Explicit Recontruction of Riemann Surface with Given Boundary in Complex Projective Space

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Notations

- $\mathbb{C}P^2$ — complex projective space with homogeneous coordinates $[w_0 : w_1 : w_2]$.
- $\mathbb{C}^2 = \{[w_0 : w_1 : w_2] \in \mathbb{C}P^2 : w_0 \neq 0\}$ with coordinates $z_1 = \frac{w_1}{w_0}$, $z_2 = \frac{w_2}{w_0}$.
- $\mathbb{C}_\xi = \{(z_1, z_2) \in \mathbb{C}^2 : \xi_0 + \xi_1 z_1 + z_2 = 0\}$, $\xi = (\xi_0, \xi_1) \in \mathbb{C}^2$.
- $\mathbb{C}_\infty = \{[0 : w_1 : w_2] \in \mathbb{C}P^2\}$.
- Fubini-Study distance in $\mathbb{C}P^2$:

$$\operatorname{dist}([a_0 : a_1 : a_2], [b_0 : b_1 : b_2]) = \arccos \frac{|a_0 \overline{b}_0 + a_1 \overline{b}_1 + a_2 \overline{b}_2|}{(|a_0|^2 + |a_1|^2 + |a_2|^2)^{1/2} (|b_0|^2 + |b_1|^2 + |b_2|^2)^{1/2}}.$$

Problem

- Let $X \subset \mathbb{C}P^2$ be a compact Riemann surface (or a complex curve) with boundary $\partial X \subset \mathbb{C}^2$. Given ∂X, reconstruct X.

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- Let $X \subset \mathbb{C}P^2$ be a compact Riemann surface (or a complex curve) with boundary $\partial X \subset \mathbb{C}^2$. Given ∂X, reconstruct X.
- **Non-uniqueness**: if $Y \subset \mathbb{C}P^2$ is a closed Riemann surface not intersecting X then $X \cup Y$ is a compact Riemann surface with boundary ∂X. This is the only source of non-uniqueness [2, 5].

Motivation

- Let (X, g) be a connected Riemannian surface with smooth boundary γ.
- Let $N\colon C^\infty(\gamma) \to C^\infty(\gamma)$ be the D2N map: $Nu_0 = \frac{\partial u}{\partial \nu}$ where ν is the unit exterior normal field to γ,

$$\begin{cases} \Delta_g u &= 0, \quad \text{on } X, \\ u|_\gamma &= u_0. \end{cases}$$

- **Inverse D2N problem:** given γ, N find (X, g).

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- **Inverse D2N problem:** given γ, N find (X, g).
- **Non-uniqueness:**
  - if $\tilde{g} = \sigma g$ for $\sigma > 0$ then $\Delta_{\tilde{g}} = \sigma^{-1}\Delta_g$.
  - if $F\colon Y \to X$ is an isometry, then $\Delta_{F^*g} \circ F^* = F^* \circ \Delta_g$.

There are no other sources of non-uniqueness [7].

Motivation

- **Problem:** reconstruct X and its conformal class up to an isometry given γ and values of N on finite number of functions.
- Let T be a nowhere vanishing tangent vector field on γ. Define $L\colon C^\infty(\gamma) \to \mathbb{C}$ by

$$Lu = \frac{1}{2}\big(Nu - iTu\big).$$

- Let u_0, u_1, $u_2 \in C^\infty(\gamma)$ be such that $Lu_0 \neq 0$ and $f(p) = \big(Lu_1(p)/Lu_0(p), Lu_2(p)/Lu_0(p)\big)$ is the embedding of γ in $\mathbb{C}^2$.
- Conformal class of g determines the complex structure J on X.
- $f(\gamma)$ turns out to bound a complex curve $Y \subset \mathbb{C}P^2$ without closed components, f extends to (X, J) defining normalization of Y [6].

$\Longrightarrow$ **New problem:** Find Y given ∂Y.

