Development of singularities for the compressible Euler equations with external force in several dimensions

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Abstract.

We consider solutions to the Euler equations in the whole space from a certain class, which can be characterized, in particular, by finiteness of mass, total energy and momentum. We prove that for a large class of right-hand sides, including the viscous term, such solutions, no matter how smooth initially, develop a singularity within a finite time. We find a sufficient condition for the singularity formation, "the best sufficient condition", in the sense that one can explicitly construct a global in time smooth solution for which this condition is not satisfied "arbitrary little".

Also compactly supported perturbation of nontrivial constant state is considered. We generalize the known theorem [1] on initial data resulting in singularities. Finally, we investigate the influence of frictional damping and rotation on the singularity formation.

1. Introduction

We are interested in the following system of balance laws in differential form

$$\rho(\partial_t \mathbf{V} + (\mathbf{V}, \nabla)\, \mathbf{V}) + \nabla P = \mathbf{F}(t, x, \rho, \mathbf{V}, S, D^{|\alpha|}\mathbf{V}), \tag{1}$$

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{V}) = 0, \tag{2}$$

$$\partial_t S + (\mathbf{V}, \nabla S) = 0, \tag{3}$$

written for unknown functions $\rho, \mathbf{V} = (V_1, ..., V_n)$ and S, density, velocity vector and entropy, respectively. The functions depend on time t and on point $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Here $P = P(t, x)$ is the pressure, $\mathbf{F} = (F_1, ..., F_n)$ is an external force, assumed to be a smooth function of all its arguments, $|\alpha| \ge 0$ is a multiindex, $\gamma > 1$ is the adiabatic exponent.

We consider $(1-3)$ together with the state equation

$$P = \rho^\gamma e^S. \tag{4}$$

For smooth solutions equations (2),(3) and (4) imply

$$\partial_t P + (\mathbf{V}, \nabla P) + \gamma P \, div \, \mathbf{V} = 0. \tag{5}$$

Set an initial-value problem for (1), (2), (3), namely

$$\rho(0, x) = \rho_0(x),\ \mathbf{V}(0, x) = \mathbf{V}_0(x),\ S(0, x) = S_0(x). \tag{6}$$

Sometime it will be more convenient for us to consider the Cauchy problem for (1),(2),(5), that is

$$\rho(0, x) = \rho_0(x),\ \mathbf{V}(0, x) = \mathbf{V}_0(x),\ P(0, x) = P_0(x) = \rho_0^\gamma(x) e^{S_0(x)}. \tag{6'}$$

It is well known that for at least $\mathbf{F} = 0$, solutions of equations (1–3), no matter how smooth initially, can develop singularities within a finite time.

In the one-dimensional (in space) case for the problem on a singularity formation for solutions to (1–3) the characteristics method can be applied. In the isoentropic case, where the system can be written in the Riemann invariants [2]), the characteristics method gives a complete answer whether the singularity (the gradient catastrophe) arises (it follows, for example, from [4]). In the non-isoentropic case also there are some advances (for example, [5]), however the results either have inexplicit character or concerns with small perturbations of a constant state. The problem on the singularity formation for the one-dimensional system of gas dynamic equations can be investigated by means of other methods ([6, 7]). However, this methods give only sufficient conditions of the gradient catastrophe, (generally speaking, with a large margin).

In [1] for the 3D case sufficient conditions on initial data perturbed from constant state with a positive density inside a compact domain were found, such that the respective solution to the Cauchy problem loses its smoothness in a finite time. The results can be partly generalized to the case of arbitrary dimension. The general sense of these conditions is that the speed of the support propagation (that is, the speed of sound in the unperturbed domain) is small, compared with the velocity of the gas inside the initial perturbation. Provided the sufficient conditions hold a breakdown occurs near the support boundary [1].

In [1] it is essential for the proof that the speed of propagation of the perturbation is finite. Therefore the result cannot be extended to the case of viscous compressible flow, with the traditional viscosity description [8], where the perturbation spreads with infinite speed.

Problems where the initial data are compactly supported can be treated separately. For this class of initial data it is significantly easier to find initial conditions producing singularities. The point is that if the solution is smooth, the boundary of perturbation does not move, that is the support does not expand. This fact allows to demonstrate that any smooth compactly supported initial data result in a singularity ([10]). Moreover, it is true for the Navier-Stokes equations as well ([11]).

In Section 2 we consider initial data without restrictions on the support, but having finite moment of mass and total energy. For these solutions the mass is conserved. If we impose some reasonable restrictions on the right-hand side of (1), then we obtain additionally conservation of angular momentum and non-increasing of total energy. If the flow is considered in all the space $\mathbb{R}^n$, rather than just inside the bounded volume of the liquid, then the density is forced to vanish rapidly as $x \to \infty$.

We will show that in this case, too, it is possible to indicate sufficient conditions to initial data, such that the solution leaves a special class of functions. For some important right-hand sides $\mathbf{F}$ it signifies that the solution loses its initial smoothness within a finite time. The role of "restraining force" preventing decay of the gradient, rather than by the finite speed of support propagation, is played by a value, that in the 3D case can be interpreted as the initial vorticity of the flow.

This result is essentially multidimensional, in the sense that in the 1D case the sufficient conditions cannot be satisfied.

It is interesting that the result can be applied to the case of viscous compressible flow, after imposing some restrictions on the velocity vanishing at $|\mathbf{x}| \to \infty$.

Further we will show that the sufficient conditions that we find are in some sense "best sufficient conditions" for a class of right hand sides. That is, there exists an explicit globally smooth in time solution, for which the sufficient conditions are not satisfied "arbitrarily little".

In Section 3 we improve the result of ([1]), and generalize it, assuming the presence of a special exterior force, that may have influence on the speed of the support propagation.

In Sections 4 and 5 we add to the right-hand side of (1) terms describing damping and rotation and find once more sufficient conditions for the finite time singularity formation.

2. Solutions with finite moment of mass

Definition. *We will say that a solution* $(\rho, \mathbf{V}, P)$ *to system (1),(2),(5) belongs to the class* $\mathfrak{K}$ *if it has the following properties:*

(i) *the solution is classical for all* $t \ge 0$*;*

(ii) $\rho|x|^2$, P, $\rho|\mathbf{V}|^2$ *are of the class* $L_1(\mathbb{R}^n)$;

(iii) $\int\limits_{\mathbb{R}^n} (\mathbf{F}, \mathbf{x})\,dx \equiv 0$, $\int\limits_{\mathbb{R}^n} (F_i x_j - F_j x_i)\,dx \equiv 0$, $i \ne j$, $i, j = 1, ..., n$, *where* $\mathbf{x}$ *is a radius-vector of point* x;

(iv) $\int\limits_{\mathbb{R}^n} (\mathbf{F}, \mathbf{V})\,dx \le 0.$

We risk, of course, that for some choice of $\mathbf{F}$ the class $\mathfrak{K}$ is empty or consists only of trivial solution. However, for $\mathbf{F} = \mathbf{0}$ this class is not trivial and we essentially seek the situation where the solutions can be treated likely to this principal case.

For $\mathbf{F} = \mathbf{F}(\mathbf{V})$, such that $\mathbf{F}(\mathbf{0}) = \mathbf{0}$,it is known that if

$$(\rho_0^{\frac{\gamma-1}{2}}, \mathbf{V}_0, S_0) \in H^m(\mathbb{R}^n), \quad m > \frac{n}{2} + 1,$$

then locally in time (1–3) has a unique solution

$$(\rho^{\frac{\gamma-1}{2}}, \mathbf{V}, S) \in \cap_{i=0}^{1} C^i([0,T], H^{m-i}(\mathbb{R}^n).$$

This result follows, for example, from [3], if the system is symmetrized by means of a new variable $P^{(\gamma-1)/2\gamma}$[12]. The classical result cannot be applied immediately, as on the solutions with finite moment of mass the density is not separated from zero and the system is not uniformely strictly hyperbolic.

Besides, in [6] it was proved if the initial data are from the class $H^m_{ul}(\mathbb{R}^n)$, then there exists a unique solution from $\cap_{i=0}^{1} C^i([0,T], H^{m-i}_{loc}(\mathbb{R}^n)$. Here H^m_{ul} is a subset in H^m_{loc} such that for all $\phi \in C_0^\infty$, if $\phi_x(y) = \phi(x-y)$, then $\sup\limits_{x\in\mathbb{R}^n} \|\phi_x u\|_{H^m(\mathbb{R}^n)} < \infty$.

For the right hand side describing viscosity, there exist results on a local in time existence of smooth solution as well, f.e. [13], [14]. In [13] the author proved the existence of classical solution, having the Hölder continuous second derivatives with respect the space variables and the first ones with respect the time. In [14] the system of equations of viscous compressible fluid is considered as a particular case of composite systems of differential equations. The consideration is proceeded in the Sobolev spaces H^l with a sufficiently large l. The uniqueness of the problem was proved earlier in [15].

For an arbitrary forcing we have to assume the existence of a local in time solution of class $\mathfrak{K}$.

Let us note that for the solutions of class $\mathfrak{K}$ we have conservation of mass $m = \int\limits_{\mathbb{R}^n} \rho\, dx$. Moreover, all integrals

$$M_k = M_{ij} = \int\limits_{\mathbb{R}^n} (V_j x_i - V_i x_j)\rho\, dx,\ i > j,\ k = 1, ..., K,\ K = \mathrm{C}_n^2,$$

are conserved. At $n = 1$ there are no integrals in this series, at $n = 2$ there is only integral $M_1 = \int\limits_{\mathbb{R}^n} (\mathbf{V}, \mathbf{x}_\perp)\rho\, dx$, where $\mathbf{x}_\perp = (x_2, -x_1)$; at $n = 3$ the integrals M_1, M_2, M_3 correspond to components of the angular momentum $\int\limits_{\mathbb{R}^n} (\mathbf{V} \times \mathbf{x})\rho\, dx$.

