Mathematics

Spectral Problems in Inverse Scattering Theory

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Motivation

- Find fast and sensitive target signatures for scatterers using acoustic, elastic, or electromagnetic wave interrogation.
- An attractive choice could be eigenvalues associated with the wave propagation phenomena for the considered model.
- The identified eigenvalues must satisfy two important properties:
  - Can be determined from scattering data
  - Are related to geometrical and physical properties of the targets in an understandable way

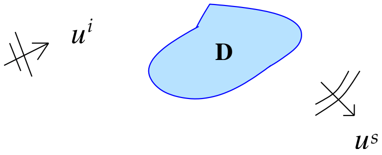
Outlook

- Part I – The transmission eigenvalues and they relation to the relative scattering operator.

- Part II – New sets of eigenvalues related to a modified relative scattering operator.

- Part III – The transmission eigenvalues in scattering theory in the hyperbolic plane for automorphic forms with respect to discrete groups and they relation to Riemann zeta function.

Scattering by an Inhomogeneous Media



$$
\begin{aligned}
\Delta u^s + k^2 u^s = 0 \qquad & \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \\
\nabla \cdot A \nabla u + k^2 n u = 0 \qquad & \text{in} \quad D \\
u = u^s + u^i \qquad & \text{on} \quad \partial D \\
\nu \cdot A \nabla u = \nu \cdot \nabla (u^s + u^i) \qquad & \text{on} \quad \partial D \\
\lim_{r \to \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 &
\end{aligned}
$$

- k is the wave number which is proportional to the frequency ω
- u^i is the incident wave solving $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^3$ (except for possibly one point) and u^s is the scattered wave.

Scattering by an Inhomogeneous Media

Particular example of incident fields are free space waves $v := v_g$ known as Herglotz wave functions of the form

$$v_{g,k}(x) = \int_{S^2} g(\hat{y}) e^{ikx\cdot\hat{y}}\, ds, \qquad g \in L^2(S^2),\ \hat{y} = y/|y|\ ,$$

or equivalently solutions to Helmholtz equation in $\mathbb{R}^3$ satisfying

$$\|v_{g,k}\|_{B^*} := \sup_{R>0} \frac{1}{\sqrt{R}} \|v_{g,k}\|_{L^2(B_R)} < \infty$$

(i.e. solutions to Helmholtz equation whose Fourier transform belongs to the Besov space $B^{-1/2}_{2,\infty}$)

Scattering by an Inhomogeneous Media

Every such Herglotz wave function $v_{g,k}$ can be uniquely decomposed

$$v_{g,k} := u_{g,k} - u^s_{g,k}$$

where $u_{g,k}$ total field $u^s_{g,k}$ the outgoing scattered field.

- The scattering operator (Lax-Phillips) maps

$$v_{g,k} \mapsto u_{g,k}$$

- The relative scattering operator (Melrose) maps

$$v_{g,k} \mapsto u^s_{g,k}$$

The scattering operator is analytic on k in the upper half complex plane and meromorphic in the lower half complex plane. Its poles are the scattering poles or resonances.

Far Field Operator & Scattering Operator

- The scattered field u^s has the asymptotic behavior

$$u^s(x,k) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x},k) + O\left(\frac{1}{|x|}\right)\right\} \qquad \text{as } |x| \to \infty$$

  uniformly in $\hat{x} = x/|x| \in \mathbb{S}^2$. u_∞ is called the far field pattern.

Now we take incident $u^i := e^{ikx\cdot\hat{d}}$ and the corresponding $u_\infty(\hat{x};\hat{d},k)$.

- We define the far field operator (or the relative scattering operator)

$$F_k : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2) \quad \text{by} \quad (F_k g)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x};\hat{d},k) g(\hat{d}) ds.$$

- F_k is related to the scattering operator $\mathcal{S}_k : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ by

$$\mathcal{S}_k = I + \frac{ik}{2\pi} F_k$$

Eigenvalues and Inverse Scattering Theory

- Spectral properties of operators associated with scattering problems provide essential information about scattering objects.
- However, the main question is whether such spectral features can be seen in the scattering data.
- Resonances constitute a fundamental part of scattering theory. Their study has led to beautiful mathematics, has shed light into deeper understanding of direct and inverse scattering phenomena.
  - R.B. Melrose (1995), Geometric Scattering Theory, Cambridge University Press.
  - S. Dyatlov - M. Zworski (2017), Mathematical Theory of Scattering Resonances, online.
- Because the resonances are complex, it is difficult to determine them from scattering data unless they are near the real axis.

