Scattering for multipoint potentials

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Exactly solvable examples

For testing direct and inverse scattering algorithms it is very important to have exactly solvable models. However, in contrast with the one-dimensional case, it is very difficult to present such exactly solvable models in multidimensions.

In the present talk we consider the model of point scatterers, which goes back to the works of Bethe and Peierls (1935), Fermi (1936), Zeldovich (1960), Berezin and Faddeev (1961) and which is exactly solvable.

We present a short review of old and recent results in this area, including works of Grinevich and Novikov (2012), (2013).

Scatting problem

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d = 2, 3, \tag{1}$$

where $v(x)$ is a real-valued sufficiently regular function on $\mathbb{R}^d$ with sufficient decay at infinity.

The classical scattering eigenfunctions ψ^+ for (1) are specified by the following asymptotics as $|x| \to \infty$:

$$\psi^+ = e^{ikx} - i\pi\sqrt{2\pi}e^{-\frac{i\pi}{4}} f\left(k, |k|\frac{x}{|x|}\right)\frac{e^{i|k||x|}}{\sqrt{|k||x|}} + o\left(\frac{1}{\sqrt{|x|}}\right), \qquad d = 2, \tag{2}$$

$$\psi^+ = e^{ikx} - 2\pi^2 f\left(k, |k|\frac{x}{|x|}\right)\frac{e^{i|k||x|}}{|x|} + o\left(\frac{1}{|x|}\right), \qquad d = 3, \tag{3}$$

$x \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, $k^2 = E > 0$, where a priori unknown function $f(k, l)$, $k, l \in \mathbb{R}^d$, $k^2 = l^2 = E$, arising in (2), (3), is the classical scattering amplitude for (1).

Scatting problem

In addition, we consider the Faddeev eigenfunctions ψ for (1) specified by

$$\psi = e^{ikx}\left(1 + o(1)\right) \quad \text{as} \quad |x| \to \infty, \tag{4}$$

$x \in \mathbb{R}^d$, $k \in \mathbb{C}^d$, $\mathrm{Im}\, k \neq 0$, $k^2 = k_1^2 + \ldots + k_d^2 = E$.

The generalized scattering data arise in more precise version of the expansion (4). The Faddeev eigenfunctions have very rich analytical properties and are quite important for inverse scattering.

The history of these eigenfunctions goes back to Faddeev (1965). These eigenfunctions are also known as multidimensional analogs of Jost eigenfunctions and as complex geometrical optics solutions.

Point-like potentials

We consider the Schrödinger equation (1), where $v(x)$ is a finite sum of point potentials in two or three dimensions. We write these potentials as:

$$v(x) = \sum_{j=1}^{n} \varepsilon_j \delta(x - z_j), \tag{5}$$

but the precise sense of these potentials will be specified below.

Strictly speaking, $\delta(x)$ is not the standard Dirac delta-function (in the physical literature the term renormalized δ-function is used).

Historically, such single-point potentials ($n = 1$) arose as a model for proton-neutron interaction – Bethe and Peierls (1935) .

A proper mathematical formalization of these potentials goes back to Berezin and Faddeev (1961).

Such scatterres arise in acoustics as well – Burov at al (2001).

By analogy with Berezin-Faddeev (1961), we understand the multipoint potentials $v(x)$ from (5) as a limit for $N \to +\infty$ of non-local potential operators

$$V_N(x,x') = \sum_{j=1}^{n} \varepsilon_j(N) u_{j,N}(x) u_{j,N}(x'), \quad u_{j,N}(x) = u_{0,N}(x - z_j), \qquad (6)$$

$$(V_N \circ \mu)(x) = \sum_{j=1}^{n} \varepsilon_j(N) \int\limits_{\mathbb{R}^d} u_{j,N}(x) u_{j,N}(x') \mu(x') dx', \qquad (7)$$

$$u_{0,N}(x) = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \hat{u}_{0,N}(\xi) e^{i\xi x} d\xi, \quad \hat{u}_{0,N}(\xi) = \begin{cases} 1 & |\xi| \le N, \\ 0 & |\xi| > N, \end{cases} \qquad (8)$$

$x, x', z_j \in \mathbb{R}^d$, $z_m \neq z_j$ for $m \neq j$, $\varepsilon_j(N)$ are normalizing constant specified by.

