J. fixed point theory appl. *Online First*

DOI 10.1007/s11784-007-0019-4

Journal of Fixed Point Theory and Applications

Asymptotic structure for solutions of the Cauchy problem for Burgers type equations

G. M. Henkin

Abstract. Large time asymptotic structure for solutions of the Cauchy problem for a generalized Burgers equation is determined. In particular, Gelfand's question about location of viscous shock waves for such equations is answered.

Mathematics Subject Classification (2000). 35K, 35L, 35Q, 35R, 39A, 76D.

Keywords. Burgers type equations, conservation laws, travelling waves, large time asymptotics, fluid mechanics, economic development.

Introduction. Main results

Large time behaviour of solutions of the Cauchy problem for the Navier–Stokes equation in $\mathbb{R}^n$, $n \geq 2$, and for its one-dimensional relative, called the Burgers equation, is one of the most interesting themes of fluid mechanics, initiated by J. Leray [L1, L2] ($n = 2, 3$) and by E. Hopf [Ho] ($n = 1$).

The (generalized) Burgers equations

$$\frac{\partial f}{\partial t} + \varphi(f)\frac{\partial f}{\partial x} = \varepsilon\frac{\partial^2 f}{\partial x^2}, \qquad \varepsilon \geq 0,\ x \in \mathbb{R}, \tag{1a}$$

have been introduced to study different models of fluids ([B], [Bu], [BL], [F], [Co], [LL], ...). The difference-differential analogs of these equations

$$\frac{df}{dt} + \varphi(f)\frac{f(x,t) - f(x-\varepsilon,t)}{\varepsilon} = 0, \qquad x = k\varepsilon,\ k \in \mathbb{Z}, \tag{1b}$$

have been proposed for some models of economic development ([HP1], [HP2]).

The following problem was formulated explicitly by I. M. Gelfand [G, p. 119]: find the asymptotics (as $t \to \infty$) of the solution f of equation (1a) with initial condition

$$f(x,0) = \begin{cases} \alpha^{\pm} & \text{if } \pm x > \pm x^{\pm}, \\ f^0(x) & \text{if } x \in [x^-, x^+], \end{cases} \tag{2}$$

where $\alpha^- \leq \alpha^+$, $x^- \leq x^+$, and f^0 is any L^∞ function on $[x^-, x^+]$.

Gelfand [G] found a solution to this problem for the inviscid case $\varepsilon = +0$ with initial conditions $f(x,0) = \alpha^{\pm}$ if $\pm x > 0$, and noted that it would be interesting to prove that the main term of the asymptotics as $t \to \infty$ of $f(x,t)$ satisfying (1a), (2) coincides with the solution of (1a), (2) for $\varepsilon = +0$. Similar problems and related conjectures were formulated later [HP3], [HP4] for equation (1b).

The purpose of this paper is to present a solution of these problems under rather general assumptions.

Assumption 1. Let $\alpha^- < \alpha^+$ and let φ be a positive twice continuously differentiable function on the interval $[\alpha^-, \alpha^+]$ such that φ' has only isolated zeros.

From [G], [O], [IO], [HP2], [S1] one can deduce the following general properties of solutions of the Cauchy problems (1), (2).

Theorem 0. *Under Assumption* 1*:*

(a) *there exists a unique (weak) solution $f(x,t)$, $x \in \mathbb{R}$, $t > 0$, of problem* (1a), (2)*; this solution is necessarily smooth for $t > 0$, and satisfies the conservation laws*

$$f(x,t) \to \alpha^{\pm}, \qquad x \to \pm\infty,$$

$$\frac{d}{dt}\left[\int_{-\infty}^{0} (\alpha^- - f(x,t))\,dx + \int_{0}^{\infty} (\alpha^+ - f(x,t))\,dx\right] = \int_{\alpha^-}^{\alpha^+} \varphi(y)\,dy, \qquad t > 0;$$

(b) *there exists a unique (weak) solution $f(x,t)$, $x \in \mathbb{R}$, $t > 0$, of problem* (1b), (2)*; this solution is uniformly bounded and satisfies the conservation laws*

$$f(x,t) \to \alpha^{\pm}, \qquad x \to \pm\infty,$$

$$\frac{d}{dt}\left[\sum_{k=-\infty}^{0} \int_{f(k\varepsilon+\theta\varepsilon,t)}^{\alpha^-} \frac{dy}{\varphi(y)} + \sum_{k=1}^{\infty} \int_{f(k\varepsilon+\theta\varepsilon,t)}^{\alpha^+} \frac{dy}{\varphi(y)}\right] = \alpha^+ - \alpha^-, \quad \forall\theta \in [0,1),\ t > 0.$$

Set

$$\psi(u) = -\int_{\alpha^-}^{u} \varphi(y)\,dy, \qquad u \in [\alpha^-, \alpha^+],\ \text{for (1.a)}, \tag{3a}$$

$$\psi(u) = \int_{\alpha^-}^{u} \frac{dy}{\varphi(y)}, \qquad u \in [\alpha^-, \alpha^+],\ \text{for (1.b)}. \tag{3b}$$

Following Gelfand's approach [G] for (1a), (2), adapted for (1b), (2) in [HP4], let us introduce for (3a) and for (3b) the concave function $\hat{\psi}(u)$ as the upper bound of the convex hull of the set

$$\{(u,v) : v \le \psi(u),\ u \in [\alpha^-, \alpha^+]\}.$$

Assumption 2. For (3a) and respectively for (3b) the set $S = \{u \in [\alpha^-, \alpha^+] : \psi(u) < \hat{\psi}(u)\}$ has the form

$$\begin{aligned} &S = (\alpha_0^-, \alpha_0^+) \cup (\alpha_1^-, \alpha_1^+) \cup \cdots \cup (\alpha_L^-, \alpha_L^+), \quad \text{where} \\ &\alpha^- = \alpha_0^- \le \alpha_0^+ < \alpha_1^- < \alpha_1^+ < \cdots < \alpha_{L-1}^- < \alpha_{L-1}^+ < \alpha_L^- \le \alpha_L^+ = \alpha^+. \end{aligned} \tag{4a, b}$$

We use here the labels (4a), (4b) in order to underline the dependence of $\alpha_l^{\pm}$, $l = 0, \dots, L$, on the choice (3a), (3b) of ψ.

Let

$$c_l = \frac{1}{\alpha_l^+ - \alpha_l^-} \int_{\alpha_l^-}^{\alpha_l^+} \varphi(y)\, dy \qquad \text{for (1a)},\ l = 0, \dots, L, \tag{5a}$$

$$c_l = (\alpha_l^+ - \alpha_l^-) \left(\int_{\alpha_l^-}^{\alpha_l^+} \frac{dy}{\varphi(y)} \right)^{-1} \qquad \text{for (1b)},\ l = 0, \dots, L. \tag{5b}$$

Assumptions 1 and 2 imply (see [G], [O], [W], [HP4]) the following important inequalities (for (1a) and respectively for (1b)):

$$\begin{aligned} &\varphi(\alpha_l^+) \le c_l \le \varphi(\alpha_l^-), \quad l = 0, \dots, L, \\ &c_l = \varphi(\alpha_l^-), \quad l = 1, \dots, L, \\ &c_l = \varphi(\alpha_l^+), \quad l = 0, \dots, L-1. \end{aligned} \tag{6}$$

Let us remark that the inequalities above are, in fact, equalities except for the cases $l = 0$ and $l = L$.

Assumption 3. For (1a) and respectively for (1b) the following inequalities are valid:

$$\begin{aligned} &\varphi'(\alpha_l^-) \ne 0, \quad l = 1, \dots, L, \\ &\varphi'(\alpha_l^+) \ne 0, \quad l = 0, \dots, L-1, \\ &\varphi'(\alpha_l^-) \ne \varphi'(\alpha_l^+), \quad l = 1, \dots, L-1, \\ &\varphi(\alpha_0^-) \ne c_0 \quad \text{if } \alpha_0^- < \alpha_0^+, \\ &\varphi(\alpha_L^+) \ne c_L \quad \text{if } \alpha_L^- < \alpha_L^+. \end{aligned}$$

The results of [G] and [O] for equation (1a) and of [HP2], [Be] for equation (1b) imply the following proposition.

Proposition 0. *Under Assumptions* 1 *and* 2, *for any* $l \in \{0, \dots, L\}$ *there exist travelling wave solutions of* (1a) *and respectively of* (1b) *of the form* $f = \tilde{f}_l(x - c_l t)$ *such that* $\tilde{f}_l(x) \to \alpha_l^{\pm}$ *as* $x \to \pm\infty$, $l = 0, \dots, L$.

For these travelling wave solutions the precise asymptotics when $x \to \pm\infty$ has been found (see (3.9), (3.14) and [Be], [HP2, Theorem 2′], [HP4, Theorems 6.1, 6.2]).

The proof of Proposition 0 for (1a) ([G], [O]) is based on the properties of the ordinary differential equation

$$-c\tilde{f}' + \varphi(\tilde{f})\tilde{f}' = \varepsilon \tilde{f}''.$$

The proof for (1b) ([HP2], [Be]) is based on the properties of the difference-differential equation

$$-c\tilde{f}' + \varphi(\tilde{f}) \frac{\tilde{f}(x) - \tilde{f}(x - \varepsilon)}{\varepsilon} = 0$$

and uses, in particular, the Schauder fixed point theorem.

The following two theorems are the main results of this work.

Theorem 1a. *Under Assumptions* 1–3 *and definitions* (3a), (4a), (5a), *for each* $A>0$ *the solution of the Cauchy problem* (1a), (2) *has the following asymptotic structure:*

$$f(x,t) \underset{t\to\infty}{\Rightarrow} \begin{cases} \tilde f_l(x - c_l t - \varepsilon\gamma_l \ln t - o_l(\ln t)) & \text{if } |x-c_l t| < A\sqrt{t},\ l=0,\dots,L,\\ \varphi^{(-1)}(x/t) & \text{if } c_l t + A\sqrt{t} \le x \le c_{l+1}t - A\sqrt{t},\\ & l=0,\dots,L-1,\\ \alpha^- & \text{if } x \le c_0 t - A\sqrt{t},\\ \alpha^+ & \text{if } x \ge c_L t + A\sqrt{t}, \end{cases}$$

where the inverse function $\varphi^{(-1)}(x/t)$ *to* φ *is well defined on the intervals* $[c_l, c_{l+1}]$, $l=0,\dots,L$; $\{\tilde f_l(x-c_l t)\}$ *are the travelling wave solutions of* (1a) *with overfalls* $[\alpha_l^-, \alpha_l^+]$; *and* $\{\gamma_l\}$ *are the constants, depending on* $\{\alpha_l^\pm, \varphi(\alpha_l^\pm), \varphi'(\alpha_l^\pm)\}$ *by explicit formulas:*

$$\gamma_0 = \begin{cases} 0 & \text{if } L=0,\\ \dfrac{1}{\alpha_0^+ - \alpha_0^-}\left(-\dfrac{2}{\varphi'(\alpha_0^+)}\right) & \text{if } L>0 \text{ and } \alpha_0^- < \alpha_0^+, \end{cases} \tag{7}$$

$$\gamma_l = \frac{1}{\alpha_l^+ - \alpha_l^-}\left(\frac{2}{\varphi'(\alpha_l^-)} - \frac{2}{\varphi'(\alpha_l^+)}\right), \qquad l=1,\dots,L-1, \tag{8}$$

$$\gamma_L = \begin{cases} \dfrac{1}{\alpha_L^+ - \alpha_L^-}\left(\dfrac{2}{\varphi'(\alpha_L^-)}\right) & \text{if } L>0 \text{ and } \alpha_L^- < \alpha_L^+,\\ 0 & \text{if } L=0. \end{cases} \tag{9}$$

Theorem 1b. *Under Assumptions* 1–3 *and definitions* (3b), (4b), (5b), *for each* $A>0$ *the solution of the Cauchy problem* (1b), (2) *has the following asymptotic structure:*

$$f(x,t) \underset{t\to\infty}{\Rightarrow} \begin{cases} \tilde f_l(x - c_l t - \frac{1}{2}\varepsilon c_l \gamma_l \ln t - o_l(\ln t)) & \text{if } |x-c_l t| < A\sqrt{t},\ l=0,\dots,L,\\ \varphi^{(-1)}(x/t) & \text{if } c_l t + A\sqrt{t} \le x \le c_{l+1}t - A\sqrt{t},\\ & l=0,\dots,L-1,\\ \alpha^- & \text{if } x \le c_0 t - A\sqrt{t},\\ \alpha^+ & \text{if } x \ge c_L t + A\sqrt{t}, \end{cases}$$

where $\{\tilde f_l(x-c_l t)\}$ *are the travelling wave solutions of* (1b) *with overfalls* $[\alpha_l^-, \alpha_l^+]$, *and* $\{\gamma_l\}$ *are the constants depending on* $\{\alpha_l^\pm, \varphi(\alpha_l^\pm), \varphi'(\alpha_l^\pm)\}$ *by formulas* (7)–(9).

Theorem 1b is motivated by applications to some models of economic development, based on J. Schumpeter's ideas (see [S] and [HP1]–[HP4]).

Theorem 1a is motivated by Gelfand's question in the theory of quasilinear equations and also by its deep relation to Theorem 1b (see [G] and [H], [HS], [HST]).

Remark 1. Theorems 1a, 1b generalize several earlier results:

- [Ho] and [Co] for (1a) and [HP1] for (1b) if $\varphi(f)$ is a linear function;
- [IO] for (1a) and [HP2] for (1b) if $L=0$;
- [W] for (1a) and [HP4] for (1b) if $x \in [c_l t + A\sqrt{t}, c_{l+1}t - A\sqrt{t}]$, $l=0,\dots,L-1$;
- [HS] if $L=1$ and the shift functions for the travelling waves have the form $O(\ln t)$ instead of the more precise form $\varepsilon\gamma_l \ln t + o_l(\ln t)$, $l=0,1$.

Remark 2. Theorem 1b proves (under Assumptions 1–3) a conjecture in [HP4, p. 718] and also more detailed conjectures [HS, p. 1463] and [H, p. 453].

Remark 3. Theorem 1a answers the question of [LMN, p. 296]: "In the Cauchy problem (for (1a)) there is a question of determining the location of viscous shock-waves".

Remark 4. Theorems 1a, 1b are still valid in the important case when $\alpha^- = \alpha^+ = \alpha$ and $\varphi'(\alpha) \neq 0$ instead of the inequality $\alpha^- < \alpha^+$ in Assumption 1. Moreover, in such a case the solutions of (1), (2) have the following asymptotic behaviour:

$$\sup_{x\in\mathbb{R}} |f(x,t) - \alpha| = O(1/\sqrt{t}), \qquad t \to \infty.$$

For equation (1a) it follows from [Ho], [La], [W].

Remark 5. In the case when we have the equality $\varphi(\alpha_0^-) = c_0$ or $\varphi(\alpha_L^+) = c_L$, instead of the corresponding inequalities in Assumption 3, Theorems 1a, 1b (and their proofs) are also valid, but with corrected constants γ_0 and γ_L:

$$\gamma_0 = \frac{1}{\alpha_0^+ - \alpha_0^-}\left(\frac{1}{\varphi'(\alpha_0^-)} - \frac{2}{\varphi'(\alpha_0^+)}\right), \qquad \text{if } L>0,\ \varphi(\alpha_0^-) = c_0 \text{ and } \varphi'(\alpha_0^+) \neq 2\varphi'(\alpha_0^-),$$

$$\gamma_L = \frac{1}{\alpha_L^+ - \alpha_L^-}\left(\frac{2}{\varphi'(\alpha_L^-)} - \frac{1}{\varphi'(\alpha_L^+)}\right), \qquad \text{if } L>0,\ \varphi(\alpha_L^+) = c_L \text{ and } 2\varphi'(\alpha_L^+) \neq \varphi'(\alpha_L^-),$$

$$\gamma_0 = \frac{1}{\alpha_0^+ - \alpha_0^-}\begin{cases} 1/\varphi'(\alpha_0^-) & \text{if } L=0,\ \varphi(\alpha_0^-) = c_0,\ \varphi(\alpha_0^+) \neq c_0, \\ -1/\varphi'(\alpha_0^+) & \text{if } L=0,\ \varphi(\alpha_0^-) \neq c_0,\ \varphi(\alpha_0^+) = c_0, \\ 1/\varphi'(\alpha_0^-) - 1/\varphi'(\alpha_0^+) & \text{if } L=0,\ \varphi(\alpha_0^-) = \varphi(\alpha_0^+) = c_0. \end{cases}$$

Under the condition $L=0$ (shock profile condition) this statement was obtained earlier in [HST].

Remark 6. Theorems 1a, 1b imply new interesting phenomena: if $L>0$ and $x \in [c_l t - A\sqrt{t}, c_l t + A\sqrt{t}]$, $l \in \{0,\dots,L\}$, then solutions of (1a), (2a) and respectively of (1b), (2b) converge to shifted travelling waves $\tilde{f}_l(x - c_l t - \varepsilon\gamma_l \ln t + o_l(\ln t))$ and respectively $\tilde{f}_l(x - c_l t - \frac{1}{2}\varepsilon c_l \gamma_l \ln t + o_l(\ln t))$, which generally do not satisfy equations (1a) or (1b) and the positions of which on the x-line, more precisely the coordinate x_l of the point x where $\tilde{f}_l(x - c_l t - \frac{1}{2}\varepsilon c_l\gamma_l \ln t + o_l(\ln t)) = (\alpha_l^+ + \alpha_l^-)/2$, depend essentially on the (viscosity) parameter $\varepsilon > 0$. These phenomena lead to the appropriate correction of Gelfand's suggestion [G] that the main term of the asymptotics ($t \to \infty$) of $f(x,t)$, satisfying (1a), coincides with the solution of (1a) for $\varepsilon = +0$ with the same initial conditions.

A similar phenomenon was observed earlier in [LY] in the special boundary value problem for the classical Burgers equation: if $u(x,t)$ satisfies the conditions

$$u_t + u \cdot u_x = u_{xx}, \qquad u(0,t) = 1, \qquad u(\infty,t) = -1, \qquad u(x,0) = -\operatorname{th}(x/2),$$

then

$$|u(x,t)| + |\operatorname{th}[(1/2)(x - \ln(1+t))]| \to 0, \qquad x \geq 0,\ t \to \infty.$$

Remark 7. One can see that the asymptotic behaviour of solutions of (1b), (2) is not the same as the asymptotic behaviour of solutions of (1a), (2) when $\varepsilon \to +0$, in spite of the fact that in the limiting case $\varepsilon = +0$ the equations (1a) and (1b) are identical. This can be explained by the fact that equation (1b) is a semi-discrete approximation of the nonphysical equation

$$\frac{\partial f}{\partial t} + \varphi(f)\frac{\partial f}{\partial x} = \frac{\varepsilon}{2}\varphi(f)\frac{\partial^2 f}{\partial x^2}.$$

In order to relate (1b) to a physically relevant equation one can use the substitution $F = \int_0^f dy/\varphi(y)$, which transforms (1b) into

$$\frac{\partial F(x,t)}{\partial t} + \frac{\psi(F(x,t)) - \psi(F(x-\varepsilon,t))}{\varepsilon} = 0,$$

where $\psi'(F) = \varphi(f)$. This equation is the so called monotone one-sided semi-discrete approximation of the viscous equation

$$\frac{\partial F}{\partial t} + \varphi(f)\frac{\partial F}{\partial x} = \frac{\varepsilon}{2}\frac{\partial}{\partial x}\left(\varphi(f)\frac{\partial F}{\partial x}\right),$$

where $F(x,0) \to \int_0^{\alpha^\pm} dy/\varphi(y)$ as $x \to \pm\infty$.

The results on finite-difference approximations for nonlinear conservation laws (see [EO], [HHL], [S2]) explain both the similarity of behaviour of solutions $F(x,t)$ of the last two equations and also some difference in the behaviour of solutions of (1a) and (1b).

Conjecture 1. Theorems 1a, 1b are also valid in the case when for some $l \in \{1,\ldots,L-1\}$ we have the equality $\varphi'(\alpha_l^-) = \varphi'(\alpha_l^+)$ instead of the corresponding inequality in Assumption 3.

Conjecture 2. Under the assumptions of Theorem 1a (respectively Theorem 1b), let $\varepsilon = +0$. Then the solution of the Cauchy problem (1a), (2) (respectively (1b), (2)) has the following asymptotic structure:

$$f(x,t) \underset{t\to\infty}{\overset{L^1(\mathbb{R})}{\Rightarrow}} \begin{cases} \alpha^- & \text{if } x < c_0 t + d_0, \\ \varphi^{(-1)}(x/t) & \text{if } c_l t + d_l \leq x < c_{l+1}t + d_{l+1},\ l = 0,\ldots,L-1, \\ \alpha^+ & \text{if } x \geq c_L t + d_L, \end{cases}$$

where the parameters c_l are determined by (5a) (respectively by (5b)) and the parameters d_l are determined by the respective equation (1a) or (1b) and initial data (2a) or (2b).

Several important results in the direction of this conjecture were obtained in [Li], [C], [KP].

The proofs of Theorems 1a, 1b combine improved versions of earlier techniques (maximum and comparison principles, Lyapunov type functions, Poisson–Green kernels for parabolic type equations) together with several new ingredients.

One of them (§3) is the discovery of "localized conservations laws", which for the problem (1a), (2) under assumption $\varepsilon = 1$ have the form

$$\int_{c_l t - A\sqrt{t}}^{c_l t + A\sqrt{t}} [f(x,t) - \tilde{f}(x - c_l t - d_l(t,A))]\, dx = 0, \tag{10a}$$

where $d_l(t,A) = \gamma_l \ln t + o_{A,l}(\ln t)$, $t > 0$, $A > 0$, $l = 0, \dots, L$, while for the problem (1b), (2) under the assumption $\varepsilon = 1$ and $x = k \in \mathbb{Z}$ they have the form

$$\begin{aligned}
&\sum_{k=[c_l t - A\sqrt{t}]+1}^{[c_l t + A\sqrt{t}]-1} (\Phi(f(k,t)) - \Phi(\tilde{f}(k - c_l t - d_l(t,A)))) \\
&\qquad \pm (c_l t \pm A\sqrt{t} - [c_l t \pm A\sqrt{t}]) \\
&\qquad \times (\Phi(f([c_l t \pm A\sqrt{t}], t)) - \Phi(\tilde{f}([c_l t \pm A\sqrt{t}] - c_l t - d_l(t,A)))) = 0,
\end{aligned} \tag{10b}$$

where $\Phi(f) = \int_f^{\alpha^+} dy/\varphi(y)$, $d_l(t,A) = \frac{1}{2} c_l \gamma_l \ln t + o_{A,l}$, $t > 0$, $A > 0$, $l = 0, \dots, L$.

Other new ingredients (§§1, 2) are precise a priori estimates of solutions $f(x,t)$ and their derivatives $f'_x(x,t)$ for (1a), (2) and (1b), (2) in the transitional domains of parameters x, t, where $x = c_l t \pm A\sqrt{t}$ and the travelling wave behaviour of solutions changes into the rarefaction behaviour. Namely, for all $A > 0$ and for all $\delta \in (0,1)$ for $x = c_l t + A\sqrt{t}$, $l = 0, \dots, L-1$ and for $x = c_l t - A\sqrt{t}$, $l = 1, \dots, L$, we have the estimates:

$$f(x,t) = \varphi^{(-1)}\left(\frac{x}{t}\right) + O\left(\frac{1}{A\sqrt{t}}\right) \quad \text{for (1a) and (1b),} \tag{11}$$

$$\frac{\partial f}{\partial x}(x,t) = \frac{\partial}{\partial x}\varphi^{(-1)}\left(\frac{x}{t}\right) + O\left(\frac{1}{A^{1-\delta} t}\right) \quad \text{for (1a),} \tag{12}$$

$$\Delta f(x,t) := f(x,t) - f(x-1,t) = \frac{\partial}{\partial x}\varphi^{(-1)}\left(\frac{x}{t}\right) + O\left(\frac{1}{A^{1-\delta} t}\right) \quad \text{for (1b).} \tag{13}$$

The main part of the proof of Theorem 1b (§§4–6) consists in proving the following estimate (see §6, inequalities (6.13), (6.14)): for all $\delta > 0$ and $l \in \{0, \dots, L\}$,

$$\begin{aligned}
\overline{\lim_{t\to\infty}} \sup_{\{n \,:\, |n - c_l t| \le \sqrt{\delta c_l t}\}} \Big| \sum_{k=[c_l t - \sqrt{\delta c_l t}]}^{n} (\Phi(f(k,t)) - \Phi(\tilde{f}_l(k - c_l t - d_l(t, \sqrt{\delta c_l})))) \Big| \\
\le O(\sqrt{\delta}).
\end{aligned} \tag{14}$$

The proof of (14) uses nonlinear parabolic type equations (see §4, (4.5)) for the functions

$$\Delta_l(n,t,d_l(\tau,\sqrt{\delta c_l})) = \sum_{k=[c_l\tau-\sqrt{\delta c_l\tau}]}^{n} (\Phi(f(k,t)) - \Phi(\tilde{f}_l(n-c_lt-d_l(\tau,\sqrt{\delta c_l}))))$$

of variables n,t in the domain $n \in [c_l\tau - \sqrt{\delta c_l \tau}, c_lt + \sqrt{\delta c_l t}] \cap \mathbb{Z}$, $t \in (\tau, \tau+\sqrt{\delta\tau})$. Localized conservation laws are used to get a priori boundary estimates

$$|\Delta_l([c_lt+\sqrt{\delta c_l t}], t, d_l(\tau,\sqrt{\delta c_l}))| = O(1/\sqrt{\tau}).$$

The estimate (14) implies uniform convergence $f(n,t) \Rightarrow \tilde{f}_l(n-c_lt-d_l(t,o(1)))$ in the intervals $|n-c_lt| \le o(\sqrt{t})$, $l = 0,\dots,L$.

