Optimal recovery of function from inaccurate information on it's Radon transform

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Integral geometry and CT study analytical and numerical methods of function recovery from different sort of tomographic information. Which is operators, that map function to it's integrals over some set of manifolds.

- radial integration;
- Radon transform;
- X-ray transform;
- Minkowski-Funk transform (integration over great circles).

For particular classes of functions there exist inverse formulas (and corresponding numerical methods), which produce an error in case the information is inaccurate. This error isn't necessarily the smallest possible. We discover methods, that minimize the error of recovery and call them optimal.

- Radon transform is given by the formula

$$Rf(\theta, s) = \int_{x\theta=s} f(x)dx.$$

It's defined on a cylinder $Z = \mathbb{S}^{d-1} \times \mathbb{R}$;

- Consider set of functions $f \in L_2(\mathbb{R}^d)$, which satisfy $|\xi|^\alpha \widehat{f}(\xi) \in L_2(\mathbb{R}^d)$ and define operator $(-\Delta)^{\alpha/2}$, $\alpha \geq 0$ by

$$\widehat{(-\Delta)^{\alpha/2}} f(\xi) = |\xi|^\alpha \widehat{f}(\xi),$$

where $\widehat{f}$ is a Fourier transform;

- Consider class of functions

$$W = \{f \in L_2(\mathbb{R}^d) : \|(-\Delta)^{\alpha/2} f\|_{L_2(\mathbb{R}^d)} \leq 1; \quad Rf \in L_2(Z)\}.$$

- Measured information is a function $g \in L_2(Z)$, such that

$$\|Rf - g\|_{L_2(Z)} \leq \delta;$$

- Methods of recovery are called arbitrary maps

$$m : L_2(Z) \to L_2(\mathbb{R}^d);$$

- Error of the method

$$e(\delta, m) = \sup_{\substack{f \in W, g \in L_2(Z) \\ \|Rf - g\|_{L_2(Z)} \leq \delta}} \|f - m(g)\|_{L_2(\mathbb{R}^d)};$$

- Error of optimal recovery

$$E(\delta) = \inf_{m : L_2(Z) \to L_2(\mathbb{R}^d)} e(\delta, m).$$

$$
\begin{tikzcd}
W \arrow[r, "id"] \arrow[d, "R"'] & L_2(\mathbb{R}^d) \\
L_2(Z) \arrow[ur, "m"'] &
\end{tikzcd}
$$

- Method m is optimal, if

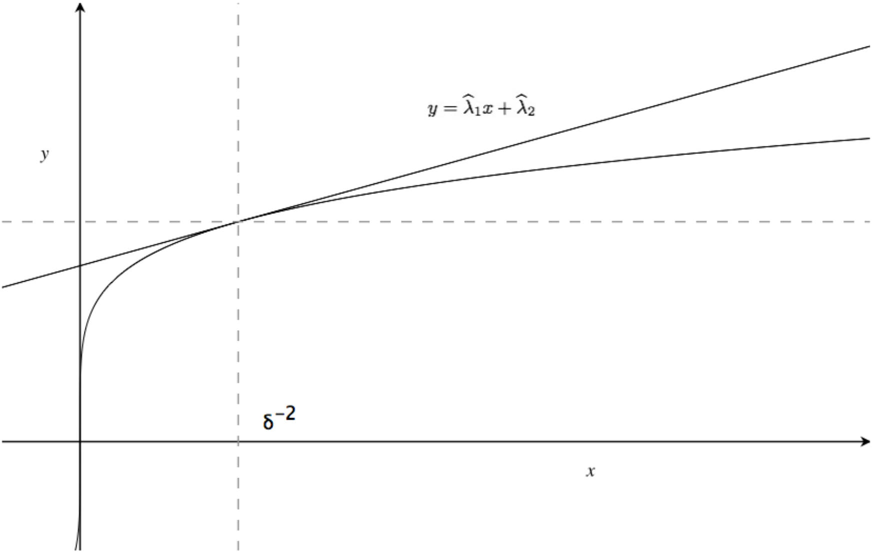
$$e(\delta, m) = E(\delta).$$

Let

- $$x(\sigma) = (2\pi)^{1-d}\sigma^{d-1+2\alpha}\chi_{[0,\infty)}(\sigma),$$
$$y(\sigma) = (2\pi)^{1-d}\sigma^{d-1}\chi_{[0,\infty)}(\sigma), \quad \sigma \in \mathbb{R}.$$
- $$\widehat{\lambda}_1 = (2\pi)^{\frac{(d-1)(d-2)}{d-1+2\alpha}} \frac{(d-1)}{d-1+2\alpha}\delta^{\frac{4\alpha}{d-1+2\alpha}},$$
$$\widehat{\lambda}_2 = (2\pi)^{\frac{(d-1)(d-2)}{d-1+2\alpha}} \frac{2\alpha}{d-1+2\alpha}\delta^{\frac{2(1-d)}{d-1+2\alpha}}$$

Then

- $$E(\delta) = \sqrt{\widehat{\lambda}_1 + \widehat{\lambda}_2\delta^2} = (2\pi)^{\frac{(d-1)(d-2)}{2(d-1+2\alpha)}}\delta^{\frac{2\alpha}{d-1+2\alpha}}.$$

