Nonlinear Regularizing Effects for Hyperbolic Conservation Laws

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Motivation

Consider Cauchy problem for the scalar conservation law

$$\left\{ \begin{array}{l} \partial_t u + \partial_x f(u) = 0 \,, \quad x \in \mathbf{R} \,, \ t > 0 \\ u\big|_{t=0} = u^{in} \end{array} \right.$$

•In the linear case $f(u) = cu$, the solution is $u(t,x) = u^{in}(x - ct)$; it has exactly the same regularity as the initial data u^{in}.

•If f is strictly convex and if u^{in} is decreasing on some nonempty open interval, the Cauchy problem has a local C^1 solution which loses C^1 regularity after some finite time.

$$\boxed{\text{Nonlinearity} \Rightarrow \text{loss of } C^1 \text{ regularity}}$$

•P. Lax (CPAM, 1954) proved that

a) for each initial data $u^{in} \in L^1(\mathbf{R})$, the Cauchy problem for a scalar conservation law with strictly convex flux f has a unique entropy solution $u \equiv u(t,x)$ defined for all $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$,

b) for each $t > 0$, the map $L^1(\mathbf{R}) \ni u^{in} \mapsto u(t,\cdot) \in L^1(\mathbf{R})$ is compact

$$\boxed{\text{Nonlinearity} \Rightarrow \text{limited regularizing effect on the solution}}$$

Regularizing effect with one entropy condition I

For $f \in C^2(\mathbf{R})$, consider the Cauchy problem

$$\left\{\begin{array}{l} \partial_t u + \partial_x f(u) = 0\,, \quad x \in \mathbf{R}\,,\ t > 0 \\ u\big|_{t=0} = u^{in} \end{array}\right.$$

Entropy condition: with C^2 convex entropy η

$$\partial_t \eta(u) + \partial_x q(u) = -\mu$$

with entropy flux q defined by the formula

$$q(u) = \int^u \eta'(v) f'(v) dv$$

and entropy production rate μ

Regularizing effect with one entropy condition II

Thm 1:
Let $\mathcal{O}$ be a convex open subset of $\mathbf{R}_+^* \times \mathbf{R}$, and $u \in L^\infty(\mathcal{O})$ satisfy the conservation law and one entropy condition

$$\begin{cases} \partial_t u + \partial_x f(u) = 0\,, \\ \partial_t \eta(u) + \partial_x q(u) = -\mu\,, \end{cases} \qquad (t,x) \in \mathcal{O}$$

Assume that μ is a signed Radon measure on $\mathcal{O}$ and that

$$f'' \text{ and } \eta'' \geq a > 0\,.$$

Then $u \in B^{1/4,4}_{\infty,loc}(\mathcal{O})$ i.e. for each $K \Subset \mathcal{O}$

$$\iint_K |u(t+s,x+h) - u(t,x)|^4 dxdt \leq C_K(|s| + |h|)$$

Comparison with known results

•Lax-Oleinik estimate $\partial_x u(t,x) \leq 1/at \Rightarrow u \in BV_{loc}(\mathbf{R}_+^* \times \mathbf{R})$ — special to scalar cons. laws, space dim. 1, $\mu \geq 0$ and $f'' \geq a > 0$)

•Lions-Perthame-Tadmor (1994), and later Perthame-Jabin (2002) prove that $u \in W^{s,p}_{loc}(\mathbf{R}_+^* \times \mathbf{R})$ for $s < \frac{1}{3}$ and $1 \leq \ p < \frac{3}{2}$. Proof is based on kinetic formulation + velocity averaging.

•DeLellis-Westdickenberg (2003): regularizing effect no better than than $B^{1/r,r}_{\infty}$ for $r \geq 3$ or $B^{1/3,r}_{r}$ for $1 \leq r < 3$, using ONLY that the entropy production is a bounded signed Radon measure (not ≥ 0)

$\Rightarrow$ Thm1 gives a regularity estimate in the DeLellis-Westdickenberg optimality class

The case of degenerate convex fluxes I

Consider the case where the flux $f \in C^2(\mathbf{R})$ is convex, but f'' is not uniformly bounded below by a positive constant. More precisely:

$$(DC) \qquad \begin{cases} f''(v) > 0 \text{ for each } v \in \mathbf{R} \setminus \{v_1, \ldots, v_n\} \\ f''(v) \geq a_k |v - v_k|^{2\beta_k} \text{ for } v \text{ near } v_k,\ k = 1, \ldots, n \end{cases}$$

for some $v_1, \ldots, v_n \in \mathbf{R}$ and $a_1, \beta_1, \ldots, a_n, \beta_n > 0$.