The total energy, $E(t)$, is a sum of its kinetic and potential components, that is

$$E(t) = E_k(t) + E_p(t) := \frac{1}{2}\int\limits_{\mathbb{R}^n} \rho|\mathbf{V}|^2\, dx + \frac{1}{\gamma - 1}\int\limits_{\mathbb{R}^n} P\, dx.$$

We get from (1), (2), (5) that $E'(t) = \int\limits_{\mathbb{R}^n} (\mathbf{F}, \mathbf{V})\, dx$. Thus, in virtue of (iv), $E(t)$ is a non-increasing function for the solutions of class $\mathfrak{K}$.

Let us introduce a functional

$$G_\phi(t) = \int\limits_{\mathbb{R}^n} \rho(t, x)\phi(|\mathbf{x}|)\, dx,$$

considered for such functions $\phi(|\mathbf{x}|) \in C^2[0, +\infty)$, that the integral converges and

$$I_{4,\phi}(t) = \lim_{R\to\infty} \int\limits_{S(R)} \rho\mathbf{V}\phi(|\mathbf{x}|)\, dS(R) = 0,$$

where $S(R)$ is the $(n-1)$ - dimensional sphere of radius R.

We denote by $\mathbf{M} = (M_1, ..., M_K)$ and $\sigma = (\sigma_1, ..., \sigma_K)$ vectors with components M_k, and $\sigma_k = V_i x_j - V_j x_i$, $i > j$, $i, j = 1, ..., n$, $k = 1, ..., K$, $K = \mathrm{C}_n^2$, respectively.

Lemma 2.1. *Provided all given integrals converge, for solutions to (1), (2), (5) of the class $\mathfrak{K}$ following equalities take place:*

$$G'_\phi(t) = \int\limits_{\mathbb{R}^n} \frac{\phi'(|\mathbf{x}|)}{|\mathbf{x}|}(\mathbf{V}, \mathbf{x})\rho\, dx,$$

$$G''_\phi(t) = I_{1,\phi}(t) + I_{2,\phi}(t) + I_{3,\phi}(t) + I_{4,\phi}(t),$$

where

$$I_{1,\phi}(t) = \int\limits_{\mathbb{R}^n} \frac{\phi''(|\mathbf{x}|)}{|\mathbf{x}|^2}|(\mathbf{V}, \mathbf{x})|^2\rho\, dx,$$

$$I_{2,\phi}(t) = \int\limits_{\mathbb{R}^n} \frac{\phi'(|\mathbf{x}|)}{|\mathbf{x}|^3}|\sigma|^2\rho\, dx,$$

$$I_{3,\phi}(t) = \int\limits_{\mathbb{R}^n} (\phi''(|\mathbf{x}|) + (n-1)\frac{\phi'(|\mathbf{x}|)}{|\mathbf{x}|})P\, dx,$$

$$I_{4,\phi}(t) = -\lim_{R\to\infty} \int\limits_{S(R)} \phi'(|\mathbf{x}|)P\, dS(R).$$

The proof is a direct calculation and an application of the general Stokes formula. For example, we get, using (2), that

$$G'_\phi(t) = \int\limits_{\mathbb{R}^n} \rho'_t(t,x)\phi(|\mathbf{x}|)\,dx = -\int\limits_{\mathbb{R}^n} \operatorname{div}(\rho\mathbf{V})\phi(|\mathbf{x}|)\,dx =$$

$$= \int\limits_{\mathbb{R}^n} (\nabla\phi(|\mathbf{x}|),\mathbf{V})\rho\,dx - \lim_{R\to\infty}\int\limits_{S(R)} \rho\mathbf{V}\phi(|\mathbf{x}|)\,dS(R) =$$

$$= \int\limits_{\mathbb{R}^n} \frac{\phi'(|\mathbf{x}|)}{|\mathbf{x}|}(\mathbf{V},\mathbf{x})\rho\,dx.$$

□

Remark 2.1 If the increase of $\phi(|\mathbf{x}|)$ as $|\mathbf{x}|\to\infty$ is no more then $const\cdot|\mathbf{x}|^2$, then the condition

$$\lim_{R\to\infty}\int\limits_{S(R)} \rho\mathbf{V}\phi(|\mathbf{x}|)\,dS(R) = 0$$

follows from (ii) without additional assumptions on the behavior of velocity at $|\mathbf{x}|\to\infty$.

In the particular case $\phi(|\mathbf{x}|) = \frac{|\mathbf{x}|^2}{2}$ we denote $G_\phi(t), I_{i,\phi}(t)$, $i = 1,...,4$, by $G(t), I_i(t)$, respectively, the derivative $G'(t)$ we denote $F(t)$.

Corollary 2.1. *For solutions to (1), (2), (5) of the class* $\mathfrak{K}$

$$F(t) = G'(t) = \int\limits_{\mathbb{R}^n} (\mathbf{V},\mathbf{x})\rho\,dx,$$

$$I_1(t) = \int\limits_{\mathbb{R}^n} \frac{|(\mathbf{V},\mathbf{x})|^2}{|\mathbf{x}|^2}\rho\,dx,$$

$$I_2(t) = \int\limits_{\mathbb{R}^n} \frac{|\sigma|^2}{|\mathbf{x}|^2}\rho\,dx,$$

$$I_3(t) = n\int\limits_{\mathbb{R}^n} P\,dx = n(\gamma-1)E_p(t),$$

$$I_4(t) = 0.$$

Moreover,

$$I_1(t) + I_2(t) = 2E_k(t).$$

The proof of Corollary 2.1 is an immediate substitution of a particular form of $\phi(|\mathbf{x}|)$; $I_4(t) = 0$ due to sufficiently rapid vanishing of P as $|\mathbf{x}|\to\infty$, forced by condition (ii). □

The following Lemma gives some useful estimates for the functionals introduced above.

Lemma 2.2. *For solutions to (1), (2), (5) of the class* $\mathfrak{K}$ *inequalities*

$$(G'(t))^2 = F^2(t) \le 4G(t)E_k(t) \le 4G(t)E(0), \tag{7}$$

$$(G'(t))^2 = F^2(t) \le 2G(t)I_1(t), \tag{8}$$

$$|\mathbf{M}|^2 \le 4G(t)E_k(t) \le 4G(t)E(0), \tag{9}$$

$$|\mathbf{M}|^2 \le 2G(t)I_2(t), \tag{10}$$

$$G(t) \le (\sqrt{E(0)}t + \sqrt{G(0)})^2 \tag{11}$$

holds.

Proof. The first four inequalities are corollaries of the Hölder inequality. Inequality (11) follows from (7) after integration.□

Let us point out that in the case $\mathbf{F} = \mathbf{0}$ the last parts in (7) and (9) is not a very strong roughening, as according to [6], for smooth solutions $E_k(t) \to E$, $t \to \infty$.

The Hölder inequality gives us also a lower estimate for the kinetic energy, namely,

$$E_k(t) \ge \frac{F^2(t)}{4G(t)}. \tag{12}$$

As for a lower estimation of the potential energy, there exists the following result.

Lemma 2.3. [6] *For solutions to (1), (2), (5) satisfying (ii)*

$$E_p(t) \ge \frac{C}{G^{\frac{(\gamma-1)n}{2}}(t)}, \tag{13}$$

with a positive constant C, depending on initial data, γ and n.

If we denote $S_0 = \inf\limits_{x\in\mathbb{R}^n} S(0,x)$, then

$$C = \frac{e^{S_0}}{\gamma - 1}(mC_{\gamma,n}^{-1})^{\frac{\gamma(n+2)-n}{2}},$$

with

$$C_{\gamma,n} = \left(\frac{2\gamma}{n(\gamma-1)}\right)^{\frac{n(\gamma-1)}{(n+2)\gamma-n}} + \left(\frac{2\gamma}{n(\gamma-1)}\right)^{\frac{-2\gamma}{(n+2)\gamma-n}}.$$

Theorem 2.1. *There are initial data (6') satisfying (ii) such that solution to (1), (2), (5) from the class $\mathfrak{K}$ exists during a finite time. Namely, it occurs if*

$$F(0) \ge L_2 \cot \frac{L_1}{2\sqrt{E(0)G(0)}}, \tag{14}$$

with constants L_1 and L_2 depending on initial data , the adiabatic exponent γ and the dimension of space only.

If $\gamma \in (1, 1+\frac{2}{n}]$, then $L_1 = L_2 = L := \left(2n(\gamma-1)C(G(0))^{1-(\gamma-1)n/2} + |\mathbf{M}|^2\right)^{1/2}$.

If $\gamma > 1 + \frac{2}{n}$, then $L_1 = L/((\gamma-1)n-1)$, $L_2 = L$.

