\Rightarrow$ We use multistatic data at fixed frequencies.

Goal: Get reliable qualitative information with a few a priori information (on physical properties).

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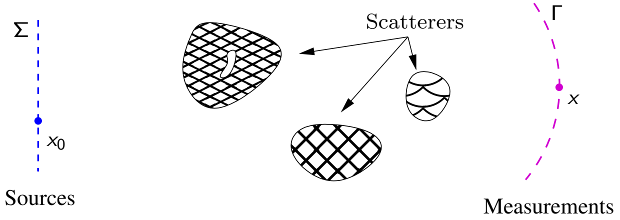


Possible Inversion Methods:

- Qualitative methods: e.g. Sampling Methods (Colton-Kirsch 1996): model-free methods but require many measurements
- Non linear optimization methods: require relatively simple (but relevant!) models.

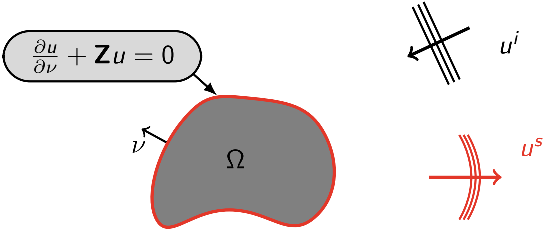
This talk: presents a mixture of both strategies in the case of Generalized Impedance Boundary Conditions (GIBC).

Outline

1. GIBC models: motivation and general settings for the scalar case
2. On the analysis of the direct problem
3. The Factorization Method for GIBC
4. Quick overview of a steepest descent method
5. Few words on the Maxwell case
6. Open problems and perspectives

General scattering model

Scalar scattering problems from obstacles can be formally written as



$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{\Omega} =: \Omega_{\text{ext}}$$

$$\lim_{R \to \infty} \int_{|x|=R} \left| \partial_r u^s - iku^s \right|^2 ds = 0$$

$$u = u^s + u^i \quad \text{in} \quad \Omega_{\text{ext}}$$

$\mathbf{Z}$: is the Dirichlet-to-Neumann operator for the wave equation inside Ω.

This operator is non local in general and exact evaluation of $\mathbf{Z}$ may be computationally expensive: typically if Ω involves a small scale $\delta \ll 2\pi/k$.

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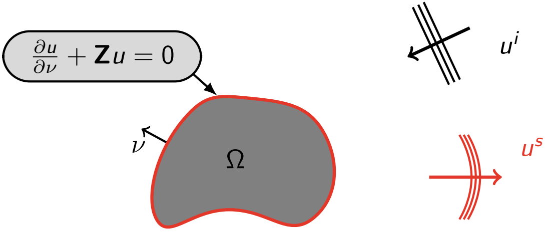

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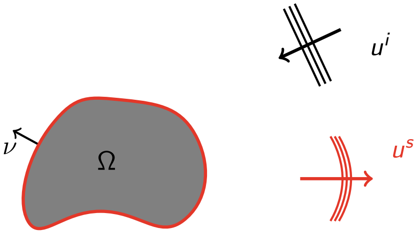
Z: is the Dirichlet-to-Neumann operator for the wave equation inside Ω.

GIBC: is an approximation of **Z** in terms of local surface operators.

Examples of GIBC

Imperfectly conducting obstacles



Inside the scatterer:

$$\Delta u + k^2 u + ik\sigma u = 0$$

Absorption: the wave decays inside Ω like $\exp(-\frac{\sqrt{2}}{2}\sqrt{k\sigma}|x \cdot \nu|)$

Small parameter for $\sigma \gg 1$: $\delta = 1/\sqrt{k\sigma}$ (skin-depth)

Examples of GIBC

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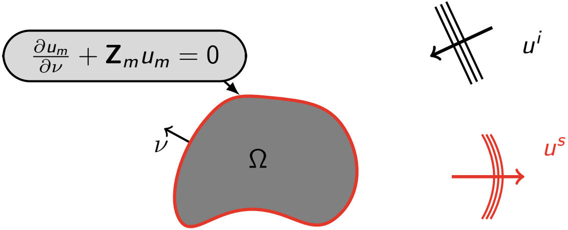


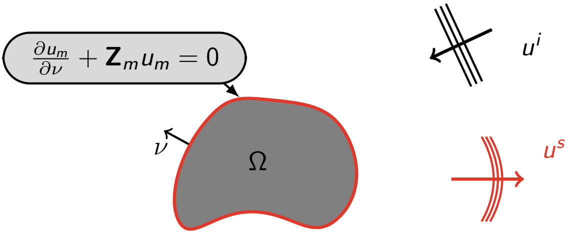
GIBC: $\mathbf{Z} = \mathbf{Z}_m \Rightarrow \|u - u_m\| = O(\delta^{m+1})$

- $\mathbf{Z}_2 = (\frac{\delta}{\sqrt{i}} + iH\delta^2)^{-1}$ (classical impedance)
- $\mathbf{Z}_3 = (\frac{\delta}{\sqrt{i}} + iH\delta^2 - \frac{\sqrt{i}\,\delta^3}{2}\left(3H^2 - G + \omega^2 + \Delta_\Gamma\right))^{-1}$

H.-Joly-Nguyen (2005).

Examples of GIBC

Imperfectly conducting obstacles



GIBC: classical model (Leontovitch 1930)

$$\mathbf{Z} = \lambda \cdot .$$

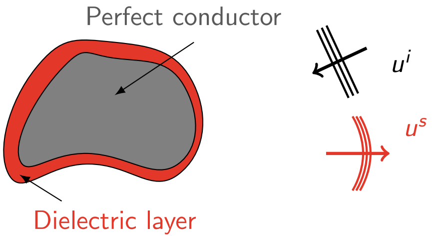
is satisfactory in general ...

