Darboux–Moutard transformations and their applications

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The Euler–Poisson–Darboux equation

The equation

$$z_{xy} - \frac{n}{x-y} z_x + \frac{m}{x-y} z_y - \frac{p}{(x-y)^2} z = 0$$

after a substitution

$$z = (x-y)^{\alpha} w$$

takes the form

$$w_{xy} - \frac{n'}{x-y} w_x + \frac{m'}{x-y} w_y - \frac{p'}{(x-y)^2} w = 0,$$

where $n' - n = m' - m = \alpha, \quad p' = p + (m+n)\alpha + \alpha(\alpha - 1).$

The Euler exact solution

Let $m' = n' = k$ are integers and $p' = 0$. The the equation is reduced to the form

$$w_{xy} - \frac{k}{x-y} w_x + \frac{k}{x-y} w_y = 0$$

and after the substitution $w = (x-y)^{-k} u$ we derive

$$u_{xy} = \frac{k(1-k)}{(x-y)^2} u.$$

A general solution of this equation is as follows

$$u(x,y) = (x-y)^k \frac{\partial^{2k-2}}{\partial x^{k-1} \partial y^{k-1}} \left(\frac{f(x) + g(y)}{x-y} \right).$$

The Laplace transformation

$$\psi_{xy} + A\psi_x + B\psi_y + C\psi = 0.$$

Replace ψ by

$$\widetilde{\psi} = \left(\frac{\partial}{\partial y} + A\right)\psi.$$

The equation on $\widetilde{\psi}$ has another coefficients: $A \to A - (\log h)_y$, $B \to B$, $C \to C - A_x + B_y - (\log h)_y B$, where $h = AB + A_x - C$. The analogous transformation is obtained after swapping $x \leftrightarrow y$, therewith h is replaced by $k = AB + B_y - C$. Under the first transformation

$$h \to 2h - k - (\log h)_{xy}, k \to h;$$

after the transformations $\psi \to \widetilde{\psi} = f(x, y)\psi$ the values of h and k are preserved. Note that $\widetilde{\psi}_x = -B\widetilde{\psi} + h\psi$, hence $h = 0$ implies the integrability.

The Darboux transformation

$$H = -\frac{d^2}{dx^2} + u(x)$$

— one-dimensional Schrödinger operator.
Let

$$H\omega = 0.$$

The Darboux transformation is defined by a solution ω and maps H into the operator $\widetilde{H}$ and solutions of the equation

$$H\psi = E\psi$$

into solutions $\widetilde{\psi}$ of the equation

$$\widetilde{H}\widetilde{\psi} = E\widetilde{\psi}.$$

Let

$$v = \frac{\omega'}{\omega} = (\log \omega)'.$$

We have

$$H\omega = 0 \quad \Leftrightarrow \quad v' + v^2 = u.$$

Define the potential $\widetilde{u}$ of $\widetilde{H}$ by the formula

$$\widetilde{u} = v^2 - v'.$$

Then to every solution ψ of $H\psi = E\psi$ there corresponds the solution

$$\widetilde{\psi} = -\psi' + v\psi$$

of $\widetilde{H}\widetilde{\psi} = E\widetilde{\psi}$.

The factorization method (Dirac, Schrödinger, Infeld–Hull)

Every solution ω of the equation $H\omega = 0$ defines a factorization of H:

$$H = A^\top A, \quad A = -\frac{d}{dx} + v, \quad A^\top = \frac{d}{dx} + v, \quad v = \frac{\omega'}{\omega}.$$