Eigenvalues and Inverse Scattering Theory

Consider scattering of $u^i = j_0(k|x|)$, by a ball B_1 of radius one with refractive index $n = 4$ and $A = I$. The total field u is given by

$$u(x) = c_1\, j_0(2k|x|),\ |x| < 1 \quad \text{and} \quad u(x) = c_2\, h_0^{(1)}(k|x|) + j_0(k|x|),\ |x| > 1$$

where $\quad c_1 j_0(2k) - c_2 h_0^{(1)}(k) = j_0(k) \quad$ and $\quad 2c_1 j_0'(2k) - c_2 h_0'^{(1)}(k) = j_0'(k)$

Example (Resonaces)

k for which $2h_0^{(1)}(k) j_0'(2k) - j_0(2k) h_0'^{(1)}(k) = 0$. Here $\Im(k) < 0$.

There exists a solution to the scattering problem where $u^i = 0$, $u^s = h_0^{(1)}(k|x|), \quad |x| > 1$.

Example (Transmission Eigenvalues)

k for which $2j_0(k) j_0'(2k) - j_0(2k) j_0'(k) = 0$. Here k is real and complex.

There exists a solution to the scattering problem where $u^s = 0$, $u^i = j_0(k|x|), \quad |x| > 1$.

TE and Non-Scattering Frequencies

Question: Is there an incident field u^i that does not scatter?

If yes, k is such that $v := u^i|_D$ and u are solutions to the transmission eigenvalue problem

$$\begin{aligned} \Delta v + k^2 v = 0 \quad & \text{in} \quad D \\ \nabla \cdot A \nabla u + k^2 n u = 0 \quad & \text{in} \quad D \\ u = v \quad & \text{on} \quad \partial D \\ \nu \cdot A \nabla u = \nu \cdot \nabla v \quad & \text{on} \quad \partial D \end{aligned}$$

Transmission Eigenvalues

Values of $k \in \mathbb{C}$ for which the transmission eigenvalue problem has a non trivial solution are called transmission eigenvalues

TE and Non-Scattering Frequencies

If k is a transmission eigenvalue and the eigenfunction v that solves $\Delta v + k^2 v = 0$ in D can be extended outside D as a solution $\tilde{v}$ of the same equation, then the scattered field due to $\tilde{v}$ as an incident wave is identically zero.

In general such an extension of v does not exist (corners!).

BLÅSTEN-PÄIVÄRINTA-SYLVESTER (2013), *Comm. Math. Phys.*

Important Fact: Superposition of plane waves or point sources are dense in

$$\left\{v \in H^1(D) : \ \Delta v + k^2 v = 0 \quad \text{in } D\right\}.$$

Thus at a transmission eigenvalue it is possible to superimpose plane waves or point sources to produce an arbitrarily small scattered field.

Far Field Operator

Consider the incident wave-to-far field mapping

$$G : v \in \left\{ H^1(D) : \Delta v + k^2 v = 0 \right\} \mapsto u_\infty(\hat{x}) \in L^2(\mathbb{S}^2)$$

where u_∞ is the far field pattern of the scattered field u^s satisfying

$$\nabla \cdot A \nabla u^s + k^2 n u^s = \nabla \cdot (I - A) \nabla v + k^2 (1 - n) v \qquad \text{in } \mathbb{R}^3$$

$A = I$ and $n = 1$ outside D.

- $F_k g = G(v_g)$
- If k is a transmission eigenvalue, the transmission eigenfunction v is such that $G(v) = 0$
- If $v_{g_\epsilon} \to v$, then $\|F_k g_\epsilon\|_{L^2} \to 0$ as $\epsilon \to 0$

Determination of **Real** Transmission Eigenvalues

CAKONI-COLTON-HADDAR (2010) *C. R. Math. Acad. Sci. Paris*.

Is $\quad e^{-ik\hat{x}\cdot z} \in \mathrm{Range}(F_k)$?

Note that $e^{-ik\hat{x}\cdot z}$ is the far field pattern of $\Phi(x,z) = \frac{e^{ik|x-z|}}{4\pi|x-z|}$.

$G(v_z) = e^{-ik\hat{x}\cdot z}$ where

$$\begin{aligned}
\Delta v_z + k^2 v_z &= 0 && \text{in} \quad D \\
\nabla \cdot A\nabla u_z + k^2 n u_z &= 0 && \text{in} \quad D \\
u_z - v_z &= \Phi(\cdot, z) && \text{on} \quad \partial D \\
\nu \cdot A\nabla u_z - \nu \cdot \nabla v_z &= \nu \cdot \nabla\Phi(\cdot, z) && \text{on} \quad \partial D
\end{aligned}$$

Then $v_g \approx v_z$ is such that $F_k g \approx e^{-ik\hat{x}\cdot z}$

Generalized linear sampling method

For $\alpha > 0$, consider $J_\alpha(g) = \alpha B(g) + \|F_k g - e^{-ik\hat{x}\cdot z}\|^2_{L^2(\mathbb{S}^2)}$

where $B : L^2(\mathbb{S}^2) \to \mathbb{R}^+$ is a functional (not necessarily convex).