A more accurate model would correspond with

$$\mathbf{Z} : L^2(\Gamma) \rightarrow L^2(\Gamma)$$

Examples of GIBC

Thin coatings



Inside the thin coating (TE mode):

$$\mathrm{div}\mu^{-1}\nabla u + k^2\epsilon u = 0$$

Small parameter: $\delta =$ (variable) width of the coating

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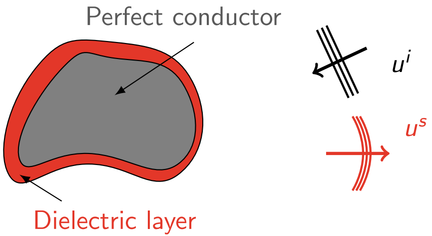

GIBC: $\mathbf{Z} = \mathbf{Z}_m \Rightarrow \|u - u_m\| = O(\delta^{m+1})$

- $\mathbf{Z}_1 = \mathrm{div}_\Gamma(\delta\mu^{-1}\nabla_\Gamma \cdot) + \delta\epsilon k^2 \cdot$
- $\mathbf{Z}_2 = \mathrm{div}_\Gamma((\delta - \delta^2 H)\mu^{-1}\nabla_\Gamma \cdot) + (\delta + \delta^2 H)\epsilon k^2 \cdot$

Aslanyureck-H.-Sahinturk (2011).

Similar expressions for periodic coatings, or periodic interfaces, with periodicity of size δ. PhD of M. Chamaillard (2011).

Examples of GIBC

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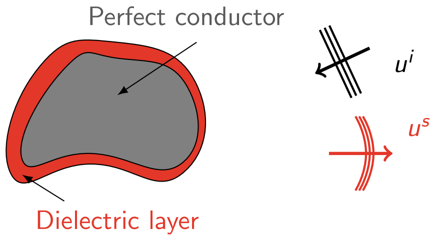

GIBC: A model of the form (Bouchitté (1990), Engquist-Nédélec (1993), Bendali-Lemrabet (1996), ...)

$$\mathbf{Z} = \mathrm{div}_\Gamma(\eta \nabla_\Gamma \cdot) + \lambda \cdot \, .$$

is satisfactory in general ...

A more accurate model would correspond with

$$\mathbf{Z} : H^1(\Gamma) \to H^{-1}(\Gamma)$$

A general framework for GIBC forward problem

- $V(\Gamma)$ an Hilbert space such that $C^\infty(\Gamma) \subset V(\Gamma)$ with dense embedding.
- $\mathbf{Z} : V(\Gamma) \longrightarrow V(\Gamma)^*$ is linear and continuous.
- $\Im m \langle \mathbf{Z}u, u \rangle_{V^*,V} \geq 0$ (compatible with the radiation condition).

The GIBC problem ($\mathcal{P}_{\text{vol}}$) can be written as :

Find $u^s \in \left\{ v \in \mathcal{D}'(\Omega_{\text{ext}}),\ \varphi v \in H^1(\Omega_{\text{ext}})\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d);\ v_{|\Gamma} \in V(\Gamma) \right\}$

$$\begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}}, \\ \dfrac{\partial u^s}{\partial \nu} + \mathbf{Z} u^s = f \text{ on } \Gamma, \qquad \left(f = -\dfrac{\partial u^i}{\partial \nu} - \mathbf{Z} u^i \right) \\ \lim\limits_{R\to\infty} \displaystyle\int_{|x|=R} |\partial_r u^s - iku^s|^2 = 0. \end{cases} ;$$

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- **Thm:** Assume that $\Re\langle \mathbf{Z}u, u\rangle_{V^*,V} \leq 0$, then $(\mathcal{P}_{\text{vol}})$ is well posed for $f \in (V(\Gamma) \cap H^{1/2}(\Gamma))^*$. Moreover, the solution is uniformly stable with respect to $\mathbf{Z}$.

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Thm: Problem $(\mathcal{P}_{\text{vol}})$ is well posed if we further assume one of the following:

1. $\mathbf{Z} = -C_{\mathbf{Z}} + K_{\mathbf{Z}}$ with $C_{\mathbf{Z}} : V(\Gamma) \to V(\Gamma)^*$ satisfying

   $$\Re\langle C_{\mathbf{Z}}u, u\rangle - \Im\langle C_{\mathbf{Z}}u, u\rangle \geq c\|u\|^2_{V(\Gamma)} \quad \text{for all } u \in V(\Gamma)$$

   is coercive and $K_{\mathbf{Z}} : V(\Gamma) \to V(\Gamma)^*$ is compact.
2. $H^{1/2}(\Gamma) \subset V(\Gamma)$ with compact embedding.
3. $V(\Gamma) \subset H^{1/2}(\Gamma)$ with compact embedding and $\mathbf{Z} : V(\Gamma) \to V(\Gamma)^*$ is Fredholm with index 0.

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The inverse problem for farfield settings

For $u^i(x, \hat{\theta}) = e^{ik\hat{\theta}\cdot x}$ define

$$(\mathbf{Z}, D) \longrightarrow u^\infty(\hat{x}, \hat{\theta})$$

where u^∞ associated with $u^s(\mathbf{Z}, D)$ is defined in dimension d by

$$u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \qquad r \longrightarrow +\infty.$$

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Factorization method: Reconstruct D without knowing $\mathbf{Z}$... using measurements of $u^\infty(\hat{x}, \hat{\theta})$ for all $\hat{x}$ and $\hat{\theta}$.

- State of the art:
  - Dirichlet and Neumann boundary conditions: Kirsch 1997,
  - Impedance boundary conditions ($\mathbf{Z} = \lambda$): Grinberg & Kirsch 2002,
- The questions we try to answer (Chamaillard-Chaulet-H. preprint 2012)
  - For which class of operators $\mathbf{Z}$ the Factorization method can be justified?
  - How accurate are the reconstructions in terms of $\mathbf{Z}$?