The time of existence for this solution can be estimated above by the constant

$$T_* = \sqrt{\frac{G(0)}{E(0)}} \frac{\frac{2\sqrt{E(0)G(0)}}{L_2}\left(\frac{\pi}{2} - \arctan\frac{F(0)}{L_1}\right)}{1 - 2\frac{\sqrt{E(0)G(0)}}{L_2}\left(\frac{\pi}{2} - \arctan\frac{F(0)}{L_1}\right)}.$$

Proof of Theorem 2.1. Let $\gamma \in (1, 1+\frac{2}{n}]$, it follows $2-(\gamma-1)n \ge 0$. Therefore, from (8), (10), (13) we have

$$F'(t) \ge \frac{F^2(t)}{2G(t)} + \frac{|\mathbf{M}|^2}{2G(t)} + \frac{n(\gamma-1)C}{(G(t))^{\frac{(\gamma-1)n}{2}}} \ge$$

$$\ge \frac{F^2(t) + |\mathbf{M}|^2 + 2n(\gamma-1)C(\sqrt{E(0)}t + \sqrt{G(0)})^{2-(\gamma-1)n}}{2(\sqrt{E(0)}t + \sqrt{G(0)})^2} \ge$$

$$\ge \frac{F^2(t) + L^2}{2(\sqrt{E(0)}t + \sqrt{G(0)})^2}. \tag{15}$$

After integration we have

$$\arctan\frac{F(t)}{L} \geq \arctan\frac{F(0)}{L} + \frac{L}{2\sqrt{E(0)}}\left(\frac{1}{\sqrt{G(0)}} - \frac{1}{\sqrt{E(0)}t + \sqrt{G(0)}}\right). \qquad (16)$$

The left hand side of (16) does not exceed $\frac{\pi}{2}$, therefore (16) cannot be true for all $t > 0$ if

$$\arctan\frac{F(0)}{L} + \frac{L}{2\sqrt{E(0)G(0)}} > \frac{\pi}{2}. \qquad (17)$$

Let us note that

$$\frac{L}{2\sqrt{E(0)G(0)}} \leq \frac{\sqrt{2n(\gamma-1)E_p(t)G(0) + 4E_k(t)G(0)}}{2\sqrt{(E_p(0)+E_k(0))G(0)}} \leq 1 < \frac{\pi}{2},$$

therefore after trigonometric transformations of (17) one can get (14).

Further, let $\gamma > 1 + \frac{2}{n}$, therefore $(\gamma-1)\frac{n}{2} - 1 \geq 0$. Analogously to the previous case we have

$$F'(t) \geq \frac{F^2(t) + |\mathbf{M}|^2}{2G(t)} + \frac{n(\gamma-1)C}{(G(t))^{\frac{(\gamma-1)n}{2}}} \geq$$

$$\geq \frac{(F^2(t) + |\mathbf{M}|^2)(\sqrt{E(0)}t + \sqrt{G(0)})^{-1+(\gamma-1)n} + n(\gamma-1)C}{2(\sqrt{E(0)}t + \sqrt{G(0)})^{(\gamma-1)n}} \geq$$

$$\geq \frac{(G(0))^{\frac{(\gamma-1)n}{2}-1}(F^2(t) + |\mathbf{M}|^2) + 2n(\gamma-1)C}{2(\sqrt{E(0)}t + \sqrt{G(0)})^{(\gamma-1)n}} =$$

$$= \frac{F^2(t) + L^2}{2(G(0))^{\frac{-(\gamma-1)n}{2}+1}(\sqrt{E(0)}t + \sqrt{G(0)})^{(\gamma-1)n}}. \qquad (18)$$

After integration we get

$$\arctan\frac{F(t)}{L} \geq \arctan\frac{F(0)}{L} +$$

$$\frac{L}{2((\gamma-1)n-1)\sqrt{E(0)}(G(0))^{\frac{-(\gamma-1)n}{2}+1}}\left(\frac{1}{(G(0))^{\frac{(\gamma-1)n-1}{2}}} - \frac{1}{(\sqrt{E(0)}t + \sqrt{G(0)})^{(\gamma-1)n-1}}\right). \qquad (19)$$

The condition (19) cannot be true for all $t > 0$ if

$$\arctan\frac{F(0)}{L} + \frac{L}{2((\gamma-1)n-1)(\sqrt{E(0)G(0)})} \geq \frac{\pi}{2}.$$

It implies (14) analogously to the previous case. Theorem 2.1 is proved.□

Remark 2.2. It seems that we can obtain an analogous nonexistence result from (8) and Lemma 2.1 using only the nonnegativity of the integrals $I_2(t)$ and $I_3(t)$. However, it is not true. Indeed, here we have

$$F'(t) \geq \frac{F^2(t)}{2G(t)} \geq \frac{F^2(t)}{2(\sqrt{E(0)}t + \sqrt{G(0)})^2}. \qquad (20)$$

So we obtain

$$-\frac{1}{F(t)} + \frac{1}{F(0)} \geq \frac{1}{2\sqrt{E(0)}}\left(\frac{1}{\sqrt{G(0)}} - \frac{1}{\sqrt{E(0)}t + \sqrt{G(0)}}\right).$$

As $F'(t) > 0$, then $F(t)$ remains positive for $F(0) > 0$. Therefore we conclude that (20) cannot hold for all t if

$$F(0) > 2\sqrt{E(0)G(0)}. \qquad (21)$$

However, (21) contradicts the inequality (7), therefore we cannot choose the initial data with such properties.

It is interesting that if $\mathbf{M} = \mathbf{0}$, then one cannot find initial data satisfying (14), either.

To show this, let us consider, for example, the case $\gamma \in (1, 1 + \frac{2}{n}]$. Let us find a necessary condition for the implementation of (14). As follows from (7) and (14)

$$\frac{2\sqrt{E_k(0)G(0)}}{L} \ge \cot \frac{L}{2\sqrt{(E_k(0) + E_p(0))G(0)}}. \tag{22}$$

We denote $z = \dfrac{2\sqrt{E_k(0)G(0)}}{L}$, $z_1 = \dfrac{2\sqrt{E_p(0)G(0)}}{L}$. Further, we introduce a function

$$f(z) := \arctan \frac{1}{z} - \frac{1}{\sqrt{z^2 + z_1^2}}, \quad z \in [0, \infty). \tag{23}$$

Since (22) signifies

$$\frac{1}{z} < \tan \frac{1}{\sqrt{z^2 + z_1^2}},$$

then for implementation of condition (22) we have to find a point z_* such that $f(z_*) \le 0$. However due to Lemma 2.3 for $\mathbf{M} = \mathbf{0}$ we have

$$z_1 = \frac{2\sqrt{E_p(0)G(0)}}{\sqrt{2n(\gamma - 1)CG^{1-(\gamma-1)n/2}(0)}} \ge \frac{2\sqrt{E_p(0)G(0)}}{\sqrt{2n(\gamma - 1)E_p(0)G(0)}} \ge 1.$$

Therefore $f(0) = \frac{\pi}{2} - \frac{1}{z_1} > 0$, and one can show that $f(z)$ will be positive for all $z > 0$.

On the other side, if $E_p(0) = 0$ (it take place in the so called "pressureless" gas dynamic, when $P \equiv 0$ [16]), then (14) can be satisfied also for $\mathbf{M} = \mathbf{0}$.

From the consideration above we can conclude that the condition (14) cannot hold in one space dimension if $E_p(0) \ne 0$, where we cannot obtain the additional positive lower bound for the integral $I_2(t)$.

As follows from (22), the necessary condition of implementation of (14) is the negativity of $f(z)$ at some points. For $\mathbf{M} \ne 0$, then it will be, for example, if $z_1 \le \frac{\pi}{2}$, that is $L \ge \pi\sqrt{E_p(0)G(0)}$. The last inequality surely holds if $|\mathbf{M}| > \frac{\sqrt{E_p(0)G(0)}}{\pi}$. It implies

$$\frac{E_k(0)}{E_p(0)} \ge \frac{\pi^2}{4},$$

that is initially the part of kinetic energy must exceed the potential one.

Now we will show that together with a large value of $|\mathbf{M}|$, the condition (14) requires a large initial divergency of the flow. We denote now $Z := \frac{L}{2\sqrt{EG(0)}}$ and point out that $Z \le 1$. Condition (14) can be re-written as

$$\lambda Z \le \tan Z,$$

with $\lambda = \frac{2\sqrt{EG(0)}}{F(0)}$, therefore $\lambda < \tan 1$, or $F(0) \ge \frac{2\sqrt{EG(0)}}{\tan 1}$. It follows from the last inequality that $\frac{E_k(0)}{E} \ge \cot 1$.

Remark 2.3. As follows from [17], the breakdown of smoothness in the compressible non-viscous flow in 3D comes from the accumulation of vorticity, divergency or compression.

REMARK 2.4. As follows from the proof of Theorem 2.1, the singularity appearance is a result of an unlimited growth of $F(t)$. For solutions from the class $\mathfrak{K}$ the Green's formula shows that

$$F(t) = -\frac{1}{2}\int\limits_{\mathbb{R}^n} |\mathbf{x}|^2 \nabla(\rho\mathbf{V})d\mathbf{x},$$

therefore the predicted appearence of a singularity can be associated with domains of "large negative divergency" or, as meteorologists say, of "large convergency."

In its turn, in the physical space 3D

$$|\mathbf{M}| = \left|\frac{1}{2}\int\limits_{\mathbb{R}^3} |\mathbf{x}|^2 \mathrm{rot}(\rho\mathbf{V})d\mathbf{x}\right|,$$

that is a large value of $|\mathbf{M}|$ corresponds to a large initial vorticity.

2.1. **On exactness of integral condition (14).** Since there is no one-to-one correspondence between solutions to the system (1), (2), (5) and the integral functionals considered, we cannot expect that the condition (14) of Theorem 2.1 are sufficient and necessary conditions for the singularity appearance in the class of solutions with a finite total energy and a finite moment of mass satisfying (iii) and (iv).

