# Complex analytic methods
# and
# the monodromy of integrable systems

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Birkhoff normal form is a power series expansion associated with the local behavior of the Hamiltonian systems near a critical point. It is known that around the critical point one can take a convergent canonical transformation which puts the Hamiltonian into Birkhoff normal form for integrable systems under some non-degeneracy conditions.

By means of an expression of the derivative for the inverse of Birkhoff normal form by a period integral, analytic continuation of the Birkhoff normal forms is considered for the free rigid body dynamics on $SO(3)$. It is shown that the monodromy of the analytic continuation for the derivative of the inverse for the Birkhoff normal forms coincides with that of an elliptic fibration introduced by Naruki and Tarama.

In analytical mechanics, the motions of rigid bodies are basic problems. Among them, the free rigid body, which stands for the rigid body under no external force, is the simplest example.

The free rigid body dynamics should first be defined as a Hamiltonian system on the cotangent bundle of the rotation group $SO(3)$. Because of the left-invariance of this system, it is essentially described by the so-called Euler equation posed on the angular momentum, which can be justified by the Lie-Poisson reduction procedure.

Moreover, by means of the Marsden-Weinstein reduction, one can reduce the original system onto the level surface of the norm of the angular momentum in the space of angular momenta, which is a two-dimensional sphere. The reduced system is of one degree of freedom, and therefore completely integrable in the sense of Liouville. It is well known that there are generically six equilibria for the reduced systems on the sphere, four of which are elliptic and the other two are hyperbolic.

Around a stationary point of a Hamiltonian system, it is possible to consider the normal form of the Hamiltonian. Historically, Birkhoff introduced the notion of Birkhoff normal forms as formal series and discussed the relation with the stability. It is known that there exists a Darboux coordinate system which makes the Hamiltonian in Birkhoff normal form, in a neighbourhood of a non-degenerate stationary point by a result of Vey for analytic completely integrable Hamiltonian systems. The differentiable case of class $\mathcal{C}^\infty$ was studied by Eliasson. The convergence of canonical transformation which makes the Hamiltonian into Birkhoff normal form for analytic integrable systems including the degenerate case was shown by Ito under the non-resonance condition.

The convergence of the canonical transformation which puts the Hamiltonian into Birkhoff normal form for the systems of one degree of freedom was essentially shown by Siegel. Nguyen Tien Zung developed another more geometric approach for the analytic completely integrable Hamiltonian systems, on the basis of the direct proof of the analytic extension through the period integrals.

The detailed structure of the Birkhoff normal form has been studied recently for the pendulum and for the free rigid body by Francoise-Garrido-Gallavotti. The Birkhoff normal forms both for the elliptic and hyperbolic stationary points are considered by using the method of relative cohomology (Brieskorn module).

On the other hand, the free rigid body dynamics is closely related to complex algebraic geometry because of its complete integrability. In fact, the integral curve of the free rigid body can be described as an intersection of two quadric surfaces in three-dimensional Euclidean space, which is a (real) elliptic curve. From the viewpoint of complex algebraic geometry, it is natural to complexify and to compactify all the settings, which is also helpful to understand the deep geometric structure of the free rigid body dynamics.

In view of this, several elliptic fibrations arising from the free rigid bodies have been considered by Naruki and Tarama. The fibrations are considered over the base space which includes not only the values of the Hamiltonian but also the principal axes of the inertia tensor, as their base coordinates. This allows a classification of the singular fibres of these elliptic fibrations. Some of these singularities do not belong to Kodaira classification.

Although the Birkhoff normal form is by definition a local object associated to a stationary point for a Hamiltonian system, it is possible to enquire its analytic extension in the integrable case. In the present contribution, the Birkhoff normal forms of the equilibria for the free rigid body dynamics and their analytic continuation are considered in relation to the simplest elliptic fibration which is discussed in Naruki-Tarama. The key to this relation is an expression of the derivative of the inverse for the Birkhoff normal form in terms of period integrals, which is closely related to a special Gauß hypergeometric differential equation. This makes the concrete calculation rather easy.

## Birkhoff normal forms for a system of one degree of freedom and period integrals

Take a real analytic symplectic manifold $(M,\omega)$ of dimension two and consider a real analytic Hamiltonian $H$. Let $(x,y)$ be a Darboux coordinates of $(M,\omega)$ such that $\omega = \mathrm{d}x \wedge \mathrm{d}y$. Assume that the origin $(x,y) = (0,0)$ is an elliptic equilibrium, where $H = 0$, and that the Hamiltonian $H$ is in Birkhoff normal form $H = \mathcal{H}\left(\dfrac{x^2+y^2}{2}\right)$, where $\mathcal{H}$ is an invertible analytic function in one variable around the origin. Denote the inverse of the function $\mathcal{H}$ by $\Phi$. Suppose that there is a one-form $\eta$ defined on $U \setminus \{(0,0)\}$, where $U$ is a neighbourhood of the origin $(x,y) =(0,0)$, such that $\omega = \eta \wedge \mathrm{d}H$.

Then, we have the following theorem.

**Theorem 1.** *The derivative of the inverse $\Phi$ of the Birkhoff normal form $\mathcal{H}$ around an elliptic stationary point, where $H = 0$, writes*

$$\Phi'(h) = -\frac{1}{2\pi}\int_{H=h} \eta.$$

*Here, the integral path is taken as the integral curve of the energy level $H = h$.*

The one-form $\eta$ can be replaced by $\eta + f\mathrm{d}H$, where $f$ is a function on $U \setminus \{(0,0)\}$. In fact, if $\eta'$ satisfies $\eta' \wedge \mathrm{d}H = \omega$, we have $(\eta - \eta') \wedge \mathrm{d}H = 0$. Thus, we have $\eta - \eta' = f\mathrm{d}H$, for a suitable function $f$, so that $\int_{H=h} \eta = \int_{H=h} \eta'$. In fact, the one-form $\eta$ is an example of the so-called Gelfand-Leray form, which can be found in [Chapter 7, Arnol'd-Gusein-Zade-Varchenko].