The case of degenerate convex fluxes II

Thm 2:
Assume that $f \in C^2(\mathbf{R})$ satisfies (DC). Let $\mathcal{O}$ be a nonempty convex open subset of $\mathbf{R}_+^* \times \mathbf{R}$, and let $u \in L^\infty(\mathcal{O})$ satisfy the conservation law and one entropy condition

$$\begin{cases} \partial_t u + \partial_x f(u) = 0\,, \\ \partial_t \eta(u) + \partial_x q(u) = -\mu\,, \end{cases} \qquad (t,x) \in \mathcal{O}$$

Assume that μ is a signed Radon measure on $\mathcal{O}$ and that

$$\eta'' \geq a > 0\,.$$

Then $u \in B^{1/p,p}_{\infty,loc}(\mathbf{R}_+^* \times \mathbf{R})$, with $p = 2 \max_{1\leq k\leq n} \beta_k + 4$.

Proof of regularizing effect I

Notation: henceforth, we denote

$$\mathbf{D}_{(s,y)}\phi(t,x) := \phi(t-s,x-y) - \phi(t,x)$$

and

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Idea: use a quantitative variant of Tartar's convergence proof of the vanishing viscosity method by compensated compactness

Step I: compensated compactness

Quantitative analogue of the Murat-Tartar div-curl lemma. Set

$$B := \begin{pmatrix} u \\ f(u) \end{pmatrix}, \qquad E := \mathbf{D}_{(s,y)} \begin{pmatrix} \eta(u) \\ q(u) \end{pmatrix}$$

One has $E, B \in L^\infty(\mathcal{O})$ and

$$\begin{cases} \operatorname{div}_{t,x} B = 0\,, & \text{(conservation law)} \\ \operatorname{div}_{t,x} E = -\mathbf{D}_{(s,y)}\mu\,, & \text{(entropy condition)} \end{cases} \quad \text{in } \mathcal{O}$$

In particular, there exists

$$\pi \in \operatorname{Lip}(\mathcal{O})\,, \qquad \text{s.t. } B = J\nabla_{t,x}\pi$$

Step I: compensated compactness (seq.)

Let $\chi \in C_c^\infty(\mathcal{O})$. Applying Green's formula shows that

$$\int_{\mathcal{O}} \chi^2 E \cdot J\mathbf{D}_{(s,y)} B dtdx = -\int_{\mathcal{O}} \chi^2 E \cdot \nabla_{t,x} \mathbf{D}_{(s,y)} \pi dtdx$$
$$= \int_{\mathcal{O}} (\nabla_{t,x} \chi^2) \cdot E\mathbf{D}_{(s,y)} \pi dtdx - \int_{\mathcal{O}} \chi^2 \mathbf{D}_{(s,y)} \pi \mathbf{D}_{(s,y)} \mu$$

Since $\pi \in \mathrm{Lip}(\mathcal{O})$, one has

$$\|\mathbf{D}_{(s,y)} \pi\|_{L^\infty} \leq \mathrm{Lip}(\pi)(|s| + |y|) \leq \|B\|_{L^\infty}(|s| + |y|)$$

Step I: compensated compactness (end)

Therefore, one has the upper bound

$$\int_{\mathcal{O}} \chi^2 E \cdot J\mathbf{D}_{(s,y)} B dtdx \leq C(|s|+|y|)$$

$$\text{with } C = C\left(\|u\|_{L^\infty(\mathcal{O})}, \int_{\text{supp}(\chi)} |\mu|, \chi\right)$$

which leads to an estimate of the form

$$\int_{\mathcal{O}} \chi^2 \left(\mathbf{D}_{(s,y)} u \, \mathbf{D}_{(s,y)} q(u) - \mathbf{D}_{(s,y)} \eta(u) \, \mathbf{D}_{(s,y)} f(u)\right) dtdx$$

$$\leq C(|s|+|y|)$$

Next we give a lower bound for the integrand in the left-hand side.

