# Plücker environments, wiring and tiling diagrams, and weakly separated set-systems

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**Abstract.** For the ordered set $[n]$ of $n$ elements, we consider the class $\mathcal{B}_n$ of bases $B$ of tropical Plücker functions on $2^{[n]}$ such that $B$ can be obtained by a series of mutations (flips) from the basis formed by the intervals in $[n]$. We show that these bases are representable by special wiring diagrams and by certain arrangements generalizing rhombus tilings on the $n$-zonogon. Based on the generalized tiling representation, we then prove that each weakly separated set-system in $2^{[n]}$ having maximum possible size belongs to $\mathcal{B}_n$, thus answering affirmatively a conjecture due to Leclerc and Zelevinsky. We also prove an analogous result for a hyper-simplex $\Delta_n^m = \{S \subseteq [n] : |S| = m\}$.

*Keywords*: Plücker relations, octahedron recurrence, wiring diagram, rhombus tiling, TP-mutation, weakly separated sets

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## 1 Introduction

For a positive integer $n$, let $[n]$ denote the ordered set of elements $1, 2, \ldots, n$. In this paper we consider a certain "class" $\mathcal{B}_n \subseteq 2^{2^{[n]}}$. The collections (set-systems) $B \subseteq 2^{[n]}$ constituting $\mathcal{B}_n$ have equal cardinalities $|B|$, and for some pairs of collections, one can be obtained from the other by a single "mutation" (or "flip") that consists in exchanging a pair of elements of a very special form in these collections. The class we deal with arises, in particular, in a study of bases of so-called tropical Plücker functions (this seems to be the simplest source; one more source will be indicated later). For this reason, we may liberally call $\mathcal{B}_n$ along with mutations on it a *Plücker environment*.

More precisely, let $f$ be a real-valued function on the subsets of $[n]$, or on the Boolean cube $2^{[n]}$. Following [1], $f$ is said to be a *tropical Plücker function*, or a *TP-function* for short, if it satisfies

$$f(Xik) + f(Xj) = \max\{f(Xij) + f(Xk),\ f(Xi) + f(Xjk)\} \qquad (1.1)$$

for any triple $i < j < k$ in $[n]$ and any subset $X \subseteq [n] - \{i, j, k\}$. Throughout, for brevity we write $Xi' \ldots j'$ instead of $X \cup \{i'\} \cup \ldots \cup \{j'\}$. The set of TP-functions on $2^{[n]}$ is denoted by $\mathcal{TP}_n$.

**Definition.** A subset $B \subseteq 2^{[n]}$ is called a *TP-basis*, or simply a *basis*, if the restriction map $res : \mathcal{TP}_n \to \mathbb{R}^B$ is a bijection. In other words, each TP-function is determined by its values on $B$, and moreover, values on $B$ can be chosen arbitrarily.

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Such a basis does exist and the simplest instance is the set $\mathcal{I}_n$ of all intervals $\{p, p+1, \ldots, q\}$ in $[n]$ (including the empty set); see, e.g., [2]. In particular, the dimension of the polyhedral conic complex $\mathcal{TP}_n$ is equal to $|\mathcal{I}_n| = \binom{n+1}{2} + 1$. The basis $\mathcal{I}_n$ is called *standard*.

(Note that the notion of a TP-function is extended to other domains, of which most popular are an *integer box* $\mathbf{B}^{n,a} := \{x \in \mathbb{Z}^{[n]} : 0 \le x \le a\}$ for $a \in \mathbb{Z}^{[n]}$ and a *hyper-simplex* $\Delta_n^m = \{S \subseteq [n] : |S| = m\}$ for $m \in \mathbb{Z}$ (in the later case, (1.1) should be replaced by a relation on quadruples $i < j < k < \ell$). Aspects involving TP-functions are studied in [1, 4, 7, 8, 10, 11, 12] and some other works. Generalizing some earlier known examples, [2] constructs a TP-basis for a "truncated integer box" $\{x \in \mathbf{B}^{n,a} : m \le x_1 + \ldots + x_n \le m'\}$, where $0 \le m \le m' \le n$. The domains different from Boolean cubes are beyond the main part of this paper; they will appear in Section 10 and Appendix.)

One can see that for a basis $B$, the collection $\{[n] - X : X \in B\}$ forms a basis as well, called the *complementary basis* of $B$ and denoted by co-$B$. An important instance is the collection co-$\mathcal{I}_n$ of co-intervals in $[n]$.

Once we are given a basis $B$ (e.g., the standard one), we can produce more bases by making a series of elementary transformations relying on (1.1). More precisely, suppose there is a cortege $(X, i, j, k)$ such that the four sets occurring in the right hand side of (1.1) and one set $Y \in \{Xj, Xik\}$ in the left hand side belong to $B$. Then the replacement in $B$ of $Y$ by the other set $Y'$ in the left hand side results in a basis $B'$ as well (and we can further transform the latter basis in a similar way). The basis $B'$ is said to be obtained from $B$ by the *flip* (or *mutation*) with respect to $X, i, j, k$. When $Xj$ is replaced by $Xik$ (thus increasing the total size of sets in the basis by 1), the flip is called *raising*. When $Xik$ is replaced by $Xj$, the flip is called *lowering*. We write $j \rightsquigarrow ik$ and $ik \rightsquigarrow j$ for such flips. The standard basis $\mathcal{I}_n$ does not admit lowering flips, whereas its complementary basis co-$\mathcal{I}_n$ does not admit raising flips.

We distinguish between two sorts of flip (mutations), which inspire consideration of two classes of bases.

**Definitions.** For a TP-basis $B$ and a cortege $(X, i, j, k)$ as above, the flip (mutation) $j \rightsquigarrow ik$ or $ik \rightsquigarrow j$ is called *strong* if both sets $X$ and $Xijk$ belong to $B$ as well, and *weak* in general. (The former (latter) is also called the flip in the presence of six (res. four) "witnesses", in terminology of [7].) A basis is called *normal* (by terminology in [2]) if it can be obtained by a series of *strong* flips starting from $\mathcal{I}_n$. A basis is called *semi-normal* if it can be obtained by a series of *any* flips starting from $\mathcal{I}_n$.

Leclerc and Zelevinsky [7] showed that the normal bases (in our terminology) are exactly the collections $C \subseteq 2^{[n]}$ of maximum possible size $|C|$ that possess the strong separation property (defined later). Also the class of normal bases admits a nice "graphical" characterization, even for a natural generalization to the integer boxes (see [2, 3]): such bases one-to-one correspond to the rhombus tilings on the related *zonogon*.

Let $\mathcal{B}_n$ denote the set of semi-normal TP-bases for the Boolean cube $2^{[n]}$; this set (together with weak flips on its members) is just the Plücker environment of our interest mentioned at the beginning. (Note that it is still open at present whether there exists a non-semi-normal (or "wild") basis; we conjecture that there is none.)

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The first goal of this paper is to characterize $\mathcal{B}_n$. We give two characterizations for semi-normal bases: via a bijection to special collections of curves, that we call *proper wirings*, and via a bijection to certain graphical arrangements, called *generalized tilings* or, briefly, *g-tilings* (in fact, these characterizations are interrelated via planar duality). We associate to a proper wiring $W$ (a g-tiling $T$) a certain collection of subsets of $[n]$ called its *spectrum*. It turns out that proper wirings and g-tilings are rigid objects, in the sense that any of these is determined by its spectrum.

(More precisely, by a wiring we mean a set of $n$ directed non-self-intersecting curves $w_1, \ldots, w_n$ in a region $R$ of the plane homeomorphic to a circle, where each $w_i$ begins at a point $s_i$ and ends at a point $s'_i$, and the points $s_1, \ldots, s_n, s'_1, \ldots, s'_n$ are different and occur in this order clockwise in the boundary of $R$. A special wiring $W$ that we deal with is defined by three axioms (W1)–(W3). Axiom (W1) is standard, it says that $W$ preserves (topologically) under small deformations, i.e., no three wires have a common point, any two wires meet at a finite number of points and they cross, not touch, at each of these points. (W2) says that the common points of $w_i, w_j$ follow in the opposed orders along these wires. The crucial axiom (W3) says that in the planar graph induced by $W$, there is a certain bijection between the faces ("chambers") whose boundary is a directed cycle and the regions ("lenses") surrounded by pieces of two wires between their neighboring common points. $W$ is called proper if none of "cyclic" faces is a whole lens. The spectrum of $W$ is the collection of subsets $X \subseteq [n]$ one-to-one corresponding to the "non-cyclic" faces $F$, where $X$ consists of the elements $i$ such that $F$ "lies on the right" from $w_i$ (according to the direction of $w_i$). When any two wires intersect exactly once, the dual planar graph is realized by a rhombus tiling, and vice versa (for a more general result of this sort, see [5]). In a general case, the construction of a g-tiling is more intricate. Axiom (W2) is encountered in [8]. Another sort of wirings, related to set-systems in a hyper-simplex $\Delta_n^m$, is studied in [8, 10].)

The characterization of semi-normal bases via generalized tilings helps answer one conjecture of Leclerc and Zelevinsky concerning weakly separated set-systems; this is the second goal of our work. Recall some definitions from [7]. Hereinafter for sets $A, B$, we write $A - B$ for $A \setminus B = \{e \colon A \ni e \notin B\}$. Let $X, Y \subseteq [n]$. We write $X \prec Y$ if $Y - X \neq \emptyset$ and $i < j$ for any $i \in X - Y$ and $j \in Y - X$ (which slightly differs from the meaning of $\prec$ in [7]); note that this relation need not be transitive. We write $X \rhd Y$ if $Y - X$ has a (unique) bipartition $\{Y_1, Y_2\}$ such that $Y_1, Y_2, X - Y \neq \emptyset$ and $Y_1 \prec X - Y \prec Y_2$.

**Definitions.** Sets $X, Y \subseteq [n]$ are called: (a) *strongly separated* if either $X \prec Y$ or $Y \prec X$, and (b) *weakly separated* if either $X \prec Y$, or $Y \prec X$, or $X \rhd Y$ and $|X| \geq |Y|$, or $Y \rhd X$ and $|Y| \geq |X|$. Accordingly, a collection $C \subseteq 2^{[n]}$ is called strongly (weakly) separated if any two members of $C$ are strongly (resp. weakly) separated.

(As is seen from a discussion in [7], an interest in studying weakly separated collections is inspired, in particular, by the problem of characterizing all families of quasicommuting quantum flag minors, which in turn comes from exploration of Lusztig's canonical bases for certain quantum groups. It is proved in [7] that, in an $n \times n$ generic $q$-matrix, the flag minors with column sets $I, J \subseteq [n]$ quasicommute if and only if the sets $I, J$ are weakly separated. See also [6].)

Important properties shown in [7] are that any weakly separated collection $C \subseteq 2^{[n]}$ has cardinality at most $\binom{n+1}{2} + 1$ and that the set of such collections is closed under weak flips (which are defined as for TP-bases above). Let $\mathcal{C}_n$ denote the set of *largest* weakly separated collections in $[n]$, i.e., having size $\binom{n+1}{2} + 1$. It turns into a poset by regarding $C$ as being less than $C'$ if $C$ is obtained from $C'$ by a series of weak lowering flips. This poset contains $\mathcal{I}_n$ and co-$\mathcal{I}$ as minimal and maximal elements, respectively, and it is conjectured in [7, Conjecture 1.8] that there are no other minimal and maximal elements in it. This would imply that $\mathcal{C}_n$ coincides with $\mathcal{B}_n$. We prove this conjecture.

The main results in this paper are summarized as follows.

**Theorem A** (main) *For $B \subseteq 2^{[n]}$, the following statements are equivalent:*

(i) *$B$ is a semi-standard TP-basis;*

(ii) *$B$ is the spectrum of a proper wiring;*

(iii) *$B$ is the spectrum of a generalized tiling;*

(iv) *$B$ is a largest weakly separated collection.*

The paper is organized as follows. Section 2 contains basic definitions and states two results involved in Theorem A. It introduces the notions of proper wirings and generalized tilings, claims the equivalence of (i) and (ii) in the above theorem (Theorem 2.1) and claims the equivalence of (i) and (iii) (Theorem 2.2). Section 3 describes some "elementary" properties of g-tilings that will be used later. The combined proof of Theorems 2.1 and 2.2 consists of four stages and is lasted throughout Sections 4–7. In fact, g-tilings are the central objects of treatment in the paper; we take advantages from their nice graphical visualization and structural features, and all implications that we explicitly prove involve just g-tilings. (Another preference of g-tilings is that they admit "local" defining axioms; see Remark 1 in Section 3.) Implication (i)→(iii) in Theorem A is proved in Section 4, (iii)→(i) in Section 5, (iii)→(ii) in Section 6, and (ii)→(iii) in Section 7. Section 8 establishes important interrelations between g-tilings in dimensions $n$ and $n-1$ (giving, as a consequence, a relation between the classes $\mathcal{B}_n$ and $\mathcal{B}_{n-1}$). Here we describe two operations, called the *n-contraction* and *n-expansion*; the former canonically transforms a g-tiling for $n$ into one for $n-1$, and the latter is applied to a pair consisting of a g-tiling for $n-1$ and a certain path in it and transforms this pair into a g-tiling for $n$. These operations are essentially used in Section 9 where we prove (iv)→(iii) by induction on $n$, thus answering Leclerc-Zelevinsky's conjecture mentioned above. This completes the proof of Theorem A, taking into account that (i)→(iv) was established in [7]. Section 10 discusses two generalizations of our theorems: to an integer box and to an arbitrary permutation of $[n]$. In Appendix we show that the equivalence (i)⟺(iv) as in Theorem A is valid when, instead of TP-bases and largest weakly separated collections in $2^{[n]}$, one considers a natural class of bases of tropical Plücker functions on a hyper-simplex $\Delta_n^m$ and the weakly separated collections of maximum possible cardinality in $\Delta_n^m$.

## 2 Wirings and tilings

Throughout the paper we assume that $n > 1$. This section gives precise definitions of the objects that we call special wiring and generalized tiling diagrams. Such diagrams live within a zonogon, which is defined as follows.

In the upper half-plane $\mathbb{R} \times \mathbb{R}_+$, take $n$ non-colinear vectors $\xi_1, \ldots, \xi_n$ so that:

(2.1)    (i) $\xi_1, \ldots, \xi_n$ follow in this order clockwise around $(0,0)$, and (ii) all integer combinations of these vectors are different.

Then the set

$$ Z = Z_n := \{\lambda_1 \xi_1 + \ldots + \lambda_n \xi_n \colon \lambda_i \in \mathbb{R},\ 0 \leq \lambda_i \leq 1,\ i = 1, \ldots, n\} $$

is a $2n$-gone. Moreover, $Z$ is a *zonogon*, as it is the sum of $n$ line-segments $\{\lambda \xi_i \colon 1 \leq \lambda \leq 1\}$, $i = 1, \ldots, n$. Also it is the image by a linear projection $\pi$ of the solid cube $conv(2^{[n]})$ into the plane $\mathbb{R}^2$, defined by $\pi(x) = x_1 \xi_1 + \ldots + x_n \xi_n$. The boundary $bd(Z)$ of $Z$ consists of two parts: the *left boundary* $lbd(Z)$ formed by the points (vertices) $p_i := \xi_1 + \ldots + \xi_i$ $(i = 0, \ldots, n)$ connected by the line-segments $p_{i-1} p_i := p_{i-1} + \{\lambda \xi_i \colon 0 \leq \lambda \leq 1\}$, and the *right boundary* $rbd(Z)$ formed by the points $p'_i := \xi_i + \ldots + \xi_n$ $(i = 0, \ldots, n)$ connected by the segments $p'_i p'_{i-1}$. So $p_0 = p'_n$ is the minimal vertex and $p_n = p'_0$ is the maximal vertex of $Z$. We orient each segment $p_{i-1} p_i$ from $p_{i-1}$ to $p_i$ and orient each segment $p'_i p'_{i-1}$ from $p'_i$ to $p'_{i-1}$. Let $s_i$ (resp. $s'_i$) denote the median point in the segment $p_{i-1} p_i$ (resp. $p'_i p'_{i-1}$).

Although the generalized tiling model will be used much more extensively later on, we prefer to start with describing the special wiring model, which looks more transparent.

### 2.1 Wiring diagrams

A *special wiring diagram*, also called a *W-diagram* or a *wiring* for brevity, is an ordered collection $W$ of $n$ wires $w_1, \ldots, w_n$ satisfying three axioms below. A *wire* $w_i$ is a continuous injective map of the segment $[0, 1]$ into $Z$ (or the curve in the plane represented by this map) such that $w_i(0) = s_i$, $w_i(1) = s'_i$, and $w_i(\lambda)$ lies in the interior of $Z$ for $0 < \lambda < 1$. We say that $w_i$ begins at $s_i$ and ends at $s'_i$, and orient $w_i$ from $s_i$ to $s'_i$. The diagram $W$ is considered up to a homeomorphism of $Z$ stable on $bd(Z)$, and up to parameterizations of the wires. Axioms (W1)–(W3) specify $W$ as follows.

(W1) No three wires $w_i, w_j, w_k$ have a common point, i.e., there are no $\lambda, \lambda', \lambda''$ such that $w_i(\lambda) = w_j(\lambda') = w_k(\lambda'')$. Any two wires $w_i, w_j$ $(i \neq j)$ intersect at a finite number of points, and at each of their common points $v$, the wires *cross*, not touch (i.e., when passing $v$, the wire $w_i$ goes from one connected component of $Z - w_j$ to the other one).

(W2) for $1 \leq i < j \leq n$, the common points of $w_i, w_j$ follow in opposed orders along these wires, i.e., if $w_i(\lambda_q) = w_j(\lambda'_q)$ for $q = 1, \ldots, r$ and if $\lambda_1 < \ldots < \lambda_r$, then $\lambda'_1 > \ldots > \lambda'_r$.

Since the order of $s_i, s_j$ in $\ell bd(Z)$ is different from the order of $s'_i, s'_j$ in $rbd(Z)$, wires $w_i, w_j$ do intersect; moreover, the number $r = r_{ij}$ of their common points is odd. Assuming that $i < j$, we denote these points as $x_{ij}(1), \ldots, x_{ij}(r)$ following the direction of $w_i$ from $w_i(0)$ to $w_i(1)$. When $r > 1$, the (bounded) region in the plane surrounded by the pieces of $w_i, w_j$ between $x_{ij}(q)$ and $x_{ij}(q+1)$ (where $q = 1, \ldots, r-1$) is denoted by $L_{ij}(q)$ and called the $q$-th *lens* for $i, j$. The points $x_{ij}(q)$ and $x_{ij}(q+1)$ are regarded as the *upper* and *lower* points of $L_{ij}(q)$, respectively. When $q$ is odd (even), we say that $L_{ij}(q)$ is an *odd* (resp. *even*) lens. Note that at each point $x_{ij}(q)$ with $q$ odd the wire with the bigger number, namely, $w_j$, crosses the wire with the smaller number ($w_i$) *from left to right* w.r.t. the direction of the latter; we call such a point *white*. In contrast, when $q$ is even, $w_j$ crosses $w_i$ at $x_{ij}(q)$ *from right to left*; in this case (which will be of especial interest), we call $x_{ij}(q)$ *black*, or *orientation-reversing*, and say that this point is the *root* of the lenses $L_{ij}(q-1)$ and $L_{ij}(q)$. In the simplest case, when any two distinct wires intersect exactly once, there are no lenses at all and all intersection points for $W$ are white. (The adjectives "white" and "black" for intersection points of wires will match terminology that we use for corresponding elements of tilings.)

The wiring $W$ is associated, in a natural way, with a planar directed graph $G_W$ embedded in $Z$. The vertices of $G_W$ are the points $p_i, p'_i, s_i, s'_i$ and the intersection points of wires. The edges of $G_W$ are the corresponding directed line-segments in $bd(Z)$ and the pieces of wires between neighboring points of intersection with other wires or with the boundary, which are directed according to the direction of wires. We say that an edge contained in a wire $w_i$ has *color $i$*, or is an *$i$-edge*. Let $\mathcal{F}_W$ be the set of (inner, or bounded) faces of $G_W$. Here each face $F$ is considered as the closure of a maximal connected component in $Z - \cup(w \in W)$. We say that a face $F$ is *cyclic* if its boundary $bd(F)$ is a directed cycle in $G_W$.

(W3) There is a bijection $\phi$ between the set $\mathcal{L}(W)$ of lenses in $W$ and the set $\mathcal{F}_W^{cyc}$ of cyclic faces in $G_W$. Moreover, for each lens $L$, $\phi(L)$ is the (unique) face lying in $L$ and containing its root.

We say that $W$ is *proper* if none of cyclic faces is a whole lens, i.e., for each lens $L \in \mathcal{L}(W)$, there is at least one wire going across $L$. An instance of proper wirings for $n = 4$ is illustrated in the picture; here the cyclic faces are marked by circles and the black rhombus indicates the black point.
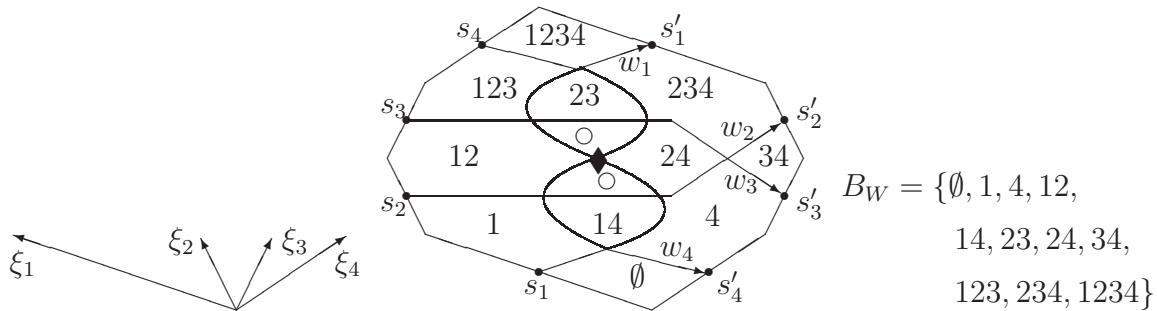


Now we associate to $W$ a set-system $B_W \subseteq 2^{[n]}$ as follows. For each face $F$, let $X(F)$ be the set of elements $i \in [n]$ such that $F$ lies *on the left* from the wire $w_i$, i.e., $F$

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and the maximal point $p_n$ lie in the same of the two connected components of $Z - w_i$. We define
$$B_W := \{X \subseteq [n]: X = X(F) \text{ for some } F \in \mathcal{F}_W - \mathcal{F}_W^{cyc}\},$$
referring to it as the *effective spectrum*, or simply the *spectrum* of $W$. Sometimes it will also be useful to consider the *full spectrum* $\widehat{B}_W$ consisting of all sets $X(F)$, $F \in \mathcal{F}_W$. (In fact, when $W$ is proper, all sets in $\widehat{B}_W$ are different; see Lemma 7.2. When $W$ is not proper, there are different faces $F, F'$ with $X(F) = X(F')$. We can turn $W$ into a proper wiring $W'$ by getting rid, step by step, of lenses forming faces (by making a series of Reidemeister moves of type II, namely, $)( \to )($ operations). This preserves the effective spectrum: $B_{W'} = B_W$, whereas the full spectrum may decrease.)

