Analytic and iterative reconstructions in SPECT

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Sommaire

Abstract

SPECT

Problem

Novikov formula

Chang formula

OAR

Numerical examples

References

Abstract

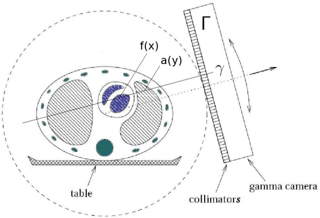
We consider analytic and iterative reconstructions in the single-photon emission computed tomography (SPECT).

As analytic techniques we use, in particular, Chang's approximate inversion formula and Novikov's exact inversion formula for the attenuated ray transform, on one hand, and Wiener type filters for data with strong Poisson noise, on other hand.

As iterative techniques we consider the least square and expectation maximization iterative reconstructions.

Different comparaisons are given.

SPECT (Single Photon Emission Computed Tomography)



In the single-photon emission computed tomography (SPECT) one considers a body containing radioactive isotopes emitting photons. The emission data p in SPECT consist in the radiation measured outside the body by a family of detectors during some fixed time. The basic problem of SPECT consists in finding the distribution f of these isotopes in the body from the emission data p and some a priori information concerning the body. Usually this a priori information consists in the photon attenuation coefficient a in the points of body, where this coefficient is found in advance by the methods of the transmission computed tomography.

$f(x)$ - density of radioactive isotopes
$a(x)$ - photon attenuation coefficient
x - point of (the space containing the) body
$p(\gamma)$ - emission data
γ - point of detector set Γ
Γ - discrete subset of the set T of all oriented straight lines in the space containing the body

More precisely, $p(\gamma)$ is the number of photons coming from (the domain containing) the body along oriented straight line γ to the detector associated with γ.

In some approximation the emission data p are modeled as follows :

$$\begin{array}{l} p(\gamma) \ \text{ is a realization of a Poisson variate } \ \mathbf{p}(\gamma) \\ \text{with the mean } \ M\mathbf{p}(\gamma) = g(\gamma) = CP_a f(\gamma) \\ \text{for any } \gamma \in \Gamma \text{ and all } \mathbf{p}(\gamma),\ \gamma \in \Gamma,\ \text{are independent,} \end{array} \tag{1}$$

where

$$P_a f(\gamma) = \int\limits_{\gamma} \exp[-\mathcal{D}a(x,\hat{\gamma})] f(x) dx, \tag{2}$$

where $\hat{\gamma}$ is the direction of γ,

$$\mathcal{D}a(x,\theta) = \int\limits_{0}^{+\infty} a(x+t\theta) dt,\ x \in \mathbb{R}^d,\ \theta \in \mathbb{S}^{d-1}, \tag{3}$$

$C = C_1 t$, where t is the detection time.
$P_a f$ - attenuated ray transform of f.
The SPECT problem $p \to Cf$ can be restricted to each fixed 2D plane Ξ intersecting the body and identified with $\mathbb{R}^2$.

We remind that $T \approx \mathbb{R} \times \mathbb{S}^1$, where T is the set of all oriented straight lines in $\mathbb{R}^2$. If $\gamma = (s, \theta) \in \mathbb{R} \times \mathbb{S}^1$, then $\gamma = \{x \in \mathbb{R}^2 : \ x = s\theta + t\theta^\perp, \ t \in \mathbb{R}\}$ (modulo orientation) and θ gives the orientation of γ, where $\theta^\perp = (-\theta_2, \theta_1)$ for $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1$.

In practice, $d = 2$ (after restriction to 2D plane),
$a \geq 0, \ f \geq 0, \ supp\, a \subset \mathcal{B}_R, \ supp\, f \subset \mathcal{B}_R$,
$\mathcal{B}_R = \{x \in \mathbb{R}^2 : \ |x| \leq R\}$,
R- radius of image support, Γ - is a uniform $n \times n$ sampling of

$$T_R = \{\gamma \in T : \ \gamma \cap B_R \neq \emptyset\} = \{(s, \theta) \in \mathbb{R} \times \mathbb{S}^1 : \ |s| \leq R\}.$$

In addition, the standard value for n is 128.