Let g_α^z be a minimizing sequence of J_α such that

$$J_\alpha(g_\alpha^z) \leq \inf_{g \in L^2(\mathbb{S}^2)} J_\alpha(g) + o(\alpha).$$

If $B(g)$ is bounded if and only if $\|v_g|_D\|_{H^1(D)}$ is bounded, then

$$e^{-ik\hat{x}\cdot z} \in \mathrm{Range}(G) \Longleftrightarrow \lim_{\alpha \to 0} B(g_\alpha) < \infty$$

The key in the choice of $B(g)$ is

- $B(g)$ must control $\|v_g|_D\|_{H^1(D)}$ but not necessarily $\approx \|g\|$
- $B(g)$ must be expressed in terms of F_k (i.e. the data)

AUDIBERT (2015) Ph.D Thesis, Ecole Polytechnique

Determination of **Real** Transmission Eigenvalues

AUDIBERT-CAKONI-HADDAR (2017), *Inverse Problems*.

If $A - I$ and $n - 1$ are one sign in a neighborhood of the boundary ∂D

$$B(g) := |(F_k g, g)| = \left|\left(T v_g, v_g\right)\right|_D$$

where T bounded linear operator bounded below.

Theorem

$$\lim_{\alpha \to 0} |(F_k g_\alpha^z, g_\alpha^z)| < +\infty \text{ for all } z \in B_\delta \subset D$$

$$\iff k > 0 \text{ is not a transmission eigenvalue}$$

where g_α^z is a minimizing sequence of

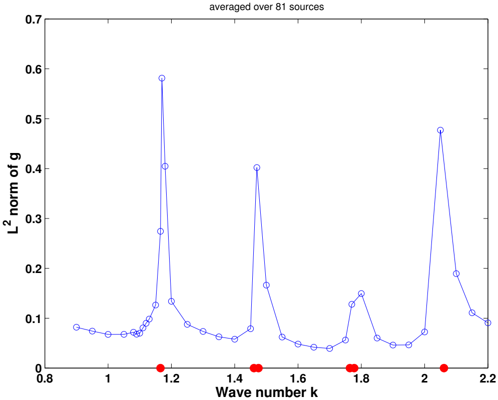
$$J_\alpha(g) = \alpha\,|(F_k g, g)| + \|F_k g - e^{-ik\hat{x}\cdot z}\|^2_{L^2(\mathbb{S}^2)}$$

Regularized-Noisy Case

$$\alpha\left(\left|\left(F_k^\delta g, g\right)\right| + \delta\|F_k^\delta\|\|g\|^2\right) + \|F_k^\delta g - e^{-ik\hat{x}\cdot z}\|^2_{L^2(\mathbb{S}^2)}$$

Computation of Real Transmission Eigenvalues

$D := B_1$, $A = I$, $n = 16$



Solving the far-field equation for several source points z gives obvious peaks at the transmission eigenvalues. Red dots indicate exact transmission eigenvalues.

Inside-Outside Duality

Transmission eigenvalues are intrinsic to scattering phenomenon!

Inside-Outside Duality

Real transmission eigenvalues can be characterized in terms of the eigenvalues of scattering operator S_k (recall $\mathcal{S}_k = I + \frac{ik}{2\pi} F_k$)

Under quite restrictive assumptions on $A - I$ and $n - 1$, one can prove that the eigenvalues of the (unitary) scattering operator S_k exhibit different phase behavior at a transmission eigenvalue $k > 0$.

- KIRSCH-LECHLEITER (2013) - *Inverse Problems* **10** no 4
- LECHLEITER-PETERS (2015) - *Commun. Math. Sci.* **13** no 7

State of the Art Results on TEP

- The transmission eigenvalue problem is non-self adjoint
- In the case when $\Im(A) < 0$ or/and $\Im(n) > 0$ all transmission eigenvalues are complex.
- If $A - I$ and $n - 1$ are one sing in D than infinite real transmission eigenvalue exists and they depend monotonically increasing on n and decreasing on A.

Open

The spectral analysis, relaxing the essential assumption that

$$\Re(A - I) \qquad \text{or} \qquad \Re(n-1) \text{ if } A = I$$

does not change sign in a neighborhood of the boundary ∂D.