The Darboux transformation of H consists in swapping $A^\top$ and A:

$$H = A^\top A \longrightarrow \widetilde{H} = AA^\top = -\frac{d^2}{dx^2} + \widetilde{u}(x),$$

and it acts on eigenfunctions as follows:

$$\psi \longrightarrow \widetilde{\psi} = A\psi.$$

The harmonic oscillator

Let $v = ax, a > 0$, then

$$v' = \mathrm{const} = a$$

and

$$AA^\top = 2H - a, \quad A^\top A = 2H + a,$$

where

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + a^2x^2\right)$$

is the harmonic oscillator operator. It follows from the commutation relation $[A^\top, A] = 2a$ that if

$$H\psi = E\psi,$$

then

$$H(A\psi) = (E + a)(A\psi), \quad H(A^\top\psi) = (E - a)(A^\top\psi).$$

Note that

$$(2E - a)(\psi, \psi) = (AA^\top \psi, \psi) = (A^\top \psi, A^\top \psi) \geq 0,$$

which implies

$$E \geq \frac{a}{2}.$$

The equality is attained on a solution of the equation

$$A^\top \psi = \left(\frac{d}{dx} + ax \right) \psi = 0,$$

which is up to a constant multiple equals

$$\psi_1(x) = e^{-\frac{ax^2}{2}}.$$

The basis of eigenfunctions has the form

$$\psi_N = A^{N-1} \psi_1, \quad N = 1, 2, 3, \dots$$

with eigenvalues

$$\frac{a}{2} + (N-1)a.$$

The inverse scattering problem (reflectionless potentials)

The Gelfand–Levitan–Marchenko equation

$$K(x,y) + F(x+y) + \int_x^{\infty} K(x,z)F(z+y)dz = 0,$$

corresponding to reflectionless potential $u(x)$ on the line with one eigenvalue $-\kappa^2, \kappa > 0$ has the form

$$F(x) = \beta e^{-\kappa x}, \quad \beta = \mathrm{const} > 0.$$

We look for a solution in the form $K(x,y) = K(x)e^{-\kappa y}$ for which the equation is written as

$$K(x)\left(1 + \beta \int_x^{\infty} e^{-2\kappa z}dz\right) + \beta e^{-\kappa x} = 0.$$

The desired potential is equal to

$$u(x) = -2\frac{d}{dx}K(x,x) = -2\frac{d^2}{dx^2}\log\left(1+\frac{\beta}{2\kappa}e^{-2\kappa x}\right) = -\frac{2\kappa^2}{\cosh^2\kappa(x-\alpha)},$$

where

$$\alpha = \frac{1}{2\kappa}\log\frac{\beta}{2\kappa}.$$

It si called one-soliton and is obtained from $u_0 = 0$ by the Darboux transformation defined by a solution

$$\omega = e^{\kappa x} + \frac{\beta}{2\kappa}e^{-\kappa x}$$

of $-\psi'' = \kappa^2\psi$.

Iterations of the Darboux transformation give N-soliton potentials.

The Crum method

Consider the problem

$$-\varphi'' + u\varphi = \lambda\varphi, \quad 0 < x < 1,$$

$$\varphi(0) = a\varphi'(0), \qquad \varphi(1) = b\varphi'(1),$$

where $u(x)$ is continuous on $[0,1]$. Denote by

$$\lambda_0 < \lambda_1 < \dots$$

the spectrum of this problem, and by $\varphi_0, \varphi_1, \dots$ — the corresponding eigenfunctions.
Let W_n be the Wronskian of φ_0, ..., φ_{n-1} and W_{ns} be the Wronskian of $\varphi_0, \dots, \varphi_{n-1}, \varphi_s$ $(s \geq n)$.

The Crum theorem:

- *the problem*

$$-\varphi'' + u_n\varphi = \lambda\varphi, 0 < x < 1, \lim_{x\to 0}\varphi(x) = 0, \qquad \lim_{x\to 1}\varphi(x) = 0,$$

where $u_n = u - 2\frac{d^2}{dx^2}\log W_n$ *has the spectrum*

$$\lambda_n < \lambda_{n+1} < \dots$$

and a complete family of corresponding eigenfunctions

$$\varphi_{ns} = \frac{W_{ns}}{W_n}, \quad s \geq n.$$

For $n \geq 2$ *the problem is not regular and*

$$u_n \sim \frac{n(n-1)}{x^2},\ x \to 0; \qquad u_n \sim \frac{n(n-1)}{(1-x)^2},\ x \to 1.$$