Step 2: a pointwise inequality

Lemma: For each $v, w \in \mathbf{R}$, assuming $f'', \eta'' \geq a > 0$, and that q is an entropy flux, i.e.

$$q(u) := \int^u \eta'(v) f'(v) dv$$

one has

$$(w-v)(q(w)-q(v)) - (\eta(w)-\eta(v))(f(w)-f(v)) \geq \frac{a^2}{12}|w-v|^4$$

Remark: Tartar noticed that the quantity above is nonnegative for a general convex flux f

Proof: WLOG, assume that $v < w$, and write

$$
\begin{aligned}
(w-v)(q(w)-q(v)) &- (\eta(w)-\eta(v))(f(w)-f(v)) \\
&= \int_v^w d\xi \int_v^w \eta'(\zeta) f'(\zeta) d\zeta - \int_v^w \eta'(\xi) d\xi \int_v^w f'(\zeta) d\zeta \\
&= \int_v^w \int_v^w (\eta'(\zeta) - \eta'(\xi)) f'(\zeta) d\xi d\zeta \\
&= \tfrac{1}{2} \int_v^w \int_v^w (\eta'(\zeta) - \eta'(\xi))(f'(\zeta) - f'(\xi)) d\xi d\zeta \\
&\geq \tfrac{a^2}{2} \int_v^w \int_v^w (\zeta - \xi)^2 d\xi d\zeta
\end{aligned}
$$

$\square$

Step 3: conclusion

The inequality in Step 2 with $v = u(t,x)$ and $w = u(t+s,x+y)$ shows that

$$\mathbf{D}_{(s,y)}u\,\mathbf{D}_{(s,y)}q(u) - \mathbf{D}_{(s,y)}\eta(u)\,\mathbf{D}_{(s,y)}f(u) \geq \tfrac{a^2}{12}|\mathbf{D}_{(s,y)}u|^4$$

Inserting this lower bound in the final estimate obtained in Step 1,

$$\tfrac{a^2}{12}\int_{\mathcal{O}}\chi^2|\mathbf{D}_{(s,y)}u|^4 dtdx \leq C(|s|+|y|)$$

which is the announced $B^{1/4,4}_{\infty,loc}$ estimate for the entropy solution u.

Regularizing effect with all convex entropies

Let $f \in C^2(\mathbf{R})$. Assume that $u \in L^\infty(\mathbf{R}_+ \times \mathbf{R})$ is a weak solution of the Cauchy problem

$$\left\{ \begin{array}{l} \partial_t u + \partial_x f(u) = 0\,, \quad x \in \mathbf{R}\,,\ t > 0 \\ u\big|_{t=0} = u^{in} \end{array} \right.$$

satisfying the entropy condition

$$\partial_t \eta(u) + \partial_x q(u) = -\int_{\mathbf{R}} \eta''(v) dm(\cdot,\cdot,v)$$

for each C^2 convex entropy η with entropy flux q, where m is a bounded signed Radon measure on $\mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}$.

Kinetic formulation

Equivalently, u satisfies the kinetic formulation of the scalar conservation law

$$(K) \qquad \left\{ \begin{array}{l} \partial_t \mathcal{M}_u + f'(v)\partial_x \mathcal{M}_u = \partial_v m\,, \quad x, v \in \mathbf{R}\,,\ t > 0 \\ \mathcal{M}_u\big|_{t=0} = \mathcal{M}_{u^{in}} \end{array} \right.$$

where $\mathcal{M}_u$ is defined by the formula

$$\mathcal{M}_u(v) := \left\{ \begin{array}{ll} +\mathbf{1}_{[0,u]}(v) & \text{if } u \geq 0 \\ -\mathbf{1}_{[u,0]}(v) & \text{if } u < 0 \end{array} \right.$$