Nevertheless, further we will see that condition (14) of Theorem 2.1 is "exact sufficient condition" for singularity appearance, at least for special right-hand sides. In other words, if it is not satisfied for certain initial data "arbitrary little", then the corresponding solution to the Cauchy problem may be globally smooth in time. Namely, the following Theorem holds:

Theorem 2.2. *Let $\mathbf{F} = \mathbf{0}$ for any velocity field of form $\mathbf{V} = \alpha(t)\mathbf{x}$. Then for an arbitrary small $\varepsilon > 0$ there exists a globally in time classical solution to system (1), (2), (5) from the class $\mathfrak{K}$, such that at $t = 0$ condition*

$$F(0) > L_1 \cot\frac{L_2}{2\sqrt{E(0)G(0)}} - \varepsilon \tag{24}$$

holds.

To prove Theorem 2.2 we give an example of solutions satisfying the properties indicated in the Theorem statement. It is known that there exists a class of globally smooth solutions with linear profile of velocity (for one-dimensional case one can find its description in [18],[19], where the Lagrangean variables are used; another approach to constructing and generalization to the case of several space dimensions there are in [20], [21], [22]). It will be sufficient for us to consider the simplest form of such fields of velocity, namely,

$$\mathbf{V} = \alpha(t)\mathbf{x}, \tag{25}$$

with a function $\alpha(t)$, taking part of the solution to system of ODE

$$G_1'(t) = -2\alpha(t)G_1(t), \quad \alpha'(t) = -\alpha^2(t) + (\gamma-1)KG_1^{\frac{(\gamma-1)n}{2}+1}(t), \tag{26}$$

Here $K = E_p(0)G^{\frac{(\gamma-1)n}{2}}(0), G_1(t) = 1/G(t)$.

The components of density and pressure can be found from the linear with respect to them equations (2) and (5) as

$$\rho(t,|\mathbf{x}|,\phi) = \exp(-2\int\limits_0^t \alpha(\tau)d\tau)\rho_0(|\mathbf{x}|\exp(-\int\limits_0^t \alpha(\tau)d\tau)),$$

$$p(t,|\mathbf{x}|,\phi)=\exp(-2\gamma\int\limits_0^t\alpha(\tau)d\tau)p_0(|\mathbf{x}|\exp(-\int\limits_0^t\alpha(\tau)d\tau)),$$

with compatible initial data $\rho_0(x), P_0(x)$. The compatibility signifies here that the condition

$$\nabla p_0(x)=-(\gamma-1)G_1(0)E_p(0)\rho_0(x)\mathbf{x} \tag{27}$$

holds.

For example, one can choose

$$p_0=\frac{1}{(1+|\mathbf{r}|^2)^a},\ a=const>\frac{n}{2},$$

$$\rho_0=\frac{2a}{(\gamma-1)G_1(0)E_p(0)}\frac{1}{(1+|\mathbf{r}|^2)^{a+1}}.$$

Let us note that system (26) takes place for all solutions with the velocity profile (25), however, compatibility condition (27), generally speaking, can do not hold, therefore we have to require the special form of $\mathbf{F}$.

For the solutions considered $F(t)=2\alpha(t)G(t)$, $\mathbf{M}=0$, the kinetic energy $E_k(t)=\alpha^2(t)G(0)$, the potential energy $E_p(t)=\dfrac{K}{(G(t))^{(\gamma-1)n/2}}$.

Let us consider, for example, the case $\gamma\le 1+\frac{2}{n}$.

Thus, (24) takes the form

$$\frac{2\alpha(0)G(0)}{\sqrt{2n(\gamma-1)CG(0)^{1-(\gamma-1)n/2}}}\ge\cot\frac{\sqrt{2n(\gamma-1)CG(0)^{1-(\gamma-1)n/2}}}{2\sqrt{G(0)}(\alpha^2(0)G(0)+KG^{-(\gamma-1)n/2})}-\varepsilon$$

or

$$\frac{2\alpha(0)(G(0))^{\frac12+\frac{(\gamma-1)n}{4}}}{\sqrt{2n(\gamma-1)C}}\ge\cot\frac{\sqrt{2n(\gamma-1)C}}{2\sqrt{\alpha^2(0)(G(0))^{1+(\gamma-1)n/2}+K}}-\varepsilon. \tag{28}$$

Let us fix $\rho_0(x)$ and $P_0(x)$. It signify that $G(0)$ and C are fixed. We are going to show that we can choose $\alpha(0)$ such that for anyhow small positive ε inequality (28) will be satisfied.

We denote

$$z(\alpha(0))=\frac{1}{\alpha(0)}\frac{\sqrt{2n(\gamma-1)C}}{2(G(0))^{\frac12+\frac{(\gamma-1)n}{4}}}$$

and

$$\lambda(\alpha(0))=\frac{\alpha(0)(G(0))^{\frac12+\frac{(\gamma-1)n}{4}}}{\sqrt{\alpha^2(0)(G(0))^{1+(\gamma-1)n/2}+K}}.$$

We note that $z(\alpha(0))\to 0$ and $\lambda(\alpha(0))\to 1$ as $\alpha(0)\to\infty$, moreover, $\lambda(\alpha(0))<1$ for any finite $\alpha(0)$.

Thus, (28) can be re-written as follows:

$$\frac{1}{z[(\alpha(0)]}\ge\cot z[(\alpha(0)]+(\cot[\lambda(\alpha(0))z(\alpha(0))]-\cot z[\alpha(0)])-\varepsilon. \tag{29}$$

We point out that if $z(\alpha(0))\in(0,\pi)$, then $\cot[\lambda(\alpha(0))z(\alpha(0))]-\cot[z(\alpha(0))]>0$, however, for any $\varepsilon>0$ we can choose $\alpha_0>0$ such that for any $\alpha(0)>\alpha_0$ the difference $(\cot[\lambda(\alpha(0))z(\alpha(0))]-\cot z[\alpha(0)])-\varepsilon<-\varepsilon_1$ for some $\varepsilon_1>0$.

Since $\dfrac{1}{z}\ge\cot z$, $z\in(0,\pi)$, then $\dfrac{1}{z}\ge\cot z-\varepsilon_1$. For $\alpha(0)>\alpha_0$ it implies (29) and, consequently, (24).

The case $\gamma>1+\frac{2}{n}$ can be treated analogously.

Thus, the proof of Theorem 2.2 is over.$\square$

Remark 2.5. Besides the trivial case $\mathbf{F} = \mathbf{0}$, the first condition of Theorem 2.2 is satisfied for $\mathbf{F} = \mathbf{F}(D^{|\alpha|}\mathbf{V})$, $|\alpha| \ge 2$.

Remark 2.6. However, we cannot assert that if condition (14) does not hold for certain initial data, then the solution to the corresponding Cauchy problem is necessarily globally smooth in time. For example, for $\mathbf{F} = \mathbf{0}$, let us consider initial data with zero velocity and compactly supported density (and pressure). These initial data always result in a singularity (f.e.[10],[23]). However, $F(0) = 0$, and as the right hand side in (14) is positive, the condition (14) of Theorem 2.1 is not satisfied. On the other hand, it seems that in this situation there exists a moment t_1 such that if it is chosen as the initial one, then (14) will be already satisfied. In others words, the hypothesis is that (14) detects singularities arising from accumulation of negative divergency, which are sufficiently close in time.

Remark 2.7. One should pay attention to the following fact: smooth initial data $\rho_0(x), P_0(0), \mathbf{V}(0)$, having compact support and satisfying the compatibility condition are not good for application of the theorem on a local in time existence and uniqueness of the Cauchy problem for the symmetric hyperbolic systems ([3]). The matter is that at the point where the density vanishes smoothly, for a compatible initial data the entropy becomes infinite, so we cannot apply the cited theorem, which require the smoothness of initial data for *symmetrized* system, where the variables are entropy, velocity and $P^{(\gamma-1)/2\gamma}$ (see [10]). Indeed, we get non- uniqueness for the compatible initial data of density and pressure as follows. According to procedure described in the Theorem 2.2 proving we can construct a global in time solution with the velocity field of form $\alpha(t)\mathbf{x}$. Let us choose a moment t_0 such that $\alpha(t_0) = 0$. Then the initial velocity $\mathbf{V}_0(x) = 0$. At the same time it is known that the solution with smooth density having a compact support cannot be globally smooth. Note that if the density and pressure are only continuous at the points of vanishing, then they can be compatible. On can construct this solution; its support spreads.

2.2. **Application to the compressible Navier-Stokes system.** For the Navier-Stokes system, describing the behavior of compressible viscous fluid, the right hand side of (1) is the following:

$$\mathbf{F} = \mathrm{div} T, \quad T_{ij} = \mu(\partial_i V_j + \partial_j V_i) + \lambda \mathrm{div} V \delta_{ij},\, i,j = 1, ..., n, \tag{30}$$

where T is the stress tensor, $\mu \ge 0$, and λ are constants ($\lambda + \frac{2}{n}\mu \ge 0$), δ_{ij} is the Kronekker symbol.

In [11] it was demonstrated that if $\mu > 0$, $\lambda + \frac{2}{n}\mu > 0$, then there exists no solution with compactly supported density to the Cauchy problem (1), (2), (5), (6') with the right-hand side of form (30) from $C^1([0,\infty), H^m(\mathbb{R}^n)), \quad m > 2 + \left[\frac{n}{2}\right]$, such that initial data (6') are in $H^m(\mathbb{R}^n)$.

As a corollary of Theorem 2.1 we obtain that if the density and velocity vanish at infinity sufficiently quickly, there are initial conditions such that the solution to the Cauchy problem exists only within a finite interval of time.

Let S_R be a sphere of radius R, the unit outer normal to S_R and the element of its surface we denote by $\mathbf{N}(N_1, ..., N_n)$ and dS_R, respectively.