Principle of the factorization method

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$$\begin{aligned} F: \quad L^2(S^{d-1}) &\longrightarrow L^2(S^{d-1}) \\ g &\longmapsto \int_{S^{d-1}} u^\infty(\hat{x},\hat{\theta}) g(\hat{\theta})\, d\hat{\theta} \end{aligned}$$

Define the self-adjoint positive operator

$$F_\# := |\Re e(F)| + |\Im m(F)|$$

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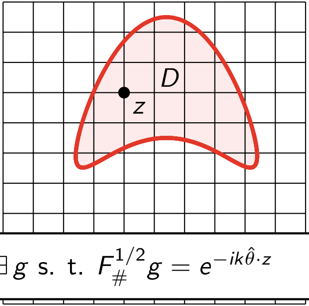

$\exists\, g$ s. t. $F_\#^{1/2} g = e^{-ik\hat\theta\cdot z}$

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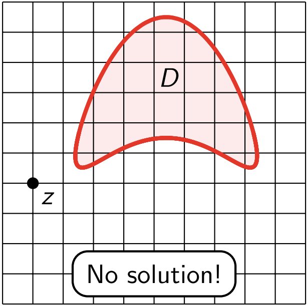

Steps of the factorization method theory

1. First step: formal factorization
   Find $\Lambda(\Gamma) \subset L^2(\Gamma)$ and two bounded operators $G : \Lambda(\Gamma)^* \to L^2(S^{d-1})$ and $T : \Lambda(\Gamma) \to \Lambda(\Gamma)^*$ such that

   $$\boxed{F = GT^*G^*}$$

   and such that the range of G is dense and characterizes D. For instance:

   $$z \in D \Longleftrightarrow \phi_z^\infty \in \mathcal{R}(G), \quad (\text{ where } \phi_z^\infty(\hat{x}) := e^{-ikz\cdot\hat{x}}).$$

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2. Second step: range identities
   Prove that

   $$\boxed{\mathcal{R}(G) = \mathcal{R}(F_\#^{1/2})}$$

   We shall rely on "classical" Grinberg-Kirsch version. The range identity holds if $\Re(T) = C + K$ with C coercive and T compact and if $\Im(T^*)$ is positive (or negative) on the closure of the range of G^*.

Formal factorization of the farfield operator

Following the case $\mathbf{Z} = \lambda.$ in Grinberg-Kirsch (2002) we define

$$G \ : \ (V(\Gamma) \cap H^{1/2}(\Gamma))^* \longrightarrow L^2(S^{d-1})$$
$$f \longmapsto u_f^\infty$$

where u_f^∞ is the farfield associated with the solution to $(\mathcal{P}_{\text{vol}})$ with boundary data f on Γ.

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Proof: ϕ_z^∞ is the farfield of the Green function

$$\Phi_z(x) := \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x-z|}.$$

- If $z \in D$, take $f = -(\partial_\nu \Phi_z + \mathbf{Z}\Phi_z)$ then $Gf = \phi_z^\infty$.
- If $z \notin D$, then $\phi_z^\infty \in \mathcal{R}(G) \Rightarrow \Phi_z$ is H^1 in the neighborhood of $z \Rightarrow$ contradiction.

Formal factorization of the farfield operator

$$F = -GT^*G^*$$

with a boundary operator T that can be formally expressed as

$$\boxed{T := \mathbf{Z}\mathcal{S}\mathbf{Z}^* + \mathcal{D} + \mathbf{Z}\mathcal{K} + \mathcal{K}'\mathbf{Z}^*}$$

with:

$$\mathcal{S} := \mathrm{SL}|_\Gamma, \quad \mathcal{K} := \mathrm{DL}|_\Gamma, \quad \mathcal{K}' := \partial_\nu \mathrm{SL}|_\Gamma, \quad \mathcal{D} := \partial_\nu \mathrm{DL}|_\Gamma.$$

and SL and DL are respectively the single and double layer potentials on Γ

$$\mathrm{SL}(q)(x) = \int_\Gamma \Phi_x(y) q(y) ds(y), \quad \mathrm{DL}(q)(x) = \int_\Gamma \frac{\partial \Phi_x(y)}{\partial \nu(y)} q(y) ds(y), \ x \in \mathbb{R}^d \backslash \Gamma.$$

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- In the case $V(\Gamma) \subset H^{1/2}(\Gamma)$ with compact embedding, the principal part is given by $\mathbf{Z}\mathcal{S}\mathbf{Z}^* \Rightarrow$ requires a careful definition of $\Lambda(\Gamma)$
- In the other cases one cannot conclude on the sign of the pricipal part of $\Re T$... Unfortunately the sign of $\mathbf{Z}$ cannot be of any help.

Function space setting for the factorization method

In the case $V(\Gamma) \subset H^{1/2}(\Gamma)$ with compact embedding...

Why $\Lambda(\Gamma) = V(\Gamma) \cap H^{1/2}(\Gamma) = V(\Gamma)$ does not fit ?

$$\boxed{T := \mathbf{Z}\mathcal{S}\mathbf{Z}^* + \mathcal{D} + \mathbf{Z}\mathcal{K} + \mathcal{K}'\mathbf{Z}^*}$$

$$\mathcal{S} \ : \ H^s(\Gamma) \to H^{s+1}(\Gamma)$$

We want $T \ : \ \Lambda(\Gamma) \to \Lambda(\Gamma)^*$.

Consider

$$\mathbf{Z} = \Delta_\Gamma,$$

$$V(\Gamma) = H^1(\Gamma),$$

then by taking $\Lambda(\Gamma) = V(\Gamma)$ *we have:*

$$T \ : \ H^1(\Gamma) \to H^{-2}(\Gamma).$$

Correct choice : $\Lambda(\Gamma) = H^{3/2}(\Gamma) = \Delta_\Gamma^{-1}(H^{-1/2}(\Gamma))$

Careful definition of T and rigorous factorization

We further assume that

- $V(\Gamma)$ is compactly embedded into $H^{1/2}(\Gamma)$.
- $\mathbf{Z} : V(\Gamma) \to V(\Gamma)^*$ (or its principal part) is symmetric.

$$\boxed{\Lambda(\Gamma) := \left\{ u \in V(\Gamma),\, \mathbf{Z}^* u \in H^{-1/2}(\Gamma) \right\}}$$

$$(u, v)_{\Lambda(\Gamma)} := (u, v)_{V(\Gamma)} + (\mathbf{Z}^* u, \mathbf{Z}^* v)_{H^{-1/2}(\Gamma)}.$$

Thm: $\Lambda(\Gamma) \subset L^2(\Gamma) \subset \Lambda(\Gamma)^*$ with dense embedding.