Plan

Lemma ([3, 4])

Let $X \subset \mathbb{C}P^2$ be a Riemann surface (with or without boundary). Denote

$$X \cap \mathbb{C}_\xi = \big\{\big(h_j(\xi_0,\xi_1), -\xi_0 - \xi_1 h_j(\xi_0,\xi_1)\big) : j = 1,\dots,N_+(\xi)\big\},$$

where $\xi = (\xi_0,\xi_1) \in \mathbb{C}^2$. Then the following equalities are valid for almost all $\xi \in \mathbb{C}^2$:

$$\frac{\partial h_j}{\partial \xi_1}(\xi) = h_j(\xi)\frac{\partial h_j}{\partial \xi_0}(\xi), \quad j = 1,\dots,N_+(\xi).$$

Proof.

- Locally there exists holomorphic $K(z_1, z_2)$, $\nabla K \neq 0$ such that X: $K(z_1, z_2) = 0$.

$$K\left(h_j(\xi_0, \xi_1), -\xi_0 - \xi_1 h_j(\xi_0, \xi_1)\right) \equiv 0.$$

- Hence

$$\frac{\partial K}{\partial z_1}\frac{\partial h_j}{\partial \xi_0} + \frac{\partial K}{\partial z_2}\left(-1 - \xi_1 \frac{\partial h_j}{\partial \xi_0}\right) = 0,$$
$$\frac{\partial K}{\partial z_1}\frac{\partial h_j}{\partial \xi_1} + \frac{\partial K}{\partial z_2}\left(-h_j - \xi_1 \frac{\partial h_j}{\partial \xi_1}\right) = 0.$$

- Hence $\exists \lambda$ such that

$$\frac{\partial h_j}{\partial \xi_0} = \lambda \frac{\partial h_j}{\partial \xi_1}, \quad 1 = \lambda h_j.$$

The Cauchy–Radon transform

Let $X \subset \mathbb{C}P^2$ be a complex curve with boundary $\partial X \subset \mathbb{C}^2$. Define

$$G_k[\partial X](\xi_0, \xi_1) = \frac{1}{2\pi i} \int\limits_{\partial X} \frac{z_1^k d(\xi_0 + \xi_1 z_1 + z_2)}{\xi_0 + \xi_1 z_1 + z_2}, \quad k \geq 0,$$

where $\xi = (\xi_0, \xi_1) \in \mathbb{C}^2$.

Notation: $\mathbb{C} \setminus \pi_2\gamma = \cup_{l=0}^{L}\Omega_k$, Ω_0 is unbounded, $\pi_2(z_1, z_2) = -z_2$.

Theorem ([1, 4])

Fix $l \in \{0, \dots, L\}$. Let $X \subset \mathbb{C}P^2 \setminus [0:1:0]$ be a complex curve with boundary $\partial X \subset \mathbb{C}^2$. Suppose that X intersects $\mathbb{C}_\infty$ and $\mathbb{C}_\xi$ transversally for all $\xi_0 \in \Omega_l$, ξ_1 near $\{\xi_1 = 0\}$. Then

$$G_k[\partial X](\xi) = \sum_{j=1}^{N_+(\xi)} h_j^k(\xi) + P_k(\xi), \quad k \geqslant 0,$$

for $\xi_0 \in \Omega_l$, ξ_1 near $\{\xi_1 = 0\}$, where $P_k(\xi_0, \xi_1)$ is a polynomial with respect to ξ_0 of degree at most k with holomorphic coefficients in ξ_1 near $\{\xi_1 = 0\}$.

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Remark: $G_0[\partial X](\xi) = N_+(\xi) - |X \cap \mathbb{C}_\infty|$.

Polynomials $P_k(\xi)$.

