Inverse Problem for Quantum Graphs: Magnetic Boundary Control

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QIPA

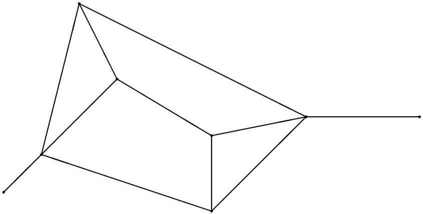
Tools are sometimes more important than results
(Jan Boman)

Mathematics is simple, one needs only to understand why
(Peter Sarnak)

Quantum graph

Combination of continuous and discrete

- **Metric graph**



- **Differential expression** on the edges

$$\ell_{q,a} = \left(i\frac{d}{dx} + a(x) \right)^2 + q(x)$$

- **Matching conditions** at every vertex
  Only standard conditions today

$$\begin{cases} \psi(x_i) = \psi(x_j),\ x_i, x_j \in V_m & \text{– continuity condition;} \\ \displaystyle\sum_{x_j \in V_m} \partial\psi(x_j) = 0 & \text{– Kirchhoff condition.} \end{cases}$$

Inverse problem

Task: reconstruct all three members from the family:

- the metric graph
- the potential(s) $q(x)$ and $\cancel{a(x)}$;
- ~~(the vertex conditions)~~

Magnetic potential can be eliminated on each edge $\Rightarrow$ only magnetic fluxes through cycles $\Phi_j = \int_{C_j} a(x)dx$ are relevant

$\Rightarrow$ DRIVING IDEA: use magnetic fluxes to solve the inverse problem

Contact set

Contact set $\partial\Gamma$ - **non-empty** set of vertices containing **all** degree one vertices, but may be also other ones of higher degree.

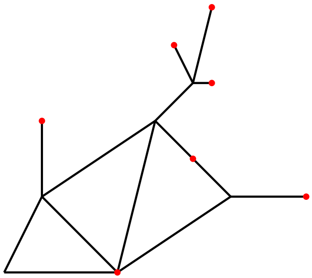


Figure: Contact set: red vertices

Spectral data

Spectral data: two equivalent sets = two different languages

- response operator = dynamical Dirichlet-to-Neumann map $\mathbf{R}^T$;
- Titchmarsh-Weyl M-function $\mathbf{M}(\lambda)$.

$$\boxed{\widehat{\left(\mathbf{R}\vec{f}\right)}(s) = \mathbf{M}(-s^2)\hat{\vec{f}}(s)}$$

where $\hat{\cdot}$ denotes the Laplace transform.

Response operator

Consider solution to the wave equation subject to **Boundary Control** on the contact set $\partial\Gamma$

$$\begin{cases} \left(i\frac{d}{dx}+a(x)\right)^2 u(x,t)+q(x)u(x,t)=-\frac{\partial^2}{\partial t^2}u(x,t) \\ \text{standard conditions at } V_m\notin\partial\Gamma \\ \text{continuity condition at } V_m\in\partial\Gamma \\ u(x,0)=u_t(x,0)=0 \text{ – zero initial data} \\ u(\cdot,t)|_{\partial\Gamma}=\vec{f}(t) \text{ – boundary control} \end{cases}$$

Response operator = dynamical Dirichlet-to-Neumann map

$$\mathbf{R}^T: \qquad \underbrace{\vec{f}(t)}_{\left\{u(V_m,t)\right\}_{V_m\in\partial\Gamma}} \qquad \mapsto \qquad \underbrace{\partial\vec{u}(\cdot,t)|_{\partial\Gamma}}_{\left\{\sum_{x_j\in V_m}\partial u(x_j,t)\right\}_{V_m\in\partial\Gamma}}$$

Response operator

The response operator is a convolution integral operator due to causality:

$$(\mathbf{R}^T \vec{f})(t) = \int_0^t r(t-\tau)\vec{f}(\tau)d\tau, \quad t \leq T$$

with the kernel

$$r(t) = -\mathrm{diag}\{\underbrace{d_m}_{\text{degree of } V_m}\} \; \delta'(t) + \underbrace{\text{smooth terms}}_{\text{reflection from } q} + \underbrace{\text{delayed terms } \delta'(t-\ell_j),\, \delta(t-\ell_j)}_{\text{reflection from other vertices}},$$

here ℓ_j are the lengths of orbits between the contact points.

Inverse problems in $\mathbb{R}$ and $\mathbb{R}^n$ – Boundary Control method = BC-method:
A.S. Blagoweschenskii, M. Belishev, Ya. Kurylev, S. Avdonin, ...

Titchmarsh-Weyl M-function

$\psi(x,\lambda)$ - solution to the Dirichlet problem for the stationary Schrödinger eq.

$$\left(i\frac{d}{dx}+a(x)\right)^2\psi(x,\lambda)+q(x)\psi(x,\lambda)=\lambda\psi(x,\lambda), \quad \Im\lambda>0$$

subject to

$$\begin{cases} \text{standard conditions at } V_m \notin \partial\Gamma \\ \text{continuity condition at } V_m \in \partial\Gamma \end{cases}$$ Kirchhoff condition is dropped!!

and prescribed values at the contact vertices: $\psi(\cdot,\lambda)|_{\partial\Gamma}=\vec{f}$.