Definition We will say that a solution to (1),(2), (5) with the right-hand side of form (30) is of class $\mathfrak{K}_1$ if it satisfies to conditions (i),(ii) and

$$(v) \quad \lim_{R\to\infty} \int\limits_{S_R} \sum_{i,j=1}^{n} (T_{ij}x_i - (2\mu + n\lambda)V_i\delta_{ij})N_j dS_R = 0,$$

$$(vi)\ \lim_{R\to\infty}\int\limits_{S_R}\sum_{i,j=1}^{n} T_{ij}V_iN_j dS_R = 0.$$

Remark 2.8. Conditions (v),(vi) are satisfied, for example, if the velocity vector decays as $|\mathbf{x}|\to\infty$ uniformly in t so that $|\mathbf{V}| = o\left(\frac{1}{|x|^{n-1}}\right)$ and $|D\mathbf{V}| = o\left(\frac{1}{|x|^{n}}\right)$.

Theorem 2.3. *For initial data (6') satisfying condition(14) of Theorem 2.1, the solution to (1), (2), (5) from the class $\mathfrak{K}_1$ exists only within a finite time.*

To prove the Theorem 2.3 it suffices to note that in this situation $\mathfrak{K}_1\subset\mathfrak{K}$, because the rate of the velocity decay implies

$$\int\limits_{\mathbb{R}^n}(\mathbf{F},\mathbf{x})\,dx = \int\limits_{\mathbb{R}^n}\sum_{i,j=1}^{n}\partial_j T_{ij}x_i dx = \lim_{R\to\infty}\int\limits_{S_R}\sum_{i,j=1}^{n} T_{ij}x_iN_j dS_R - (2\mu+n\lambda)\int\limits_{\mathbb{R}^n} div\mathbf{V}\,dx =$$

$$= \lim_{R\to\infty}\int\limits_{S_R}\sum_{i,j=1}^{n}(T_{ij}x_i - (2\mu+n\lambda)V_i\delta_{ij})N_j dS_R = 0,$$

$$\int\limits_{\mathbb{R}^n}(F_ix_j - F_jx_i)\,dx = \lim_{R\to\infty}\int\limits_{S_R}\sum_{i,j,k=1}^{n}(T_{kj}x_i - T_{ki}x_j)N_k dS_R = 0,$$

$$\int\limits_{\mathbb{R}^n}(\mathbf{F},\mathbf{V})\,dx = \int\limits_{\mathbb{R}^n}\sum_{i,j=1}^{n}\partial_j T_{ij}V_i dx =$$

$$= \lim_{R\to\infty}\int\limits_{S_R}\sum_{i,j=1}^{n} T_{ij}V_iN_j dS_R - \int\limits_{\mathbb{R}^n}\sum_{i,j=1}^{n} T_{ij}\partial_i V_j dx \le 0.$$

□

We point out that classes $\mathfrak{K}$ and $\mathfrak{K}_1$ do not coincide. For example, the solutions with linear profile of velocity belong to $\mathfrak{K}$, however, condition (vi) is not satisfied here and therefore these solutions are not of class $\mathfrak{K}_1$.

Remark 2.9. The fact that a solution with a finite moment of mass and a finite total energy leaves the class $\mathfrak{K}$ both in the forcing free Euler system and the Navier-Stokes system means that a singularity appears (provided the decay of velocity at infinity in the Navier-Stokes case is sufficiently rapid). But the nature of the singularity appearing under the same initial conditions for these systems is different. In both situations, it signifies that the integration by parts becomes prohibited. However, in the first case this integration is still allowed if the solution is only continuous along certain piece-wisely smooth curves. For Navier-Stokes system we need to require C^1 – smoothness with respect to the space variables along these curves. Thus, for the hyperbolic systems the singularity predicted is either a strong discontinuity or some week discontinuity on a complicated set. For the Navier-Stokes system even week discontinuity along smooth curves means that a singularity appears.

On the other hand, in the case of the Navier-Stokes system "the singularity appearance" may signify that the solution does not belong anymore to the class $\mathfrak{K}_1$. For example, the derivatives of velocity in this possible global-in-time smooth solution satisfying (14) cannot decay at infinity sufficiently quickly.

Remark 2.10. Condition (14) of Theorem 2.1 is "the best sufficient condition" for the leaving the class $\mathfrak{K}$, in the viscous case, too. Indeed, the viscous term "does not feel" the velocity with a linear profile, therefore the first condition of Theorem 2.2 is satisfied, and we can apply Theorem 2.2 in this situation. It is interesting that this solution belongs to $\mathfrak{K}\backslash\mathfrak{K}_1$.

3. Breakdown of compactly supported smooth perturbation of a constant state

Results of [1] for the perturbation of the constant state $(\bar\rho, \mathbf{0}, \bar S)$ having compact support $B(t)$ can be extended to the case of right-hand sides $\mathbf{F}$ with the properties, analogous to (iii) and (iv), if we suppose the finite speed of the perturbation propagation. This characteristic property of hyperbolic systems, generally speaking, does not hold for Navier-Stokes equations (at least, for $\bar\rho \ne 0$, see in this context [11]). Indeed, to obtain results analogous to [1], it is sufficient to impose certain condition to the function $R(t)$, where $R(t)$ is the minimal radius of ball containing the support of perturbation. This condition has the form:

$$R(t) < C(1+t)^{\alpha},\ \alpha \in \mathbb{R},\ C \in \mathbb{R}_+. \tag{31}$$

If $\mathbf{F} = \mathbf{0}$, then $\alpha = 1$.

Let us denote, following to [1],

$$m(t) = \int\limits_{\mathbb{R}^n} [\rho(t,x) - \bar\rho]dx = \int\limits_{B(t)} [\rho(t,x) - \bar\rho]dx,$$

$$\eta(t) = \int\limits_{\mathbb{R}^n} [\rho(t,x)\exp\left(\frac{S(t,x)}{\gamma}\right) - \bar\rho\exp\left(\frac{\bar S}{\gamma}\right)]dx =$$

$$= \int\limits_{B(t)} [\rho(t,x)\exp\left(\frac{S(t,x)}{\gamma}\right) - \bar\rho\exp\left(\frac{\bar S}{\gamma}\right)]dx,$$

$$\tilde G(t) = \frac{1}{2}\int\limits_{B(t)} |\mathbf{x}|^2 \rho dx, \quad \tilde F(t) = \int\limits_{B(t)} (\mathbf{x}, \mathbf{V})\rho dx.$$

If we recall the denotation of section 2, it occurs that $F(t) = \tilde F(t)$ (see Corollary 2.1).

We obtain the following generalization of Theorem 1 from [1]:

Theorem 3.1. *Let us suppose that (ρ, V, P) is a classical solution to (1,2,5) such that properties*

- $\int\limits_{B(t)} (\mathbf{F}, \mathbf{x})\, dx \equiv 0;$
- $\int\limits_{B(t)} (F_i x_j - F_j x_i)\, dx \equiv 0,\ i \ne j,\ i,j = 1,...,n,$ *where* $\mathbf{x}$ *is a radius-vector of the point* x;
- $\int\limits_{B(t)} (\mathbf{F}, \mathbf{V})\, dx \le 0$

hold.

We assume that the support of perturbation propagates according to condition (31)and $\eta(0) \ge 0$. Suppose also that any of following conditions takes place:

a) $\alpha \le \frac{1}{2+n}, \quad |\mathbf{M}| = 0, \quad F(0) > 0;$
b) $\alpha \le \frac{1}{2+n}, \quad |\mathbf{M}| \ne 0;$
c) $\alpha > \frac{1}{2+n}, \quad F(0) > (\alpha(2+n) - 1)A,$

d) $\alpha > \frac{1}{2+n}$, $F(0) > |\mathbf{M}| \cot \dfrac{|\mathbf{M}|}{(\alpha(2+n)-1)A}$, $|\mathbf{M}| \geq \pi(\alpha(2+n)-1)A$,

e) $\alpha > \frac{1}{2+n}$, $|\mathbf{M}| \geq \pi(\alpha(2+n)-1)A$,

where the constant $A = \max_{\mathbb{R}^n} \rho_0(x)\omega_n C^{(2+n)}$, *and* ω_n *is the volume of a unit ball in* $\mathbb{R}^n$.

Then the life span of the solution is finite.

Remark 3.1 In [1] the system under consideration is hyperbolic in physical space, therefore $n=3$, $\alpha = 1$, and $|R(t)| \leq R(0) + \sigma t$, the constant $\sigma = \left(\frac{\partial P}{\partial \rho}|_{(\bar\rho,\bar S)}\right)^{1/2}$ is the sound speed.

