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Time-asymptotic behaviour of a solution of the Cauchy initial-value problem for a conservation law with non-linear divergent viscosity

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Abstract. We study the time-asymptotic behaviour of a solution of the Cauchy initial-value problem for a conservation law with non-linear divergent viscosity. We shall prove that when a bounded measurable initial function has limits at $\pm\infty$, a solution of the Cauchy initial-value problem converges uniformly to a system of waves consisting of travelling waves and rarefaction waves, where the phase shifts of the travelling waves are allowed to depend on time. The rate of convergence is estimated under additional conditions on the initial function.

Keywords: conservation law with non-linear divergent viscosity, equation of Burgers type, asymptotics of solutions, convergence in form, convergence on the phase plane, travelling wave, rarefaction wave, system of waves, maximum principle, comparison principle (on the phase plane), inequality of Kolmogorov type.

Introduction

In the second half of the 1940s, researchers in the USSR and USA were intensively studying processes occurring in an explosion of a bomb [1], [2]. In particular, much attention was paid to the study of initial-boundary-value problems for the following equation of conservation-law type with non-linear divergent viscosity (henceforth, x is a scalar variable and y is a scalar function):

$$\frac{\partial y}{\partial t}+\frac{\partial f(y)}{\partial x}=\frac{\partial^2 \nu(y)}{\partial x^2}\,, \tag{0.1}$$

where $\nu'(y)>0$ and $f(y)$, $\nu(y)$ are sufficiently smooth functions. In the course of the next 50 years it was established that the Cauchy initial-value problem (in what follows, C. p.) for equation (0.1) also arises in the macroscopic theory of traffic flows [3], in the modelling of the convective flow of fluids in a porous medium [4], in the study of stable monotonic difference approximations of quasilinear equations

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of conservation-law type [5], and in mathematical economics in the modelling of scientific-technological progress (the Polterovich–Henkin model [6]–[10]).

At the end of the 1950s Gel'fand posed the following problem (see [11]): find the asymptotics as $t \to \infty$ of a solution $y(t,x)$ of equation (0.1) for the following initial conditions of Riemann type:

$$y(0,x) = \begin{cases} y_- & \text{if } \ x < x_-, \\ \overline{y}_0(x) & \text{if } \ x_- \leqslant x < x_+, \\ y_+ & \text{if } \ x \geqslant x_+, \end{cases} \tag{0.2}$$

where $x_- \leqslant x_+$, $y_- \leqslant y_+$ and $\overline{y}_0(x)$ is a bounded measurable function for $x \in [x_-, x_+]$.

The interest in this problem was primarily motivated by qualitative questions of gas dynamics. A solution to Gel'fand's problem would also provide answers to the following questions. How the information about a traffic jam spreads in a traffic flow described by the Lighthill–Witham model? How to explain the existence of several set-ups in an economic sector described by the Polterovich–Henkin model and its generalizations?

In 1960 Il'in and Oleinik [12] studied the case of convergence of a solution of the C. p. (0.1), (0.2) to a travelling wave (a solution of equation (0.1) of the form $\widetilde{y}(x-ct)$) or to a rarefaction wave (a solution of equation (0.1) for $\nu(y) \equiv 0$ of the form $g(x/t)$). 30 years later Weinberger [13] studied the asymptotics of the form of a system of travelling waves and rarefaction waves. He succeeded in showing the convergence of a solution of the C. p. (0.1), (0.2) to this system of waves on the regions corresponding to the asymptotic behaviour 'rarefaction wave'. In 1994 Henkin and Polterovich [9] conjectured that under certain conditions the asymptotics has the form of an alternating system of travelling waves and rarefaction waves, where the phase shifts of travelling waves should be allowed to depend on time [10]. In 2004 Henkin and Shananin [14] proved the validity of this conjecture in the important special case when the asymptotics has the form 'travelling wave – rarefaction wave – travelling wave' (see also [15]). In 2007 Henkin [16] completed the proof of the above conjecture. Note that a concrete viscosity was assumed in the conjecture. Furthermore, conditions were imposed on the flow function $f(y)$ which exclude the possibility of the presence of two or more consecutive travelling waves in the system of waves. Additional requirements on the second derivative of the flow function were also imposed in neighbourhoods of the points corresponding to a transition from a travelling wave to a rarefaction wave, and vice versa.

In recent years a tendency has arisen to take a different approach to solving Gel'fand's problem. This approach goes back to the pioneering paper of Kolmogorov, Petrovskii, and Piskunov [17], in which they studied the convergence to a single travelling wave of a solution of the initial C. p. for the heat equation with a non-linear source. The results of that paper were extended in 1977 by Fife and McLeod [18] to a system of waves consisting, for this equation, only of travelling waves (see also [19]). In 1999 Mejai and Volpert [20] adapted the approach of [17] to the study of Gel'fand's problem. In 2006 Engelberg and Schochet [21] used the 'adapted' approach to study the case of the convergence of a solution of the initial C. p. for equation (0.1) to a travelling wave. There arose a problem similar to that solved by Fife and McLeod [18] for the heat equation with a non-linear source: to use the 'adapted' approach to study the convergence of a solution of the initial C. p. for equation (0.1) to a system of waves. The need to solve this problem is caused by at least two circumstances. First, a new approach might yield new results in the solution of Gel'fand's problem, in particular, it could answer the question of how two or more consecutive travelling waves interact. Second, it might become possible to take a uniform view of the problems of studying the asymptotic behaviour of solutions of the initial C. p. for the heat equation with a non-linear source and for a conservation law with non-linear divergent viscosity. Thus, the conjecture arises that the approach of [17] is sufficiently universal.

The main purpose of the present paper is the extension to a conservation law with a non-linear divergent viscosity of the results of Fife and McLeod [18] concerning the convergence to a system of waves of a solution of the initial C. p. for the heat equation with a non-linear source.

A new approach is proposed to the study of Gel'fand's problem which does not require a detailed analysis of the behaviour of a solution on the regions corresponding to transitions from one wave to another. We now briefly describe what this approach consists of.

The real axis corresponding to the space variable is divided (depending on time) into four types of region corresponding to a given behaviour of a solution for large values of time:

1) travelling wave;

2) rarefaction wave;

3) transition from one travelling wave to another, transition from a travelling wave to a rarefaction wave and vice versa;

4) neighbourhoods of the points $x = \pm\infty$.

In the proof of the uniform convergence with respect to time of a solution of the initial C. p. for a conservation law with non-linear divergent viscosity on the regions corresponding to travelling waves, we use the construction proposed by Engelberg and Schochet [21], which is based on the results of Mejai and Volpert [20].

In the case of rarefaction waves, we sharpen the results of Weinberger [13].

The convergence on the remaining regions is derived from those already established. Using this approach we prove the convergence of a solution to a system of waves in the norm of $C(\mathbb{R}_x)$ under more general conditions on the flow function and initial function than those in [14]–[16], although with less precise estimates. In particular, the interaction of two or more consecutive travelling waves is investigated. It turns out that, in contrast to the case of a conservation law (see [22]), for a conservation law with non-linear divergent viscosity the distance between the centres of neighbouring travelling waves in a system increases unboundedly.

An important distinction between our approach and that used in [14]–[16] is the way of defining the phase shifts of travelling waves. In 2004 Henkin and Shananin [14] proposed defining the phase shifts using their 'localized conservation laws'. In the present paper, in the case when the asymptotics has the form of a system of waves, it is shown that the phase shifts can be defined in a way similar to that in [17].

We now describe the structure of the paper.

In § 1 we describe the properties of a solution of the initial C. p. for equation (0.1) that are needed in what follows. We investigate the convergence in the norm of $L_1(\mathbb{R}_x)$ of a solution of the initial C. p. for equation (0.1) to a travelling wave.

In § 2 we investigate the behaviour for large values of time of the derivative of a solution of the initial C. p. for equation (0.1) with respect to the space variable as a many-valued function of the solution itself and time. We prove the convergence on the phase plane of a solution of the initial C. p. for equation (0.1) to a system of waves. In § 2.1 we state a theorem on the convergence to a system of waves on the phase plane and discuss the extremal meaning of the system of waves. In § 2.2 we describe the maximum principle and use it to establish an assertion on the structure of the set of zeros of a linear parabolic equation with variable coefficients. Based on this assertion, we derive a comparison principle on the phase plane (in what follows, c. p. p. p.), which forms a basis for the rest of § 2. We use the c. p. p. p. in § 2.3 to estimate from above the derivative of a solution of the initial C. p. for equation (0.1) with respect to the space variable, and in § 2.4 to estimate it from below. The proof of the theorem stated in § 2.1 is completed.

In § 3 we prove the convergence in the norm of $C(\mathbb{R}_x)$ of a solution of the initial C. p. for equation (0.1) to a system of waves. The proof is based on Theorem 2.1 and the comparison principle. In § 3.1 we state a theorem on the convergence to a system of waves in the norm of $C(\mathbb{R}_x)$. In § 3.2 we describe the scheme of the proof of Theorem 3.1. Under the assumption that the uniform convergence on the regions corresponding to the asymptotic behaviour 'travelling wave' and 'rarefaction wave' is already proved, we establish the uniform convergence on the remaining regions. In § 3.3 we use Theorem 2.1 to establish the uniform convergence on the regions corresponding to the asymptotic behaviour 'travelling wave'. In § 3.4 we use the maximum principle to establish the comparison principle, based on which we show the uniform convergence of a solution on the regions corresponding to the asymptotic behaviour 'rarefaction wave'.

§ 1. Convergence in the norm of $L_1(\mathbb{R}_x)$ of a solution of the Cauchy initial-value problem for a conservation law with non-linear divergent viscosity to a travelling wave

1.1. Properties of a solution of the Cauchy initial-value problem for a conservation law with non-linear divergent viscosity. We consider the Cauchy initial-value problem

$$\frac{\partial y}{\partial t} + \frac{\partial f(y)}{\partial x} = \frac{\partial^2 \nu(y)}{\partial x^2}\,, \tag{1.1}$$

$$\forall\, R > 0 \to \lim_{t\to 0+} \|y(t,x) - y_0(x)\|_{L_1([-R,R])} = 0, \tag{1.2}$$

where $\forall\, x \in \mathbb{R} \to m \leqslant y_0(x) \leqslant M$; $\lim_{x\to\pm\infty} y_0(x) = y_\pm$ (let $y_- \leqslant y_+$ for definiteness); $y_0(x)$ is a measurable function; $f(y), \nu(y) \in C^4([m,M])$; $\forall\, y \in [m,M] \to \nu'(y) \geqslant \overline{\nu} > 0$.

By a solution of the C. p. (1.1), (1.2) we mean a bounded function $y(t,x)$ that satisfies (1.1) in the half-plane $t>0$ and takes the initial values $y_0(x)$ in the sense of (1.2). The notation introduced above will be used throughout the paper.

Throughout what follows we study the behaviour of a solution of the C. p. (1.1), (1.2) under the above assumptions. Note that many of the assertions below are valid under substantially weaker assumptions on the smoothness of the functions $f(y)$, $\nu(y)$. In order to avoid clumsiness of exposition, we impose from the outset smoothness conditions as strong as may be required later. Also note that the condition that the initial function is bounded can be replaced everywhere by the condition of being essentially bounded, that is, $\|y_0(x)\|_{L_\infty(\mathbb{R})}<\infty$. However, for greater clarity we simply assume $y_0(x)$ to be a bounded function.

We now list the properties of a solution of the C. p. (1.1), (1.2) that will be needed in what follows.

Theorem 1.1. *A solution $y(t,x)=y(t,x;y_0(x))$ of the C. p.* (1.1), (1.2) *for $t>0$ exists and is unique. Furthermore, the function $y(t,x)$ is continuous in the half-plane $t>0$ together with the derivatives involved in equation* (1.1), *and the following properties of the solution hold.*

1) *If $\forall\, x\in\mathbb{R}\to y_0(x)\leqslant \overline{y}_0(x)$, then $\forall\, t>0$, $x\in\mathbb{R}$ we have*

$$y(t,x;y_0(x))\leqslant y(t,x;\overline{y}_0(x)).$$

2) *If $\|y_0(x)-\overline{y}_0(x)\|_{L_1(\mathbb{R}_x)}<\infty$, then $\forall\, t>0$ we have*

$$\int_{-\infty}^{+\infty}\bigl(y(t,x;y_0(x))-y(t,x;\overline{y}_0(x))\bigr)\,dx\equiv\int_{-\infty}^{+\infty}(y_0(x)-\overline{y}_0(x))\,dx.$$

3) *If $\|y_0(x)-\overline{y}_0(x)\|_{L_1(\mathbb{R}_x)}<\infty$, then $\forall\, t>0$ we have*

$$\bigl\|y(t,x;y_0(x))-y(t,x;\overline{y}_0(x))\bigr\|_{L_1(\mathbb{R}_x)}\leqslant\|y_0(x)-\overline{y}_0(x)\|_{L_1(\mathbb{R}_x)}.$$

4) *$|l_0(b)-l_0(a)|\leqslant\|b(x)-a(x)\|_{L_1(\mathbb{R}_x)}$, where the functional*

$$l_0(c)=\lim_{t\to\infty}\|y(t,x;c(x))-y_-\|_{L_1(\mathbb{R}_x)}$$

is defined on functions $c(x)$ such that $\|c(x)-y_-\|_{L_1(\mathbb{R}_x)}<\infty$.

5) *If $\{\overline{y}^k(x)\}_{k=1}^\infty$ is a uniformly bounded sequence of smooth functions converging almost everywhere to the function $y_0(x)$, then $\forall\, T>t_0>0$, $R>0$ we have*

$$\lim_{k\to\infty}\bigl\|y(t,x;y_0(x))-y(t,x;\overline{y}^k(x))\bigr\|_{C^{1,3}_{t,x}([t_0,T],[-R,R])}=0.$$

6) *$\forall\, t>0$, $x\in\mathbb{R}$ we have $m\leqslant y(t,x)\leqslant M$, that is, $y(t,x)$ is independent of the behaviour of the sufficiently smooth functions $f(y)$, $\nu(y)$, $\nu'(y)>0$ for $y\in\mathbb{R}\setminus[m,M]$.*

7) *$\forall\, t_0>0$ $\exists\,\{D^r_{f,\nu}(t_0)\}_{r=1}^3>0\colon\forall\, t\geqslant t_0$, $r=1,2,3$ we have*

$$\biggl\|\frac{\partial^r y(t,x)}{\partial x^r}\biggr\|_{C(\mathbb{R}_x)}\leqslant D^r_{f,\nu}.$$

8) $\forall\, t>0$ *we have* $\lim_{x\to\pm\infty} y(t,x)=y_\pm$ *and* $\lim_{x\to\pm\infty} y_x(t,x)=0$.

9) $\forall\,\delta>0\quad \exists\, T^*(\delta)>0\colon \forall\, t\geqslant T^*(\delta)$ *we have*

$$\inf_{x\in\mathbb{R}} y(t,x)\geqslant y_- -\delta, \qquad \sup_{x\in\mathbb{R}} y(t,x)\leqslant y_+ +\delta.$$

10) *If the initial function $y_0(x)$ has uniformly bounded derivatives of up to and including the second order, its third derivative satisfies a Lipschitz condition, and*

$$q(y_0(\,\cdot\,))=\sup\bigl\{k\in\mathbb{R}\mid \exists\, x_1<\cdots<x_k\colon \forall\, i=2,\dots,k-1 \\ \to (y_0(x_{i+1})-y_0(x_i))(y_0(x_i)-y_0(x_{i-1}))<0\bigr\}=N,$$

then $\forall\, t>0$ we have $q(y(t,\,\cdot\,;y_0(x)))\leqslant N$.

It was proved in [23] that if the initial function has uniformly bounded derivatives of up to and including the second order and its third derivative satisfies a Lipschitz condition, then a solution of the C. p. (1.1), (1.2) (where in equation (1.2) the norm of $L_1([-R,R])$ can be replaced by the norm of $C^2(\mathbb{R}_x)$) for $t>0$ exists and is unique. Furthermore, $y(t,x)$ is continuous in the half-plane $t\geqslant 0$ together with the derivatives involved in equation (1.1) (for the case of an arbitrary bounded measurable initial function, see [24]). Part 6) and some of part 7) of Theorem 1.1 were also proved in [23]. More precisely, a method was proposed which generalizes Bernshtein's method of a priori estimates for the higher derivatives of a solution of a linear parabolic equation with variable coefficients [25] and which made it possible to obtain the estimates given in part 7) (see also [26]–[28]; concerning the uniformity of the estimates with respect to time, see [29]). An assertion similar to part 5) was stated in [24]. In [30] it was stated that part 5) follows from the results of [28]. An analogue of part 3) can be found in [24]; a proof follows easily from the results of [30]. An original method for proving part 3) is contained in [31] (see also [32], [33]). It turns out that part 3) is a simple consequence of parts 1), 2). Parts 1), 2), 8) were justified and used extensively in [12] under the condition that $\nu(y)=y$. The proofs of parts 1), 2), 8) for $\nu'(y)>0$ are similar to those given in [12] (for part 1), see also [34]). Note that the second relation in part 8) follows easily from the first and part 7). In [13] it was shown that part 9) is valid if $f(y)$ has no accumulation points of zeros of the second derivative on the closed interval $[m,M]$. In § 1.2 we shall get rid of this requirement by following [35] (see below Remark 1.9). Part 4) follows from part 3) and the triangle inequality (see [32]). Finally, the validity of part 10) follows from [36], Theorem 1 and Remark 2.3, and [23], Theorem 1.