More precisely, define the following family of functions:

$$F(n,t,d_1,\dots,d_L) = \begin{cases} \tilde{f}_l(n-c_lt-d_l(t)) & \text{if } |n-c_lt| < t^{1/4},\ l=0,\dots,L, \\ \varphi^{(-1)}(n/t) & \text{if } c_lt+t^{1/4} \le n \le c_{l+1}t - t^{1/4}, \\ \alpha^- & \text{if } n \le c_0t - t^{1/4}, \\ \alpha^+ & \text{if } n \ge c_Lt + t^{1/4}. \end{cases}$$

Propositions from §§1, 3, 6 imply the following result.

Theorem 2. *Let $f(n,t)$ be the solution of the Cauchy problem* (1b), (2), *where $\varepsilon = 1$, $L \ge 0$, $n \in \mathbb{Z}$, $t > 0$. Then under the assumptions and notations of Theorem* 1b *there exist shift functions $d_l(t) = \frac{1}{2}c_l\gamma_l/\ln t + o(\ln t)$, $l = 0,\dots,L$, such that $f(n,t)$ approaches $F(n,t,d_1,\dots,d_L)$ with the estimate*

$$\sup_{z\in\mathbb{Z}} |f(n,t) - F(n,t,d_1,\dots,d_L)| = O(t^{-1/4}).$$

Theorem 2 implies Theorem 1b. In this paper we give a complete proof of Theorem 2 and, as a consequence, of Theorem 1b.

A complete proof of Theorem 1a will be given in another paper. But several important steps of the latter proof, which are similar to the corresponding steps of the proof of Theorem 1b, will be indicated in this paper.

Some further questions

The problem of finding the asymptotics ($t \to \infty$) of solutions of (viscous) conservation laws has been posed originally by Gelfand not only for generalized Burgers equations but also for systems of conservation laws in one spatial variable (see [G]). In this direction many important results on existence and asymptotic stability of viscous shock profiles (continuous and discrete) have been obtained and applied (see [BHR], [S1] and references there). Results of the type of Theorems 1a, 1b above for systems of conservation laws have not been obtained yet.

It is very interesting also to study the asymptotic behaviour of scalar (viscous) conservation laws in several spatial variables (continuous or discrete), relying on the asymptotic properties of Burgers type equations. In this direction there are several important results and problems (see [BP], [HP2], [HZ], [S1], [W]) and references there).

1. Estimates of $f(x,t)$ for $x = c_l t \pm A\sqrt{t}$

For the proof of Theorem 1b we need, first, the following comparison proposition, which is an essential improvement of Theorem 7.5 from [HP4].

Proposition 1. *Under the assumptions of Theorem* 1b*, the solution* $f(x,t)$ *of* (1b), (2) *satisfies the following estimate: for every* $\gamma > 0$ *and* $b_l > c_l/\gamma$, $l = 0,\dots,L-1$, *there exists* $t_0 > 0$ *such that*

$$\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right) \le f(x,t) \le \varphi^{(-1)}\left(\frac{x+\gamma\sqrt{t}}{t}\right) \tag{1.1}$$

for $x \in [c_l t + b_l\sqrt{t}, c_{l+1}t - b_{l+1}\sqrt{t}]$ *and* $t \ge t_0$, *where the constants* c_l *are defined by* (3b), (5b), *and* $t_0 = O(b\gamma)$.

Complement. Proposition 1 (and its proof) is also valid for the solution $f(x,t)$ of (1a), (2) (under the assumptions of Theorem 1a) if in its formulation we replace $b_l > c_l/\gamma$ by $b_l > 2/\gamma$, $l = 0,\dots,L-1$.

Corollary. *Under the assumptions of Theorem* 1a *and* 1b, *for any* $A > c_l/\gamma$, $\gamma > 0$, *the solution* $f(x,t)$ *of* (1a), (2), *respectively of* (1b), (2) *satisfies estimate* (1.1) *for* $x = c_l t + A\sqrt{t}$, $l = 0,\dots,L-1$, *and for* $x = c_l t - A\sqrt{t}$, $l = 1,\dots,L$,.

The proof of Proposition 1 is based on two important lemmas: the first (Lemma 0) is a comparison principle permitting us to estimate solutions of (1b), (2) through sub(super)solutions of (1b), (2), the second (Lemma 1) gives an explicit construction of necessary sub(super)solutions for (1b), (2).

The comparison statement below (Lemma 0) is obtained in [HP4, Lemma 7.3]. It can be considered an analogue for problem (1b), (2) of the well-known comparison principle for problem (1a), (2) (see [W, Lemma 2.1]).

Lemma 0. *Let* φ *be a Lipschitz continuous function on* $\mathbb{R}$, *and* $x_+(t)$ *be a continuous function of* $t \ge 0$, *with* $x_+(t) \ge 1$. *Suppose that functions* $g(n,t)$, $\hat g(n,t)$ *with values in* $[\alpha^-,\alpha^+]$ *satisfy the inequalities*

$$\begin{aligned}
&\frac{dg(n,t)}{dt} \ge \varphi(g(n,t))(g(n-1,t)-g(n,t)),\\
&\frac{d\hat g(n,t)}{dt} \le \varphi(\hat g(n,t))(\hat g(n-1,t)-\hat g(n,t)), \qquad n \ge 0,\ t \ge t_0.
\end{aligned}$$

If

$$\begin{aligned}
&g(n,t_0) > \hat g(n,t_0) \quad \text{for } n \in [0,x(0)],\\
&g(0,t) \ge \hat g(0,t), \quad g(-1,t) \ge \hat g(-1,t) \quad \text{for } t \ge t_0,\\
&g([x_+(t)],t) > \hat g([x_+(t)],t) \quad \text{for } t \ge t_0,
\end{aligned}$$

then $g(n,t) > \hat g(n,t)$ *for all* $n \in [0,x_+(t)]$ *and* $t \ge t_0$.

The following statement essentially generalizes and specifies Proposition 1 from [HS].

Lemma 1. *Under the assumptions of Proposition* 1, *let* $\varepsilon = 1$, $L = 1$; $\alpha_0^- < \alpha_0^+ < \alpha_1^- < \alpha_1^+$; *and let* c_0, c_1 *be the parameters defined by* (3b), (4b), (5b). *Put* $\Delta_x f(x,t) = f(x,t) - f(x-1,t)$. *Consider the following functions* $f^{\pm}(x,t)$, *depending also on the parameters* $\{\alpha_l^{\pm}\}$, $\{c_l\}$, *small positive parameters* γ *and* δ *and positive bounded functions* $b_0^-(t)$, $b_1^-(t)$, $b_0^+(t)$, $b_1^+(t)$:

$$f^-(x,t) = \begin{cases} f_1^-(x,t) = \tilde{f}_0(x - c_0 t), \quad -\infty < x < c_0 t + b_0^- \sqrt{t}; \\ f_2^-(x,t) = \varphi^{(-1)}\left(\dfrac{x - \gamma\sqrt{t}}{t}\right) - \dfrac{c_0}{\varphi'(\alpha_0^+)(x - c_0 t)}, \\ \hfill c_0 t + b_0^- \sqrt{t} \le x \le c_1 t + b_1^- \sqrt{t}; \\ f_3^-(x,t) = \tilde{f}_1(x - c_1 t - (2\sqrt{c_1} + \gamma + 2\delta)\sqrt{t}), \\ \hfill c_1 t + b_1^- \sqrt{t} < x < +\infty. \end{cases} \tag{1.2}$$

$$f^+(x,t) = \begin{cases} f_1^+(x,t) = \tilde{f}_0(x - c_0 t + (2\sqrt{c_0} + \gamma + 2\delta)\sqrt{t}), \\ \hfill -\infty < x < c_0 t - b_0^+ \sqrt{t}; \\ f_2^+(x,t) = \varphi^{(-1)}\left(\dfrac{x + \gamma\sqrt{t}}{t}\right) + \dfrac{c_1}{\varphi'(\alpha_1^-)(c_1 t - x)}, \\ \hfill c_0 t - b_0^+ \sqrt{t} \le x \le c_1 t - b_1^+ \sqrt{t}; \\ f_3^+(x,t) = \tilde{f}_1(x - c_1 t), \quad c_1 t - b_1^+ \sqrt{t} < x < +\infty. \end{cases} \tag{1.3}$$

Then:

(i) *For all* $\gamma, \delta > 0$ *there exist functions*

$$\begin{aligned} b_0^-(t) &= \gamma + o(1), \\ b_1^-(t) &= \gamma + \sqrt{c_1} + \delta + \sqrt{\delta^2 + 2\delta\sqrt{c_1}} + o(1), \\ b_0^+(t) &= \gamma + \sqrt{c_0} + \delta + \sqrt{\delta^2 + 2\delta\sqrt{c_0}} + o(1), \\ b_1^+(t) &= \gamma + o(1), \end{aligned}$$

satisfying for all $t \ge t_0$ *and* $\theta \in [0,1]$ *the relations:*

$$\begin{aligned} &f_1^-(c_0 t + b_0^- \sqrt{t}, t) = f_2^-(c_0 t + b_0^- \sqrt{t}, t), \\ &\Delta_x f_1^-(c_0 t + b_0^- \sqrt{t} + \theta, t) \le \Delta_x f_2^-(c_0 t + b_0^- \sqrt{t} + \theta, t); \end{aligned} \tag{1.4}$$

$$\begin{aligned} &f_2^-(c_1 t + b_1^- \sqrt{t}, t) = f_3^-(c_1 t + b_1^- \sqrt{t}, t), \\ &\Delta_x f_2^-(c_1 t + b_1^- \sqrt{t} + \theta, t) \le \Delta_x f_3^-(c_1 t + b_1^- \sqrt{t} + \theta, t); \end{aligned} \tag{1.5}$$

$$\begin{aligned} &f_1^+(c_0 t - b_0^+ \sqrt{t}, t) = f_2^+(c_0 t - b_0^+ \sqrt{t}, t), \\ &\Delta_x f_1^+(c_0 t - b_0^+ \sqrt{t} + \theta, t) \ge \Delta_x f_2^+(c_0 t - b_0^+ \sqrt{t} + \theta, t); \end{aligned} \tag{1.6}$$

$$\begin{aligned} &f_2^+(c_1 t - b_1^+ \sqrt{t}, t) = f_3^+(c_1 t - b_1^+ \sqrt{t}, t), \\ &\Delta_x f_2^+(c_1 t - b_1^+ \sqrt{t} + \theta, t) \ge \Delta_x f_3^+(c_1 t - b_1^+ \sqrt{t} + \theta, t). \end{aligned} \tag{1.7}$$

(ii) *For all* $\gamma, \delta > 0$ *and with* b_0^-, b_1^-, b_0^+, b_1^+ *from* (i) *there exists* $t_0 \geq 0$ *such that the functions* $f^{\mp}(x,t)$, $x \in \mathbb{R}$, $t \geq t_0$, *are sub(super)solutions for* (1b), *i.e.*

$$\pm\left[\frac{df^{\pm}}{dt} + \varphi(f^{\pm})(f^{\pm}(x,t) - f^{\pm}(x-1,t))\right] \geq 0. \tag{1.8}$$

Complement. This lemma (and its proof) is also valid for equation (1a) (under the assumptions of Theorem 1a) if in the definitions of $f_2^{\mp}(x,t)$ the numerators c_0, c_1 are replaced by 2, the differences $\Delta_x f_j^{\mp}$ by the derivatives $\partial f^{\mp}/\partial x$, and inequality (1.8) by

$$\pm\left[\frac{\partial f^{\pm}}{\partial t} + \varphi(f^{\pm})\frac{\partial f^{\pm}}{\partial x} - \frac{\partial^2 f^{\pm}}{\partial x^2}\right] \geq 0.$$

Proof. (i) Let us check first the existence of $b_0^- = \gamma + o(1)$ satisfying (1.4). Relations (1.4) mean that

$$\tilde{f}_0(b_0^-\sqrt{t}) = \varphi^{(-1)}\left(\frac{c_0 t + (b_0^- - \gamma)\sqrt{t}}{t}\right) - \frac{c_0}{\varphi'(\alpha_0^+)b_0^-\sqrt{t}},$$

$$\Delta_x \tilde{f}_0(x - c_0 t)|_{x = c_0 t + b_0^-\sqrt{t} + \theta} \leq \Delta_x \varphi^{(-1)}\left(\frac{x - \gamma\sqrt{t}}{t}\right)\Big|_{x = c_0 t + b_0^-\sqrt{t} + \theta} + \frac{c_0(1 - O(1/\sqrt{t}))}{\varphi'(\alpha_0^+)(b_0^-\sqrt{t})^2}.$$

From these relations and the asymptotic formula for $\tilde{f}_0(x)$ from [HP4] (see (3.9) below) we obtain the equivalent inequalities

$$\alpha_0^+ - \frac{c_0}{\varphi'(\alpha_0^+)b_0^-\sqrt{t}} = \alpha_0^+ + \frac{1}{\varphi'(\alpha_0^+)}\frac{(b_0^- - \gamma)}{\sqrt{t}} - \frac{c_0}{\varphi'(\alpha_0^+)b_0^-\sqrt{t}} + o(1/\sqrt{t}),$$

$$\frac{c_0}{\varphi'(\alpha_0^+)(b_0^-\sqrt{t})^2} \leq \frac{1}{\varphi'(\alpha_0^+)}\frac{1}{t} + \frac{c_0}{\varphi'(\alpha_0^+)(b_0^-\sqrt{t})^2} + o(1/t).$$

These inequalities are both satisfied if $b_0^- = \gamma + o(1)$, where $o(1) = -\varphi'(\alpha_0^+)\sqrt{t} \times o(1/\sqrt{t})$ and $t \geq t_0$.

Let us check further the existence of $b_1^- = \gamma + \sqrt{c_1} + \delta + \sqrt{\delta^2 + 2\delta\sqrt{c_1}} + o(1)$ satisfying (1.5). Relations (1.5) mean that

$$\varphi^{(-1)}\left(\frac{c_1 t + (b_1^- - \gamma)\sqrt{t}}{t}\right) - \frac{c_0}{\varphi'(\alpha_0^+)(c_1 t - c_0 t + b_1^-\sqrt{t})} = \tilde{f}_1(b_1^-\sqrt{t} - (2\sqrt{c_1} + 2\delta + \gamma)\sqrt{t}),$$

$$\Delta_x \varphi^{(-1)}\left(\frac{x - \gamma\sqrt{t}}{t}\right)\Big|_{x = c_1 t + b_1^-\sqrt{t} + \theta} + \frac{c_0(1 + o(1/\sqrt{t}))}{\varphi'(\alpha_0^+)(c_1 t - c_0 t + b_1^-\sqrt{t})^2} \leq \Delta_x \tilde{f}_1(x - c_1 t - (2\sqrt{c_1} + 2\delta + \gamma)\sqrt{t})|_{x = c_1 t + b_1^-\sqrt{t} + \theta}.$$

From these relations and the asymptotic formula for $\tilde{f}_1(x)$ from [HP4] (see (3.9)) we obtain the equivalent inequalities

$$\begin{aligned}\alpha_1^- &+ \frac{1}{\varphi'(\alpha_1^-)}\frac{b_1^- - \gamma}{\sqrt{t}} - \frac{c_0}{\varphi'(\alpha_0^+)(c_1 t - c_0 t + b_1^-\sqrt{t})} \\ &= r\alpha_1^- + \frac{c_1}{\varphi'(\alpha_1^-)(-b_1^-\sqrt{t} + (2\sqrt{c_1} + 2\delta + \gamma)\sqrt{t})} + o(1/\sqrt{t}),\\ &\frac{1}{\varphi'(\alpha_1^-)}\frac{1}{t} \le \frac{c_1}{\varphi'(\alpha_1^-)(-b_1^-\sqrt{t} + (2\sqrt{c_1} + 2\delta + \gamma)\sqrt{t})^2} + o(1/t).\end{aligned}$$

These inequalities are satisfied for $t \ge t_0$ if we have the equality

$$\frac{c_1}{(2\sqrt{c_1} + 2\delta + \gamma - b_1^-)} = b_1^- - \gamma + o(1)$$

and the inequality

$$1 < \frac{c_1}{(2\sqrt{c_1} + 2\delta + \gamma - b_1^-)^2}.$$

The equality above means that

$$b_1^- = \gamma + \sqrt{c_1} + \delta + \sqrt{\delta^2 + 2\delta\sqrt{c_1}} + o(1).$$

With such b_1^- the inequality above is satisfied for $t \ge t_0$.

Let us now check the existence of

$$b_0^+ = \gamma + \sqrt{c_0} + \delta + \sqrt{\delta^2 + 2\delta\sqrt{c_0}} + o(1)$$

satisfying (1.6). Relations (1.6) mean that

$$\begin{aligned}&\tilde{f}_0((2\sqrt{c_0} + 2\delta + \gamma - b_0^+)\sqrt{t})\\ &\qquad = \varphi^{(-1)}\left(\frac{c_0 t - (b_0^+ - \gamma)\sqrt{t}}{t}\right) + \frac{c_1}{\varphi'(\alpha_1^-)((c_1 - c_0)t + b_0^+\sqrt{t})},\\ &\Delta_x \tilde{f}_0(x - c_0 t + (2\sqrt{c_0} + 2\delta + \gamma)\sqrt{t})|_{x = c_0 t - b_0^+\sqrt{t} + \theta}\\ &\qquad \ge \Delta_x \varphi^{(-1)}\left(\frac{x + \gamma\sqrt{t}}{t}\right)\Bigg|_{x = c_0 t - b_0^+\sqrt{t} + \theta} + \frac{c_1(1 + O(1/\sqrt{t}))}{\varphi'(\alpha_1^-)((c_1 - c_0)t + b_0^+\sqrt{t})^2}.\end{aligned}$$

From these relations and the asymptotic formula for $\tilde{f}_0(x)$ we obtain the equivalent inequalities:

$$\begin{aligned}\alpha_0^+ &- \frac{c_0}{\varphi'(\alpha_0^+)(2\sqrt{c_0} + 2\delta + \gamma - b_0^+)\sqrt{t}}\\ &= \alpha_0^+ - \frac{1}{\varphi'(\alpha_0^+)}\frac{b_0^+ - \gamma}{\sqrt{t}} + \frac{c_1}{\varphi'(\alpha_1^-)((c_1 - c_0)t + b_0^+\sqrt{t})} + o(1/\sqrt{t}),\\ &\frac{c_0}{\varphi'(\alpha_0^+)(2\sqrt{c_0} + 2\delta + \gamma - b_0^+)^2 t} \ge \frac{1}{\varphi'(\alpha_0^+)}\frac{1}{t} + o(1/t).\end{aligned}$$

These relations are satisfied if we have the equality

$$\frac{c_0}{(2\sqrt{c_0}+2\delta+\gamma-b_0^+)}=b_0^+-\gamma+o(1)$$

and the inequality

$$\frac{c_0}{(2\sqrt{c_0}+2\delta-(b_0^+-\gamma))^2}>1.$$

The equality above implies that

$$b_0^+=\gamma+\sqrt{c_0}+\delta+\sqrt{\delta^2+2\delta\sqrt{c_0}}+o(1).$$

With such b_0^+ the inequality above is satisfied for $t\ge t_0$.

Let us check, finally, the existence of $b_1^+=\gamma+o(1)$ satisfying (1.7). Relations (1.7) mean that

$$\begin{aligned}
&\varphi^{(-1)}\left(\frac{c_1t-b_1^+\sqrt{t}+\gamma\sqrt{t}}{t}\right)+\frac{c_1}{\varphi'(\alpha_1^-)b_1^+\sqrt{t}}=\tilde f_1(-b_1^+\sqrt{t}),\\
&\Delta_x\left(\varphi^{(-1)}\left(\frac{x+\gamma\sqrt{t}}{t}\right)+\frac{c_1}{\varphi'(\alpha_1^-)(c_1t-x)}\right)\Big|_{x=c_1t-b_1^+\sqrt{t}+\theta}\\
&\qquad\qquad\qquad\qquad\ge\Delta_x\tilde f_1(x-c_1t)\big|_{x=c_1t-b_1^+\sqrt{t}+\theta}.
\end{aligned}$$

From these relations and the asymptotic formula for $\tilde f_1(x)$ we obtain the equivalent inequalities

$$\begin{aligned}
&\alpha_1^-+\frac{1}{\varphi'(\alpha_1^-)}\frac{(\gamma-b_1^+)}{\sqrt{t}}+\frac{c_1}{\varphi'(\alpha_1^-)b_1^+\sqrt{t}}=\alpha_1^-+\frac{c_1}{\varphi'(\alpha_1^-)b_1^+\sqrt{t}}+o(1/\sqrt{t}),\\
&\frac{1}{\varphi'(\alpha_1^-)}\frac{1}{t}+\frac{c_1}{\varphi'(\alpha_1^-)(b_1^+)^2t}\ge\frac{c_1}{\varphi'(\alpha_1^-)(b_1^+)^2t}+o(1/t).
\end{aligned}$$

These relations are satisfied for $b_1^+=\gamma+o(1)$ and $t\ge t_0$.