Note that when any two wires intersect at exactly one point (i.e., when no black points exist), $B_W$ is a normal basis, and conversely, any normal basis is obtained in this way (see [2]).

Our main result on wirings is the following

**Theorem 2.1** *For any proper wiring $W$ (obeying (W1)–(W3)), the spectrum $B_W$ is a semi-normal TP-basis. Conversely, for any semi-normal TP-basis $B$, there exists a proper wiring $W$ such that $B_W = B$.*

This theorem will be obtained in Sections 6–7.

## 2.2   Generalized tilings

When it is not confusing, we identify a subset $X \subseteq [n]$ with the corresponding vertex of the $n$-cube and with the point $\sum_{i \in X} \xi_i$ in the zonogon $Z$. Due to (2.1)(ii), all points $X$ are different (concerning $Z$).

Assuming that the vectors $\xi_i$ have the same Euclidean norm, a *rhombus tiling diagram* is defined to be a subdivision $T$ of $Z$ into rhombi of the form $x + \{\lambda \xi_i + \lambda' \xi_j : 0 \le \lambda, \lambda' \le 1\}$ for some $i < j$ and a point $x$ in $Z$, i.e., the rhombi are pairwise non-overlapping (have no common interior points) and their union is $Z$. From (2.1)(ii) it follows that for $i, j, x$ as above, $x$ represents a subset in $[n] - \{i, j\}$. The diagram $T$ is also regarded as a directed planar graph whose vertices and edges are the vertices and side segments of the rhombi, respectively. An edge connecting $X$ and $Xi$ is directed from the former to the latter. It is shown in [2, 3] that the vertex set of $T$ forms a normal basis and that each normal basis is obtained in this way.

It makes no difference whether we take vectors $\xi_1, \ldots, \xi_n$ with equal or arbitrary norms (subject to (2.1)); to simplify technical details and visualization, throughout the paper we will assume that these vectors have *unit height*, i.e., each $\xi_i$ is of the form $(x, 1)$. Then we obtain a subdivision $T$ of $Z$ into parallelograms of height 2, and for convenience we refer to $T$ as a *tiling* and to its elements as *tiles*. A tile $\tau$ defined by $X, i, j$ (with $i < j$) is called an *ij-tile* at $X$ and denoted by $\tau(X; i, j)$. According to a natural visualization of $\tau$, its vertices $X, Xi, Xj, Xij$ are called the *bottom, left, right, top* vertices of $\tau$ and denoted by $b(\tau), \ell(\tau), r(\tau), t(\tau)$, respectively. Also we will say that: for a point (subset) $Y \subseteq [n]$, $|Y|$ is the *height* of $Y$; the set of vertices of tiles in

$T$ having height $h$ form $h$-th *level*; and a point $Y$ *lies on the right* from a point $Y'$ if $Y, Y'$ have the same height and $\sum_{i \in Y} \xi_i \geq \sum_{i \in Y'} \xi_i$.

In a *generalized tiling*, or a *g-tiling*, the union of tiles is again $Z$ but some tiles may overlap. It is a collection $T$ of tiles which is partitioned into two subcollections $T^w$ and $T^b$, of *white* and *black* tiles (say), respectively, obeying axioms (T1)–(T4) below. When $T^b = \emptyset$, we will obtain a tiling as before, for convenience referring to it as a *pure tiling*. Let $V_T$ and $E_T$ denote the sets of vertices and edges, respectively, occurring in tiles of $T$, not counting multiplicities. For a vertex $v \in V_T$, the set of edges incident with $v$ is denoted by $E_T(v)$, and the set of tiles having a vertex at $v$ is denoted by $F_T(v)$.

(T1) All tiles are contained in $Z$. Each boundary edge of $Z$ belongs to exactly one tile. Each edge in $E_T$ not contained in $bd(Z)$ belongs to exactly two tiles. All tiles in $T$ are different (in the sense that no two coincide in the plane).

(T2) Any two white tiles having a common edge do not overlap (in the sense that they have no common interior point). If a white tile and a black tile share an edge, then these tiles do overlap. No two black tiles share an edge.

(T3) Let $\tau$ be a black tile. None of $b(\tau), t(\tau)$ is a vertex of another black tile. All edges in $E_T(b(\tau))$ *leave* $b(\tau)$ (i.e., are directed from $b(\tau)$). All edges in $E_T(t(\tau))$ *enter* $t(\tau)$ (i.e., are directed to $t(\tau)$).

We distinguish between three sorts of vertices by saying that $v \in V_T$ is: (a) a *terminal* vertex if it is the bottom or top vertex of some black tile; (b) an *ordinary* vertex if all tiles in $F_T(v)$ are white; and (c) a *mixed* vertex otherwise (i.e. $v$ is the left or right vertex of some black tile). Note that a mixed vertex may belong, as the left or right vertex, to several black tiles.

Each tile $\tau \in T$ is associated, in a natural way, to a square in the solid $n$-cube $conv(2^{[n]})$, denoted by $\sigma(\tau)$: if $\tau = \tau(X; i, j)$ then $\sigma(\tau)$ spans the vertices (corresponding to) $X, Xi, Xj, Xij$ in the cube. In view of (T1), the interiors of these squares are disjoint, and $\cup(\sigma(\tau) : \tau \in T)$ forms a 2-dimensional surface, denoted by $D_T$, whose boundary is the preimage by $\pi$ of the boundary of $Z$; the vertices in $bd(D_T)$ correspond to the *principal intervals* $\emptyset$, $[q]$ and $[q..n]$ for $q = 1, \ldots, n$. (For $1 \leq p \leq r \leq n$, we denote the interval $\{p, p+1, \ldots, r\}$ by $[p..r]$). The last axiom is:

(T4) $D_T$ is a disc (i.e., is homeomorphic to $\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$).

The *reversed* g-tiling $T^{rev}$ of a g-tiling $T$ is formed by replacing each tile $\tau(X; i, j)$ of $T$ by the tile $\tau([n] - Xij; i, j)$ (or by changing the orientation of all edges in $E_T$, in particular, in $bd(Z)$). Clearly (T1)–(T4) remain valid for $T^{rev}$.

The *effective spectrum*, or simply the *spectrum*, of a g-tiling $T$ is the collection $B_T$ of (subsets of $[n]$ represented by) *non-terminal* vertices in $T$. The *full spectrum* $\widehat{B}_T$ is formed by all vertices in $T$. An example of g-tilings for $n = 4$ is drawn in the picture, where the unique black tile is indicated by thick lines and the terminal vertices are surrounded by circles (this is related to the wiring shown on the previous picture).

$$B_T = \{\emptyset, 1, 4, 12, 14, 23, 24, 34,$$
$$123, 234, 1234\}$$
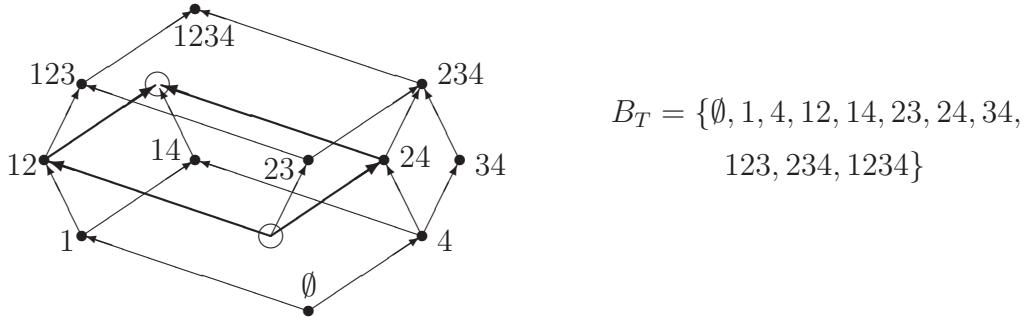
Our main result on g-tilings is the following

**Theorem 2.2** *For any generalized tiling $T$ (obeying (T1)–(T4)), the spectrum $B_T$ is a semi-normal TP-basis. Conversely, for any semi-normal TP-basis $B$, there exists a generalized tiling $T$ such that $B_T = B$.*

(In particular, the cardinalities of the spectra of all g-tilings on $Z_n$ are the same and equal to $\binom{n+1}{2} + 1$.) The first part of this theorem will be proved in Section 5, and the second one in Section 4.

We will explain in Section 7 that for each semi-normal basis $B$, there are precisely one proper wiring $W$ and precisely one g-tiling $T$ such that $B_W = B_T = B$ (see Theorem 7.5); this is similar to the one-to-one correspondence between the normal bases and pure tilings.

## 3   Elementary properties of generalized tilings

In this section we give additional definitions and notation and demonstrate several corollaries from axioms (T1)–(T4) which will be used later on. Let $T$ be a g-tiling on $Z = Z_n$.

**1.** An edge $e$ of $G_T$ is called *black* if there is a black tile containing $e$ (as a side edge); otherwise $e$ is called *white*. The sets of white and black edges incident with a vertex $v$ are denoted by $E_T^w(v)$ and $E_T^b(v)$, respectively. For a vertex $v$ of a tile $\tau$, let $C(\tau, v)$ denote the minimal cone at $v$ containing $\tau$ (i.e., generated by the pair of edges of $\tau$ incident to $v$), and let $\alpha(\tau, v)$ denote the angle of this cone taken with sign $+$ if $\tau$ is white, and $-$ if $\tau$ is black. The sum $\sum(\alpha(\tau, v)\colon \tau \in F_T(v))$ is called the *(full) rotation angle* at $v$ and denoted by $\rho(v) = \rho_T(v)$. We observe from (T1)–(T3) that terminal vertices behave as follows.

**Corollary 3.1** *Let $v$ be a terminal vertex belonging to a black $ij$-tile $\tau$. Then:*

(i) *$v$ is not connected by edge with another terminal vertex (whence $|E_T^b(v)| = 2$);*

(ii) *$|E_T(v)| \geq 3$ (whence $E_T^w(v) \neq \emptyset$);*

(iii) *each edge $e \in E_T^w(v)$ lies in the cone $C(\tau, v)$ (whence $e$ is a $q$-edge for some $i < q < j$);*

(iv) *$\rho(v) = 0$;*

(v) *$v$ does not belong to the boundary of $Z$ (whence any tile containing a boundary edge of $Z$ is white).*

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Indeed, since each edge of $G_T$ belongs to some tile, at least one of its end vertices has both entering and leaving edges, and therefore (by (T3)), this vertex cannot be terminal (yielding (i)). Next, if $|E_T(v)| = 2$, then $F_T(v)$ would consist only of the tile $\tau$ and its white copy; this is not the case by (T1) (yielding (ii)). Next, assume that $v = t(\tau)$. Then $v$ is the top vertex of all tiles in $F_T(v)$ (by (T3)). This together with the facts that all tiles in $F_T(v) - \{\tau\}$ are white and that any two white tiles sharing an edge do not overlap (by (T2)) implies (iii) and (iv). When $v = b(\tau)$, the argument is similar. Finally, $v$ cannot be a boundary vertex $p_k$ or $p'_k$ for $k \neq 0, n$ since the latter vertices have both entering and leaving edges. In case $v = p_0$, the tile $\tau$ would contain both boundary edges $p_0 p_1$ and $p_0 p'_{n-1}$ (in view of (iii)). But then the white tile sharing with $\tau$ the edge $(r(\tau), t(\tau))$ would trespass the boundary of $Z$. The case $v = p_n$ is impossible for a similar reason (yielding (v)).

Note that (iii) in this corollary implies that

(3.1) if a black $ij$-tile $\tau$ and a (white) tile $\tau'$ share an edge $e$, then $\tau'$ is either an $iq$-tile or a $qj$-tile for some $i < q < j$; also $\tau' \subset C(\tau, v)$ and $\tau \subset C(\tau', v')$, where $v$ and $v'$ are the terminal and non-terminal ends of $e$, respectively.

**2.** The following important lemma specifies the rotation angle at non-terminal vertices.

**Lemma 3.2** *Let $v \in V_T$ be a non-terminal vertex.*

(i) *If $v$ belongs to $bd(Z)$, then $\rho(v)$ is positive and equals the angle between the boundary edges incident to $v$.*

(ii) *If $v$ is inner (i.e., not in $bd(Z)$), then $\rho(v) = 2\pi$.*

**Proof** (i) For $v \in bd(Z)$, let $e, e'$ be the boundary edges incident to $v$, where $e, Z, e'$ follow clockwise around $v$. Consider the maximal sequence $e = e_0, \tau_1, e_1, \ldots, \tau_r, e_r$ of edges in $E_T(v)$ and tiles in $F_T(v)$ such that for $q = 1, \ldots, r$, $e_{q-1}, e_q$ are distinct edges of the tile $\tau_q$, and $\tau_q \neq \tau_{q+1}$ (when $q < n$). Using (3.1), one can see that all tiles in this sequence are different and give the whole $F_T(v)$; also $e_r = e'$ and the tiles $\tau_1, \tau_r$ are white. For each $q$, the ray at $v$ containing $e_q$ is obtained by rotating the ray at $v$ containing $e_{q-1}$ by the angle $\alpha(\tau_q, v)$ (where the rotation is clockwise if the angle is positive). So the sum of angles over this sequence amounts to $\rho(v)$ and is equal to the angle of $e, e'$.

To show (ii), let $V := V_T$ and $E := E_T$. Also denote the set of terminal vertices by $V^t$, and the set of *inner* non-terminal vertices by $\widehat{V}$. Since the boundary of $Z$ contains $2n$ vertices and by (i),

$$|V| = |V^t| + |\widehat{V}| + 2n \quad \text{and} \quad \sum_{v \in V \cap bd(Z)} \rho(v) = \pi \cdot 2n - 2\pi = 2\pi(n-1) \qquad (3.2)$$

Let $\Sigma := \sum(\rho(v) : v \in V)$ and $\widehat{\Sigma} := \sum(\rho(v) : v \in \widehat{V})$. The contribution to $\Sigma$ from each white (black) tile is $2\pi$ (resp. $-2\pi$). Therefore, $\Sigma = 2\pi(|T^w| - |T^b|)$. On the other hand, in view of Corollary 3.1(iv) and the second relation in (3.2), $\Sigma = \widehat{\Sigma} + 2\pi(n-1)$. Then

$$\widehat{\Sigma} = 2\pi(|T^w| - |T^b| - n + 1). \qquad (3.3)$$

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Considering $G_T$ as a planar graph properly embedded in the disc $D_T$ and applying Euler formula to it, we have $|V| + |T| = |E| + 1$. Each tile has four edges, the number of boundary edges is $2n$, and each inner edge belongs to two tiles; therefore, $|E| = 2n + (4|T| - 2n)/2 = 2|T| + n$. Then $|V|$ is expressed as

$$|V| = |E| - |T| + 1 = 2|T| + n - |T| + 1 = |T| + n + 1. \tag{3.4}$$

Also $|V| = |\widehat{V}| + 2|T^b| + 2n$ (using the first equality in (3.2) and the equality $|V^t| = 2|T^b|$). This and (3.4) give

$$|\widehat{V}| = |V| - 2|T^b| - 2n = (|T| + n + 1) - 2|T^b| - 2n = |T^w| - |T^b| - n + 1.$$

Comparing this with (3.3), we obtain $\widehat{\Sigma} = 2\pi|\widehat{V}|$. Now the desired equality $\rho(v) = 2\pi$ for each vertex $v \in \widehat{V}$ follows from the fact that $\rho(v)$ equals $2\pi \cdot d$ for some integer $d \geq 1$. The latter is shown as follows. Let us begin with a white tile $\tau_1 \in F_T(v)$ and its edges $e_0, e_1 \in E_T(v)$, in this order clockwise, and form a sequence $e_0, \tau_1, e_1, \ldots, \tau_r, e_r, \ldots$ similar to that in (i) above, until we return to the initial edge $e_0$. Let $R_q$ be the ray at $v$ containing $e_q$. Since $\alpha(\tau_q) > 0$ when $\tau_q$ is white, and $\alpha(\tau_q) + \alpha(\tau_{q+1}) > 0$ when $\tau_q$ is white and $\tau_{q+1}$ is black (cf. (3.1)), the current ray $R_\bullet$ must make at least one turn clockwise before it returns to the initial ray $R_0$. If it happens that the sequence uses not all tiles in $F_T(v)$, we start with a new white tile to form a next sequence (for which the corresponding ray makes at least one turn clockwise as well), and so one. Thus, $d \geq 1$, as required (implying $d = 1$). ∎

**Remark 1** If we postulate property (ii) in Lemma 3.2 as axiom (T4') and add it to axioms (T1)–(T3), then we can eliminate axiom (T4); in other words, (T4') and (T4) are equivalent subject to (T1)–(T3). Indeed, reversing reasonings in the above proof, one can conclude that $\widehat{\Sigma} = 2\pi|\widehat{V}|$ implies $|V| + |T| = |E| + 1$. The latter is possible only if $D_T$ is a disc. (Indeed, if $D_T$ forms a regular surface with $g$ handles and $c$ cross-caps, from which an open disc is removed, then Euler formula is modified as $|V| + |T| = |E| + 1 - 2g - c$. Also $|V|$ decreases when some vertices merge.) Note that each of axioms (T1)–(T3),(T4') is "local"; this gives rise to a local characterization for semi-normal bases (as well as for largest weakly separated set-systems); see Remark 5 in Section 7.

**3.** Considering the sequence of rotations of the edge ray $R_\bullet$ around a non-terminal vertex $v$ (like in the proof of Lemma 3.2), one can see that the sets $E_T^w(v)$ and $E_T^b(v)$ are arranged as follows.

(3.5) For an ordinary or mixed vertex $v \in V_T$, let $E_T(v)$ consists of edges $e_1, \ldots, e_p$ following counterclockwise around $v$, and let $e_1$ enter and $e_p$ leave $v$. Then:

    (i) there is $1 \leq p' < p$ such that $e_1, \ldots, e_{p'}$ enter $v$ and $e_{p'+1}, \ldots, e_p$ leave $v$;

    (ii) if $v$ is the right vertex of $r$ black tiles and the left vertex of $r'$ black tiles, then $r + r' < \min\{p', p - p'\}$ and the black edges incident to $v$ are exactly $e_{p-r+1}, \ldots, e_p, e_1, \ldots, e_r$ and $e_{p'-r'+1}, \ldots, e_{p'+r'}$;

(iii) the black tiles in $F_T(v)$ have the following pairs of edges incident to $v$: $\{e_{p-r+1}, e_1\}, \ldots, \{e_p, e_r\}$ and $\{e_{p'-r'+1}, e_{p'+1}\}, \ldots, \{e_{p'}, e_{p'+r'}\}$, while the white tiles in $F_T(v)$ have the following pairs of edges incident to $v$: (a) $\{e_{r+1}, e_{r+2}\}, \ldots, \{e_{p'-r'-1}, e_{p'-r'}\}$; (b) $\{e_{p'+r'+1}, e_{p'+r'+2}\}, \ldots, \{e_{p-r-1}, e_{p-r}\}$; (c) $\{e_{p-r}, e_1\}, \ldots, \{e_p, e_{r+1}\}$; (d) $\{e_{p'-r'}, e_{p'+1}\}, \ldots, \{e_{p'}, e_{p'+r'+1}\}$.

(If $v$ is ordinary, then $r = r' = 0$ and each (white) tile in $F_T(v)$ meets a pair of consecutive edges $e_q, e_{q+1}$ or $e_p, e_1$.) The case with $p = 9$, $p' = 5$, $r = 2$, $r' = 1$ is illustrated in the picture; here the black edges are drawn in bold and the white (black) tiles at $v$ are indicated by thin (bold) arcs.
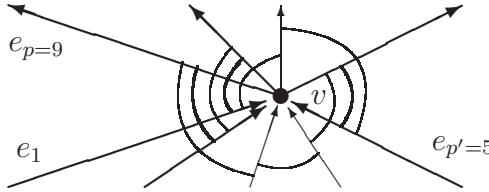


Note that (3.5) implies the following property (which will be used, in particular, in Subsection 4.3):

(3.6) for a tile $\tau \in T$ and a vertex $v \in \{\ell(\tau), r(\tau)\}$, let $e, e'$ be the edges of $\tau$ entering and leaving $v$, respectively, and suppose that there is an edge $\widetilde{e} \neq e, e'$ incident to $v$ and contained in $C(\tau, v)$; then $\widetilde{e}$ is black; furthermore: (a) $e'$ is black if $\widetilde{e}$ enters $v$; (b) $e$ is black if $\widetilde{e}$ leaves $v$.

**4.** We will often use the fact (implied by (2.1)(ii)) that for any g-tiling $T$,

(3.7) the graph $G_T = (V_T, E_T)$ is *graded* for each color $i \in [n]$, which means that for any closed path $P$ in $G_T$, the numbers of forward $i$-edges and backward $i$-edges in $P$ are equal.

Hereinafter, speaking of a path in a directed graph, we mean is a sequence $P = (\widetilde{v}_0, \widetilde{e}_1, \widetilde{v}_1, \ldots, \widetilde{e}_r, \widetilde{v}_r)$ in which each $\widetilde{e}_p$ is an edge connecting vertices $\widetilde{v}_{p-1}, \widetilde{v}_p$; an edge $\widetilde{e}_p$ is called *forward* if it is directed from $\widetilde{v}_{p-1}$ to $\widetilde{v}_p$ (denoted as $\widetilde{e}_p = (\widetilde{v}_{p-1}, \widetilde{v}_p)$), and *backward* otherwise (when $\widetilde{e}_p = (\widetilde{v}_p, \widetilde{v}_{p-1})$). The path $P$ is called: *closed* if $v_0 = v_r$, *directed* if all its edges are forward, and *simple* if all vertices $v_0, \ldots, v_r$ are different. $P^{rev}$ denotes the reversed path $(\widetilde{v}_r, \widetilde{e}_r, \widetilde{v}_{r-1}, \ldots, \widetilde{e}_1, \widetilde{v}_0)$.

# 4 From semi-normal bases to generalized tilings

In this section we prove the second assertion in Theorem 2.2, namely, the inclusion

$$\mathcal{B}_n \subseteq \mathcal{BT}_n, \tag{4.1}$$

where $\mathcal{B}_n$ is the set of semi-normal bases in $2^{[n]}$ and $\mathcal{BT}_n$ denotes the collection of the spectra of g-tilings on $Z_n$. The proof falls into three parts, given in Subsections 4.1–4.3.

## 4.1 Flips in g-tilings

Let $T$ be a g-tiling. By an *M-configuration* in $T$ we mean a quintuple of vertices of the form $Xi, Xj, Xk, Xij, Xjk$ with $i < j < k$ (as it resembles the letter "M"), which is denoted as $CM(X; i, j, k)$. By a *W-configuration* in $T$ we mean a quintuple of vertices $Xi, Xk, Xij, Xik, Xjk$ with $i < j < k$ (as resembling "W"), briefly denoted as $CW(X; i, j, k)$. A configuration is called *feasible* if all five vertices are non-terminal, i.e., they belong to $B_T$.

We know that any normal basis $B$ (in particular, $B = \mathcal{I}_n$) is expressed as $B_T$ for some pure tiling $T$, and therefore, $B \in \mathcal{BT}_n$. Thus, to conclude with (4.1), it suffices to show the following assertion, which says that the set of g-tilings is closed under transformations analogous to flips for semi-normal bases.