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Thm: $\Lambda(\Gamma) \subset L^2(\Gamma) \subset \Lambda(\Gamma)^*$ with dense embedding. Moreover: $F = -GT^*G^*$ with

- $G : \Lambda(\Gamma)^* \to L^2(S^{d-1})$ is compact with dense range.
- $T : \Lambda(\Gamma) \to \Lambda(\Gamma)^*$ is continuous. $\Re e(T) = C + K$ with C coercive and K compact.
- $-\Im m(T^*)$ compact and positive on $\overline{\mathcal{R}(G^*)}$ if k^2 is not an eigenvalue for the interior GIBC problem (the eigenvalues form a discrete set if $\mathbf{Z}$ analytically depends on k).

Quick summary

- OK if the embedding $V \subset H^{1/2}(\Gamma)$ is compact,

$$\mathbf{Z} = \mathrm{div}_\Gamma(\mu \nabla_\Gamma \cdot) + \lambda \cdot$$

$$V(\Gamma) = H^1(\Gamma)$$

Quick summary

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$$\mathbf{Z} = \lambda \cdot$$
$$V(\Gamma) = L^2(\Gamma)$$

- Open problem : none of the above compact embeddings hold.

$$\mathbf{Z} = \mathrm{div}_\Gamma(\mu \chi_{\Gamma_0} \nabla_\Gamma \cdot) + \lambda \cdot \quad \Gamma_0 \subsetneq \Gamma$$
$$V(\Gamma) = \{u \in L^2(\Gamma); \int_{\Gamma_0} \mu |\nabla_\Gamma u|^2 ds < \infty\}$$

Some numerical experiments

- $\mathbf{Z} = \mathrm{div}_\Gamma(\eta \nabla_\Gamma \cdot)$ with $\eta = 1$,
- For N=50, the synthetic data are

$$\left\{ u^\infty \left(\frac{2i\pi}{N}, \frac{2j\pi}{N} \right) \right\}_{i,j=1,\cdots,N}$$

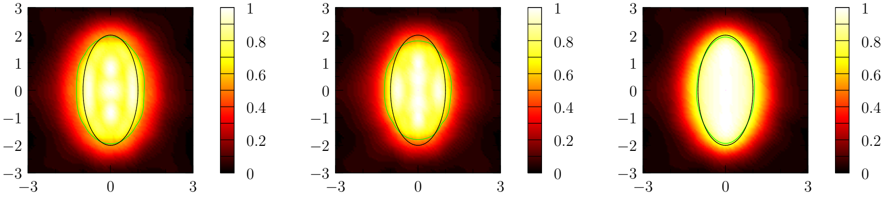
- For each z in a given sampling grid we solve a discrete version of

$$F_\#^{1/2} g_z = \phi_z^\infty$$

with Tikhonov-Morozov regularization and plot

$$z \longmapsto \frac{1}{\|g_z\|}.$$

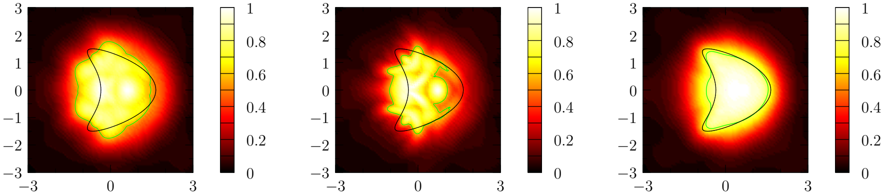
Numerical reconstructions



(a) ellipse, $\eta = 0.01$ (b) ellipse, $\eta = 0.1$ (c) ellipse, $\eta = 10$

Figure: Reconstruction of a convex geometry for several values of η. Wave number: $k = 2$. Added random noise: 1%.

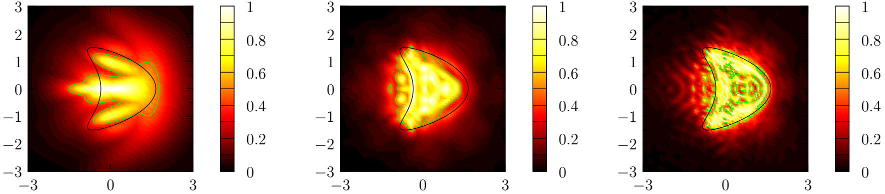
Numerical reconstructions



(a) $\eta = 0.01$ (b) $\eta = 0.1$ (c) $\eta = 10$

Figure: Reconstruction of a non-convex geometry for several values of η. Wave number: $k = 2$. Added random noise: 1%.

Numerical reconstructions

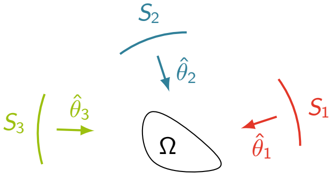


(a) $k = 1$ (b) $k = 4$ (c) $k = 8$

Figure: Reconstruction of a non-convex geometry for several of k. GIBC parameter: $\eta = 0.1$. Added random noise: 1%.

The inverse problem for finite number of measurements

Goal: Reconstruct D and/or $\mathbf{Z}$ using measurements of $u^{\infty}(\hat{x}, \hat{\theta})$ for small number of incidents $\hat{\theta}$ directions and reduced aperture of measurement $\hat{\theta}$.