Titchmarsh-Weyl ***M*-function** = energy dependent Dirichlet-to-Neumann map

$$\mathbf{M}(\lambda): \underbrace{\vec{f}}_{\vec{\psi}(\cdot,\lambda)|_{\partial\Gamma}} \mapsto \underbrace{\partial\vec{\psi}(\cdot,\lambda)|_{\partial\Gamma}}_{\left\{\sum_{x_j\in V_m}\partial\psi(x_j,t)\right\}_{V_m\in\partial\Gamma}}$$

– matrix-valued Herglotz-Nevanlinna function.

Inverse problems:

I. Gelfand - B. Levitan, V. Marchenko, ..., $\underbrace{\text{B. Simon, F. Gesztesy, A. Ramm,}}_{\text{A new approach to inverse spectral theory}}$ ...

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Local inverse problems

Theorem (Boundary control: Avdonin-Belishev-Kurylev)
The response operator $\mathbf{R}^T$ for the Schrödinger operator on $[0, \infty)$ determines the unique potential on the interval $[0, T/2]$.

Theorem (Gesztesy-Simon-Ramm)
Two Schrödinger operators $-\frac{d^2}{dx^2} + q_j(x)$ with the M-functions $M_j(\lambda)$, then

$$q_1(x) = q_2(x) \quad \text{on}(0, a) \quad \Leftrightarrow \quad M_1(-k^2) - M_2(-k^2) = \tilde{\mathcal{O}}(e^{-2ka})$$

where $f = \tilde{\mathcal{O}}(g) \quad \text{as} \;\; x \to x_0 \Leftrightarrow \lim_{x \to x_0} \frac{|f(x)|}{|g(x)|^{1-\epsilon}} = 0$ for all $\epsilon > 0$.

Equivalence
S. Avdonin, P.K., 2008

$$\widehat{\left(\mathbf{R}\vec{f}\right)}(s) = \mathbf{M}(-s^2)\hat{\vec{f}}(s)$$

A. Rybkin, 2009.

Two explicit formulas

$$\mathbf{M}_\Gamma(\lambda) = -\left(\sum_{n=1}^{\infty} \frac{\langle \psi_n^{\rm st}|_{\partial\Gamma}, \cdot \rangle_{\ell_2(\partial\Gamma)} \psi_n^{\rm st}|_{\partial\Gamma}}{\lambda_n^{\rm st} - \lambda} \right)^{-1},$$

where $\lambda_n^{\rm st}$ and $\psi_n^{\rm st}$ are the eigenvalues and orthonormalized eigenfunctions of $L^{\rm st}$ – standard conditions at all vertices.

$$\mathbf{M}_\Gamma(\lambda) - \mathbf{M}_\Gamma(\lambda') = \sum_{n=1}^{\infty} \frac{\lambda - \lambda'}{(\lambda_n^D - \lambda)(\lambda_n^D - \lambda')} \langle \partial\psi_n^D|_{\partial\Gamma}, \cdot \rangle_{\ell_2(\partial\Gamma)} \partial\psi_n^D|_{\partial\Gamma},$$

where λ_n^D and ψ_n^D are the eigenvalues and eigenfunctions of the Dirichlet Schrödinger operator L^D – Dirichlet conditions on $\partial\Gamma$ and standard conditions at all other vertices.

- These formulas determine where zeroes and singularities of M-functions may be situated.
- Existence of invisible eigenfunctions.

Two explicit formulas

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Inverse problems for trees

No magnetic potential on trees.

- Travelling times between boundary vertices determine the tree
- Peeling procedure
  1. BC-method determines the potential on all pending edges
  2. Knowing potential on a bunch of edges allows one to peel off this bunch and thus reduce the problem to a smaller tree.
- Repeating the peeling procedure solves the inverse problem in a finite number of steps

Theorem. *The response operator or the Titchmarsh-Weyl M-function uniquely determines the metric tree and potential on it.*

The reconstruction procedure is local and explicit.