**Proposition 4.1** *Let a g-tiling $T$ contain five non-terminal vertices $Xi$, $Xk$, $Xij$, $Xjk$, $Y$, where $i < j < k$ and $Y \in \{Xik, Xj\}$. Then there exists a g-tiling $T'$ such that $B_{T'}$ is obtained from $B_T$ by replacing $Y$ by the other member of $\{Xik, Xj\}$.*

**Proof** We may assume that $Y = Xik$, i.e., that we deal with the feasible W-configuration $CW(X; i, j, k)$ (since an M-configuration in $T$ turns into a W-configurations in the reversed g-tiling $T^{rev}$). We rely on the following two facts which will be proved Subsections 4.2 and 4.3.

(4.2) Any pair of non-terminal vertices $X', X'i'$ in $T$ is connected by edge.

(Therefore, for $T$ as above, $E_T$ contains edges $(Xi, Xij)$, $(Xi, Xik)$, $(Xk, Xik)$ and $(Xk, Xjk)$. Note that vertices $X', X'i'$ need not be connected by edge if some of them is terminal; e.g., in the picture before the statement of Theorem 2.2, the vertices with $X' = \emptyset$ and $i' = 2$ are not connected.)

(4.3) $T$ contains the $jk$-tile $\tau$ with $b(\tau) = Xi$ and the $ij$-tile $\tau'$ with $b(\tau') = Xk$.

Then $\ell(\tau) = Xij$, $r(\tau) = \ell(\tau') = Xik$, $r(\tau') = Xk$, and $t(\tau) = t(\tau') = Xijk$. Since the vertices $Xi, Xk$ are non-terminal, both tiles $\tau, \tau'$ are white. See the picture.
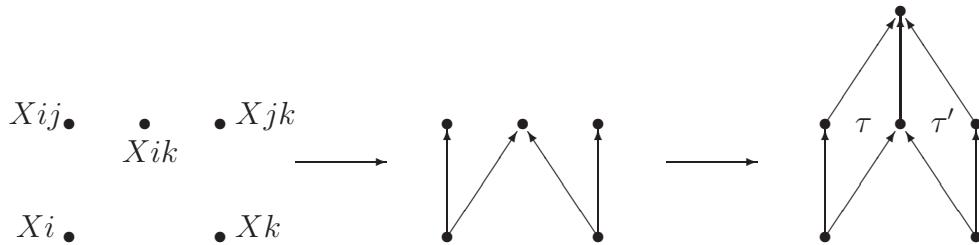


Assuming that (4.2) and (4.3) are valid, we argue as follows. First of all we observe that

(4.4) the vertex $v := Xik$ is ordinary.

Indeed, since both vertices $Xi, Xik$ are non-terminal, the edge $(Xi, Xik)$ cannot belong to a black tile. So the edge $(Xi, Xik)$, which belongs to the white tile $\tau$ and enters $v$, is white. Also the edge $(Xik, Xijk)$ of $\tau$ that leaves $v$ is white (for if it belongs to a black tile $\overline{\tau}$, then $\overline{\tau}$ should have $v' := Xijk$ as its top vertex, but then the cone of $\overline{\tau}$ at $v'$ cannot simultaneously contain both edges $(Xij, Xijk)$ and $(Xjk, Xijk)$, contrary to Corollary 3.1(iii)). Now one can conclude from (3.5) that there is no black tile having its left or right vertex at $v$. So $v$ is ordinary.

Let $e_0, \ldots, e_q$ be the sequence of edges entering $v$ in the counterclockwise order; then $e_0 = (Xi, Xik)$ and $e_q = (Xk, Xik)$. Since $v$ is ordinary, each pair $e_{p-1}, e_p$ $(p = 1, \ldots, q)$ belongs to a white tile $\tau_p$. Two cases are possible.

*Case 1*: The edges $e := (Xij, Xijk)$ and $e' := (Xjk, Xijk)$ do not belong to the same black tile. Consider two subcases.

(a) Let $q = 1$. We replace in $T$ the tiles $\tau, \tau', \tau_1$ by three new white tiles: $\tau(X; i, j)$, $\tau(X; j, k)$ and $\tau(Xj; i, k)$ (so the vertex $v$ is replaced by $Xj$). See the picture.
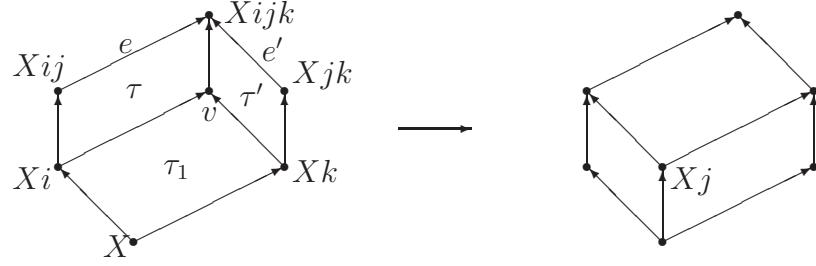


(b) Let $q > 1$. We remove the tiles $\tau, \tau'$ and add four new tiles: the white tiles $\tau(X; i, j)$, $\tau(X; j, k)$, $\tau(Xj; i, k)$ (as before) and the black tile $\tau(X; i, k)$ (so $v$ becomes terminal). See the picture for $q = 3$; here the added black tile is drawn in bold.
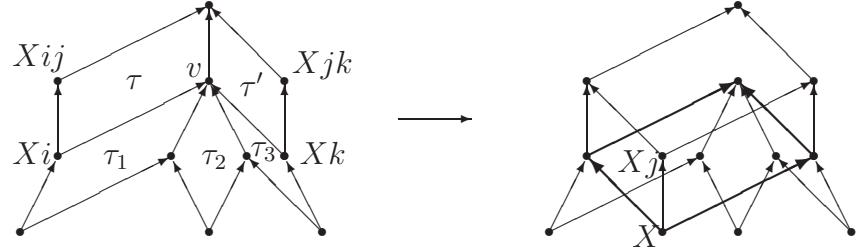


*Case 2*: Both edges $e$ and $e'$ belong to a black tile $\overline{\tau}$ (which is $\tau(Xj; i, k)$). We act as in Case 1 with the only difference that $\overline{\tau}$ is removed from $T$ and the white $ik$-tile at $Xj$ (which is a copy of $\overline{\tau}$) is not added. Then the vertex $Xijk$ vanishes, $v$ either vanishes or becomes terminal, and $Xj$ becomes non-terminal. See the picture; here (a') and (b') concern the subcases $q = 1$ and $q > 1$, respectively, and the arc above the vertex $Xj$ indicates the bottom cone of $\overline{\tau}$ in which some white edges (not indicated) are located.

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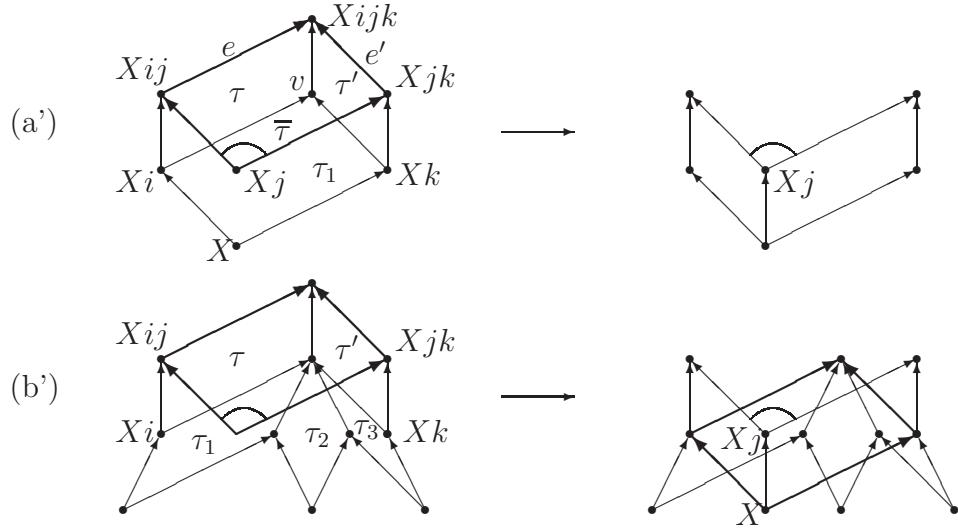
Let $T'$ be the resulting collection of tiles. It is routine to check that in all cases the transformation of $T$ into $T'$ maintains the conditions on tiles and edges involved in axioms (T1)–(T3) at the vertices $Xi, Xk, Xij, Xjk$, as well as at the vertices $Xik$ and $Xijk$ when the last ones do not vanish. Also the conditions continue to hold at the vertex $X$ in Cases 1(a) and 2(a') (with $q = 1$), and at the vertex $Xj$ in Case 2 (when the terminal vertex $Xj$ becomes non-terminal). A less trivial task is to verify for $T'$ the correctness at $Xj$ in Case 1 and at $X$ in Cases 1(b) and 2(b'). We assert that

(4.5) (i) $V_T$ does not contain $Xj$ in Case 1; and (ii) $V_T$ does not contain $X$ in Cases 1(b) and 2(b').

Then these vertices (in the corresponding cases) are indeed new in the arising $T'$, and now the required properties for them become evident by the construction. Note that this implies (T4) as well. We will prove (4.5) in Subsection 4.3.

Thus, assuming validity of (4.2), (4.3), (4.5), we can conclude that $T'$ is a g-tiling and that $B_{T'} = (B_T - \{Xik\}) \cup \{Xj\}$, as required. ∎

**Remark 2** Adopting terminology for set-systems, we say that for the g-tilings $T, T'$ as in the proof of Proposition 4.1, $T'$ is obtained from $T$ by the *lowering flip* w.r.t. the feasible W-configuration $CW(X; i, j, k)$. One can see that $Xi, Xj, Xk, Xij, Xjk$ are non-terminal vertices in $G_{T'}$; so they form a feasible M-configuration for $T'$. Moreover, it is not difficult to check that the corresponding lowering flip applied to the reverse of $T'$ results in the g-tiling $T^{rev}$. Equivalently: the *raising flip* for $T'$ w.r.t. the configuration $CM(X; i, j, k)$ returns the initial $T$. An important consequence of this fact will be demonstrated in Section 7 (see Theorem 7.5).

## 4.2 Strips in a g-tiling

In this subsection we show property (4.3). For this purpose, we introduce the following notion (which will be extensively used subsequently as well).

**Definition.** For $i \in [n]$, an *i-strip* (or a *dual i-path*) in a g-tiling $T$ is a maximal sequence $Q = (e_0, \tau_1, e_1, \ldots, \tau_r, e_r)$ of edges and tiles in it such that: (a) $\tau_1, \ldots, \tau_r$ are

different tiles, each being an $iq$- or $qi$-tile for some $q$, and (b) for $p = 1, \ldots, r$, $e_{p-1}$ and $e_p$ are the opposite $i$-edges of $\tau_p$.

(Recall that speaking of an $i'j'$-tile, we assume that $i' < j'$.) Clearly $Q$ is determined uniquely (up to reversing it and up to shifting cyclically when $e_0 = e_r$) by any of its edges or tiles. Also, unless $e_0 = e_r$, one of $e_0, e_r$ lies on the left boundary, and the other on the right boundary of $Z$; we default assume that $Q$ is directed so that $e_0 \in \ell bd(Z)$. In this case, going along $Q$, step by step, and using (T2), one can see that

(4.6) for consecutive elements $e, \tau, e'$ in an $i$-strip $Q$: (a) if $\tau$ is either a white $iq$-tile or a black $qi$-tile (for some $q$), then $e$ leaves $b(\tau)$ and $e'$ enters $t(\tau)$; and (b) if $\tau$ is either a white $qi$-tile or a black $iq$-tile, then $e$ enters $t(\tau)$ and $e'$ leaves $b(\tau)$ (see the picture where the $i$-edges $e, e'$ are drawn vertically).



Let $v_p$ (resp. $v_p'$) be the beginning (resp. end) vertex of an edge $e_p$ in $Q$. Define the *right boundary* of $Q$ to be the path $R_Q = (v_0, a_1, v_1, \ldots, a_r, v_r)$, where $a_p$ is the edge of $\tau_p$ connecting $v_{p-1}, v_p$. The *left boundary* $L_Q$ of $Q$ is defined in a similar way (regarding the vertices $v_p'$). From (4.6) it follows that

(4.7) for an $i$-strip $Q$, the forward edges of $R_Q$ are exactly those edges in it that belong to either a white $iq$-tile or a black $qi$-tile in $Q$, and similarly for the forward edges of $L_Q$.

For $I \subseteq [n]$, we call a *maximal alternating $I$-subpath* in $R_Q$ a maximal subsequence $P$ of *consecutive* elements in $R_Q$ such that each $a_p \in P$ is a $q$-edge with $q \in I$, and in each pair $a_p, a_{p+1}$, one edge is forward and the other is backward in $R_Q$ (i.e., exactly one of the tiles $\tau_p, \tau_{p+1}$ is black). A maximal alternating $I$-subpath in $L_Q$ is defined in a similar way. The following fact is of importance.

**Lemma 4.2** *A strip $Q$ cannot be cyclic, i.e., its first and last edges are different.*

**Proof** For a contradiction, suppose that some $i$-strip $Q = (e_0, \tau_1, e_1, \ldots, \tau_r, e_r)$ is cyclic ($e_0 = e_r$). One may assume that (4.6) holds for $Q$ (otherwise reverse $Q$). Take the right boundary $R_Q = (a_1, \ldots, a_r = a_0)$ of $Q$. For $q \in [n]$, let $\alpha_q$ ($\beta_q$) denote the number of forward (resp. backward) $q$-edges in $R_Q$. Since $G_T$ is graded, $\alpha_q = \beta_q$ (cf. (3.7)).

Assume that $R_T$ contains a $q$-edge with $q > i$. Put $I^> := [i + 1..n]$ and consider a maximal alternating $I^>$-subpath in $R_Q$ (regarding $Q$ up to shifting cyclically and taking indices modulo $r$). Using (3.1), we observe that if $a_p$ is an edge in $P$ such that $\tau_p$ is black, then the edges $a_{p-1}, a_{p+1}$ are contained in $P$ as well; also both tiles $\tau_{p-1}, \tau_{p+1}$ are white. This together with (4.7) implies that the difference $\Delta_P$ between the number of forward edges and the number of backward edges in $P$ is equal to 0 or 1, and that $\Delta_P = 0$ is possible only if $P$ coincides with the whole $R_Q$ (having equal numbers of

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forward and backward edges). On the other hand, the sum of numbers $\Delta_P$ over the maximal alternating $I^>$-subpaths must be equal to $\sum_{q>i}(\alpha_q - \beta_q) = 0$.

So $R_Q$ is an alternating $I^>$-cycle. To see that this is impossible, notice that if $a_{p-1}, a_p, a_{p+1}$ are $q'$-, $q$-, and $q''$-edges, respectively, and if the tile $\tau_p$ is black, then (3.1) implies that $q', q'' < q$. Therefore, taking the maximum $q$ such that $R_Q$ contains a $q$-edge, we obtain $\alpha_q = 0$ and $\beta_q > 0$; a contradiction. Thus, $R_Q$ has no $q$-edges with $q > i$ at all.

Similarly, considering maximal alternating $[i-1]$-subpaths in $R_Q$ and using (3.1) and (4.7), we conclude that $R_Q$ has no $q$-edge with $q < i$. Thus, a cyclic $i$-strip is impossible. ∎

**Corollary 4.3** *For a g-tiling $T$ and each $i \in [n]$, there is a unique $i$-strip $Q_i$. It contains all $i$-edges of $T$, begins at the edge $p_{i-1}p_i$ and ends at the edge $p'_i p'_{i-1}$ of $bd(Z)$.*

Using strip techniques, we are now able to prove property (4.3) in the assumption that (4.2) is valid (the latter will be shown in the next subsection).

**Proof of (4.3)** Let $X, i, j, k$ as in the hypotheses of Proposition 4.1 (with $Y = Xik$). We consider the part $Q$ of the $j$-strip between the $j$-edges $e := (Xi, Xij)$ and $e' := (Xk, Xjk)$ (these edges exist by (4.2) and $Q$ exists by Corollary 4.3). Suppose that $Q$ begins at $e$ and ends at $e'$ and consider the right boundary $R_Q = (a_1, \ldots, a_r)$ of $Q$. This is a (not necessarily directed) path from $Xi$ to $Xk$. Comparing $R_Q$ with the path $\widetilde{P}$ from $Xi$ to $Xk$ formed by the forward $k$-edge $(Xi, Xik)$ and the backward $i$-edge $(Xik, Xk)$, we have (since $G_T$ is graded):

$$\alpha_q - \beta_q = \begin{cases} -1 & \text{for } q = i, \\ 1 & \text{for } q = k, \\ 0 & \text{otherwise,} \end{cases} \tag{4.8}$$

where $\alpha_q$ ($\beta_q$) is the number of forward (resp. backward) $q$-edges in $R_Q$. We show that $\alpha_i = 0$, $\beta_i = 1$, $\alpha_k = 1$, $\beta_k = 0$ and $\alpha_q = \beta_q = 0$ for $q \neq i, k$, by arguing as in the proof of Lemma 4.2.

Let $P_1, \ldots, P_d$ be the sequence of maximal alternating $J^>$-subpaths in $R_Q$, where $J^> := [j + 1..n]$. Each subpath $P_h$ begins and ends with forward edges (taking into account (3.1),(4.7) and the fact that the edges $e, e'$ are white). Therefore, $\Delta_{P_h} = 1$. Then $\Delta_1 + \ldots + \Delta_{P_d} = \sum_{q>j}(\alpha_q - \beta_q) = \alpha_k - \beta_k = 1$ (cf. (4.8)) implies $d = 1$. Moreover, $|P_1| = 1$. For if $|P_1| > 1$, then $P_1$ contains a backward edge (belonging to a black tile in $Q$), and taking the maximum $q$ such that $P_1$ contains a $q$-edge, we obtain $\alpha_q = 0$ and $\beta_q > 0$, which is impossible. Hence $P_1$ consists of a unique forward edge, and now (4.8) implies that it is a $k$-edge.

By similar reasonings, there is only one maximal alternating $[j-1]$-subpath $P'$ in $R_Q$, and $P'$ consists of a unique backward $i$-edge.

Thus, $R_Q = (a_1, a_2)$, and one of $a_1, a_2$ is a forward $k$-edge, while the other is a backward $i$-edge in $R_Q$. If $R_Q = \widetilde{P}$ (i.e., $a_1$ is a $k$-edge), then the tiles in $Q$ are as required in (4.3). The case when $a_1$ is an $i$-edge is impossible. Indeed, this would imply

that the first tile $\tau$ in $Q$ is generated by the edges $a_1 = (X, Xi)$ and $e = (Xi, Xij)$; but then the cone of $\tau$ at $Xi$ contains the white edge $(Xi, Xik)$, contrary to (3.6).

Now suppose that $Q$ goes from $e'$ to $e$. Then $R_Q$ begins at $Xk$ and ends at $Xi$. Define the numbers $\alpha_q, \beta_q$ as before. Then $\sum_{q>j}(\alpha_q - \beta_q)$ (equal to the numbers of maximal alternating $J^>$-subpaths in $R_Q$) is nonnegative. But a similar value for the path reverse to $\widetilde{P}$ (going from $Xk$ to $Xi$ as well) is equal to $-1$, due to the $k$-edge $(Xi, Xik)$ which is backward in this path; a contradiction. ∎

### 4.3   Strip contractions

The remaining properties (4.2) and (4.5) are proved by induction on $n$, relying on a natural contracting operation on g-tilings (also important for purposes of Sections 8,9).

Let $T$ be a g-tiling on $Z_n$ and $i \in [n]$. We partition $T$ into three subsets $T_i^0, T_i^-, T_i^+$ conisting, respectively, of all $i*$- and $*i$-tiles, of the tiles $\tau(X; i', j')$ with $i', j' \neq i$ and $i \notin X$, and of the tiles $\tau(X; i', j')$ with $i', j' \neq i$ and $i \in X$. Then $T_i^0$ is the set of tiles occurring in the $i$-strip $Q_i$, and the tiles in $T_i^-$ are vertex disjoint from those in $T_i^+$.

**Definition.** The $i$-*contraction* of $T$ is the collection $T/i$ obtained by removing the members of $T_i^0$, keeping the members of $T_i^-$, and replacing each $\tau(X; i', j') \in T_i^+$ by $\tau(X - \{i\}; i', j')$. The image of $\tau \in T$ in $T/i$ is denoted by $\tau/i$ (regarding it as the "void tile" if $\tau \in T_i^0$). The black/white coloring of tiles in $T/i$ is inherited from $T$.

The tiles of $T/i$ live within the zonogon generated by the vectors $\xi_q$, $q \in [n] - \{i\}$ (and cover this zonogon). The regions $D_{T_i^-}$ and $D_{T_i^+}$ of the disc $D_T$ are simply connected, as they arise when the interior of (the image of) the strip $Q_i$ is removed from $D_T$. The shape $D_{T/i}$ is obtained as the union of $D_{T_i^-}$ and $D_{T_i^+} - \epsilon_i$, where $\epsilon_i$ is the $i$-th unit base vector in $\mathbb{R}^n$. In other words, $D_i^+$ is shifted by $-\epsilon_i$ and the (image of) the left boundary $L_{Q_i}$ of $Q_i$ in it merges with (the image of) $R_{Q_i}$ in $D_{T_i^-}$. In general, $D_{T_i^-}$ and $D_{T_i^+} - \epsilon_i$ may intersect at some other points, and therefore, $D_{T/i}$ need not be a disc (this happens when $G_T$ contains two vertices $X, Xi$ not connected by edge, or equivalently, such that $X \notin R_{Q_i}$ and $Xi \notin L_{Q_i}$).

For our purposes, it suffices to deal with the case $i = n$. We take advantages from the important property that $T/n$ is a feasible g-tiling, i.e., obeys (T1)–(T4). Instead of a direct proof of this property (in which verification of axiom (T4) is rather tiresome), we prefer to appeal to explanations in Section 7 where a similar property is obtained on the language of wirings; see Corollary 7.4 in Remark 4. (More precisely, by results in Sections 6,7, there is a bijection $\beta$ of the g-tilings to the proper wirings. Furthermore, one shows that removing $w_n$ from a proper $n$-wiring $W = (w_1, \ldots, w_n)$ results in a proper $(n-1)$-wiring $W'$. It turns out that the g-tiling $\beta^{-1}(W')$ is just $(\beta^{-1}(W))/n$, yielding the desired property.)

We will use several facts, exposed in (i),(ii) below.

(i) In light of explanations above, validity of (T4) for $T/n$ implies that

(4.9) any two vertices of the form $X', X'n$ in $G_T$ are connected by edge (which is an $n$-edge, and therefore, $X' \in R_{Q_n}$ and $X'n \in L_{Q_n}$).

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(ii) For a black $i'j'$-tile $\tau \in T$ with $i', j' \neq n$, none of the vertices $t(\tau), b(\tau)$ can occur in (the boundary of) $Q_n$. Indeed, all edges incident to such a vertex are $q$-edges with $q \leq j' < n$, whereas each vertex occurring in $Q_n$ is incident to an $n$-edge. Also for a vertex $X'$ not occurring in $Q_n$, the local tile structure of $T/n$ at $X' - \{n\}$ (including the white/black coloring of tiles) is inherited by that of $T$ at $X'$. It follows that

(4.10) if $X'$ is a non-terminal vertex for $T$, then $X' - \{n\}$ is such for $T/n$.