Cakoni-Gintides-Haddar (2010), Cakoni-Kirsch (2010), Bonnet-Ben Dhia-Chesnel-Haddar (2011),

Aktusun-Gintides-Papanicolaou (2011), Sylvester (2012), Leung-Colton (2012-13-15), HM. Nguyen-QH. Nguyen

(2016), Robianno (2013), Lakshtanov-Vainberg (2015-16), Petkov-Vodev (2016-17), Vodev (2017-18) etc....

Literature

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Inverse Scattering Theory and Transmission Eigenvalues

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Inverse Scattering Theory and Transmission Eigenvalues

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Inverse scattering theory is a major theme of applied mathematics and has applications to such diverse areas as medical imaging, geophysical exploration, and nondestructive testing. The inverse scattering problem is both nonlinear and ill-posed, thus presenting particular problems in the development of efficient inversion algorithms. Although linearized models continue to play an important role in many applications, an increased need to focus on problems in which multiple scattering effects cannot be ignored has led to a central role for nonlinearity, and the possibility of collecting large amounts of data over limited regions of space means that the ill-posed nature of the inverse scattering problem has become a problem of central importance.

Initial efforts to address the nonlinear and the ill-posed nature of the inverse scattering problem focused on nonlinear optimization methods. While efficient in many situations, strong a priori information is necessary for their implementation. This problem led to a qualitative approach to inverse scattering theory in which the amount of a priori information is drastically reduced, although at the expense of only obtaining limited information about the values of the constitutive parameters. This qualitative approach (the linear sampling method, the factorization method, the theory of transmission eigenvalues, etc.) is the theme of Inverse Scattering Theory and Transmission Eigenvalues.

The authors

- begin with a basic introduction to inverse scattering theory and then proceed to more recent developments, including a detailed discussion of the transmission eigenvalue problem;
- present a new generalized linear sampling method in addition to the well-known linear sampling and factorization methods; and
- focus on the inverse scattering problem for scalar homogeneous media in order to achieve clarification of presentation.

This book is for research mathematicians as well as engineers and physicists working on problems in target identification. It will also be of interest to advanced graduate students in diverse areas of applied mathematics.

Fioralba Cakoni is a professor in the Department of Mathematics at Rutgers University. She is coauthor with David Colton of A Qualitative Approach to Inverse Scattering Theory (Springer, 2014).

David Colton is a professor in the Department of Mathematical Sciences at the University of Delaware, where he was appointed Unidel Professor in 1996. He is coauthor of the aforementioned book with Fioralba Cakoni and coauthor of Inverse Acoustic and Electromagnetic Scattering Theory (Springer, 3rd edition, 2013) with Rainer Kress.

Houssem Haddar is Director of Research at INRIA and a part-time Professor at Ecole Polytechnique. He is coauthor with Ralf Hiptmair, Peter Monk, and Rodolfo Rodriguez of Computational Electromagnetism (Springer, 2015).

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Cons of Using Transmission Eigenvalues

Drawbacks of using the transmission eigenvalues as target signature

- Non-self adjointness makes it harder to relates eigenvalues to material properties.
- Multifrequency data is needed.
- The first transmission eigenvalue is determined by the material properties of the scatterer, i.e. one can not choose the range of interrogating frequencies.
- The method of transmission eigenvalues only applies to non-absorbing media or materials with very small absorption.

It is possible to modify the scattering data in such away that the same analysis yields a new eigenvalue problem whose eigenvalue parameter is not related to the interrogating frequency

$$u^{total} = u^{scattered} + u^{incident}$$

u^{total} is the physical field, thus change $u^{scattered}$ by changing $u^{incident}$.

Modified Far-Field Operator

Let $h^{s,\lambda}$ be the scattered field due a plane wave by an artificial scatterers possibly depending on $\lambda \in \mathbb{C}$, and let h_∞^λ be its far field. Define

$$\mathcal{F}g = F_k g - F_{k,\lambda} g := \int_{\mathbb{S}^2} \left[u_\infty(\cdot\,;\hat{d}) - h_\infty^\lambda(\cdot\,;\hat{d}) \right] g(\hat{d}) ds$$

- F_k is known from measurements.
- $F_{k,\lambda}$ is computed (can be precomputed).

If the above analysis is now performed to the modified far field operator $\mathcal{F}$, a new eigenvalue problem appears instead of the transmission eigenvalue problem.