Optimal regularizing effect

Thm 3. (F.G. - B. Perthame)
Let $f \in C^2(\mathbf{R})$ satisfy $f'' \geq a > 0$ and assume that $u \equiv u(t,x)$ is an element of $L^\infty(\mathbf{R}_+ \times \mathbf{R})$ that satisfies the kinetic formulation (K) with m a signed Radon measure on $\mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}$. Then

$$u \in B^{1/3,3}_{\infty,loc}(\mathbf{R}_+^* \times \mathbf{R})\,.$$

According to the counterexample of DeLellis-Westdickenberg ('03), the regularizing effect so obtained is optimal

Corollary Under the same assumptions as above, one also has

$$u \in B^{1/p,p}_{\infty,loc}(\mathbf{R}_+ \times \mathbf{R}) \qquad \text{for each } p \geq 3\,.$$

Step 1: Varadhan's interaction identity

Consider the system of PDEs

$$\begin{cases} \partial_t A + \partial_x B = C \\ \partial_t D + \partial_x E = F \end{cases}$$

with compactly supported A, B, C, D, E, F in $\mathbf{R}_+^* \times \mathbf{R}$. Define the interaction (also used by Tartar, Bony, Cercignani, Ha...)

$$I(t) := \iint_{x<y} A(t,x)D(t,y)dxdv$$

has compact support in $\mathbf{R}_+^*$ and therefore

$$\int_0^\infty I'(t)dt = 0$$

Extending A, B, C, D, E, F by 0 for $t < 0$, one finds that

$$\begin{aligned}\iint_{\mathbf{R}\times\mathbf{R}}&(AE-DB)(t,z)dzdt\\ &=-\iint_{\mathbf{R}\times\mathbf{R}} C(t,x)\left(\int_x^\infty D(t,y)dy\right)dxdt\\ &\quad-\iint_{\mathbf{R}\times\mathbf{R}} F(t,y)\left(\int_{-\infty}^y A(t,x)dx\right)dydt\end{aligned}$$

Apply this with

$$\left\{\begin{aligned}A(t,x,v)&:=\chi(t,x)\mathbf{D}_{(0,h)}\mathcal{M}_u(v)\\ D(t,x,w)&:=\chi(t,x)\mathbf{D}_{(0,h)}\mathcal{M}_u(w)\end{aligned}\right.$$

and integrate in $v < w$

Step 2: pointwise lower bound

Lemma.
For each $\bar{u}, u \in \mathbf{R}$, one has

$$\begin{aligned}\Delta(u,\bar{u}) := \iint \mathbf{1}_{\mathbf{R}_+}(v-w)(a'(v)-a'(w)) \\ \times (\mathcal{M}_u(v)-\mathcal{M}_{\bar{u}}(v))(\mathcal{M}_u(w)-\mathcal{M}_{\bar{u}}(w))dvdw \\ \geq \tfrac{1}{6}\alpha|u-\bar{u}|^3\,.\end{aligned}$$

Step 3: conclusion

Define

$$Q := \iiiint_{v<w} (A(t,z,v)E(t,z,w) - D(t,z,w)B(t,z,v))dzdtdvdw$$
$$= \iint \chi(t,z)^2 \Delta(u(t,z), u(t,z+h))dzdt$$

By step 2

$$Q \geq \tfrac{1}{6}a \iint \chi(t,z)^2 |u(t,z+h) - u(t,z)|^3 dzdt$$

while step 1 implies that

$$Q = O(|h|)$$

Presentation of the polytropic Euler system

Unknowns: $\rho \equiv \rho(t,x)$ (density) and $u \equiv u(t,x)$ (velocity field)

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x\left(\rho u^2 + \kappa\rho^\gamma\right) = 0 \end{cases}$$

•Hyperbolic system of conservation laws, characteristic speeds

$$\lambda_+ := u + \theta\rho^\theta > u - \theta\rho^\theta =: \lambda_- \,, \qquad \text{with } \theta = \sqrt{\kappa\gamma} = \tfrac{\gamma-1}{2}$$

