GLOBAL STABILITY FOR THE MULTI-CHANNEL GEL'FAND-CALDERÓN INVERSE PROBLEM IN TWO DIMENSIONS

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ABSTRACT. We prove a global logarithmic stability estimate for the multi-channel Gel'fand-Calderón inverse problem on a two-dimensional bounded domain, i.e. the inverse boundary value problem for the equation $-\Delta\psi + v\,\psi = 0$ on D, where v is a smooth matrix-valued potential defined on a bounded planar domain D.

1. INTRODUCTION

The Schrödinger equation at zero energy

$$(1.1) \qquad -\Delta\psi + v(x)\psi = 0 \ \text{ on } D \subset \mathbb{R}^2$$

arises in quantum mechanics, acoustics and electrodynamics. The reconstruction of the complex-valued potential v in equation (1.1) through the Dirichlet-to-Neumann operator is one of the most studied inverse problems (see [9], [8], [3], [10], [11], [12] and references therein).

In this article we consider the multi-channel two-dimensional Schrödinger equation, i.e. equation (1.1) with matrix-valued potentials and solutions; this case was already studied in [13, 12]. One of the motivations for studying the multi-channel equation is that it comes up as a 2D-approximation for the 3D equation (see [12, Sec. 2]).

This paper is devoted to give a global stability estimate for this inverse problem in the multi-channel case, which is highly related to the reconstruction method of [12].

Let D be an open bounded domain in $\mathbb{R}^2$ with C^2 boundary and $v \in C^1(\bar{D}, M_n(\mathbb{C}))$, where $M_n(\mathbb{C})$ is the set of the $n \times n$ complex-valued matrices. The Dirichlet-to-Neumann map associated to v is the operator $\Phi : C^1(\partial D, M_n(\mathbb{C})) \to L^p(\partial D, M_n(\mathbb{C}))$, $p < \infty$ defined by:

$$(1.2) \qquad \Phi(f) = \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial D}$$

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where $f \in C^1(\partial D, M_n(\mathbb{C}))$, ν is the outer normal of ∂D and ψ is the $H^1(\bar{D}, M_n(\mathbb{C}))$-solution of the Dirichlet problem

$$(1.3) \qquad -\Delta\psi + v(x)\psi = 0 \ \text{ on } D, \ \ \psi|_{\partial D} = f;$$

here we assume that

$$(1.4) \qquad 0 \text{ is not a Dirichlet eigenvalue for the operator } -\Delta + v \text{ in } D.$$

The following inverse boundary value problem arises from this construction: given Φ, find v.

This problem can be considered as the Gel'fand inverse boundary value problem for the multi-channel Schrödinger equation at zero energy (see [6], [9]) and can also be seen as a generalization of the Calderón problem for the electrical impedance tomography (see [4], [9]). Note also that we can think of this problem as a model for the monochromatic ocean tomography (e.g. see [2] for similar problems arising in this tomography).

In the case of complex-valued potentials the global injectivity of the map $v \to \Phi$ was firstly proved in [9] for $D \subset \mathbb{R}^d$ with $d \geq 3$ and in [3] for $d = 2$ with $v \in L^p$: in particular, these results were obtained by the use of global reconstructions developed in the same papers. The first global uniqueness result (along with an exact reconstruction method) for matrix-valued potentials was given in [12], which deals with C^1 matrix-valued potentials defined on a domain in $\mathbb{R}^2$. A global stability estimate for the Gel'fand-Calderón problem for $d \geq 3$ was found for the first time by Alessandrini in [1]; this result was recently improved in [10]. In the two-dimensional case the first global stability estimate was given in [11].

In this paper we extend the results of [11] to the matrix-valued case; we do not discuss global results for special real-valued potentials arising from conductivities: for this case the reader is referred to the references given in [1], [3], [8], [9], [10], [11].

Our main result is the following:

Theorem 1.1. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with C^2 boundary, let $v_1, v_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$ be two matrix-valued potentials which satisfy (1.4), with $\|v_j\|_{C^2(\bar{D})} \leq N$ for $j = 1, 2$, and Φ_1, Φ_2 the corresponding Dirichlet-to-Neumann operators. For simplicity we assume also that $v_j|_{\partial D} = 0$ and $\frac{\partial}{\partial \nu} v_j|_{\partial D} = 0$ for $j = 1, 2$. Then there exists a constant $C = C(D, N, n)$ such that*

$$(1.5)$$
$$\|v_2 - v_1\|_{L^\infty(D)} \leq C \left(\log(3 + \|\Phi_2 - \Phi_1\|^{-1})\right)^{-\frac{3}{4}} \left(\log(3\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))\right)^2,$$

where $\|A\|$ denotes the norm of an operator $A : L^\infty(\partial D, M_n(\mathbb{C})) \to L^\infty(\partial D, M_n(\mathbb{C}))$ and $\|v\|_{L^\infty(D)} = \max_{1 \le i,j \le n} \|v_{i,j}\|_{L^\infty(D)}$ (likewise for $\|v\|_{C^2(\bar{D})}$) for a matrix-valued potential v.

This is the first global stability result for the multi-channel ($n \ge 2$) Gel'fand-Calderón inverse problem in two dimension. In addition, Theorem 1.1 is new also for the scalar case, as the estimate obtained in [11] is weaker.

Instability estimates complementing the stability estimates of [1], [10], [11] and of the present work are given in [8], [7].

The proof of Theorem 1.1 is based on results obtained in [11], [12], which takes inspiration mostly from [3] and [1]. In particular, for $z_0 \in D$ we use the existence and uniqueness of a family of solution $\psi_{z_0}(z, \lambda)$ of equation (1.1) where in particular $\psi_{z_0} \to e^{\lambda(z-z_0)^2} I$, for $\lambda \to \infty$ (where I is the identity matrix). Then, using an appropriate matrix-valued version of Alessandrini's identity along with stationary phase techniques, we obtain the result. Note that this matrix-valued identity is one of the new results of this paper.

A generalization of Theorem 1.1 in the case where we do not assume that $v_j|_{\partial D} = 0$ and $\frac{\partial}{\partial \nu} v_j|_{\partial D} = 0$ for $j = 1, 2$, is given in section 5.

This work was fulfilled in the framework of researches under the direction of R. G. Novikov.