Now we are ready to prove (4.2) and (4.5).

**Proof of (4.2)** Let $X', X'i'$ be non-terminal vertices for $T$. If $i' = n$ then these vertices are connected by edge in $G_T$, by (4.9). Now let $i' \neq n$. Then $G_{T/n}$ contains the vertices $\widetilde{X}, \widetilde{X}i'$, where $\widetilde{X} := X' - \{n\}$, and these vertices are non-terminal, by (4.10). We assume by induction that these vertices are connected by some edge $e$ in $G_{T/n}$. Let $\tau'$ be a tile in $T/n$ containing $e$. Then the tile $\tau \in T_n^- \cup T_n^+$ such that $\tau' = \tau/n$ has an edge connecting $X'$ and $X'i'$, as required. ∎

**Proof of (4.5)** We use notation as in the proof of Proposition 4.1 and consider three possible cases.

(A) Let $k < n$ and $n \notin X$. Then all tiles in $T$ containing the vertex $v = Xik$ are tiles in $T/n$, and $Xi, Xk, Xik, Xij, Xjk$ are vertices for $T/n$ forming a feasible W-configuration in it (as they are non-terminal, by (4.10)). By induction $G_{T/n}$ contains as a vertex neither $Xj$ in Case 1, nor $X$ in Cases 1(b) and 2(b'). Then the same is true for $G_T$, as required.

(B) Let $k < n$ and $n \in X$. The argument is similar to that in (A) (taking into account that all vertices $X'$ for $T$ that we deal with contain the element $n$, and the corresponding vertices for $T/n$ are obtained by removing this element).

(C) Let $k = n$. First we consider Case 1 and show (i) in (4.5). Suppose $G_T$ contains the vertex $Xj$. Then $G_T$ contains the $n$-edge $\widetilde{e} = (Xj, Xjn)$, by (4.9). This edge lies in the cone of $\tau'$ at $Xjn$ (where $\tau'$ is the white $ij$-tile with $b(\tau') = Xn$, $r(\tau') = Xjn$ and $t(\tau') = Xijn$). By (3.6), the presence of the edge $\widetilde{e}$ (entering $Xjn$) in this cone implies that the edge $e' = (Xjn, Xijn)$ of $\tau'$ is black. Let $\widetilde{\tau}$ be the black tile containing $e'$; then $\widetilde{\tau}$ has the top vertex at $Xijn$ (since $Xjn$ is non-terminal). The $n$-edge $e = (Xij, Xijn)$ entering $Xijn$ must be the left-to-top edge of $\widetilde{\tau}$. So both $e, e'$ are edges of the same black tile $\widetilde{\tau}$, which is not the case.

Now we consider Cases 1(b) and 2(b') and show (ii) by arguing in a similar way. Suppose $G_T$ contains the vertex $X$. Then $G_T$ contains the $n$-edge $\widetilde{e} = (X, Xn)$, by (4.9). This edge lies in the cone of $\tau_q$ at $Xn$ (where, according to notation in the proof of Proposition 4.1, $\tau_q$ is the white tile in $T$ with $r(\tau_q) = Xn$ and $t(\tau') = Xin$). By (3.6), the edge $e'' := (Xn, Xin)$ of $\tau_q$ is black (since $\widetilde{e}$ enters $Xn$). But each of the end vertices $Xn, Xin$ of $e''$ has both entering and leaving edges, and therefore, it cannot be terminal; a contradiction. ∎

This completes the proof of inclusion (4.1).

# 5 From generalized tilings to semi-normal bases

In this section we complete the proof of Theorem 2.2 by proving the first assertion in it, namely, we show the inclusion

$$\mathcal{BT}_n \subseteq \mathcal{B}_n. \tag{5.1}$$

This together with the reverse inclusion (4.1) will give $\mathcal{BT}_n = \mathcal{B}_n$, as required.

Let $T$ be a g-tiling. We have to prove that $B_T$ is a semi-normal basis.

If $T$ has no black tile, then $B_T$ is a normal basis, and we are done. So assume $T^b \neq \emptyset$. Our aim is to show the existence of a feasible W-configuration $CW(X; i, j, k)$ for $T$ (formed by non-terminal vertices $Xi, Xk, Xij, Xik, Xjk$, where $i < j < k$). Then we can transform $T$ into a g-tiling $T'$ as in Proposition 4.1, i.e., with $B_{T'} = (B_T - \{Xik\}) \cup \{Xj\}$. Under such a *lowering flip* (concerning g-tilings), the sum of sizes of the sets involved in $B_\bullet$ decreases. Then the required relation $B_T \in \mathcal{B}_n$ follows by induction on $\sum(|X'|\colon X' \in B_T)$ (this sort of induction is typical when one deals with tilings or related objects, cf. [2, 3, 7]).

In what follows by the *height* $h(v)$ of a vertex $v \in V_T$ we mean the size of the corresponding subset of $[n]$. The height $h(\tau)$ of a tile $\tau \in T$ is defined to be the height of its left vertex; then $h(\tau) = h(r(\tau)) = h(b(\tau)) + 1 = h(t(\tau)) - 1$. The height of a W-configuration $CW(X; i, j, k)$ is defined to be $|X| + 2$.

In fact, we are able to show the following sharper version of the desired property.

**Proposition 5.1** *Let $h \in [n]$. If a g-tiling $T$ has a black tile of height $h$, then there exists a feasible W-configuration $CW(X; i, j, k)$ of the same height $h$. Moreover, such a $CW(X; i, j, k)$ can be chosen so that $Xijk$ is the top vertex of some black tile (of height $h$).*

**Proof** Let $\tau$ be a black tile of height $h$. Denote by $M(\tau)$ the set of vertices $v$ such that there is a white edge from $v$ to $t(\tau)$. This set is nonempty (by Corollary 3.1(ii)) and each vertex in it is non-terminal. Suppose that some $v \in M(\tau)$ is ordinary, and let $\lambda$ and $\rho$ be the (white) tiles sharing the edge $(v, t(\tau))$ and such that $v = r(\lambda) = \ell(\rho)$. Then the five vertices $b(\lambda), b(\rho), \ell(\lambda), v, r(\rho)$ form a W-configuration of height $h$ (since $h(v) = h(\tau) = h$). Moreover, this configuration is feasible. Indeed, the vertices $\ell(\lambda), v, r(\rho)$ are non-terminal (since each has an entering edge and a leaving edge). And the tile $\widetilde{\tau}$ that shares the edge $(b(\lambda), v)$ with $\lambda$ has $v$ as its top vertex (taking into account that $\widetilde{\tau}$ is white and overlaps neither $\lambda$ nor $\rho$ since $v$ is ordinary); then $b(\lambda)$ is the left vertex of $\widetilde{\tau}$, and therefore $b(\lambda)$ is non-terminal. The vertex $b(\rho)$ is non-terminal for a similar reason. For an illustration, see the left fragment on the picture.
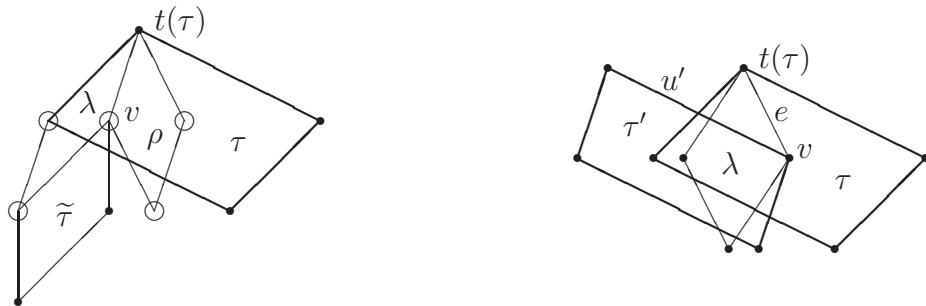
We assert that a black tile $\tau$ of height $h$ whose set $M(\tau)$ contains an ordinary vertex does exist (yielding the result).

Suppose this is not so. Let us construct an alternating sequence of white and black edges as follows. Choose a black tile $\tau$ of height $h$ and a vertex $v \in M(\tau)$. Let $e$ be the white edge $(v, t(\tau))$. Since $v$ is mixed (by the supposition), there is a black tile $\tau'$ (of height $h$) such that either (a) $v = r(\tau')$ or (b) $v = \ell(\tau')$. We say that $\tau'$ *lies on the left from* $\tau$ in the former case, and *lies on the right from* $\tau$ in the latter case. Let $u'$ be the right-to-top edge $(r(\tau'), t(\tau'))$ of $\tau'$ in case (a), and the left-to-top edge $(\ell(\tau'), t(\tau'))$ in case (b). Case (a) is illustrated on the right fragment of the above picture.

Repeat the procedure for $\tau'$: choose $v' \in M(\tau')$ (which is mixed again by the supposition); put $e' := (v', t(\tau'))$; choose a black tile $\tau''$ such that either (a) $v' = r(\tau'')$ or (b) $v' = \ell(\tau'')$; and define $u'$ to be be the edge $(r(\tau''), t(\tau''))$ in case (a), and the edge $(\ell(\tau''), t(\tau''))$ in case (b). Repeat the procedure for $\tau''$, and so on. Sooner or later we must return to a black tile that has occurred earlier in the process. Then we obtain an alternating cycle of white and black edges.

More precisely, there appear a cyclic sequence of different black tiles $\tau_1, \ldots, \tau_{r-1}, \tau_r = \tau_0$ of height $h$ and an alternating sequence of white and black edges $C = (e_0 = e_r, u_1, e_1, \ldots, u_r = u_0)$ (forming a cycle in $G_T$) with the following properties, for $q = 1, \ldots, r$: (a) $e_q$ is the edge $(v_q, t(\tau_q))$ for some $v_q \in M(\tau_q)$; (b) $\tau_{q+1}$ is a black tile whose right of left vertex is $v_q$; and (c) $u_{q+1} = (r(\tau_{q+1}), t(\tau_{q+1}))$ when $r(\tau_{q+1}) = v_q$, and $u_{q+1} = (\ell(\tau_{q+1}), t(\tau_{q+1}))$ when $\ell(\tau_{q+1}) = v_q$, where the indices are taken modulo $r$. We consider $C$ up renumbering the indices cyclically and assume that $\tau_q$ is an $i_q k_q$-tile, that $e_q$ is a $j_q$-edge, and that $u_q$ is a $p_q$-edge. Then $i_q < j_q < k_q$, $p_q = i_q$ if $\tau_q$ lies on the left from $\tau_{q-1}$, and $p_q = k_q$ if $\tau_q$ lies on the right from $\tau_{q-1}$. Note that in the former (latter) case the vertex $v_q$ lies on the left (resp. right) from $v_{q-1}$ in the horizontal line at height $h$ in $Z$. This implies that there exists a $q$ such that $\tau_q$ lies on the left from $\tau_{q-1}$, and there exists a $q'$ such that $\tau'_q$ lies on the right from $\tau_{q'-1}$.

To come to a contradiction, consider a maximal subsequence $Q$ of consecutive tiles in which each but first tile lies on the left from the previous one; one may assume that $Q = (\tau_1, \tau_2, \ldots, \tau_d)$. Then $\tau_1$ lies on the right from $\tau_0$; so $u_1$ is the left-to-top edge of $\tau_1$, whence $p_1 = k_1$. Also we observe that

$$k_1 \geq k_2 \geq \ldots \geq k_d. \tag{5.2}$$

Indeed, for $1 \leq q < d$, let $\lambda$ be the (white) tile containing the edge $e_q$ and such that $r(\lambda) = v_q$. This tile lies in the cone of $\tau_q$ at $t(\tau_q)$. So $\lambda$ is an $i'k'$-tile with $i_q \leq i' < k' \leq k_q$, and therefore, the (bottom-to-right) edge $\widetilde{e}$ of $\lambda$ entering $v_q$ has color $k' \leq k_q$. Using (3.5)(iii), we observe that $\widetilde{e}$ lies in the cone of the black tile $\tau_{q+1}$ at $v_q$ (taking into account that $v_q = r(\tau_{q+1})$). This implies that the bottom-to-right edge $e' = (b(\lambda), v_q)$ of $\tau_{q+1}$ has color at most $k'$. Since $u_{q+1}$ is parallel to $e'$, we obtain $k_{q+1} \leq k' \leq k_q$, as required.

By (5.2), we have $j_q < k_q \leq k_1 = p_1$ for all $q = 1, \ldots, d$. Also if a tile $\tau_{q'}$ lies on the right from the previous tile $\tau_{q'-1}$, then $u_{q'}$ is the left-to-top edge of $\tau_{q'}$, whence $j_{q'} < k_{q'} = p_{q'}$. Thus, the maximum of $p_1, \ldots, p_r$ is strictly greater than the maximum of $j_1, \ldots, j_r$. This is impossible since all $u_1, \ldots, u_r$ are forward edges, all $e_1 \ldots, e_r$ are backward edges in $C$, and the graph $G_T$ is graded. ∎

This completes the proof of Theorem 2.2.

**Remark 3** For black tiles $\tau, \tau' \in T^b$, let us denote $\tau' \prec^\diamond \tau$ if there is a white edge $(v, t(\tau))$ such that $v$ is the right or left vertex of $\tau'$. The proof of Proposition 5.1 gives the following additional result.

**Corollary 5.2** *The relation $\prec^\diamond$ determines a partial order on $T^b$.*

Similarly, the relation $\prec_\diamond$ determines a partial order on $T^b$, where for $\tau, \tau' \in T^b$, we write $\tau' \prec_\diamond \tau$ if there is a white edge $(b(\tau), v)$ such that $v$ is the right or left vertex of $\tau'$. It is possible that the graph on $T^b$ induced by $\prec^\diamond$ (or $\prec_\diamond$) is a forest, but we do not go in our analysis so far.

We conclude this section with one more result which easily follows from Proposition 5.1.

**Proposition 5.3** *Let a g-tiling $T$ be such that, for some $h < n$, all non-terminal vertices of height $h+1$ are intervals in $[n]$ and there is no feasible W-configuration of height $h$. Then all non-terminal vertices of height $h$ are intervals as well. Symmetrically, if for some $h > 0$, all non-terminal vertices of height $h - 1$ are co-intervals and there is no feasible M-configuration of height $h$, then all non-terminal vertices of height $h$ are co-intervals.*

**Proof** If $T$ has a black tile of height $h$, then there exists a feasible W-configuration of height $h$, by Proposition 5.1. So this is not the case.

Let $u$ be a non-terminal vertex of height $h$ and take an edge $(u, v)$. Then the vertex $v$ (of height $h + 1$) is non-terminal (for otherwise $v$ would be the top vertex of a black tile of height $h$). Let $\tau, \tau'$ be the tiles sharing the edge $(u, v)$; then both tiles are white and non-overlapping. Suppose $b(\tau) = u$. Then both vertices $\ell(\tau), r(\tau)$ lie in level $h+1$, and therefore, they are intervals. This easily implies that $u$ is an interval as well. Similarly, $u$ is an interval if $b(\tau') = u$.

Now suppose that $u$ is neither $b(\tau)$ nor $b(\tau')$. Then $t(\tau) = t(\tau') = v$. Letting for definiteness that $u = r(\tau) = \ell(\tau')$, we obtain that the vertices $u, b(\tau), b(\tau'), \ell(\tau), r(\tau')$ form a feasible W-configuration of height $h$; a contradiction (the case when $b(\tau)$ or $b(\tau')$ is terminal is impossible, otherwise it would belong to a black tile of height $h$).

The second assertion in the theorem follows from the first one by considering the reversed tiling. ∎

(Note that if all non-terminal vertices of height $h$ in a g-tiling are intervals, then there is no feasible W-configuration of height $h$. Indeed, suppose such a configuration $CW(X; i, j, k)$ exists. Then $Xik$ is a non-terminal vertex of height $h$. Since $i < j < k$ and $j \notin X$, the set $Xik$ is not an interval.)

In view of the coincidence of the set of spectra of g-tilings on $Z_n$ with the set of largest weakly separated collections in $2^{[n]}$ (proved in Section 9), Proposition 5.3 answers affirmatively Conjecture 5.5 in Leclerc and Zelevinsky [7].

Finally, analyzing the proof of Proposition 5.3, one can see that this proposition can be slightly strengthened as follows: if there is no feasible W-configuration of height $h$,

then each non-terminal vertex $Y \subset [n]$ of height $h$ not contained in the boundary of $Z_n$ is representable as $Y' \cap Y''$ for some non-terminal vertices $Y', Y'' \subset [n]$ of height $h+1$. (Similarly, if there is no feasible M-configuration of height $h$, then each non-terminal vertex $Y$ of height $h$ not contained in the boundary of $Z_n$ is representable as $Y' \cup Y''$ for some non-terminal vertices $Y', Y''$ of height $h-1$.)

# 6 From generalized tilings to proper wirings

In this section we show the following

**Proposition 6.1** *For any g-tiling $T$ on $Z_n$, there exists a proper wiring $W$ on $Z_n$ such that $B_W = B_T$.*

This and the converse assertion established in the next section will imply that the collection $\mathcal{BT}_n$ of the spectra $B_T$ of g-tilings on $Z_n$ coincides with the collection $\mathcal{BW}_n$ of the spectra $B_W$ of proper wirings $W$ on $Z_n$, and then Theorem 2.1 will follow from Theorem 2.2.

**Proof** For convenience we identify and use the same notation for vertices, edges and tiles concerning a g-tiling $T$ on $Z = Z_n$ and their corresponding points, line-segments and squares (respectively) in the disc $D_T$ (every time it will be clear from the context or explicitly indicated which of $Z$ and $D_T$ we deal with). Accordingly, the planar graph $G_T = (V_T, E_T)$ is regarded as properly embedded in $D_T$.

In order to construct the desired wiring, we first draw curves on $D_T$ associated to strips (dual paths) in $G_T$. More precisely, for each $i \in [n]$, take the $i$-strip $Q_i = (e_0, \tau_1, e_1, \ldots, \tau_r, e_r)$ for $T$ (defined in Section 4), considering it as the corresponding sequence of edges and squares in $D_T$. (Recall that $Q_i$ contains all $i*$- and $*i$-tiles in $T$, $e_0$ is the edge $p_{i-1}p_i$ on the left boundary $\ell bd(Z)$, and $e_r$ is the edge $p'_i p'_{i-1}$ on $rbd(Z)$; cf. Corollary 4.3.) For $q = 1, \ldots, r$, draw the line-segment on the square $\tau_q$ connecting the median points of the edges $e_{r-1}$ and $e_r$. This segment meets the central point of $\tau_q$, denoted by $c(\tau_q)$. The concatenation of these segments gives the desired (piece-wise linear) curve $\zeta_i$ corresponding to $Q_i$; we direct $\zeta_i$ according to the direction of $Q_i$.

Fix a homeomorphic map $\gamma : D_T \to Z$ such that each boundary edge of $D_T$ is linearly mapped onto the corresponding edge of $bd(Z)$. Then the curves ("wires") $\zeta_i$ on $D_T$ generate the wires $w_i := \gamma(\zeta_i)$ on $Z$ (where $w_i$ begins at the median point $s_i$ of $p_{i-1}p_i$ on $\ell bd(Z)$ and ends at the medial point $s'_i$ of $p'_i p'_{i-1}$ on $rbd(Z)$). We assert that the wiring $W = (w_1, \ldots, w_n)$ is as required in the proposition.

Obviously, $W$ satisfies axiom (W1) for $W$. To verify the other axioms, we first explain how the planar graphs $G_T$ and $G_\zeta$ on $D_T$ are related to each other, where $G_\zeta$ is the "preimage" by $\gamma$ of the graph $G_W$ (it is defined as in Subsection 2.1 by considering $\zeta_1, \ldots, \zeta_n$ in place of $w_1, \ldots, w_n$). The vertices of $G_\zeta$ are the central points $c(\tau)$ of squares $\tau$ (where corresponding wires cross one another) and the points $s_i, s'_i$. (We identify corresponding points on the boundaries of $D_T$ and $Z$, writing $s_i$ for $\gamma^{-1}(s_i)$.) Each vertex $v$ of $G_T$ one to-one corresponds to the face of $G_\zeta$ where $v$ is located, denoted by $v^*$. The edges of color $i$ in $G_\zeta$ (which are the pieces of $\zeta_i$ obtained by its

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subdivision by the central points of squares lying on $\zeta_i$) one-to-one correspond to the $i$-edges of $G_T$. More precisely, if an $i$-edge $e \in E_T$ belongs to squares $\tau, \tau'$ and if $\tau, e, \tau'$ occur in this order in the $i$-strip, then the $i$-edge of $G_\zeta$ corresponding to $e$, denoted by $e^*$, is the piece of $\zeta_i$ between $c(\tau)$ and $c(\tau')$, and this $e^*$ is directed from $c(\tau)$ to $c(\tau')$. Observe that $e$ crosses $e^*$ *from right to left* on the disc. (We assume that the clockwise orientation on $D_T$ is agreeable by $\gamma$ with that of $Z$.) The first and last pieces of $\zeta_i$ correspond to the boundary $i$-edges $p_{i-1}p_i$ and $p_i'p_{i-1}'$ of $G_T$, respectively.

Consider an $ij$-tile $\tau \in T$, and let $e, e'$ be its $i$-edges, and $u, u'$ its $j$-edges, where $e, u$ leave $b(\tau)$ and $e', u'$ enter $t(\tau)$. We know that: (a) if $\tau$ is white, then $e$ occurs in $Q_i$ before $e'$, while $u$ occurs in $Q_j$ after $u'$, and (b) if $\tau$ is black, then $e$ occurs in $Q_i$ after $e'$, while $u$ occurs in $Q_j$ before $u'$. In each case, in the disc $D_T$, both $e, e'$ cross the wire $\zeta_i$ from right to left (w.r.t. the direction of $\zeta_i$), and similarly both $u, u'$ cross $\zeta_j$ from right to left. Also it is not difficult to see (using (T1),(T2)) that when $\tau$ is white, the orientation of the tile $\tau$ in $Z$ coincides with that of the square $\tau$ in $D_T$, whereas when $\tau$ is black, the clockwise orientation of $\tau$ in $Z$ turns in the counterclockwise orientation of $\tau$ in $D_T$ (causing the "orientation-reversing" behavior of wires at the vertex $c(\tau)$ of $G_\zeta$). It follows that: in case (a), $\zeta_j$ crosses $\zeta_i$ at $c(\tau)$ from left to right, and therefore, the vertex $c(\tau)$ of $G_\zeta$ is white, and in case (b), $\zeta_j$ crosses $\zeta_i$ at $c(\tau)$ from right to left, and therefore, the vertex $c(\tau)$ is black. (Both cases are illustrated in the picture.) So the white (black) tiles of $T$ generate the white (resp. black) vertices of $G_\zeta$.