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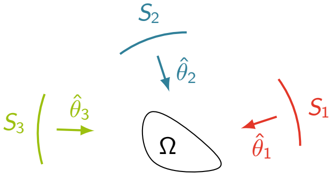


Adapted class of methods: Non linear optimization methods $\Rightarrow$ need a model for $\mathbf{Z}$:

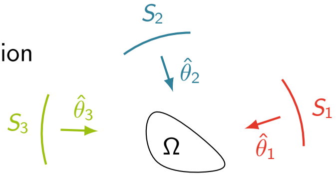
$$\boxed{\mathbf{Z} = \mathrm{div}_\Gamma(\eta \nabla_\Gamma \cdot) + \lambda \cdot}$$

$\Rightarrow$ relevance of GIBC is the most obvious for this class of methods

- reduced number of unknowns
- quicker solver for the direct problem
- $\mathbf{Z}$ contains "less unstable" information on the physical parameters and/or the geometry

The chosen method

$$
\begin{cases}
\Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}} + \text{ radiation condition} \\
u = u^s + u^i(\cdot, \hat{\theta}) \\
\dfrac{\partial u}{\partial \nu} + \operatorname{div}_\Gamma(\eta \nabla_\Gamma u) + \lambda u = 0 \text{ on } \Gamma
\end{cases}
$$



Minimize the least square cost functional

$$
\boxed{F(\lambda, \eta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \| u^\infty_{\lambda,\eta,\Gamma}(\cdot, \hat{\theta}_j) - u^\infty_{\text{obs}}(\cdot, \hat{\theta}_j) \|^2_{L^2(S_j)}}
$$

using a steepest descent method.

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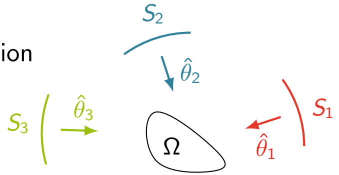

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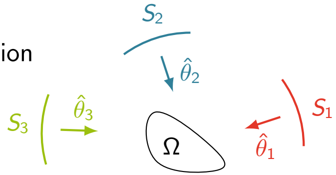
using a steepest descent method.

We found it: simple to implement, adapted to parametrization free problems, quick iteration cost by using adjoint technique... but convergence rate may be slow.

The chosen method

$$\begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}} + \text{ radiation condition} \\ u = u^s + u^i(\cdot, \hat{\theta}) \\ \dfrac{\partial u}{\partial \nu} + \text{div}_\Gamma(\eta \nabla_\Gamma u) + \lambda u = 0 \text{ on } \Gamma \end{cases}$$



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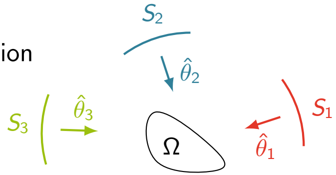
using a steepest descent method.

- we need partial derivatives with respect to λ and η (quite standard),
- we need an appropriate derivative w.r.t. the obstacle: is not uniquely defined if λ and η depends on Γ.
- we need careful regularization of the gradient

The chosen method

$$\begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}} + \text{ radiation condition} \\ u = u^s + u^i(\cdot, \hat{\theta}) \\ \dfrac{\partial u}{\partial \nu} + \text{div}_\Gamma(\eta \nabla_\Gamma u) + \lambda u = 0 \text{ on } \Gamma \end{cases}$$



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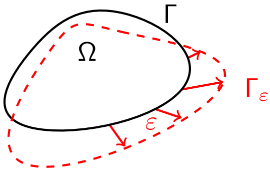
using a steepest descent method.

- Bourgeois-Chaulet-H. (2011, 2012): scalar case + stability and uniqueness issues for the boundary coefficients, Chaulet-H. (2013) the Maxwell-case.
- Shape optimization techniques: book of Allaire, Shape Optimization methods.. (2002)
- Nonlinear Integral equation method for the Laplace problem with constant coefficients Cakoni-Kress (2013)

Shape derivative

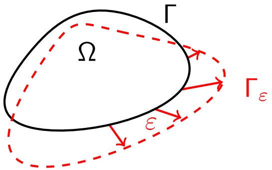


- Γ is given
- $\varepsilon \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ such that $\|\varepsilon\|_{C^1} < 1$
- $f_\varepsilon := \mathrm{Id} + \varepsilon$
- $\Gamma_\varepsilon := f_\varepsilon(\Gamma)$

Definition (constant coefficients): The shape derivative of the scattered field is given by the Fréchet derivative at 0 of

$$R : \varepsilon \longrightarrow u^s(\lambda, \eta, \Gamma_\varepsilon).$$

Shape derivative



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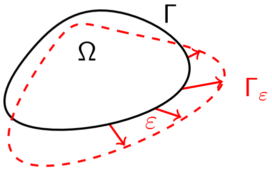
Definition (non constant coefficients): The shape derivative of the scattered field is given by the Fréchet derivative at 0 of

$$\widetilde{R} : \varepsilon \longrightarrow u^s(\lambda_\varepsilon, \eta_\varepsilon, \Gamma_\varepsilon).$$

This derivative depends on the way one defines $\lambda_\varepsilon, \eta_\varepsilon$

Shape derivative



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This derivative depends on the way one defines $\lambda_\varepsilon, \eta_\varepsilon$

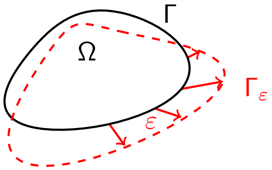
A first choice

$$\lambda_\varepsilon(x) := \lambda(x_\varepsilon)\,, \quad \eta_\varepsilon(x) := \eta(x_\varepsilon)$$

where x_ε is the projection on Γ of $x \in \Gamma_\varepsilon$ along the normal ν

$\Rightarrow$ provide the same expression of the derivative as for constant λ and η.