Proof of Theorem 3.1. From the general Stokes formula we have $m(t) = m(0)$, $\eta(t) = \eta(0)$. The Jensen inequality together with $\eta(0) \geq 0$ give us, according to [1],

$$\int\limits_{B(t)} P dx \geq (vol B(t))^{1-\gamma} \left(\int\limits_{B(t)} \rho \exp\left(\frac{S}{\gamma}\right) dx \right)^{\gamma} =$$

$$= (vol B(t))^{1-\gamma} \left(\eta(0) + vol B(t) \bar\rho \exp\left(\frac{\bar S}{\gamma}\right)\right)^{\gamma} \geq \bar P vol B(t) = \int\limits_{B(t)} \bar P dx. \tag{32}$$

Let us note that if we integrate instead of $\mathbb{R}^n$ over $B(t)$, we get an analog of Lemma 2.1 and Corollary 2.1. The only difference will be in the integral $I_4(t)$. Here $I_4(t) = -\int\limits_{S(t)} P(\mathbf{x},\mathbf{N}) dS$, where $S(t)$ is the boundary of $B(t)$. Therefore

$$\tilde G''(t) = F'(t) = I_1(t) + I_2(t) + I_3(t) + I_4(t) \geq I_{1,\phi}(t) + I_{2,\phi}(t) + n \int\limits_{B(t)} (P - \bar P) dx.$$

In this case $(\tilde G'(t))^2 = F^2(t) \geq 2\tilde G(t) I_1(t)$ and $|M|^2 \geq 2\tilde G(t) I_2(t)$. Thus, taking into account (32), we have

$$F'(t) \geq \frac{F^2(t) + |\mathbf{M}|^2}{2\tilde G(t)}. \tag{33}$$

Further, since

$$\tilde G(t) \leq \frac{1}{2}(C(1+t)^{\alpha})^2 (m(0) + \int\limits_{B(t)} \bar\rho dx) = \frac{1}{2}(C(1+t)^{\alpha})^2 \int\limits_{B(t)} \rho_0 dx =$$

$$\frac{1}{2} \max_{\mathbb{R}^n} \rho_0(x) \omega_n C^{(2+n)} (1+t)^{\alpha(2+n)},$$

we get from (33) that

$$F'(t) \geq (\max_{\mathbb{R}^n} \rho_0(x)\omega_n C^{(2+n)}(1+t)^{\alpha(2+n)})^{-1}(F^2(t) + |\mathbf{M}|^2).$$

Integrating (34) we get the following. For $|\mathbf{M}| = 0$

$$-\frac{1}{F(t)} + \frac{1}{F(0)} \geq \frac{1}{A}\frac{(1+t)^{1-\alpha(2+n)} - 1}{1 - \alpha(2+n)}, \quad \alpha(2+n) \neq 1, \tag{35}$$

$$-\frac{1}{F(t)} + \frac{1}{F(0)} \geq \frac{1}{A}\ln(1+t), \quad \alpha(2+n) = 1. \tag{36}$$

If $\alpha < \frac{1}{2+n}$, then we have from (35)

$$F(t) \geq \frac{AF(0)(1-\alpha(2+n))}{A(1-\alpha(2+n)) - F(0)((1+t)^{1-\alpha(2+n)} - 1)}.$$

Respectively, from (36)we have

$$F(t) \geq \frac{AF(0)}{A - F(0)\ln(1+t)}.$$

Thus, if condition (a) of Theorem 3.1 holds, then $F(t)$ become infinite within a finite time, whereas it follows from the Cauchy-Schwartz inequality that $F^2(t) \leq 4\tilde{G}(t)E_k(t)$, that is $F(t)$ is finite at finite t. We obtain a contradiction.

Further, for $\alpha > \frac{1}{2+n}$ from (35) we get that

$$F(t) \geq \frac{AF(0)(\alpha(2+n)-1)(1+t)^{\alpha(2+n)-1}}{F(0) - (F(0) - A(\alpha(2+n)-1))(1+t)^{\alpha(2+n)-1}},$$

it follows that provided condition (c) of Theorem 3.1 holds, $F(t)$ goes to infinity within a finite time.

Let us $|\mathbf{M}| \neq 0$. Then

$$\arctan\frac{F(t)}{|\mathbf{M}|} \geq \arctan\frac{F(0)}{|\mathbf{M}|} + \frac{|\mathbf{M}|}{A}\frac{(1+t)^{1-\alpha(2+n)}-1}{1-\alpha(2+n)}, \quad \alpha(2+n) \neq 1, \tag{37}$$

$$\arctan\frac{F(t)}{|\mathbf{M}|} \geq \arctan\frac{F(0)}{|\mathbf{M}|} + \frac{|\mathbf{M}|}{A}\ln(1+t), \quad \alpha(2+n) = 1. \tag{38}$$

One can conclude from (37) and (38), that if $\alpha \leq \frac{1}{2+n}$, then at any value of $F(0)$ in a finite time the right-hand sides of these inequalities exceed $\frac{\pi}{2}$, whereas the left-hand sides are later then this number. This contradiction shows that the solution cannot keep smoothness provided condition (b) of Theorem holds.

At last, if $\alpha > \frac{1}{2+n}$, then an analogous contradiction we can get from (37), if

$$\arctan\frac{F(0)}{|\mathbf{M}|} + \frac{|\mathbf{M}|}{A(\alpha(2+n)-1)} > \frac{\pi}{2}, \tag{39}$$

this results conditions (d) and (e).

Thus, Theorem 3.1 is proved. □

Remark 3.1. The statement of Theorem 1 from [1] is a particular case of condition (c) of Theorem 3.1 for $\mathbf{F} = 0$, $\alpha = 1, n = 3$. Let us analyze the condition (c) in the physical space. For it we decompose the velocity $\mathbf{V}$ into a sum of its radial and tangential components, $\mathbf{V}_r$ and $\mathbf{V}_\tau$. We mean that the tangential component is a projection of velocity into subspace, orthogonal to the radius-vector $\mathbf{x}$, that is $(\mathbf{V}_\tau, \mathbf{x}) = 0$. As the estimate

$$|F(0)| \leq \max_{\mathbb{R}^n}\rho_0(x)\max_{\mathbb{R}^n}|\mathbf{V}_r(0,x)|C^{n+1}\omega_n$$

is true, then from condition (c) we get that

$$(\alpha(2+n)-1)C < \max_{\mathbb{R}^n}|\mathbf{V}_r(0,x)|,$$

that is

$$4C < \max_{\mathbb{R}^3}|\mathbf{V}_r(0,x)|.$$

It signifies that the initial perturbation of radial component of velocity (or the initial divergency, according to Remark 2.4) is sufficiently large comparing with the velocity of the support expanding, C, that it with the sound speed at infinity.

However, if $|\mathbf{M}| \neq 0$, that is, according to Remark 2.4, there exists an initial vorticity, then, as condition (d) shows, initially the radial component can be later, since

$$|\mathbf{M}|\cot\frac{|\mathbf{M}|}{A(\alpha(2+n)-1)} < A(\alpha(2+n)-1),$$

$$|\mathbf{M}|\cot\frac{|\mathbf{M}|}{A(\alpha(2+n)-1)} \to A(\alpha(2+n)-1), \qquad |\mathbf{M}| \to 0.$$

Thus, condition (c) follows from (d) at $|\mathbf{M}| \to 0$.

At the same time, as condition (e) shows, the singularity may appear due to a large vorticity only. Indeed,

$$|\mathbf{M}| \le \max_{\mathbb{R}^n} \rho_0(x) \max_{\mathbb{R}^n} |\mathbf{V}_\tau(0,x)| C^{n+1}\omega_n,$$

it shows together with (e) that

$$\pi(\alpha(2+n)-1)C < \max_{\mathbb{R}^n} |\mathbf{V}_\tau(0,x)|,$$

that is

$$4\pi C < \max_{\mathbb{R}^3} |\mathbf{V}_\tau(0,x)|.$$

However, at any case, it results the increase of the radial component of velocity.

4. Influence of damping to the singularity formation

We assume that in the right hand side of (1) there arise an additional forcing, namely, the dry friction with the coefficient $\mu(t,x)$. Thus, instead of (1), we consider

$$\rho(\partial_t \mathbf{V} + (\mathbf{V},\nabla)\,\mathbf{V}) + \nabla P = \mathbf{F}(t,x,\rho,\mathbf{V},S,D^{|\alpha|}\mathbf{V}) - \mu\rho\mathbf{V}. \tag{40}$$

In this situation the total energy is not conserved, however, this function is non-increasing:

$$E(t) \le E(0).$$

It is known that if the density is initially compactly supported then the dry friction (arbitrary large) do not prevent the singularity formation, only delays it [23].

At the same time, for the initial perturbation of nontrivial constant state concentrated in bounded domain, the damping prevents the singularity formation [24].

One can prove that if the friction is small, then singularities may arise both for solutions from class $\mathfrak{K}$ and for compact perturbation of nontrivial constant state. We concentrate at the first case, since the proof in the second one is analogous (see also [24] in this context).

Let us come back to the notation of Section 2, that is we will write below $G(t), I_k(t),\ k=1,...,4$ instead of $G_2(t), I_{k,\phi}(t)$, respectively.

We suppose that $|\mu(t,x)| \le \mu_0$, with some positive constant μ_0.

Let us come back to the notation of Section 2, that is for $\phi(|\mathbf{x}|) = \frac{1}{2}|\mathbf{x}|^2$ we will write below $G(t), I_k(t),\ \ k=1,...,4$ instead of $G_\phi(t), I_{k,\phi}(t)$, respectively.