2. Preliminaries

In this section we introduce and give details about the above-mentioned family of solutions of equation (1.1), which will be used throughout all the paper.

We identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ where $(x_1, x_2) \in \mathbb{R}^2$. Let us define the function spaces $C_{\bar{z}}^1(\bar{D}) = \{u : u, \frac{\partial u}{\partial \bar{z}} \in C(\bar{D}, M_n(\mathbb{C}))\}$ with the norm $\|u\|_{C_{\bar{z}}^1(\bar{D})} = \max(\|u\|_{C(\bar{D})}, \|\frac{\partial u}{\partial \bar{z}}\|_{C(\bar{D})})$, where $\|u\|_{C(\bar{D})} = \sup_{z \in \bar{D}} |u|$ and $|u| = \max_{1 \le i,j \le n} |u_{i,j}|$; we define also $C_z^1(\bar{D}) = \{u : u, \frac{\partial u}{\partial z} \in C(\bar{D}, M_n(\mathbb{C}))\}$ with an analogous norm. Following [11], [12], we consider the functions:

$$(2.1) \quad G_{z_0}(z, \zeta, \lambda) = e^{\lambda(z-z_0)^2} g_{z_0}(z, \zeta, \lambda) e^{-\lambda(\zeta-z_0)^2},$$

$$(2.2) \quad g_{z_0}(z, \zeta, \lambda) = \frac{e^{\lambda(\zeta-z_0)^2 - \bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}}{4\pi^2} \int_D \frac{e^{-\lambda(\eta-z_0)^2 + \bar{\lambda}(\bar{\eta}-\bar{z}_0)^2}}{(z-\eta)(\bar{\eta}-\bar{\zeta})} d\text{Re}\eta \, d\text{Im}\eta,$$

$$(2.3) \quad \psi_{z_0}(z, \lambda) = e^{\lambda(z-z_0)^2} \mu_{z_0}(z, \lambda),$$

$$(2.4) \quad \mu_{z_0}(z, \lambda) = I + \int_D g_{z_0}(z, \zeta, \lambda) v(\zeta) \mu_{z_0}(\zeta, \lambda) d\text{Re}\zeta \, d\text{Im}\zeta,$$

$$(2.5) \qquad h_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} v(z) \mu_{z_0}(z,\lambda) d\mathrm{Re}z \, d\mathrm{Im}z,$$

where $z, z_0, \zeta \in D$ and $\lambda \in \mathbb{C}$ and I is the identity matrix. In addition, equation (2.4) at fixed z_0 and λ, is considered as a linear integral equation for $\mu_{z_0}(\cdot,\lambda) \in C^1_{\bar{z}}(\bar{D})$. The functions $G_{z_0}(z,\zeta,\lambda)$, $g_{z_0}(z,\zeta,\lambda)$, $\psi_{z_0}(z,\lambda)$, $\mu_{z_0}(z,\lambda)$ defined above, satisfy the following equations (see [11], [12]):

$$(2.6) \qquad 4\frac{\partial^2}{\partial z \partial \bar{z}} G_{z_0}(z,\zeta,\lambda) = \delta(z-\zeta),$$

$$(2.7) \qquad 4\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} G_{z_0}(z,\zeta,\lambda) = \delta(\zeta-z),$$

$$(2.8) \qquad 4\left(\frac{\partial}{\partial z} + 2\lambda(z-z_0)\right)\frac{\partial}{\partial \bar{z}} g_{z_0}(z,\zeta,\lambda) = \delta(z-\zeta),$$

$$(2.9) \qquad 4\frac{\partial}{\partial \bar{\zeta}}\left(\frac{\partial}{\partial \zeta} - 2\lambda(\zeta-z_0)\right) g_{z_0}(z,\zeta,\lambda) = \delta(\zeta-z),$$

$$(2.10) \qquad -4\frac{\partial^2}{\partial z \partial \bar{z}}\psi_{z_0}(z,\lambda) + v(z)\psi_{z_0}(z,\lambda) = 0,$$

$$(2.11) \qquad -4\left(\frac{\partial}{\partial z} + 2\lambda(z-z_0)\right)\frac{\partial}{\partial \bar{z}}\mu_{z_0}(z,\lambda) + v(z)\mu_{z_0}(z,\lambda) = 0,$$

where $z, z_0, \zeta \in D$, $\lambda \in \mathbb{C}$, δ is the Dirac's delta. (In addition, it is assumed that (2.4) is uniquely solvable for $\mu_{z_0}(\cdot,\lambda) \in C^1_{\bar{z}}(\bar{D})$ at fixed z_0 and λ.)

We say that the functions G_{z_0}, g_{z_0}, ψ_{z_0}, μ_{z_0}, h_{z_0} are the Bukhgeim-type analogues of the Faddeev functions (see [12]).

Now we state some fundamental lemmata. Let

$$(2.12) \qquad g_{z_0,\lambda}u(z) = \int_D g_{z_0}(z,\zeta,\lambda)u(\zeta)d\mathrm{Re}\zeta \, d\mathrm{Im}\zeta, \ z \in \bar{D}, \ z_0, \lambda \in \mathbb{C},$$

where $g_{z_0}(z,\zeta,\lambda)$ is defined by (2.2) and u is a test function.

Lemma 2.1 ([11]). *Let $g_{z_0,\lambda}u$ be defined by (2.12). Then, for $z_0, \lambda \in \mathbb{C}$, the following estimates hold:*

$$(2.13) \qquad g_{z_0,\lambda}u \in C^1_{\bar{z}}(\bar{D}), \quad for\ u \in C(\bar{D}),$$

$$(2.14) \qquad \|g_{z_0,\lambda}u\|_{C^1(\bar{D})} \leq c_1(D,\lambda)\|u\|_{C(\bar{D})}, \quad for\ u \in C(\bar{D}),$$

$$(2.15) \qquad \|g_{z_0,\lambda}u\|_{C^1_{\bar{z}}(\bar{D})} \leq \frac{c_2(D)}{|\lambda|^{\frac{1}{2}}}\|u\|_{C^1_{\bar{z}}(\bar{D})}, \quad for\ u \in C^1_{\bar{z}}(\bar{D}), \ |\lambda| \geq 1.$$