In a parallel manner, we consider an expression of the derivative of the inverse for Birkhoff normal form around a hyperbolic stationary point. In this case, we can take a Darboux coordinates $(X, Y)$ with the origin at a hyperbolic equilibrium, such that $\omega = \mathrm{d}X \wedge \mathrm{d}Y$ and the Hamiltonian $H$ with $H(0,0) = 0$ is in Birkhoff normal form $H = \mathcal{H}(XY)$, where $\mathcal{H}$ is an invertible analytic function in one variable whose inverse is again denoted by $\Phi$. We consider the complexification $M^{\mathbb{C}}$ of $M$ where the symplectic form $\omega$ and the Hamiltonian $H$ are extended as a holomorphic two-form and as a holomorphic function. Taking a suitable complex neighbourhood $U^{\mathbb{C}} \subset M^{\mathbb{C}}$ of $(X, Y) = (0, 0)$, we choose a holomorphic one-form $\eta^{\mathbb{C}}$ on $U^{\mathbb{C}} \setminus \{(0, 0)\}$, such that $\eta^{\mathbb{C}} \wedge \mathrm{d}H = \omega$. We can assume that $(X, Y)$ are holomorphic coordinates on $U^{\mathbb{C}}$. The real integral curve is naturally complexified to a complex one-dimensional curve $H(XY) = h$, $(X, Y) \in U^{\mathbb{C}} \subset M^{\mathbb{C}}$, as a level curve of the holomorphic function $H$. On the complex curve $H = h$ in

$M^{\mathbb{C}}$, we consider the real closed arc

$$\gamma : \; X = \sqrt{\epsilon} e^{\sqrt{-1}\theta}, \; Y = \sqrt{\epsilon} e^{-\sqrt{-1}\theta}, \; \theta : 0 \to 2\pi,$$

where $\epsilon := \Phi(h) = \mathcal{H}^{-1}(h)$.

**Theorem 2.** *The derivative of the inverse $\Phi$ of the Birkhoff normal form $\mathcal{H}$ for the Hamiltonian $H$ around a hyperbolic equilibrium where $H = 0$ writes*

$$\Phi'(h) = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \eta^{\mathbb{C}}.$$

## Derivative of inverse Birkhoff normal forms for free rigid body dynamics

The motion of a free rigid body can essentially be described by the Euler equation

$$\frac{\mathsf{d}}{\mathsf{d}t} P = P \times \left( \mathcal{I}^{-1}(P) \right),$$

where $P = (p_1, p_2, p_3) \in \mathbb{R}^3$ is the angular momentum and $\mathcal{I} : \mathbb{R}^3 \to \mathbb{R}^3$ is the inertia tensor which is a symmetric positive-definite linear operator with respect to the standard inner product $\cdot$ of $\mathbb{R}^3$. Without loss of generality, we can assume $\mathcal{I} = \operatorname{diag}(I_1, I_2, I_3)$ and $I_1 < I_2 < I_3$.

Then, the Euler equation writes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} p_1 &= -\left(\frac{1}{I_2} - \frac{1}{I_3}\right) p_2 p_3, \\
\frac{\mathrm{d}}{\mathrm{d}t} p_2 &= -\left(\frac{1}{I_3} - \frac{1}{I_1}\right) p_3 p_1, \\
\frac{\mathrm{d}}{\mathrm{d}t} p_3 &= -\left(\frac{1}{I_1} - \frac{1}{I_2}\right) p_1 p_2.
\end{aligned} \tag{1}
$$

An important property of the Euler equation is that the functions $H(P) = \frac{1}{2} P \cdot \mathcal{I}^{-1}(P)$ and $L(P) = \frac{1}{2} P \cdot P$ are first integrals, so that the system can be restricted to the level surface of $L$. On the level surface $L = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) = \ell$, where $\ell$ is a positive constant, there are six equilibria on the $p_1$-, $p_2$-, $p_3$-axes, and those four on the $p_1$- and $p_3$-axes are elliptic, while the other two on the $p_2$-axis are hyperbolic. The restricted system on $L(P) = \ell$ is a Hamiltonian system for the Hamiltonian $H|_{\{L=\ell\}}$ with respect to the symplectic form

$$\omega = \frac{\mathrm{d}p_1 \wedge \mathrm{d}p_2}{3p_3} = \frac{\mathrm{d}p_2 \wedge \mathrm{d}p_3}{3p_1} = \frac{\mathrm{d}p_3 \wedge \mathrm{d}p_1}{3p_2}. \tag{2}$$

We consider the derivative of the inverse Birkhoff normal form around the elliptic equilibrium $(p_1, p_2, p_3) = \left(\sqrt{2\ell}, 0, 0\right)$, where $(p_2, p_3)$ serves as the local coordinate system on $L = \ell$. As is mentioned in the previous section, the derivative for the inverse of Birkhoff normal form around an elliptic equilibrium is the period of the integral curve. By considering such a period in the free rigid body case, one can more easily consider the analytic extension than the inverse Birkhoff normal form itself because of the relation between the derivative of it and a Gauß hypergeometric differential equation.

