Existence/Uniqueness for a crystalline mean curvature flow

Antonin Chambolle

CMAP, Ecole Polytechnique, CNRS, Palaiseau, France

joint work with **M. Morini** (Parma), **M. Ponsiglione** (Roma I), **M. Novaga** (Pisa)

Introduction

- Crystalline curvature motion: definition, known results;
- A new definition of sub and supersolutions;
- Comparison;
- Construction of a solution.
- An example.

Anisotropic and Crystalline mean curvature flow

Given a set $E \subset \mathbb{R}^N$ and a *surface tension* φ, which is a convex, one-homogeneous and even norm, we can define a perimeter

$$\mathrm{Per}_\varphi(E) := \int_{\partial E} \varphi(\nu_E) d\mathcal{H}^{N-1}$$

(which can be appropriately extended to non necessarily smooth set) and an anisotropic curvature (first variation)

$$\kappa_\varphi = \mathrm{div}_\tau \nabla\varphi(\nu).$$

The *anisotropic* mean curvature motion is a family of sets $E(t)$ whose boundary evolves with a speed proportional to κ_φ (or a nondecreasing function of κ_φ).

Anisotropic and Crystalline mean curvature flow

Such evolutions are well defined

- if φ is smooth enough (Almgren-Taylor-Wang, 1993);
- in the level set sense: Chen-Giga-Goto (1991) show existence for

$$u_t + H(Du, D^2 u) = 0$$

  in the viscosity sense, where H is "geometric" (meaning that the motion of a level set does not depend on the other level sets);

What if φ is not smooth?

Crystalline curvature flow

The crystalline case is the case where $\{\varphi \leq 1\}$ is a polytope. In this case, $\nabla\varphi(\nu)$ is *a priori* not well defined. We should use a selection of the subgradient $\partial\varphi(\nu)$: "$\kappa_\varphi \in \mathrm{div}_\tau \partial\varphi(\nu)$".

- loss of ellipticity / solutions are not expected to be regular in the classical sense;
- infinite diffusion / the motion should be non-local.

Crystalline curvature flow

Let us introduce the polar $\varphi^\circ(\xi) := \sup\{\xi \cdot \eta : \varphi(\eta) \leq 1\}$, then $W = \{\varphi^\circ \leq 1\}$ is called the **Wulff shape**, for φ smooth it is a constant curvature compact set. In the crystalline case it is also a polytope.

Then, one knows how to define a motion

- in 2D if the initial set is a polygon with facets parallel to the faces of W (system of ODEs, cf Almgren-Taylor 95, Giga-Gurtin 96...);
- in 2D for a short time if the initial set has a "interior/exterior Wulff shape conditions" (C.-Novaga, 2012/15);
- in any dimension if the initial set is convex (Bellettini-Caselles-C-Novaga, 2006);

Crystalline curvature flow

- in 2D in the "viscosity sense" adapted to crystalline motions (Giga-Giga, 2001)
- in 3D, viscosity: Giga-Požar 2014 (preprint appeared on arxiv in Jan. 2016). Maybe ND?

An important advantage of the viscosity approach is that it solves **any** equation of the form $V = -F(\nu, \kappa_\varphi)$ (with F nondecreasing wr κ). A drawback is that the anisotropy must be **purely crystalline**.

This presentation

With Massimiliano Morini (Parma) and Marcello Ponsiglione (Roma I): motion with "*natural*" mobility, in **any** dimension, **any** anisotropy. That is

$$V_N = -\varphi(\nu)\kappa_\varphi$$

or

$$V_\varphi = -\kappa_\varphi$$

where V_φ is the velocity along a *Cahn-Hoffmann* vector field $\partial\varphi(\nu)$.