Given a potential $v \in C^1_{\bar{z}}(\bar{D})$ we define the operator $g_{z_0,\lambda}v$ simply as $(g_{z_0,\lambda}v)u(z) = g_{z_0,\lambda}w(z)$, $w = vu$, for a test function u. If $u \in C^1_{\bar{z}}(\bar{D})$, by Lemma 2.1 we have that $g_{z_0,\lambda}v : C^1_{\bar{z}}(\bar{D}) \to C^1_{\bar{z}}(\bar{D})$,

$$(2.16) \qquad \|g_{z_0,\lambda}v\|^{op}_{C^1_{\bar{z}}(\bar{D})} \leq 2n\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar{z}}(\bar{D})}\|v\|_{C^1_{\bar{z}}(\bar{D})},$$

where $\|\cdot\|^{op}_{C^1_{\bar{z}}(\bar{D})}$ denotes the operator norm in $C^1_{\bar{z}}(\bar{D})$, $z_0, \lambda \in \mathbb{C}$. In addition, $\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar{z}}(\bar{D})}$ is estimated in Lemma 2.1. Inequality (2.16) and Lemma 2.1 imply existence and uniqueness of $\mu_{z_0}(z, \lambda)$ (and thus also $\psi_{z_0}(z, \lambda)$) for $|\lambda| > \rho(D, K, n)$, where $\|v\|_{C^1_{\bar{z}}(\bar{D})} < K$.

Let

$$\mu_{z_0}^{(k)}(z, \lambda) = \sum_{j=0}^{k} (g_{z_0,\lambda}v)^j I,$$

$$h_{z_0}^{(k)}(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} v(z)\mu_{z_0}^{(k)}(z, \lambda)d\mathrm{Re}z\, d\mathrm{Im}z,$$

where $z, z_0 \in D$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N} \cup \{0\}$.

Lemma 2.2 ([11]). *For $v \in C^1_{\bar{z}}(\bar{D})$ such that $v|_{\partial D} = 0$ the following formula holds:*

$$(2.17) \qquad v(z_0) = \frac{2}{\pi} \lim_{\lambda \to \infty} |\lambda| h_{z_0}^{(0)}(\lambda), \ z_0 \in D.$$

In addition, if $v \in C^2(\bar{D})$, $v|_{\partial D} = 0$ and $\frac{\partial v}{\partial \nu}|_{\partial D} = 0$ then

$$(2.18) \qquad \left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}^{(0)}(\lambda) \right| \le c_3(D, n)\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar{D})},$$

for $z_0 \in D$, $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$.

Let

$$W_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} w(z)d\mathrm{Re}\, z d\mathrm{Im}\, z,$$

where $z_0 \in \bar{D}$, $\lambda \in \mathbb{C}$ and w is some $M_n(\mathbb{C})$-valued function on $\bar{D}$. (One can see that $W_{z_0} = h_{z_0}^{(0)}$ for $w = v$.)

Lemma 2.3 ([11]). *For $w \in C^1_{\bar{z}}(\bar{D})$ the following estimate holds:*

$$(2.19) \qquad |W_{z_0}(\lambda)| \le c_4(D)\frac{\log(3|\lambda|)}{|\lambda|}\|w\|_{C^1_{\bar{z}}(\bar{D})}, \ z_0 \in \bar{D}, \ |\lambda| \ge 1.$$

Lemma 2.4 ([12]). *For $v \in C^1_{\bar{z}}(\bar{D})$ and for $\|g_{z_0,\lambda}v\|^{op}_{C^1_{\bar{z}}(\bar{D})} \le \delta < 1$ we have that*

$$(2.20) \qquad \|\mu_{z_0}(\cdot, \lambda) - \mu_{z_0}^{(k)}(\cdot, \lambda)\|_{C^1_{\bar{z}}(\bar{D})} \le \frac{\delta^{k+1}}{1-\delta},$$

$$(2.21) \qquad |h_{z_0}(\lambda) - h_{z_0}^{(k)}(\lambda)| \le c_5(D, n)\frac{\log(3|\lambda|)}{|\lambda|}\frac{\delta^{k+1}}{1-\delta}\|v\|_{C^1_{\bar{z}}(\bar{D})},$$

where $z_0 \in D$, $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$, $k \in \mathbb{N} \cup \{0\}$.

The proofs of Lemmata 2.1-2.4 can be found in the references given.

We will also need the following two new lemmata.

Lemma 2.5. *Let $g_{z_0,\lambda}u$ be defined by (2.12), where $u \in C^1_{\bar z}(\bar D)$, $z_0, \lambda \in \mathbb{C}$. Then the following estimate holds:*

$$(2.22) \qquad \|g_{z_0,\lambda}u\|_{C(\bar D)} \le c_6(D)\frac{\log(3|\lambda|)}{|\lambda|}\|u\|_{C^1_{\bar z}(\bar D)}, \ |\lambda| \ge 1.$$

Lemma 2.6. *The expression*

$$(2.23) \qquad W(u,v)(\lambda) = \int_D e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}u(z)(g_{z_0,\lambda}v)(z)d\mathrm{Re}z\,d\mathrm{Im}z,$$

defined for $u,v \in C^1_{\bar z}(\bar D)$ with $\|u\|_{C^1_{\bar z}(\bar D)}, \|v\|_{C^1_{\bar z}(\bar D)} \le N_1$, $\lambda \in \mathbb{C}$, $z_0 \in D$, satisfies the estimate

$$(2.24) \qquad |W(u,v)(\lambda)| \le c_7(D,N_1,n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{1+3/4}}, \qquad |\lambda| \ge 1.$$

The proofs of Lemmata 2.5, 2.6 are given in section 4.

3. Proof of Theorem 1.1

We begin with a technical lemma, which will be useful to generalise Alessandrini's identity.

Lemma 3.1. *Let $v \in C^1(\bar D, M_n(\mathbb{C}))$ be a matrix-valued potential which satisfies condition (1.4) (i.e. 0 is not a Dirichlet eigeinvalue for the operator $-\Delta + v$ in D). Then ${}^t v$, the transpose of v, also satisfies condition (1.4).*

The proof of Lemma 3.1 is given in section 4.

We can now state and prove a matrix-valued version of Alessandrini's identity (see [1] for the scalar case).