On this coordinate neighbourhood, we consider the one-form

$$\eta_s := (1-s)\frac{\mathrm{d}p_2}{3\left(\frac{1}{I_3}-\frac{1}{I_1}\right)p_3p_1} + s\frac{\mathrm{d}p_3}{3\left(\frac{1}{I_1}-\frac{1}{I_2}\right)p_1p_2}, \tag{3}$$

where $s$ is an arbitrary parameter. It is easy to verify that $\eta_s \wedge \mathrm{d}H = \omega$ for any $s$.

Denote the inverse function of the Birkhoff normal form $H - \frac{\ell}{I_1} = \mathcal{H}_1$ around the elliptic stationary point $\left(\sqrt{2\ell}, 0, 0\right)$ by $\Phi_1$. We have an expression of $\Phi_1'$ in terms of a period integral as follows :

**Theorem 3.** *The derivative of the inverse Birkhoff normal form around* $(p_1, p_2, p_3) = \left(\sqrt{2\ell}, 0, 0\right)$ *writes*

$$\Phi_1'(h) = -\frac{1}{3\pi}\sqrt{\frac{2}{\ell}}\frac{1}{\sqrt{(d-c)(a-b)}}\mathcal{K}\left(\frac{(d-a)(b-c)}{(d-c)(b-a)}\right). \tag{4}$$

Here, $a = \frac{1}{I_1}, b = \frac{1}{I_2}, c = \frac{1}{I_3}, d = \frac{h}{\ell}$ and $\mathcal{K}(\lambda) := \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-\lambda x^2)}}$ is the complete elliptic integral of the first kind. Denote the right hand side of (4) by $S(a, b, c, d)$.

Here, we mention some covariance properties of the Euler equation (1). First, obviously, the following transformations preserve the Euler equation :

$$\delta_1 : \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \mapsto \begin{bmatrix} -p_1 \\ p_2 \\ p_3 \end{bmatrix}, \; t \mapsto -t; \quad \delta_2 : \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \mapsto \begin{bmatrix} p_1 \\ -p_2 \\ p_3 \end{bmatrix}, \; t \mapsto -t; \quad \delta_3 : \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \mapsto \begin{bmatrix} p \\ p \\ - \end{bmatrix}$$

It is clear that $\delta_1$, $\delta_2$, $\delta_3$ generate a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. On the other hand, the following transformations, where $t$ is not transformed, preserve the Euler equation :

$$\epsilon_1 : \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \mapsto - \begin{bmatrix} p_1 \\ p_3 \\ p_2 \end{bmatrix}, \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \mapsto \begin{bmatrix} I_1 \\ I_3 \\ I_2 \end{bmatrix}; \quad \epsilon_2 : \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \mapsto - \begin{bmatrix} p_3 \\ p_2 \\ p_1 \end{bmatrix}, \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \mapsto \begin{bmatrix} I_3 \\ I_2 \\ I_1 \end{bmatrix};$$
$$\epsilon_3 : \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \mapsto - \begin{bmatrix} p_2 \\ p_1 \\ p_3 \end{bmatrix}, \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \mapsto \begin{bmatrix} I_2 \\ I_1 \\ I_3 \end{bmatrix}. \tag{5}$$

These involutions $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ generate another group isomorphic to the symmetric group $\mathfrak{S}_3$ of degree three. Needless to say that the first integral $L$ is invariant with respect to the above transformations. Note that the transformations $\delta_1$, $\delta_2$, $\delta_3$, $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ act on the symplectic form $\omega$ and the Hamiltonian $H$ as $\delta_j^*\omega = -\omega$, $\delta_j^* H = H$ and $\epsilon_j^*\omega = -\omega$, $\epsilon_j^* H = H$, $j = 1, 2, 3$.

By the transformation $\delta_1$ of the Euler equation, the equilibrium $(p_1, p_2, p_3) = \left(\sqrt{2\ell}, 0, 0\right)$ is mapped to $\left(-\sqrt{2\ell}, 0, 0\right)$. The integral curves around the two equilibrium points $\left(\pm\sqrt{2\ell}, 0, 0\right)$ are mapped to each other, but their orientations are opposite, since the time is reversed by $\delta_1$. As to the period integral, the integrand $\eta_s$ is transformed as $\delta_1^* \eta_s = -\eta_s$. Thus, the integral $\displaystyle\int_{H=h} \eta_s$ is invariant with respect to $\delta_1$. As a result, we have the following corollary.

**Corollary 1**

The derivative of the inverse of the Birkhoff normal form around $(p_1, p_2, p_3) = \left(-\sqrt{2\ell}, 0, 0\right)$ is given by (4).

Similarly, the transformation $\epsilon_2$ maps the equilibrium $(p_1, p_2, p_3) = \left(\sqrt{2\ell}, 0, 0\right)$ to $\left(0, 0, -\sqrt{2\ell}\right)$. It maps the integral curves around each equilibrium to each other by $\epsilon_2$, reversing their orientations. The integrand $\eta_s$ of the period integral is transformed as $\epsilon_2^* \eta_s = -\eta_s$ and the period integral itself is transformed from (4) by the permutation $(ac) \in \mathfrak{S}_4$ with respect to $\epsilon_2$.