Remark 1.1. It follows from [24] that if $y_0(x)$ is a continuous function, then equation (1.2) can be understood in the ordinary sense:

$$\forall\, x\in\mathbb{R}\to \lim_{t\to 0+,\, x_1\to x} y(t,x_1)=y_0(x).$$

Remark 1.2. It follows from part 5) of Theorem 1.1 that we can confine ourselves to considering only continuous (sufficiently smooth) initial conditions $y_0(x)$. Indeed, from an arbitrary bounded measurable initial function $y_0(x)$, we can construct the sequence of functions

$$y_0^h(x)=\int_{-\infty}^{+\infty} y_0(x-\xi)\omega^h(\xi)\,d\xi, \qquad h>0,$$

where $\omega^h(x) \in C^\infty(\mathbb{R})$, $\omega^h(x) \geqslant 0$, $\omega^h(x) \equiv 0$ for $|x| \geqslant h$, $\exists\, C > 0 \colon \omega^h(x) \leqslant C/h$, $\int_{-\infty}^{+\infty} \omega^h(x)\,dx \equiv 1$. It follows from § 4 of Ch. 9 in [37] that the bounded sequence of functions $y_0^h(x)$ with bounded derivatives converges almost everywhere to $y_0(x)$ as $h \to 0+$.

Remark 1.3. It follows from [12], [23] that if the initial function $y_0(x)$ has uniformly bounded derivatives of up to and including the second order and its third derivative satisfies a Lipschitz condition, then parts 7), 8) can be sharpened as follows:

$$\exists\, \{D^r_{f,\nu}\}_{r=1}^3 > 0,\ D_{f,\nu} > 0\colon\ \forall\, t \geqslant 0,\ r = 1,2,3$$
$$\to \left\| \frac{\partial^r y(t,x)}{\partial x^r} \right\|_{C(\mathbb{R}_x)} \leqslant D^r_{f,\nu}, \qquad \left\| \frac{\partial y(t,x)}{\partial t} \right\|_{C(\mathbb{R}_t^+)} \leqslant D_{f,\nu},$$
$$\forall\, t \geqslant 0 \to \lim_{x \to \pm\infty} y(t,x) = y_\pm, \qquad \lim_{x \to \pm\infty} y(t,x) = 0.$$

1.2. Convergence in the norm of $L_1(\mathbb{R}_x)$ of a solution of the Cauchy initial-value problem for a conservation law with non-linear divergent viscosity to a constant and to a travelling wave. Here we establish the convergence in the norm of $L_1(\mathbb{R}_x)$ of a solution of the C. p. (1.1), (1.2) to a travelling wave. As a consequence we obtain that the convergence also holds in the norm of $C(\mathbb{R}_x)$.

We introduce the definition of a solution of equation (1.1) having the form of a travelling wave.

Definition 1.1. By a *wave solution* (*solution having the form of a travelling wave*) of equation (1.1) we mean a function $\widetilde{y}(x - ct)$ satisfying the following conditions:

1) $\exists\, \lim_{s \to -\infty} \widetilde{y}(s)$, $\exists\, \lim_{s \to +\infty} \widetilde{y}(s)$;

2) $\widetilde{y}(x - ct)$ is a solution of equation (1.1).

Necessary and sufficient conditions for the existence of a wave solution are given by the following assertion.

Assertion 1.1. *Suppose that $y_1 < y_2$ and that $f(y), \nu(y) \in C^4([y_1, y_2])$ and $\nu'(y) > 0$ on the closed interval $[y_1, y_2]$. Then for the existence of a wave solution $\widetilde{y}(x - ct)$ of equation (1.1) such that*

$$\lim_{s \to -\infty} \widetilde{y}(s) = y_1, \qquad \lim_{s \to +\infty} \widetilde{y}(s) = y_2,$$

it is necessary and sufficient that the following conditions hold:

1) $c = \frac{f(y_2) - f(y_1)}{y_2 - y_1}$, *the* (R–H), *or Rankine–Hugoniot, condition* [2];

2) $\forall\, y \in (y_1, y_2) \to \frac{f(y) - f(y_1)}{y - y_1} > \frac{f(y_2) - f(y_1)}{y_2 - y_1}$, *the* E- , *or Gel'fand–Oleinik entropy, condition* [11], [12], [38].

Remark 1.4. Suppose that there exists a solution of equation (1.1) of the form of a travelling wave. Then this solution is unique up to a shift of the argument (if $\widetilde{y}(x - ct)$ is a wave solution, then so is $\widetilde{y}(x - ct + \mathrm{const})$).

Remark 1.5. Under the conditions of Assertion 1.1, the function $\widetilde{y}(s)$ is increasing and has continuous uniformly bounded derivatives of up to and including the third order ($\widetilde{y}(s) \in C^3(\mathbb{R})$).

Remark 1.6. Under the assumptions of Assertion 1.1, condition 2) of Definition 1.1 is equivalent to the following:

$$\frac{\partial \nu(\widetilde{y}\,)}{\partial s} = f(\widetilde{y}\,) - c\widetilde{y} - (f(y_1) - cy_1). \tag{1.3}$$

Therefore if $\widetilde{y}(0) = y^*$, then

$$s = \int_{y^*}^{\widetilde{y}} \frac{\nu'(y)}{f(y) - cy - (f(y_1) - cy_1)}\, dy. \tag{1.4}$$

Remark 1.7. The E-condition has a clear geometric interpretation: the graph of the function $f(y)$ for $y \in (y_1, y_2)$ lies strictly above the line passing through the points $(y_1, f(y_1))$ and $(y_2, f(y_2))$. Furthermore, the speed c of the travelling wave is equal to the slope of this line.

The proofs of Assertion 1.1 and Remarks 1.4–1.7 are contained, for example, in [11]–[13], [39].

The main result of this section is the theorem on the convergence of a solution of the C. p. (1.1), (1.2) to a travelling wave.

Theorem 1.2. *Suppose that conditions* 1), 2) *of Assertion* 1.1 *hold for* $y_1 = y_-$, $y_2 = y_+$ *and that*

$$\forall\, d \in \mathbb{R} \to \|y_0(x) - \widetilde{y}(x+d)\|_{L_1(\mathbb{R}_x)} < \infty.$$

Then

$$\lim_{t\to\infty} \|y(t,x) - \widetilde{y}(x - ct + d)\|_{L_1(\mathbb{R}_x)} = 0, \tag{1.5}$$

where the constant d (*the phase shift*) *is determined by the following condition* (*see part* 2) *of Theorem* 1.1):

$$\int_{-\infty}^{+\infty} (y_0(x) - \widetilde{y}(x+d))\, dx = 0.$$

In 1960 Il'in and Oleinik [12] established for $\nu(y) = y$ that if, under the hypotheses of Theorem 1.2, one requires in addition the validity of the following relations (the Lax conditions [40]):

$$f'(y_-) \neq c, \qquad f'(y_+) \neq c,$$

then the following formula holds:

$$\lim_{t\to\infty} \|y(t,x) - \widetilde{y}(x - ct + d)\|_{C(\mathbb{R}_x)} = 0. \tag{1.6}$$

In 1982 Osher and Ralston showed that if in Theorem 1.2 one also requires the validity of the condition

$$\forall\, x \in \mathbb{R} \to y_- \leqslant y_0(x) \leqslant y_+,$$

then formula (1.5) holds (see also [41]).

In 1998 Freistühler and Serre [32], [35] used the results of [39] to reduce the proof of Theorem 1.2 to that of the following assertion.

Assertion 1.2. *Suppose that* $y_- = y_+$, $\int_{-\infty}^{+\infty} (y_0(x) - y_-)\,dx = 0$, *and* $\|y_0(x) - y_-\|_{L_1(\mathbb{R}_x)} < \infty$. *Then*

$$\lim_{t\to\infty} \|y(t,x) - y_-\|_{L_1(\mathbb{R}_x)} = 0. \tag{1.7}$$

In [32], [35] Assertion 1.2 was proved for $\nu(y) = y$ by two different methods. Below we extend the approach of [35] to the case $\nu'(y) > 0$, and thus prove Theorem 1.2.

Before embarking on the proof of Assertion 1.2, we state the known results on Kolmogorov-type inequalities for derivatives [42] which will be needed in the proof of Assertions 1.2, 1.3.

Kolmogorov-type inequalities for derivatives are by definition inequalities of the following form:

$$\exists\, K > 0\colon \forall\, z(x) \in W^n_{p,r}(\mathbb{R}) \to \|z^{(k)}(x)\|_{L_q(\mathbb{R})} \leqslant K \|z(x)\|^{\alpha}_{L_p(\mathbb{R})} \|z^{(n)}(x)\|^{\beta}_{L_r(\mathbb{R})},$$

where $0 \leqslant k < n$ are integers, $0 < p, q, r \leqslant \infty$, $\alpha, \beta \geqslant 0$, $W^n_{p,r}(\mathbb{R})$ is the space of functions $z(x) \in L_p(\mathbb{R})$ whose $(n-1)$st derivative is locally absolutely continuous on $\mathbb{R}$, and $z^{(n)}(x) \in L_r(\mathbb{R})$. The constant K depends on five parameters, n, k, p, q, r. The quantities α and β are uniquely determined by them:

$$\alpha = \frac{n - k - 1/r + 1/q}{n - 1/r + 1/p}\,, \qquad \beta = 1 - \alpha.$$

It was shown in [43], [44] that if $q \neq p$ for $k = 0$, then

$$K(n,k,p,q,r) < \infty \ \Leftrightarrow\ \frac{n-k}{p} + \frac{k}{r} \geqslant \frac{n}{q}\,, \qquad r \geqslant 1.$$

Furthermore, the best-possible value of the constant $K(n,k,p,q,r) < \infty$ was obtained in the cases important for us (see Table 1).

Table 1

n	k	q	p	r	Authors
1	0	$\geqslant p$	> 0	$\geqslant 1$	Sz.-Nagy
2	1	$2p$	$\geqslant 1/2$	∞	V. N. Gabushin, A. P. Buslaev
2	0; 1	∞	> 0	∞	V. N. Gabushin

Those cases where the best-possible (sharp) values of the constant K have been successfully calculated were considered in more detail in [42], [45].

Proof of Assertion 1.2. We assume without loss of generality that

$$\nu(y_-) = f(y_-) = f'(y_-) = 0. \tag{1.8}$$

We denote by $\widetilde{\mathrm{C}}$ the set of functions $y(x)$ defined on the axis $\mathbb{R}$ and satisfying the conditions

$$y(x) \in C^{\infty}_{y_-}(\mathbb{R}), \qquad \int_{-\infty}^{+\infty} (y(x) - y_-)\, dx = 0,$$

$$q(y(\,\cdot\,)) = \sup\bigl\{k \in \mathbb{N} \mid \exists\, x_1 < \cdots < x_k \colon \forall\, i = 2, \dots, k-1 \\ \to (y(x_{i+1}) - y(x_i))(y(x_i) - y(x_{i-1})) < 0\bigr\} < \infty,$$

where $C^{\infty}_{y_-}(\mathbb{R})$ is the space of infinitely differentiable functions on $\mathbb{R}$ that take the constant value y_- outside some interval.

We observe that $\widetilde{\mathrm{C}}$ is L_1-dense in the Banach space

$$\biggl\{y(x) \colon \int_{-\infty}^{+\infty} (y(x) - y_-)\, dx = 0,\ \|y(x) - y_-\|_{L_1(\mathbb{R}_x)} < \infty\biggr\}.$$

Therefore (see part 3) of Theorem 1.1) it is sufficient to prove formula (1.7) for $y_0(x) \in \widetilde{\mathrm{C}}$.

Moreover, it follows from part 4) of Theorem 1.1 that it is sufficient to prove the assertion for

$$y_0(x) \in \widetilde{\mathrm{C}} \cap B_{L_1}(\lambda), \qquad \lambda > 0,$$

where $B_{L_1}(\lambda) = \{y(x) \colon \|y(x) - y_-\|_{L_1(\mathbb{R})} \leqslant \lambda\}$. Indeed, let

$$l_0(z) = \lim_{t\to\infty} \|y(t, x; z(x)) - y_-\|_{L_1(\mathbb{R}_x)} \equiv 0, \qquad z(x) \in \widetilde{\mathrm{C}} \cap B_{L_1}(\lambda). \tag{1.9}$$

By part 4) of Theorem 1.1, the functional $l_0(y) \colon L_1(\mathbb{R}) \to \mathbb{R}$ is uniformly continuous on $\widetilde{\mathrm{C}}$. Therefore we obtain from (1.9) that

$$\exists\, \delta > 0 \colon \forall\, y(x) \in \widetilde{\mathrm{C}} \cap B_{L_1}(\lambda + \delta) \to l_0(y) \leqslant \frac{\lambda}{2}\,.$$

However, $l_0(y(0, \,\cdot\,)) \leqslant \lambda/2$ means that

$$\exists\, \tau > 0 \colon \|y(\tau, \,\cdot\,) - y_-\|_{L_1(\mathbb{R}_x)} \leqslant \lambda.$$

Consequently (see (1.9)), $l_0(y) = 0$ for $y \in \widetilde{\mathrm{C}} \cap B_{L_1}(\lambda + \delta)$. It is easy to see that the arguments given above justify the induction step: if $l_0(y) = 0$ for $y \in \widetilde{\mathrm{C}} \cap B_{L_1}(\lambda + k\delta)$, then $l_0(y) = 0$ for $y \in \widetilde{\mathrm{C}} \cap B_{L_1}(\lambda + (k+1)\delta)$, $k \in \mathbb{N}$.

Thus, by the induction hypothesis (the basis of induction is the relation (1.9)) we get $l_0(y) = 0$ for $y \in \widetilde{\mathrm{C}}$.

By part 3) of Theorem 1.1, Remark 1.3, and the inequalities of Sz.-Nagy and Gabushin–Buslaev (see Table 1), we have

$$\exists\, C_{y_0} > 0,\ C^1_{y_0} > 0 \colon\ \forall\, t \geqslant 0 \\ \to \|y(t, x) - y_-\|_{L_2(\mathbb{R}_x)} \leqslant C_{y_0},\ \|y_x(t, x)\|_{L_2(\mathbb{R}_x)} \leqslant C^1_{y_0}. \tag{1.10}$$

We multiply equation (1.1) by $y(t, x) - y_-$ and integrate with respect to $x \in \mathbb{R}$ taking into account Remark 1.3 and the relations (1.10). We obtain the so-called energy inequality

$$\frac{d\|y(t,x)-y_-\|^2_{L_2(\mathbb{R}_x)}}{dt} = -2\|\nu'(y(t,x))y_x^2(t,x)\|_{L_1(\mathbb{R}_x)} \leqslant -2\overline{\nu}\|y_x(t,x)\|^2_{L_2(\mathbb{R}_x)}. \tag{1.11}$$

We introduce the notation $\mathrm{V}(t,x) = \int_{-\infty}^{x}(y(t,\xi)-y_-)\,d\xi$. We can assume without loss of generality that $\|\mathrm{V}(0,x)\|_{C(\mathbb{R}_x)} \leqslant \lambda$. Then it follows from part 3) of Theorem 1.1 that $\forall\, t \geqslant 0 \to \|\mathrm{V}(t,x)\|_{C(\mathbb{R}_x)} \leqslant \lambda$. From part 2) of Theorem 1.1 and Remark 1.3 we have $\forall\, t \geqslant 0 \to \lim_{x\to\pm\infty}\mathrm{V}(t,x) = 0$. Using these facts, we integrate equation (1.1) from $-\infty$ to x, then multiply the resulting equation by $\mathrm{V}(t,x)$ and again integrate with respect to x, but this time over the whole real axis $\mathbb{R}$. By taking into account Remark 1.3 and relations (1.8), (1.10) we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{1}{2}\mathrm{V}^2(t,x)\,dx + \int_{-\infty}^{+\infty}\mathrm{V}(t,x)(f(y(t,x))-f(y_-))\,dx \\
&\quad= -\int_{-\infty}^{+\infty}(y(t,x)-y_-)\big(\nu(y(t,x))-\nu(y_-)\big)\,dx \leqslant -\overline{\nu}\int_{-\infty}^{+\infty}(y(t,x)-y_-)^2\,dx \\
&\qquad\Rightarrow\ \frac{d}{dt}\int_{-\infty}^{+\infty}\frac{1}{2}\mathrm{V}^2(t,x)\,dx - \frac{\lambda M_2}{2}\int_{-\infty}^{+\infty}(y(t,x)-y_-)^2\,dx \\
&\qquad\quad\leqslant -\overline{\nu}\int_{-\infty}^{+\infty}(y(t,x)-y_-)^2\,dx,
\end{aligned}$$

where $\overline{\nu} = \min_{y\in[m,M]}\nu'(y)$ and $M_2 = \max_{y\in[m,M]}|f''(y)|$. Hence,

$$\exists\,\lambda_0 = \frac{2\overline{\nu}}{M_2}\colon \forall\,\lambda_0 > \lambda > 0\ \exists\,\widetilde{K} = 2\overline{\nu} - \lambda M_2\colon \forall\, t \geqslant t_0 > 0$$

$$\to \frac{d\|\mathrm{V}(t,x)\|^2_{L_2(\mathbb{R}_x)}}{dt} \leqslant -\widetilde{K}\|\mathrm{V}_x(t,x)\|^2_{L_2(\mathbb{R}_x)} = -\widetilde{K}\|y(t,x)-y_-\|^2_{L_2(\mathbb{R}_x)}. \tag{1.12}$$

It follows from inequalities (1.11), (1.12) that

$$\lim_{t\to\infty}\|y(t,x)-\overline{y}\|_{L_2(\mathbb{R}_x)} = 0, \qquad \|\mathrm{V}(t,x)\|_{L_2(\mathbb{R}_x)} \leqslant \|\mathrm{V}(0,x)\|_{L_2(\mathbb{R}_x)}.$$

Since by the Cauchy–Bunyakovskii inequality

$$\mathrm{V}^2(t,x) = 2\int_{-\infty}^{x}(y(t,\xi)-y_-)\mathrm{V}(t,\xi)\,d\xi \leqslant 2\|y(t,x)-y_-\|_{L_2(\mathbb{R}_x)}\|\mathrm{V}(t,x)\|_{L_2(\mathbb{R}_x)},$$

we have

$$\lim_{t\to\infty}\|\mathrm{V}(t,x)\|_{C(\mathbb{R}_x)} = 0.$$

It follows from part 10) of Theorem 1.1 that $\forall\, t \geqslant 0 \to q(y(t,\,\cdot\,)) \leqslant q(y_0(\,\cdot\,))$. Thus,

$$\begin{aligned}
&\forall\, t \geqslant 0\ \exists\,\mathrm{q}(t) = q(y(t,\,\cdot\,)) \leqslant q(y_0(\,\cdot\,)),\ -\infty = x_1 < \cdots < x_{\mathrm{q}(t)+1} = +\infty\colon \\
&\|y(t,x)-y_-\|_{L_1(\mathbb{R}_x)} \leqslant \sum_{i=1}^{\mathrm{q}(t)}\left|\int_{x_i}^{x_{i+1}}y(t,x)\,dx\right| = \sum_{i=1}^{\mathrm{q}(t)}|\mathrm{V}(t,x_{i+1})-\mathrm{V}(t,x_i)| \\
&\qquad\leqslant 2\mathrm{q}(0)\|\mathrm{V}(t,x)\|_{C(\mathbb{R}_x)}.
\end{aligned}$$

The assertion is proved.