(ii) Let us check inequality (1.8) for the function $f^-(x,t)$, defined by (1.2). From (i) it follows that it is now sufficient under the conditions of (i) to prove the inequalities

$$\frac{df_j^-}{dt}+\varphi(f_j^-)(f_j^-(x,t)-f_j^-(x-1,t))\le 0,\qquad j=1,2,3,\tag{1.9}$$

on the respective intervals:

$$\begin{aligned}
&x<c_0t+b_0^-\sqrt{t} &&\text{for } j=1,\\
&c_0t+b_0^-\sqrt{t}\le x\le c_1t+b_1^-\sqrt{t} &&\text{for } j=2,\\
&x>c_1t+b_1^-\sqrt{t} &&\text{for } j=3.
\end{aligned}$$

Inequality (1.9) for $j=1$ is valid because the function $\tilde f_0(x-c_0t)$ satisfies (1b). To check (1.9) for $j=3$ we remark that by definition (1.8),

$$\frac{\partial}{\partial t}f_3^-(x,t)=\left(-c_1-(2\sqrt{c_1}+\gamma+2\delta)\frac{1}{2\sqrt{t}}\right)\frac{\partial}{\partial x}\tilde f_1(x-c_1t-(2\sqrt{c_1}+\gamma+2\delta)\sqrt{t}).$$

Applying [HP2, Theorem 2′], we obtain

$$-c_1\frac{\partial}{\partial x}f_3^- + \varphi(f_3^-(x,t))\cdot(f_3^-(x,t) - f_3^-(x-1,t)) = 0, \qquad \frac{\partial}{\partial x}f_3^- \geq 0.$$

Hence,

$$\frac{\partial}{\partial t}f_3^- + \varphi(f_3^-)\cdot(f_3^-(x,t) - f_3^-(x-1,t)) = -(2\sqrt{c_1}+\gamma+2\delta)\frac{1}{2\sqrt{t}}\frac{\partial f_3^-}{\partial x} \leq 0.$$

Let us check (1.9) for $j=2$, when $c_0t + b_0^-\sqrt{t} \leq x \leq c_1t + b_1^-\sqrt{t}$. Let φ' and $\dot\varphi$ both mean the derivative of φ. We have

$$\frac{\partial f_2^-(x,t)}{\partial t} = \frac{\partial\varphi^{(-1)}(\frac{x-\gamma\sqrt{t}}{t})}{\partial t} - \frac{c_0^2}{\varphi'(\alpha_0^+)(x-c_0t)^2}, \tag{1.10}$$

$$\begin{aligned}
&\varphi(f_2^-(x,t))(f_2^-(x,t) - f_2^-(x-1,t)) = \varphi\left(\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right) - \frac{c_0}{\varphi'(\alpha_0^+)(x-c_0t)}\right)\\
&\times\left[\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right) - \varphi^{(-1)}\left(\frac{x-1-\gamma\sqrt{t}}{t}\right) - \frac{c_0}{\varphi'(\alpha_0^+)(x-c_0t)} + \frac{c_0}{\varphi'(\alpha_0^+)(x-1-c_0t)}\right]\\
&\qquad = \left[\varphi\left(\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right)\right) - \dot\varphi\left(\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right)\right)\cdot\frac{c_0}{\dot\varphi(\alpha_0^+)(x-c_0t)}\right.\\
&\qquad\quad \left. + \frac{1}{2}\ddot\varphi(\cdot)\frac{c_0^2}{(\dot\varphi(\alpha_0^+))^2(x-c_0t)^2}\right]\\
&\qquad\quad \times\left[\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right) - \varphi^{(-1)}\left(\frac{x-1-\gamma\sqrt{t}}{t}\right) + \frac{c_0}{\dot\varphi(\alpha_0^+)(x+\theta-c_0t)^2}\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\varphi(f_2^-(x,t))(f_2^-(x,t) - f_2^-(x-1,t))\\
&\qquad = \varphi\left(\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right)\right)\left[\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right) - \varphi^{(-1)}\left(\frac{x-1-\gamma\sqrt{t}}{t}\right)\right]\\
&\qquad\quad + \left[-\dot\varphi\left(\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right)\right) + \frac{1}{2}\ddot\varphi(\cdot)\frac{c_0}{\dot\varphi(\alpha_0^+)(x-c_0t)}\right]\\
&\qquad\quad \times\frac{c_0}{\dot\varphi(\alpha_0^+)(x-c_0t)}\left[\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right) - \varphi^{(-1)}\left(\frac{x-1-\gamma\sqrt{t}}{t}\right)\right]\\
&\qquad\quad + \varphi\left(\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right)\right)\frac{c_0}{\dot\varphi(\alpha_0^+)(x+\theta-c_0t)^2}\\
&\qquad\quad + \left[-\dot\varphi\left(\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right)\right) + \frac{1}{2}\ddot\varphi(\cdot)\frac{c_0}{\dot\varphi(\alpha_0^+)(x-c_0t)}\right]\\
&\qquad\quad \times\frac{c_0}{\dot\varphi(\alpha_0^+)(x-c_0t)}\cdot\frac{c_0}{\dot\varphi(\alpha_0^+)(x+\theta-c_0t)^2}.
\end{aligned}\tag{1.11}$$

From [HP4, Lemma 5.4], we have

$$\frac{\partial\varphi^{(-1)}(\frac{x-\gamma\sqrt{t}}{t})}{\partial t} + \varphi\left(\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right)\right)\left[\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right) - \varphi^{(-1)}\left(\frac{x-1-\gamma\sqrt{t}}{t}\right)\right] \leq 0 \qquad (1.12)$$

if $c_0 t + b_0^- \sqrt{t} \leq x \leq c_1 t + b_1^- \sqrt{t},\ t \geq t_0$.

Relations (1.10)–(1.12) imply that to prove (1.9) for $j = 2$ it is now sufficient to check the following:

$$\begin{aligned} &- \frac{c_0^2}{\dot\varphi(\alpha_0^+)(x-c_0t)^2} - \left(\frac{x-\gamma\sqrt{t}}{t}\right)\frac{c_0}{\dot\varphi(\alpha_0^+)(x+\theta-c_0t)^2} \\ &\qquad + \left[\dot\varphi\left(\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right)\right) - \frac{1}{2}\ddot\varphi(\cdot)\frac{c_0}{\dot\varphi(\alpha_0^+)(x-c_0t)}\right] \\ &\qquad \times \frac{c_0}{\dot\varphi(\alpha_0^+)(x-c_0t)} \cdot \frac{1}{\dot\varphi(\alpha_0^+ + o(1))}\frac{1}{t} \\ &\qquad + \left[\dot\varphi\left(\varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right)\right) - \frac{1}{2}\ddot\varphi(\cdot)\frac{c_0}{\dot\varphi(\alpha_0^+)(x-c_0t)}\right] \\ &\qquad \times \frac{c_0^2}{(\dot\varphi(\alpha_0^+))^2(x-c_0t)^3} + O\left(\frac{1}{(x-c_0t)^3}\right) \leq 0 \end{aligned} \qquad (1.13)$$

if $c_0 t + \gamma\sqrt{t} + o(1)\sqrt{t} \leq x \leq c_1 t + (\gamma + \sqrt{c_1} + O_+(\delta))\sqrt{t}$.

In order to have (1.13) it is sufficient to have

$$\begin{aligned} &- \frac{c_0^2}{\dot\varphi(\alpha_0^+)(x-c_0t)^2} + \frac{1+o(1)}{t}\frac{c_0}{\dot\varphi(\alpha_0^+)(x-c_0t)} \\ &\qquad - \left(\frac{x-\gamma\sqrt{t}}{t}\right)\frac{c_0}{\dot\varphi(\alpha_0^+)(x-c_0t)^2} + O\left(\frac{1}{(x-c_0t)^3}\right) \leq 0. \end{aligned}$$

For this it is sufficient to prove

$$-\frac{c_0(1+o(1))}{x-c_0t} + \frac{1+o(1)}{t} - \frac{x-\gamma\sqrt{t}}{t(x-c_0t)} \leq 0 \qquad (1.14)$$

if $c_0 t + \gamma\sqrt{t} + o(1)\sqrt{t} \leq x \leq c_1 t + (\gamma + \sqrt{c_1} + O_+(\delta))\sqrt{t}$.

Let $x = ct + \lambda\sqrt{t}$, where $c \in [c_0, c_1]$, $\lambda \in [\gamma + o(1), \gamma + \sqrt{c_1} + O_+(\delta)]$. To prove (1.14) we use the fact that

$$\begin{aligned} \frac{1+o(1)}{t} - \frac{ct+(\lambda-\gamma)\sqrt{t}}{t(ct-c_0t+\lambda\sqrt{t})} &= \frac{1}{t}\left[1+o(1) - \frac{c+(\lambda-\gamma)(1/\sqrt{t})}{c-c_0+\lambda(1/\sqrt{t})}\right] \\ &\leq \frac{1}{t}\left[1+o(1) - \frac{c+o(1/\sqrt{t})}{(c-c_0)+\lambda(1/\sqrt{t})}\right] \end{aligned}$$

$$\leq \frac{1}{t}\left[1+o(1)-\frac{c+o(1/\sqrt{t})}{c-c_0+(\gamma+\sqrt{c_1}+O_+(\delta))/\sqrt{t}}\right]$$
$$\leq \frac{1}{t}\left[1+o(1)-\frac{c_1+o(1/\sqrt{t})}{c-c_0+o(1/\sqrt{t})}\right]=-\frac{1+o(1)}{t}\left[\frac{c_0}{c_1-c_0}\right].$$

Hence,

$$-\frac{c_0(1+o(1))}{x-c_0t}+\frac{1+o(1)}{t}-\frac{x-\gamma\sqrt{t}}{t(x-c_0t)}=-\frac{c_0(1+o(1))}{x-c_0t}-\frac{(1+o(1))c_0}{t(c_1-c_0)}<0$$

if $x \geq c_0t+\gamma\sqrt{t}+o(1)\sqrt{t}$ and $t \geq t_0$.

Inequality (1.14) together with (1.8) for $f^-(x,t)$ is proved. Inequality (1.8) for $f^+(x,t)$ can be obtained in a similar way. Lemma 1 is proved.

Proof of Proposition 1. We will give the proof of Proposition 1 only in the typical case when

$$\varepsilon = 1, \quad L=1; \quad \alpha^- = \alpha_0^- < \alpha_0^+ < \alpha_1^- < \alpha_1^+ = \alpha^+;$$
$$\varphi(\alpha_0^-) > c_0 = \varphi(\alpha_0^+), \quad \varphi(\alpha_1^-) = c_1 > \varphi(\alpha_1^+).$$

The general case can be treated in a similar way.

Let the function $f \mapsto \varphi(f)$ be extended outside $[\alpha^-,\alpha^+]$ keeping Assumption 1 and with the condition $\varphi'(f)<0$ if $f<\alpha^-$ or $f>\alpha^+$.

By Proposition 0 for any sufficiently small $\sigma^->0$ there exists $\sigma^+=O(\sigma^-)>0$ and σ-modified travelling wave solutions of the form

$$\tilde{f}^{\pm}_{l,\sigma}(x-c^{\mp}_{l,\sigma}t \mp d_{l,\sigma})$$

with overfalls $[\alpha_l^- \mp (-1)^l\sigma^-, \alpha_l^+ \pm (-1)^l\sigma^+]$, $l=0,1$. By (5b) we have

$$\frac{1}{c^{\mp}_{l,\sigma}}=\frac{1}{\alpha_l^+-\alpha_l^- \pm (-1)^l(\sigma^++\sigma^-)}\int_{\alpha_l^- \mp(-1)^l\sigma^-}^{\alpha_l^+ \pm(-1)^l\sigma^+}(1/\varphi(y))\,dy.$$

For (1b) we have

$$c^{\mp}_{l,\sigma}=c_l(1\pm O_+(\sigma^-)), \quad \text{where } 0<O_+(\sigma^-)\leq \text{const}\cdot\sigma^-.$$

Replacing in the definitions of $f^{\mp}(x,t)$ in the statement of Lemma 1 the travelling waves $\tilde{f}_l(x-c_lt)$, $l=0,1$, by the σ-modified travelling waves

$$\tilde{f}^-_{l,\sigma}(x-c^-_{l,\sigma}t-d_{l,\sigma}) \quad \text{and} \quad \tilde{f}^+_{l,\sigma}(x-c^+_{l,\sigma}t+d_{l,\sigma}), \quad l=0,$$

we obtain functions

$$f^{\mp}_\sigma(x,t)=\{f^{\mp}_{j,\sigma}(x,t),\ j=1,2,3\}$$

of the form (1.2), (1.3), where the parameters α_l^-, α_l^+, c_l, b_l^-, b_l^+, $l=0,1$, are replaced by the σ-modified parameters:

$$\alpha^{-\mp}_{l,\sigma}=\alpha_l^- \mp (-1)^l\sigma^-, \quad \alpha^{+\mp}_{l,\sigma}=\alpha_l^+ \pm (-1)^l\sigma^+,$$
$$b^{\mp}_{l,\sigma}=b^{\mp}_l \pm O_+(\sigma^-), \qquad c^{\mp}_{l,\sigma}=c_l(1\pm O_+(\sigma^-)).$$

Put further $\sigma^-=\exp(-t^{1/3})$ and $d_{l,\sigma}=N\ln(1/\sigma^-)=Nt^{1/3}$. For fixed $t_0 \geq 0$

and for N large enough the formulas for $f_\sigma^{\mp}(x,t)$ imply

$$f_\sigma^-(x,t) < f(x,t) < f_\sigma^+(x,t), \quad x<0,\, t\ge t_0,$$
$$f_\sigma^-(+\infty,t) < f(\infty,t) < f_\sigma^+(+\infty,t), \quad t\ge t_0.$$

From these inequalities we deduce the existence of $T>0$ such that for initial values $f(x,t_0)$ satisfying (2) we have

$$f_\sigma^-(x,t_0+T) < f(x,t_0) < f_\sigma^+(x,t_0-T).$$

Hence, from the comparison principle for solutions of (1b) (Lemma 0) we deduce

$$f_\sigma^-(x,t+T) < f(x,t) < f_\sigma^+(x,t-T), \quad t\ge t_0,\, x\in\mathbb{R}. \tag{1.15}$$

From the σ-modified versions of (1.2), (1.3) for $f_\sigma^{\mp}(x,t)$ we deduce

$$f_{2,\sigma}^-(x,t) \ge \varphi^{(-1)}\left(\frac{x-\gamma\sqrt{t}}{t}\right) - \frac{c_{0,\sigma}^-}{\varphi'(\alpha_0^{++})(x-c_{0,\sigma}^- t)}$$

if $c_{0,\sigma}^- t+\gamma+o(1) < x < c_{1,\sigma}^- t$.

Hence, for $x\in(c_{0,\sigma}^- t + B_0^-\sqrt{t}, c_{1,\sigma}^- t)$, where $B_0^->\gamma$ and $t\ge t_0$, we obtain

$$\begin{aligned} f_{2,\sigma}^-(x,t) &\ge \alpha_{0,\sigma}^{++} + \frac{1}{\varphi'(\alpha_{0,\sigma}^{++})}\left(\frac{x-c_{0,\sigma}^- t}{t} - \frac{\gamma}{\sqrt{t}} - \frac{c_{0,\sigma}^-}{B_0^-\sqrt{t}}\right) \\ &= \alpha_{0,\sigma}^{++} + \frac{1}{\varphi'(\alpha_{0,\sigma}^{++})}\left(\frac{x-c_{0,\sigma}^- t-\tilde\gamma\sqrt{t}}{t}\right), \end{aligned}$$

where $\tilde\gamma = \gamma + c_{0,\sigma}^-/B_0^-$. From this we deduce the inequality

$$f_{2,\sigma}^-(x,t) \ge \varphi^{(-1)}\left(\frac{x-\tilde\gamma\sqrt{t}}{t}\right), \tag{1.16}$$

where $c_{0,\sigma}^- t + B_0^-\sqrt{t} < x < c_{1,\sigma}^- t$, $B_0^- = c_{0,\sigma}^-/(\tilde\gamma-\gamma)$, $t\ge t_0$. In a similar way we obtain the inequality

$$f_{2,\sigma}^+(x,t) \le \varphi^{(-1)}\left(\frac{x+\tilde\gamma\sqrt{t}}{t}\right), \tag{1.17}$$

where $c_{0,\sigma}^+ t < x < c_{1,\sigma}^+ t - B_1^+\sqrt{t}$, $B_1^+ = c_{1,\sigma}^+/(\tilde\gamma-\gamma)$, $t\ge t_0$. From (1.15)–(1.17) it follows that for all $\tilde\gamma>\gamma>0$ there exist $t_0, T, N>0$ such that

$$\varphi^{(-1)}\left(\frac{x-\tilde\gamma\sqrt{t+T}}{t}\right) \le f(x,t) \le \varphi^{(-1)}\left(\frac{x+\tilde\gamma\sqrt{t-T}}{t}\right)$$

if

$$(c_0+O(\sigma^-))t + \left(\frac{c_0+O(\sigma^-)}{\tilde\gamma-\gamma}\right)\sqrt{t} < x < (c_1-O(\sigma^-))t - \left(\frac{c_1-O(\sigma^-)}{\tilde\gamma-\gamma}\right)\sqrt{t},$$

where $t\ge t_0$, $\sigma^- = N\exp(-t^{1/3})$. The last inequalities for x can be replaced by

$$c_0 t + \frac{c_0}{\tilde\gamma-\gamma}\sqrt{t} + o(1) < x < c_1 t - \frac{c_1}{\tilde\gamma-\gamma}\sqrt{t} + o(1).$$

The inequalities above imply the statement of Proposition 1.

2. Estimates of $\frac{\partial f}{\partial x}(x,t)$ for $x = c_l t \pm A\sqrt{t}$

We now formulate a proposition which for equation (1a) improves the results of the general theory of quasilinear parabolic equations (see [LSU]), and for equation (1b) the results of the recent work [HST] concerning a priori estimates of derivatives of solutions of these equations.

Proposition 2. *Under the assumptions of Theorem* 1b *and of Proposition* 1 *let* $\varepsilon = 1$, $L > 0$, $\varphi(\alpha_l^+) = c_l$, $l = 0, \dots, L-1$, $\varphi(\alpha_l^-) = c_l$, $l = 1, \dots, L$. *Let* $\tilde{b}_l > b_l > c_l/\gamma$, $l = 0, \dots, L$, $\gamma, \delta > 0$. *Then the difference* $\Delta f := f(x,t) - f(x-1,t)$ *for the solution* f *of* (1b), (2) *satisfies the following estimates:*

$$\Delta f = \frac{1}{\varphi'(\alpha_l^+)t} + O\left(\frac{\gamma^{1-\delta}}{\varphi'(\alpha_l^+)t}\right) \tag{2.1}$$

for $x \in [c_l t + b_l\sqrt{t}, c_l t + \tilde{b}_l\sqrt{t}]$, $l = 0, \dots, L-1$, $t \geq t_0$, *and*

$$\Delta f = \frac{1}{\varphi'(\alpha_l^-)t} + O\left(\frac{\gamma^{1-\delta}}{\varphi'(\alpha_l^-)t}\right) \tag{2.2}$$

for $x \in [c_l t - \tilde{b}_l\sqrt{t}, c_l t - b_l\sqrt{t}]$, $l = 1, \dots, L$, $t \geq t_0$, *where*

$$t_0 = O\left(\frac{\tilde{b}_l}{b_l} + \frac{b_l\gamma}{c_l} + \frac{1}{\delta}\right).$$

Complement. Proposition 2 is also valid for the solution $f(x,t)$ of (1a), (2) (under the assumptions of Theorem 1a and of Proposition 1 if in its formulation we replace $b_l > c_l/\gamma$ by $b_l > 2/\gamma$ and the difference Δf by the derivative $\partial f/\partial x$. The proof in this case uses some complex analysis technique and will be given in another paper.

Lemma 2. *Under the assumptions of Proposition* 2, *put* $u = f - \varphi^{(-1)}(x/t)$. *Then* u *is well defined for* $x \in [c_l t + b_l\sqrt{t}, c_{l+1}t - b_{l+1}\sqrt{t}]$, $l = 0, \dots, L-1$, *and satisfies for such* x *the inequality*

$$|u(x,t)| = O\left(\frac{\gamma}{\dot{\varphi}(\varphi^{(-1)}(x/t))\sqrt{t}}\right), \tag{2.3}$$

where $\gamma = \sup\{1/b_l, 1/b_{l+1}\}$. *Moreover, equation* (1b) *implies*

$$\frac{du}{dt} + \frac{x}{t}\Delta u + \frac{u}{t} + \frac{1}{2}\dot{\varphi}(\varphi^{(-1)}(x/t))\Delta u^2 = O(1/t^2). \tag{2.4}$$

Proof. The definition of u and estimate (1.1) from Proposition 1 imply (2.3). From (2.3) and Theorem 2 of [HST] it follows that

$$|\Delta u(x,t)| = O\left(\frac{\tilde{\gamma}}{\dot{\varphi}(\varphi^{(-1)}(x/t))t}\right), \tag{2.5}$$

where $x \in [c_l t + b_l\sqrt{t}, c_{l+1}t - b_{l+1}\sqrt{t}]$, $\tilde{\gamma} = \tilde{\gamma}(b_l, b_{l+1})$.

Equation (1b) for f, the definition of u, and inequalities (2.3) and (2.5) imply for $x \in [c_l t + b_l\sqrt{t}, c_{l+1}t - b_{l+1}\sqrt{t}]$ the following:

$$\begin{aligned}
0 &= \frac{d}{dt}(u + \varphi^{(-1)}(x/t)) + \varphi(f)(\Delta u + \Delta\varphi^{(-1)}(x/t)) \\
&= \frac{du}{dt} - \frac{1}{t^2}\frac{x}{\dot\varphi(\varphi^{(-1)}(x/t))} + \varphi(f)\Delta u \\
&\quad + \varphi(f)\left[\frac{1}{\dot\varphi(\varphi^{(-1)}(x/t))t} + O(1/t^2)\right] \\
&= \frac{du}{dt} + \varphi(f)\Delta u + \left[\frac{x}{t} + \dot\varphi(\varphi^{(-1)}(x/t))u + O(1/t)\right] \\
&\quad \times \left[\frac{1}{\dot\varphi(\varphi^{(-1)}(x/t))t} + O(1/t^2)\right] \\
&\quad - \frac{x}{\dot\varphi(\varphi^{(-1)}(x/t))t^2} = \frac{du}{dt} + \varphi(f)\Delta u + \frac{u}{t} + O(1/t^2) \\
&= \frac{du}{dt} + \frac{x}{t}\Delta u + \frac{1}{2}\dot\varphi(\varphi^{(-1)}(x/t))\Delta u^2 + \frac{u}{t} + O(1/t^2).
\end{aligned}$$

To obtain the last equality we have used in addition the relation

$$u \cdot \Delta u = \frac{1}{2}\Delta u^2 - \frac{1}{2}(\Delta u)^2.$$

In the proof of Proposition 2 we will use the Green–Poisson type formula, associated with the operator $u'_t + \Delta u$. Let $\chi_0 : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $0 \le \chi_0 \le 1$, $\chi_0|_{(-\infty,b)} \equiv 0$, $\chi_0|_{((\tilde b+b)/2),\infty)} \equiv 1$, $|\chi_0'| \le A_0/\delta$, $|\chi_0''| \le A_0/\delta^2$, $\delta = \tilde b - b$, $b < \tilde b$. Put $\chi(x,t) = \chi_0(\frac{x-t}{\sqrt{t}})$.

Lemma 3. *Let $u(x,t)$ be a function defined in the domain*

$$\Omega = \left\{(x,t) : t > 0,\, b - \varepsilon < \bar x := \frac{x-t}{\sqrt{t}} < \tilde b + \sigma\sqrt{t}\right\}, \qquad \sigma > \sigma_0 > 0,$$

and $\tilde u(x,t) = u(x,t)\cdot\chi(x,t)$. Then the function $\tilde u = u\cdot\chi$ can be represented in Ω for $\alpha t > t_0$, $\alpha \in (\frac{1+\sigma_0}{1+\sigma}, 1)$ by the following formula:

$$\begin{aligned}
\tilde u(x,t) = &\int_{-\infty}^{\infty} G(x-\xi, t-\alpha t)\tilde u(\xi,\alpha t)\,d\xi \\
&+ \int_{\alpha t}^{t} d\tau \int_{-\infty}^{\infty} G(x-\xi,t-\tau)(\tilde u'_\tau + \Delta\tilde u)(\xi,\tau)\,d\xi, \qquad (2.6)
\end{aligned}$$

where

$$G(x,t) = \sum_{n=-\infty}^{\infty} G_n(t)\cdot\delta(n-x), \qquad \delta(\cdot) \text{ the Dirac function,}$$

$$G_n(t) = \frac{t^n}{n!}e^{-t}, \quad n \ge 0; \qquad G_n(t) = 0, \quad n < 0 \quad \text{(the Poisson distribution).}$$

Proof. This statement is certainly classical. In the present form it has been formulated and proved in [HS, p. 1475] and [HST, p. 730].

Lemma 4. *Under the assumptions of Lemmas* 2, 3, *suppose that for some* $l \in \{0,\dots,L-1\}$ *we have* $\tilde{b}_l = \tilde{b} > b_l = b \geq c_l/\gamma$. *Put* $\varphi(\alpha_l^+) = c_l = 1$. *Let a function* $u(\xi,\tau)$ *satisfy* (2.4) *in variables* ξ, τ. *Then the function* $\tilde{u}(\xi,\tau) = u(\xi,\tau)\cdot\chi(\xi,\tau)$ *satisfies*

$$\begin{aligned}\tilde{u}'_\tau + \Delta\tilde{u} = &-\frac{\xi-\tau}{\tau}\Delta u\cdot\chi - \frac{1}{2}\dot{\varphi}(\varphi^{(-1)}(\xi/\tau))(\Delta u^2)\chi - \frac{u}{\tau}\chi\\ &+ u(\chi'_\tau + \Delta\chi) + O(1/\tau^2), \qquad \tau \geq t_0. \end{aligned}\tag{2.7}$$

Proof. From $\tilde{u} = u\cdot\chi$ and $\Delta\tilde{u} = \Delta u\cdot\chi + u\cdot\Delta\chi - \Delta u\cdot\Delta\chi$ it follows that

$$\tilde{u}'_\tau + \Delta\tilde{u} = (u'_\tau + \Delta u)\chi + u(\chi'_\tau + \Delta\chi) - \Delta u\cdot\Delta\chi.$$

Putting in this relation formula (2.4) in variables (ξ,τ) and using the estimate $|\Delta u\cdot\Delta\chi| = O(1/\tau^2)$ (see [HST, Th. 2]) we obtain (2.7).