**Corollary 2**

The derivative of the inverse for the Birkhoff normal form $\Phi_3$ around $(p_1, p_2, p_3) = \left(0, 0, \pm\sqrt{2\ell}\right)$ writes

$$\Phi_3'(h) = -\frac{1}{3\pi}\sqrt{\frac{2}{\ell}}\frac{1}{\sqrt{(d-a)(c-b)}}\mathcal{K}\left(\frac{(d-c)(b-a)}{(d-a)(b-c)}\right) = S(c,b,a,d). \tag{6}$$

Next, we consider the derivative of the inverse for the Birkhoff normal form around the hyperbolic stationary points $(p_1, p_2, p_3) = \left(0, \pm\sqrt{2\ell}, 0\right)$. Around the stationary point $(p_1, p_2, p_3) = \left(0, \sqrt{2\ell}, 0\right)$, we can use $(p_3, p_1)$ as a local coordinate system. Take the one-form

$$\eta_s' := (1-s)\frac{\mathrm{d}p_3}{3\left(\frac{1}{I_1} - \frac{1}{I_2}\right) p_1 p_2} + s\frac{\mathrm{d}p_1}{3\left(\frac{1}{I_2} - \frac{1}{I_3}\right) p_2 p_3},$$

where $s$ is an arbitrary parameter as before, and the real closed arc

$$\gamma:\ p_3 = \sqrt{2\ell\frac{\frac{1}{I_2} - \frac{h}{\ell}}{\frac{1}{I_2} - \frac{1}{I_3}}}\cos\theta,\ p_1 = \sqrt{2\ell\frac{\frac{1}{I_2} - \frac{h}{\ell}}{\frac{1}{I_2} - \frac{1}{I_1}}}\sin\theta, \quad \theta : 0 \to 2\pi.$$

Note that $\gamma$ is a real closed arc contained in the complexification of the real integral curve, where $(p_1, p_2, p_3) \in \mathbb{C}^3$ are regarded as complex affine coordinates.

We consider the period integral $\displaystyle\int_\gamma \eta'_s$. Note that the equilibrium $(p_1, p_2, p_3) = \left(\sqrt{2\ell}, 0, 0\right)$ is mapped to $\left(0, \sqrt{2\ell}, 0\right)$ by the transformation $\epsilon_3 \circ \delta_1$, which maps the integral curves around $\left(\sqrt{2\ell}, 0, 0\right)$ to those represented by $\gamma$. Therefore, using Theorem 2, we can calculate the derivative for the inverse of the Birkhoff normal form around $\left(0, \sqrt{2\ell}, 0\right)$. As before, this is invariant with respect to the transformation $\delta_2$, which maps $(p_1, p_2, p_3) = \left(0, \sqrt{2\ell}, 0\right)$ to $\left(0, -\sqrt{2\ell}, 0\right)$.

**Theorem 4.** *The derivative of the inverse $\Phi_2$ of the Birkhoff normal forms around $(p_1, p_2, p_3) = \left(0, \pm\sqrt{2\ell}, 0\right)$ writes*

$$\Phi_2'(h) = \frac{\sqrt{-1}}{3\pi}\sqrt{\frac{2}{\ell}}\frac{1}{\sqrt{(d-c)(b-a)}}\mathcal{K}\left(\frac{(d-b)(a-c)}{(d-c)(a-b)}\right) = -\sqrt{-1}S(b,a,c,d). \tag{7}$$

Before closing this section, we give the following formulae for $S(a,b,c,d)$ obtained through the transformations $\delta_1, \delta_2, \delta_3, \epsilon_1, \epsilon_2, \epsilon_3$ :

$$S(a,b,c,d) = S(a,c,b,d),\ S(b,a,c,d) = S(b,c,a,d),\ S(c,b,a,d) = S(c,a,b,d). \tag{8}$$

These three formulae follow from the previous theorem and the covariance with respect to $\delta_j$, $\epsilon_j$, $j = 1,2,3$. The list can be regarded as the description of the action of the symmetric group $\mathfrak{S}_3$ of degree three on $S(a,b,c,d)$. We explain the action by the symmetric group $\mathfrak{S}_4$ of degree four in Section 5.

## Elliptic fibration and explicit calculation of the cocycles

In this section, we discuss the simplest elliptic fibration considered in Naruki-Tarama and calculate the cocycles of the first cohomology group of regular fibres in relation to the period integrals in Section 3. The main results of this section are the explicit expressions, in terms of the analytic extension of the function $S$, for a basis of the first cohomology group.

We start with a basic description of the elliptic fibration. As we have seen, the integral curve of the Euler equation is given by the intersection of the two quadrics. From the viewpoint of algebraic or analytic geometry, it is natural to complexify and to compactify the integral curve by the complex projective curve

$$\begin{cases} x^2 + y^2 + z^2 + w^2 = 0, \\ ax^2 + by^2 + cz^2 + dw^2 = 0, \end{cases} \tag{9}$$

where $(x : y : z : w) \in P_3(\mathbb{C})$ are the coordinates given by $p_1 = \sqrt{-2\ell}\dfrac{x}{w}, p_2 = \sqrt{-2\ell}\dfrac{y}{w}, p_3 = \sqrt{-2\ell}\dfrac{z}{w}$, and where $a, b, c, d \in \mathbb{C}$ are parameters given by $\dfrac{1}{I_1} = a, \dfrac{1}{I_2} = b, \dfrac{1}{I_3} = c, \dfrac{h}{\ell} = d$.

The following theorem is fundamental to the geometric understanding of this projective curve.