Problem 1. Find (as well as possible) Cf from p and a.

Our analytic approach to Problem 1 is based on the scheme

$$Cf \approx P_a^{-1}\mathcal{W}p, \tag{4}$$

where $\mathcal{W}$ is a space-variant Wiener-type filter developed in [GN 2008] (or, more generally, some analytic method for approximate finding the noiseless data g of (1) from p), P_a^{-1} is a reconstruction from $\tilde{p} = \mathcal{W}p$, based on some optimal combination of the Novikov exact (see [N 2002]) and Chang approximate (see [Ch 1978], [N 2011]) inversion formulas for P_a. In addition, the aforementioned optimal combination is constructed via a Morozov-type discrepancy principle.

Novikov formula (see [N 2002])

$$Cf = \mathcal{N}_a g, \quad \text{where} \tag{5}$$

$$\begin{aligned}
\mathcal{N}_a g(x) &= \frac{1}{4\pi}\int_{\mathbb{S}^1} \theta^{\perp}\nabla_x K(x,\theta)d\theta, \\
K(x,\theta) &= \exp[-\mathcal{D}a(x,-\theta)]\,\tilde{g}(x\theta^{\perp}), \\
\tilde{g}(s) &= \exp\left[A_\theta(s)\right]\cos\left(B_\theta(s)\right)H(\exp\left[A_\theta\right]\cos\left(B_\theta\right)g_\theta)(s) + \\
&\quad \exp\left[A_\theta(s)\right]\sin\left(B_\theta(s)\right)H(\exp\left(A_\theta\right)\sin\left(B_\theta\right)g_\theta)(s),
\end{aligned}$$

$A_\theta(s) = \frac{1}{2}P_0 a(s,\theta)$, $\quad B_\theta(s) = H\,A_\theta(s)$, $\quad g_\theta(s) = g(s,\theta)$,
$g = CP_a f$ (see (1) and (2)),

$$Hu(s) = \frac{1}{\pi}p.v.\int_{\mathbb{R}} \frac{u(t)}{s-t}dt \quad \text{(Hilbert transform)} \tag{6}$$

$x = (x_1, x_2) \in \mathbb{R}^2$, $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1$, $\theta^{\perp} = (-\theta_2, \theta_1)$, $s \in \mathbb{R}$, $\mathcal{D}a$ is defined by (3).
Formula (5) is exact (for continuous case) but is not very stable for reconstruction from discrete and noisy data p of (1).

Chang formula (see [Ch 1978], [N 2011])

$$Cf \simeq \mathcal{C}h_a g, \quad \text{where} \tag{7}$$

$$\mathcal{C}h_a g(x) = \frac{1}{4\pi w_0(x)} \int_{\mathbb{S}^1} \theta^\perp \nabla_x H g_\theta(x\theta^\perp) d\theta,$$

$$w_0(x) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \exp[-\mathcal{D}a(x,\theta)] d\theta$$

$g = CP_a f$ (see (1) and (2)), $\quad g_\theta(s) = g(s,\theta)$,

where H is defined by (6), $\mathcal{D}a$ is defined by (3), $x \in \mathbb{R}^2$, $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1, \theta^\perp = (-\theta_2, \theta_1), s \in \mathbb{R}$.

Formula (7) is approximate (for continuous case) but its result is sufficiently stable for reconstruction from discrete and noisy data p of (1).

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Optimized analytic reconstruction (OAR)

$$Cf \simeq Cf_\alpha = \mathcal{N}_{a_\alpha}(\mathcal{W}p)_\alpha + \mathcal{C}h_a(\mathcal{W}p - (\mathcal{W}p)_\alpha), \quad \text{where} \qquad (8)$$

α is optimization parameter, $\mathcal{W}$ is space-variant Wiener-type filter of [GN 2008],
$(\mathcal{W}p)_\alpha$ and a_α are the low-frequency parts of $\mathcal{W}p$ and a (respectively) obtained via some standard $2D$ space-invariant filtering dependent on α.
$\mathcal{N}_a$ and $\mathcal{C}h_a$ are the inversion operators of formulas (5) and (7).
In addition, we choose α as a parameter minimizing the discrepancy $||P_a Cf - \mathcal{W}p||_{L^2(\Gamma)}$.
The ansatz Cf_α of (8) is motivated by the facts that $\mathcal{N}_a p$ of the exact formula (5) is sufficiently stable on sufficiently low frequency part of p and a, whereas $\mathcal{C}h_a p$ of the approximate formula (7) is sufficiently stable on reasonably high frequency part of p and a.