- Polynomials $P_k(\xi)$ depend on behavior of X near $\mathbb{C}_\infty$.
- Introduce $u_0 = \frac{w_0}{w_2}$, $u_1 = \frac{w_1}{w_2}$ and let

$$X \cap \mathbb{C}_\infty = \{(0, v_k) \colon k = 1, \dots, \mu_0\}.$$

If X is transversal to $\mathbb{C}_\infty$ then $\frac{d^k u_1}{du_0^k}$ is correctly defined on X near infinity and $P_0(\xi_0, 0) = -\mu_0$,

$$P_1(\xi_0, 0) = \sum_{k=1}^{\mu_0} \left(v_k \xi_0 - \frac{du_1}{du_0}(0, v_k) \right),$$

$$P_2(\xi_0, 0) = \sum_{k=1}^{\mu_0} \Bigg(-v_k^2 \xi_0^2 + 2 v_k \frac{du_1}{du_0}(0, v_k) \xi_0$$

$$-2 v_k \frac{d^2 u_1}{du_0^2}(0, v_k) - \left(\frac{du_1}{du_0}(0, v_k) \right)^2 \Bigg),$$

$$P_k(\xi_0, 0) = (-1)^{k-1} (v_1^k + \dots + v_{\mu_0}^k) \xi_0^k + \dots$$

Polynomials $P_k(\xi)$.

Example. X: $w_0^2 - w_1 w_2 = 0$, $\mathrm{dist}([w_0 : w_1 : w_2], [0:0:1]) > \varepsilon$.

- $X \cap \mathbb{C}_\infty = \{N\}$ transversally, $N = [0:1:0]$.
- $G_1[\partial X](\xi_0, 0) = \frac{1}{2\pi i} \int_{\partial X} \omega_{\xi_0}$,

$$\omega_{\xi_0} = \frac{z_1 dz_2}{z_2 + \xi_0} = \frac{w_1}{w_0} \frac{dw_2}{w_2 + w_0 \xi_0} - \frac{w_1}{w_0^2} \frac{w_2 dw_0}{w_2 + w_0 \xi}.$$

- $\widetilde{X}$: $w_0^2 - w_1 w_2 = 0$. ω_{ξ_0} is meromorhic on $\widetilde{X}$ with simple poles in N, $M = [0:0:1]$ and $K_{\xi_0} = [1 : \frac{1}{\xi}_0 : \xi_0]$. Residues are $-\frac{1}{\xi_0}$, 0 and $h_1(\xi_0, 0) = \frac{1}{\xi_0}$, hence

$$G_1[\partial X](\xi_0, 0) = h_1(\xi_0, 0) - \frac{1}{\xi_0}.$$

$\Longrightarrow$ Requirement $[0:1:0] \notin X$ is necessary.

Finding of polynomials $P_k(\xi)$.

By virtue of the Darboux lemma we have:

$$\begin{aligned}\frac{\partial P_k}{\partial \xi_1} &= \frac{\partial}{\partial \xi_1}(G_k - h_1^k - \cdots - h_{\mu_0}^k) \\ &= \frac{\partial G_k}{\partial \xi_1} - k(h_1^{k-1}\frac{\partial h_1}{\partial \xi_1} + \cdots + h_{\mu_0}^{k-1}\frac{\partial h_{\mu_0}}{\partial \xi_1}) \\ &= \frac{\partial G_k}{\partial \xi_1} - k(h_1^{k}\frac{\partial h_1}{\partial \xi_0} + \cdots + h_{\mu_0}^{k}\frac{\partial h_{\mu_0}}{\partial \xi_0}) \\ &= \frac{\partial G_k}{\partial \xi_1} - \frac{k}{k+1}\frac{\partial}{\partial \xi_0}(h_1^{k+1} + \cdots + h_{\mu_0}^{k+1}) \\ &= \frac{\partial G_k}{\partial \xi_1} - \frac{k}{k+1}\frac{\partial G_{k+1}}{\partial \xi_0} + \frac{k}{k+1}\frac{\partial P_{k+1}}{\partial \xi_0}.\end{aligned}$$

Finding of polynomials $P_k(\xi)$.