**Theorem 5.** *If $a, b, c, d$ are distinct, then the variety $C$ defined by the above equations* (9) *is a smooth elliptic curve, which has four branch points $a, b, c, d$ as a double covering of the projective line $P_1(\mathbb{C}) \cong \mathbb{C} \cup \{\infty\}$.*

For the proof, see Naruki-Tarama. Because of this proposition, we have a natural elliptic fibration. In fact, denoting by $F$ the algebraic variety in $P_3(\mathbb{C}) \times P_3(\mathbb{C}) : ((x:y:z:w),(a:b:c:d))$ defined by the equation (9), we consider the projection $\pi_F : F \ni ((x:y:z:w),(a:b:c:d)) \mapsto (a:b:c:d) \in P_3(\mathbb{C})$ to the second component of the product space, which is an elliptic fibration called the naive elliptic fibration in Naruki-Tarama. Here, an elliptic fibration means a smooth holomorphic mapping of a complex space onto another complex space whose regular fibres are elliptic curves. As basic facts of the naive elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$, it is known that the total space $F$ is smooth rational variety and that the fibration $\pi_F$ is non-flat, i.e. there is a two-dimensional fibre of $\pi_F$. In fact, the singular fibres of $\pi_F$ are classified as follows :

**Classification of the singular fibres of $\pi_F$**

1. If only two of the parameters $a, b, c, d$ are equal, the fibre consists of two smooth rational curves intersecting at two points. This is a singular fibre of type $\mathrm{I}_2$ in Kodaira's notation [Kodaira, Barth-Hulek-Peters-Vandeven]. Topologically, these singular fibres are double pinched tori of real dimension two.

2. If two of $a, b, c, d$ are equal and the other two are also equal without further coincidence, the fibre consists of four smooth rational curves intersecting cyclically. This is a singular fibre of type $\mathrm{I}_4$ in Kodaira's notation. Topologically, these singular fibres are quadruple pinched tori of real dimension two.

3. If three of $a, b, c, d$ are equal without further coincidence, the fibre is a smooth rational curve, i.e. a two-dimensional sphere as a point set, but with multiplicity two. This singular fibre is not in the list of singular fibres of elliptic surfaces by Kodaira.
4. If $a = b = c = d$, the fibre is a space quadric surface $x^2+y^2+z^2+w^2 = 0$, which is isomorphic to $P_1(\mathbb{C}) \times P_1(\mathbb{C})$.

In order to think about the monodromy of the naive elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$, let $p_0$ be a point in $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$ and $\gamma : t \mapsto \gamma(t)$, $0 \le t \le 1$, a closed path with the reference point at $p_0$ in $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$. Taking a basis $\sigma_{1,0}, \sigma_{2,0}$ of $H_1\left(\pi_F^{-1}(p_0), \mathbb{Z}\right)$, we consider the elements $\sigma_1(t), \sigma_2(t)$ of $H_1\left(\pi_F^{-1}(\gamma(t)), \mathbb{Z}\right)$ which continuously depend on $t$ and $\sigma_i(0) = \sigma_{i,1}$, $i = 1, 2$. Then, there is a matrix $A_{[\gamma]} \in SL(2, \mathbb{Z})$ such that

$$\begin{bmatrix} \sigma_1(1) \\ \sigma_2(1) \end{bmatrix} = A_{[\gamma]} \begin{bmatrix} \sigma_{1,0} \\ \sigma_{2,0} \end{bmatrix}.$$

Note that $A_{[\gamma]}$ depends only on the homotopy class $[\gamma] \in \pi_1(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D); p_0)$ of $\gamma$. The homomorphism $\rho : \pi_1(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D); p_0) \ni [\gamma] \mapsto A_{[\gamma]} \in SL(2, \mathbb{Z})$ is called the monodromy representation of $\pi_1(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D); p_0)$. Similarly, taking the dual basis $\sigma^*_{1,0}, \sigma^*_{2,0}$ of $H^1\left(\pi_F^{-1}(p_0), \mathbb{Z}\right)$ such that $\sigma^*_{i,0} \cdot \sigma_{j,0} = \delta_{ij}$, we have the representation $\rho^* : \pi_1\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D); p_0\right) \ni [\gamma] \mapsto A^{\mathrm{T}}_{[\gamma]} \in SL(2, \mathbb{Z})$. We call it also monodromy representation.

We, now, consider the period integrals which appeared in the last section and discuss the global monodromy, namely the monodromy representation itself. We start with the extension of the one-form $\eta$ such that $\omega = \eta \wedge \mathrm{d}H$ to the complex elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$. On the level surface $L(P) = \dfrac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) = \ell$, the symplectic form $\omega$ writes as (2). On the coordinate neighbourhood with the coordinates $(p_2, p_3)$, the one-form $\eta_s$ in (3) satisfies $\omega = \eta_s^{(1)} \wedge \mathrm{d}H$ for arbitrary $s$.

With this in mind, we consider the following one-form $\eta$ :

$$\eta = \frac{1}{\sqrt{2\ell}} \frac{w^2 \mathsf{d}\left(\frac{y}{w}\right)}{3(c-a)zx}.$$

According to the notation in Section 2, $\eta$ should be written as $\eta^{\mathbb{C}}$, but we use $\eta$ also for the holomorphic form for the brevity. Because of the transformation of the coordinates $p_1 = \sqrt{-2\ell}\frac{x}{w}, p_2 = \sqrt{-2\ell}\frac{y}{w}, p_3 = \sqrt{-2\ell}\frac{z}{w}$ and $a = \frac{1}{I_1}, b = \frac{1}{I_2}, c = \frac{1}{I_3}, d = \frac{h}{\ell}$, we see that $\eta$ and $\eta_s$ induce the same section in $\Omega^1_{\pi_F} := \Omega^1_F / \pi_F^* \Omega^1_{P_3(\mathbb{C})}$, which is the sheaf over $F$ of germs of relative differential forms with respect to the fibration $\pi_F : F \to P_3(\mathbb{C})$. More precisely, we have the following theorem.