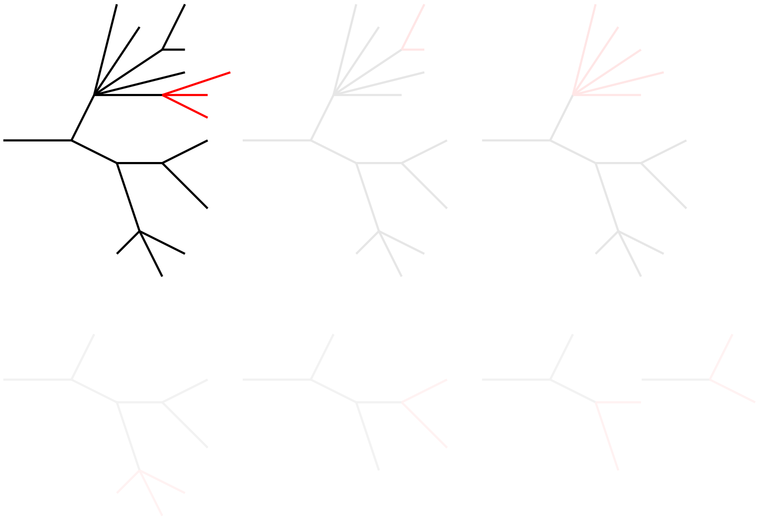


Figure: Peeling trees

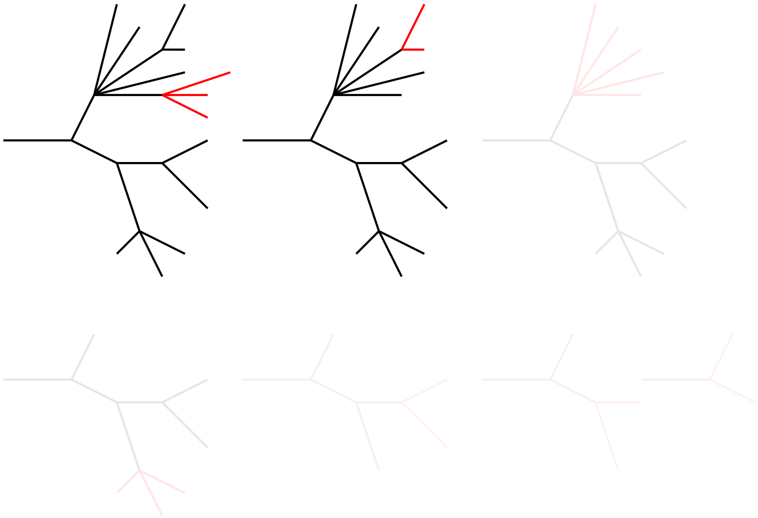
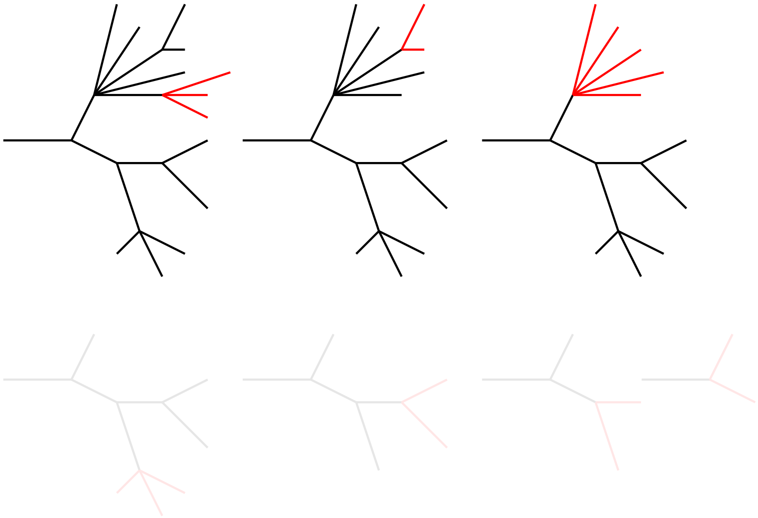


Figure: Peeling trees



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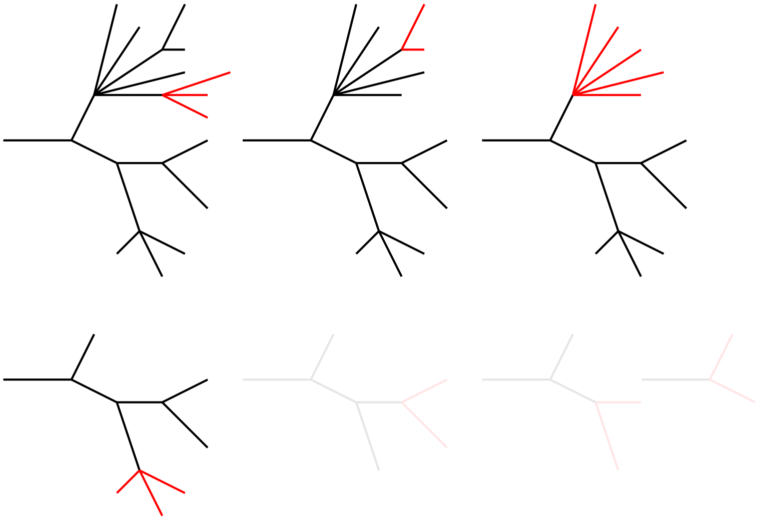