Take into account that

$$P_k(\xi_0, \xi_1) = C_{k1}(\xi_1) + C_{k2}(\xi_1)\xi_0 + \cdots + C_{k,k+1}(\xi_1)\xi_0^k,$$
$$\frac{\partial G_k}{\partial \xi_1}(\xi_0, 0) \to 0, \quad \frac{\partial G_{k+1}}{\partial \xi_0}(\xi_0, 0) \to 0, \quad \xi_0 \to +\infty,$$

and denote

$$c_{ij} = C_{ij}(0), \quad \dot{c}_{ij} = \frac{\partial C_{ij}}{\partial \xi_1}(0),$$

to obtain

$$\dot{c}_{ij} = \frac{ij}{i+1} c_{i+1,j+1}, \quad i = 1, \ldots, \mu_0 - 1; \; j = 1, \ldots, i+1.$$

Finding of polynomials $P_k(\xi)$.

Denote

$$e_k = \sum\nolimits_{1 \le i_1 < \cdots < i_k \le \mu_0} h_{i_1} h_{i_2} \cdots h_{i_k},$$
$$p_k = h_1^k + h_2^k + \cdots + h_{\mu_0}^k, \quad k = 1, \ldots, \mu_0.$$

The Newton identity reads:

$$k e_k = \sum_{i=1}^{k-1} (-1)^{i+1} e_{k-i} p_i + (-1)^{k+1} p_k.$$

Finding of polynomials $P_k(\xi)$.

The Darboux lemma implies:

$$\begin{aligned}\frac{\partial e_{\mu_0}}{\partial \xi_1} &= \frac{\partial}{\partial \xi_1}(h_1 \cdots h_{\mu_0}) \\ &= (h_1^{-1}\frac{\partial h_1}{\partial \xi_1} + \cdots + h_{\mu_0}^{-1}\frac{\partial h_{\mu_0}}{\partial \xi_1}) h_1 \cdots h_{\mu_0} \\ &= (\frac{\partial h_1}{\partial \xi_0} + \cdots + \frac{\partial h_{\mu_0}}{\partial \xi_0}) h_1 \cdots h_{\mu_0} = \frac{\partial p_1}{\partial \xi_0} e_{\mu_0}.\end{aligned}$$

Use the Newton identities to represent $\{e_k\}$ via $\{p_k\}$ and represent $p_k = G_k - P_k$

$\Longrightarrow$ Explicit formulas $\dot{c}_{\mu_0,j} = \dot{c}_{\mu_0,j}(\partial X, \{c_{ij} \colon i \leq \mu_0\})$.

$\Longrightarrow$ Polynomial in c_{ij}, $i \leq \mu_0$ with analytic in $\xi_0 \in \Omega_0$ coefficients identity for finding of c_{ij}: $Q_{\mu_0}[\partial X, \{c_{ij} \colon i \leq \mu_0\}](\xi_0) \equiv 0$.

$\Longrightarrow$ System for finding of $X \cap \mathbb{C}_\infty$:

$$v_1^k + \cdots + v_{\mu_0}^k = (-1)^{k-1} c_{k,k+1}, \quad k = 1, \ldots, \mu_0.$$

Finding of polynomials $P_k(\xi)$.

Examples:

- $\mu_0 = 0$: $c_{ij} \equiv 0$ for all i, j.
- $\mu_0 = 1$: $c_{11}\frac{\partial G_1}{\partial \xi_0} + c_{12}\left(\xi_0 \frac{\partial G_1}{\partial \xi_0} + G_1\right) = G_1 \frac{\partial G_1}{\partial \xi_0} - \frac{\partial G_1}{\partial \xi_1}$.
- $\mu_0 = 2$:

$$
\begin{gathered}
æ^{00}_{10}\left(c_{12}^2 + c_{23}\right) = \frac{\partial G_2}{\partial \xi_1} - 2\frac{\partial G_1}{\partial \xi_1}\left(G_1 - c_{11} - c_{12}\xi_0\right) \\
+\frac{\partial G_1}{\partial \xi_0}\left((G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2\right) \\
+G_1\left(c_{22} + 2c_{23}\xi_0\right) + c_{12}\left(2c_{11}G_1 - G_1^2 + 2c_{12}G_1\xi_0 + G_2\right),
\end{gathered}
$$

$$
æ^{\alpha\beta}_{\gamma\delta} = \frac{1}{2\pi i}\int\limits_{\partial X} z_1^{\alpha} z_2^{\beta}\, dz_1 + z_1^{\gamma} z_2^{\delta}\, dz_2.
$$

Finding of polynomials $P_k(\xi)$.