Lemma 3.2. *Let $v_1, v_2 \in C^1(\bar D, M_n(\mathbb{C}))$ be two matrix-valued potentials which satisfy (1.4), Φ_1, Φ_2 their associated Dirichlet-to-Neumann operators, respectively, and $u_1, u_2 \in C^2(\bar D, M_n(\mathbb{C}))$ matrix-valued functions such that*

$$(-\Delta + v_1)u_1 = 0, \quad (-\Delta + {}^t v_2)u_2 = 0 \quad on\ D,$$

where ${}^t A$ stand for the transpose of A. Then we have the identity

$$(3.1)$$
$$\int_{\partial D} {}^t u_2(z)(\Phi_2 - \Phi_1)u_1(z)|dz| = \int_D {}^t u_2(z)(v_2(z) - v_1(z))u_1(z)d\mathrm{Re}z\,d\mathrm{Im}z.$$

Proof. If $v \in C^1(\bar D, M_n(\mathbb{C}))$ is any matrix-valued potential (which satisfies (1.4)) and $f_1, f_2 \in C^1(\partial D, M_n(\mathbb{C}))$ then we have

$$(3.2) \qquad \int_{\partial D} {}^t f_2 \Phi f_1 |dz| = \int_{\partial D} {}^t\left({}^t f_1 \Phi^* f_2\right)|dz|,$$

where Φ and Φ^* are the Dirichlet-to-Neumann operators associated to v and ${}^t v$, respectively (these operators are well-defined thanks to Lemma 3.1).

Indeed, it is sufficient to extend f_1 and f_2 in D as the solutions of the Dirichlet problems $(-\Delta + v)\tilde{f}_1 = 0$, $(-\Delta + {}^t v)\tilde{f}_2 = 0$ on D and $\tilde{f}_j|_{\partial D} = f_j$, for $j = 1, 2$, so that one obtains

$$\int_{\partial D} \left({}^t f_2 \Phi f_1 - {}^t\left({}^t f_1 \Phi^* f_2\right)\right) |dz|$$

$$= \int_{\partial D} \left({}^t f_2 \frac{\partial \tilde{f}_1}{\partial \nu} - {}^t\left(\frac{\partial \tilde{f}_2}{\partial \nu}\right) f_1\right) |dz|$$

$$= \int_D \left({}^t \tilde{f}_2 \, \Delta \tilde{f}_1 - {}^t\left(\Delta \tilde{f}_2\right) \tilde{f}_1\right) d\mathrm{Re}z \, d\mathrm{Im}z$$

$$= \int_D \left({}^t \tilde{f}_2 \, v \, \tilde{f}_1 - {}^t\left({}^t v \, \tilde{f}_2\right) \tilde{f}_1\right) d\mathrm{Re}z \, d\mathrm{Im}z = 0,$$

where for the second equality we used the following matrix-valued version of the classical scalar Green's formula:

$$(3.3) \qquad \int_{\partial D} \left({}^t\left(\frac{\partial f}{\partial \nu}\right) g - {}^t f \frac{\partial g}{\partial \nu}\right) |dz| = \int_D \left({}^t(\Delta f)\, g - {}^t f \Delta g\right) d\mathrm{Re}z \, d\mathrm{Im}z,$$

for any $f, g \in C^2(D, M_n(\mathbb{C})) \cap C^1(\bar{D}, M_n(\mathbb{C}))$.

Identities (3.2) and (3.3) imply

$$\int_{\partial D} {}^t u_2(z)(\Phi_2 - \Phi_1)u_1(z)|dz|$$

$$= \int_{\partial D} \left({}^t\left({}^t u_1(z)\Phi_2^* u_2(z)\right) - {}^t u_2(z)\Phi_1 u_1(z)\right) |dz|$$

$$= \int_{\partial D} \left({}^t\left(\frac{\partial u_2(z)}{\partial \nu}\right) u_1(z) - {}^t u_2(z)\frac{\partial u_1(z)}{\partial \nu}\right) |dz|$$

$$= \int_D \left({}^t(\Delta u_2(z))\, u_1(z) - {}^t u_2(z)\Delta u_1(z)\right) d\mathrm{Re}z \, d\mathrm{Im}z$$

$$= \int_D \left({}^t\left({}^t v_2(z)\, u_2(z)\right) u_1(z) - {}^t u_2(z)\, v_1(z)\, u_1(z)\right) d\mathrm{Re}z \, d\mathrm{Im}z$$

$$= \int_D {}^t u_2(z)(v_2(z) - v_1(z))u_1(z)d\mathrm{Re}z \, d\mathrm{Im}z. \qquad \Box$$

Now let $\bar{\mu}_{z_0}$ denote the complex conjugated of μ_{z_0} (the solution of (2.4)) for a $M_n(\mathbb{R})$-valued potential v and, more generally, the solution of (2.4) with $g_{z_0}(z, \zeta, \lambda)$ replaced by $\overline{g_{z_0}(z, \zeta, \lambda)}$ for a $M_n(\mathbb{C})$-valued potential v. In order to make use of (3.1) we define

$$u_1(z) = \psi_{1,z_0}(z, \lambda) = e^{\lambda(z-z_0)^2}\mu_1(z, \lambda),$$

$$u_2(z) = \overline{\psi}_{2,z_0}(z, -\lambda) = e^{-\bar{\lambda}(\bar{z}-\bar{z}_0)^2}\bar{\mu}_2(z, -\lambda),$$

for $z_0 \in D$, $\lambda \in C$, $|\lambda| > \rho$ (ρ is mentioned in section 2), where we called for simplicity $\mu_1 = \mu_{1,z_0}$, $\mu_2 = \mu_{2,z_0}$ and μ_{1,z_0}, μ_{2,z_0} are the solutions of (2.4) with v replaced by v_1, ${}^t v_2$, respectively.

Equation (3.1), with the above-defined u_1, u_2, now reads

$$(3.4) \quad \int_{\partial D} \int_{\partial D} e^{-\bar{\lambda}(\bar{z}-\bar{z}_0)^2} {}^t\bar{\mu}_2(z, -\lambda)(\Phi_2 - \Phi_1)(z, \zeta) e^{\lambda(\zeta-z_0)^2} \mu_1(\zeta, \lambda) |d\zeta||dz|$$

$$= \int_D e_{\lambda,z_0}(z) \, {}^t\bar{\mu}_2(z, -\lambda)(v_2 - v_1)(z)\mu_1(z, \lambda) d\mathrm{Re}z \, d\mathrm{Im}z.$$

with $e_{\lambda,z_0}(z) = e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2}$ and $(\Phi_2 - \Phi_1)(z, \zeta)$ is the Schwartz kernel of the operator $\Phi_2 - \Phi_1$.