The Moutard transformation

Let H be a two-dimensional potential Schrödinger operator and ω be a solution of the equation

$$H\omega = (-\Delta + u)\omega = 0,$$

where Δ is the two-dimensional Laplace operator:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The Moutard transformation of H is defined as

$$\widetilde{H} = -\Delta + u - 2\Delta \log \omega = -\Delta - u + 2\frac{\omega_x^2 + \omega_y^2}{\omega^2}.$$

If ψ satisfies $H\psi = 0$, then the function θ, defined via the system

$$(\omega\theta)_x = -\omega^2 \left(\frac{\psi}{\omega}\right)_y, \quad (\omega\theta)_y = \omega^2 \left(\frac{\psi}{\omega}\right)_x,$$

satisfies $\widetilde{H}\theta = 0$.

REMARKS:

1) the Moutard transformation describes deformations only of "eigenfunctions" with zero "eigenvalue";

2) the action of the Moutard transformation on "eigenfunctions" ψ is multi-valued and is defined modulo multiples of $\frac{1}{\omega}$;

3) if $u = u(x)$ and $\omega = f(x)e^{\kappa y}$, the the Moutard transformation reduces to the Darboux transformation defined by f.

Two-dimensional rational solitons

Let

$$u_0(x,y)=0, \quad \omega_1 = p_1(z)+\overline{p_1(z)}, \quad \omega_2 = p_2(z)+\overline{p_2(z)},$$

where p_1 and p_2 are holomorphic functions of $z = x+iy$.
The double iteration, of the Moutard transformation, defined by ω_1 and ω_2 gives the potential

$$u = -2\Delta \log i[(p_1\bar{p}_2 - p_2\bar{p}_1)+$$

$$+\int ((p_1' p_2 - p_1 p_2')dz + (\bar{p}_1\bar{p}_2' - \bar{p}_1'\bar{p}_2)d\bar{z})].$$

A two-dimensional Schrödinger operator with nontrivial kernel (T.–Tsarev, 2007)

Let

$$\omega_1 = x + 2(x^2 - y^2) + xy, \quad \omega_2 = x + y + \frac{3}{2}(x^2 - y^2) + 5xy.$$

Then the double iteration of the Moutard transformation gives the potential

$$u^* = -\frac{5120(1 + 8x + 2y + 17x^2 + 17y^2)}{(160 + (4 + 16x + 4y)(x^2 + y^2) + 17(x^2 + y^2)^2)^2}$$

and the eigenfunctions with $E = 0$:

$$\psi_1 = \frac{x + 2x^2 + xy - 2y^2}{160 + (4 + 16x + 4y)(x^2 + y^2) + 17(x^2 + y^2)^2}$$

$$\psi_2 = \frac{2x + 2y + 3x^2 + 10xy - 3y^2}{160 + (4 + 16x + 4y)(x^2 + y^2) + 17(x^2 + y^2)^2}.$$

The Novikov–Veselov equation

The Novikov–Veselov equation:

$$U_t = \partial^3 U + \bar{\partial}^3 U + 3\partial(UV) + 3\bar{\partial}(\bar{V} U) = 0,$$

$$\bar{\partial} V = \partial U.$$

The one-dimensional reduction

$$U = U(x), \quad U = V = \bar{V}$$

leads to the Korteweg–de Vries equation

$$U_t = \frac{1}{4} U_{xxx} + 6UU_x.$$

The Novikov–Veselov equation is the compatibility condition for the system

$$\begin{aligned} H\psi = (\partial\bar{\partial} + U)\psi = 0, \\ \partial_t\psi = -A\psi = (\partial^3 + \bar{\partial}^3 + 3V\partial + 3\bar{V}\bar{\partial})\psi \end{aligned} \qquad (1)$$