Eigenvalues and Inverse Scattering Theory

Example (Steklov Eigenvalues)

$F_{k,\lambda} : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is defined by

$$(F_{k,\lambda})g = \int_{\mathbb{S}} h_\infty^\lambda(\hat{x}, d)g(d)\, ds(d)$$

where h_∞^λ is the far field pattern of the scattered field $h^s := h^{s,\lambda}$

$$\begin{aligned}
\Delta h^s + k^2 h^s = 0 & \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D_b} \\
h = h^s + e^{ik\hat{d}\cdot x} & \\
\frac{\partial h}{\partial \nu} + \lambda h = 0 & \quad \text{in} \quad \partial D_b \\
\lim_{r\to\infty} r\left(\frac{\partial h^s}{\partial r} - ikh^s\right) = 0 & \quad r = |x|
\end{aligned}$$

and D_b is such that $D \subseteq D_b$.

Steklov Eigenvalues

The question if there is a $g \in L^2(\mathbb{S}^2)$ s.th. $\mathcal{F}g = F_k g - F_{k,\lambda} g = 0$ yield

$$\begin{aligned} \nabla \cdot A \nabla u + k^2 n u &= 0 \qquad \text{in } D_b \\ \nu \cdot A \nabla u + \lambda u &= 0 \qquad \text{on } \partial D_b. \end{aligned}$$

with $A = I$, $n = 1$ in $D_b \setminus D$. If k is fixed then this is the Steklov eigenvalue problem for λ

Theorem (Audibert-Cakoni-Haddar Inverse Problems (2017))

$$\lambda \in \mathbb{C} \text{ is a Steklov eigenvalue} \iff$$

$$\lim_{\alpha \to 0} |(F_\lambda g_\alpha^z, g_\alpha^z)| < +\infty \text{ for all } z \in B_\delta \subset D_b$$

where g_α^z is a minimizing sequence of

$$J_\alpha(g) = \alpha\, |(F_\lambda g, g)| + \|\mathcal{F}g - e^{-ik\hat{x}\cdot z}\|^2_{L^2(\mathbb{S}^2)}$$

Modified Far Field Operator

Example (New eigenvalue problems)

$F_{k,\lambda} : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ is defined by

$$F_{k,\lambda} g = \int_S h_\infty^\lambda(\hat{x}, d) g(d)\, ds(d)$$

where h_∞^λ is the far field pattern of the scattered field $h^s := h^{s,\lambda}$

$$\begin{aligned}
\Delta h^s + k^2 h^s &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D_b} \\
a\Delta h + k^2 \lambda h &= 0 && \text{in } D_b \\
h &= h^s + e^{ik\hat{d}\cdot x} && \text{on } \partial D_b \\
a\frac{\partial h}{\partial \nu} &= \frac{\partial (h^s + e^{ik\hat{d}\cdot x})}{\partial \nu} && \text{on } \partial D_b \\
\lim_{r\to\infty} r\left(\frac{\partial h^s}{\partial r} - ikh^s\right) &= 0 && r = |x|
\end{aligned}$$

The chosen parameter a is a fixed. The only requirement is that the above scattering problem is well-posed.

Eigenvalues and Inverse Scattering Theory

The question if there is a $g \in L^2(\mathbb{S}^2)$ s.th. $\mathcal{F}g = F(-F_{k,\lambda})g = 0$ yield

$$\begin{aligned} \nabla \cdot A\nabla u + k^2 n(x) u = 0 \qquad & \text{in} \quad D_b \\ a\Delta v + k^2 \lambda v = 0 \qquad & \text{in} \quad D_b \\ u = v \qquad & \text{on} \quad \partial D_b \\ \nu \cdot A\nabla u = \nu \cdot a\nabla v \qquad & \text{on} \quad \partial D_b \end{aligned}$$

with $A = I$, $n = 1$ in $D_b \setminus D$.

Zero Refractive Index (Audibert-Chesnel-Haddar)

For the case $A = I$, choose $\lambda = 0$, $a = 1$ to obtain

$$\Delta(n^{-1}\Delta w) = -k^2 \Delta w, \; w := u - v \in H_0^2(D_p).$$

- If n is real valued, this is an eigenvalue problem for a positive selfadjoint compact operator with eigen-parameter k.
- The eigenvalues k can be computed using multifrequency data.

Eigenvalues and Inverse Scattering Theory

The question if there is a $g \in L^2(\mathbb{S}^2)$ s.th. $\mathcal{F}g = (F - F_{k,\lambda})g = 0$ yield

$$\begin{aligned} \nabla \cdot A\nabla u + k^2 n(x) u &= 0 && \text{in} \quad D_b \\ a\Delta v + k^2 \lambda v &= 0 && \text{in} \quad D_b \\ u &= v && \text{on} \quad \partial D_b \\ \nu \cdot A\nabla u &= \nu \cdot a\nabla v && \text{on} \quad \partial D_b \end{aligned}$$

with $A = I$, $n = 1$ in $D_b \setminus D$.