Cf. [Bellettini-Paolini, 96]

Cahn Hoffmann vector field

We define the φ°-*signed distance function*:

$$d_E(x) = \mathrm{dist}\,(x, E) - \mathrm{dist}\,(x, E^c) = \min_{y \in E} \varphi^\circ(x - y) - \min_{y \notin E} \varphi^\circ(y - x)$$

Then one has $\varphi(\nabla d_E) = 1$ a.e., and we define a Cahn-Hoffmann vector field as a field $n_\varphi \in \partial\varphi(\nabla d_E)$. In particular, $n_\varphi \cdot \nabla d_E = 1$ a.e. (thanks to Euler's identity).

A curvature can be (non uniquely) defined as

$$\kappa_\varphi = \mathrm{div}\, n_\varphi.$$

Super/subsolutions

We consider an initial set E^0.
Definition A (closed) "tube" $E \subseteq \mathbb{R}^N \times [0, +\infty)$ is a **supersolution** starting from E^0 if

a. $E(0) \subseteq \overline{E^0}$;
b. $E(t) = \emptyset \Rightarrow E(s) = \emptyset$ if $s > t$;
c. E is (Kuratowski) left-continuous;
d. For $d = \mathrm{dist}^{\varphi^\circ}(x, E)$, there exists $z \in \partial\varphi(\nabla d)$ with

$$\partial_t d \geq \operatorname{div} z$$

in the **distributional sense** in $\mathbb{R}^N \times (0, T^*) \setminus E$ where T^* is the extinction time of E, moreover (for $t \leq T^*$)

$$(\operatorname{div} z)^+ \in L^\infty(\{d > \delta\})$$

for any $\delta > 0$

A **subsolution** is an open tube A such that A^c is a supersolution starting from $(E^0)^c$.

Remarks

- A kind of mixture of Barles-Soner-Souganidis (93), Ambrosio-Soner (96), however in the distributional sense (a *regression*!). Equivalent if φ, φ° C^2.
- d is a supersolution of the φ-total variation flow in E^c;
- we build a motion with $\operatorname{div} z \leq (N-1)/d$;
- if E is a supersolution and its interior a subsolution, then if $|\partial E| = 0$ one expects that $\partial_t d = \operatorname{div} z$ on ∂E, which means that $V_\varphi = -\kappa_\varphi$ and it is a *solution*;
- we will prove existence and uniqueness of such solutions, up to "fattening".

Comparison

Theorem Let E be a supersolution with initial datum E^0 and F be a subsolution with initial datum $F^0 \supset E^0$, and assume $\Delta = \mathrm{dist}^{\varphi^\circ}(E^0, (F^0)^c) > 0$. Then $\mathrm{dist}\,(E(t), F(t)^c) \geq \Delta$ for any $t \geq 0$.

The proof is by parabolic comparison. Indeed, if $d(x,t) = \mathrm{dist}\,(x, E(t))$ and $d'(x,t) = \mathrm{dist}\,(x, F^c(t))$ then between E and F

$$\partial_t d \geq \mathrm{div}\,\partial\varphi^\circ(\nabla d), \quad \partial_t d' \geq \mathrm{div}\,\partial\varphi^\circ(\nabla d'),$$

and one has $d + d' \geq \Delta$ at $t = 0$. Using a priori estimate on the speed at which d, d' decrease, one can control also $d + d'$ on a parabolic boundary of a small tube, and obtain the comparison inside this tube. As d, d' are distance function it yields global comparison.

Existence

Basic idea: Almgren-Taylor-Wang 93 (Taylor-Wang 95), Luckhaus-Sturzenhecker 95. We pick a time step $h > 0$, and for E_0 (temporarily a compact set), we define E^{n+1} from E^n, $n \geq 0$ by solving

$$\min_E P(E) \; + \; \frac{1}{h} \int_E d_{E^n}$$

Then the Euler-Lagrange equation is

$$d_{E^n} = -h\kappa_\varphi(E^{n+1})$$

hence it is an implicit discretization of the flow.