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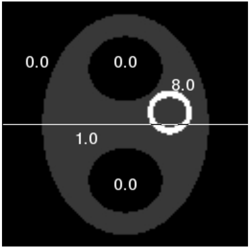
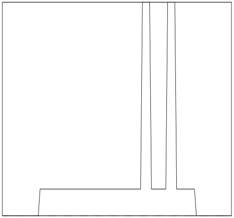
Numerical example 1

Attenuation map a, (128×128)

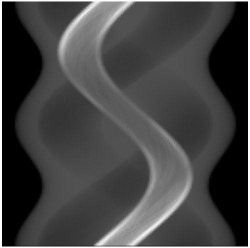


Emission activity f, (128×128)

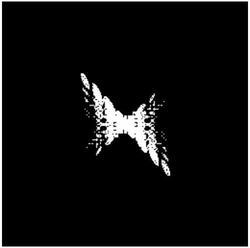


Emission profile

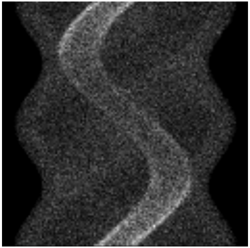
Noiseless and noisy projections



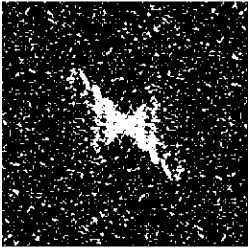
Projections g (128×128)



Spectrum $|\hat{g}|$

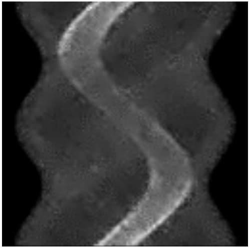


Projections $\|p - g\|_2 / \|g\|_2 = 30\%$

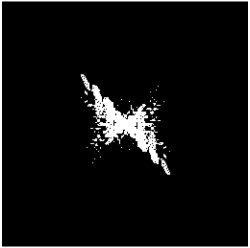


Spectrum $|\hat{p}|$

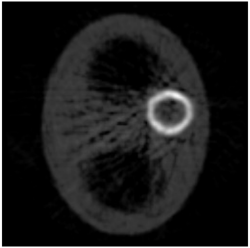
Space-variant Wiener filter $A^{sym}_{8,8}$ and O.A. reconstruction



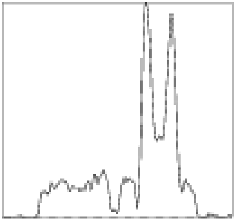
Projections $\|\tilde{p} - g\|_2 / \|g\|_2 = 11\%$



Spectrum

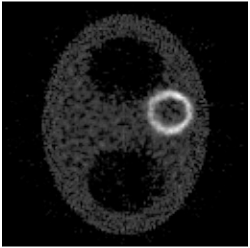


OAR : $\|r - r_0\|_2 / \|r_0\|_2 = 36\%$

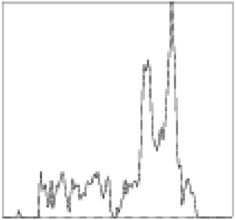


Profile

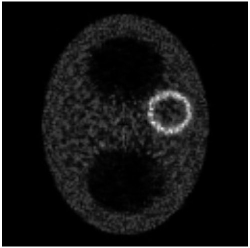
Iterative reconstructions (60 It)



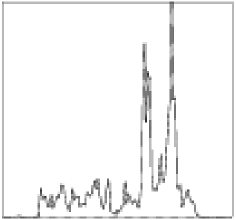
Gradient : $\|r - r_0\|_2 / \|r_0\|_2 = 43\%$