Shape derivative



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This derivative depends on the way one defines $\lambda_\varepsilon, \eta_\varepsilon$

Our choice

$$\lambda_\varepsilon := \lambda \circ f_\varepsilon^{-1}\,, \quad \eta_\varepsilon := \eta \circ f_\varepsilon^{-1}$$

$\Rightarrow$ One may find f_ε such that $\Gamma = f_\varepsilon(\Gamma)$ and $\boxed{F'_{\lambda,\eta}(\Gamma) \cdot \varepsilon \neq 0.}$

$\Rightarrow$ $F'_{\lambda,\eta}(\Gamma)$ does not satisfy the classical shape derivative's properties!

Expression of the shape derivative

Adapting the technique in H.-Kress (2004) for constant impedance... and after tedious technical calculations...

$$\boxed{u^s(\lambda_\varepsilon, \eta_\varepsilon, \Gamma_\varepsilon) - u^s(\lambda, \eta, \Gamma_\varepsilon) = v_\varepsilon + o(||\varepsilon||),}$$

where $v_\varepsilon (= \widetilde{R}'(0).\varepsilon)$ is the outgoing solution of the scattering problem with

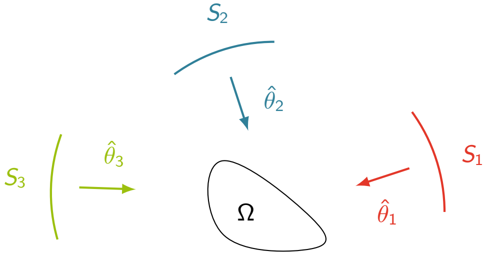
$$\frac{\partial v_\varepsilon}{\partial \nu} + \mathbf{Z} v_\varepsilon = B_\varepsilon u \quad \text{on} \quad \Gamma$$

$$\begin{aligned} B_\varepsilon u =& (\varepsilon \cdot \nu)(k^2 - 2H\lambda)u + \mathrm{div}_\Gamma\left((1 + 2\eta(R - H))(\varepsilon \cdot \nu)\nabla_\Gamma u\right) \\ &+ (\nabla_\Gamma \lambda \cdot \varepsilon)u + \mathrm{div}_\Gamma\left((\nabla_\Gamma \eta \cdot \varepsilon)\nabla_\Gamma u\right) \\ &+ \mathbf{Z}\left((\varepsilon \cdot \nu)\mathbf{Z} u\right), \end{aligned}$$

with $2H := \mathrm{div}_\Gamma \nu$, $R := \nabla_\Gamma \nu$ and $\mathbf{Z}\cdot = \mathrm{div}_\Gamma(\eta \nabla_\Gamma \cdot) + \lambda \cdot$

:) Indeed it is compatible with the cases $\eta = 0$ and λ constant (Hettlich, Kress-Païvaïrinta, H.-Kress, Potthast, Kirsch, ...).

A steepest descent algorithm to solve the inverse problem



$$F(\lambda, \eta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \| u^{\infty}_{\lambda,\eta,\Gamma}(\cdot, \hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot, \hat{\theta}_j) \|^2_{L^2(S_j)}$$

Numerical procedure:

- update alternatively λ, η and Γ with a direction given by the partial derivative of the cost function
- use the adjoint state technique $\Rightarrow$ $2I$ forward problems to solve at each iteration.

The regularization procedure

$$F(\lambda, \eta, \Gamma) = \frac{1}{2} \sum_{j=1}^{I} \| u^{\infty}_{\lambda,\eta,\Gamma}(\cdot, \hat{\theta}_j) - u^{\infty}_{\text{obs}}(\cdot, \hat{\theta}_j) \|^2_{L^2(S_j)}$$

We regularize the gradient by choosing $H^1(\Gamma)$ representation of partial derivatives.

- Descent direction for λ: $\delta\lambda \in H^1(\Gamma)$ that solves for every $\phi \in H^1(\Gamma)$ in

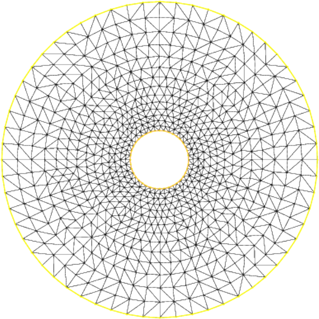
$$\beta_\lambda \int_\Gamma \nabla_\Gamma(\delta\lambda) \cdot \nabla_\Gamma \phi \, ds + \int_\Gamma \delta\lambda \phi \, ds = -\alpha_\lambda \, F'_{\eta,\Gamma}(\lambda) \cdot \phi$$

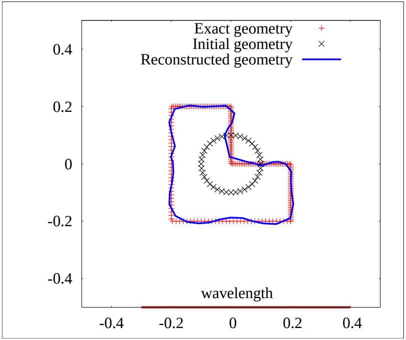
where β_λ is the regularization coefficient and α_λ is the descent coefficient.

- Do the same for $\delta\eta$ and $\delta(\Gamma)$.

Numerical reconstruction

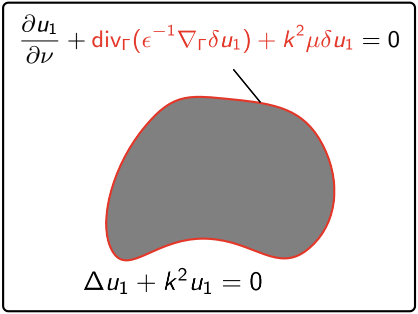
Finite elements method and remeshing procedure using FreeFem++

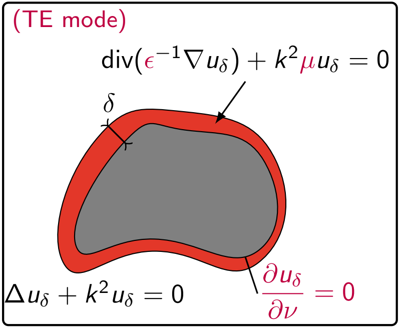




Reconstruction of the geometry with 2 incident waves and 1% noise on the farfield, $\lambda = ik/2$ and $\eta = 2/k$ being known

Application to the reconstruction of a coated obstacle





Reconstruction of an obstacle using the generalized impedance boundary condition model of order 1 minimizing

$$F(\epsilon, \delta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \| u_1^\infty(\epsilon, \delta, \Gamma, \hat{\theta}_j) - u_{\delta,\mathrm{obs}}^\infty(\cdot, \hat{\theta}_j) \|_{L^2(S_j)}^2$$

with $\mu = 0.1$ known.