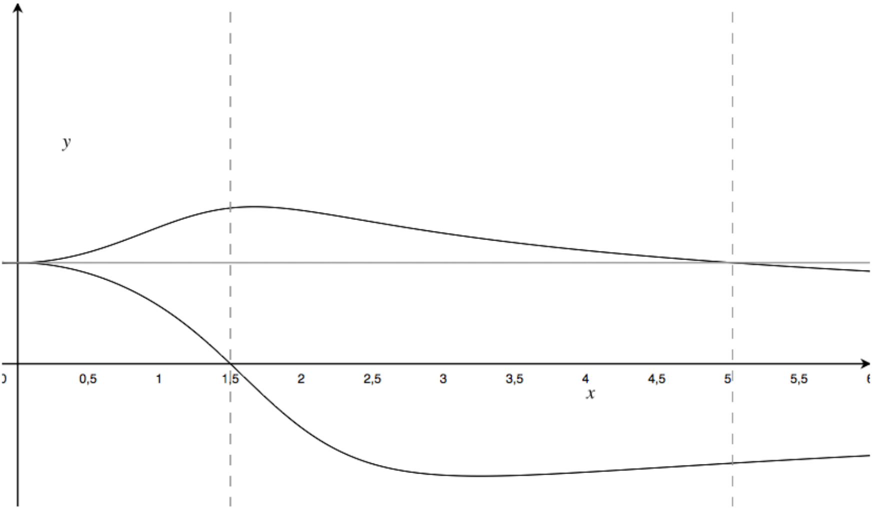


$d = 2$, $\alpha = 2$, $\delta = 1$

Set of optimal methods is given by

$$\widehat{m_a(g)}(\sigma\theta) = (2\pi)^{(1-d)/2} a(\sigma)\widehat{g_\theta}(\sigma),$$

- $g_\theta(s) = g(\theta, s)$
- $a(\sigma) = \left(\frac{\widehat{\lambda}_2}{\widehat{\lambda}_1 x(\sigma)+\widehat{\lambda}_2} + \epsilon(\sigma)\frac{\sigma^\alpha\sqrt{\widehat{\lambda}_1\widehat{\lambda}_2}}{\widehat{\lambda}_1 x(\sigma)+\widehat{\lambda}_2}\sqrt{x(\sigma)\widehat{\lambda}_1 + \widehat{\lambda}_2 - y(\sigma)}\right)\chi_{[0,\infty)}(\sigma)$
- $\|\epsilon(\cdot)\|_{L_\infty(\mathbb{R})} \leq 1$



$d = 2,\ \alpha = 2,\ \delta = 1$

Let $d > 2$ or $d = 2$ and $\alpha \geq \alpha^*$, where $(1 + 2\alpha^*)^{1+1/2\alpha^*} = 4\pi\alpha^*$.
Then method $\widehat{m_a(g)}(\sigma\theta) = (2\pi)^{(1-d)/2} a(\sigma)\widehat{g_\theta}(\sigma)$,

$$a(\sigma) = \begin{cases} 1 & , \sigma \in [0, \widehat{\lambda}_1^{-1/2\alpha}], \\ 0 & , \sigma \in [\widehat{\lambda}_1^{-1/2\alpha}, \infty) \end{cases}$$

is optimal.

For functions $f \in L_2(\mathbb{R}^d)$ we have the folowing inequality

$$\|f\|_{L_2(\mathbb{R}^d)} \leq (2\pi)^{\frac{(d-1)(d-2)}{2(d-1+2\alpha)}} \|Rf\|_{L_2(Z)}^{\frac{2\alpha}{d-1+2\alpha}} \|(-\Delta)^{\alpha/2} f\|_{L_2(\mathbb{R}^d)}^{\frac{d-1}{d-1+2\alpha}}, \quad \alpha > 0.$$

Consider the dual problem

$$\|f\|_{L_2(\mathbb{R}^d)} \to \max, \quad \|(-\Delta)^{\alpha/2} f\|_{L_2(\mathbb{R}^d)} \le 1, \quad \|Rf\|_{L_2(Z)} \le \delta$$

and it's Lagrange function

$$L(f, \lambda_1, \lambda_2) = -\lambda_1 - \lambda_2\delta^2 +$$

$$\lambda_1\|(-\Delta)^{\alpha/2} f\|_{L_2(\mathbb{R}^d)} + \lambda_2\|Rf\|_{L_2(Z)} - \|f\|_{L_2(\mathbb{R}^d)}.$$

Find $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ and admissible function $\widehat{f}$, which satisfies the complementary slackness conditions

$$\widehat{\lambda}_1 \left(\|(-\Delta)^{\alpha/2}\widehat{f}\|_{L_2(R^d)} - 1\right) + \widehat{\lambda}_2 \left(\|R\widehat{f}\|_{L_2(Z)} - \delta\right) = 0$$

and minimizes Lagrange function

$$\min L(f, \widehat{\lambda}_1, \widehat{\lambda}_2) = L(\widehat{f}, \widehat{\lambda}_1, \widehat{\lambda}_2).$$

Then $\widehat{f}$ is an extremum in dual problem.