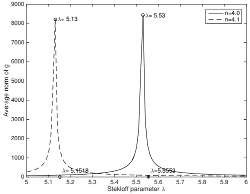
Negative Refractive Index (Audibert-Cakoni-Haddar)

Fix an interrogating frequency k, and choose $-1 \neq a < 0$.

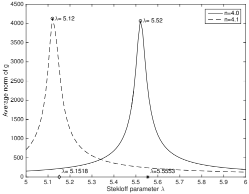
- If A and n are real valued, this is an eigenvalue problem for a selfadjoint compact operator (not necessary sign-definite) with eigen-parameter λ.
- The eigenvalues $\lambda \in \mathbb{C}$ can be computed using data at fixed frequency k.

Steklov Eigenvalues

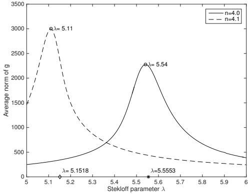
D unit disk, $k = 1$, $n(x)$ changes form 4 to 4.1, 51 direction all around.



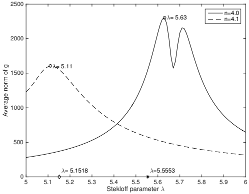
$\epsilon = 0(0\%)$



$\epsilon = 0.01(.57\%)$



$\epsilon = 0.05(2.9\%)$

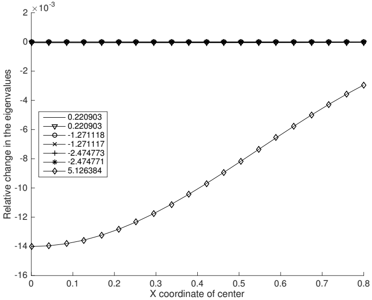


$\epsilon = 0.15(8.6\%)$

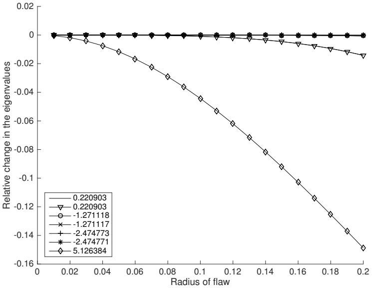
Noise added point-wise. Percentage is the relative ℓ^2 norm

Sensitivity of Eigenvalues: Unit Disk with Flaw

The “flaw” is a circular region of radius r_c centered at $(x_c, 0)$ with $n(x) = 1$ inside the flaw. Noise $\epsilon = 0.01$. Wavenumber $k = 1$.



Changing x_c, $r_c = 0.05$

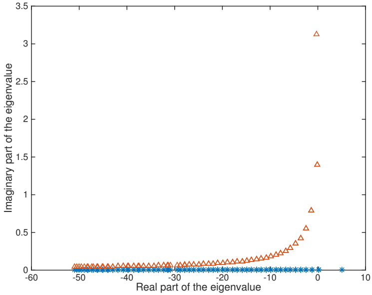


Changing r_c, $x_c = 0.3$.

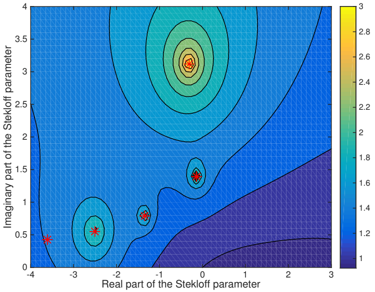
Plot $(\lambda_{j^*}^c - \lambda_j)/|\lambda_j|$, $j = 1, \cdots, 7$ against geometric parameters.

Complex Eigenvalues: Unit Disk $n(x) = 4 + 4i$

Complex eigenvalues can be detected by the same procedure as before but now searching in a region in the complex plane.



Comparison of eigenvalues for $n(x) = 4$ (blue) and $n(x) = 4 + 4i$ (red)



Contours of $\log_{10}(\|g\|)$. Exact Steklov eigenvalues are shown as *.

New Eigenvalues Problems

- We introduced a general framework of modifying the far field operator involving a family of computable scattering problems.
- The injectivity of the modified far field operator relates to the existence of non-trivial solutions to a family of homogeneous problems, hence leading to new eigenvalue problems.
- A broad question is how to design modifications of the data operator, in other words define artificial background scattering problems appropriate to a specific application.