and is represented by a "Manakov triple" of the form

$$H_t = [H, A] + BH.$$

Equations represented by such triples preserve the "spectrum on the zero energy level" deforming "eigenfunctions" via

$$(\partial_t + A)\psi = 0.$$

The extended Moutard transformation

The system (1) is invariant under the transformation

$$\varphi \to \theta = \frac{i}{\omega} \int (\varphi \partial \omega - \omega \partial \varphi) dz - (\varphi \bar{\partial} \omega - \omega \bar{\partial} \varphi) d\bar{z} +$$

$$+[\varphi \partial^3 \omega - \omega \partial^3 \varphi + \omega \bar{\partial}^3 \varphi - \varphi \bar{\partial}^3 \omega + 2(\partial^2 \varphi \partial \omega - \partial \varphi \partial^2 \omega) -$$

$$-2(\bar{\partial}^2 \varphi \bar{\partial} \omega - \bar{\partial} \varphi \bar{\partial}^2 \omega) + 3V(\varphi \partial \omega - \omega \partial \varphi) + 3\bar{V}(\omega \bar{\partial} \varphi - \varphi \bar{\partial} \omega)] dt,$$

$$U \to U + 2\partial \bar{\partial} \log \omega, \quad V \to V + 2\partial^2 \log \omega.$$

Therefore if two holomorphic in z functions $p_1(z,t)$ and $p_2(z,t)$ satisfy the equation

$$\frac{\partial p}{\partial t} = \frac{\partial^3 p}{\partial z^3},$$

then the double iteration of the extended Moutard transformation defined by them and applied to $U = 0$ gives a solution of the Novikov–Veselov equation:

Blowing up solution of the Novikov–Veselov equation (T.–Tsarev, 2008)

Apply this construction to a pair of polynomials $p_k = p_k(z,0)$:

$$p_1 = i\,z^2, \qquad p_2 = z^2 + (1+i)z$$

and obtain a solution

$$U = \frac{H_1}{H_2},$$

where

$$H_1 = -12\Big(12t(2(x^2+y^2)+x+y) + x^5 - 3x^4y + 2x^4 - 2x^3y^2 -$$

$$-4x^3y - 2x^2y^3 - 60x^2 - 3xy^4 - 4xy^3 - 30x + y^5 + 2y^4 - 60y^2 - 30y\Big),$$

$$H_2 = (3x^4 + 4x^3 + 6x^2y^2 + 3y^4 + 4y^3 + 30 - 12t)^2.$$

It decays as r^{-3}, is nonsingular for $0 \le t < T_* = \frac{29}{12}$ and is singular for $t \ge T_* = \frac{29}{12}$.

Two-dimensional von Neumann–Wigner potentials (R. Novikov–T.–Tsarev, 2014)

First example of the Schrödinger operator with a positive eigenvalue is due to von Neumann and Wigner: a three-dimensional rotation-symmetric nonsingular potential $U(r)$ with the asymptotic

$$U(r) = -\frac{8 \sin 2r}{r} + O(r^{-2}) \quad \text{as } r = |x| \to \infty,\ x \in \mathbb{R}^3.$$

The Schrödinger operators with $U(x) = o(1/|x|)$ as $x \to \infty$ have no positive eigenvalues (Kato).

Let $U = -1$ and

$$\omega_1 = x^2 \cos y - y \sin y + y^2 \sin x + x \cos x,$$

$$\omega_2 = 4(y \cos x + x \sin y), \quad x, y \in \mathbb{R}.$$

Then

$$\widehat{U} = \frac{P}{Q^2}$$

where

$$Q = \omega_1 \theta_1 = -x^4 - y^4 - 4x^2 y \sin x \sin y + \dots,$$

$$P = 16\big(x^6 y \sin x \sin y - x^5 y^2 \cos x \cos y +$$

$$+ x^2 y^5 \sin x \sin y - x y^6 \cos x \cos y\big) + \dots,$$

$$\psi_1 = \frac{\omega_1}{Q}, \qquad \psi_2 = \frac{\omega_2}{Q}$$

$$\widehat{H}\psi = \psi \quad \text{with } \widehat{H} = -\Delta + \widehat{U}.$$

$$\widehat{U} = O\left(\frac{1}{r}\right), \quad \psi_1 = O\left(\frac{1}{r^2}\right), \quad \psi_2 = O\left(\frac{1}{r^3}\right), \quad \text{as } r = \longrightarrow \infty.$$