•Along C^1 solutions (ρ, u), Euler's system has diagonal form

$$\begin{cases} \partial_t w_+ + \lambda_+ \partial_x w_+ = 0\,, \\ \partial_t w_- + \lambda_- \partial_x w_- = 0\,, \end{cases}$$

where $w_\pm \equiv w_\pm(\rho, u)$ are the Riemann invariants

$$w_+ := u + \rho^\theta > u - \rho^\theta =: w_-$$

DiPerna's existence result

•R. DiPerna (1983): for each initial data (ρ^{in}, u^{in}) satisfying

$$(\rho^{in} - \bar{\rho}, u^{in}) \in C_c^2(\mathbf{R}) \text{ and } \rho^{in} > 0$$

there exists an entropy solution (ρ, u) of polytropic Euler s.t.

$$0 \leq \rho \leq \rho^* = \sup_{x \in \mathbf{R}} \left(\tfrac{1}{2}(w_+(\rho^{in}, u^{in}) - w_-(\rho^{in}, u^{in})\right)^{1/\theta}$$

$$\inf_{x \in \mathbf{R}} w_-(\rho^{in}, u^{in}) =: u_* \leq u \leq u^* := \sup_{x \in \mathbf{R}} w_+(\rho^{in}, u^{in})$$

•DiPerna's argument applies to $\gamma = 1 + \frac{2}{2n+1}$, for each $n \geq 1$;
•Improved by G.Q. Chen, by P.-L. Lions, B. Perthame, E. Tadmor, and P. Souganidis by using a kinetic formulation of Euler's system

Admissible solutions

Def: Let $\mathcal{O} \subset \mathbf{R}_+^* \times \mathbf{R}$ open. A weak solution $U = (\rho, \rho u)$ s.t.

$$0 < \rho_* \leq \rho \leq \rho^* \quad \text{and} \quad u_* \leq u \leq u^* \text{ for } (t,x) \in \mathcal{O}$$

is called an <u>admissible solution on $\mathcal{O}$</u> iff for each entropy ϕ,

$$\partial_t \phi(U) + \partial_x \psi(U) = -\mu[\phi]$$

is a Radon measure such that

$$\|\mu[\phi]\|_{\mathcal{M}(\mathcal{O})} \leq C \|D^2 \phi\|_{L^\infty([\rho_*, \rho^*] \times [u_*, u^*])}$$

•<u>Example</u>: any DiPerna weak solution whose viscous approximation $U_\epsilon = (\rho_\epsilon, \rho_\epsilon u_\epsilon)$ satisfies the uniform lower bound $\rho_\epsilon \geq \rho_* > 0$ on $\mathcal{O}$ for each $\epsilon > 0$ is admissible on $\mathcal{O}$.

•Existence of admissible solutions in the large?

Regularizing effect for polytropic Euler

Thm 4: Assume that $\gamma \in (1,3)$ and let $\mathcal{O} \subset \mathbf{R}_+^* \times \mathbf{R}$ be open. Any admissible solution of Euler's system on $\mathcal{O}$ satisfies

$$\iint_{\mathcal{O}} |(\rho,u)(t+s,x+y)-(\rho,u)(t,x)|^2 dxdt \leq \frac{\text{Const.}}{|\ln(|s|+|y|)|^2}$$

whenever $|s|+|y|<\frac{1}{2}$.

Rmk: For $\gamma = 3$, the same method shows that $(\rho,u) \in B^{1/4,4}_{\infty,loc}(\mathcal{O})$

Previous results

•For $\gamma = 3$, by using the kinetic formulation and velocity averaging, one has (Lions-Perthame-Tadmor 1994, and Jabin-Perthame 2002)

$$\rho, \rho u \in W^{s,p}_{loc}(\mathbf{R}_+ \times \mathbf{R}) \text{ for all } s < \tfrac{1}{4}\,,\ 1 \leq p \leq \tfrac{8}{5}$$

•The kinetic formulation for $\gamma \in (1,3)$ is of the form

$$\partial_t \chi + \partial_x[(\theta\xi + (1-\theta)u(t,x))\chi] = \partial_{\xi\xi} m \qquad \text{with } m \geq 0$$

$$\text{and } \chi = [(w_+ - \xi)(\xi - w_-)]_+^\alpha \qquad \text{for } \alpha = \tfrac{3-\gamma}{2(\gamma-1)}$$

The presence of $u(t,x)$ in the advection velocity — only bounded, not smooth — forbids using classical velocity averaging lemmas (Agoshkov, G-Lions-Perthame-Sentis, DiPerna-Lions-Meyer, . . .)