Remark 1.8. We can derive formula (1.6) from Theorem 1.2 by using part 7) of Theorem 1.1. The reasoning repeats word for word the proof of Corollary 1 in [35].

Remark 1.9. We can deduce the validity of part 9) of Theorem 1.1 from Assertion 1.1 using part 7) of Theorem 1.1. The reasoning repeats word for word the proof of Corollary 1 in [35].

In the next section (see the proof of Lemma 2.3), we shall need a more precise version of Assertion 1.2 in the case when $y_- = y_+$.

Assertion 1.3. *Suppose that $y_- = y_+$ and $\|y_0(x) - y_-\|_{L_2(\mathbb{R}_x)} < \infty$. Then for any $t \geqslant t_0 > 0$ we have the inequalities*

$$\|y(t,x) - y_-\|_{C(\mathbb{R}_x)} \leqslant C^1(\overline{\nu}, D^2_{f,\nu})\|y_0(x) - y_-\|^{4/5}_{L_1(\mathbb{R}_x)} t^{-1/5},$$

$$\|y_x(t,x)\|_{C(\mathbb{R}_x)} \leqslant C^2(\overline{\nu}, D^2_{f,\nu})\|y_0(x) - y_-\|^{2/5}_{L_1(\mathbb{R}_x)} t^{-1/10}.$$

Note that these inequalities are similar to the dispersion inequality in [46] (see also [32]).

Proof of Assertion 1.3. By Sz.-Nagy's inequality (see Table 1) we have

$$\|y(t,x) - y_-\|_{L_2(\mathbb{R}_x)} \leqslant \mathrm{SN}\,\|y(t,x) - y_-\|^{2/3}_{L_1(\mathbb{R}_x)} \left\|\frac{\partial(y(t,x) - y_-)}{\partial x}\right\|^{1/3}_{L_2(\mathbb{R}_x)}.$$

Hence, by part 3) of Theorem 1.1 we have

$$\|y(t,x) - y_-\|_{L_2(\mathbb{R}_x)} \leqslant \mathrm{SN}\,\|y_0(x) - y_-\|^{2/3}_{L_1(\mathbb{R}_x)} \,\|y_x(t,x)\|^{1/3}_{L_2(\mathbb{R}_x)}.$$

It follows from inequality (1.11) that

$$2K\|y_0(x) - y_-\|^4_{L_1(\mathbb{R}_x)} \frac{d\|y(t,x) - y_-\|^2_{L_2(\mathbb{R}_x)}}{dt} + \left(\|y(t,x) - y_-\|^2_{L_2(\mathbb{R}_x)}\right)^3 \leqslant 0,$$

where $K = \mathrm{SN}^6/(4\overline{\nu})$. Hence we have

$$\|y(t,x) - y_-\|_{L_2(\mathbb{R}_x)} \leqslant \|y_0(x) - y_-\|_{L_1(\mathbb{R}_x)} \sqrt[4]{\frac{K}{t}}\,.$$

Using Gabushin's inequalities (see Table 1):

$$\begin{aligned}\|y(t,x) - y_-\|_{C(\mathbb{R}_x)} &= \|y(t,x) - y_-\|_{L_\infty(\mathbb{R}_x)} \\ &\leqslant \mathrm{G}_1\|y(t,x) - y_-\|^{4/5}_{L_2(\mathbb{R}_x)} \left\|\frac{\partial^2(y(t,x) - y_-)}{\partial x^2}\right\|^{1/5}_{L_\infty(\mathbb{R}_x)}, \\ \|(y(t,x) - y_-)_x\|_{C(\mathbb{R}_x)} &= \left\|\frac{\partial(y(t,x) - y_-)}{\partial x}\right\|_{L_\infty(\mathbb{R}_x)} \\ &\leqslant \mathrm{G}_2\|y(t,x) - y_-\|^{2/5}_{L_2(\mathbb{R}_x)} \left\|\frac{\partial^2(y(t,x) - y_-)}{\partial x^2}\right\|^{3/5}_{L_\infty(\mathbb{R}_x)} \\ &= G_2\|y(t,x) - y_-\|^{4/5}_{L_2(\mathbb{R}_x)} \,\|y_{xx}(t,x)\|^{1/5}_{L_\infty(\mathbb{R}_x)},\end{aligned}$$

we obtain the validity of the assertion.

We observe that the equations

$$\|y(t,x)-y_-\|_{C(\mathbb{R}_x)} = \|y(t,x)-y_-\|_{L_\infty(\mathbb{R}_x)},$$
$$\|(y(t,x)-y_-)_x\|_{C(\mathbb{R}_x)} = \|(y(t,x)-y_-)_x\|_{L_\infty(\mathbb{R}_x)}$$

hold by Theorem 1.1.

§ 2. Convergence on the phase plane of a solution of the Cauchy initial-value problem for a conservation law with non-linear divergent viscosity to a system of waves

2.1. Generalization of the Mejai–Volpert theorem. The extremal meaning of a system of waves. In order to state the main result of this section we need the following definition.

Definition 2.1. We define $H(y)$ to be the *infimum of the convex hull of the set* $\{(y,v)\colon y\in Y,\ v\geqslant f(y)\}$, where $Y=[y_-,y_+]$.

Thus, the function $H(y)$ is defined on the set Y. We put

$$S=\{y\in Y\colon f(y)>H(y)\}.$$

All the subsequent main results of the paper will be established under the condition that the following two assumptions hold.

Assumption 2.1. *The set S can be written in the form*

$$S=(\alpha_0,\beta_0)\cup(\alpha_1,\beta_1)\cup(\alpha_2,\beta_2)\cup\dots\cup(\alpha_{n-1},\beta_{n-1})\cup(\alpha_n,\beta_n),$$

where

$$y_-=\alpha_0\leqslant\beta_0\leqslant\alpha_1\leqslant\beta_1\leqslant\dots\leqslant\alpha_{n-1}\leqslant\beta_{n-1}\leqslant\alpha_n\leqslant\beta_n=y_+,\qquad n\in\mathbb{N},$$

that is, the number of intervals in the representation of S is finite.

Assumption 2.2. *The boundary points of the set S, except for the two extreme points y_-, y_+, are not accumulation points of zeros of $f''(y)$. Furthermore, y_- is not an accumulation point of zeros of $f''(y)$ if $\alpha_0<\beta_0$ and $H'(y_-+0)=f'(y_-)$, and y_+ is not an accumulation point of zeros of $f''(y)$ if $\alpha_n<\beta_n$ and $H'(y_+-0)=f'(y_+)$.*

We set

$$y_x(y,t)=\left\{\frac{\partial y(t,x)}{\partial x}\colon y(t,x)=y\right\},$$

where $y(t,x)$ is a solution of the C. p. (1.1), (1.2). Note that for a fixed t, several values of $y_x(y,t)$ may correspond to one value of y.

In what follows we shall come across inequalities of the form $a\leqslant y_x(y,t)\leqslant b$, where a, b are some numbers but, possibly, $a,b=\pm\infty$. We interpret such inequalities in such a way that all the values of the many-valued function $y_x(y,t)$ belong to the closed interval $[a,b]$. In particular, the inequality $y_x(y,t)\geqslant 0,\ y\in[y_1,y_2]$, means that $\exists\, x_1,x_2\colon x_1\leqslant x_2,\ y(x_1)=y_1,\ y(x_2)=y_2$ such that

1) $y(t,x)$ is a single-valued function of x for $x\in[x_1,x_2]$ and fixed t,

2) for $x\notin[x_1,x_2]$ the values of $y(t,x)$ lie outside the interval (y_1,y_2).

Theorem 2.1. *Suppose that Assumptions 2.1, 2.2 hold. Then the graph of the many-valued function $y_x(t,x)$ converges as $t \to \infty$ uniformly with respect to $y \in (y_-, y_+)$ to the graph of the function $R_0(y)/\nu'(y)$, where $R_0(y) = f(y) - H(y)$, that is,*

$$\exists\,\sigma_0 > 0\colon \forall\,\sigma_0 > \sigma > 0\ \exists\,T(\sigma) > 0\colon \forall\,t \geqslant T(\sigma),\ y \in (y_-, y_+)$$
$$\to \left| y_x(y,t) - \frac{R_0(y)}{\nu'(y)} \right| \leqslant \sigma.$$

For $\nu(y) = y$ and $y_0(x)$ 'squeezed' between two monotonic functions that have the same limits at $x = \pm\infty$ and $y_\pm$, respectively, Theorem 2.1 was established by Mejai and Volpert [20]. The convergence in Theorem 2.1 was referred to in [20] as *convergence on the phase plane*.

Remark 2.1. Essential use of Assumption 2.2 is made in the proof of Theorem 2.1 (see the derivation of formula (2.8)). However, somewhat more cumbersome arguments in § 2.4 enable us to get rid of the requirement that Assumption 2.2 should hold in Theorem 2.1.

Setting $u(t,x) = \nu(y(t,x))$, we rewrite the C. p. (1.1), (1.2) as follows:

$$\frac{\partial \eta(u)}{\partial t} + \frac{\partial \psi(u)}{\partial x} = \frac{\partial^2 u}{\partial x^2}\,, \tag{2.1}$$
$$u(0,x) = u_0(x), \tag{2.2}$$

where $y(t,x) = \eta(u(t,x))$, $\psi(\eta^{-1}(y)) = f(y)$, $u_0(x) = \nu(y_0(x))$. Here we can assume without loss of generality (see Remark 1.2) that $u_0(x)$ is a continuous function. Therefore equation (2.2) can be understood in the ordinary sense (see Remark 1.1).

The new notation enables us to reformulate Theorem 2.1:

$$\exists\,\sigma_0 > 0\colon \forall\,\sigma_0 > \sigma > 0\ \exists\,T(\sigma)\colon \forall\,t \geqslant T(\sigma),\ y \in (y_-, y_+)$$
$$\to R_0(y) - \sigma \leqslant u_x(y,t) \leqslant R_0(y) + \sigma, \tag{2.3}$$

where

$$u_x(y,t) = \left\{ \frac{\partial u(t,x)}{\partial x} \colon \eta(u(t,x)) = y \right\}.$$

Let $w(x - ct)$ be a solution of equation (2.1) of the form of a travelling wave, and $v(x - ct)$ the corresponding wave solution of equation (1.1), that is, $v = \eta(w)$. We set $p = w'$. Then

$$-\,c\eta'(w)w' + \psi'(w)w' = w'' \ \Leftrightarrow\ \frac{dp}{dw} = \psi'(w) - c\eta'(w)$$
$$\Leftrightarrow\ \frac{dp}{dv}\,\frac{dv}{dw} = \left(\frac{\psi'(w)}{\eta'(w)} - c \right)\eta'(w) \ \Leftrightarrow\ p'(v) = f'(v) - c. \tag{2.4}$$

It is easy to see (formula (1.3)) that for the existence of a wave solution $v(x - ct)$ of equation (1.1) with limits $v_\pm$ at $\pm\infty$ it is necessary and sufficient that equation (2.4) have a solution $p(v)$ for $v \in [v_-, v_+]$ satisfying the conditions

$$p(v_-) = p(v_+) = 0, \qquad \forall\,v \in (v_-, v_+) \to p(v) > 0.$$

Definition 2.2. We call a function $R(y)$ a *phase system of waves* if

1) $R(y_-) = R(y_+) = 0$;

2) $\forall\, y \in (y_-, y_+) \to R(y) \geqslant 0$;

3) on the intervals where the function $R(y)$ is non-zero, it coincides with the trajectory (2.4) for some c.

Definition 2.3. A phase system of waves $R^0(y)$ is called *maximal* if for any other phase system of waves $R(y)$ we have

$$\forall\, y \in (y_-, y_+) \to R^0(y) \geqslant R(y).$$

The following assertion clarifies the extremal meaning of the function $R^0(y)$ involved in Theorem 2.1.

Assertion 2.1. *There exists a maximal phase system of waves $R^0(y)$. Furthermore,*

$$R^0(y) = R_0(y) = f(y) - H(y).$$

Proof. Let $R(y)$ be an arbitrary phase system of waves, that is,

$$R(y_-) = R(y_+) = 0, \qquad \forall\, y \in (y_-, y_+) \to R(y) \geqslant 0,$$

and on the intervals where the function $R(y)$ is non-zero it must coincide with the trajectory (2.4) for some c. Suppose that

$$R(y_1) = R(y_2) = 0, \qquad \forall\, y \in (y_1, y_2) \to R(y) > 0,$$

where $(y_1, y_2) \subseteq Y$. Then it follows from (2.4) that for $y \in [y_1, y_2]$, $R(y)$ is the difference between the function $f(y)$ and the line through the points $(y_1, f(y_1))$ and $(y_2, f(y_2))$. Therefore the assertion follows from Definition 2.1 and Remark 1.7.

Remark 2.2. If $y_1 = \alpha_k$ in formula (1.4), then

$$s = \int_{y^*}^{\widetilde{y}} \frac{\nu'(y)}{R_0(y)}\, dy.$$

2.2. The comparison principle on the phase plane. An important role in the study of equations of parabolic type is played by various modifications of the maximum principle (see, for example, [12], [13], [17], [19], [23], [25]–[28], [34], [36]). Following [19], we state the maximum principle and use it to establish the comparison principle on the phase plane (c. p. p. p.) on the basis of which the inequality (2.3) will be proved in the next two subsections.

We denote by Z_T the class of continuous functions $z(t,x)$ that are bounded in the strip $0 \leqslant t \leqslant T$, and for $0 < t \leqslant T$ have continuous partial derivatives $\partial z(t,x)/\partial x$, $\partial^2 z(t,x)/\partial x^2$, $\partial z(t,x)/\partial t$. For $0 < t \leqslant T$ we define an action of a linear operator L on the class of functions Z_T as follows:

$$Lz(t,x) = \frac{\partial}{\partial x}\left(\varepsilon(t,x)\frac{\partial z(t,x)}{\partial x}\right) - \frac{\partial z(t,x)}{\partial t} - a(t,x)\frac{\partial z(t,x)}{\partial x} - b(t,x)z(t,x),$$

where $\varepsilon(t,x)>0$ and $a(t,x)$, $b(t,x)$, $\partial\varepsilon(t,x)/\partial x$, $\partial a(t,x)/\partial x$ are bounded continuous functions in the strip $0<t\leqslant T$.

Let D be a domain (an open connected set) contained in the strip $0\leqslant t\leqslant T$. We denote by Γ_D the part of ∂D (the boundary of D) that lies in the strip $0\leqslant t<T$, and by Γ_D^T the part of ∂D lying on the line $t=T$.

Maximum principle. *Suppose that $Lz(t,x)\leqslant 0$ (or $Lz(t,x)\geqslant 0$) in a domain D, where $z(t,x)\in \mathrm{Z}_T$. If $z(t,x)\geqslant 0$ (respectively, $z(t,x)\leqslant 0$) on Γ_D, then $z(t,x)\geqslant 0$ (respectively, $z(t,x)\leqslant 0$) in $\overline{D}$.*

Proof. Following [23], Lemma 1, we set $\widetilde{z}(t,x)=z(t,x)e^{-t\mu}$, where $\mu=2+\sup_{0<t\leqslant T,\ x\in\mathbb{R}}(-b(t,x))<\infty$, and $\tilde{L}\widetilde{z}\overset{\text{def}}{=}L\widetilde{z}-\mu\widetilde{z}=e^{t\mu}Lz$. By hypothesis, $\tilde{L}\widetilde{z}\leqslant 0$ in D.

Suppose the opposite, that is, suppose that the condition $z(t,x)\geqslant 0$ in $\overline{D}$ does not hold. Then there exists a point $(t_0,x_0)\in\overline{D}\setminus\overline{\Gamma}_D$ such that $\widetilde{z}(t_0,x_0)<0$.