More precisely

It is possible to show that this problem can be solved equivalently by solving

$$\min_u \int \varphi(Du) + \int \frac{(u - d_{E^n})^2}{2h} dx$$

and letting then $E^{n+1} = \{u \leq 0\}$. If E^0 is not compact, one can consider the Euler-Lagrange equation of this problem in $\mathbb{R}^N$. It solves

$$\begin{aligned} - h \operatorname{div} z + u = d_{E^n}, \\ z \in \partial\varphi(\nabla u) \quad \text{a.e.} \end{aligned}$$

Then one lets $E^{n+1} = \{u \leq 0\}$.

The limit is a solution

We then consider $E_h(t) = E^{[t/h]}$ and consider a subsequence such that $E_{h_i} \to E$ and $(E_{h_i})^c \to A^c$ in the Kuratowski sense, with therefore $A \subset E$. One can show then:

Theorem E is a supersolution and A a subsolution. In particular if $\partial E = \partial A$, they are a solution.

Why does it work?

- The advantage of the scheme using "u" is that at each time, one has not only E (and d_E, u) but also a candidate field z.
- One can easily show that u is Lipschitz and $\varphi(\nabla u) \leq \varphi(\nabla d_{E^n}) = 1$ a.e.: it follows that $u \leq d_{E^{n+1}}$ in $\{u > 0\}$ and $u \geq d_{E^{n+1}}$ in $\{u < 0\}$.
- As a consequence, in $\{u > 0\}$,

$$\frac{\partial d_E}{\partial t} \approx \frac{d_{E^{n+1}} - d_{E^n}}{h} \geq \operatorname{div} z\,.$$

- It will be easy to pass to the limit in this equation in the distributional sense. The difficult part is to show that d and z converge to what we expect.

An estimate for d

Lemma Assume $d_{E_h(t)}(x) = R > 0$. Then $d_{E_h(t+s)}(x) \gtrapprox \sqrt{R^2 - cs}$ for $s \geq 0$.

This is proved by comparison with an explicit construction for the Wulff shape. It shows that d_{E_h} does not decrease too fast ($d_{E_h}^2 + ct$ is nondecreasing). It allows to use a Helly-like argument to show convergence for all time t of a subsequence.

The limit of d_{E_h}

More precisely, we can build a set $N \subset [0, +\infty)$ which is at most countable and such that if $t \notin N$

$$d^+_{E_{h_i}(s)} \to \operatorname{dist}(x, E(s)), \qquad d^-_{E_{h_i}(s)} \to \operatorname{dist}(x, A(s)^c)$$

locally uniformly. We can show similar convergence for the functions u.

It is elementary to deduce

$$\frac{\partial d_E}{\partial t} \geq \operatorname{div} z$$

in the sense of distributions, in E^c, where z is a weak-$*$ limit of (z_{h_i}) (appropriately defined).

The limit for z_h

We have in E^c

$$\frac{\partial d_E}{\partial t} \geq \operatorname{div} z$$

so what remains to show is that actually, $z \in \partial\varphi(\nabla d)$ a.e., or, equivalently, as $z \in \partial\varphi(0)$ a.e.,

$$1 = \varphi(\nabla d) \leq z \cdot \nabla d \quad \text{a.e.}$$

This is shown using an upper estimate for $\operatorname{div} z_h$, which follows from comparison results for the variational problem defining (u_h, z_h).

An estimate for z_h

One can show from the equation defining z_h that if $d_{E_h(t)}(x) > R$ then

$$\operatorname{div} z_h(x, t+h) \leq \frac{N-1}{R}$$

(this is obtained by comparison of u, with the explicit solutions obtained by replacing d_{E^n} with $\varphi^\circ(\cdot - y)$, $y \in E^n$, in the equation defining z, u).

• Here the fact that d is computed from φ° is crucial.
• We can extend to other distances, however with a compatibility condition.