Remarks on the proof

We use two important features of Euler's polytropic system.

•Fact no.1: with $\theta = \frac{\gamma-1}{2}$,

$$\begin{pmatrix} \lambda_+ \\ \lambda_- \end{pmatrix} = \mathcal{A} \begin{pmatrix} w_+ \\ w_- \end{pmatrix} \text{ with } \mathcal{A} = \frac{1}{2} \begin{pmatrix} 1+\theta & 1-\theta \\ 1-\theta & 1+\theta \end{pmatrix}$$

and for $\gamma \in (1,3)$ one has $\theta \in (0,1)$, leading to the coercivity estimate

$$\begin{pmatrix} \sinh(a) \\ \sinh(b) \end{pmatrix} \cdot \mathcal{A} \begin{pmatrix} a \\ b \end{pmatrix} \geq \theta\,(a\sinh(a) + b\sinh(b)) + (1-\theta) \times (\geq 0)$$

This replaces the pointwise inequality (Step 2) in the scalar case

Remarks on the proof II

•Fact no.2: Euler's polytropic system satisfies the relation

$$\partial_+\left(\frac{\partial_-\lambda_+}{\lambda_+-\lambda_-}\right)=\partial_-\left(\frac{\partial_+\lambda_-}{\lambda_--\lambda_+}\right)$$

Hence there exists a function $\Lambda\equiv\Lambda(w_+,w_-)$ such that

$$(\partial_+\Lambda,\partial_-\Lambda)=\left(\frac{\partial_+\lambda_-}{\lambda_--\lambda_+},\frac{\partial_-\lambda_+}{\lambda_+-\lambda_-}\right)$$

so that one can take

$$A_0^+(w_+,w_-)=A_0^-(w_+,w_-)=e^{\Lambda(w_+,w_-)}=(w_+-w_-)^{\frac{1-\theta}{2\theta}}$$

in Lax entropies given in Riemann invariant coordinates by

$$\phi_\pm(w,k)=e^{kw_\pm}\left(A_0^\pm(w)+\frac{A_1^\pm(w)}{k}+\dots\right)\,,\quad k\to\pm\infty$$

The case $\gamma = 3$

The kinetic formulation of the isentropic Euler for $\gamma = 3$ is

$$\partial_t \chi + \xi \partial_x \chi = \partial_{\xi\xi} m \quad \text{with } m \geq 0$$

with

$$\begin{cases} \chi(t,x,\xi) = \mathbf{1}_{[w_-(t,x), w_+(t,x)]}(\xi) \\ w_\pm(t,x) = u(t,x) \pm \frac{1}{2}\rho(t,x) \end{cases}$$

Pbm: applying the interaction identity requires a lower bound of

$$\iint \phi(\xi-\eta)(\xi-\eta)^2 \mathbf{D}_{s,y}\chi(t,x,\xi)\mathbf{D}_{s,y}\chi(t,x,\eta)d\xi d\eta$$

for $\phi \in C^\infty(\mathbf{R})$ such that $\mathbf{1}_{(-\epsilon,\epsilon)} \leq \phi \leq \mathbf{1}_{(-2\epsilon,2\epsilon)}$

The case $\gamma = 3$ (sequel)

For (t, x) and (s, y) given, there are 3 cases

$$(1) \quad w_-(t-s, x-y) < w_-(t, x) < w_+(t, x) < w_+(t-s, x-y)$$

$$(2) \quad w_-(t, x) < w_-(t-s, x-s) < w_+(t, x) < w_+(t-s, x-y)$$

$$(3) \quad w_-(t, x) < w_+(t, x) < w_-(t-s, x-y) < w_+(t-s, x-y)$$

With $\{a, b, c, d\} = \{w_\pm(t, x), w_\pm(t+s, x+y)\}$ and $a < b < c < d$

$$\pm \mathbf{D}_{s,y}\chi(t, x, \xi) = \begin{cases} \mathbf{1}_{[c,d]}(\xi) + \mathbf{1}_{[a,b]}(\xi) & \text{in case (1)} \\ \mathbf{1}_{[c,d]}(\xi) - \mathbf{1}_{[a,b]}(\xi) & \text{in case (2-3)} \end{cases}$$