We set $M_{\widetilde{z}}=\sup_{0\leqslant t\leqslant T,\ x\in\mathbb{R}}(-\widetilde{z}(t,x))>0$. We choose $\omega>0$ so small that

$$\begin{aligned}\forall\,(t,x)\in D\cup\Gamma_D^T\to\varepsilon(t,x)(M_{\widetilde{z}}+t)\omega^2-1&\\ +\big(\varepsilon_x(t,x)-a(t,x)\big)(M_{\widetilde{z}}+t)\,\mathrm{th}(\omega(x-x_0))\omega&\leqslant 0.\end{aligned}$$

We choose $\rho>0$ so large that

$$\mathrm{ch}(\omega\rho)\geqslant\rho,\qquad \frac{M_{\widetilde{z}}+T}{\rho}+\widetilde{z}(t_0,x_0)<0.$$

Following [12], Lemma 1, we introduce the auxiliary functions

$$w(t,x)=(M_{\widetilde{z}}+t)\frac{\mathrm{ch}(\omega(x-x_0))}{\rho},\qquad \widetilde{v}(t,x)=w(t,x)+\widetilde{z}(t,x).$$

The parameters $\omega>0$ and $\rho>0$ were chosen in such a way that

$$\begin{gathered}\tilde{L}w(t,x)\leqslant-(\mu+b(t,x))w(t,x)\leqslant-2w(t,x)<0\quad\text{in }D,\\ \widetilde{v}(t_0,x_0)<0,\quad \widetilde{v}(t,x)\geqslant 0\quad\text{on }\Gamma_{\Pi\cap D},\end{gathered}$$

where $\Pi=\{(t,x)\colon 0<t<T,\ x_0-\rho<x<x_0+\rho\}$. Hence there exists a point $(\bar{t}_0,\overline{x}_0)\in(\overline{\Pi}\cap\overline{D})\setminus\Gamma_{\Pi\cap D}$ at which the function $\widetilde{v}(t,x)$ attains a (negative) minimum on the set $\overline{\Pi}\cap\overline{D}$, that is,

$$\widetilde{v}_x(\bar{t}_0,\overline{x}_0)=0,\qquad \widetilde{v}_{xx}(\bar{t}_0,\overline{x}_0)\geqslant 0,\qquad \widetilde{v}_t(\bar{t}_0,\overline{x}_0)\leqslant 0.$$

Consequently, there exists $\varepsilon>0$ such that

$$\tilde{L}\widetilde{v}(t,x)\geqslant-\widetilde{v}(\bar{t}_0,\overline{x}_0)>0$$

for any $t\in(\bar{t}_0-\varepsilon,\bar{t}_0+\varepsilon)\cap(0,T]$ and $x\in(\overline{x}_0-\varepsilon,\overline{x}_0+\varepsilon)$.

Since $\tilde{L}\widetilde{z}\leqslant 0$ in D, we have $\tilde{L}\widetilde{v}(t,x)=\tilde{L}(w(t,x)+\widetilde{z}(t,x))<0$ in D, a contradiction.

Corollary 2.1. *Suppose that $Lz(t,x)=0$ in a domain D, where $z(t,x)\in\mathrm{Z}_T$. If $z(t,x)=0$ on Γ_D, then $z(t,x)=0$ in $\overline{D}$.*

In what follows, we write $D \in \Upsilon$ to mean that a domain D has the form $\{(t,x)\colon 0 < t < T,\ \gamma_-(t) < x < \gamma_+(t)\}$, where $\gamma_\pm(t)$ is a continuous curve for $0 \leqslant t \leqslant T$ or is equal to $\pm\infty$.

Corollary 2.2. *Suppose that* $Lz(t,x) \leqslant 0$ (*or* $Lz(t,x) \geqslant 0$) *in a domain* $D \in \Upsilon$, *where* $z(t,x) \in \mathrm{Z}_T$. *If* $z(t,x) \geqslant 0$ (*respectively*, $z(t,x) \leqslant 0$) *on* Γ_D *and there exists a point* $(t^*,x^*) \in \Gamma_D$ *such that* $z(t^*,x^*) > 0$ (*respectively*, $z(t^*,x^*) < 0$), *then*

$$\forall\,(t,x) \in \overline{D} \setminus \overline{\Gamma}_D \to z(t,x) > 0 \quad (\textit{respectively},\ z(t,x) < 0).$$

In [19] the maximum principle and Corollaries 2.1, 2.2 were established for $\varepsilon(t,x) \equiv 1$ (see Theorems 4.6–4.8 in Ch. 1 in [19], respectively). The proof of the maximum principle and Corollaries 2.1, 2.2 for $\varepsilon(t,x) > 0$ can be carried out as in [19] (see also [28], Ch. 2).

We define two classes of functions A and A_0:

$$\begin{aligned} a(s) \in A \ &\Leftrightarrow\ \bigl\{\forall\, s_0 \in \mathbb{R} \to a(s_0) > 0 \ \Rightarrow\ \{\forall\, s > s_0 \to a(s) > 0\}\bigr\} \\ &\qquad \wedge \bigl\{a(s_0) < 0 \ \Rightarrow\ \{\forall\, s < s_0 \to a(s) < 0\}\bigr\}, \\ a(s) \in A_0 \ &\Leftrightarrow\ \bigl\{\forall\, s_0 \in \mathbb{R} \to a(s_0) > 0 \ \Rightarrow\ \{\forall\, s > s_0 \to a(s) \geqslant 0\}\bigr\} \\ &\qquad \wedge \bigl\{a(s_0) < 0 \ \Rightarrow\ \{\forall\, s < s_0 \to a(s) \leqslant 0\}\bigr\}. \end{aligned}$$

The proof of the c. p. p. p. makes essential use of an assertion on the structure of the set of zeros of a solution of a linear parabolic equation with variable coefficients ([19], Ch. 1, Theorem 4.9).

Assertion 2.2. *Suppose that* $Lz(t,x) = 0$ *for* $0 < t \leqslant T$, *where* $z(t,x) \in \mathrm{Z}_T$. *Then*

$$z(0,\,\cdot\,) \in A_0 \ \Rightarrow\ \{\forall\, 0 < t \leqslant T \to z(t,\,\cdot\,) \in A\}.$$

Proof. Suppose that $z(t_0,x_0) > 0$, where $P_0 = (t_0,x_0) \in \{(t,x)\colon 0 < t \leqslant T,\ x \in \mathbb{R}\}$. We denote by $S_+(P_0)$ the set of all points $Q \in \overline{D} = \{(t,x)\colon 0 \leqslant t \leqslant t_0,\ x \in \mathbb{R}\}$ that can be joined to P_0 by a simple continuous curve contained in $\overline{D}$ along which the coordinate t does not decrease from Q to P_0 and $z(t,x) > 0$. The set of all such curves is denoted by Λ.

We claim that $S_+(P_0) \cap D$ is a domain (it is sufficient to establish that $S_+(P_0) \cap D$ is an open set). Let $P_1 = (t_1,x_1) \in S_+(P_0) \cap D$. Consider the rectangle

$$\Pi_\varepsilon(t_1,x_1) = \{(t,x)\colon t_1 - \varepsilon < t < t_1 + \varepsilon,\ x_1 - \varepsilon < x < x_1 + \varepsilon\}.$$

We choose $\varepsilon > 0$ so small that $\varepsilon < \min\{t_1, t_0 - t_1\}$ and $z(t,x) > 0$ for $(t,x) \in \overline{\Pi}_\varepsilon(t_1,x_1)$. If a curve $l \in \Lambda$ joining P_0 to P_1 has points in common with the upper boundary of the rectangle $\Pi_\varepsilon(t_1,x_1)$, then $\Pi_\varepsilon(t_1,x_1) \subset S_+(P_0) \cap D$. In the opposite case, consider the domain D_1 bounded by the curve l and the lines $t = t_1 + \varepsilon$ and $x = x_1$ (here we assume without loss of generality that l is a line inside $\Pi_\varepsilon(t_1,x_1)$). Since $z(t,x) > 0$ on Γ_{D_1}, it follows from the maximum principle that $z(t,x) \geqslant 0$ in $\overline{D}_1$. Therefore there exists a domain $\widetilde{D}_1 \subseteq D_1$ such that $\widetilde{D}_1 \in \Upsilon$, $z(t,x) \geqslant 0$ on $\Gamma_{\widetilde{D}_1}$, $P_1 \in \Gamma_{\widetilde{D}_1}$, $\Gamma^{t_1+\varepsilon}_{\widetilde{D}_1} = \Gamma^{t_1+\varepsilon}_{D_1}$. Hence by Corollary 2.2 we have $z(t,x) > 0$ on $\Gamma^{t_1+\varepsilon}_{D_1}$. Therefore, $\Pi_\varepsilon(t_1,x_1) \subset S_+(P_0) \cap D$.

Note that by proving that the set $S_+(P_0) \cap D$ is open, we have established as a by-product that $z(t,x) = 0$ on $\Gamma_{S_+(P_0)} \cap D$ (where $\Gamma_{S_+(P_0)}$ is the part of the boundary of the set $S_+(P_0)$ contained in the strip $0 \leqslant t < t_0$).

If $z(t,x) = 0$ on the set $\Gamma_{S_+(P_0)}$, then by Corollary 2.1 we have $z(t,x) = 0$ in $S_+(P_0)$. Therefore there exist a point $P^* = (0, x^*)$ and a curve $l^* \in \Lambda$ joining P_0 and P^* such that $z(t,x) > 0$ on l^* (here we assume without loss of generality that l^* intersects the line $t = t_0$ in a single point P^*). Since $z(0,\,\cdot\,) \in A_0$ by hypothesis, it follows that $z(t,x) \geqslant 0$ for any $x > x^*$. Consider the domain D_2 contained in the strip $0 \leqslant t \leqslant t_0$ to the right of the curve l^*. By the maximum principle we have $z(t,x) \geqslant 0$ in $\overline{D}_2$. Therefore there exists a domain $\widetilde{D}_2 \subseteq D_2$ such that $\widetilde{D}_2 \in \Upsilon$, $z(t,x) \geqslant 0$ on $\Gamma_{\widetilde{D}_2}$, $P^* \in \Gamma_{\widetilde{D}_2}$, $\Gamma^{t_0}_{\widetilde{D}_2} = \Gamma^{t_0}_{D_2}$. Hence by Corollary 2.2 we have $z(t,x) > 0$ on $\Gamma^{t_0}_{D_2}$, that is,

$$z(t_0,x_0) > 0 \ \Rightarrow\ \forall\, x > x_0 \to z(t_0,x) > 0.$$

It can be shown in similar fashion that

$$z(t_0,x_0) < 0 \ \Rightarrow\ \forall\, x < x_0 \to z(t_0,x) < 0.$$

Thus, the assertion is proved.

Comparison principle on the phase plane. *Suppose that the* $u^i(t,x)$, $i = 1,2$, *satisfy equation* (2.1) *in the half-plane* $t > 0$ *and the* $y^i(t,x) = \eta(u^i(t,x))$, $i = 1,2$, *are continuous functions bounded in the half-plane* $t \geqslant 0$ *such that*

$$y^1(0,x) - y^2(0,x) \in A_0, \qquad y^1(0,x), y^2(0,x) \in C^3(\mathbb{R}).$$

Then

$$\forall\, t > 0,\ x(t) \in \{x\colon y^1(t,x) = y^2(t,x)\} \to u^1_x(t,x(t)) \geqslant u^2_x(t,x(t)).$$

Proof. We set $z(t,x) = u^1(t,x) - u^2(t,x)$. Then $Lz(t,x) = 0$, where

$$\varepsilon(t,x) = \frac{1}{\eta'(u^1(t,x))}\,, \qquad a(t,x) = \frac{\psi'(u^1(t,x))}{\eta'(u^1(t,x))} - \frac{\eta''(u^1(t,x))}{(\eta'(u^1(t,x)))^2}\,\frac{\partial u^1(t,x)}{\partial x}\,,$$

$$b(t,x) = \frac{1}{\eta'(u^1(t,x))}\biggl(\frac{\partial u^2(t,x)}{\partial x}\,\frac{\psi'(u^1(t,x)) - \psi'(u^2(t,x))}{u^1(t,x) - u^2(t,x)} + \frac{\partial u^2(t,x)}{\partial t}\,\frac{\eta'(u^1(t,x)) - \eta'(u^2(t,x))}{u^1(t,x) - u^2(t,x)}\biggr);$$

at the points $\{(t,x)\colon u^1(t,x) = u^2(t,x)\}$ we extend the definition of $b(t,x)$ by continuity by setting

$$b(t,x) = \frac{1}{\eta'(u^2(t,x))}\biggl(\frac{\partial u^2(t,x)}{\partial x}\psi''(u^2(t,x)) + \frac{\partial u^2(t,x)}{\partial t}\eta''(u^2(t,x))\biggr).$$

It follows from Remark 1.3 and Assertion 2.2 that

$$\forall\, t > 0 \to u^1(t,\,\cdot\,) - u^2(t,\,\cdot\,) \in A. \tag{2.5}$$

The validity of the comparison principle on the phase plane follows from (2.5) by Theorem 1.1.

2.3. An upper bound for the derivative with respect to the space variable of a solution of the Cauchy initial-value problem for a conservation law with non-linear divergent viscosity. Let $\sigma > 0$ be a fixed sufficiently small number. In this and the following subsections we assume that time starts flowing not from zero but from $T^*(\delta(\sigma))$. This is equivalent to saying that the initial condition at $t = 0$ is $y_0(x) := y(T^*(\delta(\sigma)), x)$. Note that according to parts 7)–9) of Theorem 1.1, we then have $y_0(x) \in C^3(\mathbb{R})$; $\lim_{x\to\pm\infty} y_0(x) = y_\pm$, $\lim_{x\to\pm\infty} y_0'(x) = 0$; $y_0(x)$ cannot take values outside the closed interval $[y_- - \delta(\sigma),\, y_+ + \delta(\sigma)]$. The dependence $\delta(\sigma)$ will be chosen later.

We now use the c. p. p. p. to prove the second inequality in (2.3). For $\lambda > 0$ we set

$$\mathrm{P}(\lambda, C) = \bigl\{p(y)\colon \forall\, y \in [y_- - \delta(\sigma), y_+ + \delta(\sigma)] \to p'(y) = f'(y) - C,\ p(y) \geqslant \lambda\bigr\},$$
$$x_-(\lambda) = \sup\bigl\{x\colon \forall\, x_1 < x \to y_0'(x_1) \leqslant \lambda\bigr\} > -\infty,$$
$$x_+(\lambda) = \inf\bigl\{x\colon \forall\, x_1 > x \to y_0'(x_1) \leqslant \lambda\bigr\} < +\infty.$$

Lemma 2.1. *Let $p_1(y) \in \mathrm{P}(\lambda, C_1)$, $p_2(y) \in \mathrm{P}(\lambda, C_2)$, $C_1 > C_2$. Then there is a*

$$\mathrm{T}^* = \frac{1}{C_1 - C_2}\left(\int_{y_-}^{y_+ + \delta(\sigma)} \frac{dy}{p_1(y)} + x_+(\lambda) - x_-(\lambda) + \int_{y_- - \delta(\sigma)}^{y_+} \frac{dy}{p_2(y)}\right)$$

such that

$$\forall\, t \geqslant \mathrm{T}^*,\ y \in (y_-, y_+) \to u_x(y, t) \leqslant \max_{i=1,2} p_i(y).$$

Proof. Let $v_1(x - C_1 t)$ and $v_2(x - C_2 t)$ be the wave solutions of equation (1.1) corresponding to $p_1(y)$ and $p_2(y)$ such that

$$v_1(x_-(\lambda)) = y_+ + \delta(\sigma), \qquad v_2(x_+(\lambda)) = y_- - \delta(\sigma).$$

Note that in order for $v_1(x - C_1 t)$ and $v_2(x - C_2 t)$ to exist, we may need to extend $f(y)$, $\nu(y)$ in a certain way beyond the closed interval $[m, M]$ (see part 6) of Theorem 1.1). We set

$$\zeta_1^{k_1}(t, x) = v_1(x - C_1 t - k_1) - y(t, x), \qquad \zeta_2^{k_2}(t, x) = v_2(x - C_2 t - k_2) - y(t, x).$$

By construction,

$$\forall\, k_1 \leqslant 0,\ k_2 \geqslant 0 \to \zeta_1^{k_1}(0, x) \in A,\ \zeta_2^{k_2}(0, x) \in A.$$

Furthermore, after the expiration of time T^* the point with ordinate y_- of the wave $v_1(x - C_1 t)$ overtakes the point with ordinate y_+ of the wave $v_2(x - C_2 t)$. This means that for $t \geqslant \mathrm{T}^*$ the families of functions $\{v_1(x - C_1 t - k_1)\}_{k_1 \leqslant 0}$, $\{v_2(x - C_2 t - k_2)\}_{k_2 \geqslant 0}$ will tile the entire strip $\{(x, y)\colon x \in \mathbb{R},\, y \in (y_-, y_+)\}$. The lemma follows from Remark 1.5 and the above two facts about the c. p. p. p.