The limit for z_h

$$1 = \varphi(\nabla d) \overset{?}{\leq} z \cdot \nabla d \quad \text{a.e.}$$

Given $\eta(x,t)$ a nonnegative test function with support in E^c, one has on one hand (using $z_h \in \partial\varphi(\nabla u_h)$ and $u_{h_i} \to d$)

$$\int\int \varphi(\nabla d)\eta\, dxdt \leq \liminf_i \int\int \varphi(\nabla u_{h_i})\eta\, dxdt = \liminf_i \int\int (z_{h_i} \cdot \nabla u_{h_i})\eta\, dxdt$$

and on the other hand (using $z_{h_i} \overset{*}{\rightharpoonup} z$)

$$\int\int (z_{h_i} \cdot \nabla u_{h_i})\eta\, dxdt = \int\int (z_{h_i} \cdot \nabla d)\eta\, dxdt + \int\int z_{h_i} \cdot \nabla(u_{h_i} - d)\eta\, dxdt$$
$$\to \int\int (z \cdot \nabla d)\eta\, dxdt + \lim_i \int\int z_{h_i} \cdot \nabla(u_{h_i} - d)\eta\, dxdt$$

so one just needs to show

$$\lim_i \int\int z_{h_i} \cdot \nabla(u_{h_i} - d)\eta\, dxdt = 0$$

The limit for z_h

Thanks to the divergence estimate it is easy to show

$$\lim_i \int\int z_{h_i} \cdot \nabla(u_{h_i} - d)\eta \, dxdt = 0$$

Indeed if $m_i(t) := \sup_{\mathrm{spt}\eta}(u_{h_i} - d)$ ($\to 0$ as $i \to \infty$), then

$$\begin{aligned}
\int\int z_{h_i} \cdot \nabla(u_{h_i} - d)\eta \, dxdt &= \int\int z_{h_i} \cdot \nabla(u_{h_i} - d - m_i)\eta \, dxdt \\
&= -\int\int (u_{h_i} - d - m_i)(z_{h_i} \cdot \nabla\eta + \eta \mathrm{div}\, z_{h_i}) \, dxdt. \\
&\geq o(1) - \frac{N-1}{\delta} \int\int (u_{h_i} - d - m_i)\eta \, dxdt \xrightarrow{i\to\infty} 0.
\end{aligned}$$

as $i \to \infty$, where δ is a lower bound for $\mathrm{dist}\,(\mathrm{spt}\eta, E)$. The reverse inequality is obtained in the same way.

Conclusion

- It is classical that such results lead to existence and uniqueness in the level-set setting, or uniqueness as long as the solution does not develop "fattening";
- Hence "generic" uniqueness;
- As usual, no fattening if initial set is star-shaped, or if it is a graph (hence uniqueness in these cases).

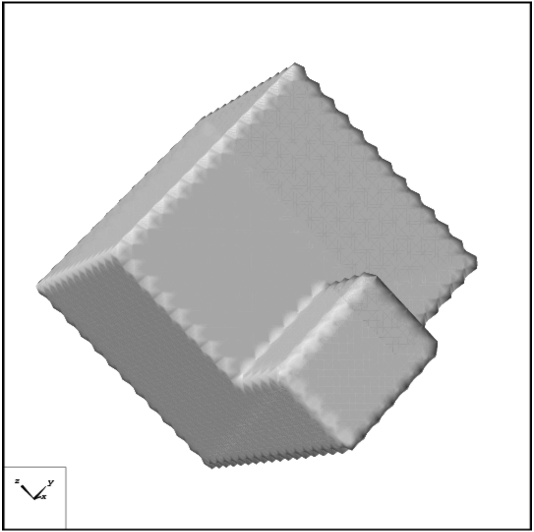
Extensions and Perspectives

- Forcing term $g(t,x)n_\varphi$ (Lipschitz in space/continuous);
- "Mobilities" ($V = \psi(\nu)(\kappa_\varphi + g(t,x))$) with some restriction *(compatibility condition between ψ and g, although this can be partially released)*;
- Possible perspective: nonlinearity in the speed: difficult with the distributional formulation... Viscosity approach of Giga-Pozaar is much better for this...