Point-like potentials

$$\varepsilon_j(N) = \alpha_j \left(1 - \frac{\alpha_j N}{2\pi^2}\right)^{-1}, \quad \alpha_j \in \mathbb{R}, \quad j = 1, \ldots, n, \quad \text{for} \quad d = 3, \tag{9}$$

$$\varepsilon_j(N) = \alpha_j \left(1 - \frac{\alpha_j}{2\pi} \ln(N)\right)^{-1}, \quad \alpha_j \in \mathbb{R}, \quad j = 1, \ldots, n, \quad \text{for} \quad d = 2. \tag{10}$$

Here we use the fact that for non-local separable potentials of the form (9) the Schrödiger equation (1) is exactly solvable.

In addition, the existence of proper limits of scattering functions ψ_N, ψ_N^+ when $N \to +\infty$ implies formulas (9), (10).

Theorem 1 (see Albeverio, Gesztesy, Høegh-Krohn, Holden (1988) for a similar result).
For ψ^+ the following formula holds:

$$\psi^+(x,k) = e^{ikx}\left[1 + \sum_{j=1}^{n} c_j^+(k) g^+(x - z_j, k)\right], \tag{11}$$

$$g^+(x,k) = -\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}}{\xi^2 + 2(k + i0k)\xi} d\xi, \quad x \in \mathbb{R}^d \ \ k \in \mathbb{R}^d \backslash 0, \tag{12}$$

$c^+(k) = (c_1^+(k), \ldots, c_n^+(k))$ is the solution of the following linear equation:

$$\tilde{A}^+(k) c^+(k) = \tilde{b}^+(k), \tag{13}$$

the $n \times n$ matrix $\tilde{A}^+(k)$ and the n-component vector $\tilde{b}^+(k)$ are defined as follows.

For $d = 3$:

$$\tilde{A}^+_{m,j}(k) = \begin{cases} 1 & m = j \\ -\alpha_m \left(1 + \frac{i\alpha_m}{4\pi}|k|\right)^{-1} g^+(z_m - z_j, k), & m \neq j, \end{cases} \tag{14}$$

$$\tilde{b}^+_m(k) = \alpha_m \left(1 + \frac{i\alpha_m}{4\pi}|k|\right)^{-1}; \tag{15}$$

For $d = 2$:

$$\tilde{A}^+_{m,j}(k) = \begin{cases} 1 & m = j \\ -\alpha_m \left(1 + \frac{\alpha_m}{4\pi}(\pi i - 2\ln|k|)\right)^{-1} g^+(z_m - z_j, k), & m \neq j, \end{cases} \tag{16}$$

$$\tilde{b}^+_m(k) = \alpha_m \left(1 + \frac{\alpha_m}{4\pi}(\pi i - 2\ln|k|)\right)^{-1}. \tag{17}$$

Classical eigenfunctions

In addition, for the scattering amplitude f the following formula holds:

$$f(k,l) = \frac{1}{(2\pi)^d} \sum_{j=1}^{n} c_j^+(k) e^{i(k-l)z_j}, \tag{18}$$

$$k, l \in \mathbb{R}^d, \quad k^2 = l^2 = E \in \mathbb{R},$$

where $c_j^+(k)$ are the same as in (11), (13).
In addition, the classical scattering functions ψ^+ and f for $d = 3$ are expressed in terms of elementary functions via (11)-(18) and the formula

$$g^+(x,k) = -\frac{1}{4\pi} \frac{e^{-ikx} e^{i|k||x|}}{|x|} \tag{19}$$

for $d = 3$.