For an arbitrary $\overline{y} \in (y_-, y_+)$ we set

$$p_1^{\overline{y}} = f(y) - \left(H'(\overline{y}) + \frac{\sigma}{\mu}\right)(y - \overline{y}) - H(\overline{y}) + \sigma, \qquad \mu = 3(y_+ - y_- + \delta(\sigma)),$$
$$p_2^{\overline{y}} = f(y) - H'(\overline{y})(y - \overline{y}) - H(\overline{y}) + \sigma.$$

We observe that $p_1^{\overline{y}}(y)$ and $p_2^{\overline{y}}(y)$ satisfy equation (2.4) for $c = C_1 = H'(\overline{y}) + \sigma/\mu$ and $c = C_2 = H'(\overline{y})$, respectively. We choose $\delta(\sigma)$ so small that $\delta(\sigma)D_1 \leqslant \sigma/3$, where $D_r \overset{\text{def}}{=} D^r_{f,\nu}(T^*(\delta(\sigma)))$ (see parts 7), 9) of Theorem 1.1). Then

$$\exists\, \lambda = \frac{\sigma}{3}\colon \forall\, \overline{y} \in (y_-, y_+),\ y \in [y_- - \delta(\sigma), y_+ + \delta(\sigma)],\ i = 1,2 \to p_i^{\overline{y}}(y) \geqslant \lambda.$$

We obtain from Lemma 2.1 the validity of the second inequality in (2.3).

2.4. A lower bound for the derivative with respect to the space variable of a solution of the Cauchy initial-value problem for a conservation law with non-linear divergent viscosity. We now use the c. p. p. p. to prove the first inequality in (2.3) and thus complete the proof of Theorem 2.1.

If we assume that $\delta(\sigma) < \sigma$, then $\exists\, y_1(x), y_2(x) \in C^3(\mathbb{R})$, $\{M_i^j\}_{i,j=1}^{2,3} > 0$ such that for any $x \in \mathbb{R}$ we have

$$0 < y_i'(x) \leqslant M_i^1, \quad |y_i''(x)| \leqslant M_i^2, \quad |y_i'''(x)| \leqslant M_i^3, \qquad i = 1,2,$$
$$\lim_{x\to\pm\infty} y_1(x) = y_\pm - \frac{\sigma}{2}\,, \qquad \lim_{x\to\pm\infty} y_2(x) = y_\pm + \frac{\sigma}{2}\,,$$
$$y_1(x) < y_0(x) < y_2(x).$$

For an arbitrary $x_0 \in \mathbb{R}$ we set

$$y_{x_0}(x) = \begin{cases} y_2(x) & \text{if } \ x < x_0, \\ -D_1(x - x_0) + y_2(x_0) & \text{if } \ x_0 \leqslant x < \rho(x_0), \\ y_1(x) & \text{if } \ x \geqslant \rho(x_0), \end{cases}$$

where $\rho(x_0)$ is determined by the equation

$$-D_1(\rho(x_0) - x_0) + y_2(x_0) = y_1(\rho(x_0))$$

and D_1 was defined in § 2.3: $D_1 = D^1_{f,\nu}(T^*(\delta(\sigma)))$.

Let $y_1(\rho(x_M)) = y_+ - \sigma$ and $y_2(x_m) = y_- + \sigma$. For an arbitrary $x_0 \in \mathbb{R}$ we 'smooth' the function $y_{x_0}(x)$ in small neighbourhoods of the points x_0 and $\rho(x_0)$ in such a way that

1) x_0 and $\rho(x_0)$ are extrema of the function $y_{x_0}(x)$, $y_2(x_0) = y_{x_0}(x_0)$, $y_1(\rho(x_0)) = y_{x_0}(\rho(x_0))$ for $x_0 \in [x_m, x_M]$,

2) the family of functions $\{y_{x_0}(x)\}_{x_0\in\mathbb{R}}$ that continuously depend on $x_0 \in \mathbb{R}$ in the norm of $C(\mathbb{R}_x)$ completely tiles the set $\{(x,y)\colon x \in \mathbb{R},\ y_1(x) \leqslant y \leqslant y_2(x)\}$,

3) $\forall\, x_0 \in \mathbb{R} \to y_{x_0}(x) \in C^3(\mathbb{R})$, $y_0(x) - y_{x_0}(x) \in A_0$,

4) $\exists\, \{M^j(\sigma)\}_{j=1}^3 > 0\colon \forall\, x_0 \in [x_m, x_M],\ x \in \mathbb{R},\ j = 1,2,3 \to |y_{x_0}^{(j)}(x)| \leqslant M^j(\sigma)$,

5) $\forall\, x_0 \in [x_m, x_M] \to q(y_{x_0}(\,\cdot\,)) \overset{\text{def}}{=} \sup\{\, k \in \mathbb{N} \mid \exists\, x_1 < \cdots < x_k\colon \forall\, i = 2,\dots,k-1 \to (y_{x_0}(x_{i+1}) - y_{x_0}(x_i))(y_{x_0}(x_i) - y_{x_0}(x_{i-1})) < 0\} = 3$.

We assume without loss of generality that $f(y), \nu(y) \in C^4([y_- - \sigma, y_+ + \sigma])$ and $\nu'(y) > 0$ for $y \in [y_- - \sigma, y_+ + \sigma]$.

It follows from part 10) of Theorem 1.1 (we smoothed $y_{x_0}(x)$ to satisfy the hypotheses of part 10) of Theorem 1.1 and the c. p. p. p.) that

$$\forall\, x_0 \in [x_m, x_M],\ t \geqslant 0 \to q(y(t,x;y_{x_0}(x))) \leqslant 3. \tag{2.6}$$

Recall (see § 2.3) that $y(t,x;y_{x_0}(x))$ is a solution of the C. p. (1.1), (1.2), but with the initial function $y_{x_0}(x)$ rather than $y_0(x)$.

By the c. p. p. p. and by the construction of $y_{x_0}(x)$ we have

$$\begin{aligned}\forall\, t>0,\ x_0\in\mathbb{R},\ x\in\bigl\{x\colon y(t,x)=y(t,x;y_{x_0}(x))\bigr\}\\ \to u_x(t,x)\geqslant u_x(t,x;u_{x_0}(x)),\end{aligned}\tag{2.7}$$

where $u=\nu(y)$ (here $u_x(t,x;u_{x_0}(x))$ is the partial derivative with respect to the space variable of a solution of the C. p. (2.1), (2.2) with the initial function $u_{x_0}(x)=\nu(y_{x_0}(x))$ rather than $u_0(x)=\nu(y_0(x))$).

An important role in the proof of the first inequality in (2.3) is played by the two lemmas below.

Lemma 2.2. *For any $x_0\in\mathbb{R}$ and $t>0$ we have $u_x(t,x;u_{x_0}(x))\geqslant 0$ for $y_1(\rho(x_0))\geqslant y(t,x;y_{x_0}(x))$ and $y_2(x_0)\leqslant y(t,x;y_{x_0}(x))$.*

Proof. To prove this, we take

$$y^1(t,x)=y(t,x;y_{x_0}(x)),\qquad y^2(t,x)\equiv\mathrm{const}\in\bigl(-\infty,y_1(\rho(x_0))\bigr]\cup[y_2(x_0),+\infty)$$

in the hypotheses of the c. p. p. p.

We now set

$$u_x(y,t;u_{x_0}(x))=\biggl\{\frac{\partial u(t,x;u_{x_0}(x))}{\partial x}:\eta(u(t,x;u_{x_0}(x)))=y\biggr\}.$$

Lemma 2.3. *For any $x_0\in[x_m,x_M]$ there exist $\overline{y}_{x_0}\in(y_1(\rho(x_0)),y_2(x_0))$ and $\widetilde{T}(x_0,\sigma)>0$ such that for any $t\geqslant\widetilde{T}(x_0,\sigma)$ we have*

1) $\forall\, y\in(y_-+\sigma/2,y_+-\sigma/2)\to u_x(y,t;u_{x_0}(x))\geqslant-\sigma$,

2) $\forall\, y\in(y_-+\sigma/2,y_+-\sigma/2)\setminus(\overline{y}_{x_0}-\sigma,\overline{y}_{x_0}+\sigma)\to u_x(y,t;u_{x_0}(x))\geqslant 0$.

Proof. We set

$$y^1(t,x)=y(t,x;y_{x_0}(x)),\qquad y^2(t,x)=\begin{cases}\widetilde{y} & \text{if } x<x_1,\\ y(t,x;y_{x_0}(x)) & \text{if } x_1\leqslant x<x_2,\\ \widetilde{y} & \text{if } x\geqslant x_2,\end{cases}$$

where x_1 is the minimal, and x_2 the maximal, root of the equation

$$y(t,x;y_{x_0}(x))=\widetilde{y},$$

and $\widetilde{y}\in(y_1(\rho(x_0)),y_2(x_0))$. Then part 1) of the lemma follows from the c. p. p. p. and Assertion 1.3, which describes the behaviour of $y^2(t,x)$ for large values of t.

We now prove part 2). Suppose the opposite. Then by Lemma 2.2, formula (2.6), and part 8) of Theorem 1.1 we have

$$\begin{gathered}\exists\,\sigma_0>0,\ \mathrm{T}_0>0\colon\forall\, t\geqslant\mathrm{T}_0\ \exists\,\breve{x}_-(t),\ \breve{x}_+(t)\ (\breve{x}_-(t)<\breve{x}_+(t))\colon\\ \forall\, x\in(-\infty,\breve{x}_-(t)]\cup[\breve{x}_+(t),+\infty)\to y^1_x(t,x)\geqslant 0,\\ \forall\, x\in[\breve{x}_-(t),\breve{x}_+(t)]\to y^1_x(t,x)\leqslant 0,\\ y_1(\rho(x_0))\leqslant y^1(t,\breve{x}_+(t))<y^1(t,\breve{x}_-(t))\leqslant y_2(x_0),\\ y^1(t,\breve{x}_-(t))-y^1(t,\breve{x}_+(t))\geqslant 2\sigma_0.\end{gathered}$$

We obtain from Assertion 1.3 that

$$\exists\,\widetilde{\mathrm{T}}(\sigma_0) > 0\colon \forall\, t \geqslant \widetilde{\mathrm{T}}(\sigma_0),\ \ \widetilde{y} \in \big(y_1(\rho(x_0)), y_2(x_0)\big) \to \|y^2(t,x) - \widetilde{y}\|_{C(\mathbb{R}_x)} < \sigma_0.$$

Therefore the set $\{x\colon y^1(t,x) = y^2(t,x)\} \neq \varnothing$ is disconnected for $t \geqslant \widetilde{\mathrm{T}}(\sigma_0)$ and $\widetilde{y} \in [y_1(\rho(x_0)) + \sigma_0,\, y_2(x_0) - \sigma_0]$. However, it follows from Assertion 2.2 that for $t > 0$ this set is necessarily connected, a contradiction.

Under Assumptions 2.1, 2.2 we can choose σ so small that

$$\begin{gathered}\forall\, k = 0,\dots,n\colon \alpha_k < \beta_k, \quad \forall\, 0 \leqslant \sigma_1, \sigma_2 \leqslant 5\sigma \\ \exists\, p(y)\colon \forall\, y \in (\alpha_k + \sigma_1, \beta_k - \sigma_2) \to p(y) > 0,\end{gathered}$$

$p(\alpha_k + \sigma_1) = p(\beta_k - \sigma_2) = 0$, and $p(y)$ satisfies (2.4) for $y \in [\alpha_k + \sigma_1,\, \beta_k - \sigma_2]$. Let $\alpha_k < \beta_k$ and $x_0 \in [x_m, x_M]$. It follows from part 2) of Lemma 2.3 that if $\overline{y}_{x_0} \notin (\alpha_k, \alpha_k + 3\sigma] \cup [\beta_k - 3\sigma, \beta_k)$, then

$$\exists\, \frac{3}{2}\sigma \leqslant \sigma_1, \sigma_2 \leqslant 2\sigma\colon u_x(\alpha_k + \sigma_1, \widetilde{T}(x_0,\sigma)) > 0,\ \ u_x(\beta_k - \sigma_2, \widetilde{T}(x_0,\sigma)) > 0,$$

for otherwise,

$$\exists\, \frac{9}{2}\sigma \leqslant \sigma_1, \sigma_2 \leqslant 5\sigma\colon u_x(\alpha_k + \sigma_1, \widetilde{T}(x_0,\sigma)) > 0,\ \ u_x(\beta_k - \sigma_2, \widetilde{T}(x_0,\sigma)) > 0.$$

We construct the following solutions of equation (2.4):

$$\begin{gathered}p_1\bigg(\alpha_k + \sigma_1 - \frac{\sigma}{2}\bigg) = 0, \qquad p_1(\beta_k - \sigma_2) = 0, \\ \forall\, y \in \bigg(\alpha_k + \sigma_1 - \frac{\sigma}{2}, \beta_k - \sigma_2\bigg) \to p_1(y) > 0, \\ p_2(\alpha_k + \sigma_1) = 0, \qquad p_2\bigg(\beta_k - \sigma_2 + \frac{\sigma}{2}\bigg) = 0, \\ \forall\, y \in \bigg(\alpha_k + \sigma_1, \beta_k - \sigma_2 + \frac{\sigma}{2}\bigg) \to p_2(y) > 0.\end{gathered}$$

By arguing as in the proof of Lemma 2.1 we obtain

$$\begin{gathered}\forall\, k = 0,\dots,n\colon \alpha_k < \beta_k,\ \forall\, x_0 \in [x_m, x_M]\ \exists\, \overline{T}(x_0,\sigma) \geqslant \widetilde{T}(x_0,\sigma)\colon \\ \forall\, y \in [\alpha_k + 5\sigma, \beta_k - 5\sigma],\ t \geqslant \overline{T}(x_0,\sigma) \to u_x(y,t;u_{x_0}(x)) \geqslant \min_{i=1,2} p_i(y).\end{gathered} \tag{2.8}$$

Indeed, let $v_1(x - C_1 t)$ and $v_2(x - C_2 t)$ be the wave solutions of equation (1.1) corresponding to $p_1(y)$ and $p_2(y)$ such that

$$\forall\, k_1 \leqslant 0,\ k_2 \geqslant 0 \to \zeta_1^{k_1}(\widetilde{T}(x_0,\sigma), x) \in A,\ \zeta_2^{k_2}(\widetilde{T}(x_0,\sigma), x) \in A,$$

where

$$\zeta_1^{k_1}(t,x) = y(t,x) - v_1(x - C_1 t - k_1), \qquad \zeta_2^{k_2}(t,x) = y(t,x) - v_2(x - C_2 t - k_2).$$

Since $C_1 > C_2$, there exists $\overline{T}(x_0,\sigma) \geqslant \widetilde{T}(x_0,\sigma)$ such that for $t \geqslant \overline{T}(x_0,\sigma)$ the families of functions $\{v_1(x - C_1 t - k_1)\}_{k_1 \leqslant 0}$ and $\{v_2(x - C_2 t - k_2)\}_{k_2 \geqslant 0}$ tile the entire strip $\{(x,y)\colon x \in \mathbb{R},\ y \in (\alpha_k + \sigma_1, \beta_k - \sigma_2)\}$. We obtain from Remark 1.5 and the two facts about the c. p. p. p. mentioned above that formula (2.8) holds.

It follows from (2.8) that

$$\forall\, x_0 \in [x_m, x_M]\ \exists\, T(x_0,\sigma) > 0\colon \forall\, t \geqslant T(x_0,\sigma),\ y \in [y_- + \sigma, y_+ - \sigma] \\ \to u_x(y,t;u_{x_0}(x)) \geqslant R_0(y) - \sigma.$$

We use Lemma 2.2 to show in similar fashion that

$$\forall\, x_0 \in \mathbb{R} \setminus (x_m, x_M),\ t \geqslant \max\{T(x_m,\sigma), T(x_M,\sigma)\},\ y \in [y_- + \sigma, y_+ - \sigma] \\ \to u_x(y,t;u_{x_0}(x)) \geqslant R_0(y) - \sigma.$$

It follows from part 5) of Theorem 1.1 that

$$\forall\, x_0 \in [x_m, x_M]\ \exists\, \varepsilon(\widetilde{x}_0) > 0\colon \forall\, \widetilde{x}_0 \geqslant (x_0 - \varepsilon, x_0 + \varepsilon) \to T(\widetilde{x}_0,\sigma) \leqslant T(x_0,\sigma) + 1.$$

Hence, in view of (2.7), we obtain

$$\exists\, \widehat{T}(\sigma) > 0\colon \forall\, t \geqslant \widehat{T}(\sigma),\ y \in [y_- + \sigma, y_+ - \sigma] \to u_x(y,t) \geqslant R_0(y) - \sigma. \tag{2.9}$$

It remains to observe that

$$\forall\, y \in (y_-, y_- + \sigma] \cup [y_+ - \sigma, y_+),\ t \geqslant \widehat{T}(\sigma) \to u_x(y,t) \geqslant -\sqrt{8D_2\sigma}\,.$$

This follows from formula (2.9) and the fact that (see § 2.3)

$$\forall\, t \geqslant 0,\ x \in \mathbb{R} \to y_- - \sigma < y(t,x) < y_+ + \sigma,\ \|y_{xx}(t,x)\|_{C(\mathbb{R}_x)} \leqslant D_2.$$

§ 3. Convergence in the norm of $C(\mathbb{R}_x)$ of a solution of the Cauchy initial-value problem for a conservation law with non-linear divergent viscosity to a system of waves

3.1. The theorem on uniform convergence to a system of waves. Throughout what follows we suppose that Assumption 2.1 holds. In order to state the main result of the paper, we need to give definitions of a rarefaction wave and a system of waves, as well as to make another additional assumption.