Examples:

- $\mu_0 = 3$: $\dot{c}_{31} + \dot{c}_{32}\xi_0 + \dot{c}_{33}\xi_0^2 + \dot{c}_{34}\xi_0^3 - \frac{\partial G_3}{\partial \xi_1} =$
  $\frac{3}{4}\left(p_1^2 - p_2\right)\frac{\partial p_2}{\partial \xi_0} - p_1\frac{\partial p_3}{\partial \xi_0} - \frac{1}{2}\left(p_1^3 - 3p_1p_2 + 2p_3\right)\frac{\partial p_1}{\partial \xi_0}$, where

$$\dot{C}_{31}(0) = \frac{1}{2}æ_{10}^{00}\left(3c_{11}c_{12}^2 + 3c_{12}c_{22} + 3c_{11}c_{23} + 2c_{33}\right)$$
$$-\frac{1}{2}æ_{11}^{00}\left(c_{12}^3 + 3c_{12}c_{23} + 2c_{34}\right) - \frac{3}{2}æ_{00}^{11}\left(c_{12}^2 + c_{23}\right)$$
$$-\frac{1}{2}c_{11}^3c_{12} - \frac{3}{2}c_{11}c_{12}c_{21} - \frac{3}{4}c_{11}^2c_{22} - \frac{3}{4}c_{21}c_{22} - c_{12}c_{31} - c_{11}c_{32},$$
$$\dot{C}_{32}(0) = æ_{10}^{00}\left(c_{12}^3 + 3c_{12}c_{23} + 2c_{34}\right) - \frac{3}{2}c_{11}^2c_{12}^2 - \frac{3}{2}c_{12}^2c_{21} - 3c_{11}c_{12}c_{22}$$
$$-\frac{3}{4}c_{22}^2 - \frac{3}{2}c_{11}^2c_{23} - \frac{3}{2}c_{21}c_{23} - 2c_{12}c_{32} - 2c_{11}c_{33},$$
$$\dot{C}_{33}(0) = -\frac{3}{2}c_{11}c_{12}^3 - \frac{9}{4}c_{12}^2c_{22} - \frac{9}{2}c_{11}c_{12}c_{23} - \frac{9}{4}c_{22}c_{23} - 3c_{12}c_{33} - 3c_{11}c_{34},$$
$$\dot{C}_{34}(0) = -\frac{1}{2}c_{12}^4 - 3c_{12}^2c_{23} - \frac{3}{2}c_{23}^2 - 4c_{12}c_{34}.$$

Plan

Algorithm for reconstruction: $\mu_0 = 0$.

Input: ∂X, $\xi_0 \notin \pi_2(\partial X)$, where $\pi_2(z_1, z_2) = -z_2$.
Output: $X \cap \pi_2^{-1}(\xi_0)$.

1. Find $\mu_0 = |X \cap \mathbb{C}_\infty|$. Suppose that $\mu_0 = 0$.
2. Find $N_+(\xi_0, 0) = \frac{1}{2\pi i} \int_{\partial X} \frac{dz_2}{z_2 + \xi_0}$.
3. Compute $G_1[\partial X](\xi_0, 0), \ldots, G_{N_+(\xi_0,0)}[\partial X](\xi_0, 0)$.
4. Find $h_1(\xi_0, 0), \ldots, h_{N_+(\xi_0,0)}(\xi_0, 0)$ from system

$$h_1^k(\xi_0, 0) + \cdots + h_{N_+(\xi_0,0)}^k(\xi_0, 0) = G_k[\partial X](\xi_0, 0),$$
$$k = 1, \ldots, N_+(\xi_0, 0).$$

Solution: $X \cap \pi_2^{-1}(\xi_0) = \left\{(h_j(\xi_0, 0), -\xi_0) : j = 1, \ldots, N_+(\xi_0, 0)\right\}$.