Literature

- Cakoni-Colton-Meng-Monk (2016) Steklov eigenvalues in inverse scattering, *SIAM J. Appl. Math*.
- Audibert-Cakoni-Haddar (2017) New sets of eigenvalues in inverse scattering for inhomogeneous media and their determination from scattering data, *Inverse Problems*.
- Cogar-Colton-Meng-Monk (2017) The modified transmission eigenvalue problem in inverse scattering, *Inverse Problems*.
- Audibert-Chesnel-Haddar (2018) Transmission eigenvalues with artificial background for explicit material index identification, *C. R. Acad. Sci. Paris, Ser. I*.

Trenasmission Eigenvalues for Spherical Media

Consider scattering of $v = j_\ell(k|x|)Y_\ell(\hat{x})$, by a ball B_1 and $n(r)$, $A = I$.

$$u^s(x) := \frac{C(k;n,\ell)}{W(k;n,\ell)} h_\ell^{(1)}(k|x|)Y_\ell(\hat{x}), \quad u^\infty(x) := \frac{C(k;n,\ell)}{W(k;n,\ell)} \frac{1}{k} Y_\ell(\hat{x})$$

$$C(k;n,\ell) = \mathrm{Det}\begin{pmatrix} y_\ell(1;k,n) & j_\ell(k) \\ y_\ell'(1;k,n) & kj_\ell'(k) \end{pmatrix}$$

$$W(k;n,\ell) = \mathrm{Det}\begin{pmatrix} y_\ell(1;k,n) & h_\ell^{(1)}(k) \\ y_\ell'(1;k,n) & kh_\ell^{(1)'}(k) \end{pmatrix}$$

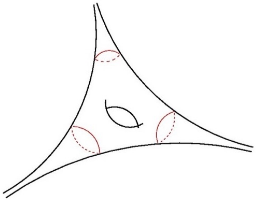
with $y_\ell(r;k,n)$ the solution (regular at $r = 0$) of

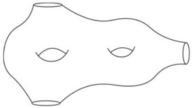
$$y'' + \frac{2}{r} + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2}\right) y = 0.$$

If k is such that $C(k;n,\ell) = 0$ then $v = j_\ell(k|x|)Y_\ell(\hat{x})$ does not scatter.

TE and Riemann Zeta Function

FADDEEV-PAVLOV (1972), LAX-PHILLIPS (1972) established a connection between the harmonic analysis of automorphic functions with respect to the subgroups of $SL_2(\mathbb{R})$, and the scattering theory for non-Euclidean wave equation together with Selberg's pioneering work on spectral theory for Riemann surfaces.





TE and Riemann Zeta Function

$\mathbb{H}$ is the compactified upper half complex plane equipped with the Rimanienn metric. Given the large isometry group of $\mathbb{H}$ a natural way to obtain a hyperbolic surface is by a quotient $\Gamma \setminus \mathbb{H}$ for some group Γ.

More specifically Γ be a group acting discontinuously on $\mathbb{H}$, i.e. the orbit $\Gamma z := \{\gamma z : \gamma \in \Gamma\}$ of any $z \in \mathbb{H}$ has no limit point in $\mathbb{H}$.

Fundamental domain $F := \Gamma \setminus \mathbb{H}$ is a region in $\mathbb{H}$, whose distinct points are not equivalent (different modulo Γ) and such that any orbit of Γ contains points in the closure of F in the $\hat{\mathbb{C}}$ topology.

$f : \mathbb{H} \to \mathbb{C}$ is called automorphic with respect to Γ if

$$f(\gamma z) = f(z) \text{ for all } \gamma \in \Gamma,$$

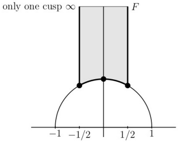
i.e. f lives on $F := \Gamma \setminus \mathbb{H}$.

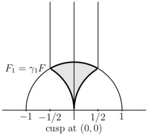
TE and Riemann Zeta Function

- $SL_2(\mathbb{R})$ is the group of 2×2 real matrices of determinant one. $PSL_2(\mathbb{R}) := SL_2(\mathbb{R}) \setminus (\pm I)$ i.e. of fractional linear transformations $g(z) = (az + b)/(cz + d)$.
- A Fuchsian group is a discrete subgroup of $PSL_2(\mathbb{R})$ (here discreteness $\equiv$ acting discontinuously). A Fuchsian group is of first kind if its fundamental domain is of finite volume and is non-compact if the closure of a fundamental domain in $\hat{\mathbb{C}}$ is non-compact.
- A cusp is formed by the two sides of a fundamental domain F meeting at a vertex in $\hat{\mathbb{R}}$ (extended reals) orthogonally to $\hat{\mathbb{R}}$.

HENRYK IWANIEC (2002), Spectral Methods of Automorphic Forms, *AMS, V. 53*.