Lemma 5. *Under the assumptions of Lemmas* 2–4, *for every* $k = 1,2,\dots$ *we have the following representation formula for* $\Delta^k u(x,t) := \Delta(\Delta^{k-1}u(x,t))$ *if* $(x,t) \in \Omega := \{(x,t) : \alpha t \geq t_0,\ x \geq t + \frac{1}{2}(b+\tilde{b})\sqrt{t}\}$*:*

$$\Delta^k u = I_0^k u + I_1^k u + I_2^k u + I_3^k u + I_4^k u + I_5^k u, \tag{2.8}$$

where

$$\begin{aligned}
I_0^k u &= -\int_{\alpha t}^t d\tau \int_{\bar{\xi}\geq b} \Delta_x^k G(x-\xi,t-\tau)\frac{\xi-\tau}{\tau}\Delta_\xi u(\xi,\tau)\chi(\xi,\tau)\,d\xi,\\
I_1^k u &= -\int_{\alpha t}^t d\tau \int_{\bar{\xi}\geq b} \Delta_x^k G(x-\xi,t-\tau)\frac{1}{2}\dot{\varphi}(\varphi^{(-1)}(\xi/\tau))\Delta_\xi u^2(\xi,\tau)\chi(\xi,\tau)\,d\xi,\\
I_2^k u &= \int_{\bar{\xi}\geq b} \Delta_x^k G(x-\xi,t-t^*)u(\xi,t^*)\,d\xi,\\
I_3^k u &= \int_{\alpha t}^t d\tau \int_{\bar{\xi}\geq b} \Delta_x^k G(x-\xi,t-\tau)(u\chi'_\tau + u\cdot\Delta\chi)(\xi,\tau)\,d\xi,\\
I_4^k u &= -\int_{\alpha t}^t d\tau \int_{\bar{\xi}\geq b} \Delta_x^k G(x-\xi,t-\tau)\frac{u(\xi,\tau)}{\tau}\chi\,d\xi,\\
I_5^k u &= \int_{\alpha t}^t d\tau \int_{\bar{\xi}\geq b} \Delta_x^k G(x-\xi,t-\tau)O(1/\tau^2)\chi(\xi,\tau)\,d\xi.
\end{aligned}$$

Proof. Putting formula (2.4) in (2.6) and applying Δ_x^k to the left- and right-hand sides of (2.6) we obtain (2.8).

Lemma 6.

$$\int_\xi |\Delta_\xi^k G(k-\xi,t-\tau)|\,d\xi = \min\{2^k, O(1/(t-\tau)^{k/2})\}.$$

Proof. For $k = 1, 2$ this has been proved in [HST, Lemma 6(iv)]. This proof admits a straightforward extension to the case $k \geq 3$.

Lemma 7. *Put $\delta = \tilde{b} - b = 1/\gamma$, $b = 1/\gamma$, $\gamma \leq 1$. Under the assumptions and notations of Lemmas 2–5, for $k \geq 3$ we have the following estimate of $\Delta^k u$:*

$$\begin{aligned}|\Delta^k u| &= \frac{(1-\alpha)^{-k/2}}{\dot{\varphi} t^{(k+1)/2}} O\biggl(\frac{\gamma}{\sqrt{\alpha}} + \frac{\gamma(1-\alpha)}{\alpha^{3/2}}\biggl(1 + \frac{1}{\delta^2} + \frac{\tilde{b}}{\delta} + \frac{\gamma}{\delta}\biggr) \\ &\quad + \frac{1-\alpha}{\delta\alpha^{3/2}} + \biggl(\frac{\gamma^2}{\alpha} + \frac{1}{\alpha}\biggr)\sqrt{1-\alpha}\biggr) \\ &\leq \frac{\mathrm{const}(\alpha)}{\dot{\varphi} t^{(k+1)/2}}, \qquad \text{where } k \geq 3,\ \alpha t \geq t_0,\ \frac{b+\tilde{b}}{2} \leq \frac{x-t}{\sqrt{t}} \leq \tilde{b}. \end{aligned} \tag{2.9}$$

Remark. For $k = 1, 2$ the method below gives only the estimate

$$|\Delta^k u| = \frac{\ln t}{\dot{\varphi} t^{(k+1)/2}} O(1), \qquad t \geq t_0,\ k = 1, 2.$$

Proof. The definition of $I_5^k u$ implies

$$|I_5^k u| = O(1/(\alpha t)^2) \int_{\alpha t}^{t} d\tau \int_{\bar{\xi} \geq b} |\Delta_x^k G(x-\xi, t-\tau)|\, d\xi.$$

The definition of $I_4^k u$ and (2.3) imply

$$|I_4^k u| = O\biggl(\frac{\gamma}{\dot{\varphi}(\alpha t)^{3/2}}\biggr) \int_{\alpha t}^{t} d\tau \int_{\bar{\xi} \geq b} |\Delta_x^k G(x-\xi, t-\tau)|\, d\xi.$$

The definition of $I_3^k u$, (2.3) and the estimate $|\chi'_\tau + \Delta\chi| = O(1/\delta^2 + \tilde{b}/\delta)\frac{1}{\tau}$ from Lemma 4 of [HST] imply

$$|I_3^k u| = O\biggl(\frac{\gamma}{\dot{\varphi}(\alpha t)^{3/2}}\biggl(\frac{1}{\delta^2} + \frac{\tilde{b}}{\delta}\biggr)\biggr) \int_{\alpha t}^{t} d\tau \int_{\bar{\xi} \geq b} |\Delta_x^k G(x-\xi, t-\tau)|\, d\xi.$$

The definition of $I_2^k u$ and (2.3) imply

$$|I_2^k u| = O\biggl(\frac{\gamma}{\dot{\varphi}\sqrt{\alpha t}}\biggr) \int_{\bar{\xi} \geq b} |\Delta_x^k G(x-\xi, t-t^*)|\, d\xi.$$

The definition of $I_1^k u$ and the relation $\Delta(v \cdot u^2) = v \cdot \Delta u^2 + u^2(\xi-1, \tau)\Delta v$ imply

$$\begin{aligned} I_1^k u &= (-1)^k \int_{\alpha t}^{t} d\tau \int_{\bar{\xi} \geq b} \Delta_\xi^k G(x-\xi, t-\tau) \frac{1}{2} \dot{\varphi}(\varphi^{(-1)}(\xi/\tau)) \Delta_\xi u^2(\xi, \tau) \chi(\xi, \tau)\, d\xi \\ &= (-1)^{k+1} \int_{\alpha t}^{t} d\tau \int_{\bar{\xi} \geq b} \Delta\biggl(\Delta_\xi^k G(x-\xi, t-\tau) \frac{1}{2} \dot{\varphi}(\varphi^{(-1)}(\xi/\tau)) \chi(\xi, \tau)\biggr) u^2(\xi-1, \tau)\, d\xi \\ &= \frac{(-1)^{k+1}}{2} \int_{\alpha t}^{t} d\tau \int_{\bar{\xi} \geq b} [\Delta_\xi^k G(x-\xi, t-\tau) \cdot \Delta(\dot{\varphi} \cdot \chi) + (\dot{\varphi}\chi)(\xi-1, \tau) \Delta_\xi^{k+1} G(x-\xi, t-\tau)] \\ &\quad \times u^2(\xi-1, \tau)\, d\xi. \end{aligned}$$

This together with (2.3) and $\Delta\chi = O(\frac{1}{\delta\sqrt{\tau}})$ and $\Delta\dot\varphi(\varphi^{(-1)}(\xi/\tau)) = O(\frac{\ddot\varphi}{\dot\varphi\tau})$ implies

$$|I_1^k u| = O\left(\frac{\gamma^2}{\dot\varphi\alpha t}\right)\int_{\alpha t}^t d\tau \int_{\bar\xi\geq b} |\Delta_\xi^{k+1}G(x-\xi,t-\tau)|\,d\xi$$
$$+ O\left(\frac{\gamma^2\ddot\varphi}{\dot\varphi^3(\alpha t)^2} + \frac{\gamma^2}{\dot\varphi(\alpha t)^{3/2}\delta}\right)\int_{\alpha t}^t d\tau \int_{\bar\xi\geq b} |\Delta_\xi^{k}G(x-\xi,t-\tau)|\,d\xi.$$

The definition of $I_0^k u$ and the relation $\Delta v\cdot u = v\cdot\Delta u + u(\xi-1,\tau)\Delta v$ imply

$$\begin{aligned}
I_0^k u &= (-1)^k \int_{\alpha t}^t d\tau \int_{\bar\xi\geq b} \Delta_\xi^k G(x-\xi,t-\tau)\frac{\xi-\tau}{\tau}\Delta_\xi u(\xi,\tau)\chi(\xi,\tau)\,d\xi \\
&= (-1)^{k+1}\int_{\alpha t}^t d\tau \int_{\bar\xi\geq b} \Delta\left(\Delta_\xi^k G(x-\xi,t-\tau)\frac{\xi-\tau}{\tau}\chi\right)u(\xi-1,\tau)\,d\xi \\
&= (-1)^{k+1}\int_{\alpha t}^t d\tau \int_{\bar\xi\geq b}\bigg[\Delta_\xi^k G(x-\xi,t-\tau)\cdot\Delta\left(\frac{\xi-\tau}{\tau}\chi\right) \\
&\quad + \frac{\xi-\tau-1}{\tau}\chi(\xi-1,\tau)\Delta_\xi^{k+1}G(x-\xi,t-\tau)\bigg]u(\xi-1,\tau)\,d\xi.
\end{aligned}$$

This together with the relations $\Delta(\frac{\xi-\tau}{\tau}\chi) = \frac{\xi-1}{\tau}\Delta\chi + \chi(\xi-1,\tau)\cdot\frac{1}{\tau}$, $|\Delta\chi| = O(\frac{1}{\delta\sqrt{\tau}})$, $|\frac{\xi-1}{\tau}u(\xi-1,\tau)| = O(\frac{1}{\dot\varphi\tau})$ and estimate $|\frac{u}{\tau}| = O(\frac{\gamma}{\dot\varphi\tau^{3/2}})$ implies

$$|I_0^k u| = \left[O\left(\frac{1}{\dot\varphi\alpha t\cdot\delta\sqrt{\alpha t}}\right) + O\left(\frac{\gamma}{\dot\varphi(\alpha t)^{3/2}}\right)\right]\int_{\alpha t}^t d\tau\int_{\bar\xi\geq b}|\Delta_\xi^k G(x-\xi,t-\tau)|\,d\xi$$
$$+ O\left(\frac{1}{\dot\varphi\alpha t}\right)\int_{\alpha t}^t d\tau \int_{\bar\xi\geq b}|\Delta_\xi^{k+1}G(x-\xi,t-\tau)|\,d\xi.$$

From Lemma 6 we have

$$\int_{\alpha t}^t d\tau\int_{\bar\xi\geq b}|\Delta_\xi^k G(x-\xi,t-\tau)|\,d\xi = O((t-\alpha t)^{-k/2+1}) \quad \text{if } k\neq 2.$$

Putting the estimates above into the representation (2.8) for $\Delta^k u(x,t)$ we obtain (2.9).

Lemma 8. *Under the conditions of Lemmas* 2–5 *and* 7, *for every* $\delta\in(0,1)$ *the following estimate is valid:*

$$|\Delta u| = O\left(\frac{\gamma^{1-\delta}}{\dot\varphi t}\right), \qquad x\in[t+b\sqrt{t}, t+\tilde b\sqrt{t}],\ t\geq t_0.$$

Proof. We give the proof for $\delta = 3/4$, because this is sufficient for current applications.

Let us estimate $\Delta^2 u$ for $k^*\in(t+b\sqrt{t}+3, t+\tilde b\sqrt{t}-1)\cap\mathbb{Z}$, relying on estimates of u and $\Delta^2 u$ on $(t+b\sqrt{t}, t+\tilde b\sqrt{t})\cap\mathbb{Z}$. Put $\tilde u(k,t) = \psi_0(k,t)u(k,t)$, where $\psi_0(k,t)$

is a function of (k,t), $k \in \mathbb{Z}$ with support in k in the interval (k^*-3, k^*+1) such that $\psi_0|_{[k^*-2,k^*]} = 1$ and $\sum_{k=0}^{4} |\Delta^k \psi_0| \leq$ absolute constant. We have

$$\begin{aligned}\Delta^2 u(k^*,t)\cdot\Delta^2 u(k^*,t) &\leq \sum_{k=k^*-1}^{k^*+1} \Delta^2\tilde{u}(k,t)\cdot\Delta^2\tilde{u}(k,t)\\ &= \sum_k [\Delta(\Delta\tilde{u}\Delta^2\tilde{u}) - \Delta\tilde{u}(k-1,t)\cdot\Delta^3\tilde{u}]\\ &= -\sum_k \Delta\tilde{u}(k-1,t)\Delta^3\tilde{u}(k,t)\\ &= -\sum_k [\Delta(\tilde{u}(k-1,t)\Delta^3\tilde{u}(k,t)) - \tilde{u}(k-2,t)\Delta^4\tilde{u}(k,t)]\\ &= \sum_k \tilde{u}(k-2,t)\Delta^4\tilde{u}(k,t).\end{aligned}$$

Using the further estimates $|u(k,t)| = O(\frac{\gamma}{\dot\varphi\sqrt{t}})$ and $|\Delta^4 u| = O(\frac{1}{\dot\varphi t^{2,5}})$ we obtain from the relations above the inequality

$$(\Delta^2 u(k^*,t))^2 = O\left(\frac{\gamma}{\dot\varphi^2 t^3}\right)$$

and finally

$$|\Delta^2 u(k^*,t)| = O\left(\frac{\sqrt{\gamma}}{\dot\varphi t^{3/2}}\right).$$

Relying on estimates of $\Delta^2 u$ on $(t + b\sqrt{t} + 3, t + \tilde{b}\sqrt{t} - 1) \cap \mathbb{Z}$ we can obtain in a similar way estimates of $\Delta u(k,t)$ for $k^* \in (t + b\sqrt{t} + 4, t + \tilde{b}\sqrt{t} - 2)$. We have

$$\Delta u(k^*,t)\cdot\Delta u(k^*,t) \leq \sum_{k=k^*-1}^{k^*+1} \Delta\tilde{u}(k,t)\cdot\Delta\tilde{u}(k,t) = \sum_k -\tilde{u}(k-1,t)\cdot\Delta^2\tilde{u}(k,t).$$

Using the estimates $|u(k,t)| = O(\frac{\gamma}{\dot\varphi\sqrt{t}})$ and $|\Delta^2 u| = O(\frac{\sqrt{\gamma}}{\dot\varphi t^{3/2}})$ we deduce from the relation above the inequality $\Delta u = O(\frac{\gamma^{1/4}}{\dot\varphi t})$.

Proof of Proposition 2. Under the assumptions of Proposition 2, we have

$$\begin{aligned}\varphi(\alpha_l^+) &= c_l, \qquad l = 0,\dots,L-1,\\ \varphi(\alpha_l^-) &= c_l, \qquad l = 1,\dots,L,\end{aligned}$$

$$f(x,t) = \varphi^{(-1)}(x/t) + u(x,t), \qquad x \in [c_l t + b_l\sqrt{t}, c_{l+1}t - b_{l+1}\sqrt{t}],\ l = 0,\dots,L-1.$$

This equality and Lemma 8 give us

$$\Delta f = \frac{1}{\dot\varphi(\alpha_l^+)t} + \Delta u = \frac{1}{\dot\varphi(\alpha_l^+)t} + O\left(\frac{\gamma^{1-\delta}}{\dot\varphi(\alpha_l^+)t}\right)$$

if $x \in [c_l t + b_l\sqrt{t}, c_l t + \tilde{b}_l\sqrt{t}]$, and

$$\Delta f = \frac{1}{\dot\varphi(\alpha_l^-)t} + \Delta u = \frac{1}{\dot\varphi(\alpha_l^-)t} + O\left(\frac{\gamma^{1-\delta}}{\dot\varphi(\alpha_l^-)t}\right)$$

if $x \in [c_{l+1}t - \tilde{b}_{l+1}\sqrt{t}, c_{l+1}t - b_{l+1}\sqrt{t}]$.

Proposition 2 is proved (at least with $\delta = 3/4$).

3. Localized conservation laws

The main tools for Theorems 1a and 1b are the following two important generalizations of classical conservation laws for Burgers type equations. We will call these generalizations *localized conservation laws.* Their origin is in [HS].

Let $B_l^\pm$ be positive constants.

Proposition 3a. *Under the assumptions of Theorem* 1a, *let* $\varepsilon = 1$ *and* $d_l(t)$ *be the function defined by the equation*

$$\int_{c_l t - B_l^- \sqrt{t}}^{c_l t + B_l^+ \sqrt{t}} [f(x,t) - \tilde{f}_l(x - c_l t - d_l(t)]\, dx = 0, \tag{3.1a}$$

where $f(x,t)$ *is the solution of* (1a), (2), *and* $\tilde{f}_l(x - c_l t)$ *is the travelling wave solution of* (1a) *with overfall* $[\alpha_l^-, \alpha_l^+]$. *Then the shift function* $d_l(t)$ *satisfies the estimate*

$$d_l(t) = \gamma_l \ln t + o(\ln t), \tag{3.2a}$$

if

$$\begin{aligned} &l = 1, \dots, L-1 && \textit{and} \quad B_l^\pm > 0, \quad \textit{or} \\ &l = 0,\, \varphi(\alpha_0^-) \neq c_0 && \textit{and} \quad B_l^+ > 0,\, B_l^- = \infty, \quad \textit{or} \\ &l = L,\, \varphi(\alpha_L^+) \neq c_L && \textit{and} \quad B_l^+ = \infty,\, B_l^- > 0, \end{aligned}$$

where γ_l *are the constants defined in the statement of Theorem* 1a.

Proposition 3b. *Under the assumptions of Theorem* 1b, *let* $\varepsilon = 1$ *and* $d_l(t)$ *be the function defined by the equation*

$$\begin{aligned} 0 = &\sum_{k=[c_l t - B_l^- \sqrt{t}]+1}^{[c_l t + B_l^+ \sqrt{t}]-1} (\Phi(f(k,t)) - \Phi(\tilde{f}(k - c_l t - d_l(t))) \\ &\pm (c_l t \pm B_l^\pm \sqrt{t} - [c_l t \pm B_l^\pm \sqrt{t}]) \\ &\times (\Phi(f[c_l t \pm B_l^\pm \sqrt{t}], t) - \Phi(\tilde{f}[c_l t \pm B_l^\pm \sqrt{t}] - c_l t - d_l(t)))), \end{aligned} \tag{3.1b}$$

where $f(x,t)$ *is the solution of* (1b), (2), $\tilde{f}_l(x - c_l t)$ *is the travelling wave solution of* (1b) *with overfall* $[\alpha_l^-, \alpha_l^+]$, *and* $\Phi(f) = \int_f^{\alpha_l^+} dy/\varphi(y)$. *Then the shift function* $d_l(t)$ *satisfies the estimate*

$$d_l(t) = \frac{c_l}{2}\gamma_l \ln t + o(\ln t) \tag{3.2b}$$

if

$$
\begin{array}{lll}
l = 1, \dots, L-1 & \textit{and} & B_l^{\pm} > 0, \quad \textit{or} \\
l = 0,\ \varphi(\alpha_0^-) \neq c_0 & \textit{and} & B_0^+ > 0,\ B_0^- = \infty, \quad \textit{or} \\
l = L,\ \varphi(\alpha_L^+) \neq c_L & \textit{and} & B_l^+ = \infty,\ B_l^- > 0,
\end{array}
$$

where γ_l *are the constants defined in the statement of Theorems* 1a*,* 1b*.*

We give here only the proof of Proposition 3b. The proof of Proposition 3a is analogous, but simpler. Let $l \in \{0, \dots, L\}$ and $d_l(t)$ be the function satisfying (3.1b). In order to prove (3.2b) we first prove the following generalization of Proposition 3 from [HS].

Lemma 10. *Under the assumptions of Theorem* 1b*, for every* $l \in \{0, \dots, L\}$ *the function* $d_l(t)$ *defined by* (3.1b) *satisfies the relation*

$$
\begin{aligned}
&\frac{\alpha_l^- - \alpha_l^+}{c_l} d_l'(t)(1 + O(1/\sqrt{t})) \\
&\qquad = (1-\kappa^+)(f_1^+ - f^+ + \tilde f^+ - \tilde f_1^+) \\
&\qquad\quad + \frac{(B_l^+\sqrt{t})'}{c_l}(f_1^+ - \tilde f_1^+)\frac{\varphi'(\alpha_l^+)}{2c_l}((\tilde f_1^+ - \alpha_l^+)^2 - (f_1^+ - \alpha_l^+)^2) \\
&\qquad\quad - (1-\kappa^-)(f_1^- - f^- + \tilde f^- - \tilde f_1^-) + \frac{(B_l^-\sqrt{t})'}{c_l}(f_1^- - \tilde f_1^-) \\
&\qquad\quad - \frac{\varphi'(\alpha_l^-)}{2c_l}((\tilde f_1^- - \alpha_l^-)^2 - (f_1^- - \alpha_l^-)^2),
\end{aligned}
\tag{3.3}
$$

where

$$
\begin{aligned}
f^{\pm} &= f([c_l t \pm B_l^{\pm}\sqrt{t}] - 1, t), \\
f_1^{\pm} &= f([c_l t \pm B_l^{\pm}\sqrt{t}], t), \\
\tilde f^{\pm} &= \tilde f_l([c_l t \pm B_l^{\pm}\sqrt{t}] - 1 - c_l t - d_l(t)), \\
\tilde f_1^{\pm} &= \tilde f_l([c_l t \pm B_l^{\pm}\sqrt{t}] - c_l t - d_l(t)), \\
\kappa^{\pm}(t) &= \{c_l t \pm B_l^{\pm}\sqrt{t}\} := c_l t \pm B_l^{\pm}\sqrt{t} - [c_l t \pm B_l^{\pm}\sqrt{t}],
\end{aligned}
$$

Proof. Equation (1b) implies

$$
\begin{aligned}
&\frac{d\Phi(f(k,t))}{dt} = f(k,t) - f(k-1,t), \\
&\frac{d\Phi(\tilde f(k - c_l t - d_l(t)))}{dt} \\
&\quad = \left(1 + \frac{1}{c_l}\frac{d}{dt}(d_l(t))\right)(\tilde f(k - c_l t - d_l(t)) - \tilde f(k - 1 - c_l t - d_l(t))).
\end{aligned}
\tag{3.4}
$$

Let us differentiate (3.1b), taking into account that the right-hand side of (3.1b) is Lipschitz continuous and using (3.4). We obtain

$$
\begin{aligned}
&f([c_lt+B_l^+\sqrt{t}]-1,t)-\tilde f([c_lt+B_l^+\sqrt{t}]-1-c_lt-d_l(t))\cdot\left(1+\frac{1}{c_l}\frac{d}{dt}(d_l(t))\right)\\
&-f([c_lt-B_l^-\sqrt{t}],t)+\tilde f([c_lt-B_l^-\sqrt{t}]-c_lt-d_l(t))\cdot\left(1+\frac{1}{c_l}\frac{d}{dt}(d_l(t))\right)\\
&+(c_lt+B_l^+\sqrt{t}-[c_lt+B_l^+\sqrt{t}])(f([c_lt+B_l^+\sqrt{t}],t)-f([c_lt+B_l^+\sqrt{t}]-1,t))\\
&-(c_lt+B_l^+\sqrt{t}-[c_lt+B_l^+\sqrt{t}])\cdot\left(1+\frac{1}{c_l}\frac{d}{dt}(d_l(t))\right)\\
&\times(\tilde f([c_lt+B_l^+\sqrt{t}]-c_lt-d_l(t))-\tilde f([c_lt+B_l^+\sqrt{t}]-1-c_lt-d_l(t)))\\
&-(c_lt-B_l^-\sqrt{t}-[c_lt-B_l^-\sqrt{t}])(f([c_lt-B_l^-\sqrt{t}],t)-f([c_lt-B_l^-\sqrt{t}]-1,t))\\
&+(c_lt-B_l^-\sqrt{t}-[c_lt-B_l^-\sqrt{t}])\cdot\left(1+\frac{1}{c_l}\frac{d}{dt}(d_l(t))\right)\\
&\times(\tilde f([c_lt-B_l^-\sqrt{t}]-c_lt-d_l(t))-\tilde f([c_lt-B_l^-\sqrt{t}]-1-c_lt-d_l(t)))\\
&+(c_lt+(B_l^+\sqrt{t})')\cdot(\Phi(f([c_lt+B_l^+\sqrt{t},t))-\Phi(\tilde f([c_lt+B_l^+\sqrt{t}]-c_lt-d_l(t))))\\
&-(c_lt-(B_l^-\sqrt{t})')\cdot(\Phi(f([c_lt-B_l^-\sqrt{t},t))-\Phi(\tilde f([c_lt-B_l^-\sqrt{t}]-c_lt-d_l(t))))=0.
\end{aligned}
$$

From this we deduce

$$
\begin{aligned}
&(f^+-\tilde f^+)-\tilde f^+\frac{d_l'}{c_l}-(f_1^--\tilde f_1^-)+\tilde f_1^-\frac{d_l'}{c_l}\\
&+\kappa^+(f_1^+-f^+)-\kappa^+(\tilde f_1^+-\tilde f^+)-\kappa^+\frac{1}{c_l}d_l'(\tilde f_1^+-\tilde f^+)\\
&-\kappa^-(f_1^--f^-)+\kappa^-(\tilde f_1^--\tilde f^-)+\kappa^-\frac{1}{c_l}d_l'(\tilde f_1^--\tilde f^-)\\
&+(c_l+(B_l^+\sqrt{t})')(\Phi(f^+)-\Phi(\tilde f^+))-(c_l-(B_l^-\sqrt{t})')(\Phi(f_1^-)-\Phi(\tilde f_1^-))=0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{d_l'}{c_l}(\tilde f_1^-+\kappa^-(\tilde f_1^--\tilde f^-)-\tilde f^+-\kappa^+(\tilde f_1^+-\tilde f^+))\\
&\qquad=-f^+-\kappa^+(f_1^+-f^+)+\tilde f^++\kappa^+(\tilde f_1^+-\tilde f^+)\\
&\qquad\quad+f_1^-+\kappa^-(f_1^--f^-)-\tilde f_1^--\kappa^-(\tilde f_1^--\tilde f^-)\\
&\qquad\quad-(c_l+(B_l^+\sqrt{t})')(\Phi(f^+)-\Phi(\tilde f^+))+(c_l-(B_l^-\sqrt{t})')(\Phi(f_1^-)-\Phi(\tilde f_1^-)).
\end{aligned}
$$

From this we obtain

$$
\begin{aligned}
\frac{d_l'}{c_l}(\tilde\alpha_l^--\tilde\alpha_l^+)&=(1-\kappa^+)\Delta f^+-(1-\kappa^-)\Delta\tilde f^+\\
&\quad+(\tilde f^+-f^+)-(c_l+(B_l^+\sqrt{t})')(\Phi(f^+)-\Phi(\tilde f^+))\\
&\quad-(1-\kappa^-)\Delta f^-+(1-\kappa^-)\Delta\tilde f^--\tilde f_1^-+f_1\\
&\quad+(c_l-(B_l^-\sqrt{t})')(\Phi(f_1^-)-\Phi(\tilde f_1^-)),
\end{aligned}\tag{3.5}
$$

where $\tilde\alpha_l^+=\tilde f^++\kappa^+(\tilde f_1^+-\tilde f^+)$ and $\tilde\alpha_l^-=\tilde f_1^-+\kappa^-(\tilde f_1^--\tilde f^-)$.