The right side $I(\lambda)$ of (3.4) can be written as the sum of four integrals, namely

$$I_1(\lambda) = \int_D e_{\lambda,z_0}(z)(v_2 - v_1)(z) d\mathrm{Re}z \, d\mathrm{Im}z,$$

$$I_2(\lambda) = \int_D e_{\lambda,z_0}(z) \, {}^t(\bar{\mu}_2 - I)(v_2 - v_1)(z)(\mu_1 - I) d\mathrm{Re}z \, d\mathrm{Im}z,$$

$$I_3(\lambda) = \int_D e_{\lambda,z_0}(z) \, {}^t(\bar{\mu}_2 - I)(v_2 - v_1)(z) \, d\mathrm{Re}z \, d\mathrm{Im}z,$$

$$I_4(\lambda) = \int_D e_{\lambda,z_0}(z) \, (v_2 - v_1)(z)(\mu_1 - I) d\mathrm{Re}z \, d\mathrm{Im}z,$$

for $z_0 \in D$.

The first term, I_1, can be estimated using Lemma 2.2 as follows:

$$(3.5) \quad \left| \frac{2}{\pi} |\lambda| I_1 - (v_2(z_0) - v_1(z_0)) \right| \le c_3(D, n) \frac{\log(3|\lambda|)}{|\lambda|} \|v_2 - v_1\|_{C^2(\bar{D})},$$

for $|\lambda| \ge 1$. The other terms, I_2, I_3, I_4, satisfy, by Lemmata 2.1 and 2.4,

$$(3.6) \quad |I_2| \le \left| \int_D e_{\lambda,z_0}(z) \, {}^t(\overline{g_{z_0,\lambda}}{}^t v_2)(v_2 - v_1)(z)(g_{z_0,\lambda}v_1) d\mathrm{Re}z \, d\mathrm{Im}z \right|$$

$$+ O\left(\frac{\log(3|\lambda|)}{|\lambda|^2} \right) c_8(D, N, n),$$

$$(3.7) \quad |I_3| \le \left| \int_D e_{\lambda,z_0}(z) \, {}^t(\overline{g_{z_0,\lambda}}{}^t v_2)(v_2 - v_1)(z) d\mathrm{Re}z \, d\mathrm{Im}z \right|$$

$$+ O\left(\frac{\log(3|\lambda|)}{|\lambda|^2} \right) c_9(D, N, n),$$

$$(3.8) \quad |I_4| \le \left| \int_D e_{\lambda,z_0}(z) \, (v_2 - v_1)(z)(g_{z_0,\lambda}v_1) d\mathrm{Re}z \, d\mathrm{Im}z \right|$$

$$+ O\left(\frac{\log(3|\lambda|)}{|\lambda|^2} \right) c_{10}(D, N, n),$$

where N is the costant in the statement of Theorem 1.1 and $|\lambda|$ is sufficiently large, for example for λ such that

$$(3.9) \quad 2n\frac{c_2(D)}{|\lambda|^{\frac{1}{2}}} \le \frac{1}{2}, \qquad |\lambda| \ge 1.$$

Lemmata 2.5, 2.6, applied to (3.6)-(3.8), give us

$$(3.10) \qquad |I_2| \leq c_{11}(D, N, n)\frac{(\log(3|\lambda|))^2}{|\lambda|^2},$$

$$(3.11) \qquad |I_3| \leq c_{12}(D, N, n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{1+3/4}},$$

$$(3.12) \qquad |I_4| \leq c_{13}(D, N, n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{1+3/4}}.$$

The left side $J(\lambda)$ of (3.4) can be estimated as follows:

$$(3.13) \qquad |\lambda||J(\lambda)| \leq c_{14}(D, n)e^{(2L^2+1)|\lambda|}\|\Phi_2 - \Phi_1\|,$$

for λ which satisfies (3.9), and $L = \max_{z \in \partial D, \, z_0 \in D} |z - z_0|$.

Putting together estimates (3.5)-(3.13) we obtain

(3.14)
$$|v_2(z_0) - v_1(z_0)| \leq c_{15}(D, N, n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{3/4}} + \frac{2}{\pi}c_{14}(D, n)e^{(2L^2+1)|\lambda|}\|\Phi_2 - \Phi_1\|$$

for any $z_0 \in D$. We call $\varepsilon = \|\Phi_2 - \Phi_1\|$ and impose $|\lambda| = \gamma \log(3 + \varepsilon^{-1})$, where $0 < \gamma < (2L^2 + 1)^{-1}$ so that (3.14) reads

(3.15)
$$|v_2(z_0) - v_1(z_0)| \leq c_{15}(D, N, n)(\gamma \log(3 + \varepsilon^{-1}))^{-\frac{3}{4}} \left(\log(3\gamma \log(3 + \varepsilon^{-1}))\right)^2$$
$$+ \frac{2}{\pi}c_{14}(D, n)(3 + \varepsilon^{-1})^{(2L^2+1)\gamma}\varepsilon,$$

for every $z_0 \in D$, with

$$(3.16) \qquad 0 < \varepsilon \leq \varepsilon_1(D, N, \gamma, n),$$

where ε_1 is sufficiently small or, more precisely, where (3.16) implies that $|\lambda| = \gamma \log(3 + \varepsilon^{-1})$ satisfies (3.9).

As $(3 + \varepsilon^{-1})^{(2L^2+1)\gamma}\varepsilon \to 0$ for $\varepsilon \to 0$ more rapidly then the other term, we obtain that

$$(3.17) \quad \|v_2 - v_1\|_{L^\infty(D)} \leq c_{16}(D, N, \gamma, n)\frac{\left(\log(3\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))\right)^2}{(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{\frac{3}{4}}}$$

for any $\varepsilon = \|\Phi_2 - \Phi_1\| \leq \varepsilon_1(D, N, \gamma, n)$.