Figure: Peeling trees

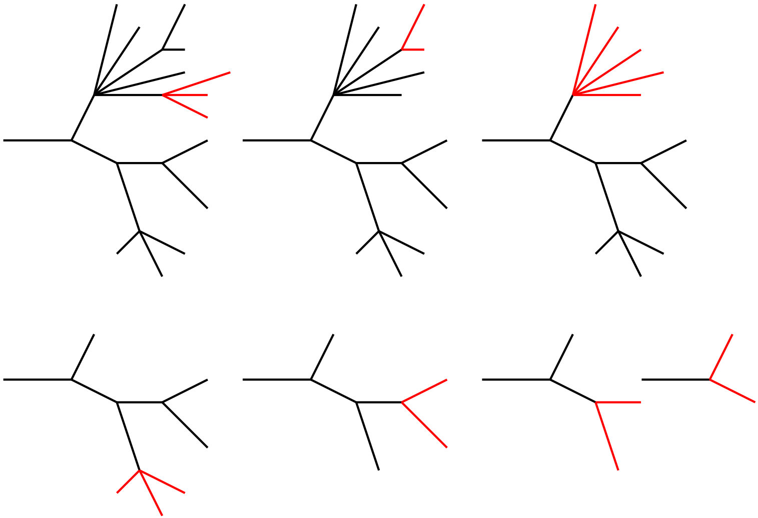


Figure: Peeling trees

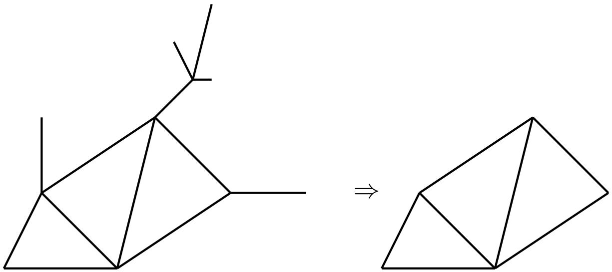
Inverse problems for graphs with cycles

- Magnetic potential on each cycle can be removed up to a phase $\Rightarrow$ magnetic fluxes Φ_j
- The response operator and M-matrices are considered as functions of the set of fluxes Φ_j, fixed each time.
  Magnetic Boundary Control - MBC-method

$$\mathbf{R}(\Phi_j) \mapsto q(x) \quad \text{or} \quad \mathbf{M}(\lambda, \Phi_j) \mapsto q(x)$$

- **Driving idea**: reduction to trees.
  - Dismantling graphs
  - Dissolving vertices using dependence on the magnetic fluxes
- The inverse problems will be solved **generically** – under mild generically satisfied conditions always explicitly formulated.

Peeling pending edges and branches



Cutting branches

Conclusion: It is enough to study graphs without degree one vertices.

Dismantling graphs

No magnetic potential involved

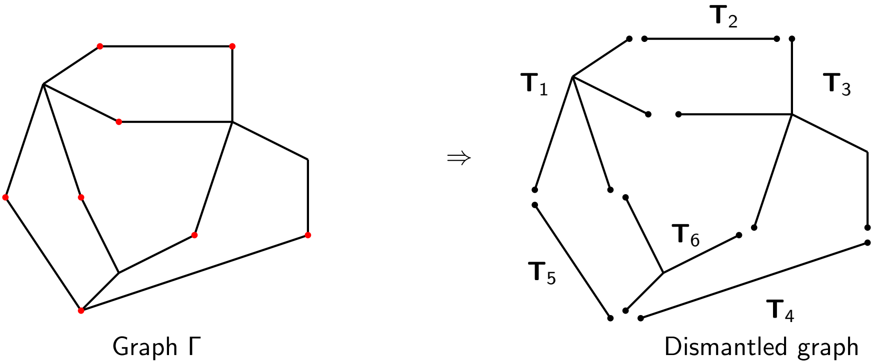


Figure: Dismantling graphs

Lemma. *Let* $\mathbf{T}$ *be a metric tree, then it holds:*

1. *every Dirichlet eigenfunction on* $\mathbf{T}$ *is visible, i.e. the corresponding eigenvalue* λ_n^{D} *is a singularity of the M-function;*
2. *every Dirichlet eigenfunction* ψ_n^{D} *on* $\mathbf{T}$ *has non-zero derivatives at (at least) two pendent vertices,*
3. *for every Dirichlet eigenvalue* λ_n^{D} *at least two diagonal elements of* $\mathbf{M}(\lambda)$ *are singular;*
4. *the following asymptotic representation holds*

$$\mathbf{M}(-s^2) = -s\mathbf{I} + o(1), \quad s \to \infty.$$

Definition. *The set of subtrees* $\{\mathbf{T}_j\}$ *of a metric graph* Γ *is called* **independent** *if any pair has at most one common vertex.*

Theorem. *Assume that the contact set* $\partial\Gamma$ *dismantles the metric graph* Γ *into a set of subtrees* $\{\mathbf{T}_j\}$*, such that*

1. *no subtree* $\mathbf{T}_j$ *has two pendent vertices coming from the same vertex in* Γ*;*
2. *the subtrees* $\mathbf{T}_j$ *are independent, i.e. no two subtrees have more than one common vertex.*

Then the M-function generically determines the metric graph Γ *and the potential* q*, i.e. provided*

a) *the Schrödinger operators* $L_q^{\mathrm{st,D}}(\mathbf{T}_j),\ j = 1, 2, \dots$ *with Dirichlet conditions at the pendent vertices and standard vertex conditions at all internal vertices have disjoint spectra*

$$\lambda_n^{\mathrm{D}}(\mathbf{T}_j) \neq \lambda_m^{\mathrm{D}}(\mathbf{T}_i), \quad j \neq i.$$

Idea of the proof:
The Dirichlet eigenfunctions for Γ and T_j are the same.