The Weierstrass (spinor) representation

To every solution $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ of the Dirac equation

$$\mathcal{D}\psi = 0$$

with

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$$

and U real-valued there corresponds a surface in $\mathbb{R}^3$ as follows

$$x^1(P) = \frac{i}{2} \int_{P_0}^{P} \left((\psi_1^2 + \bar{\psi}_2^2)dz - (\bar{\psi}_1^2 + \psi_2^2)d\bar{z}\right) + x^1(P_0),$$

$$x^2(P) = \frac{1}{2} \int_{P_0}^{P} \left((\bar{\psi}_2^2 - \psi_1^2)dz + (\psi_2^2 - \bar{\psi}_1^2)d\bar{z}\right) + x^2(P_0),$$

$$x^3(P) = \int_{P_0}^{P} \left(\psi_1\bar{\psi}_2 dz + \bar{\psi}_1\psi_2\bar{z}\right) + x^3(P_0).$$

The induced metric is equal to

$$ds^2 = e^{2\alpha} dz d\bar{z} = (|\psi_1|^2 + |\psi_2|^2)^2 dz d\bar{z}$$

and the potential of the surface (with a fixed conformal parameter z) is

$$U = \frac{1}{2} e^{\alpha} H$$

where H is the mean curvature.
Every surface admits such a representation in which ψ_1 and ψ_2 are sections of a spinor bundle.

There is a version of the Moutard transformation for Dirac operator $\mathcal{D}$:
if $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ which meets the Dirac equation, then $\psi^* = \begin{pmatrix} -\bar{\psi}_2 \\ \bar{\psi}_1 \end{pmatrix}$ also satisfies it. Hence we have a matrix-valued solution

$$\Psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$$

of the Dirac equation. To every pair Ψ and Φ of such matrix functions we correspond a matrix-valued 1-form ω

$$\omega(\Phi, \Psi) = \Phi^\top \Psi dy - i\Phi^\top \sigma_3 \Psi dx =$$

$$-\frac{i}{2}\left(\Phi^\top \sigma_3 \Psi + \Phi^\top \Psi\right) dz - \frac{i}{2}\left(\Phi^\top \sigma_3 \Psi - \Phi^\top \Psi\right) d\bar{z}$$

and a matrix-valued function

$$S(\Phi,\Psi)(z,\bar{z},t)=\Gamma\int_0^z\omega(\Phi,\Psi),$$

which is defined up to constant matrices from $su(2)$ formed by integration constants. Here $\Gamma=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. Let us also put

$$K(\Psi_0)=\Psi_0 S^{-1}(\Psi_0,\Psi_0)\Gamma\Psi_0^\top\Gamma^{-1}=\begin{pmatrix}iW & a\\ -\bar{a} & -iW\end{pmatrix}.$$

with W real-valued.

Then for every solution

$$\Psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$$

of the Dirac equation the function $\widetilde{\Psi}$ of the form

$$\widetilde{\Psi} = \Psi - \Psi_0 S^{-1}(\Psi_0, \Psi_0) S(\Psi_0, \Psi)$$

satisfies the equation

$$\widetilde{\mathcal{D}}\widetilde{\Psi} = 0$$

for the Dirac operator $\widetilde{\mathcal{D}}$ with potential

$$\widetilde{U} = U + W.$$

We showed that this transformation has a very simple geometrical meaning which is as follows.
By definition, U is the potential of the surface constructed from Ψ_0 and this surface is exactly

$$S = \begin{pmatrix} ix^3 & -x^1 - ix^2 \\ x^1 - ix^2 & -ix^3 \end{pmatrix}$$

(notice that such a surface is defined up to translations which correspond to the integration constanst in the definition of S). Let us take the inverted surface where in these terms the inversion has a simple form

$$S \to S^{-1}$$

and therewith U is mapped into $\widetilde{U}$ which is the potential of the inverted surface S^{-1}.