Algorithm for reconstruction: $\mu_0 > 0$.

Input: ∂X, $\xi_0 \notin \pi_2(\partial X)$, where $\pi_2(z_1, z_2) = -z_2$.

Output: $X \cap \pi_2^{-1}(\xi_0)$.

1. Find $\mu_0 = |X \cap \mathbb{C}_\infty|$. Suppose that $\mu_0 > 0$.
2. Choose $R > 0$ such that $\pi_2(\partial X) \subset \mathrm{B}_R(0)$ and $|\xi_0| < R$.
3. Find polynomials $P_1(\xi_0, 0), \ldots, P_{\mu_0}(\xi_0, 0)$, or, equivalently, constants c_{ij} for $i = 1, \ldots, \mu_0$ and $j = 1, \ldots, i+1$.
4. Compute $G_1[\partial X](\eta_0, 0), \ldots, G_{\mu_0}[\partial X](\eta_0, 0)$ for all $\eta_0 \in \mathrm{S}_R(0)$.

Algorithm for reconstruction: $\mu_0 > 0$.

5. Find functions $h_1(\eta_0, 0)$, ..., $h_{\mu_0}(\eta_0, 0)$ for all $\eta_0 \in \mathrm{S}_R(0)$ from system

$$h_1^k(\eta_0, 0) + \cdots + h_{\mu_0}^k(\eta_0, 0) = G_k[\partial X](\eta_0, 0) - P_k(\eta_0, 0),$$

where $k = 1, \ldots, \mu_0$. We have

$$X \cap \pi_2^{-1} S_R(0) = \{ (h_j(\eta_0, 0), -\eta_0) : |\eta_0| = 1,\ j = 1, \ldots, \mu_0 \}.$$

6. Now consider the Riemann surface $X_R = X \cap \pi_2^{-1} B_R(0)$ with known boundary $\partial X_R = \partial X - X \cap \pi_2^{-1} S_R(0)$ and with $X_R \cap \mathbb{C}_\infty = \varnothing$. Since we have $X \cap \pi_2^{-1}(\xi_0) = X_R \cap \pi_2^{-1}(\xi_0)$ it is sufficient to recover $X_R \cap \pi_2^{-1}(\xi_0)$. The problem is reduced to the case $\mu_0 = 0$.

Infinitesimal neighborhood of order one.

- Let $\Omega \subseteq \mathbb{C}$ be a domain with coordinate ξ_0. Let $f_i \in C^\infty(D_i)$, $i = 1, 2$, where D_1, D_2 are open sets in $\Omega \times \mathbb{C}$ (with coordinates ξ_0, ξ_1) containing $\Omega \times 0$. We say that $f_1 \sim f_2$ if there exists open $D \subset D_1 \cap D_2$, $\Omega \times 0 \subset D$ such that $f_1 - f_2 = g_1 g_2$ for some $g_1, g_2 \in C^\infty(D)$ with $g_1|_{\Omega\times 0} = g_2|_{\Omega\times 0} = 0$. Denote the set of classes of equivalence by $C^\infty(\Omega^{(1)})$.
- **Important:** $\frac{\partial}{\partial \xi_1}$, $\frac{\partial}{\partial \bar{\xi}_1}$ are correctly defined on $C^\infty(\Omega^{(1)})$.
- We say that $f \in C^\infty(\Omega^{(1)})$ is analytic if $\frac{\partial f}{\partial \bar{\xi}_0} = 0$, $\frac{\partial f}{\partial \bar{\xi}_1} = 0$.