Reconstruct the M-functions for the subtrees from the M-function for the original graph

$$\mathbf{M}(\lambda) = \mathbf{M}_1(\lambda) + \mathbf{M}_2(\lambda) + \cdots + \mathbf{M}_N(\lambda)$$

Step 1 For every subtree we have explicit formula through the traces of Dirichlet eigenfunctions

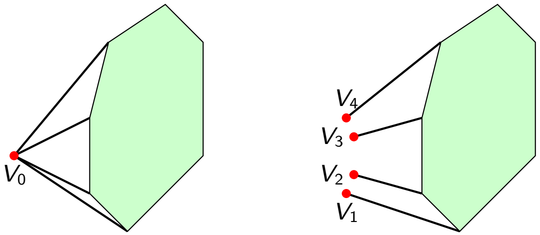
$$\mathbf{M}_\Gamma(\lambda) - \mathbf{M}_\Gamma(\lambda') = \sum_{n=1}^{\infty} \frac{\lambda - \lambda'}{(\lambda_n^D - \lambda)(\lambda_n^D - \lambda')} \langle \partial \psi_n^D|_{\partial\Gamma}, \cdot \rangle_{\ell_2(\partial\Gamma)} \partial \psi_n^D|_{\partial\Gamma},$$

Step 2 Using condition *a)* we can sort the singular terms into classes associated with the subtrees (without knowing the subtree a priori)
Step 3 The regular term $\mathbf{M}_j(\lambda)$ is determined from the asymptotics

$$\mathbf{M}(-s^2) = -s\mathbf{I} + o(1), \quad s \to \infty.$$

Dissolving vertices

Metric graph Γ with a vertex V_0 of degree d_0. **Dissolving the vertex** V_0 into d_0 degree one vertices $\Rightarrow$ metric graph Γ'.



Assume Γ' - connected $\Rightarrow$ $d_0 - 1$ cycles are broken.

The Dirichlet eigenfunctions on Γ and Γ' coincide!

Dissolving vertices

The M-functions for Γ and Γ' are related as:

$$\underbrace{\mathbb{M}(\lambda, \vec{\Phi}'')}_{= \mathbf{M}_{00}(\lambda, \vec{\Phi}'')} = \sum_{i,j=1}^{d_0} e^{i(F_i - F_j)} \underbrace{\mathbb{M}'_{ij}(\lambda, \vec{0})}_{= \mathbf{M}'_{ij}(\lambda, \vec{0})}$$

$$F_i - F_j = \Phi_{ij} - \text{the fluxes}$$

In particular for the singularities

$$\mathbf{M}_{00}(\lambda, \vec{\Phi}'') \underset{\lambda \to \lambda_n^{\mathrm{D}}}{\sim} \frac{1}{\lambda_n^{\mathrm{D}} - \lambda} \sum_{i,j=1}^{d_0} e^{i(F_i - F_j)} \partial\psi_n^{\mathrm{D}}(V_i) \overline{\partial\psi_n^{\mathrm{D}}(V_j)}$$

$$= \frac{1}{\lambda_n^{\mathrm{D}} - \lambda} \Big(\sum_{i=1}^{d_0} |\partial\psi_n^{\mathrm{D}}(V_i)|^2 + \sum_{\substack{i,j=1, \\ i<j}}^{d_0} 2\cos(\underbrace{F_i - F_j}_{\Phi_{ij}}) \partial\psi_n^{\mathrm{D}}(V_i) \overline{\partial\psi_n^{\mathrm{D}}(V_j)} \Big).$$

$$\mathbf{M}_{00}(\lambda, \vec{\Phi}''), \Phi''_j = 0, \pi \quad \Rightarrow \left\{ \begin{array}{l} \sum_{i=1}^{d_0} |\partial\psi_n^{\mathrm{D}}(V_i)|^2 \\ \partial\psi_n^{\mathrm{D}}(V_i) \overline{\partial\psi_n^{\mathrm{D}}(V_j)} \end{array} \right.$$

Dissolving vertices

Theorem. *Assume that*

1. *the graph Γ' and hence Γ is connected;*
2. *the degree d_0 of the contact vertex V_0 is at least three: $d_0 \geq 3$.*

Divide the magnetic fluxes $\vec{\Phi} = (\vec{\Phi}', \vec{\Phi}'')$ so that $\vec{\Phi}'$ correspond to preserved cycles.
Then the diagonal entry $\mathbf{M}_{00}(\vec{\Phi}', \vec{\Phi}'')$ **generically** *determines the diagonal $d_0 \times d_0$ block of $\mathbf{M}'(\vec{\Phi}')$ associated with the new pendent vertices, i.e. provided*

a) *the spectrum of the Dirichlet operator on Γ' is simple;*

b) *every Dirichlet eigenfunction ψ_n^{D} on Γ' has at least three non-zero normal derivatives at the pendent vertices.*

Why $d_0 \geq 3$?
It is simple:
$a^2 + b^2$ and ab do not determine a and b.
$a^2 + b^2 + c^2$, ab, bc and ac do determine a, b, c (up to multiplication by -1).