Faddeev eigenfunctions

Theorem 2 (Grinevich, Novikov (2013)).
For the Faddeev (complex geometrical optics) eigenfunctions the following formula holds:

$$\psi(x,k) = e^{ikx}\left[1 + \sum_{j=1}^{n} c_j(k) g(x - z_j, k)\right], \quad k \in \mathbb{C}^d \backslash \mathbb{R}^d, \quad k^2 = E \in \mathbb{R}, \tag{20}$$

$$g(x,k) = -\frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}}{\xi^2 + 2k\xi} d\xi,$$

where $c(k) = (c_1(k), \ldots, c_n(k))$ is the solution of the following linear equation:

$$\tilde{A}(k)c(k) = \tilde{b}(k), \tag{21}$$

the $n \times n$ matrix $\tilde{A}^+(k)$ and the n-component vector $\tilde{b}^+(k)$ are defined as follows.

Faddeev eigenfunctions

For $d = 3$:

$$\tilde{A}_{m,j}(k) = \begin{cases} 1, & m = j \\ -\alpha_m \left(1 - \frac{\alpha_m}{4\pi}|\operatorname{Im} k|\right)^{-1} g(z_m - z_j, k), & m \neq j, \end{cases} \tag{22}$$

$$\tilde{b}_m(k) = \alpha_m \left(1 - \frac{\alpha_m}{4\pi}|\operatorname{Im} k|\right)^{-1}. \tag{23}$$

For $d = 2$:

$$\tilde{A}_{m,j}(k) = \begin{cases} 1, & m = j \\ -\alpha_m \left(1 - \frac{\alpha_m}{2\pi}(\ln(|\operatorname{Re} k| + |\operatorname{Im} k|)\right)^{-1} g(z_m - z_j, k), & m \neq j, \end{cases} \tag{24}$$

$$\tilde{b}_m(k) = \alpha_m \left(1 - \frac{\alpha_m}{2\pi}(\ln(|\operatorname{Re} k| + |\operatorname{Im} k|)\right)^{-1}. \tag{25}$$

In addition, the Faddeev generalized scattering amplitude for the limiting potential $v = \lim_{N\to+\infty} V_N$, associated with the limiting eigenfunction ψ is given by:

$$h(k,l) = \frac{1}{(2\pi)^d}\sum_{j=1}^{n} c_j(k)e^{i(k-l)z_j}, \tag{26}$$

$$k,l \in \mathbb{C}^3, \ \ \mathrm{Im}\, k = \mathrm{Im}\, l \neq 0, \ \ k^2 = l^2 = E \in \mathbb{R},$$

where $c_j(k)$ are the same as in (20), (21).

Comments

Formulas of Theorem 1 can be used, in particular, for testing different monochromatic inverse scattering algorithms.

In addition, formulas of Theorem 2, can be used, in particular, for testing monochromatic inverse scattering algorithms based on the properties of the Faddeev functions ψ and h.

Formulas of Theorems 1 and 2 can be used also for checking different conjectures concerning the properties of scattering functions ψ^+, f and ψ, h in dimensions $d = 2, 3$, in general.

In particular, formulas of Theorem 2 are quite useful for studies of the poles of the Faddeev functions ψ and h.