Estimate (3.17) for general ε (with modified c_{16}) follows from (3.17) for $\varepsilon \leq \varepsilon_1(D, N, \gamma, n)$ and the assumption that $\|v_j\|_{L^\infty(D)} \leq N$, $j = 1, 2$. This completes the proof of Theorem 1.1. $\qquad \square$

4. Proofs of Lemmata 2.5, 2.6, 3.1.

Proof of Lemma 2.5. We decompose the operator $g_{z_0,\lambda}$, defined in (2.12), as the product $\frac{1}{4}T_{z_0,\lambda}\bar{T}_{z_0,\lambda}$, where

$$(4.1) \qquad T_{z_0,\lambda}u(z) = \frac{1}{\pi}\int_D \frac{e^{-\lambda(\zeta-z_0)^2+\bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}}{z-\zeta}u(\zeta)d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta,$$

$$(4.2) \qquad \bar{T}_{z_0,\lambda}u(z) = \frac{1}{\pi}\int_D \frac{e^{\lambda(\zeta-z_0)^2-\bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}}{\bar{z}-\bar{\zeta}}u(\zeta)d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta,$$

for $z_0, \lambda \in \mathbb{C}$. From the proof of [11, Lemma 3.1] we have the estimate

$$(4.3) \qquad \|\bar{T}_{z_0,\lambda}u\|_{C(\bar{D})} \leq \frac{\eta_1(D)}{|\lambda|^{1/2}}\|u\|_{C(\bar{D})} + \eta_2(D)\frac{\log(3|\lambda|)}{|\lambda|}\left\|\frac{\partial u}{\partial\bar{z}}\right\|_{C(\bar{D})},$$

for $u \in C^1_{\bar{z}}(\bar{D})$, $z_0 \in D$, $|\lambda| \geq 1$. As the kernel of $T_{z_0,\lambda}$ and $\bar{T}_{z_0,\lambda}$ are conjugated each other we deduce immediately

$$(4.4)\ \ \|T_{z_0,\lambda}u\|_{C(\bar{D})} \leq \frac{\eta_1(D)}{|\lambda|^{1/2}}\|u\|_{C(\bar{D})} + \eta_2(D)\frac{\log(3|\lambda|)}{|\lambda|}\left\|\frac{\partial u}{\partial z}\right\|_{C(\bar{D})},\ |\lambda| \geq 1,$$

for $u \in C^1_z(\bar{D})$. Combining the two estimates we obtain

$$\begin{aligned}
\|g_{\lambda,z_0}u\|_{C(\bar{D})} &= \frac{1}{4}\|T_{z_0,\lambda}\bar{T}_{z_0,\lambda}u\|_{C(\bar{D})} \\
&\leq \frac{1}{4}\left(\eta_1(D)\frac{\|\bar{T}_{z_0,\lambda}u\|_{C(\bar{D})}}{|\lambda|^{1/2}} + \eta_2(D)\frac{\log(3|\lambda|)}{|\lambda|}\left\|\frac{\partial}{\partial z}\bar{T}_{z_0,\lambda}u\right\|_{C(\bar{D})}\right) \\
&\leq \eta_3(D)\left(\frac{\|u\|_{C(\bar{D})}}{|\lambda|} + \frac{\log(3|\lambda|)}{|\lambda|^{3/2}}\left\|\frac{\partial u}{\partial\bar{z}}\right\|_{C(\bar{D})} + \frac{\log(3|\lambda|)}{|\lambda|}\|u\|_{C(\bar{D})}\right) \\
&\leq \eta_4(D)\frac{\log(3|\lambda|)}{|\lambda|}\|u\|_{C^1_{\bar{z}}(\bar{D})}, \qquad |\lambda| \geq 1,
\end{aligned}$$

where we used the fact that $\|\frac{\partial}{\partial z}\bar{T}_{z_0,\lambda}u\|_{C(D)} = \|u\|_{C(D)}$. $\qquad\square$

Proof of Lemma 2.6. For $0 < \varepsilon \leq 1$, $z_0 \in D$, let $B_{z_0,\varepsilon} = \{z \in \mathbb{C} : |z-z_0| \leq \varepsilon\}$. We write $W(u,v)(\lambda) = W^1(\lambda) + W^2(\lambda)$, where

$$W^1(\lambda) = \int_{D\cap B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{z}-\bar{z}_0)^2}u(z)g_{z_0,\lambda}v(z)d\mathrm{Re}z\,d\mathrm{Im}z,$$

$$W^2(\lambda) = \int_{D\backslash B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{z}-\bar{z}_0)^2}u(z)g_{z_0,\lambda}v(z)d\mathrm{Re}z\,d\mathrm{Im}z.$$

The first term, W^1, can be estimated as follows:

$$(4.5) \qquad |W^1(\lambda)| \leq \sigma_1(D,n)\|u\|_{C(\bar{D})}\|v\|_{C^1_{\bar{z}}(\bar{D})}\frac{\varepsilon^2\log(3|\lambda|)}{|\lambda|}, \qquad |\lambda| \geq 1,$$

where we used estimates (2.16) and (2.22).

For the second term, W^2, we proceed using integration by parts, in order to obtain

$$W^2(\lambda) = \frac{1}{4i\bar{\lambda}} \int_{\partial(D \backslash B_{z_0,\varepsilon})} e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} \frac{u(z)g_{z_0,\lambda}v(z)}{\bar{z} - \bar{z}_0} dz$$
$$- \frac{1}{2\bar{\lambda}} \int_{D \backslash B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} \frac{\partial}{\partial \bar{z}} \left(\frac{u(z)g_{z_0,\lambda}v(z)}{\bar{z} - \bar{z}_0} \right) d\mathrm{Re}z \, d\mathrm{Im}z.$$