Assumption 3.1. *For any $y \in (\beta_0, \alpha_1) \cup (\beta_1, \alpha_2) \cup \dots \cup (\beta_{n-1}, \alpha_n)$ we have*

$$f''(y) > 0.$$

Definition 3.1. By the *rarefaction waves* for the C. p. (1.1), (1.2) we mean the following system of functions:

$$g_k(x/t) = \begin{cases} \beta_k & \text{if } x < f'(\beta_k)t, \\ (f')^{-1}(x/t) & \text{if } f'(\beta_k)t \leqslant x < f'(\alpha_{k+1})t, \quad k = 0, \dots, n-1, \\ \alpha_{k+1} & \text{if } x \geqslant f'(\alpha_{k+1})t, \end{cases}$$

where $(f')^{-1}(\cdot)$ is the inverse function of $f'(y)$. When $\beta_k = \alpha_{k+1}$, we set $g_k(x/t) \equiv \beta_k$.

By Assumption 3.1, for $\beta_k < \alpha_{k+1}$, $k = 0, \dots, n-1$, the function $f'(y)$ increases monotonically on the intervals (β_k, α_{k+1}), and so the rarefaction waves $g_k(x/t)$, $k = 0, \dots, n-1$, are well defined.

Definition 1.1, Assumption 2.1, and Remark 1.7 imply that there exist solutions of equation (1.1) of the form of a travelling wave $\{\widetilde{y}_k(x - c_k t + d_k)\}_{k=0}^n$, where $\widetilde{y}_k(s) \to \alpha_k$ as $s \to -\infty$, and $\widetilde{y}_k(s) \to \beta_k$ as $s \to +\infty$, and $c_k = \frac{f(\beta_k) - f(\alpha_k)}{\beta_k - \alpha_k}$, $k = 0, \dots, n$, for $\alpha_k < \beta_k$. When $\alpha_k = \beta_k$, we set $\widetilde{y}_k(x - c_k t + d_k) \equiv \alpha_k$.

We determine $d_k(t)$ (the phase shift of the kth travelling wave) and the $\widetilde{y}_k(0)$, $k = 0, \dots, n$, from the condition

$$\forall\, t \geqslant t_0^* > 0 \to y(t, c_k t - d_k(t)) \equiv \widetilde{y}_k(0) = y_k^* \overset{\text{def}}{=} \frac{\alpha_k + \beta_k}{2}\,. \tag{3.1}$$

Recall that $y(t, x)$ is a solution of the C. p. (1.1), (1.2). As will be seen later, we can assume here only that y_k^* is an interior point of the interval (α_k, β_k), not necessarily the midpoint.

We obtain the following assertion from Theorem 2.1 and the implicit function theorem.

Assertion 3.1. *Suppose that Assumptions 2.1, 2.2 hold and $\alpha_k < \beta_k$. Then there exists a $t_0^* > 0$ independent of $k = 0, \dots, n$ such that for $t \geqslant t_0^*$ formula (3.1) uniquely defines a smooth function $d_k(t)$.*

For $t \geqslant t_0^*$ we define a system of waves $\widetilde{y}(t, x, \{d_k(t)\}_{k=0}^n)$ by the formula

$$\widetilde{y}(t, x, \{d_k(t)\}_{k=0}^n) = \alpha_0 + \sum_{k=0}^{n} \big(\widetilde{y}_k(x - c_k t + d_k(t)) - \alpha_k\big) + \sum_{k=0}^{n-1} \left(g_k\left(\frac{x}{t}\right) - \beta_k\right). \tag{3.2}$$

The main result of the paper is the following theorem on the uniform convergence of a solution of the C. p. (1.1), (1.2) to a system of waves.

Theorem 3.1. *Suppose that Assumptions 2.1, 2.2, 3.1 hold. Then*

1) $\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |y(t, x) - \widetilde{y}(t, x, \{d_k(t)\}_{k=0}^n)| = 0$,

2) $\forall\, k = 0, \dots, n \to \lim_{t\to\infty} d_k'(t) = 0$,

3) *if* $\alpha_k < \beta_k = \alpha_{k+1} < \beta_{k+1}$, *then* $d_k(t) - d_{k+1}(t) \to \infty$ *as* $t \to \infty$.

Remark 3.1. Since $d_k(t) = o(t)$, $k = 0, \dots, n$, and in view of Theorem 2.1, it follows from part 2) of Theorem 3.1 that the function $y(t, x)$ converges on the phase plane to the system of waves $\widetilde{y}(t, x, \{d_k(t)\}_{k=0}^n)$, that is,

$$y_x(y, t) \xrightarrow[t\to\infty]{L_\infty(Y)} \frac{R_0(y)}{\nu'(y)} \xleftarrow[t\to\infty]{L_\infty(Y)} \widetilde{y}_x(y, t, \{d_k(t)\}_{k=0}^n).$$

Remark 3.2. The following result was stated in [16]. Let $f(y) \in C^3([m, M])$, suppose that $f''(y)$ has no accumulation points of zeros on the closed interval $[m, M]$ and the following conditions hold.

1) Assumptions 2.1, 3.1.

2) $\alpha_0 \leqslant \beta_0 \leqslant \alpha_1 < \beta_1 < \dots < \alpha_{n-1} < \beta_{n-1} < \alpha_n \leqslant \beta_n$ (the equality $\beta_0 = \alpha_1$ is possible only for $n = 1$ when $\alpha_0 = \beta_0 = \alpha_1$).

3) $f''(\beta_{k-1}) > 0$, $f''(\alpha_k) > 0$, $k = 1, \dots, n$; $f''(\alpha_0) > 0$ if $\alpha_0 < \beta_0$ and $f'(\alpha_0) = H'(\alpha_0 + 0)$; $f''(\beta_n) > 0$ if $\alpha_n < \beta_n$ and $f'(\beta_n) = H'(\beta_n - 0)$ (conditions of general position).

4) $\nu(y) = \varepsilon y$, $\varepsilon > 0$.

5) The initial function $y_0(x)$ has the form (0.2).

Then there exist $d_k(t) = C(\alpha_k, \beta_k, f'(\alpha_k), f'(\beta_k))\varepsilon \ln t + o(\ln t)$ (if $\alpha_k < \beta_k$), $k = 0, \dots, n$, such that part 1) of Theorem 3.1 holds. Note that the form of the function $C(\,\cdot\,)$ was explicitly written out.

Remark 3.3. Suppose that Assumptions 2.1, 3.1 and the conditions of general position hold and that the monotonic initial function $y_0(x)$ has the form (0.2). Then (see [47], [48]) we have

1) $\sup_{x\in\mathbb{R}} |y(t,x) - \widetilde{y}(t,x,\{d_k(t)\}_{k=0}^n)| = O(1/\sqrt[3]{t}\,)$,

2) if $\{(\beta_k < \alpha_{k+1}, k < n) \vee (k = n)\} \wedge \{(\beta_{k-1} < \alpha_k, k > 1) \vee (k = 0)\}$, then $|d_k'(t)| = O(1/t)$,

3) if $\alpha_k < \beta_k = \alpha_{k+1} < \beta_{k+1}$, then $\exists\, \chi_2 > \chi_1 > 0$, $T > 0$: $\forall\, t \geqslant T \to \chi_1 \sqrt[3]{t} \leqslant d_k(t) - d_{k+1}(t) \leqslant \chi_2 \sqrt[3]{t}$, $|d_k(t)| = O(\sqrt[3]{t})$, $|d_{k+1}(t)| = O(\sqrt[3]{t})$.

Remark 3.4. The requirement of the validity of Assumption 2.2 in the hypotheses of Theorem 3.1 is needed only to enable us to use the result of Theorem 2.1 (see § 2.3). Therefore in view of Remark 2.1, it is possible to drop Assumption 2.2 in the hypotheses of Theorem 3.1.

Remark 3.5. An assertion similar to Theorem 3.1 holds in the non-viscous case:[1] $\nu(y) = \varepsilon y$, $\varepsilon \to 0+$. Suppose that Assumptions 2.1, 3.1 hold and the function $f''(y)$ has no accumulation points of zeros on the closed interval $[m, M]$. In formula (3.2) for the system of waves we set

$$\widetilde{y}_k(x - c_k t + d_k(t)) = \begin{cases} \beta_k & \text{if } \ x - c_k t + d_k(t) < 0, \\ \alpha_k & \text{if } \ x - c_k t + d_k(t) \geqslant 0, \end{cases} \tag{3.3}$$

with $d_k(t) = o(t)$, and if $\exists\, r \geqslant k$: $\beta_r < \alpha_{r+1}$ and $\exists\, l \leqslant k$: $\beta_{l-1} < \alpha_l$, then $d_k(t) \equiv \mathrm{const}_k$. Then a solution of the Cauchy problem (1.1), (0.2) (or (1.1), (1.2) but with $y_0(x)$ a monotonic function) with $\nu(y) = \varepsilon y$, $\varepsilon \to 0+$, converges for large values of time to the system of waves (3.2) consisting of the shock waves (3.3) (where $d_k(t) \equiv \mathrm{const}_k$, $k = 0, \dots, n$, if the initial condition is of the form (0.2)) and rarefaction waves uniformly in x, except for arbitrarily small neighbourhoods of the shock waves in the system of waves (if the initial condition has the form (0.2), then we also have convergence in the norm of $L_1(\mathbb{R}_x)$); see [11], [12], [16], [22], [33], [51]–[53]. In [11] the problem of the decay of the discontinuity (the initial condition of Riemann type (0.2) had the form of a jump: $x_- = x_+$) was solved and the system of waves (3.2) consisting of the shock waves (3.3) with $d_k(t) \equiv 0$ and rarefaction waves was introduced, apparently for the first time. Furthermore, a system

[1]The justification of this passage to the limit (the method of vanishing viscosity) for arbitrary bounded measurable initial conditions was studied intensively in the USSR and USA in the 1950s (Oleinik, Ladyzhenskaya; Hopf, Lax). In Kruzhkov's paper [30] the method of vanishing viscosity was justified under the most general conditions. Note that although the viscosity of the special form $\nu() = \varepsilon y$ was considered in [30], it is easy to show that all the results remain valid in the case $\nu(y) = \varepsilon\widetilde{\nu}(y)$ when $\widetilde{\nu}\,'(y) > 0$ is a sufficiently smooth function. The further development and generalization of this method are described, for example, in the monographs [33], [49], [50].

of waves emerged not as an asymptotic but as an exact solution of the problem. It was shown in [12], [16], [51]–[53] that a solution of the problem (1.1), (0.2) for $\nu(y)=\varepsilon y$, $\varepsilon\to 0+$, has the asymptotics (3.2) consisting of the shock waves (3.3) with $d_k(t)\equiv \mathrm{const}_k$ and rarefaction waves. It was pointed out in [54] (where the asymptotics of the form of a single shock wave was studied) that, because of the insufficiently rapid convergence of the initial function $y_0(x)$ to its limit values at $x=\pm\infty$, the phase shifts of the shock waves for $k=0$ and $k=n$ should be allowed to depend on time.[2] The results of [53], [54] were extended in [22] to the case of an arbitrary monotonic bounded initial function $y_0(x)$.

3.2. Scheme of proof. We divide the set $X^\sigma(t)=\{x\colon y(t,x)\in[\alpha_0+\sigma,\beta_n-\sigma]\}$ into finitely many disjoint subsets depending on $t\geqslant 0$ and σ, $\sigma_0\geqslant\sigma>0$:

$$\begin{aligned}X^\sigma(t)=\Xi_0^\sigma(t)\cup\Theta_1^\sigma(t)\cup\Delta_0^\sigma(t)\cup\Theta_2^\sigma(t)\cup\Xi_1^\sigma(t)\cup\Theta_3^\sigma(t)\cup\Delta_1^\sigma(t)\cup\cdots\\ \cdots\cup\Theta_{2n-2}^\sigma(t)\cup\Xi_{n-1}^\sigma(t)\cup\Theta_{2n-1}^\sigma(t)\cup\Delta_{n-1}^\sigma(t)\cup\Theta_{2n}^\sigma(t)\cup\Xi_n^\sigma(t),\end{aligned}$$

where

$$\Xi_k^\sigma(t)=\begin{cases}\bigl\{x\colon y(t,x)\in[\alpha_k+\sigma,\beta_k-\sigma]\bigr\} & \text{if }\ \alpha_k<\beta_k,\\ \varnothing & \text{if }\ \alpha_k=\beta_k,\end{cases}$$

$$\Delta_k^\sigma(t)=\begin{cases}\bigl\{x\colon y(t,x)\in[\beta_k+\sigma,\alpha_{k+1}-\sigma]\bigr\} & \text{if }\ \alpha_k<\beta_k,\\ \varnothing & \text{if }\ \beta_k=\alpha_{k+1},\end{cases}$$

$$\Theta_{2k+1}^\sigma(t)=\bigl\{x\colon y(t,x)\in(\beta_k-\sigma,\beta_k+\sigma)\cap[\alpha_0+\sigma,\beta_n-\sigma]\bigr\},$$

$$\Theta_{2k}^\sigma(t)=\bigl\{x\colon y(t,x)\in(\alpha_k-\sigma,\alpha_k+\sigma)\cap[\alpha_0+\sigma,\beta_n-\sigma]\bigr\}.$$

By Lemmas 3.1, 3.2 below,

$$\begin{gathered}\forall\,\sigma_0>\sigma>0\ \exists\,\breve T(\sigma)=\max\{T_\Xi(\sigma),T_\Delta(\sigma)\}\colon\forall\,t\geqslant\breve T(\sigma),\\ \forall\,k=0,\dots,n\to\sup_{x\in\Xi_k^\sigma(t)}\bigl|y(t,x)-\widetilde y_k(x-c_kt+d_k(t))\bigr|\leqslant\sigma,\end{gathered}\tag{3.4}$$

$$\forall\,k=0,\dots,n-1\to\sup_{x\in\Delta_k^\sigma(t)}|y(t,x)-g_k(x/t)|\leqslant\sigma,\tag{3.5}$$

where by definition the supremum over the empty set is equal to zero. We note that

1) for $n=1$, $\alpha_0=\alpha_1$, $f'(\alpha_0)\neq H'(\alpha_0+0)=c_1$, $f'(\beta_1)\neq H'(\beta_1-0)=c_1$, formula (3.4) was established in [21], Theorem 8,

2) formula (3.5) generalizes Theorem 3.3 of [13], where the supremum was taken over the set $\Delta_k^{\sigma_0}$, where $\sigma_0>0$ is a fixed number, and $T_\Delta(\sigma)\sim\sigma^{-2}$.

It follows from Theorem 2.1 that

$$\forall\,\sigma_0>\sigma>0\ \exists\,\lambda(\sigma)>0\colon\forall\,t\geqslant T(\lambda(\sigma)),\ x\in\bigcup_{k=0}^n\Xi_k^\sigma(t)\to y_x(t,x)>0.\tag{3.6}$$

[2]In contrast to the non-viscous case, the dependence of the phase shifts of travelling waves arises not only because of insufficiently rapid convergence to zero: $|y_0(x)-\widetilde y_0(x)|\xrightarrow[x\to-\infty]{}0$ or $|y_0(x)-\widetilde y_n(x)|\xrightarrow[x\to+\infty]{}0$ [15], [21], [48] (see also § 1.2), but also due to the interaction of the waves within a system [14], [16], [47], [48].