Fixed energy level

For example, at fixed $E > 0$, for $d = 2$, we have

$$\psi = \psi(x, k), \text{ where } x \in \mathbb{R}^2, \ k \in \Sigma_E \backslash S^1_{\sqrt{E}},$$

$$\Sigma_E = \{k = (k_1, k_2) \in \mathbb{C}^2 : k^2 = k_1^2 + k_2^2 = E\},$$
$$S^1_r = \{k = (k_1, k_2) \in \mathbb{R}^2 : k^2 = k_1^2 + k_2^2 = r^2\}.$$

Note that $\Sigma_E \approx \mathbb{C}\backslash 0$, $S^1_r \approx T = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

$$k \in \Sigma_E \quad \Rightarrow \lambda(k) = \frac{k_1 + ik_2}{\sqrt{E}} \in \mathbb{C}\backslash 0, \qquad k \in S^1_{\sqrt{E}} \Rightarrow \quad \lambda(k) \in T,$$
$$\lambda \in \mathbb{C}\backslash 0 \quad \Rightarrow k(\lambda) = (k_1(\lambda), k_2(\lambda)) \in \Sigma_E, \qquad \lambda(k) \in T \Rightarrow \quad k \in S^1_{\sqrt{E}},$$

$$k_1(\lambda) = \left(\frac{1}{\lambda} + \lambda\right)\frac{\sqrt{E}}{2}, \quad k_2(\lambda) = \left(\frac{1}{\lambda} - \lambda\right)\frac{i\sqrt{E}}{2}.$$

Poles of Faddeev's (complex geometrical optics) eigenfucntions ψ for $E > 0$, $d = 2$

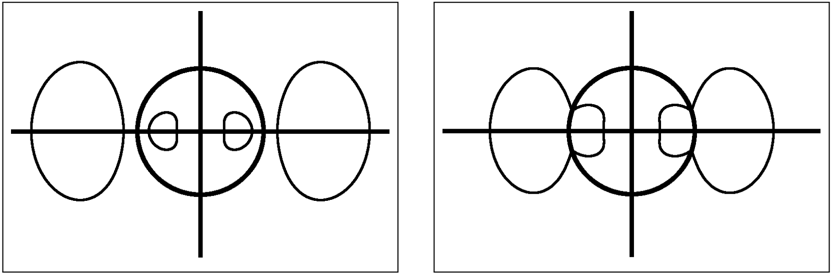

Fig. 1

$E = 4, \;\; z_2 - z_1 = (0.5\,,\,0),$
$\alpha_1 = 5, \;\; \alpha_2 = 6$

Fig. 2

$E = 6, \;\; z_2 - z_1 = (0.5\,,\,0),$
$\alpha_1 = 5, \;\; \alpha_2 = 6$

Figure 2 replies the Faddeev's question of 1965-1974. We see that the spectral singularities intersect the real space T.

Poles of Faddeev's (complex geometrical optics) eigenfucntions ψ for $E > 0$, $d = 2$

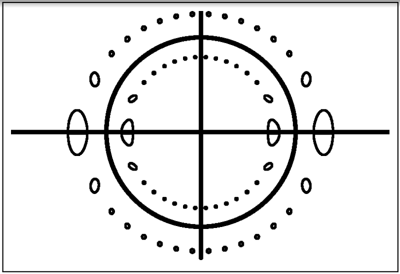

Fig. 3

$$E = 5, \quad z_2 - z_1 = (10\,,\,0), \qquad \alpha_1 = 6, \quad \alpha_2 = 6$$

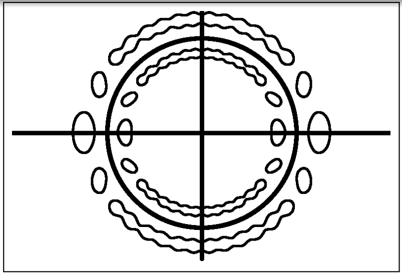

Fig. 4

$$E = 5, \quad z_2 - z_1 = (10\,,\,0), \qquad \alpha_1 = 6, \quad \alpha_2 = 6.8$$

Figures 3-4 reply the question by Grinevich and S.P. Novikov 1988. We see that the nest property

$$[\Gamma_{-J} \subset \Gamma_{-J+1} \subset \ldots \subset \Gamma_{-1} \subset S^1 \subset \Gamma_1 \subset \ldots \subset \Gamma_J],$$

for spectral singularities is not fulfilled.

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