This imply

$$(4.6) \quad |W^2(\lambda)| \leq \frac{1}{4|\lambda|} \int_{\partial(D \backslash B_{z_0,\varepsilon})} \frac{\|u(z)g_{z_0,\lambda}v(z)\|_{C(\bar{D})}}{|\bar{z} - \bar{z}_0|} |dz|$$
$$+ \frac{1}{2|\lambda|} \left| \int_{D \backslash B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} \frac{\partial}{\partial \bar{z}} \left(\frac{u(z)g_{z_0,\lambda}v(z)}{\bar{z} - \bar{z}_0} \right) d\mathrm{Re}z \, d\mathrm{Im}z \right|,$$

for $\lambda \neq 0$. Again by estimates (2.16) and (2.22) we obtain

$$(4.7) \quad |W^2(\lambda)| \leq \sigma_2(D,n) \|u\|_{C^1_{\bar{z}}(\bar{D})} \|v\|_{C^1_{\bar{z}}(\bar{D})} \frac{\log(3\varepsilon^{-1})\log(3|\lambda|)}{|\lambda|^2}$$
$$+ \frac{1}{8|\lambda|} \left| \int_{D \backslash B_{z_0,\varepsilon}} u(z) \frac{\bar{T}_{z_0,\lambda}v(z)}{\bar{z} - \bar{z}_0} d\mathrm{Re}z \, d\mathrm{Im}z \right|, \qquad |\lambda| \geq 1,$$

where we used the fact that $\frac{\partial}{\partial \bar{z}} g_{z_0,\lambda}v(z) = \frac{1}{4} e^{-\lambda(z-z_0)^2 + \bar{\lambda}(\bar{z}-\bar{z}_0)^2} \bar{T}_{z_0,\lambda}v(z)$, with $\bar{T}_{z_0,\lambda}$ defined in (4.2).

The last term in (4.7) can be estimated independently on ε by

$$(4.8) \qquad \sigma_3(D,n) \|u\|_{C(\bar{D})} \|v\|_{C^1_{\bar{z}}(\bar{D})} \frac{\log(3|\lambda|)}{|\lambda|^{1+3/4}}.$$

This is a consequence of (4.3) and of the estimate

$$(4.9) \qquad |\bar{T}_{z_0,\lambda}u(z)| \leq \frac{\log(3|\lambda|)(1 + |z - z_0|)\tau_1(D)}{|\lambda||z - z_0|^2} \|u\|_{C^1_{\bar{z}}(\bar{D})}, \ |\lambda| \geq 1,$$

for $u \in C^1_{\bar{z}}(\bar{D})$, $z, z_0 \in D$ (a proof of (4.9) can be found in the proof of [11, Lemma 3.1]).

Indeed, for $0 < \delta \leq \frac{1}{2}$ we have

$$\left| \int_D u(z) \frac{\bar{T}_{z_0,\lambda} v(z)}{\bar{z} - \bar{z}_0} d\mathrm{Re}z \, d\mathrm{Im}z \right|$$

$$\leq \int_{B_{z_0,\delta} \cap D} |u(z)| \frac{|\bar{T}_{z_0,\lambda} v(z)|}{|z - z_0|} d\mathrm{Re}z \, d\mathrm{Im}z + \int_{D \setminus B_{z_0,\delta}} |u(z)| \frac{|\bar{T}_{z_0,\lambda} v(z)|}{|z - z_0|} d\mathrm{Re}z \, d\mathrm{Im}z$$

$$\leq \|u\|_{C(\bar{D})} \|v\|_{C^1_{\bar{z}}(\bar{D})} \frac{\tau_2(D,n)}{|\lambda|^{1/2}} \int_{B_{z_0,\delta} \cap D} \frac{d\mathrm{Re}z \, d\mathrm{Im}z}{|z - z_0|}$$

$$+ \|u\|_{C(\bar{D})} \|v\|_{C^1_{\bar{z}}(\bar{D})} \frac{\log(3|\lambda|)}{|\lambda|} \tau_3(D,n) \int_{D \setminus B_{z_0,\delta}} \frac{d\mathrm{Re}z \, d\mathrm{Im}z}{|z - z_0|^3}$$

$$\leq 2\pi \|u\|_{C(\bar{D})} \|v\|_{C^1_{\bar{z}}(\bar{D})} \tau_2(D,n) \frac{\delta}{|\lambda|^{\frac{1}{2}}} + \|u\|_{C(\bar{D})} \|v\|_{C^1_{\bar{z}}(\bar{D})} \tau_4(D,n) \frac{\log(3|\lambda|)}{|\lambda|\delta},$$

for $|\lambda| \geq 1$. Putting $\delta = \frac{1}{2}|\lambda|^{-1/4}$ in the last inequality gives (4.8).

Finally, defining $\varepsilon = |\lambda|^{-1/2}$ in (4.7), (4.5) and using (4.8), we obtain the main estimate (2.24), which thus finishes the proof of Lemma 2.6. $\qquad\square$

Proof of Lemma 3.1. Take $u \in H^1(D, M_n(\mathbb{C}))$ such that $(-\Delta + {}^t v)u = 0$ on D and $u|_{\partial D} = 0$. We want to prove that $u \equiv 0$ on D.

By our hypothesis, for any $f \in C^1(\partial D, M_n(\mathbb{C}))$ there exists a unique $\tilde{f} \in H^1(D, M_n(\mathbb{C}))$ such that $(-\Delta + v)\tilde{f} = 0$ on D and $\tilde{f}|_{\partial D} = f$. Thus we have, using Green's formula (3.3),

$$\int_{\partial D} {}^t \left(\frac{\partial u}{\partial \nu} \right) f |dz| = \int_D \left({}^t(\Delta u) \, \tilde{f} - {}^t u \Delta \tilde{f} \right) d\mathrm{Re}z \, d\mathrm{Im}z$$

$$= \int_D \left({}^t({}^t v \, u) \, \tilde{f} - {}^t u \, v \, \tilde{f} \right) d\mathrm{Re}z \, d\mathrm{Im}z = 0$$

which yields $\frac{\partial u}{\partial \nu}|_{\partial D} = 0$. Now consider the following straightforward generalization of Green's formula (3.3),

(4.10)
$$\int_{\partial D} \left({}^t \left(\frac{\partial f}{\partial \nu} \right) g - {}^t f \frac{\partial g}{\partial \nu} \right) |dz| = \int_D {}^t((\Delta - {}^t v)f) \, g - {}^t f \, ((\Delta - v)g) \, d\mathrm{Re}z \, d\mathrm{Im}z,$$

which holds (weakly) for any $f, g \in H^1(D, M_n(\mathbb{C}))$. If we put $f = u$ we obtain

(4.11)
$$\int_D {}^t u \, (-\Delta + v) g \, d\mathrm{Re}z \, d\mathrm{Im}z = 0$$

for any $g \in H^1(D, M_n(\mathbb{C}))$. By Fredholm alternative (see [5, Sec. 6.2]), for each $h \in L^2(D, M_n(\mathbb{C}))$ there exists a unique $g \in H^1_0(D, M_n(\mathbb{C})) = \{g \in H^1(D, M_n(\mathbb{C})) : g|_{\partial D} = 0\}$ such that $(-\Delta + v)g = h$: this yields $u \equiv 0$ on D. Thus Lemma 3.1 is proved. $\qquad\square$

5. AN EXTENSION OF THEOREM 1.1

As an extension of Theorem 1.1 for the case when we do not assume that $v_j|_{\partial D} \equiv 0$, $\frac{\partial}{\partial \nu} v_j|_{\partial D} \equiv 0$, $j = 1, 2$, we give the following result.