**Theorem 6.** *The one-form $\eta$ induces a globally-defined meromorphic section of $\Omega^1_{\pi_F}$, which is a holomorphic non-zero one-form on $\pi_F^{-1}\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)\right)$.*

Note that the integral $\int_\sigma \eta$ of the one-form $\eta$ over a cycle $\sigma$ included in a fibre of $\pi_F$ depends only on the class of the relative differential one-form to which $\eta$ belongs. Note that $\eta$ can be regarded as a holomorphic (hence closed) one-form on each regular fibre, so that $\displaystyle\int_\sigma \eta$ depends only on the homology class of $\sigma$. It is clear that $\int_{\sigma_{1,0}} \eta$ and $\int_{\sigma_{2,0}} \eta$ form a basis of $H^1\left(\pi_F^{-1}(p_0), \mathbb{C}\right)$ and it suffices to consider the monodromy of these period integrals in order to calculate the monodromy representation of $\pi_1\left(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D), p_0\right)$ with respect to the fibration $\pi_F : F \to P_3(\mathbb{C})$.

We choose a basis of the first homology group $H_1\left(\pi_F^{-1}(p_0), \mathbb{Z}\right)$ of the regular fibre near the elliptic and hyperbolic stationary points of the free rigid body dynamics. Assuming the same condition $I_1 < I_2 < I_3$ for the inertia tensor $\mathcal{I} = \mathrm{diag}(I_1, I_2, I_3)$ as in Section 3, we start with a regular fibre around the elliptic stationary points on the $p_1$-axis. Note that these two points are in the same intersection of two quadrics $H = h, L = \ell$, such that $\dfrac{h}{\ell} = \dfrac{1}{I_1}$, i.e. $a = d$. For the simplicity, we assume that $\dfrac{1}{I_1} > \dfrac{h}{\ell} > \dfrac{1}{I_2} > \dfrac{1}{I_3}$.

The real integral curves are parameterized as

$$\sigma_{\pm}^{(1)} : (p_1, p_2, p_3) = \left( \pm \sqrt{ -\frac{2\ell\left(\frac{1}{I_2} - \frac{h}{\ell}\right)}{\frac{1}{I_1} - \frac{1}{I_2}} \left\{ 1 - \frac{\left(\frac{h}{\ell} - \frac{1}{I_1}\right)\left(\frac{1}{I_3} - \frac{1}{I_2}\right)}{\left(\frac{h}{\ell} - \frac{1}{I_2}\right)\left(\frac{1}{I_3} - \frac{1}{I_1}\right)} \sin^2\theta \right\} }, \right.$$

$$\sqrt{\frac{2\ell\left(\frac{1}{I_1} - \frac{h}{\ell}\right)}{\frac{1}{I_1} - \frac{1}{I_2}}} \cos\theta, \sqrt{\frac{2\ell\left(\frac{1}{I_1} - \frac{h}{\ell}\right)}{\frac{1}{I_1} - \frac{1}{I_3}}}$$

where $\theta : 0 \to 2\pi$, near $(p_1, p_2, p_3) = \left(\pm\sqrt{2\ell}, 0, 0\right)$. If $h$ is near to $\dfrac{\ell}{I_1}$, i.e. if $d$ is near to $a$, these real closed arcs are vanishing cycles around two different $A_1$ singular points and are included in the same fibre of $\pi_F : F \to P_3(\mathbb{C})$.

By Theorem 3, the period integrals of $\eta$ along $\sigma_{\pm}^{(1)}$ are the same :

$$\int_{\sigma_{\pm}^{(1)}} \eta = S(a,b,c,d). \tag{10}$$

This reflects the fact that $\sigma_{\pm}^{(1)}$ are homologous to each other in $H_1\left(\pi_F^{-1}(p_0), \mathbb{Z}\right)$. Here, $S(a,b,c,d)$ is considered as a multi-valued holomorphic function over $P_3(\mathbb{C}) \setminus \mathrm{Supp}(D)$. We take this period integral as one of the basis of $\mathrm{H}^1\left(\pi_F^{-1}(p_0), \mathbb{C}\right)$.

To obtain the other basis element, we consider the following real one-dimensional arcs in the same regular fibre. We take the arcs

$$\tau_{\pm}^{(1)} : (p_1, p_2, p_3) = \left( \pm \sqrt{ -\frac{2\ell\left(\frac{1}{I_2} - \frac{h}{\ell}\right)}{\frac{1}{I_1} - \frac{1}{I_2}} \left\{ 1 + \frac{\left(\frac{h}{\ell} - \frac{1}{I_1}\right)\left(\frac{1}{I_3} - \frac{1}{I_2}\right)}{\left(\frac{h}{\ell} - \frac{1}{I_2}\right)\left(\frac{1}{I_3} - \frac{1}{I_1}\right)} \sinh^2 \varphi \right\}}, \right.$$

$$\sqrt{\frac{2\ell\left(\frac{1}{I_1} - \frac{h}{\ell}\right)}{\frac{1}{I_1} - \frac{1}{I_2}}} \cosh \varphi, \pm\sqrt{-1}\sqrt{\frac{2\ell\,(}{I}}$$

where $\varphi$ moves from $-\infty$ to $+\infty$. It is easy to check that these two arcs are included in the same fibre of $\pi_F : F \to P_3(\mathbb{C})$ near the singular locus $a = d$. Note that $\sigma_{\pm}^{(1)}$ meet $\tau_{\pm}^{(1)}$ respectively at single points $(\pm\sqrt{b-d} : \sqrt{d-a} : 0 : \sqrt{a-b})$, while $\sigma_{\pm}^{(1)}$ do not meet $\tau_{\mp}^{(1)}$. We also need to take the following arcs :