In order to deduce (3.3) from (3.5) we use the following relations:

$$\begin{aligned}\Phi(f^+) - \Phi(\tilde f^+) &= \int_{f^+}^{\tilde f^+} \frac{dz}{c_l + \varphi'(\xi^+)(z-\alpha_l^+)}, \qquad \xi^+ \in [f^+, \tilde f^+],\\ \Phi(f_1^-) - \Phi(\tilde f_1^-) &= \int_{f_1^-}^{\tilde f_1^-} \frac{dz}{c_l + \varphi'(\xi^-)(z-\alpha_l^-)}, \qquad \xi^- \in [f_1^-, \tilde f_1^-].\end{aligned} \tag{3.6}$$

Moreover, using [HP2, Proposition 1], [HP4, Theorems 2, 2′], and [HS, Lemmas 2, 8] we have

$$\tilde f^\pm = \alpha_l^\pm \mp O(1/\sqrt{t}) \quad \text{and} \quad f^\pm = \alpha_l^\pm \pm O(1/\sqrt{t}). \tag{3.7}$$

From (3.6), (3.7) and the definition of $\tilde\alpha_l^\pm$ (see (3.5)) we obtain

$$\tilde\alpha_l^\pm = \alpha_l^\pm + O(1/\sqrt{t}), \qquad \varphi'(\xi_\pm) = \varphi'(\alpha_l^\pm) + O(1/\sqrt{t}),$$

$$\Phi(f^+) - \Phi(\tilde f^+) = \frac{1}{\varphi'(\alpha_l^+)} \ln \frac{c_l + \varphi'(\alpha_l^+)(\tilde f^+ - \alpha_l^+)}{c_l + \varphi'(\alpha_l^+)(f^+ - \alpha_l^+)} + O(1/t^{3/2}),$$

$$\Phi(f_1^-) - \Phi(\tilde f_1^-) = \frac{1}{\varphi'(\alpha_l^-)} \ln \frac{c_l + \varphi'(\alpha_l^-)(\tilde f_1^- - \alpha_l^-)}{c_l + \varphi'(\alpha_l^-)(f_1^- - \alpha_l^-)} + O(1/t^{3/2}).$$

Let us put these relations into the formula (3.5), replacing $\ln(1+y)$ by $y - y^2/2 + O(y^3)$, where

$$y = \frac{\varphi'(\alpha_l^\pm)}{c_l}(\tilde f^\pm - \alpha_l^\pm) \quad \text{or} \quad y = \frac{\varphi'(\alpha_l^\pm)}{c_l}(f^\pm - \alpha_l^\pm).$$

We obtain

$$\begin{aligned} d_l'(t)&\frac{(\alpha_l^- - \alpha_l^+)(1 + O(1/\sqrt{t}))}{c_l}\\ &= O(1/t^{3/2}) + (1-\kappa^+)(f_1^+ - f^+ + \tilde f^+ - \tilde f_1^+)\\ &\quad + \frac{(B_l^+\sqrt{t})'}{c_l}(f_1^+ - \alpha_l^+) - \frac{(B_l^+\sqrt{t})'}{c_l}(\tilde f_1^+ - \alpha_l^+)\\ &\quad + \frac{\varphi'(\alpha_l^+)}{2c_l}((\tilde f_1^+ - \alpha_l^+)^2 - (f_1^+ - \alpha_l^+)^2) - (1-\kappa^-)(f_1^- - f^- + \tilde f^- - \tilde f_1^-)\\ &\quad + \frac{(B_l^-\sqrt{t})'}{c_l}(f_1^- - \alpha_l^-) - \frac{(B_l^-\sqrt{t})'}{c_l}(\tilde f_1^- - \alpha_l^-)\\ &\quad - \frac{\varphi'(\alpha_l^-)}{2c_l}((\tilde f_1^- - \alpha_l^-)^2 - (f_1^- - \alpha_l^-)^2).\end{aligned}$$

Lemma 10 is proved.

Lemma 11. *Under the assumptions of Theorems* 1a *and* 1b *and notations of Propositions* 1–3, *for all* $b > 0$, $B^\pm > b$, $\delta > 0$, $\theta \in [0, 1)$ *and* $l \in \{0, \dots, L\}$, *the shift* *functions* $d_l(t) = d_l(t, B^+, B^-)$*, defined by* (3.1a) *and respectively* (3.1b)*, satisfy the estimates*

$$d_l'(t) = \frac{\gamma_l}{t} + O\biggl(\frac{1}{b^{1-\delta}t}\biggr) \tag{3.8a}$$

for $d_l(t)$ *defined by* (3.1a)*, and*

$$\int_{t-\theta}^{t+1-\theta} d_l'(\tau)d\tau = \frac{c_l}{2}\frac{\gamma_l}{t} + O\biggl(\frac{1}{b^{1-\delta}t}\biggr) \tag{3.8b}$$

for $d_l(t)$ *defined by* (3.1b)*, where* γ_l *are the constants defined in the statement of Theorem* 1*, and*

$$t \geq t_0 = O\biggl(\frac{B^+ + B^-}{b} + \frac{1}{\delta}\biggr).$$

We give only the proof of Lemma 11 in the part concerning equation (1b). By Theorem 6.2 from [HP4] and its slight improvements, we have the estimates

$$\begin{aligned} \alpha_l^\pm - \tilde{f}_l(x) &= \frac{\varphi(\alpha_l^\pm)}{x\varphi'(\alpha_l^\pm)} + O(\ln x/x^2), \\ \frac{d}{dx}\tilde{f}_l(x) &= \frac{1}{x^2}\frac{\varphi(\alpha_l^\pm)}{\varphi'(\alpha_l^\pm)} + o(1/x^2). \end{aligned} \tag{3.9}$$

From Lemma 10 and estimate (3.9) we deduce

$$\begin{aligned} d_l'(t)&\frac{(\alpha_l^- - \alpha_l^+)(1 + O(1/\sqrt{t}))}{c_l} \\ &= O(d_l(t)/t^{3/2}) + (1-\kappa^+)(f_1^+ - f^+) \\ &\quad - (1-\kappa^+)\frac{c_l}{\varphi'(\alpha_l^+)(B_l^+)^2 t} + \frac{(B_l^+\sqrt{t})'}{c_l}(f_1^+ - \alpha_l^+) \\ &\quad + \frac{(B_l^+\sqrt{t})'}{c_l}\frac{1}{B_l^+\sqrt{t}}\frac{c_l}{\varphi'(\alpha_l^+)} \\ &\quad + \frac{\varphi'(\alpha_l^+)}{2c_l}\frac{c_l^2}{(\varphi'(\alpha_l^+))^2(B_l^+)^2 t} - \frac{\varphi'(\alpha_l^+)}{2c_l}(f_1^+ - \alpha_l^+)^2 \\ &\quad - (1-\kappa^-)(f_1^- - f^-) + (1-\kappa^-)\frac{c_l}{\varphi'(\alpha_l^-)(B_l^-)^2 t} \\ &\quad + \frac{(B_l^-\sqrt{t})'}{c_l}(f_1^- - \alpha_l^-) - \frac{(B_l^-\sqrt{t})'}{c_l}\frac{1}{B_l^-\sqrt{t}}\frac{c_l}{\varphi'(\alpha_l^-)} \\ &\quad - \frac{\varphi'(\alpha_l^-)}{2c_l}\frac{c_l^2}{(\varphi'(\alpha_l^-))^2(B_l^-)^2 t} + \frac{\varphi'(\alpha_l^-)}{2c_l}(f_1^- - \alpha_l^-)^2. \end{aligned} \tag{3.10}$$

Let us consider first the case when $l \in \{1, \dots, L-1\}$. Using the estimates for $\Delta f^\pm := f_1^\pm - f^\pm$ from Proposition 2 and for $f^\pm - \alpha^\pm$ from Proposition 1 we obtain from (3.10),

$$
\begin{aligned}
d_l'(t)&\frac{(\alpha_l^- - \alpha_l^+)(1+O(1/\sqrt{t}))}{c_l}\\
&= O\biggl(\frac{1}{b^{\delta-1}t}\biggr) + O(d_l(t)/t^{3/2}) + (1-\kappa^+)\frac{1}{\varphi'(\alpha_l^+)t}\\
&\quad - (1-\kappa^+)\frac{c_l}{\varphi'(\alpha_l^+)(B_l^+)^2 t} + \frac{B_l^+}{2\sqrt{t}c_l}\cdot\frac{B_l^+}{\varphi'(\alpha_l^+)\sqrt{t}}\\
&\quad + \frac{B_l^+}{2\sqrt{t}c_l}\cdot\frac{1}{B_l^+\sqrt{t}}\cdot\frac{c_l}{\varphi'(\alpha_l^+)} + \frac{c_l}{2\varphi'(\alpha_l^+)(B_l^+)^2 t} - \frac{(B_l^+)^2}{2c_l\varphi'(\alpha_l^+)t}\\
&\quad - (1-\kappa^-)\frac{1}{\varphi'(\alpha_l^-)t} + (1-\kappa^-)\frac{c_l}{\varphi'(\alpha_l^-)(B_l^-)^2 t} - \frac{B_l^-}{2\sqrt{t}c_l}\cdot\frac{B_l^-}{\varphi'(\alpha_l^-)\sqrt{t}}\\
&\quad - \frac{B_l^-}{2\sqrt{t}c_l}\cdot\frac{1}{B_l^-\sqrt{t}}\cdot\frac{c_l}{\varphi'(\alpha_l^-)} - \frac{c_l}{2\varphi'(\alpha_l^-)(B_l^-)^2 t} + \frac{(B_l^-)^2}{2c_l\varphi'(\alpha_l^-)t}\\
&= \frac{1}{\varphi'(\alpha_l^+)t}\biggl(\frac{1}{2} + (1-\kappa^+)\biggr) - \frac{1}{\varphi'(\alpha_l^-)t}\biggl(\frac{1}{2} + (1-\kappa^-)\biggr)\\
&\quad + \frac{c_l}{\varphi'(\alpha_l^+)(B_l^+)^2 t}\biggl(\frac{1}{2} - (1-\kappa^+)\biggr) - \frac{c_l}{\varphi'(\alpha_l^-)(B_l^-)^2 t}\biggl(\frac{1}{2} - (1-\kappa^-)\biggr)\\
&\quad + O(d_l(t)/t^{3/2}) + O\biggl(\frac{1}{b^{1-\delta}t}\biggr).
\end{aligned}
$$

From the last equality and the elementary equality

$$
\int_{t-\theta}^{t+1-\theta} \frac{(1-\kappa^{\pm}(\tau))}{\tau}d\tau = \frac{1}{2t} + O(1/t^{3/2}) \tag{3.11}
$$

we deduce

$$
\frac{\alpha_l^- - \alpha_l^+}{c_l}\int_{t-\theta}^{t+1-\theta} d_l'(\tau)d\tau = \biggl(\frac{1}{\varphi'(\alpha_l^+)} - \frac{1}{\varphi'(\alpha_l^-)}\biggr)\frac{1}{t} + O\biggl(\frac{1}{b^{1-\delta}t}\biggr). \tag{3.12}
$$

This implies (3.8) for $l \in \{1,\dots,L-1\}$.

Let us now consider the cases when $l=0$, $\varphi(\alpha_0^-) \neq c_0$, $B_0^- = \infty$ and $l=L$, $\varphi(\alpha_L^+) \neq c_L$, $B_L^+ = \infty$. By Lemma 3 from [HP2] and Theorem 6.1 from [HP4] in these cases we have the estimates

$$
\begin{aligned}
&\sum_{k=-\infty}^{n} |f(k,t) - \alpha_0^-| \le \delta \quad \text{ if } n \le ct - x^-(\delta),\\
&\sum_{k=n}^{\infty} |\alpha_L^+ - f(k,t)| \le \delta \quad \text{ if } n \ge ct + x^+(\delta),
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
&\lim_{x\to-\infty}\frac{1}{x}\ln(\tilde f_0(x) - \alpha_0^-) = \lambda_1, \quad \lambda_1 = \frac{\varphi(\alpha_0^-)}{c_0}(1-e^{-\lambda_1}), \quad \varphi(\alpha_0^-) > c_0,\\
&\lim_{x\to+\infty} -\frac{1}{x}\ln(\tilde f_L(x) - \alpha_L^+) = \lambda_2, \quad \lambda_2 = \tfrac{\varphi(\alpha_L^+)}{c_L}(e^{\lambda_2}-1), \quad \varphi(\alpha_L^+) < c_L.
\end{aligned} \tag{3.14}
$$

Inequalities (3.13), (3.14) imply the correctness of definition (3.1b) for $d_0(t)$ and $d_L(t)$ when $B_0^- = \infty$ and $B_L^+ = \infty$. If $l = 0$, then from (3.1b), Lemma 10 and estimate (3.9) (for $l = 0$, $x \to +\infty$) we obtain

$$\begin{aligned}
&\frac{\alpha_0^- - \alpha_0^+}{c_0} d_0'(t) \cdot (1 + O(1/\sqrt{t}))\\
&\quad = O(d_0(t)/t^{3/2}) + (1-\kappa^+)(f_1^+ - f^+) - (1-\kappa^+)\frac{c_0}{\varphi'(\alpha_0^+)(B_0^+)^2 t}\\
&\qquad + \frac{B_0^+}{2\sqrt{t}c_0}(f_1^+ - \alpha_0^+) + \frac{B_0^+}{2\sqrt{t}c_0} \cdot \frac{1}{B_0^+\sqrt{t}} \cdot \frac{c_0}{\varphi'(\alpha_0^+)} + \frac{\varphi'(\alpha_0^+)}{2c_0} \cdot \frac{c_0^2}{\varphi'(\alpha_0^+)(B_0^+)^2 t}\\
&\qquad - \frac{\varphi'(\alpha_0^+)}{2c_0}(f_1^+ - \alpha_0^+)^2. \qquad (3.15)
\end{aligned}$$

If $l = L$, then from (3.1b), Lemma 10 and estimate (3.9) (for $l = L$, $x \to -\infty$) we obtain

$$\begin{aligned}
&\frac{\alpha_L^- - \alpha_L^+}{c_L} d_L'(t) \cdot (1 + O(1/\sqrt{t}))\\
&\quad = O(d_L(t)/t^{3/2}) - (1-\kappa^-)(f_1^- - f^-) + (1-\kappa^-)\frac{c_L}{\varphi'(\alpha_L^-)(B_L^-)^2 t}\\
&\qquad + \frac{B_L^-}{2\sqrt{t}c_L}(f_1^- - \alpha_L^-) - \frac{B_L^-}{2\sqrt{t}c_L} \cdot \frac{1}{B_L^-\sqrt{t}} \cdot \frac{c_L}{\varphi'(\alpha_L^-)} - \frac{\varphi'(\alpha_L^-)}{2c_L} \cdot \frac{c_L^2}{\varphi'(\alpha_L^-)(B_L^-)^2 t}\\
&\qquad + \frac{\varphi'(\alpha_L^-)}{2c_L}(f_1^- - \alpha_L^-)^2. \qquad (3.16)
\end{aligned}$$

From (3.11), (3.15), (3.16) we deduce estimate (3.12) for $l = 0, L$ exactly in the same way as from (3.10) we deduced it for $l = 1, \ldots, L-1$. This implies estimate (3.8)$'$ for $l = 0$, $\varphi(\alpha_0^-) \neq c_0$ and $l = L$, $\varphi(\alpha_L^+) \neq c_L$. Lemma 11 is proved.

The definitions (3.1a) and (3.1b) of the shift functions $d_l(t)$ depend on the parameters $B_l^\pm$, $l = 0, \ldots, L$. For the proof of Propositions 3a, 3b the following statements about the dependence of $d_l(t, B_l^\pm)$ on $B_l^\pm$ turn out to be important.

Lemma 12. *Let f be the solution of* (1a), (2) *or respectively of* (1b), (2). *Let $L > 0$ and $S = \bigcup_{l=0}^L (\alpha_l^-, \alpha_l^+)$ be the set defined by* (3a), (4) *or by* (3b), (4). *Let $\tilde{f}_l(x - c_l t)$ be the travelling wave solution of* (1a), (2) *or of* (1b), (2) *with overfall (α_l^-, α_l^+). Let $d_l(t) = d_l(t, B^\pm)$ be defined by* (3.1a) *or by* (3.1b) *with*

$$B_l^+ = B^+,\ l < L, \qquad B_L^+ = \infty, \qquad B_l^- = B^-,\ l > 0, \qquad B_0^- = \infty.$$

Then under Assumptions 1–3 *the following estimates hold:*

$$\begin{aligned}
\left|\frac{d}{dB^+} d_l(t, B^+, B^-)\right| &= O\left(B^+ + \frac{1}{B^+}\right), \qquad l < L,\\
\left|\frac{d}{dB^-} d_l(t, B^+, B^-)\right| &= O\left(B^- + \frac{1}{B^-}\right), \qquad l > 0.
\end{aligned} \qquad (3.17)$$

Proof. We first consider the much simpler case of equation (1a).

Let us consider the case when $l = 0$, $\varphi(\alpha_0^-) \neq c_0$, $\varphi(\alpha_0^+) = c_0$, $B^- = \infty$. From the definition (3.1a) of $d_0(t, B^+, B^-)$ we deduce

$$
\begin{aligned}
0 &= \frac{d}{dB^+} \int_{c_0 t - B^- \sqrt{t}}^{c_0 t + B^+ \sqrt{t}} (f(x,t) - \tilde{f}_0(x - c_0 t - d_0(t, B^+, B^-))\, dx \\
&= \int_{c_0 t - B^- \sqrt{t}}^{c_0 t + B^+ \sqrt{t}} \tilde{f}_0'(x - c_0 t - d_0(t, B^+, B^-)) \cdot \frac{d}{dB^+} d_0(t, B^+, B^-)\, dx \\
&\quad + [f(c_0 t + B^+ \sqrt{t}, t) - \tilde{f}_0(B^+ \sqrt{t} - d_0(t, B^+, B^-))] \cdot \sqrt{t}.
\end{aligned}
$$

Hence

$$
\frac{d}{dB^+} d_0(t, B^+, B^-) = \frac{[\tilde{f}_0(B^+ \sqrt{t} - d_0(t, B^+, B^-)) - f(c_0 t + B^+ \sqrt{t}, t)]\sqrt{t}}{\tilde{f}_0(B^+ \sqrt{t} - d_0(t, B^+, B^-)) - \tilde{f}_0(-B^- \sqrt{t} - d_0(t, B^+, B^-))}.
$$

Using estimate (1.1) for $f(x,t)$, analogues for (1a) of estimates (3.9), (3.14) for $\tilde{f}_0(x)$, and estimate (3.8) for $d_0(t, B^\pm)$ we obtain

$$
\begin{aligned}
\frac{d}{dB^+} d_0(t, B^\pm) &= \frac{\sqrt{t}[\alpha_0^+ - \frac{c_0}{(B^+ \sqrt{t} - d_0(t, B^\pm))\varphi'(\alpha_0^+)} - \varphi^{(-1)}(\frac{c_0 t + B^+ \sqrt{t}}{t}) + O(\frac{1}{B^+ \sqrt{t}})]}{\alpha_0^+ - \alpha_0^- - \frac{c_0}{(B^+ \sqrt{t} - d_0(t, B^\pm))\varphi'(\alpha_0^+)} + o(1/\sqrt{t})} \\
&= \frac{\sqrt{t}[\frac{B^+ \sqrt{t}}{\varphi'(\alpha_0^+) t} + O(\frac{1}{B^+ \sqrt{t}}) - \frac{c_0}{(B^+ \sqrt{t} - d_0(t, B^\pm))\varphi'(\alpha_0^+)}]}{\alpha_0^+ - \alpha_0^- - \frac{c_0}{(B^+ \sqrt{t} - d_0(t, B^\pm))\varphi'(\alpha_0^+)} + o(1/\sqrt{t})} \\
&= \frac{1}{\varphi'(\alpha_0^+)} \frac{B^+ + O(1/B^+)}{\alpha_0^+ - \alpha_0^- + O(1/\sqrt{t})} = O\left(B^+ + \frac{1}{B^+}\right).
\end{aligned}
$$

The other cases ($l \geq 1$) can be considered in the same way. Estimate (3.17) for equation (1a) is proved.

We now consider the more difficult case of equation (1b).