Let us note that here Lemmas 2.2 and 2.3 are true, however, in the expression for $G''(t) = F'(t)$ from Lemma 2.1 we have to add in this new situation one more term. Namely, taking into account Corollary 2.1 we can write this expression in the form

$$F'(t) = I_1(t) + I_2(t) + n(\gamma-1)E_p(t) - \int\limits_{\mathbb{R}^n} \mu(t,x)(\mathbf{V},x)\rho dx. \tag{41}$$

If μ is constant, then

$$\int\limits_{\mathbb{R}^n} \mu(t,x)(\mathbf{V},x)\rho dx = \mu F(t),$$

in general case, using the estimate

$$|\int\limits_{\mathbb{R}^n} \mu(t,x)(\mathbf{V},x)\rho dx| \le 2\mu_0\sqrt{E(0)G(t)},$$

we obtain only the inequality

$$F'(t) \geq I_1(t) + I_2(t) + n(\gamma - 1)E_p(t) - 2\mu_0\sqrt{E(0)G(t)}. \tag{42}$$

The value of $|\mathbf{M}(t)|$ is not conserved here. If μ is constant, then

$$M_k(t) \leq M_k(0)e^{-\mu t},\ k = 1, ..., K.$$

In general case we note that $(M_k'(t))^2 \leq 4\mu_0^2 E_k(t)G(t)$ and apply estimate (11). Integrating the two-sided inequality for $M_k'(t)$ we obtain that

$$M_k(0) - \mu_0((\sqrt{E(0)}t + \sqrt{G(0)})^2 - G(0)) \leq M_k(t) \leq$$
$$M_k(0) + \mu_0((\sqrt{E(0)}t + \sqrt{G(0)})^2 - G(0)). \tag{43}$$

Theorem 4.1. *For sufficiently small μ_0 there are initial data (6') such that the solution to the Cauchy problem (40),(2),(5),(6') cannot belong to the class $\mathfrak{K}$ for all $t \geq 0$.*

Proof. As follows from (41), (8), (10)

$$F'(t) \geq \frac{F^2(t) + |\mathbf{M}(t)|^2}{2G(t)} + \frac{n(\gamma-1)C}{(G(t))^{\frac{(\gamma-1)n}{2}}} - 2\mu_0\sqrt{E(0)G(t)}.$$

For $\gamma \leq 1 + \frac{2}{n}$ it results inequality

$$F'(t) \geq \frac{F^2(t) + L^2\Psi(\mu_0, t)}{2(\sqrt{E(0)}t + \sqrt{G(0)})^2}. \tag{44}$$

Here we use the notation of Theorem 2.1 ($|\mathbf{M}|$ denotes now $|\mathbf{M}(0)|$ in the expression for L), and we introduce a function

$$\Psi(\mu_0, t) = 1 - \frac{1}{L^2}(4\mu_0\sqrt{E(0)}(\sqrt{E(0)}t + \sqrt{G(0)})^3 +$$
$$nE(0)\mu_0^2 t^2(\sqrt{E(0)}t + 2\sqrt{G(0)})^2 + 2\mu_0\sqrt{n}|\mathbf{M}(0)|\sqrt{E(0)}t(\sqrt{E(0)}t + 2\sqrt{G(0)})).$$

Let us fix a positive constant $\Psi_*^2 < 1$. Choosing μ_0 sufficiently small we can obtain that the inequality $\Psi(t) > \Psi_*^2$ will hold for all $t \in [0, T(\mu_0)),$ moreover, $T(\mu_0) \to \infty$ as $\mu_0 \to 0$. Thus, for such t we have from (44) that

$$F'(t) \geq \frac{F^2(t) + L^2\Psi_*^2}{2(\sqrt{E(0)}t + \sqrt{G(0)})^2},$$

the integration results

$$\arctan\frac{F(t)}{L\Psi_*} \geq \arctan\frac{F(0)}{L\Psi_*} + \frac{L\Psi_*}{2\sqrt{E(0)}}\left(\frac{1}{\sqrt{G(0)}} - \frac{1}{\sqrt{E(0)}t + \sqrt{G(0)}}\right).$$

Let us denote t_* a unique positive solution (in t) to equation

$$\arctan\frac{F(0)}{L\Psi_*} + \frac{L\Psi_*}{2\sqrt{E(0)}}\left(\frac{1}{\sqrt{G(0)}} - \frac{1}{\sqrt{E(0)}t + \sqrt{G(0)}}\right) = \frac{\pi}{2},$$

which always exists if

$$F(0) \geq L\Psi_*\cot\frac{L\Psi_*}{2\sqrt{E(0)G(0)}}.$$

It suffices to choose μ_0 such small that $T(\mu_0) > t_*$.

The case $\gamma > 1 + \frac{2}{n}$ can be treated analogously.

Thus, Theorem 5.1 is proved. □

Remark 4.1. The function $L\Psi_* \cot \frac{L\Psi_*}{2\sqrt{E(0)G(0)}}$ is decreasing in Ψ_* for $0 < \Psi_* \leq 1$, therefore

$$L\Psi_* \cot \frac{L\Psi_*}{2\sqrt{E(0)G(0)}} > L \cot \frac{L}{2\sqrt{E(0)G(0)}},$$

and if condition (14) is satisfied, the condition

$$F(0) \geq L\Psi_* \cot \frac{L\Psi_*}{2\sqrt{E(0)G(0)}},$$

sufficient for the formation of singularity in the system with damping, generally speaking, does not hold. In this sense the damping prevents the singularity formation.

5. Influence of rotation to the singularity formation

In many meteorological problems for the modelling of the atmospherical processes they use systems of equations, analogous to (1–3). However, in this problems the air motion is considered under rotating Earth, therefore it needs to take into account the Coriolis force. It is defined as $\mathbf{V} \times 2\mathbf{\Omega}$, where $\mathbf{\Omega}$ is a constant vector of the Earth angular rotation (see, for example, [25],[26]).

The question on sufficient conditions of singularity formation from smooth initial data is very important for the meteorology, where the discontinuity is associated with the atmospherical front.

The vertical scale of atmospherical processes, considered in the problems of weather forecast, is small compared with the horizontal one (no more then 10 km in vertical against several hundred kilometers by horizontal), therefore equations describing horizontal and vertical processes, are not equal in rights. In the vertical direction the so called hydrostatic approximation is usually assumed. Its general sense is that vertical velocity and acceleration are assumed to be small with respect to horizontal ones (see, for example, [25],[26]). (This approach is not acceptable, of course, for a description of small scales processes such that spouts, convection near frontal zones, typhoons generation, and so on.)

However, if the scale is such that the assumption on a smallness of vertical processes is acceptable, then it is convenient to average the primitive tree dimensional system of equation over hight and to deal with a simpler two dimensional in space system of equation where both space equations are already equal in rights [27],[28]. Vertical processes are hidden now, however, they are not eliminated from consideration. One can see it, for example, in the fact that the value of adiabatic exponent changes in the averaged over high system.

We also restrict ourself to the case $n = 2$.

Thus, instead of (1) we consider

$$\rho(\partial_t \mathbf{V} + (\mathbf{V}, \nabla)\, \mathbf{V}) + \nabla P = \mathbf{F}(t, x, \rho, \mathbf{V}, S, D^{|\alpha|}\mathbf{V}) + l\rho \mathbf{V}_\perp. \tag{45}$$

We denote $\mathbf{V}_\perp(v_\perp^1, v_\perp^2)$, a vector with components $v_\perp^i = e^i_{.j} v^j$, $i, j = 1, 2$ where e_{ij} is the Levi-Civita tensor, $l = l(x)$ is the Coriolis parameter.

We consider as before the solutions from the class $\mathfrak{K}$.

Among compactly supported solutions for $l = 0$ there exist no globally smooth ones [10].

In contrast to the case for $l \neq 0$ we can construct explicitly a stationary nontrivial compactly supported solution.

Let us $\mathbf{F}=\mathbf{0}$ and l is constant. We consider the isentropic gas, that is the state equation is $P=A\rho^\gamma$, $A=const$). We will seek a solution of the form $(\rho,\mathbf{V},P)$, where

$$\mathbf{V}=f(\theta)\mathbf{r}_\perp,\ \rho=(g(\theta))^{1/(\gamma-1)},\ P=A\rho^\gamma.$$

Here $\theta=|\mathbf{x}|^2/2$, $f(\theta)$ is an arbitrary smooth function supported on a segment $[a,b]\in[0,\infty)$,

$$g(\theta)=C+\frac{1}{2K}\int_0^\theta (f^2(\xi)-lf(\xi))d\xi,$$

with the constant $K=\frac{A\gamma}{\gamma-1}$. We can always choose the constant C such that $g(\theta)$ will vanish as $|\mathbf{x}|\to\infty$. For example, if $f(\theta)=\frac{l}{\nu}(\nu-\theta)$ for $0\le\theta\le\nu<\infty$ and $f(\theta)=0$ for $\theta>\nu$ we have $g(\theta)=\frac{l^2}{12K\nu^2}(\nu^3+\theta^2(2\theta-3\nu))$, $0\le\theta\le\nu<\infty$, and $g(\theta)=0$, $\theta>\nu$.

These solutions correspond to compactly supported stationary divergence-free flows.

For the constant Coriolis parameter we can also construct a non-stationary periodic in time global solution from the class $\mathfrak{K}$ with linear profile of velocity

$$\mathbf{V}=\alpha(t)\mathbf{x}+\beta(t)\mathbf{x}_\perp,$$

acting in the spirit of Subsection 2.1 (see [20], [21], [22]for detail).

However, in the rotational case, too, there exists initial data, resulting in a singularity formation in the class of solutions with a finite moment of mass and total energy.

If $|l(x)|\le l_0$, these data can be found exactly as in the case of small damping, described in Section 4. It is sufficient to note that

$$|\int\limits_{\mathbb{R}^2} l(x)(\mathbf{V}_\perp,x)\rho dx|\le 2l_0\sqrt{E(0)G(t)},$$

and to proceed as before changing μ to l. The relative conclusion is that if the rotation is small we can still find initial conditions resulting singularity satisfying to an analog of (14).

It will be convenient for us to consider l as a constant.

Remark 5.1 In [29] the question was investigated whether the rotation prevents the singularity formation for the pressureless gas dynamics. Here the basic equation is the nonlinear transport equation with rotational forcing, namely

$$\partial_t\mathbf{V}+(\mathbf{V},\nabla)\,\mathbf{V}=l\rho\mathbf{V}_\perp.$$

The answer is "conditionally yes". Specifically, in the case $l\ne 0$ the solution is globally smooth if and only if

$$\forall x\in\mathbb{R}^2\quad 2l\omega_0(x)+\eta_0(x)^2<l^2,$$

where $\omega_0(x)=(V_{02}(x))'_{x_1}-(V_{01}(x))'_{x_2}$, $\quad\eta_0(x)=\lambda_2(0)-\lambda_1(0)$, $\quad\lambda_j(0)$ are eigenvalues of Jacobian of initial velocity.