By the definition of the system of waves (see formula (3.2)) we have

$$\begin{aligned}
&\sup_{x\in\Xi_k^\sigma(t)}\bigl|y(t,x)-\widetilde{y}(t,x,\{d_k(t)\}_{k=0}^n)\bigr| \\
&\quad\leqslant \sup_{x\in\Xi_k^\sigma(t)}\bigl|y(t,x)-\widetilde{y}(x-c_kt+d_k(t))\bigr| + \sum_{\substack{p=0\\ k\geqslant 1}}^{k-1}\sup_{x\in\Xi_k^\sigma(t)}\bigl(\beta_p-\widetilde{y}_p(x-c_pt+d_p(t))\bigr) \\
&\qquad + \sum_{\substack{p=k+1\\ k\leqslant n-1}}^{n}\sup_{x\in\Xi_k^\sigma(t)}\bigl(\widetilde{y}_p(x-c_pt+d_p(t))-\alpha_p\bigr) + \sum_{\substack{p=0\\ k\geqslant 1}}^{k-1}\sup_{x\in\Xi_k^\sigma(t)}\left(\alpha_{p+1}-g_p\left(\frac{x}{t}\right)\right) \\
&\qquad + \sum_{\substack{p=k\\ k\leqslant n-1}}^{n-1}\sup_{x\in\Xi_k^\sigma(t)}\left(g_p\left(\frac{x}{t}\right)-\beta_p\right). \qquad (3.7)
\end{aligned}$$

We obtain from (3.4), (3.5) that

$$\forall\,\sigma_0>\sigma>0,\ t\geqslant \max\{\breve{T}(\sigma),T(\lambda(\sigma))\}\overset{\text{def}}{=}\widetilde{T}(\sigma),$$

$$\forall\,p=0,\dots,k-1\to \sup_{x\in\Xi_k^\sigma(t)}\bigl(\beta_p-\widetilde{y}_p(x-c_pt+d_p(t))\bigr)\leqslant 2\sigma\quad\text{for } k\geqslant 1, \qquad (3.8)$$

$$\forall\,p=k+1,\dots,n\to \sup_{x\in\Xi_k^\sigma(t)}\bigl(\widetilde{y}_p(x-c_pt+d_p(t))-\alpha_p\bigr)\leqslant 2\sigma\quad\text{for } k\leqslant n-1, \qquad (3.9)$$

$$\forall\,p=0,\dots,k-1\to \sup_{x\in\Xi_k^\sigma(t)}\left(\alpha_{p+1}-g_p\left(\frac{x}{t}\right)\right)\leqslant 2\sigma\quad\text{for } k\geqslant 1, \qquad (3.10)$$

$$\forall\,p=k,\dots,n-1\to \sup_{x\in\Xi_k^\sigma(t)}\left(g_p\left(\frac{x}{t}\right)-\beta_p\right)\leqslant 2\sigma\quad\text{for } k\leqslant n-1. \qquad (3.11)$$

For example, let us prove the inequalities (3.8); (3.9)–(3.11) are proved in similar fashion. We set $x_{p,\Xi}^{\sigma,+}(t)=\max\{x\colon x\in\Xi_p^\sigma(t)\}$ for $\Xi_p^\sigma(t)\neq\varnothing$. It follows from (3.4) that

$$\forall\,\sigma_0>\sigma>0,\ t\geqslant T_\Xi(\sigma)\to \bigl|\beta_p-\sigma-\widetilde{y}_p(x_{p,\Xi}^{\sigma,+}(t)-c_pt+d_p(t))\bigr|\leqslant\sigma.$$

By Remark 1.5, the function $\widetilde{y}_p(x-c_pt+d_p(t))$ is increasing with respect to x. Therefore,

$$\forall\,\sigma_0>\sigma>0,\ t\geqslant T_\Xi(\sigma),\ x\geqslant x_{p,\Xi}^{\sigma,+}(t)\to \beta_p-\widetilde{y}_p(x-c_pt+d_p(t))\leqslant 2\sigma.$$

It remains to observe that

$$\forall\,\sigma_0>\sigma>0,\ t\geqslant\max\{T_\Xi(\sigma),T(\lambda(\sigma))\},\ x\in\Xi_k^\sigma(t)\to x\geqslant x_{p,\Xi}^{\sigma,+}(t).$$

Thus, it follows from (3.8)–(3.11) that

$$\begin{aligned}
&\forall\,\sigma_0>\sigma>0,\ t\geqslant\widetilde{T}(\sigma),\ \forall\,k=0,\dots,n \\
&\to \sup_{x\in\Xi_k^\sigma(t)}\bigl|y(t,x)-\widetilde{y}(t,x,\{d_k(t)\}_{k=0}^n)\bigr|\leqslant(4n+1)\sigma. \qquad (3.12)
\end{aligned}$$

Similarly, we obtain from the inequality

$$\begin{aligned}
&\sup_{x\in\Delta_k^\sigma(t)}\left|y(t,x)-\widetilde{y}(t,x,\{d_k(t)\}_{k=0}^n)\right| \\
&\quad\leqslant \sup_{x\in\Delta_k^\sigma(t)}\left|y(t,x)-g_k\left(\frac{x}{t}\right)\right| + \sum_{p=0}^{k}\sup_{x\in\Delta_k^\sigma(t)}\left(\beta_p-\widetilde{y}_p(x-c_pt+d_p(t))\right) \\
&\quad+\sum_{p=k+1}^{k}\sup_{x\in\Delta_k^\sigma(t)}\left(\widetilde{y}_p(x-c_pt+d_p(t))-\alpha_p\right)+\sum_{\substack{p=0\\k\geqslant 1}}^{k-1}\sup_{x\in\Delta_k^\sigma(t)}\left(\alpha_{p+1}-g_p\left(\frac{x}{t}\right)\right) \\
&\quad+\sum_{\substack{p=k+1\\k\leqslant n-1}}^{n-1}\sup_{x\in\Delta_k^\sigma(t)}\left(g_p\left(\frac{x}{t}\right)-\beta_p\right)
\end{aligned}$$

that

$$\begin{gathered}
\forall\,\sigma_0>\sigma>0,\ t\geqslant\widetilde{T}(\sigma),\ \forall\,k=0,\dots,n-1 \\
\to\sup_{x\in\Delta_k^\sigma(t)}\left|y(t,x)-\widetilde{y}(t,x,\{d_k(t)\}_{k=0}^n)\right|\leqslant(4n+1)\sigma.
\end{gathered}\tag{3.13}$$

We observe that the system of waves $\widetilde{y}(t,x,\{d_k(t)\}_{k=0}^n)$ defined in (3.2) is a non-decreasing function with respect to x since it can be represented in the form of a sum of travelling waves (increasing functions in view of Remark 1.5), rarefaction waves (non-decreasing functions by Definition 3.1), and a constant. Therefore we have from (3.12), (3.13) that

$$\begin{gathered}
\forall\,\sigma_0>\sigma>0,\ t\geqslant\widetilde{T}(\sigma),\ \forall\,l=1,\dots,2n \\
\to\sup_{x\in\Theta_l^\sigma(t)}\left|y(t,x)-y(t,x,\{d_k(t)\}_{k=0}^n)\right|\leqslant(4n+3)\sigma.
\end{gathered}\tag{3.14}$$

The validity of part 1) of Theorem 3.1 follows from (3.12)–(3.14) and part 9) of Theorem 1.1.

Remark 3.6. When Assumptions 2.1, 2.2, 3.1 hold and the initial condition is monotonic, the arguments above imply that (see [48])

$$\begin{gathered}
\forall\,\sigma_0>\sigma>0,\ t\geqslant\max\{T_\Xi(\sigma),T_\Delta(\sigma)\}=\breve{T}(\sigma) \\
\to\sup_{x\in\mathbb{R}}\left|y(t,x)-\widetilde{y}(t,x,\{d_k(t)\}_{k=0}^n)\right|\leqslant(4n+3)\sigma.
\end{gathered}$$

Thus, we have reduced the proof of part 1) of Theorem 3.1 to that of formulae (3.4), (3.5). In the course of establishing these formulae, we shall prove the validity of part 2) of Theorem 3.1 as a by-product (see Corollary 3.1 below).

To prove part 3) of Theorem 3.1 it is sufficient to observe that (by formulae (3.1), (3.6) and Theorem 2.1) when $\alpha_k<\beta_k=\alpha_{k+1}<\beta_{k+1}$, for $\sigma_0>\sigma>0$

and $t \geqslant T(\lambda(\sigma))$ we have

$$d_k(t) - d_{k+1}(t) > \int_{(\alpha_k+\beta_k)/2}^{\beta_k-\sigma} \frac{dy}{y_x(y,t)} + \int_{\alpha_{k+1}+\sigma}^{(\alpha_{k+1}+\beta_{k+1})/2} \frac{dy}{y_x(y,t)}$$
$$\geqslant \int_{(\alpha_k+\beta_k)/2}^{\beta_k-\sigma} \frac{dy}{R_0(y)/\nu'(y)+\sigma} + \int_{\alpha_{k+1}+\sigma}^{(\alpha_{k+1}+\beta_{k+1})/2} \frac{dy}{R_0(y)/\nu'(y)+\sigma}.$$

However, the last two integrals increase unboundedly as $\sigma > 0$ decreases. This follows from the facts that $\nu'(y) \geqslant \overline{\nu} > 0$ for $y \in [m, M]$, $R_0(\beta_k) = 0$, and $R_0(y)$ is a smooth non-negative function.

3.3. Investigation of uniform convergence to a system of waves on the regions corresponding to the asymptotic behaviour 'travelling wave'. We now use Theorem 2.1 to prove formula (3.4) and as a corollary establish the validity of part 2) of Theorem 3.1.

Lemma 3.1. *Suppose that Assumptions 2.1, 2.2 hold. Then*

$$\exists\,\sigma_0 > 0\colon \forall\, k = 0,\dots,n,\ \ \sigma_0 > \sigma > 0\ \exists\,\omega_k(\sigma) > 0\colon \forall\, t \geqslant T(\omega_k(\sigma)) \overset{\mathrm{def}}{=} T_{\Xi}^k(\sigma)$$
$$\to \sup_{x\in\Xi_k^\sigma(t)} \bigl|y(t,x) - \widetilde{y}_k(x - c_k t + d_k(t))\bigr| \leqslant \sigma.$$

Proof. Let $\alpha_k < \beta_k$. It follows from Theorem 2.1 that

$$\exists\,\sigma_0 > 0\colon \forall\,\sigma_0 > \sigma > 0\ \exists\,\breve{\omega}_k(\sigma) > 0\colon t \geqslant T(\breve{\omega}_k(\sigma)),\ y \in \left[\alpha_k + \frac{\sigma}{2},\ \beta_k - \frac{\sigma}{2}\right]$$
$$\to \left(\frac{R_0(y)}{\nu'(y)} + \breve{\omega}_k(\sigma)\right)^{-1} y_x(y,t) \leqslant 1 \leqslant \left(\frac{R_0(y)}{\nu'(y)} - \breve{\omega}_k(\sigma)\right)^{-1} y_x(y,t). \tag{3.15}$$

Setting

$$H_{\pm\sigma,k}(y) = \int_{y_k^*}^{y} \left(\frac{R_0(w)}{\nu'(w)} \pm \breve{\omega}_k(\sigma)\right)^{-1} dw,$$

we integrate (3.15) with respect to dx from $c_k t - d_k(t)$ to x taking into account formula (3.1). It follows that for any $t \geqslant T(\breve{\omega}_k(\sigma))$ and $x \in \Xi_k^{\sigma/2}(t)$ such that

$$H_{\sigma,k}(y(t,x)) \leqslant x - (c_k t - d_k(t)) \leqslant H_{-\sigma,k}(y(t,x)),$$

we have

$$\widetilde{y}_k^- \overset{\mathrm{def}}{=} H_{-\sigma,k}^{-1}(x - c_k t + d_k(t)) \leqslant y(t,x) \leqslant H_{\sigma,k}^{-1}(x - c_k t + d_k(t)) \overset{\mathrm{def}}{=} \widetilde{y}_k^+. \tag{3.16}$$

By Remark 2.2 we have

$$H_{0,k}^{-1}(x - c_k t + d_k(t)) = \widetilde{y}_k(x - c_k t + d_k(t)) \overset{\mathrm{def}}{=} \widetilde{y}_k.$$

Assuming for definiteness that $\widetilde{y}_k \geqslant y_k^*$, we obtain $\widetilde{y}_k^+ \geqslant \widetilde{y}_k \geqslant \widetilde{y}_k^- \geqslant y_k^*$.

We estimate $\widetilde{y}_k - \widetilde{y}_k^-$ and $\widetilde{y}_k^+ - \widetilde{y}_k$ from above on the set

$$\widetilde{\Xi}_k^{\sigma/2}(t) = \left\{x \colon \widetilde{y}_k(x - c_k t + d_k(t)) \in \left[\alpha_k + \frac{\sigma}{2},\, \beta_k - \frac{\sigma}{2}\right]\right\}$$

depending on σ. To this end we observe that

$$\begin{aligned}
\int_{y_k^*}^{\widetilde{y}_k^-} \frac{dw}{R_0(w)/\nu'(w) - \breve{\omega}_k(\sigma)} &= x - c_k t + d_k(t) \\
&= \int_{y_k^*}^{\widetilde{y}_k^-} \frac{dw}{R_0(w)/\nu'(w)} + \int_{\widetilde{y}_k^-}^{\widetilde{y}_k} \frac{dw}{R_0(w)/\nu'(w)} \\
&\geqslant \int_{y_k^*}^{\widetilde{y}_k^-} \frac{dw}{R_0(w)/\nu'(w)} + \frac{(\widetilde{y}_k - \widetilde{y}_k^-)}{\max_{y \in [\widetilde{y}_k^-, \widetilde{y}_k]} (R_0(y)/\nu'(y))}\,.
\end{aligned}$$

Therefore for $x \in \widetilde{\Xi}_k^{\sigma/2}(t)$ (that is, for $y_k^* \leqslant \widetilde{y}_k^- \leqslant \widetilde{y}_k \leqslant \beta_k - \sigma/2$) we have

$$\begin{gathered}
\exists\, \breve{\omega}_k(\sigma) \geqslant \omega_k(\sigma) > 0 \colon \\
\widetilde{y}_k - \widetilde{y}_k^- \leqslant \max_{y \in [\widetilde{y}_k^-, \widetilde{y}_k]} \left(\frac{R_0(y)}{\nu'(y)}\right) \int_{y_k^*}^{\widetilde{y}_k^-} \frac{\omega_k(\sigma)\, dw}{(R_0(w)/\nu'(w) - \omega_k(\sigma))(R_0(w)/\nu'(w))} \leqslant \frac{\sigma}{2}\,.
\end{gathered}$$

It is shown in similar fashion that

$$\begin{aligned}
&\exists\, \breve{\omega}_k(\sigma) \geqslant \omega_k(\sigma) > 0 \colon \ \forall\, x \in \widetilde{\Xi}_k^{\sigma/2}(t) \\
&\to \widetilde{y}_k^+ - \widetilde{y}_k \leqslant \max_{y \in [\widetilde{y}_k, \widetilde{y}_k^+]} \left(\frac{R_0(y)}{\nu'(y)} + \omega_k(\sigma)\right) \\
&\qquad \times \int_{y_k^*}^{\widetilde{y}_k} \frac{\omega_k(\sigma)\, dw}{(R_0(w)/\nu'(w) + \omega_k(\sigma))(R_0(w)/\nu'(w))} \leqslant \frac{\sigma}{2}\,.
\end{aligned}$$

We use these estimates to rewrite formula (3.16) in the form

$$\begin{gathered}
\forall\, t \geqslant T(\omega_k(\sigma)),\ x \in \Xi_k^{\sigma/2}(t) \cap \widetilde{\Xi}_k^{\sigma/2}(t) \\
\to \widetilde{y}_k(x - c_k t + d_k(t)) - \frac{\sigma}{2} \leqslant y(t,x) \leqslant \widetilde{y}_k(x - c_k t + d_k(t)) + \frac{\sigma}{2} \\
\Rightarrow\ y(t,x) - \frac{\sigma}{2} \leqslant \widetilde{y}_k(x - c_k t + d_k(t)) \leqslant y(t,x) + \frac{\sigma}{2}\,.
\end{gathered} \tag{3.17}$$

Consequently,

$$x \in \Xi_k^{\sigma/2}(t) \cap \widetilde{\Xi}_k^{\sigma/2}(t) \ \Rightarrow\ x \in \Xi_k^{\sigma/2}(t) \cap \Xi_k^{\sigma}(t). \tag{3.18}$$

Formulae (3.17), (3.18) imply the validity of the lemma.

It is easy to see that formula (3.4) for $T_\Xi(\sigma) = \max_{k=0,\dots,n} T_\Xi^k(\sigma)$ follows from Lemma 3.1.

Corollary 3.1. *Suppose that Assumptions 2.1, 2.2 hold. Then*

$$\forall\, k = 0, \dots, n\ \exists\, \rho_k > 0 \colon \forall\, \sigma_0 > \sigma > 0,\ t \geqslant T((\omega_k(\rho_k \sigma))^4) \to |d_k'(t)| \leqslant \sigma.$$

Proof. Let $\alpha_k < \beta_k$. We set $z(t,x) = y(t,x) - \widetilde{y}_k(x - c_k t + d_k(t))$. It follows from Lemma 3.1 that

$$\forall\, \sigma_0 > \sigma > 0,\ t \geqslant T(\omega_k(\sigma)) \to \|z(t,x)\|_{C(\Xi_k^\sigma(t))} \leqslant \sigma.$$

By applying the inequality

$$\begin{gathered}\forall\, D > 0,\ b,\ a\colon b > a\ \exists\, K(b-a,D) > 0\colon \forall\, c \leqslant a,\ d \geqslant b,\ z(x) \in C^2([c,d])\colon\\ \|z(x)\|_{C([c,d])} \leqslant 1,\ \|z_{xx}(x)\|_{C([c,d])} \leqslant D\\ \to \|z_x(x)\|_{C([c,d])} \leqslant K(b-a,D)\|z(x)\|^{1/2}_{C([c,d])}\end{gathered}$$

to the functions $z(t,x)$ and $z_x(t,x)$ of the argument $x \in \Xi_k^\sigma(t)$, we obtain, in view of part 7) of Theorem 1.1, that

$$\begin{gathered}\exists\, \breve{\rho}_k > 0\colon \forall\, t \geqslant T((\omega_k(\breve{\rho}_k\sigma))^4),\ r = 1,2\\ \to \max_{x\in\Xi_k^\sigma(t)}\left|\frac{\partial^r y(t,x)}{\partial x^r} - \widetilde{y}_k^{(r)}(x - c_k t + d_k(t))\right| \leqslant \sigma.\end{gathered} \tag{3.19}$$

Since $y(t,x)$ satisfies (1.1) and $d_k(t)$ is determined by (3.1), we have

$$\begin{aligned}&- y_x(t, c_k t - d_k(t))(c_k - d_k'(t)) + f'\bigl(y(t, c_k t - d_k(t))\bigr) y_x(t, c_k t - d_k(t))\\ &\quad = \nu'\bigl(y(t, c_k t - d_k(t))\bigr) y_{xx}(t, c_k t - d_k(t))\\ &\qquad + \nu''\bigl(y(t, c_k t - d_k(t))\bigr)\bigl(y_x(t, c_k t - d_k(t))\bigr)^2.\end{aligned} \tag{3.20}$$

Since

$$-c_k\widetilde{y}_k'(0) + f'(\widetilde{y}_k(0))\widetilde{y}_k'(0) = \nu'(\widetilde{y}_k(0))\widetilde{y}_k''(0) + \nu''(\widetilde{y}_k(0))(\widetilde{y}_k'(0))^2, \qquad \widetilde{y}_k'(0) > 0,$$

we obtain the corollary from (3.19), (3.20).