(a) ... (in $Z$) ... (in $D_T$) ... (b) ... (in $Z$) ... (in $D_T$)
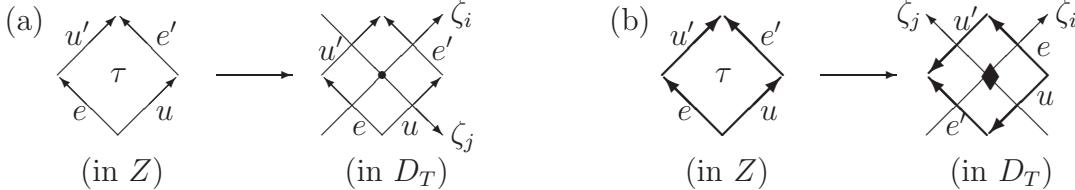
Consider a vertex $v$ of $G_T$ and an edge $e \in E_T(v)$. Then the edge $e^*$ belongs to the boundary of the face $v^*$ of $G_\zeta$. As mentioned above, $e$ crosses $e^*$ from right to left on $D_T$. This implies that $e^*$ is directed clockwise around $v^*$ if $e$ leaves $v$, and counterclockwise if $e$ enters $v$. In view of axiom (T3), we obtain that

(6.1) the terminal vertices of $G_T$ and only these generate cyclic faces of $G_W \simeq G_\zeta$; moreover, for $\tau \in T^b$, the boundary cycle of $(t(\tau))^*$ is directed counterclockwise, while the boundary cycle of $(b(\tau))^*$ is directed clockwise.

(We use the fact that any non-terminal vertex $v \neq p_0, p_n$ has both entering and leaving edges, and therefore, the boundary of the face $v^*$ has edges in both directions. When $v = p_0$ ($v = p_n$), a similar fact for $v^*$ is valid as well, since $v^*$ contains the edges $(p_0, s_1)$ and $(p_0, s_n')$ (resp. $(s_n, p_n)$ and $(s_1', p_n)$) lying on the boundary of $Z$.)

Next, for each $i \in [n]$, removing from $D_T$ the interior of the $i$-strip $Q_i$ (i.e., the relative interiors of all edges and tiles in it) results in two closed regions $\Omega_1, \Omega_2$, the former containing the vertex $\emptyset$, and the latter containing the vertex $[n]$ (regarding the vertices as subsets of $[n]$). The fact that all edges in $Q_i$ (which are the $i$-edges of $G_T$) go from $\Omega_1$ to $\Omega_2$ implies that each vertex $v$ of $G_T$ occurring in $\Omega_1$ (in $\Omega_2$) is a subset

in $[n]$ not containing (resp. containing) the element $i$. So $i \notin X(v^*)$ if $v \in \Omega_1$, and $i \in X(v^*)$ if $v \in \Omega_2$. This implies the desired equality for spectra: $B_W = B_T$.

A less trivial task is to check validity of (W2) for $W$. One can see that axiom (W2) is equivalent to the condition that if wires $w_i, w_j$ intersect at a point $x$ and this point is white, then the parts of $w_i, w_j$ after $x$ do not intersect. So it suffices to show the following

**Claim** Let wires $\zeta_i, \zeta_j$ with $i < j$ intersect at a white point $x$. Then the part $\zeta$ of $\zeta_i$ from $x$ to $s'_i$ and the part $\zeta'$ of $\zeta_j$ from $x$ to $s'_j$ have no other common points.

**Proof of the Claim** Suppose this is not so and let $y$ be the common point of $\zeta, \zeta'$ next to $x$ in $\zeta$. Since $x$ is white, $y$ is black. Therefore, the $ij$-tile $\tau$ such that $x = c(\tau)$ is white, and the $ij$-tile $\tau'$ such that $y = c(\tau')$ is black. Also in both strips $Q_i$ and $Q_j$, the tile $\tau$ occurs earlier than $\tau'$. One can see (cf. (4.6)) that in the strip $Q_i$, the edge succeeding $\tau$ is $(r(\tau), t(\tau))$ and the edge preceding $\tau'$ is $(r(\tau'), t(\tau'))$, whereas in the strip $Q_j$, the edge succeeding $\tau$ is $(b(\tau), r(\tau))$ and the edge preceding $\tau'$ is $(b(\tau'), r(\tau'))$. So the right boundary of $Q_i$ passes the vertices $r(\tau)$ and $r(\tau')$, in this order, and similarly for the left boundary of $Q_j$.

Consider the part $R$ of $R_{Q_i}$ from $r(\tau)$ to $r(\tau')$ and the part $L$ of $L_{Q_j}$ from $r(\tau)$ to $r(\tau')$. For $q \in [n]$, let $\alpha_q, \alpha'_q, \beta_q, \beta'_q$ be the numbers of $q$-edges that are forward in $R$, forward in $L$, backward in $R$, and backward in $L$, respectively. Since $G_T$ is graded, we have (∗) $\alpha_q - \beta_q = \alpha'_q - \beta'_q$.

Next we argue in a similar spirit as in the proof of Lemma 4.2. Define $I := [i+1..j-1]$, $\Delta := \sum_{q \in I}(\alpha_q - \beta_q)$, and $\Delta' := \sum_{q \in I}(\alpha'_q - \beta'_q)$. We assert that $\Delta > 0$ and $\Delta' < 0$, which leads to a contradiction with (∗) above.

To see $\Delta > 0$, consider a $q$-edge $e$ in $R$ with $q \in I$, and let $\tau^e$ denote the tile in $Q_i$ containing $e$. Since $q > i$, $\tau^e$ is white if $e$ is forward, and $\tau^e$ is black if $e$ is backward in $R$ (cf. (4.7)). Using this, one can see that:

(i) for a $q$-edge $e \in R$ such that $q \in I$ and $\tau^e$ is black, the next edge $e'$ in $R_{Q_i}$ is a forward $q'$-edge in $R$ with $q' \in I$ (since the fact that $\tau^e$ is a black $iq$-tile implies that $\tau^{e'}$ is a white $q'q$-tile with $i < q' < q$, in view of (3.1)); a similar property holds for the edge in $R_{Q_i}$ preceding $e$;

(ii) the last edge $e$ of $R$ is a forward $q$-edge with $q \in I$ (since the tile $\tau^e$ shares an edge with the black $ij$-tile $\tau'$);

(iii) if the first edge $e$ of $R$ is backward, then it is a $q$-edge with $q \notin I$ (since $\tau^e$ is black and shares an edge with the white $ij$-tile $\tau$).

These observations show that the first and last edges of any maximal alternating $I$-subpath $P$ in $R$ are forward, and therefore, $P$ contributes $+1$ to $\Delta$. Also at least one such $P$ exists, by (ii). So $\Delta > 0$, as required.

The inequality $\Delta' < 0$ is shown in a similar way, by considering $L$ and swapping "forward" and "backward" in the above reasonings (due to replacing $q > i$ by $q < j$). More precisely, for a $q$-edge $e$ in $L$ with $q \in I$, the tile $\tau^e$ in $Q_j$ containing $e$ is black if $e$ is forward, and white if $e$ is backward (in view of $q < j$ and (4.7)). This implies that:

(i') for a $q$-edge $e \in L$ such that $q \in I$ and $\tau^e$ is black, the next edge $e'$ in $L_{Q_j}$ is a backward $q'$-edge in $L$ with $q' \in I$; and similarly for the previous edge in $L_{Q_j}$;

(ii') the last edge $e$ of $L$ is a backward $q$-edge with $q \in I$;

(iii') if the first edge $e$ of $R$ is forward, then it is a $q$-edge with $q \notin I$.

Then the first and last edges of any maximal alternating $I$-subpath $P$ in $L$ are backward, and therefore, $P$ contributes $-1$ to $\Delta'$. Also at least one such $P$ exists, by (ii'). Thus, $\Delta' < 0$, obtaining a contradiction with $\Delta = \Delta'$. ∎

Thus, (W2) is valid. Considering lenses formed by a pair of wires and using (6.1) and (W2), one can easily obtain (W3). Finally, since $|E_T(v)| \geq 3$ holds for each terminal vertex in $G_T$ (by Corollary 3.1(ii)), each cyclic face in $G_W$ is surrounded by at least three edges, and therefore, this face cannot be a lens. So the wiring $W$ is proper.

This completes the proof of Proposition 6.1. ∎

# 7 From proper wirings to generalized tilings

In this section we complete the proof of Theorem 2.1 by showing the converse to Proposition 6.1.

**Proposition 7.1** *For a proper wiring $W$ on $Z = Z_n$, there exists a g-tiling $T$ on $Z$ such that $B_T = B_W$.*

**Proof** The construction of the desired $T$ is converse, in a sense, to that described in the proof of Proposition 6.1; it combines planar duality techniques and geometric arrangements.

We associate to each (inner) face $F$ of the graph $G_W$ the point (viz. the subset) $X(F)$ in the zonogon, also denoted as $F^*$. These points are just the vertices of tiles in $T$. The edges concerning $T$ are defined as follows. Let faces $F, F' \in \mathcal{F}_W$ have a common edge $e$ formed by a piece of a wire $w_i$, and let $F$ lie on the right from $w_i$ according to the direction of this wire (and $F'$ lies on the left from $w_i$). Then the vertices $F^*, F'^*$ are connected by edge $e^*$ going from $F^*$ to $F'^*$. Note that in view of the evident relation $X(F') = X(F) \cup \{i\}$, the direction of $e^*$ matches the edge direction for g-tilings.

The tiles in $T$ one-to-one correspond to the intersection points of wires in $W$. More precisely, let $v$ be a common point of wires $w_i, w_j$ with $i < j$. Then the vertex $v$ of $G_W$ has four incident edges $e_i, \overline{e}_i, e_j, \overline{e}_j$ such that: $e_i, \overline{e}_i \subset w_i$; $e_j, \overline{e}_j \subset w_j$; $e_i, e_j$ enter $v$; and $\overline{e}_i, \overline{e}_j$ leave $v$. Also one can see that for the four faces $F$ containing $v$, the subsets $X(F)$ are of the form $X, Xi, Xj, Xij$ for some $X \subset [n]$. The tile surrounded by the edges $e_i^*, \overline{e}_i^*, e_j^*, \overline{e}_j^*$ connecting these subsets (regarded as points) is just the $ij$-tile in $T$ corresponding to $v$, denoted as $v^*$. Observe that the edges $e_i, e_j, \overline{e}_i, \overline{e}_j$ follow in this order counterclockwise around $v$ if $v$ is black (orientation-reversing), and clockwise otherwise (when $v$ is "orientation-respecting"). The tile $v^*$ is regarded as black in $T$ if $v$ is black, and white otherwise. Both cases are illustrated in the picture where the right fragment concerns the orientation-reversing case.

Next we examine properties of the obtained collection $T$ of tiles. The first and second conditions in (T2) (concerning overlapping and non-overlapping tiles with a common edge) follow from the above construction and explanations.

1) Consider an $i$-edge $e = (u, v)$ in $G_W$ (a piece of the wire $w_i$). If $u \neq s_i$ and $v \neq s'_i$, then the dual edge $e^*$ belongs to exactly two tiles, namely, $u^*$ and $v^*$. If $u = s_i$, then $e^*$ belongs to the unique tile $v^*$. Furthermore, for the faces $F, F' \in \mathcal{F}_W$ containing $e$, the sets $X(F), X(F')$ are the principal intervals $[i-1]$ and $[i]$ (letting $[0] := \emptyset$). This implies that $e^*$ is the boundary edge $p_{i-1} p_i$ of $Z$, and this edge belongs to a unique tile in $T$ (which is, obviously, white). Considering $v = s'_i$, we obtain a similar property for the edges in $rbd(Z)$. This gives the first and second condition in (T1).

The proper wiring $W$ possesses the following important property, which will be proved later (see Lemma 7.2): ($*$) each face $F$ in $G_W$ has at most one $i$-edge for each $i$, and all sets $X(F)$ among $F \in \mathcal{F}_W$ are different. This implies that $T$ has no tile copies, yielding the third condition in (T1). Also property ($*$) and the planarity of $G_W$ imply validity of axiom (T4).

2) For a face $F \in \mathcal{F}_W$, let $E(F)$ denote the set of its edges not contained in $bd(Z)$. By the construction and explanations above,

(7.1) the edges in $E(F)$ one-to-one correspond to the edges incident to the vertex $v = F^*$ of $G_T$; moreover, for $e \in E(F)$, the corresponding edge $e^*$ enters $v$ if $e$ is directed counterclockwise (around $F$), and leaves $v$ otherwise.

This implies that $v$ has both entering and leaving edges if and only if $F$ is non-cyclic, unless $v = p_0$ or $p_n$. (Here we also use an easy observation that if $F$ contains a vertex $p_i$ or $p'_i$ for some $1 \leq i < n$, then $E(F)$ has edges in both directions.)

Consider a cyclic face $F \in \mathcal{F}_W^{cyc}$, and let $C = (v_0, e_1, v_1, \ldots, e_r, v_r = v_0)$ be its boundary cycle, where for $p = 1, \ldots, r$, the edge $e_p$ goes from $v_{p-1}$ to $v_p$. Denote the color of $e_p$ by $i_p$. Suppose $C$ is directed clockwise. Then for each $p$, we have $i_p < i_{p+1}$ if $v_p$ is white, and $i_p > i_{p+1}$ if $v_p$ is black (taking the indices modulo $r$). Hence $C$ contains at least one black point (for otherwise we would have $i_1 < \ldots < i_r < i_1$). Moreover,

(7.2) $C$ contains exactly one black point.

Indeed, let $v_p$ be black. Then $v_p$ is the root of the (even) lens $L$ of wires $w_{i_{p-1}}$ and $w_{i_p}$ such that $F \subseteq L$. By axiom (W3), $L$ is bijective to $F$. The existence of another black vertex in $C$ would cause the appearance of another lens bijective to $F$, which is impossible. So (7.2) is valid. This implies that the vertex $F^*$ (which has leaving edges only, by (7.1)) is the bottom vertex of exactly one black tile. When $C$ is directed counterclockwise, we have $i_p > i_{p+1}$ if $v_p$ is white, and $i_p < i_{p+1}$ if $v_p$ is black, implying (7.2) again, which in turn implies that $F^*$ is the top vertex of exactly one black tile. Thus, $T$ obeys (T3).

3) If a cyclic face $F$ and another face $F'$ in $G_W$ have a common edge $e = (u, v)$, then $F'$ is non-cyclic. Indeed, the edge $e'$ preceding $e$ in the boundary cycle of $F$ enters the vertex $u$. The wire in $W$ passing through $e'$ leaves $u$ by an edge $e''$. Obviously, $e''$ belongs to $F'$. Since the edges $e, e''$ of $F'$ have the same beginning vertex, $F'$ is

non-cyclic. Hence the cyclic faces in $G_W$ are pairwise disjoint, implying that no pair of black tiles in $T$ share an edge (the third condition in (T2)).

Thus, $T$ is a g-tiling. If a face $F$ of $G_W$ lies on the left from a wire $w_i$, then the vertices $F^*$ and $[n]$ occur in the same region when the interior of the $i$-strip is removed from the disc $D_T$. This implies that the sets $X(F)$, $F \in \mathcal{F}_W$, are just the vertices of $T$, i.e., the full spectra for $T$ and $W$ are the same. Now the correspondence between cyclic faces for $W$ and terminal vertices for $T$ yields $B_T = B_W$, as required. ∎

It remains to show the following (cf. ($*$) in the above proof).

**Lemma 7.2** *Let $W$ be a proper wiring. Then:*
*(i) for each face $F$ in $G_W$, all edges surrounding $F$ belong to different wires;*
*(ii) there are no different faces $F, F' \in \mathcal{F}_W$ such that $X(F) = X(F')$.*

**Proof** Suppose that a face $F$ contains two $i$-edges $e, e'$ for some $i$. One can see that: (a) $e, e'$ have the same direction in the boundary of $F$, and (b) the face $F' \neq F$ containing $e$ is different from the face $F'' \neq F$ containing $e'$. Property (a) implies $X(F') = X(F'')$. Therefore, (i) follows from (ii).

To show (ii), we use induction on $n$ (the assertion is obvious if $n = 2$). Let $W' := (w_1, \ldots, w_{n-1})$. Clearly $W'$ obeys axioms (W1),(W2). One can see that if none of cyclic faces of $G_{W'}$ is separated by $w_n$, then (W3) is valid for $W'$ as well. So suppose that some cyclic face $F$ of $G_{W'}$ is separated by $w_n$. Let $v$ be a black vertex in $F$ (such a vertex exists, for otherwise the edge colors along the boundary cycle of $F$ would be monotone increasing or monotone decreasing, which is impossible). Then there is a lens $L$ for $W'$ such that $F \subseteq L$ and $v$ is the root of $L$. Take the face $\widetilde{F}$ of $G_W$ such that $v \in \widetilde{F} \subset F$. By (W3) for $W$, $\widetilde{F}$ is cyclic (since $L$ and $v$ continue to be a lens and its root when $w_n$ is added to $W'$). Assume that the boundary cycle $C$ of $\widetilde{F}$ is directed clockwise (when $C$ is directed counterclockwise, the argument is similar). Since $\widetilde{F} \neq F$, $C$ should contain consecutive edges $e' = (u', u), e'' = (u, u'')$ with colors $n$ and $i < n$, respectively. Then the wire $w_i$ crosses $w_n$ at $u$ from left to right. This implies that the vertex $u$ is black and there is an $in$-lens $L'$ rooted at $u$ and containing $\widetilde{F}$. But $\widetilde{F}$ is bijective to $L$; a contradiction.

Thus, $W'$ is a wiring (obeying (W1)–(W3)). We first prove (ii) in the assumption that $W'$ is proper. Then by induction all sets $X(F)$, $F \in \mathcal{F}_{W'}$, are different. Suppose that there are different faces $\widetilde{F}, \widetilde{F'} \in \mathcal{F}_W$ such that $X(\widetilde{F}) = X(\widetilde{F'})$. Let $F$ and $F'$ be the faces for $W'$ containing $\widetilde{F}$ and $\widetilde{F'}$, respectively. Then $X(F) = X(\widetilde{F}) - \{n\}$ and $X(F') = X(\widetilde{F'}) - \{n\}$. This implies $X(F) = X(F')$, and therefore $F = F'$. Furthermore, $w_n$ separates $F$ at most twice (for otherwise we would have $F = \widetilde{F} \cup \widetilde{F'}$, which implies $X(\widetilde{F}) \neq X(\widetilde{F'})$).

It follows that $w_n$ and the boundary of $F$ have two common points $u, v$ such that: (a) $u$ occurs in $w_n$ earlier than $v$, and (b) the piece $P$ of $w_n$ between $u$ and $v$ (not including $u, v$) lies outside $F$. Let $Q$ be the part of the boundary of $F$ between $u$ and $v$ such that the simply connected region $\Omega$ surrounded by $P$ and $Q$ is disjoint from the interior of $F$. Consider the case when $P$ goes clockwise around $\Omega$; see the picture.
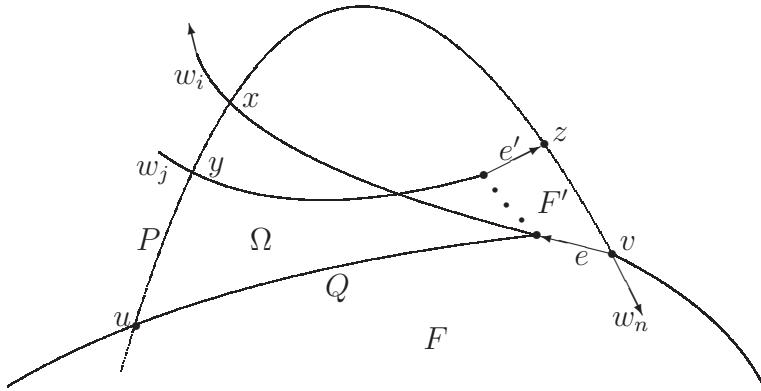
Let $e$ be the edge in $E_W$ contained in $Q$ and incident to $v$; then $e$ has color $i < n$. Take the maximal connected piece $Q'$ of $w_i$ lying in $\Omega$ and containing $e$. Since $w_i$ does not meet the interior of $F$, the end $x$ of $Q'$ different from $v$ lies on $P$. Then $Q$ and the piece $P'$ of $P$ from $x$ to $v$ form an *in*-lens $L$ for $W$. Since $P'$ is directed from $x$ to $v$, $Q'$ must be directed from $v$ to $x$ (by (W2)); in particular, $e$ leaves $v$. So $w_n$ crosses $w_i$ at $v$ from right to left, and therefore, the vertex $v$ is black and is the root of $L$. Let $F'$ be the cyclic face in $G_W$ lying in $L$ and containing $v$, and let $C$ be its boundary cycle. Since $W$ is proper, $F' \neq L$, whence $C \neq P' \cup Q'$. Then $C$ contains an edge $e'$ with color $j \neq i, n$ (one can take as $e'$ the edge of $C$ that either succeeds $e$ or precedes the last edge on $P$). Take the maximal connected piece $R$ of $w_j$, from a point $y$ to a point $z$ say, that lies in $\Omega$ and contains $e'$. It is not difficult to realize that $y$ occurs in $P$ earlier than $z$. This violates (W2) for $w_j, w_n$.

When $P$ goes counterclockwise around $\Omega$, a contradiction is shown in a similar way. (In this case, we take as $e$ the edge on $Q$ incident to $u$; one shows that $e$ enters $u$, whence the vertex $u$ is black.)

Finally, we assert that the wiring $W'$ is always proper. Indeed, suppose this is not so and consider an "empty" $ij$-lens $L$ (where $i < j$), i.e., forming a face in $G_{W'}$. One may assume that $L$ is an odd lens with lower vertex $u$ and upper vertex $v$ (the case of an even "empty" lens is examined in a similar way). Then $v$ is black (the root of $L$), and the boundary of $L$ is formed by the piece $P$ of $w_i$ from $u$ to $v$ and the piece $Q$ of $w_j$ from $v$ to $u$ (giving the edges in $G_{W'}$ connecting $u$ and $v$). Consider the (cyclic) face $F$ in $G_W$ lying in $L$ and containing $v$, and let $C$ be its boundary cycle. Since $W$ is proper, $F \neq L$. Let $e = (x, v)$ be the $i$-edge in $P$ entering $v$, and $e' = (v, y)$ the $j$-edge in $Q$ leaving $v$. Besides these, the cycle $C$ contains some $n$-edge $e''$. Note that $e''$ cannot connect two points on $P$ or two points on $Q$, for otherwise there would appear an "empty" *in*- or $jn$-lens. This implies that $e''$ goes from $y$ to $x$ (respecting the direction on $C$). But then $w_n$ crosses $w_j$ from right to left, whence the vertex $y$ is black. So the face $F$ contains two black vertices, contradicting (7.2).

Thus, Lemma 7.2 is proven, and this completes the proof of Proposition 7.1. ∎

Propositions 6.1 and 7.1 imply the desired equality $\mathcal{BT}_n = \mathcal{BW}_n$, and now Theorem 2.1 follows from Theorem 2.2. Analyzing the transformation of a g-tiling into a proper wiring described in Section 6 and the converse transformation described above, one can conclude that their composition returns the initial g-tiling (or the initial proper

wiring). This implies the following result (where, as before, $B_T$ and $\widehat{B}_T$ stand for the effective and full spectra of a g-tiling $T$, respectively, and similarly for wirings).

**Theorem 7.3** *There is a bijection $\beta$ of the set $\mathbf{T}_n$ of g-tilings to the set $\mathbf{W}_n$ of proper wirings on $Z_n$ such that $B_T = B_{\beta(T)}$ holds for each $T \in \mathbf{T}_n$. Furthermore, for each proper wiring $W$, all subsets $X(F) \subseteq [n]$ determined by the faces $F$ for $W$ are different, and one holds $\widehat{B}_W = \widehat{B}_T$, where $T = \beta^{-1}(W)$.*

We conclude this section with several remarks and additional results.