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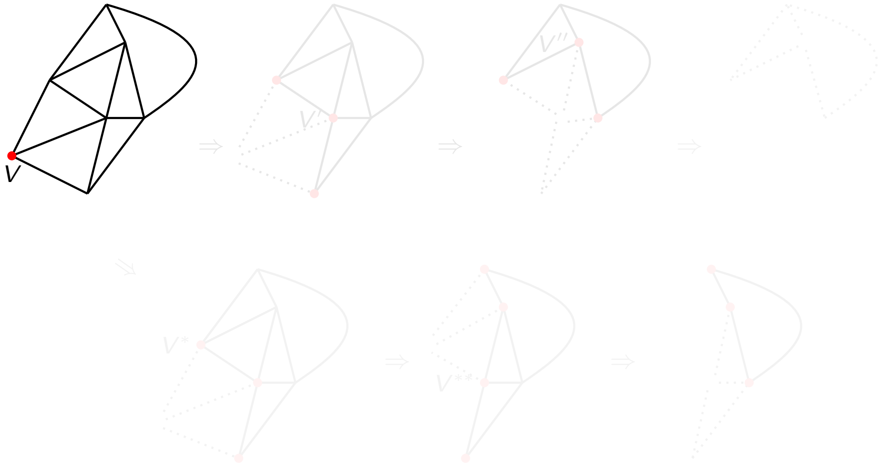
Reconstruction by dissolving vertices



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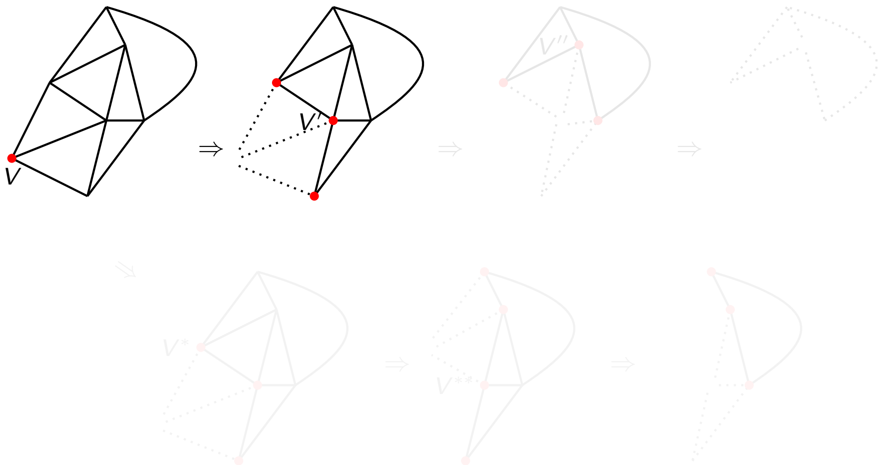

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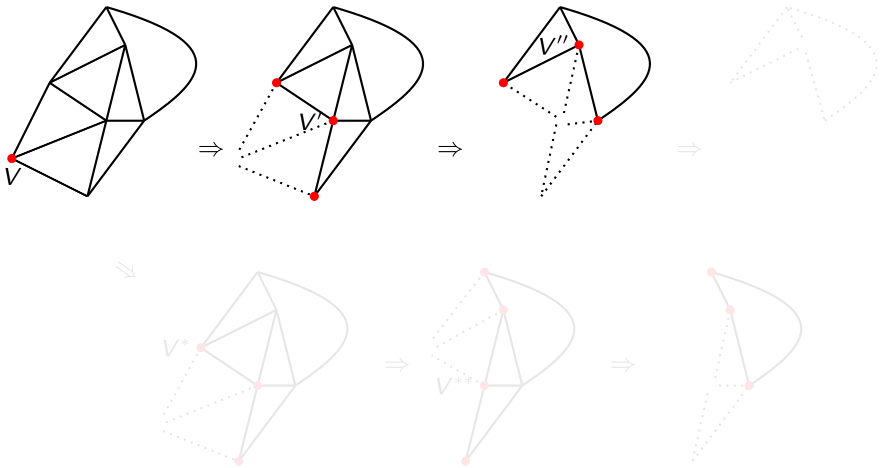

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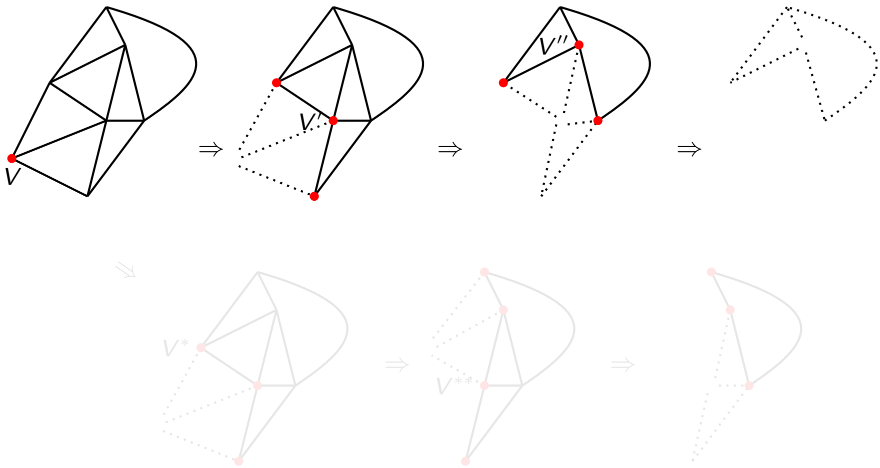