Example of a facet breaking

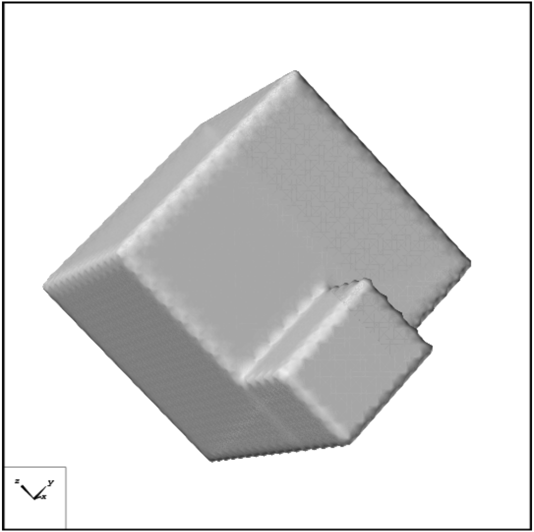
The following example is due to Bellettini, Novaga, Paolini (99). (The iterative variational problem can be solved numerically—with a parametric maximal flow approach.)

Facet breaking



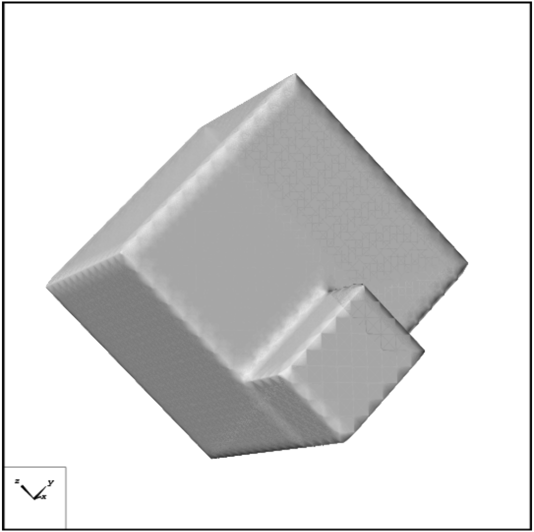
$t = 0$

Facet breaking



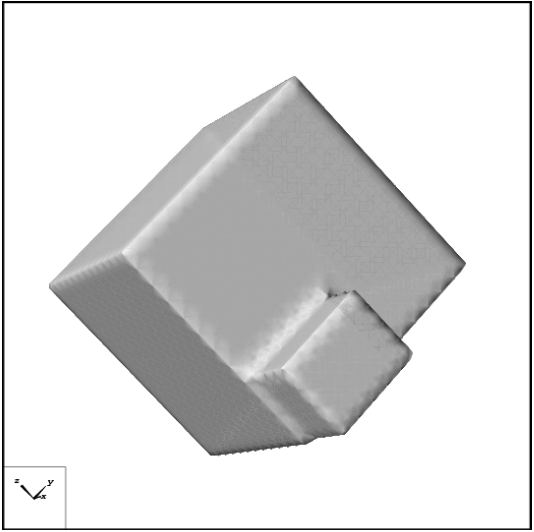