Infinitesimal neighborhoods: general situation

- Let Y be a complex manifold and $X \subset Y$ be its complex submanifold.
- $C^\infty(Y)$ — sheaf of C^∞ functions on Y,

$$\mathcal{I}_X = \{f \in C^\infty(Y) \colon f|_X = 0\},$$
$$C^\infty(X^{(k)}) := C^\infty(Y)/\mathcal{I}_X^{k+1}, \quad k \geq 1.$$

  The ringed space $\left(X, C^\infty(X^{(k)})\right)$ is called k-th infinitesimal neighborhood of X in Y.
- Clearly, $\overline{\partial} \colon C^\infty(X^{(k)}) \to C^\infty(X^{(k-1)})$. We say that $f \in C^\infty(X^{(k)})$ is analytic if $\overline{\partial} f = 0$ in $C^\infty(X^{(k-1)})$.

Finding of μ_0.

Theorem ([1])

Suppose that analytic functions $\widehat{h}_j \in C^\infty(\Omega_0^{(1)})$, $j = 1, \ldots, N$ *satisfy the system of* $(N+1)$ *equations*

$$\frac{\partial \widehat{h}_j}{\partial \xi_1}(\xi) = \widehat{h}_j(\xi)\frac{\partial \widehat{h}_j}{\partial \xi_0}(\xi), \quad \xi \in \Omega_l^{(0)}, \quad j = 1, \ldots, N, \qquad (1)$$

$$\frac{\partial^2}{\partial \xi_0^2}\left(G_1[\partial X](\xi) - \widehat{h}_1(\xi) - \cdots - \widehat{h}_N(\xi)\right) = 0, \quad \xi \in \Omega_l^{(1)} \qquad (2)$$

with minimal $N \geq 1$. *Then* $N = \mu_0$, $\widehat{h}_j = h_j$.

Finding of μ_0.

Let c_{ij}, $i = 1, \dots, N$; $j = 1, \dots, i+1$ satisfy $Q_N[\partial X, \{c_{ij} \colon i \le N\}](\xi_0) = 0$, $\xi_0 \in \Omega_0$. Define

$$C_{ij}(\xi_1) = c_{ij} + \dot{c}_{ij}(\partial X, \{c_{ij} \colon i \le \mu_0\})\xi_1,$$

and find $h_i(\xi_0, \xi_1)$, $i = 1, \dots, N$ from system

$$h_1^k(\xi_0, \xi_1) + \cdots + h_N^k(\xi_0, \xi_1) = G_k[\partial X](\xi_0, \xi_1) - \sum_{j=1}^{k+1} C_{kj}(\xi_1)\xi_0.$$

where $k = 1, \dots, N$. Then functions $h_i \in C^\infty(\Omega_0^{(1)})$, $i = 1, \dots, N$ are analytic and satisfy (1), (2).

$\Longrightarrow$ $\mu_0 \le N$ by the last theorem.

Remark. If $G_1(\xi_0, \xi_1) = 0$ for $|\xi_0| \ge \mathrm{const}(X)(1 + |\xi_1|)$ and X doesn't contain algebraic subdomains then $\mu_0 = 0$.

Example.

$$X_1 : (z_1 - 1)z_2 = \exp(z_1^2), \quad |z_1| \le 2.$$

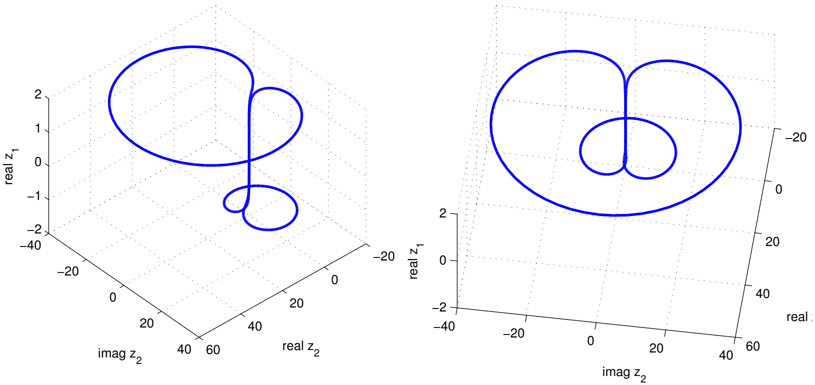

Figure: Input: $\gamma_1 = \partial X_1$.