**Remark 4** As is shown in the proof of Lemma 7.2, for any proper wiring $W = (w_1, \ldots, w_n)$, the set $W' = (w_1, \ldots, w_{n-1})$ forms a proper wiring as well (concerning the zonogon $Z_{n-1}$). Clearly a similar result takes place when we remove the wire $w_1$. As a generalization, we obtain that for any $1 \le i < j \le n$, the set $(w_i, \ldots, w_j)$ forms a proper wiring on the corresponding subzonogon. One can see that removing $w_n$ from $W$ corresponds to the contracting operation concerning $n$ in the g-tiling $\beta^{-1}(W)$, and this results in the set of tiles corresponding to $W'$. This gives the following important result to which we have appealed in Section 4.

**Corollary 7.4** *For a g-tiling $T$ on $Z_n$, its $n$-contraction $T/n$ is a (feasible) g-tiling on $Z_{n-1}$.*

**Remark 5** Properties of g-tilings and proper wirings established during the proofs of Theorems 2.1 and 2.2 enable us to obtain the following result saying that these objects are determined by their spectra.

**Theorem 7.5** *For each semi-normal basis $B$, there are a unique g-tiling $T$ and a unique proper wiring $W$ such that $B = B_T = B_W$.*

**Proof** Due to Theorem 7.3, it suffices to prove this uniqueness property for g-tilings. We apply induction on $h(B) := \sum(|X| : X \in B)$. Suppose there are different g-tiles $T, T'$ with $B_T = B_{T'} =: B$. This is impossible when none of $T, T'$ has black tiles. Indeed, the vertices of $G_T$ and $G_{T'}$ (which are the sets in $B$) are the same and they determine the edges of these graphs, by (4.2). So $G_T = G_{T'}$. This graph is planar and subdivides $Z_n$ into little parallelograms, which are just the tiles in $T$ and the tiles in $T'$. Then $T = T'$. Now let $T$ (say) have a black tile. By Proposition 5.1, $T$ has a feasible W-configuration $CW(X; i, j, k)$, and we can make the corresponding lowering flip for $T$, obtaining a g-tiling $\widetilde{T}$ with $B_{\widetilde{T}} = (B - \{Xik\}) \cup \{Xj\}$. Since $B_T = B_{T'}$, $CW(X; i, j, k)$ is a feasible W-configuration for $T'$ as well, and making the corresponding lowering flip for $T'$, we obtain a g-tiling $\widetilde{T}'$ such that $B_{\widetilde{T}'} = B_{\widetilde{T}}$. We have $h(B_{\widetilde{T}}) < h(B)$, whence, by induction, $\widetilde{T} = \widetilde{T}'$. But the raising flip in $\widetilde{T}$ w.r.t. the (feasible) M-configuration $CM(X; i, j, k)$ returns $T$, as mentioned in Remark 2 in Subsection 4.1. Hence $T = T'$; a contradiction. ∎

One can develop an efficient procedure that, given the spectrum $B_T$ of a g-tiling $T$, restores $T$ itself (in essence, the procedure uses only "local" operations). This is provided by the possibility of constructing the graph $G_T$, as follows. We know that $B_T$ is the set of non-terminal vertices of $G_T$, and the edges connecting these vertices are of the form

$(X, Xi)$ for all corresponding $X, i$; let $G'$ be the graph formed by these vertices and edges. The goal is to construct the terminal vertices (if any) and the remaining edges of $G_T$ (in particular, obtaining the full spectrum $\widehat{B}_T$). This relies on the observation that each terminal vertex $X$ one-to-one corresponds to a maximal collection $\mathcal{Y} \subset B_T$ such that $|\mathcal{Y}| \geq 3$ and: either (a) each $Y \in \mathcal{Y}$ satisfies $Y = Xi$ for some $i$ and at least one member of $\mathcal{Y}$ has no entering edge in $G'$; or (b) each $Y \in \mathcal{Y}$ satisfies $Y = X - \{i\}$ for some $i$ and at least one member of $\mathcal{Y}$ has no leaving edge in $G'$. In case (a), $X$ is the bottom vertex of a black tile $\tau$, and $E_T(X) = \{(X, Y) : Y \in \mathcal{Y}\}$. In case (b), $X$ is the top vertex of a black tile $\tau$, and $E_T(X) = \{(Y, X) : Y \in \mathcal{Y}\}$. So, by extracting all such collections $\mathcal{Y}$, we are able to obtain the whole $G_T$. Now to construct the tiles of $T$ is easy (using (3.5)). A slightly modified efficient procedure can be applied to an arbitrary $B \subset 2^{[n]}$ to decide whether or not $B$ is a semi-normal basis.

**Remark 6** The authors can propose an alternative method of proving Theorems 2.1 and 2.2 in which the former theorem is proved directly and then the latter is obtained via the relationship of g-tilings and proper wirings established in this and previous sections. (Other possible methods: prove the first (second) assertion in Theorem 2.1 and the second (resp. first) assertion in Theorem 2.2.) The alternative method is based on ideas and techniques different from those applied in Sections 4,5: the former extensively exploit Jordan curve theorem, while the latter typically appeal to the fact that the graph of a g-tiling is graded for each color. For some illustration, we outline how the lowering flip is viewed on the language of wirings, for simplicity considering the situation corresponding to Case 1(a) in the proof of Proposition 4.1. Here we handle three wires $w_i, w_j, w_k$ of a wiring $W$ such that $i < j < k$ and there are five non-cyclic faces $A, B, C, D, E$ whose local configuration is as illustrated on the left fragment of the picture below. The sets $X(A), X(B), X(C), X(D), X(E)$ are, respectively, $Xij, Xi, Xik, Xk, Xjk$, the face $C$ looks like a triangle, and no other wire in $W$ traverses some open neighborhood $\Omega$ of $C$.
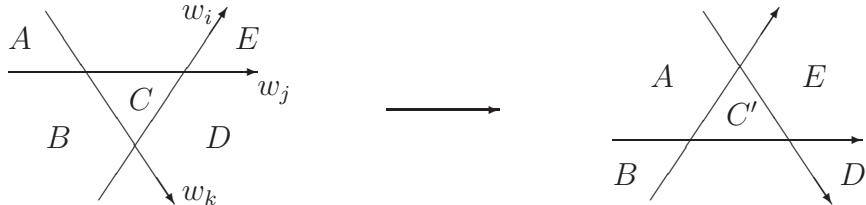


The lowering flip replaces $Xik$ by $Xj$. This corresponds to a deformation of the wire $w_j$ within $\Omega$ which makes it pass below the intersection point of $w_i$ and $w_k$, as illustrated on the right fragment of the picture. The triangle-shaped face $C'$ arising instead of $C$ satisfies $X(C') = Xj$ and is non-cyclic (as well as the modified $A, B, D, E$).

Note that if, in the initial wiring $W$, the face above $C$ is also triangle-shaped (i.e., the wires $w_i, w_k$ form a lens $L$ containing $C$ and such that $w_j$ is the unique wire going across $L$), then the above flip turns $L$ into a face and the wiring becomes non-proper. In this case the flip should be followed by the corresponding operation $() \to )($ which eliminates $L$ and makes the wiring proper again. Such a transformation of $W$ corresponds to that in Case 2(a') in the proof of Proposition 4.1.

# 8 $n$-contraction and $n$-expansion

In Subsection 4.3 we introduced the $n$-contraction operation for a g-tiling on the zonogon $Z_n$. In this section we examine this operation more systematically. Then we introduce and study a converse operation that transforms a g-tiling on $Z_{n-1}$ into a g-tiling on $Z_n$. The results obtained here (which are interesting by its own right) will be essentially used in Section 9.

Consider a g-tiling $T$ on $Z_n$. Let $P$ be the reversed path to the right boundary $R_Q$ of the $n$-strip $Q$. It possesses a number of important features, as follows:

(8.1) For the path $P = (v_0, e_1, v_1, \ldots, e_r, v_r)$ as above and the colors $i_1, \ldots, i_r$ of its edges $e_1, \ldots, e_r$, respectively, the following hold:

    (i) $P$ begins at the minimal point $p_0$ of $Z_n$ and ends at $p_{n-1}$;

    (ii) none of $v_0, \ldots, v_r$ is the top or bottom vertex of a black $ij$-tile with $i, j < n$;

    (iii) $P$ has no pair of consecutive backward edges;

    (iv) if $e_q = (v_{q-1}, v_q)$ and $e_{q+1} = (v_{q+1}, v_q)$ (i.e., $e_q$ is forward and $e_{q+1}$ is backward in $P$), then $i_q > i_{q+1}$;

    (v) if $e_q = (v_q, v_{q-1})$ and $e_{q+1} = (v_q, v_{q+1})$ (i.e., $e_q$ is backward and $e_{q+1}$ is forward in $P$), then $i_q < i_{q+1}$.

Indeed, the first and last edges of $Q$ are $p_{n-1}p_n$ and $p_0 p'_{n-1}$, yielding (i). Property (ii) follows from the facts that each vertex $v_q$ has an incident $n$-edge (which belongs to $Q$) and that all edges incident to the top or bottom vertex of a black $ij$-tile have colors between $i$ and $j$ (see Corollary 3.1(iii)). The forward (backward) edges of $P$ are the backward (resp. forward) edges of $R_Q$. Therefore, each forward (backward) edge $e_q$ of $P$ belongs to a white (resp. black) $i_q n$-tile, taking into account the maximality of color $n$; cf. (4.7). Then for any two consecutive edges $e_q, e_{q+1}$, at least one of them is forward, yielding (iii) (for otherwise the vertex $v_q$ is terminal and has both entering and leaving edges, which is impossible). Next, let $\tau$ be the $i_q n$-tile (in $Q$) containing $e_q$, and $\tau'$ the $i_{q+1} n$-tile containing $e_{q+1}$. If $e_q$ is forward and $e_{q+1}$ is backward in $P$, then $\tau'$ is black, $v_q$ is the left vertex of $\tau'$, and the $i_q$-edge $e$ opposite to $e_q$ in $\tau$ enters the top vertex of $\tau'$. Since $e$ lies in the cone of $\tau'$ at $t(\tau')$, we have $i_{q+1} < i_q < n$, as required is (iv). And if $e_q$ is backward and $e_{q+1}$ is forward, then $\tau$ is black and $v_q$ is its bottom vertex. Since $e_{q+1}$ lies in the cone of $\tau$ at $b(\tau)$, we have $i_q < i_{q+1} < n$, as required in (v).

Recall that the $n$-contraction operation applied to $T$ shrinks the $n$-strip in such a way that $R_Q$ merge with the left boundary $L_Q$ of $Q$. From (8.1)(ii) it follows that in the resulting g-tiling $T'$ on $Z_{n-1}$, the path $P$ as above no longer contains terminal vertices at all.

Next we describe the converse operation that transforms a pair consisting of an arbitrary g-tiling $T'$ on $Z_{n-1}$ and a certain path in $G_{T'}$ into a g-tiling on $Z_n$. To explain the construction, we first consider an arbitrary simple path $P$ in $G_{T'}$ which begins at $p_0$, ends at the maximal point $p_{n-1}$ of $Z_{n-1}$, and may contain backward edges.

Since the graph $G_{T'}$ has a planar layout (without intersection of non-adjacent edges) in the disc $D_{T'}$), the path $P$ subdivides $G_{T'}$ into two connected subgraphs $G' = G'_P$ and $G'' = G''_P$ such that: $G' \cup G'' = G_{T'}$, $G' \cap G'' = P$, $G'$ contains $\ell bd(Z_{n-1})$, and $G''$ contains $rbd(Z_{n-1})$; we call $G'$ $(G'')$ the *left* (resp. *right*) *subgraph* w.r.t. $P$. Then each tile of $T'$ becomes a face of exactly one of $G', G''$ (and all inner faces of $G', G''$ are such), and for an edge $e$ of $P$ not in $bd(Z_{n-1})$, the two tiles sharing $e$ occur in different subgraphs. So $T'$ is partitioned into two subsets, one being the set of faces of $G'$, and the other of $G''$.

The *n-expansion operation* for $(T', P)$ disconnects $G', G''$ by cutting $G_{T'}$ along $P$ and then glue them by adding the corresponding $n$-strip. More precisely, we shift the vertices of $G''$ by the vector $\xi_n$, i.e., each vertex $X$ in it changes to $Xn$; this induces the corresponding shift of edges and tiles in $G''$. The vertices of $G'$ preserve. So each vertex $X$ occurring in the path $P$ produces two vertices, namely, $X$ and $Xn$. As a result, for each edge $e = (X, Xi)$ of $P$, there appears its copy $\tilde{e} = (Xn, Xin)$ in the shifted $G''$; we connect $e$ and $\tilde{e}$ by the corresponding (new) *in*-tile, namely, by $\tau(X; i, n)$. This added tile is colored white if $e$ is a forward edge of $P$, and black if $e$ is backward. The colors of all old tiles preserve.

We refer to the resulting set $T$ of tiles, with the partition into white and black ones, as the *n-expansion of $T'$ along $P$*. Since the right boundary of the shifted $G''$ becomes the part of $rbd(Z_n)$ from the point $p'_{n-1}$ $(=\{n\})$ to $p_n$ $(=[n])$, it follows that the union of the tiles in $T$ is $Z_n$. Also it is easy to see that the shape $D_T$ in $conv(2^{[n]})$ associated to $T$ is again a disc (as required in axiom (T4)), and that $T$ obeys axiom (T1). The path $P$ generates the $n$-strip $Q$ for $T$ (consisting of the added $*n$-tiles and the edges of the form $(X, Xn)$), and we observe that $R_Q = P^{rev}$ and that $L_Q$ is the shift of $P^{rev}$ by $\xi_n$. Therefore, the $n$-contraction operation applied to $T$ returns $T'$.

To ensure validity of the remaining axioms (T2) and (T3), we have to impose additional conditions on the path $P$. In fact, they are similar to those exposed in (8.1). Moreover, these conditions are necessary and sufficient.

**Lemma 8.1** *Let $P = (p_0 = v_0, e_1, v_1, \ldots, e_r, v_r = p_{n-1})$ be a simple path in $G_{T'}$. Then the following are equivalent:*

(i) *the n-expansion $T$ of $T'$ along $P$ is a (feasible) g-tiling on $Z_n$;*

(ii) *$P$ contains no terminal vertices for $T'$ and satisfies (8.1)(iii),(iv),(v).*

**Proof** Let $P$ be as in (ii). We have to verify axioms (T2),(T3) for $T$. Let $P' = (v'_0, e'_1, v'_1, \ldots, e'_r, v'_r)$ and $P'' = (v''_0, e''_1, v''_1, \ldots, e''_r, v''_r)$ be the copies of $P$ in the graphs $G'$ and $G''$ (taken apart), respectively. It suffices to check conditions in (T2),(T3) for objects involving elements of $P', P''$ (since for any vertex of $G'$ not in $P'$, the structure of its incident edges and tiles, as well as the white/black coloring of tiles, is inherited from $G_{T'}$, and similarly for $G''$).

Consider a vertex $v_q$ with $1 \leq q < r$. Let $E^L_q$ $(E^R_q)$ denote the set of edges in $E_{T'}(v_q)$ lying on the left (resp. right) when we move along $P$ and pass through $e_q, v_q, e_{q+1}$; we include $e_q, e_{q+1}$ in both $E^L_q$ and $E^R_q$. Let $F^L_q$ $(F^R_q)$ denote the set of tiles in $F_{T'}(v_q)$ of which both edges incident to $v_q$ belong to $E^L_q$ (resp. $E^R_q$). Note that each tile $\tau \in F_{T'}(v_q)$ must occur in either $F^L_q$ or $F^R_q$, i.e., $\tau$ is not separated by $e_q$ or $e_{q+1}$

(taking into account that all vertices of $P$ are non-terminal, and therefore the edges of $P$ are white, and considering the behavior of edges and tiles at a non-terminal vertex exhibited in (3.5)). By the construction of $G', G''$, any two tiles of $T'$ that share an edge not in $P$ are faces of the same graph among $G', G''$, and if a tile $\tau \in T'$ has an edge contained in $lbd(Z_{n-1}) - P$ (resp. $rbd(Z_{n-1}) - P$), then $\tau$ is a face of $G'$ (resp. $G''$). Using these observations, one can conclude that

(8.2) for $1 \le q < r$, $E_q^L$ and $F_q^L$ are entirely contained in $G'$, while $E_q^R$ and $F_q^R$ are entirely contained in $G''$.

For $q = 1, \ldots, r$, let $\tau_q^L$ ($\tau_q^R$) denote the tile in $T'$ (if exists) that contains the edge $e_q$ and lies on the left (resp. right) when we traverse $e_q$ from $v_{q-1}$ to $v_q$. By (8.2), $\tau_q^L$ is in $G'$ and $\tau_q^R$ is in $G''$. Also each of $\tau_q^L, \tau_q^R$ is white. Let $\tau_q$ be the $i_q n$-tile in $T$ that was added to connect the edges $e_q'$ and $e_q''$. Then

$$e_q' = (b(\tau_q), \ell(\tau_q)) \quad \text{and} \quad e_q'' = (r(\tau_q), t(\tau_q)). \tag{8.3}$$

Suppose that $e_q$ is forward in $P$. Then $\tau_q$ is white. Since $e_q$ is directed from $v_{q-1}$ to $v_q$ and $\tau_q^L$ lies on the left from $e_q$ when moving from $v_{q-1}$ to $v_q$, $e_q$ belongs to the right boundary of $\tau_q^L$. This and (8.3) imply that $\tau_q$ and $\tau_q^L$ do not overlap. In its turn, $\tau_q^R$ contains $e_q$ in its left boundary; this together with (8.3) implies that $\tau_q$ and the shifted $\tau_q^R$ (sharing the edge $e_q''$) do not overlap as well. Now suppose that $e_q$ is backward in $P$. Then $\tau_q$ is black. Since $e_q$ is directed from $v_q$ to $v_{q-1}$ and $\tau_q^L$ lies on the left from $e_q$ when moving from $v_{q-1}$ to $v_q$, $e_q$ belongs to the left boundary of $\tau_q^L$. This implies that $\tau_q$ and $\tau_q^L$ overlap. Similarly, $\tau_q$ and $\tau_q^R$ overlap. Thus, (T2) holds for $\tau_q, \tau_q^L$ and for $\tau_q, \tau_q^R$, as required. Also the non-existence of pairs of consecutive reverse edges in $P$ implies that no two black tiles in $T$ share an edge.

To verify (T3), consider a black tile $\tau_q$. Then $1 < q < r$, the edges $e_{q-1}, e_{q+1}$ are forward, and $e_q$ is backward in $P$. Also $i_{q-1}, i_{q+1} > i_q$ (by (8.1)(iv),(v)). Observe that the set $E_{q-1}^R$ consists of the edges in $E_{T'}(v_{q-1})$ that enter $v_{q-1}$ and have color $j$ such that $i_q \le j \le i_{q-1}$ (including $e_{q-1}, e_q$). All these edges are white (as is seen from (3.5)). The second copies of these edges (shifted by $\xi_n$) plus the $n$-edge $(v_{q-1}', v_{q-1}'')$ are exactly those edges of $G_T$ that are incident to the top vertex $v_{q-1}''$ of $\tau_q$. It its turn, the set $E_q^L$ consists of the edges in $E_{T'}(v_q)$ that leave $v_q$ and have color $j$ such that $i_q \le j \le i_{q+1}$, and these edges are white. Exactly these edges plus the $n$-edge $(v_q', v_q'')$ form the set of edges of $G_T$ incident to $b(\tau_q)$. (See the picture.) This gives (T3) for $T$.
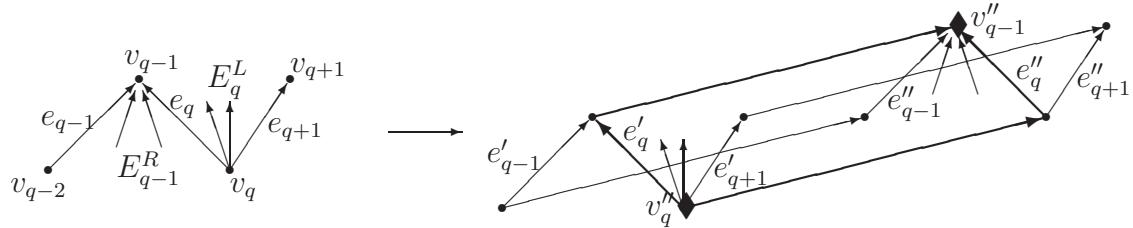


Thus, (ii) implies (i) in the lemma. The converse implication (i)→(ii) follows from (8.1) and the fact (mentioned earlier) that for the $n$-expansion $T$ of $T'$ along $P$, the $n$-contraction operation applied to $T$ produces $T'$, and under this operation the $n$-strip for $T$ shrinks into $P^{rev}$. This completes the proof of the lemma. ∎

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Let us call a path $P$ as in (ii) of Lemma 8.1 *legal*. It is the concatenation of $P_1, \ldots, P_{n-1}$, where $P_h$ is the maximal subpath of $P$ whose edges connect levels $h-1$ and $h$, i.e., are of the form $(X, Xi)$ with $|X| = h-1$. We refer to $P_h$ as $h$-th *subpath* of $P$ and say that this subpath is *ordinary* if it has only one edge, and *zigzag* otherwise. The beginning vertices of these subpaths together with $h_{n-1}$ are called *critical* in $P$ (so there is exactly one critical vertex in each level); these vertices will play an important role in what follows. Note that the critical vertices of a legal path $P = (v_0, e_1, v_1, \ldots, e_r, v_r)$ are $v_0 = p_0$, $v_r = p_{n-1}$ and the intermediate vertices $v_q$ such that $e_q$ enters and $e_{q+1}$ leaves $v_q$. We distinguish between two sorts of non-critical vertices $v_q$ by saying that $v_q$ is a $\vee$-*vertex* if both $e_q, e_{q+1}$ leave $v_q$, and a $\wedge$-*vertex* if both $e_q, e_{q+1}$ enter $v_q$. Observe that

(8.4) regarding a vertex of $P$ as a subset $X$ of $[n-1]$, the following hold: (a) if $X$ is critical, then both $X, Xn$ are in $B_T$; (b) if $X$ is a $\wedge$-vertex, then $X \in B_T$ and $Xn \notin B_T$; and (c) if $X$ is a $\vee$-vertex, then $X \notin B_T$ and $Xn \in B_T$ (where $T$ is the $n$-expansion of $T'$ along $P$).

Indeed, from the proof of Lemma 8.1 one can see that: if $X$ is critical, then both vertices $X, Xn$ of $G_T$ have entering and leaving edges, so they are non-terminal; if $X$ is a $\wedge$-vertex, then $Xn$ is terminal while $X$ is not; and if $X$ is a $\vee$-vertex, then $X$ is terminal while $Xn$ is not (see the above picture).

It follows that

(8.5) (i) $B_T = B' \cup B''$, where $B'$ consists of all non-terminal vertices $X$ in $G'_P$ that are not $\vee$-vertices in $P$, and $B''$ consists of all $Xn$ such that $X$ is a non-terminal vertex in $G''_P$ that is not a $\wedge$-vertex of $P$;

(ii) for each $h = 0, \ldots, n-1$, there is exactly one set $X \subseteq [n-1]$ with $|X| = h$ such that both $X$ and $Xn$ belong to $B_T$; moreover, this $X$ is just the unique critical vertex of $P$ in level $h$.