$t = 1$

Facet breaking



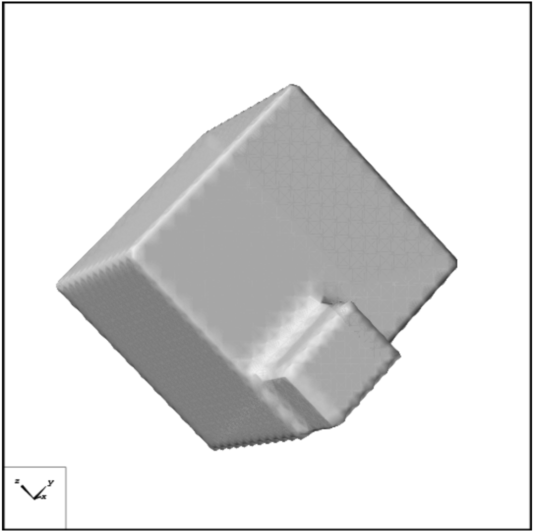
$t = 2$

Facet breaking



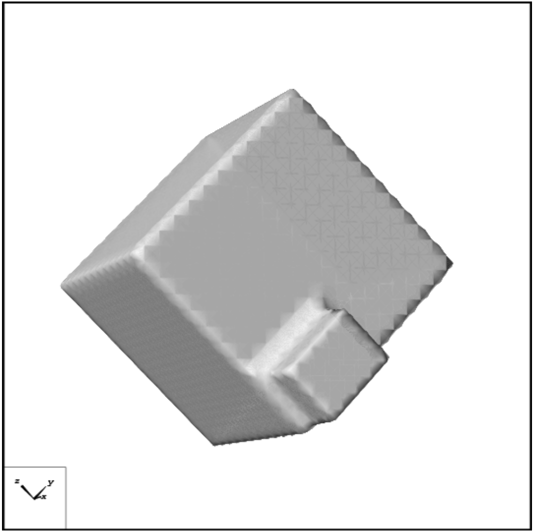
$t = 3$

Facet breaking



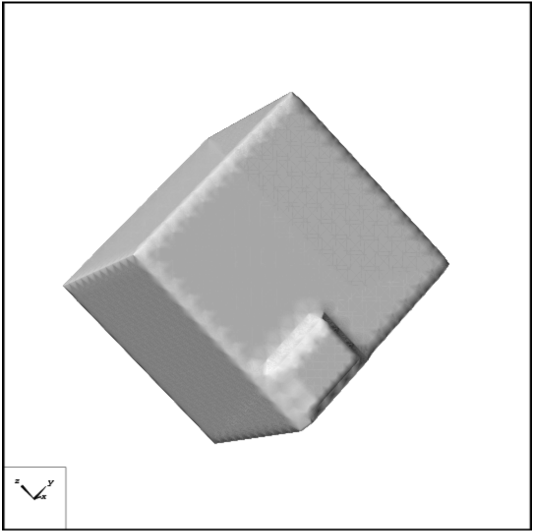
$t = 4$

Facet breaking



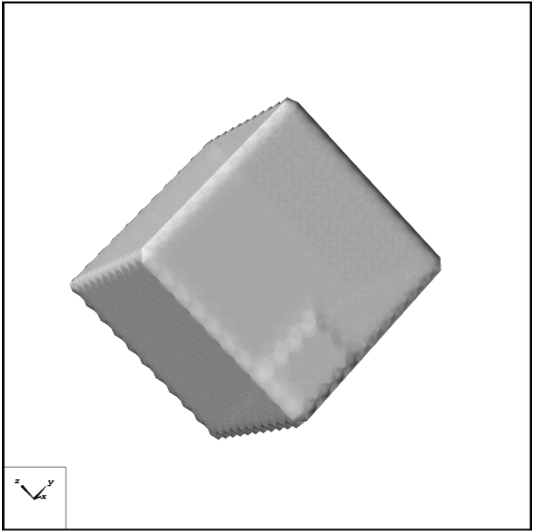
$t = 5$

Facet breaking



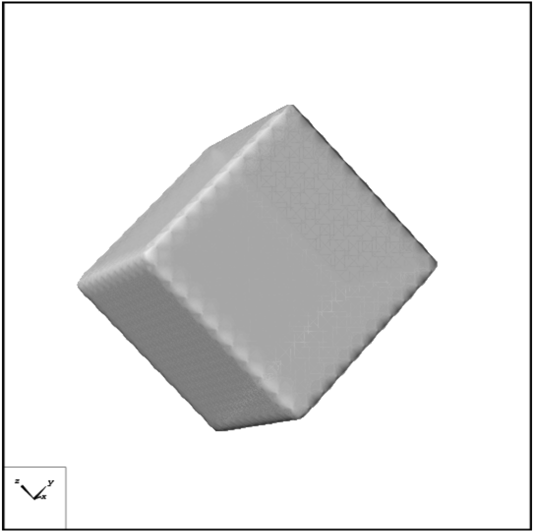
$t = 6$