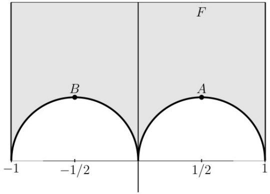
TE and Riemann Zeta Function





Example (Modular group)

$SL_2(\mathbb{Z})$ is the subgroup of 2×2 matrices with integer entries. For example $\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$, which acts according to $z \to -\frac{1}{z}$. One non-equivalent cusp.



Example (Conguent group)

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{matrix} a \equiv d \equiv 1\,(mod\ N) \\ b \equiv c \equiv 0\,(mod\ N) \end{matrix} \right\}$$

Left is the fundamental domain for $\Gamma(2)$. Two non-equivalent cusps.

F is Ford fundamental domain, the image of F under Γ tesselate $\mathbb{H}$

TE and Riemann Zeta Function

$\mathbb{H}$ is a Riemannian manifold with the complete metric

$$ds^2 = y^{-2}(dx^2 + dy^2).$$

Hence the Laplace-Beltrami operator in this case is

$$\Delta_{\mathbb{H}} u := y^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Governing equation of wave propagation on the hyperbolic planeis

$$\Delta_{\mathbb{H}} u + s(1-s)u = 0 \qquad \text{or} \qquad y^2 \Delta u + s(1-s)u = 0.$$

Solutions y^s and y^{1-s}, $s \in \mathbb{C}$ are invariant under $z \mapsto z + 1$.

If $\Im(s) > 0$, $\Im(1-s) < 0$ thus y^s is outgoing (away from the cusp) and y^{1-s} is incoming (toward the cusp).

TE and Riemann Zeta Function

Given fundamental domain $\Gamma \setminus \mathbb{H}$ and $\Re(s) > 1$, and incident wave sent at a cusp **a**, the scattering problem for $u := y^s + u_{scat}$

$$
\begin{aligned}
y^2 \Delta u + s(1-s)u = 0, \qquad & z = (x, y) \in F := \Gamma \setminus \mathbb{H} \\
u(\gamma z) = u(z), \qquad & z \in \partial F, \ \gamma \in \Gamma \\
\frac{\partial u}{\partial \nu}(\gamma z) = \frac{\partial u}{\partial \nu}(z) \qquad & z \in \partial F, \ \gamma \in \Gamma.
\end{aligned}
$$

The solution is given by the Eisenstein series, and satisfies as $y \to \infty$ within the cusp **a** uniformly in $z \in \mathbb{H}$

$$u \sim \delta_{\mathbf{ab}} y^s + \varphi_{\mathbf{ab}}(s) y^{1-s}$$

$\delta_{\mathbf{ab}}$ is Kronecker delta, vanishing when **a**, **b** are inequivalent cusps.

In a similar manner as for the far field operator F here the relative "incoming-outgoing" scattering matrix is

$$\Phi(s) := (\varphi_{\mathbf{ab}}(s)), \quad \text{where } \mathbf{a} \text{ and } \mathbf{b} \text{ run over all cusps,}$$

TE and Riemann Zeta Function

Example (Explicit Relative Scattering Matrix)

- For $F := SL_2(\mathbb{Z}) \setminus \mathbb{H}$

$$\varphi_{\infty\infty} = \pi^{1/2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}$$

- For $F := \Gamma(p) \setminus \mathbb{H}$, p-prime,

$$\Phi(s) = \begin{pmatrix} \varphi_{\infty\infty} & \varphi_{\infty 0} \\ \varphi_{0\infty} & \varphi_{00} \end{pmatrix} = \psi(s) N_p(s)$$

$$\text{where} \quad \psi(s) = \pi^{1/2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}$$

  and $N_p(s)$ is 2×2 matrix with non-vanishing entries.

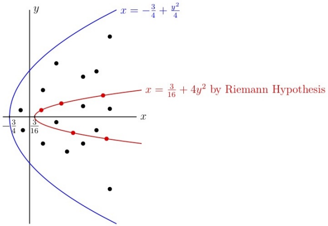
$\zeta(s)$ is the Riemann zeta function.

Transmission Eigenvalues and the Riemann Hypothesis

Definition

A transmission eigenvalue $k := s(1-s)$ for the cusps **a** to **b** is such that $s \in \mathbb{C}$ satisfy

$$\varphi_{\mathbf{ab}}(s) = 0.$$



In the context of both examples, the Riemann hypothesis is equivalent to the statement that all transmission eigenvalues lie on the parabola

$$x = 3/16 + 4y^2$$

except for the trivial eigenvalues $\lambda = 0$ and $\lambda = 1/4$

F. Cakoni and S. Chanillo (2018). Transmission eigenvalues