Proposition 5.1. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with C^2 boundary, let $v_1, v_2 \in C^2(\bar{D}, M_n(\mathbb{C}))$ be two matrix-valued potentials which satisfy (1.4), with $\|v_j\|_{C^2(\bar{D})} \leq N$ for $j = 1, 2$, and Φ_1, Φ_2 the corresponding Dirichlet-to-Neumann operators. Then, for any $0 < \alpha < \frac{1}{5}$, there exists a constant $C = C(D, N, n, \alpha)$ such that*

$$(5.1) \qquad \|v_2 - v_1\|_{L^\infty(D)} \leq C \left(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}) \right)^{-\alpha},$$

where $\|A\|_1$ is the norm for an operator $A : L^\infty(\partial D, M_n(\mathbb{C})) \to L^\infty(\partial D, M_n(\mathbb{C}))$, with kernel $A(x, y)$, defined as $\|A\|_1 = \sup_{x,y \in \partial D} |A(x, y)|(\log(3 + |x-y|^{-1}))^{-1}$ and $|A(x, y)| = \max_{1 \leq i,j \leq n} |A_{i,j}(x, y)|$.

The only properties of $\| \ \|_1$ we will use are the following:

   i) $\|A\|_{L^\infty(\partial D) \to L^\infty(\partial D)} \leq const(D, n)\|A\|_1$;
   ii) In a similar way as in formula (4.9) of [9] one can deduce

$$\|v\|_{L^\infty(\partial D)} \leq const(n)\|\Phi_v - \Phi_0\|_1,$$

   for a matrix-valued potential v, Φ_v its associated Dirichlet-to-Neumann operator and Φ_0 the Dirichlet-to-Neumann operator of the 0 potential.

We recall a lemma from [11], which generalize Lemma 2.2 to the case of potentials without boundary conditions. We define $(\partial D)_\delta = \{z \in \mathbb{C} : dist(z, \partial D) < \delta\}$.

Lemma 5.2. *For $v \in C^2(\bar{D})$ we have that*

$$(5.2) \qquad \left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}^{(0)}(\lambda) \right| \leq \kappa_1(D, n)\delta^{-4} \frac{\log(3|\lambda|)}{|\lambda|} \|v\|_{C^2(\bar{D})}$$
$$+ \kappa_2(D, n) \log(3 + \delta^{-1})\|v\|_{C(\partial D)},$$

for $z_0 \in D \setminus (\partial D)_\delta$, $0 < \delta < 1$, $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$.

The proof of Lemma 5.2 for the scalar case can be found in [11]: the generalization to the matrix-valued case is straightforward.

Proof of Proposition 5.1. Fix $0 < \alpha < \frac{1}{5}$ and $0 < \delta < 1$. We have the following chain of inequalities

$$
\begin{aligned}
\|v_2 &- v_1\|_{L^\infty(D)} \\
&= \max(\|v_2 - v_1\|_{L^\infty(D \cap (\partial D)_\delta)}, \|v_2 - v_1\|_{L^\infty(D \setminus (\partial D)_\delta)}) \\
&\leq C_1 \max \left(2N\delta + \|\Phi_2 - \Phi_1\|_1, \frac{\log(3\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))}{\delta^4 \log(3 + \|\Phi_2 - \Phi_1\|^{-1})} \right. \\
&\qquad \left. + \log(3 + \frac{1}{\delta})\|\Phi_2 - \Phi_1\|_1 + \frac{\left(\log(3\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))\right)^2}{(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{\frac{3}{4}}} \right) \\
&\leq C_2 \max \left(2N\delta + \|\Phi_2 - \Phi_1\|_1, \frac{1}{\delta^4} \left(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})\right)^{-5\alpha} \right. \\
&\qquad \left. + \log(3 + \frac{1}{\delta})\|\Phi_2 - \Phi_1\|_1 + \frac{\left(\log(3\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))\right)^2}{(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))^{\frac{3}{4}}} \right),
\end{aligned}
$$

where we followed the scheme of the proof of Theorem 1.1 with the following modifications: we made use of Lemma 5.2 instead of Lemma 2.2 and we also used i)-ii); note that $C_1 = C_1(D, N, n)$ and $C_2 = C_2(D, N, n, \alpha)$.

Putting $\delta = \left(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})\right)^{-\alpha}$ we obtain the desired inequality

$$
(5.3) \qquad \|v_2 - v_1\|_{L^\infty(D)} \leq C_3 \left(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})\right)^{-\alpha},
$$

with $C_3 = C_3(D, N, n, \alpha)$, $\|\Phi_2 - \Phi_1\|_1 = \varepsilon \leq \varepsilon_1(D, N, n, \alpha)$ with ε_1 sufficiently small or, more precisely when $\delta_1 = \left(\log(3 + \varepsilon_1^{-1})\right)^{-\alpha}$ satisfies:

$$
\delta_1 < 1, \qquad \varepsilon_1 \leq 2N\delta_1, \qquad \log(3 + \frac{1}{\delta_1})\varepsilon_1 \leq \delta_1.
$$

Estimate (5.3) for general ε (with modified C_3) follows from (5.3) for $\varepsilon \leq \varepsilon_1(D, N, n, \alpha)$ and the assumption that $\|v_j\|_{L^\infty(\bar{D})} \leq N$ for $j = 1, 2$. This completes the proof of Proposition 5.1. $\qquad\square$

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