Let us consider again the crucial case when $l = 0$, $\varphi(\alpha_0^-) \neq c_0$, $\varphi(\alpha_0^+) = c_0$. The other cases can be proved similarly. We have

$$
\begin{aligned}
&\Phi(f) = \int_f^{\alpha_0^+} \frac{dy}{\varphi(y)}, \qquad \frac{d}{dB^+} \Phi(f(k,t)) = 0, \\
&\frac{d}{dB^+} \Phi(\tilde{f}_0(k - c_0 t - d_0(t, B^\pm)) = \frac{d}{dB^+} d_0'(t, B^\pm) \cdot \frac{\tilde{f}_{0,x}'(k - c_0 t - d_0(t, B^\pm))}{\varphi(\tilde{f}_0(k - c_0 t - d_0(t, B^\pm)))}.
\end{aligned}
$$

From these relations and (3.1b) we deduce, taking into account the Lipschitz continuity of the right-hand side of (3.1b) in B^+,

$$
\begin{aligned}
0 = &\sum_{k=[c_0t-B^-\sqrt{t}]+1}^{[c_0t+B^+\sqrt{t}]-1} -\frac{d}{dB^+}\Phi(\tilde{f}_0(k-c_0t-d_0(t,B^\pm)))\\
&+(c_0t+B^+\sqrt{t}-[c_0t+B^+\sqrt{t}])\\
&\times\left[\frac{d}{dB^+}\Phi(f([c_0t+B^+\sqrt{t}],t))-\frac{d}{dB^+}\Phi(\tilde{f}_0([c_0t+B^+\sqrt{t}]-c_0t-d_0(t,B^\pm)))\right]\\
&+\sqrt{t}(\Phi(f([c_0t+B^+\sqrt{t}],t))-\Phi(\tilde{f}_0([c_0t+B^+\sqrt{t}]-c_0t-d_0(t,B^\pm))). \qquad (3.18)
\end{aligned}
$$

From (3.6), (3.7) we deduce

$$
\begin{aligned}
&\Phi(f([c_0t+B^+\sqrt{t}],t))-\Phi(\tilde{f}_0([c_0t+B^+\sqrt{t}]-c_0t-d_0(t,B^\pm)))\\
&\quad=\frac{1}{\varphi'(\alpha_0^+)}\\
&\qquad\times\ln\frac{c_0+\dot{\varphi}(\alpha_0^+)(\tilde{f}_0(B^+\sqrt{t}-d_0(t,B^\pm))-\alpha_0^+)}{c_0+\dot{\varphi}(\alpha_0^+)(f(c_0t+B^+\sqrt{t},t)-\alpha_0^+)}+O(1/t^{3/2})\\
&\quad=\frac{1}{\varphi'(\alpha_0^+)}\ln\frac{1-O_+(\frac{1}{B^+\sqrt{t}})}{1+O_+(B^+/\sqrt{t})}+O(1/t^{3/2})=-O_+\left(\frac{B^++1/B^+}{\sqrt{t}}\right). \qquad (3.19)
\end{aligned}
$$

From (3.18) and (3.19) we obtain

$$
\begin{aligned}
0=&\frac{d}{dB^+}d_0(t,B^\pm)\cdot\sum_{k=[c_0t-B^-\sqrt{t}]+1}^{[c_0t+B^+\sqrt{t}]-1}\Phi(\tilde{f}_0(k-c_0t-d_0(t,B^\pm)))\\
&-(c_0t+B^+\sqrt{t}-[c_0t+B^+\sqrt{t}])\cdot\frac{\sqrt{t}\Delta_xf([c_0t+B^+\sqrt{t},t])}{\varphi(f([c_0t+B^+\sqrt{t},t]))}\\
&+(c_0t+B^+\sqrt{t}-[c_0t+B^+\sqrt{t}])\cdot\left(\sqrt{t}-\frac{d}{dB^+}d_0(t,B^\pm)\right)\\
&\times\frac{\Delta_x\tilde{f}_0([c_0t+B^+\sqrt{t}]-c_0t-d_0(t,B^\pm))}{\varphi(\tilde{f}_0([c_0t+B^+\sqrt{t}]-c_0t-d_0(t,B^\pm)))}-O_+(B^++1/B^+).
\end{aligned}
$$

Using the estimate for f from Proposition 1, the estimate for $\Delta_x f$ from Proposition 2, the estimate (3.9) for $\tilde{f}_0$ and $\Delta_x\tilde{f}_0$ and the estimate for $d_0'(t,B^\pm)$ from Lemma 11, we obtain

$$
\begin{aligned}
&\frac{d}{dB^+}d_0(t,B^\pm)\times\\
&[\Phi(\tilde{f}_0([c_0t+B^+\sqrt{t}]-1-c_0t-d_0(t,B^\pm)))-\Phi(\tilde{f}_0([c_0t-B^-\sqrt{t}]-c_0t-d_0(t,B^\pm)))]\\
&-(c_0t+B^+\sqrt{t}-[c_0t+B^+\sqrt{t}])\\
&\times\left[\frac{\sqrt{t}O(\frac{1}{\varphi'(\alpha_0^+)t})}{c_0+O(B^+/\sqrt{t})}-\frac{(\sqrt{t}+O(1/t))O(1/t)}{c_0-O(\frac{1}{B^+\sqrt{t}})}\right]-O_+(B^++1/B^+)=0.
\end{aligned}
$$

Hence,

$$(\alpha_0^+ - \alpha_0^-)\frac{d}{dB^+}d_0(t, B^\pm)(1 + O(1/\sqrt{t})) = O_+(B^+ + 1/B^+).$$

Lemma 12 is proved.

Proof of Proposition 3b. Let $0 < B^+ < B$, $0 < B^- < B$. By Lemma 12 we have

$$d_l(t) := d_l(t, B^+, B^-) = d_l(t, B, B) + O(B^2).$$

By Lemma 11, for every $\delta \in (0, 1)$ we have

$$d_l(t, B, B) = \frac{c_l}{2}\gamma_l \ln t + O\left(\frac{\ln t}{B^{1-\delta}}\right).$$

Hence,

$$d_l(t) = \frac{c_l}{2}\gamma_l \ln t + O\left(\frac{\ln t}{B^{1-\delta}}\right) + O(B^2).$$

This implies that

$$d_l(t) = \frac{c_l}{2}\gamma_l \ln t + O((\ln t)^{2/(3-\delta)}), \qquad t \geq t_0.$$

Proposition 3b is proved.

4. Estimates of $\sum_{k=-\infty}^{n}(\Phi(f(k,t)) - \Phi(\tilde{f}_l(k - c_l t - d_l(\tau))))$

Let f be the solution of (1b), (2), where $\varepsilon = 1$, and $d_l(t) = d_l(t, B_l^+, B_l^-)$ be the function defined by (3.1b). The convergence of $f(n,t)$, $t \to \infty$, to the travelling waves $\tilde{f}_l(n - c_l t - d_l(t))$ on the intervals $n \in [c_l t - B_l^- \sqrt{t}, c_l t + B_l^+ \sqrt{t}]$, $l = 0, \ldots, n$, will be obtained further from estimates of the following functions:

$$\Delta_0(n, t, d_0(\tau)) = \sum_{k=-\infty}^{n} (\Phi(f(k,t)) - \Phi(\tilde{f}_0(k - c_0 t - d_0(\tau)))), \tag{4.1}$$

where $l = 0$, $t \in [\tau, (1+\delta)\tau]$, $n < c_0 t + B_0^+ \sqrt{t}$;

$$\Delta_l(n, t, d_l(\tau)) = \sum_{[c_l\tau - B_l^- \sqrt{\tau}]}^{n} (\Phi(f(k,t)) - \Phi(\tilde{f}_l(k - c_l t - d_l(\tau)))), \tag{4.2}$$

where $l = 1, \ldots, L-1$, $t \in [\tau, (1+\delta)\tau]$, $n \in [c_l\tau - B_l^- \sqrt{\tau}, c_l t + B_l^+ \sqrt{t}]$; and

$$\tilde{\Delta}_L(n, t, d_L(\tau)) = \sum_{n}^{\infty} (\Phi(f(k,t)) - \Phi(\tilde{f}_L(k - c_L t - d_L(\tau)))), \tag{4.3}$$

where $l = L$, $t \in [\tau, (1+\delta)\tau]$, $n > c_L t - B_L^- \sqrt{t}$.

The following simple, but important statement having its origin in [HP2] shows that the functions $\Delta_l(n, t, d_l(\tau))$ for fixed l and τ satisfy appropriate parabolic type equations with respect to n, t.

Lemma 13. *Put*

$$\Theta_l(n,t,d_l(\tau)) := \frac{f(n,t)-\tilde{f}_l(n-c_lt-d_l(\tau))}{\Phi(\tilde{f}_l(n-c_lt-d_l(\tau)))-\Phi(f(n,t))}, \tag{4.4}$$

where $d_l(\tau)=d_l(\tau,B_l^+,B_l^-)$. Then under Assumptions 1–3 we have

$$\begin{aligned}
&\frac{d\Delta_l(n,t,d_l(\tau))}{dt} = \Theta_l(n,t,d_l(\tau))(\Delta_l(n-1,t,d_l(\tau))-\Delta_l(n,t,d_l(\tau))),\\
&\text{where } n<c_0t+B_0^+\sqrt{t}, \text{ if } l=0 \text{ and}\\
&c_l\tau-B_l^-\sqrt{\tau}<n<c_lt+B_l^+\sqrt{t}, \text{ if } l=1,\dots,L-1,\\
&\frac{d\tilde{\Delta}_L(n,t,d_l(\tau))}{dt} = \Theta_L(n-1,t,d_L(\tau))(\tilde{\Delta}_L(n-1,t,d_L(\tau))-\tilde{\Delta}_L(n,t,d_L(\tau))),\\
&\text{where } n\ge c_Lt-B_L^-\sqrt{t}, \ l=L.
\end{aligned} \tag{4.5}$$

Moreover, for $n<c_0t+B_0^+\sqrt{t}$ if $l=0$, for $n\in[c_l\tau-B_l^-\sqrt{\tau},c_lt+B_l^+\sqrt{t}]$ if $l=1,\dots,L-1$, and for $n\ge c_Lt-B_L^-\sqrt{t}$ if $l=L$, where $t\ge\tau>t_0$, we have the equality

$$\Theta_l(n,t,d_l(\tau))=\varphi(\xi) \quad \text{for some } \xi\in[\alpha_l^- - O(1/\sqrt{t}),\alpha_l^+ + O(1/\sqrt{t})],$$

depending on the values of $\Delta_l(n-1,t,d_l(\tau))$, $\Delta_l(n,t,d_l(\tau))$ and $\tilde{f}_l(n-c_lt-d_l(\tau))$, which implies

$$\frac{1}{b_1}-O(1/\sqrt{t}) \le \Theta_l(n,t,d_l(\tau)) \le \frac{1}{b_0}+O(1/\sqrt{t}), \tag{4.6}$$

where $1/b_1=\min_\xi\varphi(\xi)$ and $1/b_0=\max_\xi\varphi(\xi)$.

Proof. In the case $l=0$ Lemma 13 is a small variation of Lemma 10 of [HS]. The case $l\le L-1$ can be treated in a similar way. Let us prove the slightly new case $l=L$. From (4.3) and (3.4) we deduce that

$$\frac{d\tilde{\Delta}_L(n,t,d_L(\tau))}{dt} = \tilde{f}_L(n-1-c_Lt-d_L(\tau))-f_L(n-1,t).$$

This gives (4.6) because

$$\begin{aligned}
\tilde{\Delta}_L(n-1,t,d_L(\tau))-\tilde{\Delta}_L(n,t,d_L(\tau)) &= \Phi(f(n-1,t))-\Phi(\tilde{f}_L(n-1-c_Lt-d_L(\tau)))\\
&:= -\frac{f(n-1,t)-\tilde{f}_L(n-1-c_Lt-d_L(\tau))}{\Theta_L(n-1,t,d_L(\tau))}.
\end{aligned}$$

We have further

$$\begin{aligned}
&f(n-1,t)-\tilde{f}_L(n-1-c_Lt-d_L(\tau))\\
&\qquad = \Phi^{(-1)}(\tilde{\Delta}_L(n-1,t,d_L(\tau))-\tilde{\Delta}_L(n,t,d_L(\tau))\\
&\qquad\quad + \Phi(\tilde{f}_L(n-1-c_Lt-d_L(\tau))))-\tilde{f}_L(n-1-c_Lt-d_L(\tau))\\
&\qquad = -\Theta_L(n-1,t,\tau)(\tilde{\Delta}_L(n-1,t,d_L(\tau))-\tilde{\Delta}_L(n,t,d_L(\tau))),
\end{aligned}$$

where

$$\Theta_L(n-1,t,d_L(\tau)) = -\frac{d\Phi^{(-1)}}{d\Phi}(\kappa\Phi(f(n-1,t)) + (1-\kappa)\Phi(\tilde f(n-1-c_Lt-d_L(\tau)))),$$

where $\kappa = \kappa(n,t,\tau)$ is some function with values in $[0,1]$. Because $-\frac{d\Phi^{(-1)}}{d\Phi}(h) = \varphi(\Phi^{(-1)}(h))$, we obtain

$$\Theta_L(n-1,t,d_L(\tau)) = \lambda f(n-1,t) + (1-\lambda)\tilde f(n-1-c_Lt-d_L(\tau)),$$

where $\lambda = \lambda(n,t,\tau)$ takes values in $[0,1]$. By (1.15) we have

$$\alpha^- - O(1/\sqrt{t}) \le f \le \alpha^+ + O(1/\sqrt{t}).$$

Hence,

$$\frac{1}{b_1} - O(1/\sqrt{t}) \le \Theta_L \le \frac{1}{b_0} + O(1/\sqrt{t}),$$

and the lemma is proved.

The following statement giving important estimates of $\Theta_l(k,l,d_l(\tau))$ generalizes and improves Proposition 4 from [HS].

Lemma 14. *Let $d_l(\tau) = d_l(\tau, B^\pm)$, $\Delta_l(n,t,d_l(\tau))$, $\Theta_l(n,t,d_l(\tau))$ be the functions determined by* (3.1b), (4.1)–(4.4), *where $B_0^+ = A\sqrt{c_0}$, $B_0^- = \infty$; $B_L^+ = \infty$, $B_L^- = A\sqrt{c_L}$; $B_l^\pm = A\sqrt{c_l}$, $l = 1,\dots,L-1$. Then under the assumptions of Theorem* 1b, *for all $A, \varepsilon > 0$ there are $\Gamma_0, t_0 > 0$ with the following properties:*

$$\Theta_l(k,t,d_l(\tau)) \le c_l + \frac{(k-c_lt+\varepsilon\sqrt{t})(1+\varepsilon)}{2t} - \frac{c_l(1-\varepsilon)}{2(k-c_lt-d_l(\tau))} \tag{4.7}$$

if $k \in (c_lt+\Gamma, c_lt+A\sqrt{c_lt})$, $\Gamma \ge \Gamma_0$, $t \ge \tau \ge t_0$, $l = 0,\dots,L-1$, and

$$\Theta_l(k,t,d_l(\tau)) \ge c_l + \frac{(k-c_lt-\varepsilon\sqrt{t})(1-\varepsilon)}{2t} - \frac{c_l(1+\varepsilon)}{2(k-c_lt-d_l(\tau))} \tag{4.8}$$

if $k \in (c_lt-A\sqrt{c_lt}, c_lt-\Gamma)$, $\Gamma \ge \Gamma_0$, $t \ge \tau \ge t_0$, $l = 1,\dots,L$. In particular,

$$\Theta_l(k,t,d_l(\tau)) \le c_l + \frac{(A\sqrt{c_l}+\varepsilon)(1+\varepsilon)}{2\sqrt{t}} - \frac{c_l(1-\varepsilon)}{2A\sqrt{c_l}} \tag{4.9}$$

if $k \in (c_lt+\Gamma, c_lt+A\sqrt{c_lt})$, $l = 0,\dots,L-1$, and

$$\Theta_l(k,t,d_l(\tau)) \ge c_l + \frac{(-A\sqrt{c_l}-\varepsilon)(1-\varepsilon)}{2\sqrt{t}} + \frac{c_l(1+\varepsilon)}{2A\sqrt{c_l}} \tag{4.10}$$

if $k \in (c_lt-A\sqrt{c_lt}, c_lt-\Gamma)$, $l = 1,\dots,L$.

Comments. Statements (4.7), (4.9) for $L = 1$ have been obtained in [HS] with a proof which requires a correction in the choice of the supersolution $F^+(n,t)$ on page 1490 of [HS]. The correct choice is $F^+(n,t) = f_2^+(n,t)$, where $f_2^+(n,t)$ is defined by formula (1.3) of our Lemma 1. Statements (4.8), (4.10) for $L = 1$ can be obtained in a similar way, using the subsolution $f_2^-(n,t)$ given by formula (1.2) of Lemma 1. Statements (4.7)–(4.10) for arbitrary $L \ge 1$ can be obtained in the same way, using a natural generalization of Lemma 1 to the case of arbitrary $L \ge 1$.

The following simple lemma is a variation (with the same proof) of Lemma 11 from [HS].

Lemma 15. *Let $\Delta_l(n,t,d_l(\tau))$ be of the form* (4.2)*. Then*

$$|\Delta_l(n,t,d_l(t)) - \Delta_l(n,t,d_l(\tau))| \le \frac{|\alpha_l^+ - \alpha_l^-|}{c_l}|d_l(t) - d_l(\tau)|,$$

where $n \in \mathbb{Z}$, $t,\tau \in \mathbb{R}_+$ and (α_l^-, α_l^+) is the overfall of the travelling wave $\tilde{f}_l(n - c_l t)$, $l = 0, \dots, L$.

The following proposition is essentially a corrected version of Proposition 5 from [HS]. It gives decreasing positive supersolutions of Lyapunov type for equations (4.5), (4.6) for the cases $l = 0$ and $l = L$, $L \ge 1$.

Proposition 4. *Under the assumptions of Theorem* 1b*, let $L \ge 1$. Let $d_l(\tau)$ and $\Theta_l(n,t) = \Theta_l(n,t,d_l(\tau))$ be as in Lemmas* 13*,* 14 *for $l = 0, L$. Let $\beta(B_0)$ and $\tilde{\beta}(B_1)$ be the positive roots of the equations*

$$B_0(1 - e^{-\beta}) - \beta = 0 \quad \text{and} \quad B_1(e^{\tilde{\beta}} - 1) - \tilde{\beta} = 0.$$

Let a and α be such that $e^a/(c_0 b_0) > 1$ and $\alpha > \beta(e^a/(c_0 b_0))$ if $l = 0$, and $e^{-a}/(c_1 b_1) < 1$ and $\alpha > \tilde{\beta}(e^{-a}/(c_1 b_1))$, if $l = L$, where $b_0 < b_1$ are defined in (4.6)*. Put*

$$\sigma := \frac{c_0 \alpha - e^a}{b_0 \cdot (1 - e^{-\alpha})} \quad \text{and} \quad \tilde{\sigma} := e^{-a/b_1} \cdot (e^\alpha - 1) - c_L \alpha.$$

Then for all $b, \Gamma > 0$ and A with $A^2 + Ab/\sqrt{c_0} < 1$ there exists $t_0 > 0$ and positive functions $\omega(n,\tau,t)$ and $\tilde{\omega}(n,\tau,t)$ with the following properties for $t \ge \tau > t_0$:

$$\frac{\partial \omega(n,\tau,t)}{\partial t} \ge \Theta_0(n,t)(\omega(n-1,\tau,t) - \omega(n,\tau,t)) \tag{4.11}$$

if $c_0 t + d_0(t) - \Gamma \le n \le c_0 t + A\sqrt{c_0 t}$,

$$\frac{\partial \tilde{\omega}(n,\tau,t)}{\partial t} \ge \Theta_L(n-1,t)(\tilde{\omega}(n-1,\tau,t) - \tilde{\omega}(n,\tau,t)) \tag{4.12}$$

if $c_L t - A\sqrt{c_L t} \le c_L t + d_L(t) + \Gamma$, and

$$\frac{\max\{\omega(n,\tau,t) : d_0(t) - \Gamma \le n - c_0 t \le A\sqrt{c_0 t}\}}{\min\{\omega(n,t,t) : d_0(t) - \Gamma \le n - c_0 t \le A\sqrt{c_0 t}\}} \le (1 + o(1)) \exp\left[1 - \frac{b\sqrt{t}e^{-2\alpha\Gamma}\sigma \ln(t/\tau)}{c_0 \alpha}\right], \tag{4.13}$$

$$\frac{\max\{\tilde{\omega}(n,\tau,t) : -A\sqrt{c_L t} \le n - c_L t \le d_L(t) + \Gamma\}}{\min\{\tilde{\omega}(n,t,t) : -A\sqrt{c_L t} \le n - c_L t \le d_L(t) + \Gamma\}} \le (1 + o(1)) \exp\left[1 - \frac{b\sqrt{t}e^{-2\alpha\Gamma}\tilde{\sigma} \ln(t/\tau)}{c_L \alpha}\right]. \tag{4.14}$$

Proof. We follow the proof of Proposition 5 in [HS] with several important corrections, in particular, restrictions on the parameter A. We give detailed proofs of (4.11) and (4.13). The proofs of (4.12) and (4.14) are quite similar. We put in this proof $c = c_0$, $d(\tau) = d_0(\tau)$. The proof needs several steps.

Step 1. Put

$$\omega_0(n,\tau,t) = \exp[-\hat{\varepsilon}(\tau)(e^{\alpha(n-ct-d(\tau)-\Gamma)} + Ke^{-\alpha\Gamma}(1-e^{\alpha(d(t)-d(\tau))}))].$$

From the condition $\alpha > \beta(e^a/(cb_0))$ it follows that $\sigma > 0$. Let us show that under the assumptions

$$\hat{\varepsilon}(\tau)(1-e^{-\alpha}) \leq a \tag{4.15}$$

and

$$0 < -\alpha K d'(t) \leq e^{-\alpha\Gamma}\sigma, \tag{4.16}$$

the following inequality is valid:

$$\frac{\partial\omega_0(n,\tau,t)}{\partial t} - \Theta(n,t)(\omega_0(n-1,\tau,t) - \omega_0(n,\tau,t)) \geq 0 \tag{4.17}$$

if $n \in [ct + d(t) - \Gamma, ct + d(\tau) + \Gamma]$. Indeed, using the inequality $e^x - 1 \leq e^a x$, $x \in (0,a)$, from (5.5) we obtain

$$\begin{aligned}
&\frac{\partial\omega_0(n,\tau,t)}{\partial t} - \Theta(n,t)(\omega_0(n-1,\tau,t) - \omega_0(n,\tau,t)) \\
&\quad = \omega_0(n,\tau,t)[\alpha c\hat{\varepsilon}e^{\alpha(n-ct-d(\tau)-\Gamma)} + \hat{\varepsilon}K\alpha e^{-\alpha\Gamma}e^{\alpha(d(t)-d(\tau))}\cdot d'(t) \\
&\qquad - \Theta(n,t)(\exp((1-e^{-\alpha})\hat{\varepsilon}e^{\alpha(n-ct-d(\tau)-\Gamma)}) - 1)] \\
&\quad \geq \omega_0(n,\tau,t)[\alpha c\hat{\varepsilon}e^{\alpha(n-ct-d(\tau)-\Gamma)} + \hat{\varepsilon}K\alpha e^{-\alpha\Gamma}e^{\alpha(d(t)-d(\tau))}\cdot d'(t) \\
&\qquad - e^a\Theta(n,t)\hat{\varepsilon}(1-e^{-\alpha})e^{\alpha(n-ct-d(\tau)-\Gamma)}].
\end{aligned}$$

Using the inequality $\Theta \leq 1/b_0$ we obtain

$$\begin{aligned}
&\frac{\partial\omega_0(n,\tau,t)}{\partial t} - \Theta(n,t)[\omega_0(n-1,\tau,t) - \omega_0(n,\tau,t)] \\
&= \hat{\varepsilon}(\tau)\omega_0(n,\tau,t)\left[e^{\alpha(n-ct-d(\tau)-\Gamma)}\left(\alpha c - \frac{e^a}{b_0}(1-e^{-\alpha}) + K\alpha e^{-\alpha\Gamma}e^{\alpha(d(t)-d(\tau))}d'(t)\right]\right. \\
&\geq \hat{\varepsilon}(\tau)\omega_0(n,\tau,t)\left[e^{\alpha(d(t)-d(\tau)-2\Gamma)}\left(\alpha c - \frac{e^a}{b_0}(1-e^{-\alpha}) + K\alpha e^{-\alpha\Gamma}e^{\alpha(d(t)-d(\tau))}d'(t)\right]\right. \\
&= \hat{\varepsilon}(\tau)\omega_0(n,\tau,t)e^{\alpha(d(t)-d(\tau)-\Gamma)}[e^{-\alpha\Gamma}\cdot\sigma + K\alpha d'(t)] \geq 0.
\end{aligned}$$

For the last inequality we have used (4.16).