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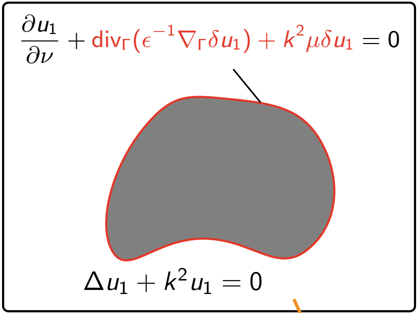

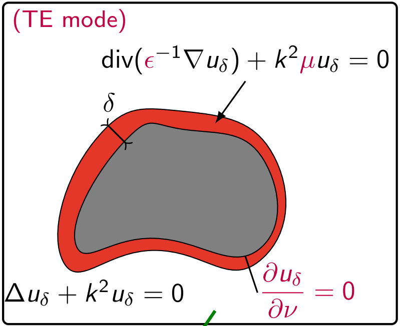

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with $\mu = 0.1$ known.

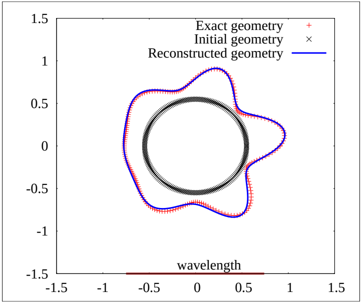
Application to the reconstruction of a coated obstacle

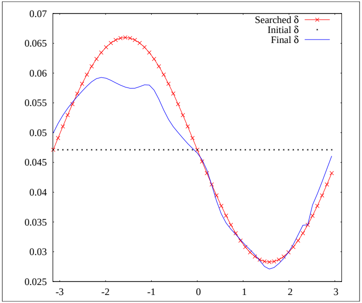
Numerical results

Synthetic data created with

- $\mu = 0.1$ is known,
- $\delta = 0.04l(1 - 0.4\sin(\theta))$ is unknown; l being the wavelength,
- $\epsilon = 2.5$ is unknown.

Reconstructed ϵ: 2.3.





Fails with a classical impedance boundary condition model!

Extension to Maxwell equations

Prototype GIBC model:

Find $(\mathbf{E}^s, \mathbf{H}^s) \in H_{\text{loc}}(\mathbf{rot}, \Omega_{\text{ext}}) \times H_{\text{loc}}(\mathbf{rot}, \Omega_{\text{ext}})$ such that

$$(\mathcal{P}_{Max}) \quad \begin{cases} \mathbf{rot}\mathbf{H}^s + ik\mathbf{E}^s = 0 & \text{in } \Omega_{\text{ext}}, \\ \mathbf{rot}\mathbf{E}^s - ik\mathbf{H}^s = 0 & \text{in } \Omega_{\text{ext}}, \\ \nu \times \mathbf{E}^s + \mathbf{Z}\mathbf{H}^s_T = -(\nu \times \mathbf{E}^i + \mathbf{Z}\mathbf{H}^i_T) & \text{on } \Gamma, \\ \lim\limits_{R\to\infty} \displaystyle\int_{\partial B_R} |\mathbf{H}^s \times \hat{x} - (\hat{x} \times \mathbf{E}^s) \times \hat{x}|^2 ds = 0 \end{cases} ;$$

$$\boxed{\mathbf{Z}\mathbf{H}_T := \mathbf{rot}_\Gamma(\eta \operatorname{rot}_\Gamma \mathbf{H}_T) + \lambda \mathbf{H}_T}$$

$$\mathbf{H}_T := (\nu \times \mathbf{H}) \times \nu$$

$$\operatorname{rot}_\Gamma = \nu \cdot \mathbf{rot}$$

$$\mathbf{rot}_\Gamma = -\nu \times \nabla_\Gamma$$

Extension to Maxwell equations

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$$\boxed{\mathbf{Z}\mathbf{H}_T := \mathbf{rot}_\Gamma(\eta \,\mathrm{rot}_\Gamma \mathbf{H}_T) + \lambda \mathbf{H}_T}$$

Thm: Assume that

$$\Re e(\lambda) \geq 0 \ , \quad \Re e(\eta) \geq 0,$$

$$\Im m(\lambda) < 0 \ , \quad \Im m(\eta) < 0.$$

Then $(\mathcal{P}_{Max})$ has a unique solution.

The inverse problem

For incident plane waves

$$\boxed{\begin{aligned}&\mathbf{E}^i(\cdot,\hat{\theta},\mathbf{p}) = ik[(\hat{\theta}\times\mathbf{p})\times\hat{\theta}]e^{ik\hat{\theta}\cdot z}\\ &\mathbf{H}^i(\cdot,\hat{\theta},\mathbf{p})) = ik(\hat{\theta}\times\mathbf{p})e^{ik\hat{\theta}\cdot z}\end{aligned}}$$

we define the corresponding far-field

$$\mathbf{E}^\infty(\hat{x},\hat{\theta},\mathbf{p}) \in L^2(S^2,S^2,S^2).$$

and we minimize

$$\boxed{F(\Gamma) := \frac{1}{2}\sum_{i=1}^{I}\left\|\mathbf{E}^\infty_{\lambda,\eta,\Gamma}(\cdot,\hat{\theta}_i,\mathbf{p}_i) - \mathbf{E}^\infty_\delta(\cdot,\hat{\theta}_i,\mathbf{p}_i)\right\|^2_{\mathbf{L}^2_t(S^2)}}$$

for noisy data $\mathbf{E}^\infty_\delta(\cdot,\hat{\theta}_i,\mathbf{p}_i)$.