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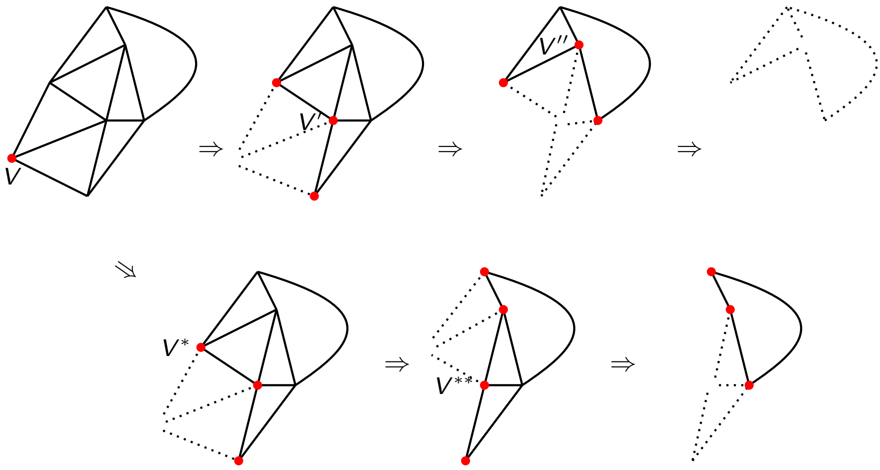
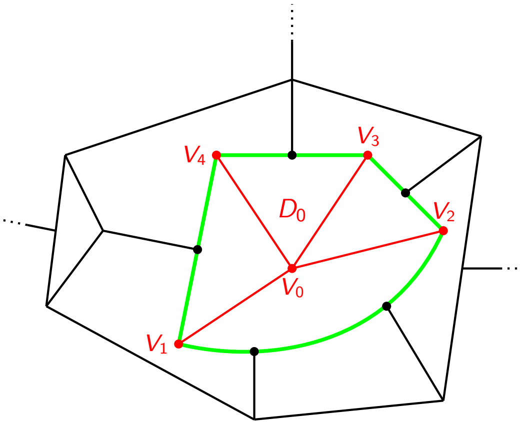
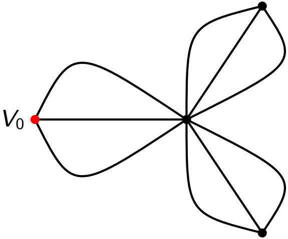


Infiltration domains: walls



V_0 – the original contact vertex
red colour – infiltration domain
green colour – domain's wall

Infiltration domains: bottlenecks



$\Rightarrow$

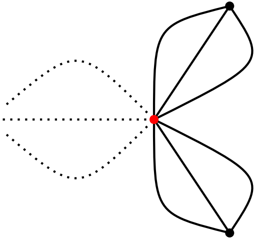
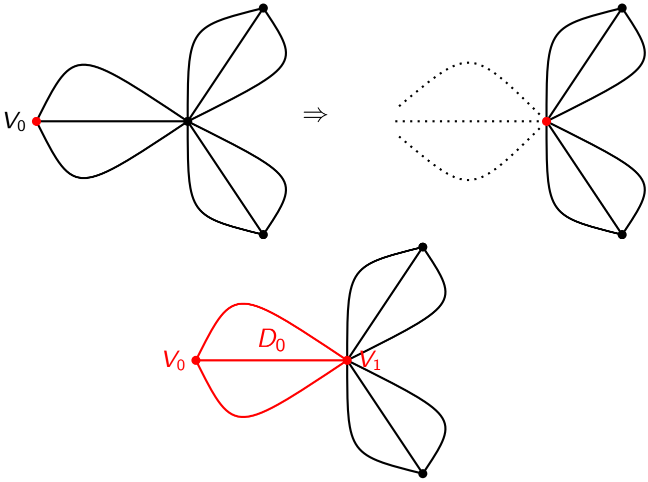