$$\tau_{\pm}'^{(1)} : (p_1, p_2, p_3) = \left( -\sqrt{-1}\sqrt{\frac{2\ell\left(\frac{1}{I_2} - \frac{h}{\ell}\right)}{\frac{1}{I_2} - \frac{1}{I_1}}} \sinh\varphi, \right.$$

$$\pm\sqrt{-\frac{2\ell\left(\frac{1}{I_3} - \frac{h}{\ell}\right)}{\frac{1}{I_2} - \frac{1}{I_3}}\left\{1 + \frac{\left(\frac{h}{\ell} - \frac{1}{I_2}\right)\left(\frac{1}{I_1} - \frac{1}{I_3}\right)}{\left(\frac{h}{\ell} - \frac{1}{I_3}\right)\left(\frac{1}{I_1} - \frac{1}{I_2}\right)}\sinh^2\varphi\right\}}, \pm\sqrt{-1}\sqrt{\frac{2\ell\left(\frac{1}{I_2} - \right.}{\frac{1}{I_2} - }}$$

where $\varphi$ moves from $-\infty$ to $+\infty$. It is easy to see that $\tau_{\pm}'^{(1)}$ are included in the same regular fibre of $\pi_F : F \to P_3(\mathbb{C})$ near the singular locus $a = d$

as $\tau_{\pm}^{(1)}$. By considering the behavior of these arcs when the parameters approaches infinity, we can check that the arcs $\tau_{\pm}^{(1)}$, ${\tau'}_{\pm}^{(1)}$ are connected at these infinity points.

Note that there is no other intersection among $\tau_{\pm}^{(1)}$ and ${\tau'}_{\pm}^{(1)}$ than these four points and that ${\tau'}_{\pm}^{(1)}$ do not meet $\sigma_{\pm}^{(1)}$. We denote the multiplication $\tau_{+}^{(1)} \cdot {\tau'}_{+}^{(1)} \cdot \tau_{-}^{(1)} \cdot {\tau'}_{-}^{(1)}$ by $\tau^{(1)}$. In particular, $\tau^{(1)}$ is a closed arc in the same regular fibre of $\pi_F : F \to P_3(\mathbb{C})$ as $\sigma_{\pm}^{(1)}$ and $\sigma_{+}^{(1)}$, homologous to $\sigma_{-}^{(1)}$, form a basis of $H_1\left(\pi_F^{-1}(p_0), \mathbb{Z}\right)$. Thus, calculating the period integrals $\displaystyle\int_{\tau^{(1)}} \eta$, together with (10), we can obtain a concrete basis of $H^1\left(\pi_F^{-1}(p_0), \mathbb{C}\right)$.

In fact, we can express the period integral $-\frac{1}{2\pi}\int_{\tau^{(1)}} \eta$ in terms of the analytic continuation of the function $S$.

**Theorem 7.** *The following formula holds for the analytic continuation of the function $S$ :*

$$S(a, b, c, d) + S(b, a, c, d) = S(c, b, a, d).$$

This is a consequence of the connexion formula satisfied by the complete elliptic integral of the first kind $\mathcal{K}(\lambda) = \dfrac{\pi}{2} F\left(\dfrac{1}{2}, \dfrac{1}{2}, 1; \lambda\right)$, which satisfies the Gauß hypergeometric differential equation

$$(1-\lambda)\lambda \frac{\mathrm{d}^2 f}{\mathrm{d}\lambda^2} + (1-2\lambda)\frac{\mathrm{d}f}{\mathrm{d}\lambda} - \frac{1}{4} f = 0. \tag{11}$$

See H. Zoladek, Ch. 12, p.494

**Theorem 8.** *The period integral of the one-form $\eta$ along the cycle $\tau^{(1)}$ is calculated as*

$$-\frac{1}{2\pi}\int_{\tau^{(1)}}\eta = S(c,b,a,d), \tag{12}$$

## Monodromy and global behavior of Birkhoff normal forms

In this section, we calculate the global monodromy of the naive elliptic fibration $\pi_F$, which is connected to the global behavior of Birkhoff normal forms. Starting with the symmetry of the naive elliptic fibration and of the function $S$, whose relation to the cohomology group of regular fibres is discussed in Section 4, we first calculate the local monodromy with respect to the basis associated to the analytic continuation of $S$, by using the connection formulae of the Gauß hypergeometric differential equation (11).

Next, we give a description of the fundamental group of the regular locus of the base space $P_3(\mathbb{C})$ for the elliptic fibration $\pi_F$, by using the arguments of the topology of the complement of hyperplane arrangements. Combining these results by the connecting formula, we finally obtain the explicit global monodromy of the elliptic fibration. This theorem implies that the monodromy of the analytic continuation for the derivative of the inverse of Birkhoff normal forms coincides with the global monodromy of the elliptic fibration of Naruki-Tarama.