Expression of the shape derivative for Maxwell

$$\widetilde{R} : \varepsilon \longrightarrow \mathbf{E}^s(\lambda_\varepsilon, \eta_\varepsilon, \Gamma_\varepsilon).$$

$$\widetilde{R}'(0) \cdot \varepsilon = \mathbf{v}^s_\varepsilon$$

Where $(\mathbf{v}^s_\varepsilon, \mathbf{w}^s_\varepsilon)$ is an outgoing solution to the Maxwell equations outside Ω and

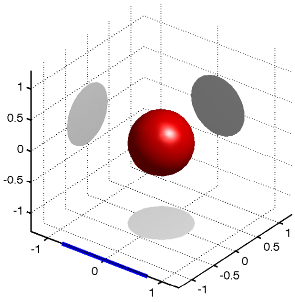
$$\nu \times \mathbf{v}^s_\varepsilon + \mathbf{Z}\mathbf{w}^s_{T,\varepsilon} = B_\varepsilon(\mathbf{E}, \mathbf{H}) \quad \text{on } \Gamma$$

$$\begin{aligned} B_\varepsilon(\mathbf{E}, \mathbf{H}) := & -ik(\nu \cdot \varepsilon)\mathbf{H}_T + \mathbf{rot}_\Gamma[(\nu \cdot \varepsilon)(\nu \cdot \mathbf{E})] + \lambda(\nu \cdot \varepsilon)\left(2R - 2H\right)\mathbf{H}_T \\ & - \lambda\nabla_\Gamma[(\nu \cdot \varepsilon)(\nu \cdot \mathbf{H})] + 2\mathbf{rot}_\Gamma[H(\nu \cdot \varepsilon)\eta\mathrm{rot}_\Gamma(\mathbf{H}_T)] \\ & + (\nabla_\Gamma\lambda \cdot \varepsilon)\mathbf{H}_T + \mathbf{rot}_\Gamma[(\nabla_\Gamma\eta \cdot \varepsilon)\mathrm{rot}_\Gamma(\mathbf{H}_T) \\ & + ik\mathbf{Z}[(\nu \cdot \varepsilon)\mathbf{Z}\mathbf{H}_T]. \end{aligned}$$

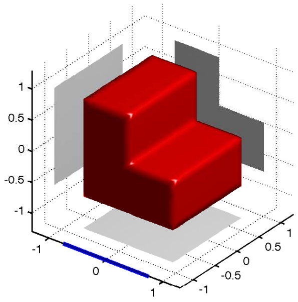
compatible with the scalar case and the Maxwell case $\eta = 0$ and λ constant H.-Kress (2004).

Numerical results

- $\lambda = 0$, $\eta = -0.25i$, $k = 4$, $\delta = 2\%$
- 4 incident plane waves



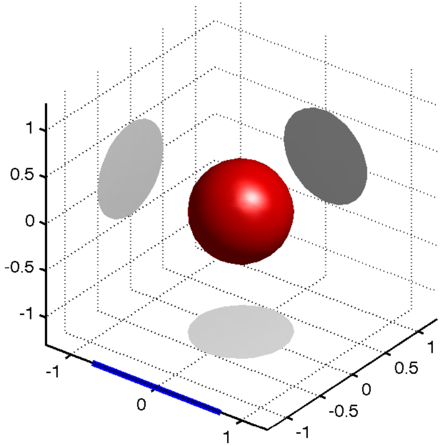
(a) Initial shape



(b) Target

Numerical results

- $\lambda = 0$, $\eta = -0.25i$, $k = 4$, $\delta = 2\%$
- 4 incident plane waves



Conclusions and perspectives

- Broader application of the Factorization method for GIBC
- Steepest descent methods are capable of providing accurate reconstructions.
- Possibility of identifying coated obstacles.

- Enlarge the applicability of the Factorization method... by using different Factorizations (inspired by the dielectric cases)
- Factorization method for Maxwell's equations.
- Propose a (steepest) decent method valid for a general symmetric operator on the boundary.
- Extension to other models (elasticity fro example).

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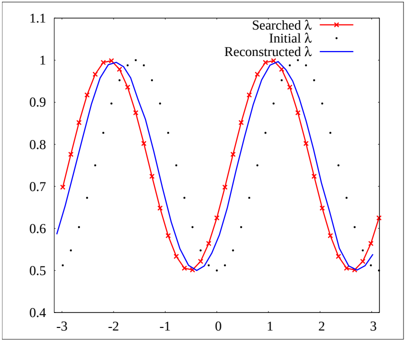
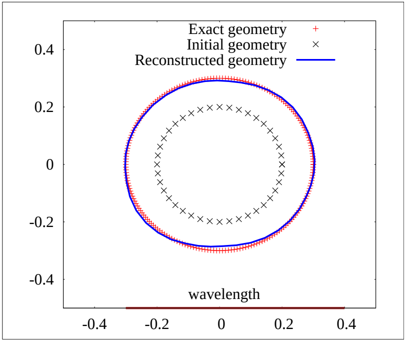
Thank you, and special thanks to Nicolas who helped me in preparing these slides.

Numerical reconstruction

Simultaneous reconstruction of λ, Γ with $\eta = 0$



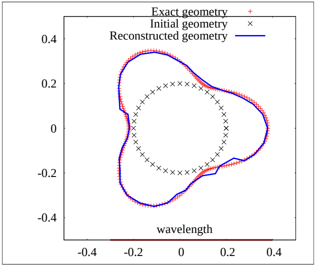


8 incident waves, 5% of noise on far–field data.
We iterate only on the geometry.

$$B_\varepsilon u = (\nabla_\Gamma \lambda \cdot \varepsilon) u + \cdots$$

Numerical reconstruction

Simultaneous reconstruction of λ, η and Γ



8 incident waves, 5% *of noise on far–field data.*