Optimal recovery of harmonic function from inaccurate information on it's Radon transform

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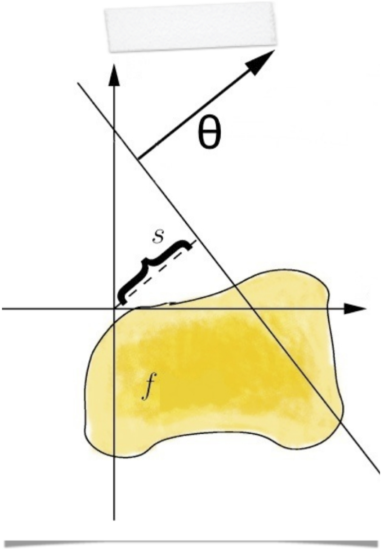
Hardy space of harmonic functions

- $f : \mathbb{R}^d \to \mathbb{R}, \quad d \geq 2$
- $suppf \subset \mathbb{B}^d$
- $\Delta f(x) = 0, \quad x \in \mathbb{B}^d$
- $$\|f\|_{h_2} = \sup_{0 \leq r < 1} \left(\int_{\mathbb{S}^{d-1}} |f(r, \phi)|^2 d\phi \right)^{1/2} < \infty$$
- $Bh_2 = \{f \in h_2 | \|f\|_{h_2} \leq 1\}$

Radon transform

$$Rf : \mathbb{S}^{d-1} \times \mathbb{R} \to \mathbb{R}$$

$$Rf(\theta, s) = \int_{x\theta = s} f(x)dx$$



Information and methods

- $\delta > 0$ is a given error
- $g \in L_2(\mathbb{S}^{d-1} \times \mathbb{R})$
- $\|Rf - g\|_{L_2(\mathbb{S}^{d-1})} \leq \delta$

Call method an arbitrary mapping

$$m : L_2(\mathbb{S}^{d-1} \times \mathbb{R}) \to L_2(\mathbb{B}^d)$$

$$\begin{tikzcd}
Bh_2 \arrow[r, "id"] \arrow[d, "R"'] & L_2(\mathbb{B}^d) \\
L_2(\mathbb{S}^{d-1} \times \mathbb{R}) \arrow[ur, "m"'] &
\end{tikzcd}$$

Problem

Method error

$$e(\delta, m) = \sup_{\substack{f \in Bh_2 \\ \|Rf - g\|_{L_2(\mathbb{S}^{d-1} \times \mathbb{R})} \le \delta}} ||m(g) - f||_{L_2(\mathbb{B}^d)}$$

Error of optimal recovery

$$E(\delta) = \inf_{m: L_2(\mathbb{S}^{d-1} \times \mathbb{R}) \to L_2(\mathbb{B}^d)} e(\delta, m)$$

Method m is optimal, if

$$e(\delta, m) = E(\delta)$$

Error of optimal recovery

Let

$$x_l = \frac{\Gamma^2(\frac{d+1}{2})\Gamma(d + l + \frac{1}{2})}{\pi^{d-1}\Gamma(d)\Gamma(l + \frac{1}{2})}, \quad y_l = \frac{x_l}{2l + d}$$

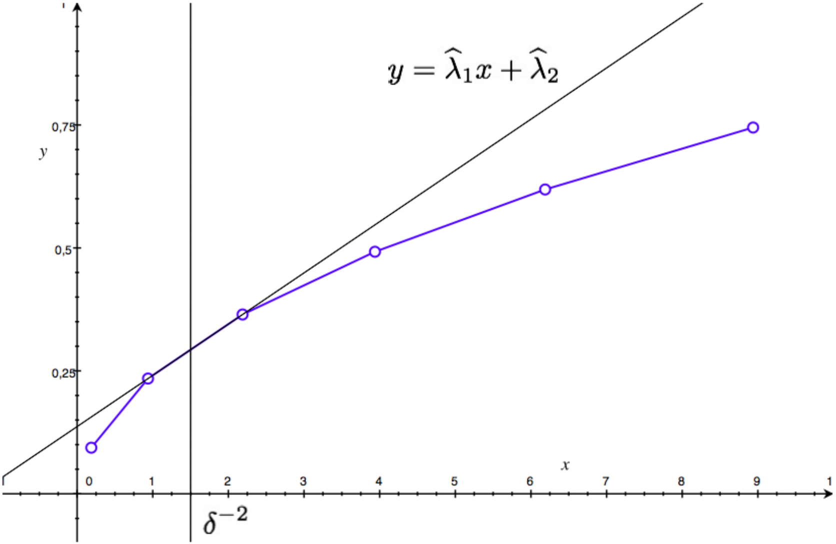
$$x_s < \delta^{-2} \le x_{s+1}$$

$$\widehat{\lambda}_1 = \frac{y_{s+1} - y_s}{x_{s+1} - x_s}, \quad \widehat{\lambda}_2 = \frac{y_s x_{s+1} - y_{s+1} x_s}{x_{s+1} - x_s}$$

Then

$$E(\delta) = \sqrt{\widehat{\lambda}_1 + \widehat{\lambda}_2 \delta^2}$$

Error of optimal recovery



Optimal methods

$$m_a(g)(x) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(l)} a_{kl} \frac{(\widehat{g}_{kl}, \psi_l)}{(\psi_l, \psi_l)} |x|^l Y_k^l \left(\frac{x}{|x|} \right)$$

- $\{Y_k^l\}$ – spherical harmonics
- $$g_{kl}(s) = \int_{\mathbb{S}^{d-1}} g(\theta, s) Y_k^l(\theta) d\theta$$
- $$\widehat{g}_{kl}(\sigma) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-s \cdot \sigma} g_{kl}(s) ds$$
- J_l – Bessel fuction of the first kind of order l
- $\psi_l(\sigma) = (2\pi)^{(d-1)/2} i^{-l} \sigma^{-d/2} J_{l+d/2}(\sigma)$
- $$(\widehat{g}_{kl}, \psi_l) = \int_{\mathbb{R}} \widehat{g}_{kl}(\sigma) \bar{\psi}_l(\sigma) d\sigma$$

Filter

$$a_{kl} = \frac{\widehat{\lambda}_2}{\widehat{\lambda}_1 x_l + \widehat{\lambda}_2} + \epsilon_{kl} \frac{\sqrt{\widehat{\lambda}_1 \widehat{\lambda}_2 (2l+d)}}{\widehat{\lambda}_1 x_l + \widehat{\lambda}_2} \sqrt{\widehat{\lambda}_1 x_l + \widehat{\lambda}_2 - \frac{x_l}{2l+d}}$$

$$\epsilon_{kl} \in [-1, 1]$$

Consequence