Summing up the above results, we can conclude with the following

**Corollary 8.2** *The correspondence* $(T', P) \mapsto T$, *where* $T'$ *is a g-tiling on* $Z_{n-1}$, $P$ *is a legal path for* $T'$, *and* $T$ *is the $n$-expansion of* $T'$ *along* $P$, *gives a bijective map* $\epsilon$ *of the set of such pairs* $(T', P)$ *to the set of g-tilings on* $Z_n$. *Moreover, for any g-tiling* $T$ *on* $Z_n$, *the pair* $\epsilon^{-1}(T)$ *consists of the $n$-contraction* $T'$ *of* $T$ *and the legal path for* $T'$ *corresponding to the $n$-strip in* $T$.

We conclude this section with an additional result which will be important for purposes of the next section. For a g-tiling $T$ on $Z_n$ and for $1 \le h \le n$, let $H_h$ denote the subgraph of $G_T$ induced by the set of *white* edges connecting levels $h-1$ and $h$.

**Lemma 8.3** *For each* $h = 1, \ldots, n$, *the graph* $H_h$ *is a forest. Furthermore:*

(i) *there exists a (connected) component* $K$ *of* $H_h$ *that contains both boundary edges* $p_{h-1}p_h$ *and* $p'_h p'_{h-1}$ *and such that all vertices of* $K$ *are non-terminal;*

(ii) *any other component* $K'$ *of* $H_h$ *contains exactly one terminal vertex* $v$ *and all edges of* $K'$ *are incident to* $v$ *(i.e.,* $K'$ *is a star).*

**Proof** We observe that

(8.6) any 3-edge path $(v_0, e_1, v_1, e_2, v_2, e_3, v_3)$ in $H_h$ satisfies either $i_1, i_3 < i_2$ or $i_1, i_3 > i_2$, where $i_1, i_2, i_3$ are the colors of $e_1, e_2, e_3$, respectively; equivalently: the edges $e_1$ and $e_3$ (regarding as line-segments in $Z_n$) do not intersect.

Indeed, suppose this is not so. W.l.o.g., one may assume that $i_1 < i_2 < i_3$ and that the edges $e_1, e_2$ leave $v_1$. Then $v_1$ is in level $h-1$, $v_2$ is in level $h$, and $e_3$ enters $v_2$. Take the edge $e$ leaving $v_1$ whose color $j$ is maximum subject to $i_1 \le j < i_2$, and the edge $e'$ entering $v_2$ whose color $j'$ is minimum subject to $i_2 < j' \le i_3$. By (3.5) (applied to $v_1$ and to $v_2$), both edges $e, e'$ are white, the edges $e, e_2$ belong to a white tile $\tau$ with the bottom vertex $v_1$, and $e', e_2$ belong to a white tile $\tau'$ with the top vertex $v_2$. But such tiles $\tau, \tau'$ (sharing the edge $e_2$) overlap, contrary to (T2).

In view of (8.6), no path in $H_h$ can be cyclic (it monotonically goes in one direction, either from left to right or from right to left). Hence $H_h$ is a forest in which any component $K$ (a tree) has a planar layout on $Z_n$, i.e., non-adjacent edges in $K$ do not intersect. Suppose that $H_h$ contains a terminal vertex $X$ in level $h-1$, and consider an edge in $H_h$ incident to $X$, say, $e = (X, Xi)$ (taking into account that $e$ goes from level $h-1$ to level $h$). Since $X$ is terminal, each of the two white tiles $\tau', \tau''$ containing $e$ has the bottom vertex at $X$ and the right of left vertex at $Xi$. By (3.6), there is no white edge incident to $Xi$ and lying strictly inside the cone $C(\tau', Xi)$, and similarly for $\tau''$. This implies that $e$ is the unique edge of $H_h$ incident to $Xi$. Hence the component of $H_h$ containing $X$ is a star of which all edges are incident to $X$. A similar property holds for the components of $H_h$ meeting a terminal vertex in level $h$.

Finally, consider a component $K$ without terminal vertices (it exists since the boundary edge $p_{h-1}p_h$ is white and both of its ends are non-terminal). We assert that $K$ contains $p_{h-1}p_h$. Indeed, take the leftmost edge in $K$, say, $e = (X, Xi)$, and suppose that $e \ne p_{h-1}p_h$. Then there is a white tile $\tau$ containing $e$ on its right boundary. Assume that $b(\tau) = X$; then $r(\tau) = Xi$. Then the edge $e' := (b(\tau), \ell(\tau))$ is black (as $e'$ connects levels $h-1$ and $h$ and lies on the left from $e$). Since $\ell(\tau)$ has both entering and leaving edges, it cannot be terminal. So $b(\tau)$ is terminal, contradicting the choice of $K$. The case $t(\tau) = Xi$ leads to a similar contradiction. Thus, $K$ contains $p_{h-1}p_h$. Considering the rightmost edge of $K$ and arguing similarly, we conclude that $K$ contains the boundary edge $p'_h p'_{h-1}$ as well. ∎

We will refer to the component $K$ as in (i) of this lemma as the *principal* one. Considering a legal path $P$ for $T$ and taking into account that all vertices of $P$ are non-terminal and that $h$-th subpath in it is contained in $H_h$, for each $h$, we obtain the following property as a consequence of Lemma 8.3.

**Corollary 8.4** *Any legal path for a g-tiling is determined by the set of its critical vertices.*

Finally, let $LP_T$ denote the subgraph of $G_T$ that is the union of all legal paths for $T$. One can construct $LP_T$ as follows. At the beginning, put $G'$ to be the union of principal components in $H_h$, $h = 1, \ldots, n$. If $G'$ contains a vertex $v$ having exactly one incident edge in $G'$, then we remove this vertex and edge from $G'$, repeat the procedure

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for the new $G'$, and so on until no such $v$ exists. Then the final $G'$ is just $LP_T$. This graph gives a nice compact representation for the set of legal paths for $T$. (It contains every legal path for $T$. Conversely, starting from $p_0$ and forming a maximal path in $LP_T$ as the concatenation of ordinary or zigzag paths going to the right, we always reach $p_n$, obtaining a legal path. Note that the same graph $LP_T$ appears when we are interested in expansions w.r.t. the new minimal color, say, 0; in this case one should consider zigzag paths in $LP_T$ going to the left.)

## 9  Weakly separated set-systems

The goal of this section is to prove the following theorem answering Leclerc–Zelevinsky's conjecture mentioned in the Introduction.

**Theorem 9.1** *Any largest weakly separated collection $C \subseteq 2^{[n]}$ is a semi-normal TP-basis.*

For brevity we will abbreviate "weakly separated collection" as "ws-collection". Recall that a ws-collection $C \subseteq 2^{[n]}$ is *largest* if its cardinality $|C|$ is maximum among all ws-collections in $2^{[n]}$; this maximum is equal to $\binom{n+1}{2} + 1$ [7]. An important example is the set $\mathcal{I}_n$ of intervals in $[n]$ (including the empty set). Also it was shown in [7] that a (lowering or raising) flip in a ws-collection produces again a ws-collection. Moreover, its cardinality preserves under a flip since it replaces one set in some pair $\{Xj, Xik\}$ (say) by the other. Due to these facts, the set $\mathcal{C}_n$ of largest ws-collections includes $\mathcal{B}_n$ (the set of semi-normal bases for $\mathcal{TP}_n$). Theorem 9.1 says that the converse inclusion takes place as well. As a result, we will conclude with the following

**Corollary 9.2** $\mathcal{C}_n = \mathcal{B}_n$.

In view of Theorem 2.2, to obtain Theorem 9.1, it suffices to show the following

**Theorem 9.3** *Any $C \in \mathcal{C}_n$ is the spectrum $B_T$ of some g-tiling $T$ on $Z_n$.*

This theorem is proved by combining additional facts established in [7] and results from the previous sections. Let $C \in \mathcal{C}_n$. To construct the desired tiling for $C$, we consider the projection $C'$ of $C$ into $2^{[n-1]}$, i.e., the collection of subsets $X \subseteq [n-1]$ such that either $X \in C$ or $Xn \in C$ or both. Partition $C'$ into three subcollections $M, N, S$, where

$$M := \{X \colon X \in C \not\ni Xn\}, \quad N := \{X \colon Xn \in C \not\ni X\}, \quad S := \{X \colon X, Xn \in C\}.$$

Also for $h = 0, \ldots, n-1$, define

$$C'_h := \{X \in C' \colon |X| = h\}, \quad M_h := M \cap C'_h, \quad N_h := N \cap C'_h.$$

It is shown in [7] that

(9.1)  for each $h = 0, \ldots, n-1$, $S \cap C'_h$ contains exactly one element.

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We call $S$ the *separator* of $C'$ and denote its elements by $S_0, \ldots, S_{n-1}$, where $|S_h| = h$. Property (9.1) implies that $|C'| = |C| - |S| = \binom{n+1}{2} + 1 - n = \binom{n}{2} + 1$, and as is shown in [7],

(9.2) $C'$ is a ws-collection, and therefore it is a *largest* ws-collection in $2^{[n-1]}$.

Two more observations in [7] are:

(9.3) (i) $S_0 \prec S_1 \prec \cdots \prec S_{n-1}$;

      (ii) for each $h = 0, \ldots, n-1$, any sets $Y \in M_h$ and $Y' \in N_h$ satisfy $Y \prec S_h$ and $S_h \prec Y'$.

An important consequence of (9.3) is that the collection $C$ can be uniquely restored from the pair $C', S$. Indeed, $C$ consists of the sets $X, Xn$ such that $X \in S$, the sets $X \subseteq [n-1]$ such that $X \prec S_{|X|}$, and the sets $Xn$ such that $X \subseteq [n-1]$ and $X \succ S_{|X|}$.

The proof of Theorem 9.3 is led by induction on $n$. The result is trivial for $n \leq 2$. Let $n > 2$ and assume by induction that there is a g-tiling $T'$ on $Z_{n-1}$ such that $B_{T'} = C'$. Our aim is to transform $T'$ into a g-tiling on $Z_n$ whose spectrum is $C$. The crucial claim is the following

**Lemma 9.4** *There exists a legal path $P$ for $T'$ whose set of critical vertices coincides with the separator $S$.*

**Proof** It uses the following fact from [7]:

(9.4) if sets $A, A', A'' \subseteq [n']$ are weakly separated, and if $|A| \leq |A'| \leq |A''|$, $A \prec A'$ and $A' \prec A''$, then $A \prec A''$.

The desired path $P$ is constructed by relying on Lemma 8.3. For $h = 1, \ldots, n-1$, let $H_h$ be the subgraph of $G_{T'}$ induced by the white edges connecting levels $h-1$ and $h$, and let $K_h$ be the principal component of $H_h$. By Lemma 8.3, all vertices $X$ of $K_h$ are non-terminal, whence $X \in C'$.

Consider two vertices $X, Y$ with $|X| \leq |Y|$ in $K_h$. If they are connected by edge, then $Y = Xi$ for some $i \in [n]$ and, obviously, $X \prec Y$. If the path $P$ from $X$ to $Y$ in $K_h$ is such that its length (number of edges) $|P|$ is at least two and $P$ goes from left to right, then $X \prec Y$ as well. (When $X, Y$ belong to different levels, we say that $P$ goes from left to right if the vertex $Y'$ preceding $Y$ in $P$ lies on the right from $X$ in the level containing $X, Y'$.) Indeed, for any three consecutive vertices $Z, Z', Z''$ in $P$ (occurring in this order) either $Z = Z'i$ and $Z'' = Z'j$, or $Z = Z' - \{j\}$ and $Z'' = Z - \{i\}$ for some $i < j$ (since $Z''$ lies on the right from $Z$), which implies $Z \prec Z''$. Then $X \prec Y$ follows from (9.4) by the transitivity.

We assert that the separating vertex $S_{h-1}$ belongs to $K_h$. Indeed, suppose this is not so. Then $S_{h-1}$ belongs to a star component $K'$ of $H_h$, and therefore, the white edge $e$ in $H_h$ leaving $S_{h-1}$ enters the top vertex of some black tile $\tau = \tau(X; p, q)$. We have $S_{h-1} = Xpq - \{i\}$ and $p < i < q$, where $i$ is the color of $e$. On the other hand, the bottom vertex $X$ of $\tau$ has a leaving $j$-edge $(X, Xj)$ for some $p < j < q$ (cf. Corollary 3.1).

The vertex $Xj$ is non-terminal, and we have $|Xj| = |S_{h-1}|$, $S_{h-1} - Xj = \{p, q\}$ and $Xj - S_{h-1} = \{i, j\}$. Since $p < i, j < q$, we come to a contradiction with the fact that $S_{h-1}$ is comparable by $\prec$ with any non-terminal vertex in level $h - 1$. Thus, $S_{h-1}$ is in $K_h$. Arguing similarly, one shows that $S_h$ is in $K_h$ as well. Let $P_h$ be the path from $S_{h-1}$ to $S_h$ in $K_h$. Since $S_{h-1} \prec S_h$ (by (9.3)), $P_h$ goes from left to right (when $|P_h| > 1$).

Concatenating $P_1, \ldots, P_{n-1}$, we obtain a legal path $P$ for $T'$ of which $h$-th subpath is $P_h$, and the critical vertices are $S_0, \ldots, S_{n-1}$, as required. ∎

Now we finish the proof of Theorem 9.3 as follows. Let $T$ be the $n$-expansion of $T'$ along $P$. Then $B_T$ is a ws-collection, moreover, it is a largest ws-collection since $|B_T| = \binom{n+1}{2} + 1$. By (8.5) and Corollary 8.2, the projection of $B_T$ into $2^{[n-1]}$ (defined by $X \mapsto X - \{n\}$) is just $B_{T'} = C'$ and, moreover, the set of $X \subseteq [n-1]$ such that $X, Xn \in B_T$ is exactly the set of critical vertices in $P$, i.e., $S$. Since the pair $C', S$ generates the corresponding largest ws-collection in $2^{[n]}$ in a unique way, we obtain $B_T = C$, and Theorem 9.3 follows. ∎∎

This completes the proof of Theorem 9.1.

# 10   Generalizations

In this concluding section we outline two generalizations, omitting proofs. They will be discussed in full, with details and related topics, in a separate paper.

**A.** The obtained relationships between semi-normal bases, proper wirings and generalized tilings are extendable to the case of an integer $n$-box $\mathbf{B}^{n,a} = \{x \in \mathbb{Z}^{[n]} \colon 0 \le x \le a\}$, where $a \in \mathbb{Z}_+^n$. Recall that a function $f$ on $\mathbf{B}^{n,a}$ is a TP-function if it satisfies

$$f(x + 1_i + 1_k) + f(x + 1_j) \tag{10.1}$$
$$= \max\{f(x + 1_i + 1_j) + f(x + 1_k), \; f(x + 1_i) + f(x + 1_j + 1_k)\}$$

for any $x$ and $1 \le i < j < k \le n$, provided that all six vectors occurring as arguments in this relation belong to $\mathbf{B}^{n,a}$, where $1_q$ denotes $q$-th unit base vector. In this case the standard basis of the TP-functions consists of the vectors $x$ such that $x_i, x_j > 0$ for $i < j$ implies $x_q = a_q$ for $q = i + 1, \ldots, j - 1$ (see [2], where such vectors are called *fuzzy-intervals*). Normal and semi-normal bases are corresponding collections of integer vectors in $\mathbf{B}^{n,a}$, defined by a direct analogy with the Boolean case.

The semi-normal bases in the box case admit representations via natural generalizations of proper wiring and g-tiling diagrams for the Boolean case. They are described as follows.

The zonogon for a given $a$ is the set $Z_{n,a} := \{\lambda_1 \xi_1 + \ldots + \lambda_n \xi_n \colon \lambda_i \in \mathbb{R}, \; 0 \le \lambda_i \le a_i, \; i = 1, \ldots, n\}$, where the vectors $\xi_i$ are chosen as above. For each $i \in [n]$ and $q = 0, 1, \ldots, a_i$, define the point $p_{i,q} := a_1 \xi_1 + \ldots + a_{i-1} \xi_{i-1} + q \xi_i$ (on the left boundary of $Z_{n,a}$) and the point $p'_{i,q} := a_n \xi_n + \ldots a_{i+1} \xi_{i+1} + q \xi_i$ (on the right boundary). These points are regarded as the vertices on the boundary of $Z_{n,a}$, and the edges in it are the directed line-segments $p_{i,q-1} p_{i,q}$ and $p'_{i,q} p'_{i,q-1}$ When $q \ge 1$, we define $s_{i,q}$ ($s'_{i,q}$) to be the median point on the edge $p_{i,q-1} p_{i,q}$ (resp. $p'_{i,q} p'_{i,q-1}$).

A generalized tiling $T$ on $Z = Z_{n,a}$ is defined by essentially the same axioms (T1)–(T4) from Subsection 2.2. A wiring $W$ on $Z$ consists of wires $w_{i,q}$ going from $s_{i,q}$ to $s'_{i,q}$, $i = 1, \ldots, n$, $q = 1, \ldots, a_i$. Again, it is defined by the same axioms (W1)–(W3) from Subsection 2.1.

Note that for any $i$ and $1 \le q < q' \le a_i$, the point $s'_{i,q}$ occurs earlier than $s'_{i,q'}$ in the right boundary of $Z$ (beginning at $p_0$), which corresponds to the order of $s_{i,q}, s_{i,q'}$ in the left boundary of $Z$. This and axiom (W2) imply that the wires $w := w_{i,q}$ and $w' := w_{i,q'}$ are always disjoint. Indeed, suppose that $w$ and $w'$ meet and take the first point $x$ of $w'$ that belongs to $w$. Let $\Omega_0, \Omega_1$ be the connected components of $Z - (P \cup P')$, where $P$ is the part of $w$ from $x$ to $s'_{i,q}$, $P'$ is the part of $w'$ from $s_{i,q'}$ to $x$, and $\Omega_0$ contains $p_0$. Then the end point $s'_{i,q'}$ of $w'$ is in $\Omega_1$. Furthermore, $w'$ crosses $w$ at $x$ from left to right (since $x$ is the first point of $w'$ where it meets $w$); this implies that when passing $x$, the wire $w'$ enters the region $\Omega_0$. Therefore, the part of $w'$ from $x$ to $s'_{i,q'}$ must intersect $P \cup P'$ at some point $y \ne x$. But $y \in P$ is impossible by (W2) and $y \in P'$ is impossible because $w'$ is not self-intersecting.

Like the Boolean case, for a g-tiling $T$, the spectrum $B_T$ is defined to be the set of non-terminal vertices (viz. $n$-vectors) for $T$. For a wiring $W$ and an (inner) face $F$ of its associated planar graph, let $x(F)$ denote the $n$-vector whose $i$-th entry is the number of wires $w_{i,q}$ such that $F$ lies on the left from $w_{i,q}$. Then $B_W$ is defined to be the collection of vectors $x(F)$ over all non-cyclic faces $F$.

Theorems 2.1 and 2.2 remain valid for these extended settings (where $Z_n$ is replaced by $Z_{n,a}$), and proving methods are essentially the same as those in Sections 4–7, with minor refinements on some steps. (E.g., instead of a unique dual $i$-path ($i$-strip) for each $i$, we now deal with $a_i$ dual $i$-paths $Q_{i,1}, \ldots, Q_{i,a_i}$, each $Q_{i,q}$ connecting a boundary edge $p_{i,q-1}p_{i,q}$ to $p'_{i,q-1}p'_{i,q}$, which does not cause additional difficulty in the proof.)

**B.** The second generalization involves an arbitrary permutation $\omega$ on $[n]$. (In fact, so far we have dealt with the longest permutation $\omega_0$, where $\omega_0(i) = n + 1 - i$.) For $i, j \in [n]$, we write $i \prec_\omega j$ if $i < j$ and $\omega(i) < \omega(j)$. This relation is transitive and gives a partial order on $[n]$. Let $\mathcal{X}_\omega \subseteq 2^{[n]}$ be the set (lattice) of ideals $X$ of $([n], \prec_\omega)$, i.e., $i \prec_\omega j$ and $j \in X$ implies $i \in X$. In particular, $\mathcal{X}_\omega$ is closed under taking a union or intersection of its members. Below we specify settings and outline how results concerning $\omega_0$ can be extended to $\omega$.

(i) By a TP-function for $\omega$, or an $\omega$-*TP-function*, we mean a function $f$ defined on the set $\mathcal{X}_\omega$ (rather than $2^{[n]}$) and satisfying (1.1) when all six sets in it belong to $\mathcal{X}_\omega$. Note that $Xi, Xk, Xij, Xjk \in \mathcal{X}_\omega$ implies that each of $X, Xj, Xik, Xijk$ is in $\mathcal{X}_\omega$ as well (since each of the latter is obtained as the intersection or union of a pair among the former). The notion of TP-basis is extended to the set $\mathcal{TP}_\omega$ of $\omega$-TP-functions in a natural way. It turns out that the role of standard basis is now played by the set $\mathcal{I}_\omega$ of $\omega$-*dense* sets $X \in \mathcal{X}_\omega$, which means that there are no triples $i < j < k$ such that $i, k \in X \not\ni j$ and each of the sets $X - \{i\}$, $X - \{k\}$ and $(X - \{i, k\}) \cup \{j\}$ belongs to $\mathcal{X}_\omega$. In particular, $\mathcal{I}_\omega$ contains the sets $[i]$, $\{i' : i' \preceq_\omega i\}$ and $\{i' : \omega(i') \le \omega(i)\}$ for each $i \in [n]$; when $\omega = \omega_0$, $\mathcal{I}_\omega$ turns into the set $\mathcal{I}_n$ of intervals in $[n]$. (It is rather easy to prove that any $\omega$-TP-function is determined by its values on $\mathcal{I}_\omega$; this is done by exactly the same method as applied in [2] to show a similar fact for $\mathcal{TP}_n$ and $\mathcal{I}_n$. The fact that

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the restriction map $\mathcal{TP}_\omega \to \mathbb{R}^{\mathcal{I}_\omega}$ is surjective (which is more intricate) can be shown by extending a flow approach developed in [2] for the cases of TP-functions on Boolean cubes and integer boxes.) Normal and semi-normal bases for the $\omega$-TP-functions are defined via flips from the standard basis $\mathcal{I}_\omega$, by analogy with those for $\omega_0$.

(ii) Instead of the zonogon $Z_n$, we now should consider the region in the plane bounded by two paths: the left boundary of $Z_n$ and the path $P_\omega$ formed by the points $p'_{0,\omega} := p_0$ and $p'_{i,\omega} := \xi_{\omega^{-1}(1)} + \ldots + \xi_{\omega^{-1}(i)}$ for $i = 1, \ldots, n$ connected by the (directed) line-segments $e'_{1,\omega}, \ldots, e'_{n,\omega}$, where $e'_{j,\omega}$ begins at $p'_{\omega(j)-1,\omega}$ and ends at $p'_{\omega(j),\omega}$. (Then $e'_{j,\omega}$ is a parallel translation of $\xi_j$. When $\omega = \omega_0$, each $p'_{i,\omega}$ is just the point $p'_{n-i}$ on the right boundary of $Z_n$. When $\omega$ is the identity, $p'_{i,\omega}$ coincides with the point $p_i$ on $\ell bd(Z_n)$.) We denote this region as $Z_\omega$ and call it the $\omega$-*deformation* of the zonogon $Z_n$, or, liberally, the $\omega$-*zonogon*. A wiring for $\omega$ is a collection $W$ of wires $w_1, \ldots, w_n$ in $Z_\omega$ satisfying axioms (W1)–(W3) and such that each $w_i$ begins at the point $s_i$ (as before) and ends at the median point $s'_{i,\omega}$ of $e'_{i,\omega}$ (a wire $w_i$ degenerates into a point if the boundary edges $p_{i-1}p_i$ and $e'_{j,\omega}$ coincide). Note that if $i \prec_\omega j$ then $s'_{i,\omega}$ occurs earlier than $s'_{j,\omega}$ in the right boundary $P_\omega$ of $Z_\omega$, and therefore, the wires $w_i$ and $w_j$ does not meet (as explained in part A above). This implies that all sets in the full spectrum of $W$ belong to $\mathcal{X}_\omega$.