The Moutard transformation is iextended onto solutions to the modified Novikov–Veselov equation

$$U_t = \left(U_{zzz} + 3U_z V + \frac{3}{2}UV_z\right) + \left(U_{\bar z\bar z\bar z} + 3U_{\bar z}\bar V + \frac{3}{2}U\bar V_{\bar z}\right),$$

$$V_{\bar z} = (U^2)_z,$$

$z = x + iy \in \mathbb{C}$, U is a real-valued function. It takes the form of Manakov's L, A, B–triple:

$$\mathcal{D}_t + [\mathcal{D}, \mathcal{A}] - \mathcal{B}\mathcal{D} = 0,$$

where $\mathcal{D}$ is a two-dimensional Dirac operator,

$$\mathcal{A} = \partial^3 + \bar\partial^3 +$$

$$+3\begin{pmatrix} V & 0 \\ U_z & 0 \end{pmatrix}\partial + 3\begin{pmatrix} 0 & -U_{\bar z} \\ 0 & \bar V \end{pmatrix}\bar\partial + \frac{3}{2}\begin{pmatrix} V_z & 2U\bar V \\ -2UV & \bar V_{\bar z} \end{pmatrix},$$

$$\mathcal{B} = 3\begin{pmatrix} -V & 0 \\ -2U_z & V \end{pmatrix}\partial + 3\begin{pmatrix} \bar V & 2U_{\bar z} \\ 0 & -\bar V \end{pmatrix}\bar\partial + \frac{3}{2}\begin{pmatrix} \bar V_{\bar z} - V_z & 2U_{\bar z\bar z} \\ -2U_{zz} & V_z - \bar V_{\bar z} \end{pmatrix}.$$

A blowing up solution to mNV (T., 2014))

Let us take $\psi_1 = z, \psi_2 = 1$ and $x_0^1 = x_0^3 = 0, x_0^2 = C > 0$. The corresponding surface is the minimal Enneper surface which intersects the x^2-axis only at $(0, C, 0)$. The solution corresponding to the stable solution given by $U = 0$ is as follows: we have the translation of the Enneper surface along X^2-axis with a constant speed and at every moment we invert the surface and construct from it the potential $\widetilde{U}(z, \bar{z}, t)$. Then the moving Enneper surface pass through the origin we obtain a blow up.The resulted solution is as follows

$$\widetilde{U}(x, y, t) = -\frac{3((x^2+y^2+3)(x^2-y^2) - 6x(C-t))}{Q(x, y, t)},$$

$$Q(x, y, t) = (x^2+y^2)^3 + 3(x^4+y^4) + 18x^2y^2 + 9(x^2+y^2) +$$
$$+9(C-t)^2 + (6x^3 - 18xy^2 - 18x)(C-t).$$

A blowing up solution to mNV

It has the following properties:

- *it is infinitely differentiable (and even really-analytical) everywhere outside a single point* $x = y = 0, t = C = \mathrm{const}$ *at which it is not defined and has different finite limit values along the rays* $x/y = \mathrm{const}, t = C$*, going into this point;*
- *its restrictions onto all planes* $t = \mathrm{const}$ *decay as* $O(1/r^2)$*, and, in particular, have finite* L_2*-norms;*
- *the first integral (conservation law)* $\int_{\mathbb{R}^2} \widetilde{U}^2 dx\, dy$ *has the same value equal to* 3π *for all times* $t \neq C$ *and jumps to* 2π *for* $t = C$*.*