Step 2. Let $g(n,t)$ be the Poisson distribution, i.e. the solution of the equation

$$\frac{\partial g}{\partial t} + c(g(n-1,t) - g(n,t)) = 0 \tag{4.18}$$

with initial conditions

$$g(n,0) = \begin{cases} 0 & \text{if } n \neq 0, \\ 1 & \text{if } n = 0. \end{cases}$$

Then it is known that for $n = [ct + A\sqrt{ct}]$ and $t \geq t_0$ we have the approximation

$$g(n,t) = \frac{1}{\sqrt{2\pi ct}} e^{-A^2/2}\left(1 + O\left(\frac{A + A^3}{\sqrt{tc}}\right)\right), \tag{4.19}$$

where $A = (n - ct)/\sqrt{ct}$ (see, for example, [HST]).

Put further

$$\begin{aligned} \omega_1(n,\tau,t) &:= \rho(\tau) g\left(n, t + \frac{\Gamma}{c} - \frac{b\sqrt{t}}{c}\right) \\ &= \rho(\tau)\frac{1}{\sqrt{2\pi ct}} \exp\left(-\frac{(n - ct - \Gamma + b\sqrt{t})^2}{2ct}\right)(1 + O(1/\sqrt{t})), \end{aligned}$$

where $n \in (ct + \Gamma - b\sqrt{t}, ct + A\sqrt{ct})$; the parameters $\rho(\tau)$ and b will be chosen later.

From the definition of $\omega_1(n,\tau,t)$ we obtain

$$\frac{\partial \omega_1}{\partial t}(n,\tau,t) = c\left(1 - \frac{b}{2c\sqrt{t}}\right)(\omega_1(n-1,\tau,t) - \omega_1(n,\tau,t)). \tag{4.20}$$

By the assumption of Proposition 4 the parameter A satisfies $A^2 + Ab/\sqrt{c_0} < 1$. This implies that for some $\varepsilon > 0$,

$$c - \frac{b}{2\sqrt{t}} \geq c + \frac{(A\sqrt{c} + \varepsilon)(1 + \varepsilon)}{2\sqrt{t}} - \frac{c(1 - \varepsilon)}{2A\sqrt{ct}}, \quad t \geq t_0, \tag{4.21}$$

where the right-hand side is taken from inequality (4.9). From (4.9), (4.20), (4.21) we deduce

$$\frac{\partial \omega_1}{\partial t}(n,\tau,t) \geq \Theta(n,t)(\omega_1(n-1,\tau,t) - \omega_1(n,\tau,t)) \tag{4.22}$$

if $n \in (ct + \Gamma, ct + A\sqrt{ct})$.

Step 3. The necessary function $\omega(n,\tau,t)$ can be constructed in the form

$$\omega(n,\tau,t) = \begin{cases} \omega_0(n,\tau,t) & \text{if } [ct + d(t) - \Gamma] \leq n \leq [ct + d(\tau) + \Gamma], \\ \omega_1(n,\tau,t) & \text{if } [ct + d(t) + \Gamma] < n \leq [ct + A\sqrt{ct}]. \end{cases}$$

In order that $\omega(n,\tau,t)$ satisfies property (4.11) we now choose parameters $\hat{\varepsilon}(\tau)$ and $\rho(\tau)$ in the expressions for ω_0 and ω_1 in such a way that

$$\omega_0(n,\tau,t) = \omega_1(n,\tau,t) \tag{4.23}$$

if $n = [ct + d(\tau) + \Gamma]$, and

$$\Delta\omega_0(n,\tau,t) \geq \Delta\omega_1(n,\tau,t) \tag{4.24}$$

if $n = [ct + d(\tau) + \Gamma]$ or $n = [ct + d(\tau) + \Gamma] + 1$. Inserting in (4.23) the definitions of ω_0 and ω_1 we obtain

$$\begin{aligned}\exp[-\hat{\varepsilon}(1 + Ke^{-\alpha\Gamma}(1 - e^{\alpha(d(t)-d(\tau))}))] \\ &= \frac{\rho(\tau)}{\sqrt{2\pi ct}} \exp\left(-\frac{(d(\tau) + b\sqrt{t})^2}{2ct}\right)(1 + O(1/\sqrt{t})) \\ &= \frac{\rho(\tau)}{\sqrt{2\pi ct}} \exp\left(-\frac{b^2}{2c}\right)\left(1 + O\left(\frac{\ln t}{\sqrt{t}}\right)\right). \end{aligned} \tag{4.25}$$

Here we have used the fact that by Proposition 3b,

$$d(t) = \frac{1}{2} c\gamma \ln t + o(\ln t).$$

Let us choose an expression for $-K$ satisfying (4.16) in the form

$$-K = \frac{1}{\alpha \cdot d'(t)} e^{-\alpha\Gamma} \cdot \sigma, \quad \text{where} \quad \sigma = c\alpha - \frac{e^{\varepsilon}}{b_0}(1 - e^{-\alpha}).$$

Putting the expressions above for $d(t)$, K and σ in (4.25), we find that (4.23) is equivalent to

$$\begin{aligned}\exp[-\hat{\varepsilon}(1 + t(1 + o(1)) \ln(t/\tau)e^{-2\alpha\Gamma}\sigma] \\ &= \frac{\rho(\tau)}{\sqrt{2\pi ct}} \exp\left(-\frac{b^2}{2c}\right)\left(1 + O\left(\frac{\ln t}{\sqrt{t}}\right)\right). \end{aligned} \tag{4.26}$$

Inserting in (4.24) the expressions for $\Delta\omega_0$ and $\Delta\omega_1$ we obtain the existence of functions $\theta_0(t,\tau)$ and $\theta_1(t,\tau)$ with values in $[0,1]$ such that

$$\begin{aligned}&-(\hat{\varepsilon}\alpha)e^{-\alpha\theta_0(t,\tau)} \cdot \omega_0|_{n=[ct+d(\tau)+\Gamma]} \\ &= -\frac{d(\tau) + b\sqrt{t} - \theta_1(t,\tau)}{ct}(1 + O(1/\sqrt{t})) \cdot \omega_1|_{n=[ct+d(\tau)+\Gamma],[ct+d(\tau)+\Gamma]+1}. \end{aligned} \tag{4.27}$$

Because of (4.23), inequality (4.27) can be transformed into the form compatible with (5.5):

$$\hat{\varepsilon}(\tau)\alpha \le e^{\alpha\theta_0(t,\tau)} \frac{d(\tau) + b\sqrt{t}}{ct}(1 + O(1/\sqrt{t})), \tag{4.28}$$

where $\tau \le t \le (1 + \delta)\tau$.

Let $\hat{\varepsilon}(\tau)$ and $\rho(\tau)$ be such that (4.15), (4.26), (4.28) are valid. Then $\omega(n,\tau,t)$ satisfies (4.11) for $n \in (ct + d(t) - \Gamma, ct + A\sqrt{ct})$ and $t \in (\tau, (1 + \delta)\tau)$, $\tau > t_0$.

Step 4. Now we will verify property (4.13). We have

$$\begin{aligned}&\max\{\omega(n,\tau,t) : d(t) - \Gamma \le n - ct \le A\sqrt{ct}\} \\ &\quad = \omega_0(n,\tau,t)|_{n=[ct+d(t)-\Gamma]} \\ &\quad = \exp[-\hat{\varepsilon}(e^{-2\alpha\Gamma+\alpha\gamma\ln(t/\tau)(1+o(1))} + t\ln(t/\tau)(1 + o(1))e^{-2\alpha\Gamma}\sigma_0)] \end{aligned} \tag{4.29}$$

and

$$\begin{aligned}\min\{\omega(n,t,t) : d(t)-\Gamma \le n-ct \le A\sqrt{ct}\} \\ &= \omega_1(n,t,t)|_{n=[ct+A\sqrt{ct}]} \\ &= \frac{\rho(t)}{\sqrt{2\pi ct}}\exp\left(-\frac{(A\sqrt{ct}+b\sqrt{t})^2}{2ct}\right)\left(1+O\left(\frac{\ln t}{\sqrt{t}}\right)\right).\end{aligned} \tag{4.30}$$

Putting in (4.26) $\tau = t$ we obtain the following expression for $\rho(t)$:

$$\rho(t) = \exp\left(-\hat{\varepsilon}(t)+\frac{b^2}{2c}\right)\sqrt{2\pi ct}\left(1+O\left(\frac{\ln t}{\sqrt{t}}\right)\right).$$

Inequality (5.18) implies the following estimate for $\hat{\varepsilon}(\tau)$:

$$\frac{b}{\alpha c\sqrt{t}} < \hat{\varepsilon}(t) \le e^{\alpha}\frac{b}{\alpha c\sqrt{t}}\left(1+O\left(\frac{\ln t}{\sqrt{t}}\right)\right), \quad t\in(\tau,(1+\delta)\tau).$$

From (4.29), (4.30), the expressions for $\rho(t)$ and the estimate for $\hat{\varepsilon}(\tau)$ we deduce

$$\begin{aligned}&\frac{\max\{\omega(n,\tau,t) : d(t)-\Gamma \le n-ct \le A\sqrt{ct}\}}{\min\{\omega(n,t,t) : d(t)-\Gamma \le n-ct \le A\sqrt{ct}\}} \\ &\quad\le \left(1+O\left(\frac{\ln t}{\sqrt{t}}\right)\right)\exp\left[\frac{A^2}{2}+\frac{Ab}{\sqrt{c}}-\sqrt{t}e^{-2\alpha\Gamma}\sigma\frac{b(1+o(1))}{c\alpha}\ln(t/\tau)\right] \\ &\quad\le (1+o(1))e\exp\left[-\frac{b\sqrt{t}e^{-2\alpha\Gamma}\sigma\ln(t/\tau)}{c\alpha}\right], \quad t\ge\tau\ge t_0\end{aligned}$$

(using the restriction on A from Proposition 4). Proposition 4 is proved.

Corollary to Proposition 4. *Under the assumptions of Proposition* 4, *for all* $A \in (0,1)$, $b < (1-A^2)\sqrt{c}/A$ *and* $\delta \in (0,1)$ *there exists* $t_0 > 0$ *such that the function* $\omega(n,\tau,t)$ *constructed in Proposition* 4 *satisfies*

$$\begin{aligned}&\frac{\max\{\omega(n,\tau,(1+\delta)\tau) : d_0(\tau(1+\delta))-\Gamma \le n-c_0\tau(1+\delta) < A\sqrt{c_0\tau(1+\delta)}\}}{\min\{\omega(n,(1+\delta)\tau,(1+\delta)\tau) : d_0(\tau(1+\delta))-\Gamma \le n-c_0\tau(1+\delta) < A\sqrt{c_0\tau(1+\delta)}\}} \\ &\qquad\le e\exp\left[-\frac{b\sqrt{(1+\delta)\tau}e^{-2\alpha\Gamma}\sigma\delta}{c_0\alpha}\right], \quad \tau\ge t_0.\end{aligned}$$

Proof. Fix $A\in(0,1)$ and take $b>0$ so small that the condition $A^2+Ab/\sqrt{c}<1$ from Proposition 4 be valid. Put $t=(1+\delta)\tau$ in inequality (4.13). Then the corollary is a consequence of (4.13).

Proposition 4 will be used in the proof of Theorem 1b in combination with the following important statement, giving estimates for

$$\Delta_0([x_-(t)],t,d_0(\tau)) \quad\text{and}\quad \tilde{\Delta}_L([x_+(t)],t,d_L(\tau)),$$

where $x_-(t) = c_0t+d_0(t)-\Gamma_0(\delta)$ and $x_+(t) = c_Lt+d_L(t)+\Gamma_L(\delta)$.

Lemma 16. *Under the assumptions of Proposition* 3b, *let* $d_0(t) = d_0(t,A,\infty)$, $d_L(t) = d_L(t,\infty,A)$, $A>0$, $\Gamma_0(\delta) = (1/\lambda_0)\ln(1/\delta)$, $\Gamma_L(\delta) = (1/\lambda_L)\ln(1/\delta)$, *where λ_0 and λ_L are defined by* (3.14)*:*

$$c_L\lambda_L = \varphi(\alpha_L^+)(e^{\lambda_L} - 1), \qquad c_0\lambda_0 = \varphi(\alpha_0^-)(1 - e^{-\lambda_0}).$$

Then for any $\tau > 0$ and $\delta \in [0,1]$ the following estimates are valid:

$$|\Delta_0([c_0t + d_0(t) - \Gamma_0(\delta)], t, d_0(\tau))| \le \frac{2b_1}{b_0}\frac{|\alpha_0^+ - \alpha_0^-|}{c_0}\ln(1+\delta) + O(1/\sqrt{t}) := O_0(\delta),$$

$$|\tilde{\Delta}_L([c_Lt + d_L(t) + \Gamma_L(\delta)], t, d_L(\tau))| \le \frac{2b_1}{b_0}\frac{|\alpha_L^+ - \alpha_L^-|}{c_L}\ln(1+\delta) + O(1/\sqrt{t}) := O_L(\delta),$$

where $t \in [\tau, (1+\delta)\tau]$, $b_0^{-1} = \max\varphi(\xi)$, $b_1^{-1} = \min\varphi(\xi)$.

Lemma 16 is a version of Lemma 12 from [HS, pp. 1485–1487], with improved values for $\Gamma_0(\delta)$ and $\Gamma_L(\delta)$.

5. Decreasing supersolutions for (4.5) for $l = 1, \dots, L-1$

The next proposition gives positive decreasing supersolutions for equation (4.5) for $l = 1, \dots, L-1$ ($L \ge 2$).

Proposition 5. *Under the assumptions of Theorem* 1b*, let $L \ge 2$. Let $d_l(\tau)$ and $\Theta_l(n,t) = \Theta_l(n,t,d_l(\tau))$ be as in Lemmas* 13, 14 *with $d_l'(\tau) \ne 0$, $\tau \ge t_0$, $l = 1, \dots, L-1$. Let $\beta(B_0)$ and $\tilde{\beta}(B_1)$ be as in Proposition* 4*. Let $a, \alpha > 0$*

$$\sup\left\{\beta\left(\frac{e^a}{c_lb_0}\right), \tilde{\beta}\left(\frac{e^{-a}}{c_lb_1}\right)\right\}, \tag{5.1}$$

where $b_0 < b_1$ are defined by (4.6)*. Put $\sigma = c_l\alpha - (e^a/b_0)(1 - e^{-\alpha})$, $\tilde{\sigma} = (e^{-a}/b_1)(e^{\alpha} - 1) - c_l\alpha)$ and $\hat{\sigma} = \min(\sigma, \tilde{\sigma})$. Then for all $b > 0$, $\Gamma > 2$ and A with $A^2 + Ab/\sqrt{c_l} < 1$ there exist $t_0 > 0$ and a positive function $\omega(n,\tau,t)$ with the following properties for $t \ge \tau > t_0$:*

$$\frac{\partial\omega(n,\tau,t)}{\partial t} \ge \Theta_l(n,t)(\omega(n-1,\tau,t) - \omega(n,\tau,t)) \tag{5.2}$$

if $c_l\tau - A\sqrt{c_l\tau} \le n \le c_lt + A\sqrt{c_lt}$, and

$$\frac{\max\{\omega(n,\tau,t) : -A\sqrt{c_lt} \le n - c_lt \le A\sqrt{c_lt}\}}{\min\{\omega(n,t,t) : -A\sqrt{c_lt} \le n - c_lt \le A\sqrt{c_lt}\}}$$
$$\le (1 + o(1))\exp\left[\frac{A^2}{2} + \frac{Ab}{\sqrt{c_l}} - \frac{b\sqrt{t}e^{-\alpha\Gamma}\hat{\sigma}\ln(t/\tau)}{c_l\alpha}\right]. \tag{5.3}$$

Proof. The proof is longer than but similar to the proof of Proposition 4. We will indicate the main steps. Put in this proof $c = c_l$, $d(\tau) = d_l(\tau)$.

Step 1. Put

$$\omega_0^+(n,\tau,t) = \exp[-\hat{\varepsilon}(\tau)(e^{\alpha(n-ct-d(\tau))} + Ke^{-\alpha\Gamma}(1 - e^{\alpha(d(t)-d(\tau))}))]$$

if $n \in [ct + d(\tau), ct + d(\tau) + \Gamma]$, and

$$\omega_0^-(n, \tau, t) = \exp[-\hat{\varepsilon}(\tau)(e^{-\alpha(n-ct-d(\tau))} + Ke^{-\alpha\Gamma}(1 - e^{\alpha(d(t)-d(\tau))}))]$$

if $n \in [ct + d(\tau) - \Gamma, ct + d(\tau)]$.

We have, in particular, the important relations

$$\begin{aligned} &\omega_0^-([ct + d(\tau)], \tau, t) = \omega_0^+([ct + d(\tau)], \tau, t), \\ &\Delta\omega_0^-(n, \tau, t) \geq \Delta\omega_0^+(n, \tau, t) \text{ if } n = [ct + d(\tau)] \text{ or } n = [ct + d(\tau)] + 1. \end{aligned} \tag{5.4}$$

From condition (5.1) on α in Proposition 5 it follows that

$$\hat{\sigma} = \min(\sigma, \tilde{\sigma}) > 0.$$

Exactly as in Step 1 of the proof of Proposition 4, we can show that if

$$\hat{\varepsilon}(\tau)(1 - e^{-\alpha}) \leq a, \quad \hat{\varepsilon}(\tau)(e^{\alpha} - 1) \leq ae^{-\alpha\Gamma}, \tag{5.5}$$

$$0 < -\alpha K d'(t) \leq \hat{\sigma}, \tag{5.6}$$

then

$$\frac{\partial\omega_0^+(n, \tau, t)}{\partial t} - \Theta_l(n, t)(\omega_0^+(n - 1, \tau, t) - \omega_0^+(n, \tau, t)) \geq 0 \tag{5.7}$$

for $n \in [ct + d(\tau), ct + d(\tau) + \Gamma]$, and

$$\frac{\partial\omega_0^-(n, \tau, t)}{\partial t} - \Theta_l(n, t)(\omega_0^-(n - 1, \tau, t) - \omega_0^-(n, \tau, t)) \geq 0 \tag{5.8}$$

for $n \in [ct + d(\tau) - \Gamma, ct + d(\tau)]$.

Step 2. Let $g(n, t)$ be the solution of (4.18) with $c = c_l$ and with initial conditions $g(n, 0) = 0$ if $n \neq 0$, and $g(n, 0) = 1$ if $n = 0$. Put

$$\omega_1^+(n, \tau, t) = \rho^+(\tau) g\left(n, t + \frac{\Gamma}{c} - \frac{b\sqrt{t}}{c}\right)$$

for $n \in (ct + d(\tau) + \Gamma, ct + A\sqrt{ct})$,

$$\omega_1^-(n, \tau, t) = \rho^-(\tau) g\left(n, t - \frac{\Gamma}{c} + \frac{b\sqrt{t}}{c}\right)$$

for $n \in (ct - A\sqrt{ct}, ct + d(\tau) - \Gamma)$.

From the definition of $\omega_1^\pm(n, \tau, t)$ we obtain

$$\frac{\partial\omega_1^\pm}{\partial t}(n, \tau, t) = c\left(1 \mp \frac{b}{2c\sqrt{t}}\right)(\omega_1^\pm(n - 1, \tau, t) - \omega_1^\pm(n, \tau, t)). \tag{5.9}$$

If A satisfies the assumption of Proposition 5, then for some $\varepsilon > 0$ we have

$$c - \frac{b}{2\sqrt{t}} \geq c + \frac{(A\sqrt{c} + \varepsilon)(1 + \varepsilon)}{2\sqrt{t}} - \frac{c(1 - \varepsilon)}{2A\sqrt{ct}}, \tag{5.10}$$

$$c + \frac{b}{2\sqrt{t}} \leq c - \frac{(A\sqrt{c} + \varepsilon)(1 - \varepsilon)}{2\sqrt{t}} + \frac{c(1 + \varepsilon)}{2A\sqrt{ct}}, \tag{5.11}$$

for $t \geq t_0$.

Inequalities (5.9)–(5.11) and (4.9), (4.10) imply

$$\frac{\partial \omega_1^{\pm}(n,\tau,t)}{\partial t} \geq \Theta_l(n,t,d_0(\tau))(\omega_1^{\pm}(n-1,\tau,t)-\omega_1^{\pm}(n,\tau,t))$$

respectively for $n \in [ct+d(\tau)+\Gamma, ct+A\sqrt{ct}]$ and for $n \in [ct-A\sqrt{ct}, ct+d(\tau)-\Gamma]$.

Step 3. The required function $\omega(n,\tau,t)$ can be constructed in the form

$$\omega(n,\tau,t) = \begin{cases} \omega_1^-(n,\tau,t) & \text{if } n \in [ct-A\sqrt{ct}, ct+d(\tau)-\Gamma], \\ \omega_0^-(n,\tau,t) & \text{if } n \in [ct+d(\tau)-\Gamma, ct+d(\tau)], \\ \omega_0^+(n,\tau,t) & \text{if } n \in [ct+d(\tau), ct+d(\tau)+\Gamma], \\ \omega_1^+(n,\tau,t) & \text{if } n \in [ct+d(\tau)+\Gamma, ct+A\sqrt{ct}]. \end{cases}$$

In order that $\omega(n,\tau,t)$ satisfies property (5.2) we choose $\hat{\varepsilon}(\tau)$ and $\rho^{\pm}(\tau)$ in the expressions for $\omega_0^{\pm}(n,\tau,t)$ and $\omega_1^{\pm}(n,\tau,t)$ in such a way that

$$\omega_0^+(n,\tau,t) = \omega_1^+(n,\tau,t) \qquad \text{if } n = [ct+d(\tau)+\Gamma], \tag{5.12}$$

$$\Delta\omega_0^+(n,\tau,t) \geq \Delta\omega_1^+(n,\tau,t) \qquad \text{if } n = [ct+d(\tau)+\Gamma], \text{ or } n = [ct+d(\tau)+\Gamma+1]. \tag{5.13}$$

$$\omega_0^-(n,\tau,t) = \omega_1^-(n,\tau,t) \qquad \text{if } n = [ct+d(\tau)-\Gamma], \tag{5.14}$$

$$\Delta\omega_1^-(n,\tau,t) \geq \Delta\omega_0^-(n,\tau,t) \qquad \text{if } n = [ct+d(\tau)-\Gamma], \text{ or } n = [ct+d(\tau)-\Gamma+1]. \tag{5.15}$$

Putting (5.12), (5.14) in the definitions of $\omega_0^{\pm}$, $\omega_1^{\pm}$ we obtain

$$\begin{aligned} &\exp[-\hat{\varepsilon}(e^{\alpha\Gamma} + Ke^{-\alpha\Gamma}(1-e^{\alpha(d(t)-d(\tau))}))] \\ &= \frac{\rho^+(\tau)}{\sqrt{2\pi ct}} e^{-b^2/(2c)}\left(1+O\left(\frac{\ln t}{\sqrt{t}}\right)\right) = \frac{\rho^-(\tau)}{\sqrt{2\pi ct}} e^{-b^2/(2c)}\left(1+O\left(\frac{\ln t}{\sqrt{t}}\right)\right). \end{aligned} \tag{5.16}$$

We choose $-K$ satisfying (5.16) in the form

$$-K = \frac{1}{\alpha d'(t)}\hat{\sigma}.$$

By Proposition 3b we have

$$d(t) = \frac{c}{2}\gamma \ln t + o(\ln t).$$

Putting these expressions in (5.16) we see that equalities (5.12), (5.14) are equivalent to

$$\begin{aligned} &\exp[-\hat{\varepsilon}(e^{\alpha\Gamma} + \hat{\sigma} t \ln(t/\tau)(1+o(1))] \\ &= \frac{\rho^+(\tau)}{\sqrt{2\pi ct}} e^{-b^2/(2c)}\left(1+O\left(\frac{\ln t}{\sqrt{t}}\right)\right) = \frac{\rho^-(\tau)}{\sqrt{2\pi ct}} e^{-b^2/(2c)}\left(1+O\left(\frac{\ln t}{\sqrt{t}}\right)\right). \end{aligned} \tag{5.17}$$