In its turn, a generalized tiling $T$ for $\omega$ is defined in the same way as for $\omega_0$, with the only differences that now the union of tiles in $T$ is $Z_\omega$ and that the corresponding shape $D_T$ is required to be simply connected (then $D_T$ is homeomorphic to $Z_\omega$). (Depending on $\omega$, points $p_{i,\omega}$ and $p'_{\omega(i),\omega}$ may coincide for some $i$, so $D_T$ need not be a disc in general.) The constructions and arguments in Sections 6,7, based on planar duality, can be transferred without essential changes to the $\omega$ case, giving a natural one-to-one correspondence between the g-tilings and proper wirings for $\omega$. (In particular, the fact that $i$-th wire $w_i$ in a proper wiring $W$ for $\omega$ turns into the $i$-strip $Q_i$ in the corresponding g-tiling $T$ (which begins with the $i$-edge $p_{i-1}p_i$ in $\ell bd(Z_\omega)$ and ends with the $i$-edge $e'_{i,\omega}$ in $rbd(Z_\omega)$) implies that all vertices of $G_T$ represent sets in $\mathcal{X}_\omega$.) The arguments in Sections 4,5 continue to work in the $\omega$ case as well. As a result, we obtain direct generalizations of Theorems 2.1 and 2.2 to an arbitrary permutation $\omega$.

(iii) A majority of results from Sections 8 and 9 can be extended, with a due care, to the $\omega$ case as well. Below we give a brief commentary, not coming into particular details. Let $\mathcal{C}_\omega$ denote the set of weakly separated collections $C \subseteq \mathcal{X}_\omega$ whose cardinality $|C|$ is maximum; let us call them *largest $\omega$-ws-collections*, and denote the maximum $|C|$ by $c_\omega$. It is shown in [7] that $c_\omega$ equals $\ell(\omega) + n + 1$ (where $\ell(\omega)$ is the minimum number of inversions for $\omega$); it follows that

$$c_\omega = c_{\omega'} + n - \omega(n) + 1,$$

where $\omega'$ is the permutation on $[n-1]$ defined by $\omega'(i) := \omega(i) - 1$ if $\omega(i) > \omega(n)$, and $\omega'(i) := \omega(i)$ otherwise. Due to this, the projection $C'$ of $C \in \mathcal{C}_\omega$ into $2^{[n-1]}$ (defined by $X \mapsto X - \{n\}$) is a largest $\omega'$-ws-collection, and the corresponding "separator" $S := \{X: X, Xn \in C\}$ consists of $n - \omega(n) + 1$ sets $S_q$ of size $|S_q| = q$ for $q = \omega(n) - 1, \ldots, n - 1$. The sets $M_q, N_q$ for $q = 0, \ldots, n - 1$ are defined accordingly (in particular, all sets $X \in C$ with $|X| = q$ belong to $M_q$ when $q < \omega(n) - 1$).

Given a g-tiling $T'$ for $\omega'$, we define a legal path $P$ in $G_{T'}$ in the same way as before (i.e., so as to satisfy conditions (iii),(iv),(v) in (8.1) and the requirement that $P$ has no terminal vertices) with the only difference that now $P$ should begin at the vertex $p'_{\omega(n)-1,\omega}$ on the right boundary of the $\omega'$-zonogon.

The desired generalizations ($\omega$-analogs) of Theorems 9.1, 9.3 and Lemma 9.4 are formulated in a natural way, and their proofs remain essentially the same as before, relying on the above definition of a legal path and corresponding direct extensions of results on $n$-contractions and $n$-expansions to the $\omega$ case.

**Remark 7** In fact, the generalization in part A is a special case of the one in part B. More precisely, given $a \in \mathbb{Z}_+^n$, define $\overline{a}_i := a_1 + \ldots + a_i$, $i = 0, \ldots, n$ (letting $\overline{a}_0 := 0$). Let us form a permutation $\omega'$ on $[\overline{a}_n]$ as follows: for $i = 1, \ldots, n$ and $q = 1, \ldots, a_i$,

$$\omega'(\overline{a}_{i-1} + q) := \overline{a}_n - \overline{a}_i + q,$$

i.e., $\omega'$ permutes the blocks $B_1, \ldots, B_n$, where $B_i := \{\overline{a}_{i-1}+1, \ldots, \overline{a}_i\}$, according to the permutation $\omega_0$ on $[n]$, and preserves the order of elements within each block. Then there is a one-to-one correspondence between the vectors $x \in \mathbf{B}^{n,a}$ and the ideals $X$ of $([\overline{a}_n], \prec_{\omega'})$, namely: $X \cap B_i$ consists of the first $x_i$ elements of $B_i$, for each $i$. Under this correspondence, (10.1) is equivalent to (1.1). Although the shape of the zonogon $Z_{n,a}$ looks somewhat different compared with $Z_{\omega'}$ (since the generating vectors $\xi_\bullet$ for different elements in a block are non-colinear), it is easy to see that the wirings for the former and the latter are, in fact, the same. (This implies an equivalence of the g-tilings for these two cases, which is not seen immediately.) So the integer box case is reduced, in all aspects we deal with, to the permutation one.

# Appendix: TP-bases and weakly separated set-systems on a hyper-simplex

Our results on TP-bases, generalized tilings and weakly separated set-systems and techniques elaborated in previous sections enable us to obtain an analog of the equivalence (i)$\Longleftrightarrow$(iv) in Theorem A to hyper-simplexes. Let us start with basic definitions and backgrounds.

When dealing with a hyper-simplex $\Delta_n^m = \{S \subseteq [n] : |S| = m\}$ rather than the Boolean cube $2^{[n]}$, the notion of TP-functions and TP-bases are modified as follows. Let $f : \Delta_n^m \to \mathbb{R}$. Instead of relation (1.1) involving triples $i < j < k$, one considers relation

$$f(Xik) + f(Xj\ell) = \max\{f(Xij) + f(Xk\ell), \ f(Xi\ell) + f(Xjk)\} \qquad (11.1)$$

for a quadruple $i < j < k < \ell$ in $[n]$ and a subset $X \subseteq [n] - \{i, j, k, \ell\}$ of size $m - 2$. When this holds for all such $X, i, j, k, \ell$, $f$ is said to be a *TP-function* on $\Delta_n^m$. Let $\mathcal{TP}_n^m$ denote the set of such functions $f$. By an analogy with the Boolean cube case, a subset $B \subseteq \Delta_n^m$ is called a *TP-basis* if the restriction map $\mathcal{TP}_n^m \to \mathbb{R}^B$ is bijective.

An important instance of TP-bases for $\Delta_n^m$ is the collection $\mathcal{IS}_n^m = \mathcal{I}^m \cup \mathcal{S}^m$, where $\mathcal{I}^m = \mathcal{I}_n^m$ consists of the intervals of size $m$ and $\mathcal{S}^m = \mathcal{S}_n^m$ consists of the sets of size

$m$ representable as the union of two nonempty intervals $[1..p]$ and $[q..r]$ with $q > p+1$ (see [2], where the elements of $\mathcal{S}^m$ are called *sesquialteral intervals*).

When a TP-basis contains four sets $Xij, Xk\ell, Xi\ell, Xjk$ as above and one set $Y \in \{Xik, Xj\ell\}$, the replacement of $Y$ by the other set $Y'$ in $\{Xik, Xj\ell\}$ gives another TP-basis $B'$. We call such a transformation $Y \rightsquigarrow Y'$ (or $B \mapsto B'$) a *raising (lowering) 4-flip* if $Y = Xik$ (resp. $Y = Xj\ell$). One can see that $\mathcal{IS}_n^m$ does not admit lowering flips and we call this TP-basis *standard* for $\Delta_n^m$ (analogously to the basis $\mathcal{I}_n$ for $2^{[n]}$ where weak lowering flips are absent as well).

The object of our interest is the class $\mathcal{B}_n^m$ of TP-bases that can be obtained by making a series of 4-flips starting from $\mathcal{IS}_n^m$. It is analogous to the class $\mathcal{B}_m$ of semi-normal bases for the Boolean cube case (and $\mathcal{B}_n^m$ along with the 4-flips on its members represents another interesting sample of Plücker environments).

A direct calculation shows that $|\mathcal{IS}_n^m| = m(n-m) + 1$; so all TP-bases for $\Delta_n^m$ have this cardinality. Besides, one can associate to $B \in \mathcal{B}_n^m$ the number $\eta(B) := \sum_{X \in B} \sum_{i \in X} i$. Clearly any lowering 4-flip decreases $\eta$; we shall see later that any $B \in \mathcal{B}_n^m$ is reachable from $\mathcal{IS}_n^m$ by a series of merely raising 4-flips, and therefore, $\eta(B) > \eta(\mathcal{IS}_n^m)$ unless $B = \mathcal{IS}_n^m$.

Our goal is to show that $\mathcal{B}_n^m$ coincides with the set $\mathcal{C}_n^m$ of *largest* weakly separated collections $C \subseteq \Delta_n^m$, i.e., having maximum possible cardinality $|C|$. We rely on two known facts.

First, Scott [9] showed that if a ws-collection $C \subseteq \Delta_n^m$ contains four sets $Xij, Xk\ell, Xi\ell, Xjk$ (with $X, i, j, k, \ell$ as above), then each of $Xik, Xj\ell$ is weakly separated from any member of $C$. (Note that $Xik, Xj\ell$ are not weakly separated from each other.)

This implies that any TP-basis in $\mathcal{B}_n^m$ is weakly separated, since the standard basis is such (which is easy to check; cf. [7]).

Second, a simple, but important, fact noticed in [7] is that: for $0 \le m' \le m \le n$ and a ws-collection $C \subseteq 2^{[n]}$ whose members have size at least $m'$ and at most $m$, if we add to $C$ all intervals of size $> m$ and all co-intervals of size $< m'$, then the resulting collection is again weakly separated. Let us call the latter collection the *straight extension* of $C$ and denote it by $C^*$. Note that the numbers of added intervals and co-intervals are $\binom{n-m+1}{2}$ and $\binom{m'+1}{2}$, respectively. When $m' = m$, the fact that the maximum cardinality of a ws-collection in $2^{[n]}$ amounts to $\binom{n+1}{2} + 1$ and the identity $\binom{n+1}{2} = \binom{n-m+1}{2} + \binom{m+1}{2} + m(n-m) + 1$ imply $|C| = |C^*| - \binom{n-m+1}{2} - \binom{m+1}{2} \le m(n-m) + 1 = |\mathcal{IS}_n^m|$.

Thus, $\mathcal{IS}_n^m$ is a largest ws-collection in $\Delta_n^m$, implying that all members of $\mathcal{B}_n^m$ are such, i.e., $\mathcal{B}_n^m \subseteq \mathcal{C}_n^m$. We show the following analog of a result from Section 9 to hyper-simplexes.

**Theorem 11.1** *Let $C \in \mathcal{C}_n^m$ and $C \ne \mathcal{IS}_n^m$. Then $C$ admits a lowering 4-flip (defined in the same way as for TP-bases). In particular, all members of $\mathcal{C}_n^m$ belong to one and the same orbit w.r.t. 4-flips.*

In light of the above discussion, this gives the desired result:

**Corollary 11.2** $\mathcal{B}_n^m = \mathcal{C}_n^m$.

**Proof of Theorem 11.1** Note that $C$ contains all intervals and co-intervals of size $m$ (since they are weakly separated from any member of $C^*$).

We use induction on $n$, assuming w.l.o.g that $0 < m < n$. The ws-collection $C^*$ is largest in $2^{[n]}$ and its projection $C'$ (defined as in Section 9) is a largest ws-collection in $2^{[n-1]}$. Let $T$ be a g-tiling on $Z_{n-1}$ whose spectrum $B_T$ is $C'$. Clearly for $h > m$ (resp. $h < m - 1$), the set $C'_h := \{X \in C' \colon |X| = h\}$ consists of all intervals (resp. all co-intervals) of size $h$ in $[n-1]$.

As to level $m$, all sets $X \in C'_m$ are exactly the members of $C$ not containing the element $n$ (since there exists only one set $Y \in C^*$ with $|Y| > m$, namely, the interval $[n-m..n]$, whose projection occurs in $C'_m$, but the latter is the interval $[n-m..n-1]$, which belongs to $C \cap C'_m$). And in level $m - 1$, all members of $C'_{m-1}$ are exactly the projections of those members of $C$ that contain $n$ (since there exists only one set $Y \in C^*$ with $|Y| < m$, namely, the "co-interval" $[1..m-1]$, that occurs in $C'_{m-1}$, but it is the projection of the co-interval $[1..m-1] \cup \{n\}$, which belongs to $C$).

Now consider two possible cases.

*Case 1.* Let $T$ have no feasible M-configuration of height $m - 1$ (defined as in Subsection 4.1). Then, by Proposition 5.3, $C'_{m-1}$ contains only co-intervals. It is easy to see that, under the above projection map, the preimages in $C$ of these co-intervals $X$ (which are exactly the members of $C$ containing $n$) are only intervals and sesquialteral intervals. Also the facts that $C'$ is the straight extension of $C'_m$ and that $C'$ is a largest ws-collection in $2^{[n-1]}$ (by Corollary 9.2) imply $C'_m \in \mathcal{C}_{n-1}^m$. Since $C \neq \mathcal{IS}_n^m$, the subcollection $C'_m$ of $C$ differs from $\mathcal{IS}_{n-1}^m$. By induction $C'_m$ admits a lowering 4-flip; this is just a required 4-flip for $C$.

*Case 2.* Let $T$ have a feasible M-configuration $CM(X; i, j, k)$ of height $m - 1$, i.e., $C' = B_T$ contains sets $Xi, Xj, Xk, Xij, Xjk$ with $|X| = m-2$ and $i < j < k$. Since the first three sets among them belong to level $m - 1$, $C$ contains the sets $Xin, Xjn, Xkn$. These together with the sets $Xij, Xjk$ contained in $C$ (since the latter ones are in level $m$ of $T$) give the desired configuration (involving the quadruple $i < j < k < n$) for performing a lowering 4-flip in $C$.

This completes the proof of the theorem. ∎

**Remark 8.** We can give an alternative method of proving the equality $\mathcal{B}_n^m = \mathcal{C}_n^m$ relying on results of Postnikov [8] on alternating strand diagrams, or, briefly, *as-diagrams*. One can outline the idea of this method as follows (omitting details). Given $C \in \mathcal{C}_n^m$, let $T$ and $W$ be, respectively, the g-tiling and proper wiring with $B_T = B_W = C^*$. Take the maximum zigzag paths $P, P'$ in the principal components $H_{m-1}, H_m$ of the graph $G_T$, respectively (see Section 8), both going from the vertex $[m]$ to the vertex $[n-m+1..n]$. Then $P'$ passes (as vertices) all interval of sizes $m$ and $m + 1$, and one can see that the sequence of colors of its edges is $m+1, 1, m+2, 2, \ldots, n, n-m$ (in this order on $P'$). In its turn, the reverse path $P^{-1}$ of $P$ passes all co-intervals of sizes $m-1$ and $m$, and the sequence of colors of its edges is $n-m+1, 1, n-m+2, 2, \ldots, n, m$. When considering a planar layout of $G_T$ on the disc $D_T$, the concatenation (circuit) $Q$ of $P'$ and $P^{-1}$ cuts out a smaller disc $\widetilde{D}$ in $D_T$; it is the union of (the squares

representing) the tiles of height $m$ in $T$.

Take the parts $w_i'$ of wires $w_i \in W$ going across $\widetilde{D}$, and denote the beginning and end points of $w_i'$ by $u_i$ and $v_i$, respectively (both $u_i, v_i$ are interior points of edges of color $i$ in $Q$, by the construction in Section 6). Let $S$ be the sequence of $2m$ points $u_i, v_j$ along $P^{-1}$, and $S'$ the sequence of $2(n-m)$ points $u_i, v_j$ along $P'$. Partition $S$ (resp. $S'$) into $m$ (resp. $n-m$) consecutive pairs; then each pair contains one "source" $u_i$ and one "sink" $v_j$ of the wiring $W' = (w_1', \ldots, w_n')$. Now extend each wire in $W'$ within the boundary $Q$ of $\widetilde{D}$ so that, for the pairs $\{u_i, v_j\}$ as above, the beginning of $w_i'$ coincide with the end of $w_j'$. One shows (a key) that the resulting wiring forms an as-diagram $\mathcal{D}$ of [8]. Moreover, the sequences of edge colors in $P^{-1}, P'$ indicated above provide that the corresponding permutation on $[n]$ associated to the set of directed chords $(u_i, v_i)$ (when the above $n$ pairs are numbered clockwise) is the Grassmann permutation $\omega$ for $m, n$, namely, $\omega(i) = m + i$ for $i = 1, \ldots, n-m$, and $\omega(i) = i - n + m$ otherwise. Finally, using properties of non-terminal vertices of $T$ exhibited in (3.5), one shows that Postnikov's operations (M1) ("square moves") on $\mathcal{D}$ correspond to 4-flips on $C$. Then Theorem 11.1 can be derived from a result (Theorem 13.4) in [8].

Next, the fact that the poset $(\mathcal{B}_n^m, \prec)$ (where $B \prec B'$ if $B$ is obtained from $B'$ by lowering 4-flips) has a unique minimal element (namely, $\mathcal{IS}_n^m$) easily implies that this poset has a unique maximal element. This is $\mathcal{I}_n^m \cup \text{co-}\mathcal{S}_n^m$, where $\text{co-}\mathcal{S}_n^m$ consists of the $m$-sized sets representable as the union of two nonempty intervals $[p..q]$ and $[r..n]$ with $r > q + 1$. Let us call this basis *co-standard* for $\Delta_n^m$.

One more useful construction concerns an embedding of the set $\mathcal{B}_n$ of semi-normal TP-bases for the Boolean cube $2^{[n]}$ into TP-bases for some hyper-simplex. More precisely, consider a hyper-simplex $\Delta_{n+n'}^n$ with $n' \geq n$. For $X \subseteq [n]$, define $X^\Delta$ to be the union of $X$ and the interval $[q..n+n']$ of size $n - |X|$ (which is empty when $X = [n]$). In addition, let $\mathcal{A}$ be the collection of all $n$-sized intervals $[p..q]$ with $n < q < n + n'$ and all $n$-sized sets $[p..q] \cup [r..n+n']$ with $n+1 < q+1 < r \leq n+n'$. Then the union of $\mathcal{A}$ and the collection $\{I^\Delta \colon I \in \mathcal{I}_n\}$ forms the co-standard basis for $\Delta_{n+n'}^n$. In a similar fashion, we associate to any $\mathcal{X} \subseteq 2^{[n]}$ the collection $\mathcal{A} \cup \{X^\Delta \colon X \in \mathcal{X}\}$, denoted as $\mathcal{X}^\Delta$.

**Proposition 11.3** *For any semi-standard TP-basis $B \in \mathcal{B}_n$, one holds $B^\Delta \in \mathcal{B}_{n+n'}^n$.*

**Proof** Let $B \neq \mathcal{I}_n$. Then $B$ contains sets $Xi, Xk, Xij, Xik, Xjk$ for some $i < j < k$ and $X \subseteq [n] - \{i, j, k\}$. Under the transformation $Y \mapsto Y^\Delta$, these sets turn into the sets (respectively) $Xi \cup [\ell..n+n'], Xk \cup [\ell..n+n'], Xij \cup [\ell+1..n+n'], Xik \cup [\ell+1..n+n'], Xjk \cup [\ell+1..n+n']$ in $\Delta_{n+n'}^m$, where $\ell := n' + |X| + 2$. Then the lowering flip $Xik \rightsquigarrow Xj$ in $B$ corresponds to the raising flip $X'ik \rightsquigarrow X'j\ell$ in $B^\Delta$, where $X' =: X \cup [\ell+1..n+n']$, and the result follows. ∎

The embedding $B \mapsto B^\Delta$ of the TP-bases for $2^{[n]}$ into TP-bases for $\Delta_{n+n'}^n$ can be regarded as a certain dual analog of the straight extension map $B' \mapsto (B')^*$ of the TP-bases for $\Delta_n^m$ to TP-bases for $2^{[n]}$.

We conclude this section with a generalization to a truncated Boolean cube $\Delta_n^{m',m} := \{S \subseteq [n] \colon m' \leq |S| \leq m\}$, where $0 \leq m' \leq m \leq n$. In this case, one considers the

class $\mathcal{TP}_n^{m',m}$ of functions $f : \Delta_n^{m',m} \to \mathbb{R}$ obeying relations (1.1) and (11.1) for all corresponding corteges where the six sets occurring as arguments belong to $\Delta_n^{m',m}$. (In fact, it suffices to require that (1.1) be imposed everywhere but that (11.1) be explicitly imposed only for the corteges related to the lowest level, i.e., when $|X| = m' - 2$. Then (11.1) for the other corteges will follow; see [2].)

The class $\mathcal{TP}_n^{m',m}$ has as a basis the set $\mathcal{IS}_n^{m',m} := \mathcal{S}_n^{m'} \cup \mathcal{I}_n^{m'} \cup \mathcal{I}_n^{m'+1} \cup \ldots \cup \mathcal{I}_n^m$ [2], called the standard basis for this case. We define $\mathcal{B}_n^{m',m}$ to be the set of bases obtained from the standard one by a series of weak 3-flips (i.e., related to (1.1)) and 4-flips. (From the theorem below it follows that the latter ones are important only in level $m'$.) On the other hand, we can consider the set $\mathcal{C}_n^{m,m'}$ of largest ws-collections in $\Delta_n^{m',m}$ (whose cardinality is $\binom{n+1}{2} - \binom{n-m+1}{2} - \binom{m'+1}{2} = |\mathcal{IS}_n^{m',m}|$). By explanations above, $\mathcal{B}_n^{m,m'} \subseteq \mathcal{C}_n^{m,m'}$.

**Theorem 11.4** *Let $C \in \mathcal{C}_n^{m,m'}$. Then $\mathcal{IS}_n^{m',m}$ can be obtained from $C$ by a series of lowering weak 3-flips followed by a series of lowering 4-flips in level $m'$. Therefore, $\mathcal{B}_n^{m,m'} = \mathcal{C}_n^{m,m'}$.*

**Proof** If $C$ admits a lowering 3-flip, then performing such a flip produces a collection in $\mathcal{C}_n^{m,m'}$ with a smaller total size of its members. If such a flip is impossible, then all sets $X \in C$ with $|X| > m'$ are intervals, by Proposition 5.3. Then $\{X \in C : |X| = m'\}$ is a largest ws-collection for the hyper-simplex $\Delta_n^{m'}$, and the result follows from Theorem 11.1. ∎

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