The case $\gamma = 3$ (sequel)

Lemma: Let $\epsilon > 0$ and assume that $\mathbf{1}_{(-\epsilon,\epsilon)} \le \phi \le \mathbf{1}_{(-2\epsilon,2\epsilon)}$. Then

$$\iint \phi(\xi-\eta)(\xi-\eta)^2(\mathbf{1}_{[c,d]}(\xi)-\mathbf{1}_{[a,b]}(\xi))(\mathbf{1}_{[c,d]}(\eta)-\mathbf{1}_{[a,b]}(\eta))d\xi d\eta \ge \tfrac{1}{6}((\epsilon\wedge(d-c))^4+(\epsilon\wedge(b-a))^4)$$

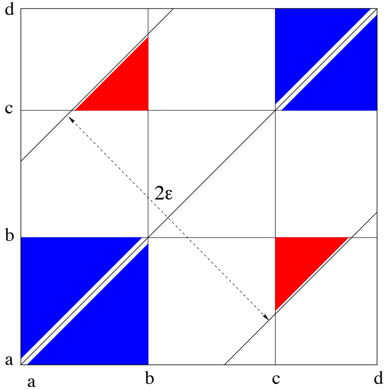
whenever

$$d-b > 11\epsilon \quad \text{and } c-a > 11\epsilon\,.$$

Rmk: the truncation ϕ and the lower bound on $d-b$ and $c-a$ are essential; in general

$$\iint (\xi-\eta)^2(\mathbf{1}_{[c,d]}(\xi)-\mathbf{1}_{[a,b]}(\xi))(\mathbf{1}_{[c,d]}(\eta)-\mathbf{1}_{[a,b]}(\eta))d\xi d\eta$$

may take negative values



Sign of $(\mathbf{1}_{[c,d]}(\xi) - \mathbf{1}_{[a,b]}(\xi))(\mathbf{1}_{[c,d]}(\eta) - \mathbf{1}_{[a,b]}(\eta))$ for $|\xi - \eta| < 2\epsilon$:
blue=positive, red=negative, white=0

The case $\gamma = 3$ (sequel)

Therefore

$$\rho(t,x) > 11\epsilon \quad \text{and } \rho(t-s,x-y) > 11\epsilon$$

imply that

$$\iint \phi(\xi-\eta)(\xi-\eta)^2 \mathbf{D}_{s,y}\chi(t,x,\xi)\mathbf{D}_{s,y}\chi(t,x,\eta)d\xi d\eta$$
$$\geq \tfrac{1}{6}((\epsilon \wedge \mathbf{D}_{s,y}w_+)^4 + (\epsilon \wedge \mathbf{D}_{s,y}w_-)^4)$$

provided that

$$\mathbf{1}_{(-\epsilon,\epsilon)} \leq \phi \leq \mathbf{1}_{(-2\epsilon,2\epsilon)} \, .$$

The case $\gamma = 3$ (end)

Thm 5: Assume that $\gamma = 3$ and let $\mathcal{O} \subset \mathbf{R}_+^* \times \mathbf{R}$ be open. Any weak entropy solution of Euler's system on $\mathcal{O}$ such that

$$\inf_{(t,x)\in\mathcal{O}} \rho(t,x) > 0$$

satisfies

$$\rho, u \in B^{1/4,4}_{\infty,loc}(\mathcal{O})$$

i.e. for each $K \Subset \mathcal{O}$

$$\iint_K |u(t+s,x+h) - u(t,x)|^4 dxdt \le C_K(|s| + |h|)$$

Final remarks

•the Tartar-DiPerna compensated compactness method, which has been used to prove the existence of weak solutions with large data, can also give new regularity estimates for hyperbolic (systems of) conservation laws

•there remain many open questions in this direction:

(a) extensions to scalar equations in space dimension > 1

(b) handle solutions of polytropic Euler without excluding cavitation

(c) handle a more general class of systems than only the polytropic Euler system with power pressure law