Let us point out that this condition is violated if in a certain point x_0 the value of $\omega_0(x_0)>>1$ (meteorologists would say that near x_0 there exists a significant cyclonic vorticity; in the northern hemisphere, where the Coriolis parameter l is positive, the cyclonic rotation is anticlockwise).

Let us note also that we cannot do here the limit pass as $l\to 0$, since if $l=0$, then the solution is globally smooth if and only if $\lambda_j(0)\ge 0$, $j=1,2$.

It is especially interesting that the presence of rotation in the real gas dynamic is in some sense convenient for the singularity formation. More precisely, there are situations where only significant initial vorticity provokes the singularity at any initial divergency. As we have seen, it is impossible in the rotation free case, where the divergency must be significantly positive.

In in the rotational case the mass m is conserved, the total energy $E(t)$ is non-increasing (it is conserved for $\mathbf{F} = \mathbf{0}$). Corollary 2.1 and inequalities (7–11), estimates (12) and (13) are true. The angular momentum balance law has now the form

$$\mathcal{M} = lG(t) + F_{\perp}(t) = const, \tag{46}$$

where

$$F_{\perp}(t) = \int\limits_{\mathbb{R}^2} (\mathbf{V}_{\perp}, x)\rho dx.$$

In this new situation the following Lemma holds:

Lemma 5.1. *For the solution of class $\mathfrak{K}$ to system (45), (2), (5) equalities*

$$G'(t) = F(t), \tag{47}$$

$$F'(t) = I_1(t) + I_2(t) + I_3(t) + lF_{\perp}(t), \tag{48}$$

$$F'_{\perp}(t) = -lF(t), \tag{49}$$

$$G''(t) + l^2 G(t) = 2(2-\gamma)E_k(t) + \Theta_{\gamma}(t), \tag{50}$$

where $\Theta_{\gamma}(t) = 2(\gamma-1)E(t) + l\mathcal{M}$, take place.

The proof of Lemma 5.1 is absolutely analogous to the Lemma 2.1 proof, the general Stokes formula in this two-dimensional case looks like the Green' formula. □

Remark 5.2. The balance law (46) follows from (47) and (49).

A new important circumstance in this situation is a boundedness of $G(t)$ above.

Lemma 5.2. *For the solution of class $\mathfrak{K}$ to system (45), (2), (5) the inequality*

$$0 < G_- \le G(t) \le G_+ \tag{51}$$

takes place, where G_- and G_+ are positive constants. For example, on can take

$$G_- = \left(\frac{C}{E(0)}\right)^{1/(\gamma-1)}, \quad G_+ = \frac{1}{l^2}(\sqrt{\Theta_2(0)} + \sqrt{2E(0)})^2.$$

Remark 5.3. Let us denote $J_i = \int\limits_{\mathbb{R}^2} V_i \rho dx$, $i = 1, 2$. The value $J_1^2(t) + J_2^2(t)$ is conserved for $\mathbf{F} = \mathbf{0}$ (for example, [30]). The estimates (51) can be refined (see [31]), if this conserved values is positive.

Proof of Lemma 6.2. It follows from (50) that

$$G''(t) + l^2 G(t) \le \Theta_2(0),$$

however due to the resonance phenomenon we cannot prove the boundedness of the solution to this inequality without taking into account additional properties of $G(t)$.

Suppose that there exists a point t_1 such that $G'(t_1) > 0$ (otherwise, $G(0)$ is the upper bound of $G(t)$). Let us denote $\epsilon(t_1) = G(t_1) - \frac{\Theta_2(0)}{l^2}$. If for all t_1 it occurs that $\epsilon(t_1) \le 0$, then the upper bound of $G(t)$ is $\frac{\Theta_2(0)}{l^2}$. Let us suppose that one can find t_1 such that $\epsilon := \epsilon(t_1) > 0$. Then there exists such $\tau > 0$, that $G(t) \ge \frac{\Theta_2(0)}{l^2} + \epsilon$ at $t \in [t_1, t_1 + \tau)$.

Thus, it follows from (52) that at $t \in [t_1, t_1 + \tau)$

$$G''(t) \leq -\epsilon l^2.$$

We integrate this inequality twice with respect to t from t_1 to $t \in [t_1, t_1 + \tau)$, and we get that

$$G(t) \leq -\epsilon \frac{l^2}{2}(t - t_1)^2 + G'(t_1)(t - t_1) + G(t_1) := G^+(t).$$

The quadratic function $G^+(t)$ is maximal at $t = t_* = t_1 + \frac{G'(t_1)}{\epsilon l^2} > t_1$. Therefore $G(t) \leq G^+(t_*)$ at $t_* \leq t_1 + \tau$, and $G(t) \leq G^+(t_1 + \tau) < G^+(t_*)$, at $t_* > t_1 + \tau$. At any case, taking into account (7), we get that

$$G(t) \leq G^+(t_*) = G(t_1) + \frac{(G'(t_1))^2}{2\epsilon l^2} \leq G(t_1) + \frac{4EG(t_1)}{2\epsilon l^2} =$$

$$= \frac{\Theta_2(0)}{l^2} + \epsilon + \frac{4E(\frac{\Theta_2(0)}{l^2} + \epsilon)}{2\epsilon l^2} = \frac{\Theta_2(0) + 2E}{l^2} + \epsilon + \frac{2E\Theta_2(0)}{l^4 \epsilon}.$$

Minimizing the right hand side in ϵ we get the estimation of $G(t)$ from above.

The lower bound indicated in the Lemma statement can be obtained from Lemma 2.3. $\square$

Corollary 5.1. *In the situation of Lemma 6.2 $|F(t)|$ is bounded for all $t > 0$.*

Indeed, it follows from (7) and Lemma 6.2. $\square$

Let us denote $\mathcal{K} = \mathcal{M}^2 - l^2 G_+^2 + \delta$, where $\delta = C G_-^{1 - \frac{(\gamma-1)n}{2}}$, if $\gamma \leq 1 + \frac{2}{n}$, and $\delta = C G_+^{1 - \frac{(\gamma-1)n}{2}}$, otherwise. The following Theorem takes place.

Theorem 5.1. *Solutions to (45), (2), (5) cannot belong to the class $\mathfrak{K}$ for all $t \geq 0$ if the initial data are such that*

$$\mathcal{K} > 0, \tag{53}$$

or

$$\mathcal{K} \leq 0, \quad F(0) > \sqrt{-\mathcal{K}} \tag{54}$$

hold.

Proof of the Theorem. It follows from (8), (10), (13), (46), (48) and Lemma 2.3 that

$$F'(t) \geq \frac{F^2(t) + F_\perp^2(t)}{2G(t)} + \frac{C}{G^{(\gamma-1)n/2}} + l\mathcal{M} - l^2 G(t) \geq$$

$$\geq \frac{F^2(t) + \mathcal{K}}{2G_+}. \tag{55}$$

We integrate (55) and see that if (53) or (54) are satisfied, then $F(t)$ become unbounded in a finite time. This contradicts to Corollary 6.1. $\square$

Remark 5.4. The estimate from above of the time of the singularity formation can be easily obtained from inequality (55).

Remark 5.5. Let us analyze, for example, condition (53), that is

$$(\mathcal{M} - lG_+)(\mathcal{M} + lG_+) > -\delta. \tag{56}$$

Taking into account the expression for G_+, given by Lemma 5.2, we can re-write inequality (56) as

$$(\Theta_2(0) - l\mathcal{M} + \sqrt{2E\Theta_2(0)})(\Theta_2(0) + 2\sqrt{2E\Theta_2(0)}) < \frac{\delta}{4} l^2.$$

Let us note that since $\Theta_2(0) > 0$, (56) does not hold for $\delta = 0$, in the case of pressure free gas dynamics.

If $\delta > 0$, inequality (56) is satisfied for $\Theta_2(0) < \Theta_*(\delta, l)$, with a constant $\Theta_*(\delta, l)$.

Thus, coming back to inequality (53), we see that it is true, if $l\mathcal{M} < \Theta_*(\delta, l) - 2E$, that is

$$lF_\perp(0) < l^2 G(0) - 2E + \Theta_*(\delta, l).$$

For large E and small (however not equal to zero!) $|l|$ (this is in the real meteorological situation) the value of $F_\perp(0)$ is negative. Since

$$F_\perp(t) = -\frac{1}{2}\int\limits_{\mathbb{R}^2} ((\rho V_2)'_{x_1} - (\rho V_1)'_{x_2})dx,$$

it signifies that there exists initially a cyclonic vorticity. Possibly, this observation would help to explain the well known for meteorologists fact that in the extratropical zone always inside of cyclone an atmospherical front exists.

Remark 5.6 The results of this section can be with respective modification extended to the case of non-constant Coriolis parameter, which, however, differs little from a constant.

Remark 5.7 One can consider in the rotational case, too, the viscid term of form (30). Then we can obtain as a corollary from Theorem 5.1 the following result. A solution from the class $\mathfrak{K}$ with the conditions of decay at infinity for the velocity and its derivatives (see Remark 2.8), loses the initial smoothness provided (53) or (54) hold.

Remark 5.8 A certain result concerning sufficient condition for the singularity formation for the Euler equations on a rotating plane demonstrating another approach can be found in[32]. I is possible to consider other exterior forces then ones mentioned in this paper. For example, [33] deals with sufficient conditions of the smoothness loss for solutions to the gas dynamic equation with exterior force of geopotential type (besides the Coriolis force).

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