Example.

- **Remark:** $\mu_0 = 1$.
- Take arbitrary large ξ_0^1 and ξ_0^2 and find $c_{11} = 1$ and $c_{12} = 0$ from

$$c_{11}\frac{\partial G_1}{\partial \xi_0}(\xi_0^k, 0) + c_{12}\Big(\xi_0^k\frac{\partial G_1}{\partial \xi_0}(\xi_0^k, 0) + G_1(\xi_0^k, 0)\Big)$$
$$= G_1(\xi_0^k, 0)\frac{\partial G_1}{\partial \xi_0}(\xi_0^k, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0^k, 0), \quad k = 1, 2.$$

- From $v_1 = c_{12}$ we have $X_1 \cap \mathbb{C}_\infty = [0 : 0 : 1]$.

Example.

- For $|\eta_0| = 60$ find $h_1(\eta_0, 0)$ from

$$h_1(\eta_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{z_1 dz_2}{z_2 + \eta_0} - 1.$$

Put $\Gamma_1 = \{(h_1(\eta_0, 0), -\eta_0) \colon |\eta_0| = 60\}$.

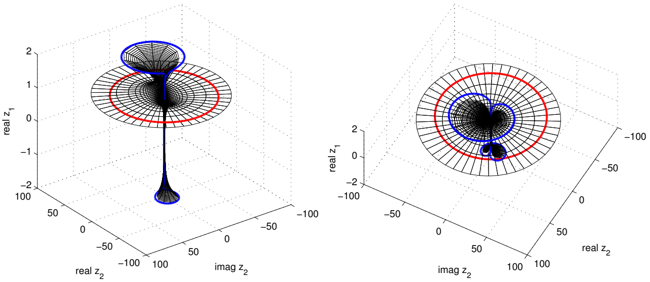

Figure: γ_1 (blue), X_1 (black), Γ_1 (red).

Example.

- Now let $\widetilde{X}_1$: $(z_1, z_2) \in X_1$, $|z_1| \le 2$, $|z_2| \le 60$. Then $\widetilde{X}_1 \cap \mathbb{C}_\infty = \varnothing$.
- For all $\xi_0 \in \pi_2(\widetilde{X}_1 \setminus (\Gamma \cup \gamma))$ find $N_+(\xi_0, 0)$ and $h_j(\xi_0, 0)$ from

$$N_+(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma \cup \Gamma} \frac{dz_2}{\xi_0 + z_2},$$

$$h_1^k(\xi_0, 0) + \cdots + h^k_{N_+(\xi_0,0)} = \frac{1}{2\pi i} \int\limits_{\gamma \cup \Gamma} \frac{z_1^k \, dz_2}{\xi_0 + z_2},$$

$$k = 1, \ldots, N_+(\xi_0, 0).$$

Example.

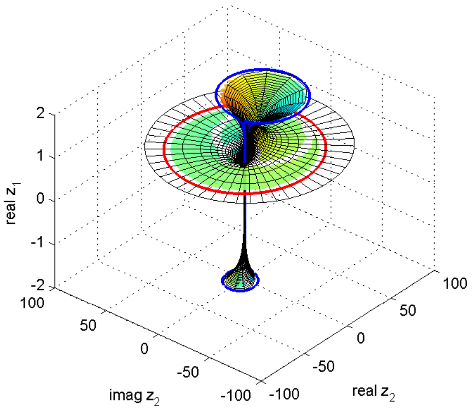

Figure: Reconstructed surface $\widetilde{X}_1$.

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