The two period integrals $-\dfrac{1}{2\pi}\displaystyle\int_{\sigma_{\pm}^{(1)}}\eta = S(a,b,c,d)$ and $-\dfrac{1}{2\pi}\displaystyle\int_{\tau^{(1)}}\eta = S(c,b,a,d)$ of the one-form $\eta$ along the basis $\sigma_{\pm}^{(1)}$ and $\tau^{(1)}$ of $H_1\left(\pi_F^{-1}(p_0),\mathbb{Z}\right)$ for a regular fibre $\pi_F^{-1}(p_0)$, where $p_0$ is near to the singular locus $a=d$, form a basis of $H^1\left(\pi_F^{-1}(p_0),\mathbb{C}\right)$. We take such a basis of the cohomology group $H^1\left(\pi_F^{-1}(p_0),\mathbb{C}\right)$ for a regular fibre $\pi_F^{-1}(p_0)$ over $p_0$ which is near to each irreducible component of the singular locus $D:\{a=b\}+\{a=c\}+\{a=d\}+\{b=c\}+\{b=d\}+\{c=d\}$. To do this, we use the symmetry of the Naruki-Tarama elliptic fibration and that of the function $S(a,b,c,d)$ with respect to the symmetric group $\mathfrak{S}_4$ acting on the base space $P_3(\mathbb{C}):(a:b:c:d)$ as the permutations of the four letters $a,b,c,d$.

In fact, we have the following list of equalities among the analytic continuation of $S\left(\sigma(a,b,c,d)\right)$, for $\sigma \in \mathfrak{S}_4$, which can be obtained by using the equalities of the Gauß hypergeometric function $F$ :

$$
\begin{array}{lllll}
S_1 & = S(a,b,c,d) & = S(a,c,b,d) & = S(b,a,d,c) & = S(b,d,a,c) \\
 & = S(c,a,d,b) & = S(c,d,a,b) & = S(d,b,c,a) & = S(d,c,b,a), \\
S_2 & = S(b,a,c,d) & = S(b,c,a,d) & = S(a,b,d,c) & = S(a,d,b,c) \\
 & = S(c,b,d,a) & = S(c,d,b,a) & = S(d,a,c,b) & = S(d,c,a,b), \\
S_3 & = S(c,b,a,d) & = S(c,a,b,d) & = S(b,c,d,a) & = S(b,d,c,a) \\
 & = S(a,c,d,b) & = S(a,d,c,b) & = S(d,a,b,c) & = S(d,b,a,c).
\end{array}
\tag{13}
$$

This means that the function $S(a,b,c,d)$ is invariant with respect to the dihedral group generated by $(bc)$ and $(abdc)$, which is isomorphic to $\mathbb{Z}_2\mathbb{Z}_4$.

The regular locus of the fibration $\pi_F$ is the open set $R := P_3(\mathbb{C}) \setminus \{a = b, a = c, a = d, b = c, b = d, c = d\}$.

The fundamental group $\pi_1(R, *)$ of the regular locus for the fibration $\pi_F$ is calculated by means of the arguments on the fundamental groups of the complements of hyperplane arrangements (cf. Orlik-Terao). We denote the generators $h_{12}$, $h_{13}$, $h_{14}$, $h_{23}$, $h_{24}$, $h_{34}$ of $\pi_1(R, *)$ which are respectively represented by real closed arcs enclosing the irreducible components $a = b$, $a = c$, $a = d$, $b = c$, $b = d$, $c = d$ of the singular locus.

**Theorem 9.** *The fundamental group $\pi_1(R, *)$ of the regular locus for the fibration $\pi_F$ is generated by $h_{12}$, $h_{13}$, $h_{14}$, $h_{23}$, $h_{24}$, $h_{34}$, with the relations*

$$\begin{aligned}
h_{12}h_{23}h_{13} &= h_{23}h_{13}h_{12} = h_{13}h_{12}h_{23}, \\
h_{23}h_{34}h_{24} &= h_{34}h_{24}h_{23} = h_{24}h_{23}h_{34}, \\
h_{12}h_{24}h_{14} &= h_{24}h_{14}h_{12} = h_{14}h_{12}h_{24}, \\
h_{34}h_{14}h_{13} &= h_{14}h_{13}h_{34} = h_{13}h_{34}h_{14}, \\
h_{12}h_{34} &= h_{34}h_{12}, \\
h_{13}h_{23}^{-1}h_{24}h_{23} &= h_{23}^{-1}h_{24}h_{23}h_{13}, \\
h_{23}h_{14} &= h_{14}h_{23}, \\
h_{13}h_{12}h_{23}h_{34}h_{24}h_{14} &= 1.
\end{aligned}$$

**Theorem 10.** *The basis of the first cohomology group for the regular fibre of the fibration $\pi_F : F \to P_3(\mathbb{C})$ is generated by the analytic continuations of $S_3$ and of $S_1$, which are proportional to the derivative of the inverse Birkhoff normal forms around the $p_3$- and $p_1$-axes, respectively. The monodromy of the fibration $\pi_F$ with respect to $S_3$ and $S_1$ is given by the correspondence of the generators $h_{12}$, $h_{13}$, $h_{14}$, $h_{23}$, $h_{24}$, $h_{34}$ of the fundamental group $\pi_1(P_3(\mathbb{C}) \setminus \mathrm{Supp}(D))$ to the matrices in $SL(2, \mathbb{Z})$ as follows :*

$$h_{14}, h_{23} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, h_{13}, h_{24} \mapsto \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, h_{12}, h_{34} \mapsto \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

## Conclusion and perspectives

Analytic extensions of the Birkhoff normal form has been used to compute the global monodromy of the Naruki-Tarama elliptic fibration.

Possible extensions to other algebraically integrable systems ?

Recently, several researches have been done on the relation between elliptic fibrations and the compactifications of string theory in view of the F-theory. Among them, **Esole-Fullwood-Yau** deals with the elliptic fibrations modeled by quadrics intersections in $P_3(\mathbb{C})$, called $D_5$ elliptic fibrations, and puts emphasis on the appearance of non-Kodaira singular fibres.