Facet breaking



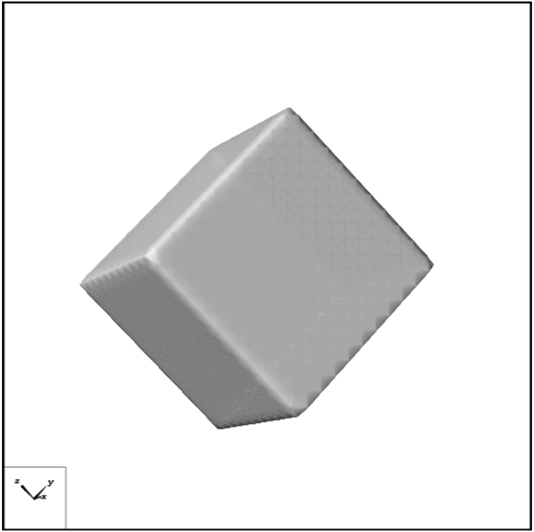
$t = 7$

Facet breaking



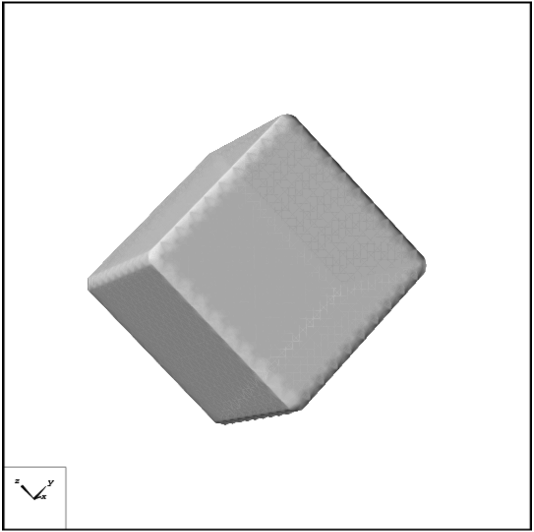
$t = 8$

Facet breaking



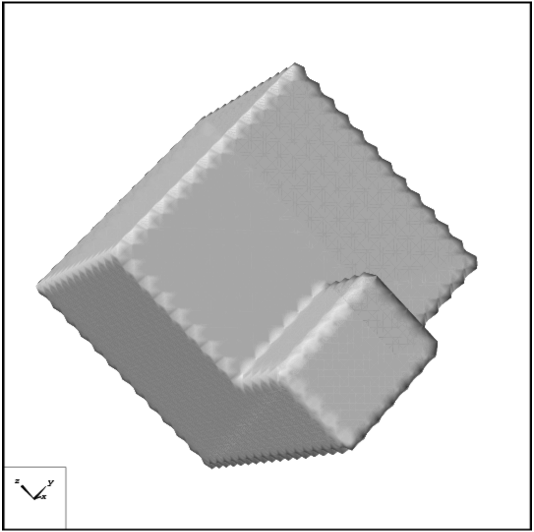
$t = 9$

Facet breaking



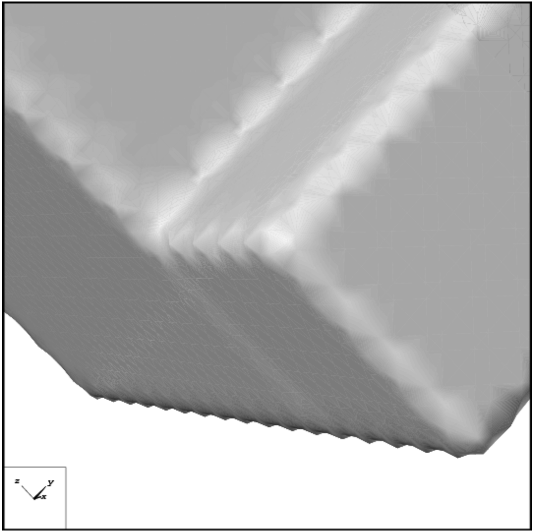
$t = 10$

Facet breaking



$t = 0$

Facet breaking



$t = 1$

Thank you for your attention