Figure: Bottleneck prevents dissolution

Infiltration domains: bottlenecks



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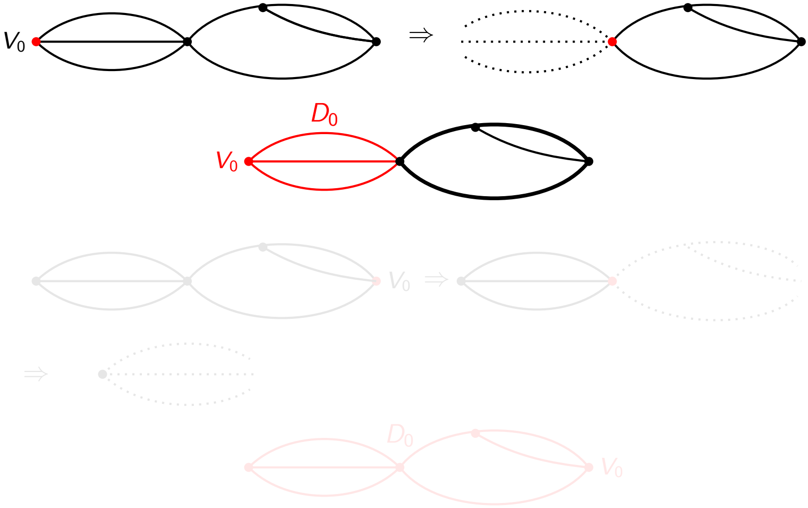

Figure: Degree two bottlenecks may allow dissolution

Infiltration domains: bottlenecks

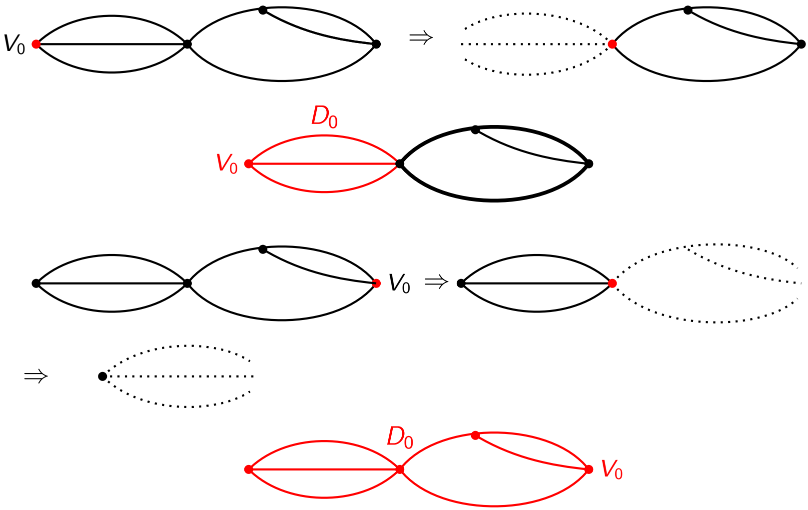

Figure: Degree two bottlenecks may allow dissolution

Magnetic Boundary Control

Theorem. *Assume that*

1. *the infiltration domains corresponding to $V_j \in \partial\Gamma$ cover the original graph Γ*

$$\bigcup_{V_j \in \partial\Gamma} D_j = \Gamma.$$

Then the $M(\Phi_j)$, $\Phi_i = 0, \pi$, generically determines the graph Γ and potential q, i.e. provided that

a) *the Dirichlet eigenfunctions on proper subgraphs of Γ do not vanish identically on any edge.*

Skeleton

$$\mathbb{S} := \Gamma \setminus \bigcup_{V_j \in \partial\Gamma} D_j$$

The skeleton is empty in this case.

Magnetic Boundary Control

Idea of the proof.

1. Determine distances between the contact points.
2. Recover infiltration domains (keeping in mind possible inclusion of other contact points)
3. Connect different infiltration domains together.

Note that:

- no a priori knowledge of the metric graph;
- procedure can start from one contact point adding new points if necessary if it is seen that obtained data are not sufficient (degree two vertices, the distances between contact points do not match reconstructed graph);
- procedure is local allowing to determine a part of an infinite graph, or even some infinite graphs.

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Magnetic Boundary Control

Theorem. *Assume that*

1. *the infiltration domains and their walls cover the original graph* Γ

$$\bigcup_{V_j \in \partial\Gamma} (D_j \cup W_j) \supset \Gamma \supset \bigcup_{V_m \in \mathbf{V}} V_m. \tag{1}$$

Then it holds:

- *the skeleton* $\mathbb{S} = \Gamma \setminus \left(\bigcup_{V_j \in \partial\Gamma} D_j \right)$ *is a union of star graphs joined at skeleton's contact vertices* $\partial\mathbb{S}$*;*
- *assuming in addition that*

2. *the skeleton contains no cycles of discrete length less or equal to* 4*,*

$M(\Phi_j)$, $\Phi_i = 0, \pi$, *generically determines the graph* Γ *and potential* q*, i.e. provided that*

a) *the Dirichlet eigenfunctions on the subgraphs of* Γ *do not vanish identically on any edge;*

b) *the spectra of the Dirichlet operators on the star graphs forming the skeleton are disjoint.*

Magnetic Boundary Control

Idea of the proof

- Repeat the proof of the previous theorem and reconstruct all infiltration domains and eventual connections between them.
  It remains to reconstruct the skeleton and potential on it.
- The contact points on the skeleton (including newly formed ones) dismantle the skeleton into star graphs – special case of trees.
- Apply our first theorem on dismantling graphs:
  - No cycles of discrete length 4 $\Rightarrow$
    - no two trees are dependent,
    - no two pendent vertices come from the same vertex in Γ.
  - The spectra of the Dirichlet operators on the star graphs are disjoint – assumption of this theorem.

THANK YOU!