3.4. Investigation of uniform convergence to a system of waves on the regions corresponding to the asymptotic behaviour 'rarefaction wave'. We now use the comparison principle to prove formula (3.5) and thus complete the proof of Theorem 3.1.

We encountered a version of the comparison principle in § 1 (see part 1) of Theorem 1.1). However, in what follows we shall need the following stronger assertion [13], [34].

Comparison principle. *Suppose that $W(t,x)$, $V(t,x)$ are continuous bounded functions in the strip $0 \leqslant t \leqslant T$ satisfying the conditions*

$$\begin{aligned}L_{f,\nu}W(t,x) &= \frac{\partial W(t,x)}{\partial t} + \frac{\partial f(W(t,x))}{\partial x} - \frac{\partial^2 \nu(W(t,x))}{\partial x^2} \geqslant 0,\\ L_{f,\nu}V(t,x) &= \frac{\partial V(t,x)}{\partial x} + \frac{\partial f(V(t,x))}{\partial x} - \frac{\partial^2 \nu(V(t,x))}{\partial x^2} \leqslant 0\end{aligned}$$

everywhere in the strip $0 < t \leqslant T$ except possibly for finitely many curves of the form $x = x(t)$ that are continuous on $[0,T]$. Suppose further that the partial *derivatives of the functions $W(t,x)$, $V(t,x)$ involved in $L_{f,\nu}$ are uniformly bounded and continuous in the strip $0<t\leqslant T$ outside these curves and satisfy*

$$\frac{\partial(W(t,x)-V(t,x))}{\partial x}\bigg|_{x=x(t)-0} \geqslant \frac{\partial(W(t,x)-V(t,x))}{\partial x}\bigg|_{x=x(t)+0} \tag{3.21}$$

on each such curve. If $W(0,x)\geqslant V(0,x)$, then $W(t,x)\geqslant V(t,x)$ for $0\leqslant t\leqslant T$.

Proof. Setting $z(t,x)=W(t,x)-V(t,x)$, we obtain

$$\begin{aligned}
Lz &\overset{\mathrm{def}}{=} L_{f,\nu}V - L_{f,\nu}W = (\nu'(V)z_x)_x + \bigl(\nu'(W)-\nu'(V)\bigr)W_{xx} + \nu''(V)W_x z_x \\
&\qquad + \bigl(\nu''(W)-\nu''(V)\bigr)(W_x)^2 - f'(V)z_x - \bigl(f'(W)-f'(V)\bigr)W_x - z_t \\
&= (\nu'(V)z_x)_x - z_t - \bigl(f'(V)-\nu''(V)W_x\bigr)z_x \\
&\qquad - \biggl(\int_0^1 \bigl(f''(V+\theta(W-V)) - \nu''(V+\theta(W-V))W_{xx} \\
&\qquad\qquad - \nu'''(V+\theta(W-V))(W_x)^2\bigr)\,d\theta\biggr) z \leqslant 0
\end{aligned}$$

everywhere in the strip $0\leqslant t\leqslant T$ except possibly for finitely many curves of the form $x=x(t)$ that are continuous on $[0,T]$. Consider the function $\widetilde{z}(t,x)=z(t,x)e^{-t\mu}$, where

$$\begin{aligned}
\mu = 2 + \sup_{\substack{0\leqslant t\leqslant T\\ x\in\mathbb{R}}} \int_0^1 &\bigl(\nu''(V+\theta(W-V))W_{xx} + \nu'''(V+\theta(W-V))(W_x)^2 \\
&- f''(V+\theta(W-V))\bigr)\,d\theta < \infty.
\end{aligned}$$

We set

$$\tilde{L}\widetilde{z} \overset{\mathrm{def}}{=} L\widetilde{z} - \mu\widetilde{z} = e^{t\mu}Lz \leqslant 0, \qquad 0<t\leqslant T, \quad x\neq x(t). \tag{3.22}$$

By the maximum principle (see § 2.2), the minimum of the function $\widetilde{z}(t,x)$ in the strip $0\leqslant t\leqslant T$ is attained either on the line $t=0$ or on one of the curves $x=x(t)$.

Suppose the opposite, that is, suppose that the condition $z(t,x)\geqslant 0$ does not hold in the strip $0\leqslant t\leqslant T$. Then the minimum of the function $\widetilde{z}(t,x)$ is attained at a point $(t_0,x(t_0))$ and $\widetilde{z}(t_0,x(t_0))<0$ (and so $0<t_0\leqslant T$). In view of condition (3.21) we have $\widetilde{z}_x(t_0,x(t_0))=0$. Since $(t_0,x(t_0))$ is a minimum point of the function $\widetilde{z}(t,x)$ in the strip $0\leqslant t\leqslant T$, we have

$$\widetilde{z}_{xx}(t_0,x(t_0)\pm 0)\geqslant 0, \qquad \widetilde{z}_t(t_0,x(t_0)\pm 0)\leqslant 0.$$

Therefore,

$$\begin{gathered}
\exists\,\varepsilon>0\colon \forall\, t\in(t_0-\varepsilon,t_0+\varepsilon)\cap(0,T],\ x\in\bigl(x(t_0)-\varepsilon,x(t_0)\bigr)\cup\bigl(x(t_0),x(t_0)+\varepsilon\bigr) \\
\to \tilde{L}\widetilde{z}(t,x)\geqslant -\widetilde{z}(t_0,x(t_0))>0.
\end{gathered}$$

We have arrived at a contradiction to condition (3.22).

Lemma 3.2. *Suppose that Assumptions 2.1, 3.1 hold. Then*

$$\forall\, k = 0, \dots, n-1\ \exists\, \sigma_0 > 0\colon \forall\, \sigma_0 > \sigma > 0\ \exists\, T^k_\Delta(\sigma) > 0\colon \forall\, t \geqslant T^k_\Delta(\sigma)$$
$$\to \sup_{x \in \Delta^\sigma_k(t)} \left| y(t,x) - g_k\left(\frac{x}{t}\right)\right| \leqslant \sigma.$$

Proof. Let a sufficiently small $\sigma > 0$ be fixed and let $\beta_k < \alpha_{k+1}$. We assume without loss of generality that

$$\forall\, x \in \mathbb{R} \to y_- - \delta(\sigma) \leqslant y_0(x) \leqslant y_+ + \delta(\sigma)$$

and $y_0(x)$ is a continuous function (see the beginning of § 2.3). Recall that we have assumed from the outset (see § 1.1) that

$$\forall\, x \in \mathbb{R} \to m \leqslant y_0(x) \leqslant M.$$

First we construct a function $W_k(t,x)$ that bounds $V(t,x) = y(t,x)$ from above on the interval $\widetilde{\Delta}^{2\sigma}_k(t) = \{x\colon g_k(x/t) \in [\beta_k + 2\sigma,\, \alpha_{k+1} - 2\sigma]\}$. To do this we define on the closed interval $[\beta_k, \beta_n + 2]$ a fourfold smooth function $\tilde{f}(y)$ that coincides with $f(y)$ for $y \in [\beta_k + \sigma,\, \beta_n + \delta(\sigma)]$ and is such that

1) $f'(y) \geqslant \tilde{f}'(y)$ for $y \in [\beta_k + \sigma/2, \beta_k + \sigma]$, $\tilde{f}'(y) = f'(y)$ for $y \in [\beta_k, \beta_k + \sigma/2]$,

2) $\tilde{f}''(y) > 0$ for $y \in [\beta_k + 2\sigma/3, \beta_k + \sigma]$,

3) $\exists\, 1/2 \geqslant \chi > 0$, $y^-_k(x - c^-_k t)\colon \widetilde{c}^-_k = \tilde{f}'(\beta_k + 3\sigma/4)$, $L_{\tilde{f},\nu} y^-_k(x - c^-_k t) \equiv 0$, $y^-_k(0) = \beta_k + 2\sigma/3$, $\widetilde{y}^-_k(s) \to \beta_k + \chi\sigma$ as $s \to -\infty$, and $\widetilde{y}^-_k(s) \to \beta_n + 2$ as $s \to +\infty$,

4) $\exists\, y^+_k(x - c^+_k t)\colon \widetilde{c}^+_k = \tilde{f}'(\alpha_{k+1} - \sigma)$, $L_{\tilde{f},\nu} y^+_k(x - c^+_k t) \equiv 0$, $y^+_k(0) = \alpha_{k+1} - \sigma/2$, $y^+_k(s) \to \alpha_{k+1} - \sigma$ as $s \to -\infty$, and $y^+_k(s) \to \beta_n + 1$ as $s \to +\infty$.

If $M < \beta_n + 2$, then in order that the action of the operator $L_{\tilde{f},\nu}$ be defined on functions taking values in the closed interval $[m, \beta_n + 2]$, we also need to extend $\nu(y)$ fourfold smoothly to the closed interval $[M, \beta_n + 2]$ while preserving the property $\nu'(y) > 0$. We can construct a function $\tilde{f}(y)$ with properties 1)–4) by setting $\tilde{f}''(y) \gg f''(y) > 0$ for $y \in [\beta_k + 3\sigma/4, \beta_k + 0.99\sigma]$ and choosing $\delta(\sigma)$ to be sufficiently small. This can easily be shown using Remark 1.7. In view of part 6) of Theorem 1.1, we can assume without loss of generality that $\tilde{f}(y) = f(y)$ for $y \in [\beta_n + \delta(\sigma), \beta_n + 2]$.

We define a function $H_k(\gamma)$ on the closed interval $[\tilde{f}'(\beta_k + \sigma/2), \tilde{f}'(\alpha_{k+1} - \sigma/2)]$ as follows:

$$\forall\, y \in \left[\beta_k + \frac{\sigma}{2}, \alpha_{k+1} - \frac{\sigma}{2}\right] \to H_k(\tilde{f}'(y)) = y.$$

We set

$$\breve{W}_k(t,x) = \begin{cases} y^-_k(x - c^-_k t) & \text{if } \ x < x^-_k(t), \\ H_k((x + \theta_k(t))/t) & \text{if } \ x^-_k(t) \leqslant x < x^+_k(t), \\ y^+_k(x - c^+_k t) & \text{if } \ x \geqslant x^+_k(t). \end{cases}$$

The functions $(x_k^-(t), x_k^+(t), \theta_k(t)) \overset{\text{def}}{=} \xi(t)$ are determined by the condition of continuity of $\breve{W}_k(t,x)$ in x and the two inequalities (see formula (3.21))

$$\begin{aligned} \frac{\partial \breve{W}_k(t,x)}{\partial x}\bigg|_{x=x_k^-(t)-0} &\geqslant \frac{\partial \breve{W}_k(t,x)}{\partial x}\bigg|_{x=x_k^-(t)+0}, \\ \frac{\partial \breve{W}_k(t,x)}{\partial x}\bigg|_{x=x_k^+(t)-0} &\geqslant \frac{\partial \breve{W}_k(t,x)}{\partial x}\bigg|_{x=x_k^+(t)+0}, \end{aligned} \tag{3.23}$$

where we regard the second inequality as an equation (it can be shown that this choice ensures a sharper upper bound for $y(t,x)$). It turns out (see [48]) that if $t \geqslant t_k(\sigma)$, where $t_k(\sigma)$ is sufficiently large, then there exists a (unique) solution $\vec{\xi}(t)$ of the three equations

$$y_k^-(x_k^- t - c_k^- t) = H_k\biggl(\frac{x_k^- t + \theta_k}{t}\biggr), \qquad H_k\biggl(\frac{x_k^+ t + \theta_k}{t}\biggr) = y_k^+(x_k^+ t - c_k^+ t),$$
$$\frac{\partial \breve{W}_k(t,x)}{\partial x}\bigg|_{x=x_k^+-0} = \frac{\partial \breve{W}_k(t,x)}{\partial x}\bigg|_{x=x_k^++0},$$

and the remaining inequality (3.23) holds on this solution. Furthermore, for $t \geqslant t_k(\sigma)$ we have

a) $y_k^-(x_k^-(t) - c_k^- t) \leqslant \beta_k + \sigma$,
b) $y_k^+(x_k^+(t) - c_k^+ t) \geqslant \alpha_{k+1} - \sigma$,
c) $\vec{\xi}(t)$ is a smooth vector-function and the estimate

$$\theta_k'(t) = \frac{\omega_k(\sigma)}{\sqrt{t}} + \rho_k(\sigma) O\biggl(\frac{1}{t}\biggr)$$

holds, where $\omega_k(\sigma) > 0$ takes large values for small σ.

From properties 3), 4) of the function $\tilde{f}(y)$, properties a)–c) of the solution $\vec{\xi}(t)$, and the definition of the function $H_k(\,\cdot\,)$, it is easy to show that if $t_k(\sigma)$ is sufficiently large, then

$$\forall\, t \geqslant t_k(\sigma),\ x \neq x_k^-(t),\ x \neq x_k^+(t) \to L_{\tilde{f},\nu} \breve{W}_k(t,x) \geqslant 0.$$

Since $\breve{W}_k(t,x)$ is an increasing function of x, it follows from property 1) of the function $\tilde{f}(y)$ that

$$\forall\, t \geqslant t_k(\sigma),\ x \neq x_k^-(t),\ x \neq x_k^+(t) \to L_{f,\nu} \breve{W}_k(t,x) \geqslant L_{\tilde{f},\nu} \breve{W}_k(t,x) \geqslant 0.$$

We set

$$W_k(t,x) = \breve{W}_k(t + t_k(\sigma), x + b_k(\sigma)), \qquad b_k(\sigma) = \widetilde{x}_k^+(\sigma) - \widetilde{x}_k^-(\sigma),$$
$$\widetilde{x}_k^-(\sigma) = \sup\{x \colon \forall\, x' < x \to y_0(x') < \beta_k + \chi\sigma\} > -\infty,$$
$$W_k(0, \widetilde{x}_k^+(\sigma)) = \beta_n + \delta(\sigma), \qquad \widetilde{x}_k^+(\sigma) < +\infty.$$

Then

$$\forall\, x \in \mathbb{R} \to y_0(x) \leqslant W_k(0,x),$$
$$\forall\, t \geqslant 0,\ x \neq x_k^-(t+t_k(\sigma)) - b_k(\sigma),\ x \neq x_k^+(t+t_k(\sigma)) - b_k(\sigma)$$
$$\to L_{f,\nu} W_k(t,x) = L_{f,\nu} \breve{W}_k(t+t_k(\sigma), x+b_k(\sigma)) \geqslant 0.$$

If we require in addition that $x \in \breve{\Delta}_k^{\sigma}(t)$, where

$$\breve{\Delta}_k^{\sigma}(t) = \{x\colon W_k(t,x) \in [\beta_k+\sigma,\ \alpha_{k+1}-\sigma]\},$$

then the last inequality can be rewritten in the form

$$L_{f,\nu} H_k(\gamma(t,x,\sigma)) \geqslant 0,$$

where

$$\gamma(t,x,\sigma) = \frac{x+\theta_k(t+t_k(\sigma))+b_k(\sigma)}{t+t_k(\sigma)}\,.$$

By the comparison principle, we then have

$$\forall\, t \geqslant t_k(\sigma),\ x \in \breve{\Delta}_k^{\sigma}(t) \to y(t,x) \leqslant H_k(\gamma(t,x,\sigma)) = g_k(\gamma(t,x,\sigma)). \tag{3.24}$$

But according to property c) of the solution $\vec{\xi}(t)$, there exists a sufficiently large $t_k(\sigma)$ such that

$$\forall\, t \geqslant t_k(\sigma),\ x \in \breve{\Delta}_k^{\sigma}(t) \to g_k\left(\frac{x}{t}\right) - \sigma \leqslant H_k(\gamma(t,x,\sigma)) \leqslant g_k\left(\frac{x}{t}\right) + \sigma. \tag{3.25}$$

Inequalities (3.24), (3.25) imply the final upper bound for $y(t,x)$:

$$\forall\, t \geqslant t_k(\sigma),\ x \in \widetilde{\Delta}_k^{2\sigma}(t) \to y(t,x) \leqslant g_k\left(\frac{x}{t}\right) + \sigma. \tag{3.26}$$

In order to bound the function $y(t,x)$ from below, we need to bound from above (as was done earlier) the function $\overline{y}(t,x) = -y(t,-x)$ that is a solution of the C. p.

$$\frac{\partial \overline{y}}{\partial t} + \frac{\partial \bar{f}(\overline{y})}{\partial x} = \frac{\partial^2 \overline{\nu}(\overline{y})}{\partial x^2}\,, \qquad \overline{y}(0,x) = -y_0(-x),$$

where $\bar{f}(\overline{y}) = f(-\overline{y})$ and $\overline{\nu}(\overline{y}) = -\nu(-\overline{y})$. As a result we obtain a similar inequality,

$$\forall\, t \geqslant t_k(\sigma),\ x \in \widetilde{\Delta}_k^{2\sigma}(t) \to y(t,x) \geqslant g_k\left(\frac{x}{t}\right) - \sigma. \tag{3.27}$$

It follows from (3.26), (3.27) that

$$\forall\, t \geqslant t_k(\sigma) \stackrel{\text{def}}{=} T_{\Delta}^k(3\sigma) \to \sup_{x \in \Delta_k^{3\sigma}(t)} \left| y(t,x) - g_k\left(\frac{x}{t}\right) \right| \leqslant \sigma.$$

Thus, the lemma is proved.

It is easy to see that formula (3.5) for $T_\Delta(\sigma) = \max_{k=0,\dots,n-1} T^k_\Delta(\sigma)$ follows from Lemma 3.2.

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