Profile

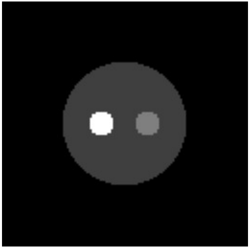


Em : $\|r - r_0\|_2 / \|r_0\|_2 = 42\%$

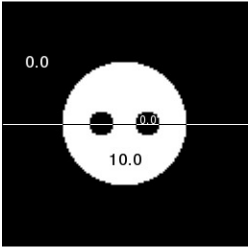


Profile

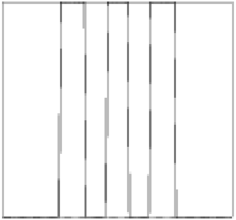
Numerical example 2



Attenuation map a, (128×128)

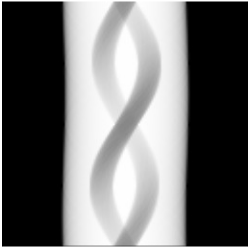


Emission activity f, (128×128)

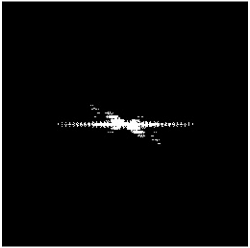


Emission profile

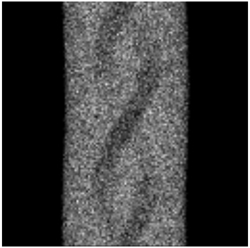
Noiseless and noisy projections



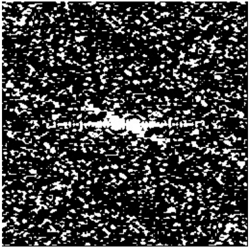
Projections g (128×128)



Spectrum $|\hat{g}|$

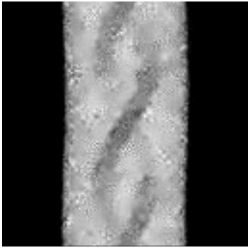


Projections $\|p - g\|_2 / \|g\|_2 = 30\%$

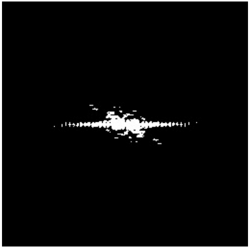


Spectrum $|\hat{p}|$

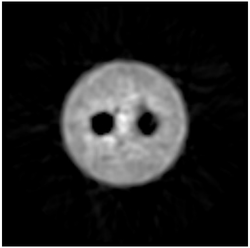
Space-variant Wiener filter $A^{sym}_{8,8}$ and O.A. reconstruction



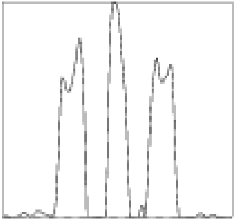
Projections $\|\tilde{p} - g\|_2 / \|g\|_2 = 10\%$



Spectrum

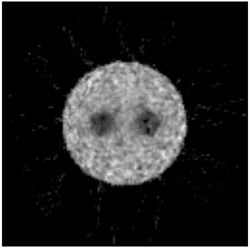


OAR : $\|r - r_0\|_2 / \|r_0\|_2 = 22\%$

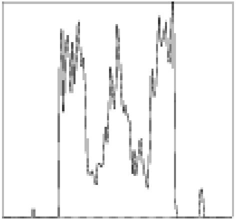


Profile

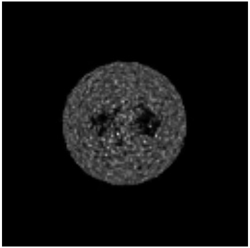
Iterative reconstructions (60 it)



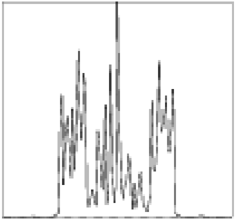
Gradient : $\|r - r_0\|_2 / \|r_0\|_2 = 24\%$



Profile



Em : $\|r - r_0\|_2 / \|r_0\|_2 = 52\%$



Profile

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