Inserting in (5.13), (5.15) the expressions for $\Delta\omega_0^{\pm}$ and $\Delta\omega_1^{\pm}$ we conclude that there exists some function $\theta^{\pm}(t,\tau)$ with values in $[0,1]$ such that

$$\begin{aligned}\hat{\varepsilon}(\tau)\alpha &\le e^{\alpha\theta^+(t,\tau)}\left(\frac{d(\tau)+b\sqrt{t}}{ct}\right)(1+O(1/\sqrt{t})),\\ -\hat{\varepsilon}(\tau)\alpha &\ge e^{\alpha\theta^-(t,\tau)}\left(\frac{d(\tau)-b\sqrt{t}}{ct}\right)(1+O(1/\sqrt{t})),\end{aligned} \tag{5.18}$$

where $\tau \le t$. Let $\hat{\varepsilon}(\tau)$ and $\rho^\pm(\tau)$ be such that (5.5), (5.16) and (5.18) hold. Then we have (5.2) and, in addition, $\rho^+(\tau) = \rho^-(\tau)$ and

$$\frac{b}{c\alpha\sqrt{t}}(1+O(1/\sqrt{t})) \le \hat{\varepsilon}(\tau) \le \frac{be^\alpha}{c\alpha\sqrt{t}}(1+O(1/\sqrt{t})). \tag{5.19}$$

Step 4. Now we can verify (5.3). We have

$$\begin{aligned}\max\{\omega(n,\tau,t) : -A\sqrt{ct} \le n-ct \le A\sqrt{ct}\}\\ &= \omega_0^\pm(n,\tau,t)|_{n=ct+d(\tau)}\\ &= \exp[-\hat{\varepsilon}(\tau)(1+Ke^{-\alpha\Gamma}(1-e^{\alpha(d(t)-d(\tau))}))]\\ &= \exp\left[-\hat{\varepsilon}(\tau)\left(1+\frac{\hat{\sigma}}{\alpha d'(t)}e^{-\alpha\Gamma}\frac{c}{2}\alpha\gamma\ln(t/\tau)\right)\right]\\ &= \exp[-\hat{\varepsilon}(\tau)(1+te^{-\alpha\Gamma}\hat{\sigma}\ln(t/\tau))].\end{aligned} \tag{5.20}$$

$$\begin{aligned}\min\{\omega(n,t,t) : -A\sqrt{ct} \le n-ct \le A\sqrt{ct}\}\\ &= \min\omega_1^\pm(n,t,t)|_{n=[ct\pm A\sqrt{ct}]}\\ &= \frac{\rho^\pm(t)}{\sqrt{2\pi ct}}\exp\left[-\frac{(n-ct\pm b\sqrt{t})^2}{2ct}\right]\left(1+O\left(\frac{\ln t}{\sqrt{t}}\right)\right)\Big|_{n=[ct\pm A\sqrt{ct}]}\\ &= \frac{\rho^\pm(t)}{\sqrt{2\pi ct}}\exp\left[-\frac{(\pm A\sqrt{ct}\pm b\sqrt{t})^2}{2ct}\right]\left(1+O\left(\frac{\ln t}{\sqrt{t}}\right)\right).\end{aligned} \tag{5.21}$$

Putting $\tau = t$ in (5.17) we obtain the following expression for $\rho^\pm(t)$:

$$\frac{\rho^\pm(t)}{\sqrt{2\pi ct}}\exp\left[-\frac{b^2}{2c}\right] = \exp[-\hat{\varepsilon}\cdot e^{\alpha\Gamma}(1+O(\ln t/t))]. \tag{5.22}$$

From (5.19)–(5.22) we deduce

$$\begin{aligned}&\frac{\max\{\omega(n,\tau,t) : -A\sqrt{ct} \le n-ct \le A\sqrt{ct}\}}{\min\{\omega(n,t,t) : -A\sqrt{ct} \le n-ct \le A\sqrt{ct}\}}\\ &\le \left(1+O\left(\frac{\ln t}{\sqrt{t}}\right)\right)\exp\left[-\hat{\varepsilon}(1+te^{-\alpha\Gamma}\hat{\sigma}\ln(t/\tau)-e^{-\alpha\Gamma})-\frac{b^2}{2c}+\frac{(A\sqrt{ct}+b\sqrt{t})^2}{2ct}\right]\\ &\le (1+o(1))\exp\left[\frac{A^2}{2}+\frac{Ab}{\sqrt{c}}-\frac{b\sqrt{t}e^{-\alpha\Gamma}\hat{\sigma}\ln(t/\tau)}{c\alpha}\right],\end{aligned}$$

where $A^2 + Ab/\sqrt{c} < 1$, $t \ge \tau \ge t_0$. Proposition 5 is proved.

Corollary to Proposition 5. *Under the assumptions of Proposition* 5*, put* $c = c_l$*. Then for all* $b > 0$*,* $\delta > 0$*,* $\Gamma > 2$*,* $\lambda > 0$ *and* $A = \sqrt{\delta}$ *with* $A^2 + Ab/\sqrt{c} < 1$ *there exist* $t_0 > 0$ *and a positive function* $\omega(n, \tau, t)$ *satisfying* (5.2) *for* $t \geq \tau \geq t_0$ *and*

$$\frac{\max\{\omega(n,\tau,\tau(1+\frac{\delta}{\sqrt{\lambda\tau}})) : |n - c\tau(1+\frac{\delta}{\sqrt{\lambda\tau}})| \leq A\sqrt{c\tau(1+\frac{\delta}{\sqrt{\lambda\tau}})}\}}{\min\{\omega(n,\tau(1+\frac{\delta}{\sqrt{\lambda\tau}}),\tau(1+\frac{\delta}{\sqrt{\lambda\tau}})) : |n - c\tau(1+\frac{\delta}{\sqrt{\lambda\tau}})| \leq A\sqrt{c\tau(1+\frac{\delta}{\sqrt{\lambda\tau}})}\}}$$

$$\leq \exp\left[\frac{\delta}{2} + \frac{b\sqrt{\lambda\delta}}{\sqrt{c}} - \frac{be^{-\alpha\Gamma}\hat{\sigma}}{c\alpha}\delta\right] \tag{5.23}$$

for $\tau \geq t_0$*.*

Proof. In (1b) with $\varepsilon = 1$, put $\tilde{t} = t\lambda$, $\tilde{\tau} = \tau\lambda$. Then the function $\tilde{f}(n, \tilde{t}) := f(n, \tilde{t}/\lambda)$ is the solution of the equation

$$\frac{d\tilde{f}}{d\tilde{t}} + \tilde{\varphi}(\tilde{f})(\tilde{f}(n,\tilde{t}) - \tilde{f}(n-1,\tilde{t})) = 0, \tag{5.24}$$

where $\tilde{\varphi}(\tilde{f}) = \frac{1}{\lambda}\varphi(f(n, \frac{\tilde{t}}{\lambda}))$. Applying Proposition 5 to (5.24) we find that for all $b, \Gamma > 0$ and $A \in (0, \sqrt{1 + b^2/(4\tilde{c})} - b/(2\sqrt{\tau}))$ with $\tilde{c} = c/\lambda$ there exist $\tilde{t}_0 > 0$ and a positive function $\tilde{\omega}(n, \tilde{\tau}, \tilde{t})$ which satisfies

$$\frac{\partial\tilde{\omega}(n,\tilde{\tau},\tilde{t})}{\partial\tilde{t}} \geq \tilde{\Theta}(n,\tilde{t})(\tilde{\omega}(n-1,\tilde{\tau},\tilde{t}) - \tilde{\omega}(n,\tilde{\tau},t)), \tag{5.25}$$

where $\tilde{\Theta}(n,\tilde{t}) = \Theta_l(n,\tilde{t}/\lambda)$, $\tilde{c}\tilde{\tau} - A\sqrt{\tilde{c}\tilde{\tau}} \leq n \leq \tilde{c}\tilde{t} + A\sqrt{\tilde{c}\tilde{t}}$, $\tilde{t} \geq \tilde{\tau} \geq \tilde{t}_0$, and also

$$\frac{\max\{\tilde{\omega}(n,\tilde{\tau},\tilde{t}) : -A\sqrt{\tilde{c}\tilde{t}} \leq n - \tilde{c}\tilde{t} \leq A\sqrt{\tilde{c}\tilde{t}}\}}{\min\{\tilde{\omega}(n,\tilde{t},\tilde{t}) : -A\sqrt{\tilde{c}\tilde{t}} \leq n - \tilde{c}\tilde{t} \leq A\sqrt{\tilde{c}\tilde{t}}\}}$$

$$\leq (1+o(1))\exp\left[\frac{A^2}{2} + \frac{Ab}{\sqrt{\tilde{c}}} - \frac{b\sqrt{\tilde{t}}e^{-\alpha\Gamma}\tilde{\hat{\sigma}}\ln(\tilde{t}/\tilde{\tau})}{\tilde{c}\alpha}\right], \tag{5.26}$$

where

$$\tilde{\hat{\sigma}} = \min(\tilde{c}\alpha - e^a(1-e^{-\alpha})/\tilde{b}_0, e^a(e^{-\alpha}-1)/\tilde{b}_1 - \tilde{c}\alpha).$$

From the definitions of $\tilde{t}$, $\tilde{\tau}$, $\tilde{c}$, $\tilde{b}_0$, $\tilde{b}_1$, $\tilde{\hat{\sigma}}_1$ we obtain $\tilde{c}\tilde{t} = ct$, $\tilde{c}\tilde{\tau} = c\tau$, $\tilde{c}\tilde{b}_0 = cb_0$, $\tilde{c}\tilde{b}_1 = cb_1$, $\tilde{\hat{\sigma}}/(\tilde{c}\alpha) = \hat{\sigma}/(c\alpha)$. Put $\tilde{t} = \tilde{\tau}(1 + \delta/\sqrt{\tilde{\tau}})$ in (5.25), (5.26) and denote by $\omega(n, \tau, t)$ the function $\tilde{\omega}(n, \tau\lambda, t\lambda)$. Then $\omega(n, \tau, t)$ satisfies (5.2) because of (5.25). From (5.26) we deduce

$$\frac{\max\{\omega(n,\tau,\tau(1+\frac{\delta}{\sqrt{\lambda\tau}})) : |n - c\tau(1+\frac{\delta}{\sqrt{\lambda\tau}})| \leq A\sqrt{c\tau(1+\frac{\delta}{\sqrt{\lambda\tau}})}\}}{\min\{\omega(n,\tau(1+\frac{\delta}{\sqrt{\lambda\tau}}),\tau(1+\frac{\delta}{\sqrt{\lambda\tau}})) : |n - c\tau(1+\frac{\delta}{\sqrt{\lambda\tau}})| \leq A\sqrt{c\tau(1+\frac{\delta}{\sqrt{\lambda\tau}})}\}}$$

$$\leq (1+o(1))\exp\left[\frac{A^2}{2} + \frac{Ab\sqrt{\lambda}}{\sqrt{c}} - \frac{be^{-\alpha\Gamma}\hat{\sigma}\sqrt{\tilde{\tau}(1+\delta/\sqrt{\tilde{\tau}})}\ln(1+\delta/\sqrt{\tilde{\tau}})}{c\alpha}\right]$$

$$= (1+o(1))\exp\left[\frac{A^2}{2} + \frac{Ab\sqrt{\lambda}}{\sqrt{c}} - be^{-\alpha\Gamma}\hat{\sigma}\cdot\delta\right] \leq \exp\left[\frac{\delta}{2} + \frac{b\sqrt{\lambda\delta}}{\sqrt{c}} - \frac{be^{-\alpha\Gamma}\hat{\sigma}\cdot\delta}{c\alpha}\right]$$

for $\tau \geq t_0$. Thus the Corollary to Proposition 5 is proved.

Proposition 5 will be used in the proof of Theorem 1b in combination with the following statement which gives an important estimate of $\Delta_l([x_-(t)], t, d_l(\tau))$, where $l = 1, \ldots, L-1$ and $x_-(t) = c_l t - A\sqrt{c_l t}$.

Lemma 17. *Under the assumptions and notations of Proposition* 3b, *let* $d_l(t) = d_l(t, A\sqrt{c_l})$, $A > 0$, $l = 1, \ldots, L-1$, $x_-(t) = c_l t - A\sqrt{c_l t}$. *Then for any* $\lambda, \tau > 0$ *and* $\delta \in (0,1)$ *there exists* $t_0 > 0$ *such that for all* $t \in [\tau, \tau(1+\delta/\sqrt{\lambda\tau})]$ *and* $\tau > t_0$,

$$|\Delta_l([x_-(t)], t, d_l(\tau))| \leq \frac{\sqrt{c_l \lambda}}{\varphi'(\alpha_l^-)}\left(\frac{2}{A} + A + \delta\sqrt{c_l/\lambda}\right)O(\delta), \tag{5.27}$$

where $|O(\delta)| \leq (\text{abs.constant}) \cdot \delta$.

Proof. Put $c = c_l$, $d(\tau) = d_l(\tau)$, $\alpha^- = \alpha_l^-$. As in the proof of Corollary to Proposition 5, we put in (1b) $\tilde{t} = t\lambda$, $\tilde{\tau} = \tau\lambda$, $\tilde{c} = c/\lambda$. Then inequality (5.27) for solutions of (1b) will be transformed into the following inequality for solutions of (5.24):

$$|\Delta([\tilde{x}_-(\tilde{t})], \tilde{t}, \tilde{d}_l(\tilde{\tau}))| \leq \frac{\sqrt{\tilde{c}}}{\tilde{\varphi}'(\alpha^-)}\left(\frac{2}{A} + A + \delta\sqrt{\tilde{c}}\right)O(\delta), \tag{5.28}$$

for $\tilde{t} \in [\tilde{\tau}, \tilde{\tau}(1+\delta/\sqrt{\tilde{\tau}})]$, where $\tilde{x}_-(\tilde{t}) = x_-(\tilde{t}/\lambda)$, $\tilde{d}(\tau) = d(\tilde{\tau}/\lambda)$, $\tilde{c} = c/\lambda$, $\tilde{\varphi} = \varphi/\lambda$.

Hence, in order to prove (5.27) with $\lambda > 0$ it is sufficient to prove it for $\lambda = 1$. So, set $\lambda = 1$. If $t \in [\tau, \tau(1+\delta/\sqrt{\tau})]$, $\tau \geq t_0$, then for all $k \in ([c\tau - A\sqrt{c\tau}], [ct - A\sqrt{ct}])$ we have $k = ct - A_\delta\sqrt{ct}$, where $A_\delta = A + O(\delta\sqrt{c})$. Using this together with estimates (1.1) and (3.9) we obtain the asymptotic relations

$$\begin{aligned} f(k,t) &= \varphi^{(-1)}(k/t) + O\left(\frac{c}{\varphi'(\alpha^-)A\sqrt{ct}}\right) \\ &= \alpha^- - \frac{1}{\varphi'(\alpha^-)}\sqrt{c/t}(A_\delta + O(1/A)), \end{aligned} \tag{5.29}$$

$$\begin{aligned} \tilde{f}(k - ct - d(\tau)) &= \alpha^- - \frac{c}{\varphi'(\alpha^-)(k - ct - d(\tau))} + o\left(\frac{1}{k - ct - d(\tau)}\right) \\ &= \alpha^- + \frac{c}{A_\delta\sqrt{ct}\varphi'(\alpha^-)} + o\left(\frac{1}{A_\delta\sqrt{ct}}\right). \end{aligned} \tag{5.30}$$

By the definition (4.2) of $\Delta(n, t, d(\tau))$ we have

$$\Delta([x_-(t)], t, d(\tau)) = \sum_{k=[c\tau - A\sqrt{c\tau}]}^{[ct - A\sqrt{ct}]} \int_{f(k,t)}^{\tilde{f}(k-ct-d(\tau))} \frac{dy}{\varphi(y)}. \tag{5.31}$$

Putting in (5.31) estimates (5.29), (5.30) we obtain

$$\begin{aligned} &|\Delta([x_-(t)], t, d(\tau))| \\ &= (1 + O(1/\sqrt{\tau}))\frac{1}{c}O(\delta\sqrt{c})\sqrt{ct} \sup_{k \in ([c\tau - A\sqrt{c\tau}], [ct - A\sqrt{ct}])} |f(x,t) - \tilde{f}(k - ct - d(\tau))| \end{aligned}$$

$$= O(\delta\sqrt{c})\frac{\sqrt{ct}}{c\varphi'(\alpha^-)}\left(\frac{c}{(A+O(\delta\sqrt{c}))\sqrt{ct}} + \sqrt{c/t}(A+O(\delta\sqrt{c})) + O(1/A)\right)$$
$$= O\left(\frac{\delta\sqrt{c}}{\varphi'(\alpha^-)}\right)(2/A + A + \delta\sqrt{c}), \quad t \geq t_0.$$

Lemma 17 is proved.

Corollary to Lemma 17. *Under the assumptions of Lemma* 17*, let* $\lambda = \rho\delta$*,* $A = \sqrt{\delta}$*,* $x_-(t) = c_l t - \sqrt{\delta c_l t}$*,* $l = 1, \ldots, L-1$*. Then for* $t \in [\tau, \tau(1 + \sqrt{\delta}/\sqrt{\rho\tau})]$*,* $\rho > 0$*,* $\tau \geq t_0$*,*

$$|\Delta_l([x_-(t)], t, d_l(\tau, \sqrt{\delta c_l}))| \leq O(\delta)\frac{c_l}{\varphi'(\alpha_l^-)}\left(\frac{(2+\delta)\sqrt{\rho}}{\sqrt{c}} + \delta\right) =: O_l(\delta). \tag{5.32}$$

Proof. From Lemmas 15, 17 and the formula for $d_l(t, A\sqrt{c_l})$ from Proposition 3b we obtain, for all $A, \lambda > 0$, $t \in [\tau, \tau(1 + \delta/(\lambda t))]$ and $\tau \geq t_0$,

$$\begin{aligned}
&|\Delta_l([x_-(t)], t, d_l(\tau, A\sqrt{c_l}))| \\
&\quad \leq O\left(\frac{\delta\sqrt{c_l\lambda}}{\varphi'(\alpha_l^-)}\right)(2/A + A + \delta\sqrt{c_l/\lambda}) + \frac{c_l}{2}\frac{|\alpha_l^+ - \alpha_l^-|}{c_l}\gamma_l\frac{\delta(1+o(1))}{\sqrt{\tau\lambda}} \\
&\quad = O(\delta)\frac{\sqrt{c_l\lambda}}{\varphi'(\alpha_l^-)}(2/A + A + \delta\sqrt{c_l/\lambda}).
\end{aligned}$$

Putting in the right-hand side $A = \sqrt{\delta}$ and $\lambda = \rho\delta$ we obtain (5.32).

6. Proof of the main result

Under the assumptions of Theorem 1b, let $f(n,t)$ be the solution of the Cauchy problem (1b), (2) with $\varepsilon = 1$. Let $A > 0$ be fixed.

The uniform convergence of $f(n,t)$ to $\varphi^{(-1)}(n/t)$ in the intervals $[c_l t + A\sqrt{c_l t}] \leq n \leq [c_{l+1}t - A\sqrt{c_{l+1}t}]$, $l = 0, \ldots, L-1$, with the estimate

$$|f(n,t) - \varphi^{(-1)}(n/t)| = O(1/(A\sqrt{t}))$$

has been proved in [HP4, Theorem 7.5], under the condition that A is large enough.

A similar result has been obtained earlier in [W] for solutions of the Cauchy problem (1a), (2). For any $A > 0$ these results follow from Proposition 1 of this article. So, to prove Theorem 1 we must for some A prove the uniform convergence of $f(n,t)$ to the shifted travelling waves $\tilde{f}_l(n - c_l t - d_l(t, A))$ in the intervals $[c_l t - A\sqrt{c_l t}] \leq n \leq [c_l t + A\sqrt{c_l t}]$, $l = 0, \ldots, L$.

In the case $L = 0$ this statement has been obtained first by A. M. Il'in and O. A. Oleĭnik [IO] for the Cauchy problem (1a), (2) and later in [HP2] for the Cauchy problem (1b), (2). In this case $l = 0 = L$ and $d_0(t, A) \equiv \text{const} + o(1)$, where the constant is determined explicitly from the initial data by classical conservation laws (see Proposition 0).

Let now $L > 0$. We will restrict ourselves to the problem (1b), (2). We will use the following semidiscrete analogue of the classical maximum principle [PW] for differential parabolic equations.

Lemma 18. *Let $E(t) = \{k \in \mathbb{Z} : [x_-(t)] \le k \le [x_+(t)]\}$, where $x_-(t)$ and $x_+(t)$ are continuous functions with values in $[-\infty, +\infty)$ and $(-\infty, +\infty]$ respectively such that $1 + x_-(t) < x_+(t)$ for $t \ge t_0$. Let the function $\Theta(n, t)$ satisfy the inequalities*

$$0 < \beta_1 \le \Theta(n, t) \le \beta_0 < \infty.$$

Let finally $V(n, t)$ be a bounded function satisfying

(i) $\frac{dV(n,t)}{dt} \le \Theta(n, t)(V(n-1, t) - V(n, t)), \quad n \in E(t),\ t \in [t_0, T]$,
(ii) $V(n, t_0) \le 0$ *for* $n \in E(t_0)$,
(iii) $V([x_-(t)], t) \le 0$ *and* $V([x_+(t)], t) \le 0$ *for* $t \in [t_0, T]$.

Then $\max_{n \in E(t)} V(n, t) \le 0$ *for* $t \in [t_0, T]$.

This lemma has been proved as part of [HP2, Lemma 2, p. 574].

Let $c = c_l$, $d(t) = d_l(t, A\sqrt{c_l})$, $\Delta(n, t, d(\tau)) = \Delta_l(n, t, d_l(\tau))$.

Put $x_-(t) = ct + d(t) - (1/\lambda_0)\ln(1/\delta)$ if $l = 0$, $x_-(t) = ct - A\sqrt{ct}$ if $l > 0$, $x_+(t) = ct + A\sqrt{ct}$ if $l < L$, and $x_+(t) = ct + d(t) + (1/\lambda_L)\ln(1/\delta)$ if $l = L$, and let $E(t) = \{k \in \mathbb{Z} : [x_-(t)] \le k \le [x_+(t)]\}$.

We give the proof of Theorem 1b for two principal cases: $l = 0$ and $l \in \{1, \dots, L-1\}$. The case $l = L$ can be considered similarly to the case $l = 0$. Let $t_0, \delta, A > 0$ satisfy the conditions of the Corollary to Proposition 4 if $l = 0$, and of Proposition 5 if $l \in \{1, \dots, L-1\}$.

Let

$$t_{\nu+1} = (1 + \delta_{\nu+1})t_\nu, \quad \nu = 0, 2, \tag{6.1}$$

where

$$\delta_{\nu+1} = \begin{cases} \delta & \text{if } l = 0, \\ \sqrt{\delta}/\sqrt{\rho t_\nu} & \text{if } l \in \{1, \dots, L-1\}. \end{cases}$$

Let $\omega_0(n, \tau, t)$ be the function $\omega(n, \tau, t)$ considered in Proposition 4 with $\Gamma = \Gamma(\delta) = \Gamma_0(\delta) = (1/\lambda_0)\ln(1/\delta)$ as in Lemma 16. Let $\omega_l(n, \tau, t)$, $l = 1, \dots, L-1$, be the function $\omega(n, \tau, t)$ constructed in Proposition 5 with $\Gamma > 2$ independent of δ. Let $\Delta_l(n, t, d_l(\tau))$ be the function defined by (4.1) if $l = 0$, and by (4.2) if $l \in \{1, \dots, L-1\}$. Put

$$V_l^\pm(n, t) = \pm\Delta_l(n, t, d(t_{\nu+1})) - s\omega_l(n, t_\nu, t) - O_l(\delta), \tag{6.2}$$

where $O_0(\delta)$ is defined (together with $\Gamma_0(\delta)$) in Lemma 16, and $O_l(\delta)$, $l = 1, \dots, L-1$, is defined in the Corollary to Lemma 17, $t \in [t_\nu, t_{\nu+1}]$, $n \in E(t)$. The parameter $s > 0$ will be chosen later. By Proposition 4 for $l = 0$ and by Proposition 5 for $l = 1, \dots, L-1$ we have

$$\frac{dV_l^\pm(n, t)}{dt} \le \Theta_l(n, t)(V_l^\pm(n-1, t) - V_l^\pm(n, t)), \tag{6.3}$$

where $\Theta_l(n, t) = \Theta_l(n, t, d_l(\tau))$ is defined in Lemma 13, $t \in [t_\nu, t_{\nu+1}]$, $n \in E(t)$.

From definitions (4.1), (4.2) of $\Delta_l(n,t,d_l(\tau))$, the definition (3.1b) of $d_l(t)$ and the Corollary to Lemma 17 we obtain

$$\Delta_0([x_+(t)],t,d_0(t)) = O(1/\sqrt{t}) \quad \text{if } l=0,\ t\geq t_0, \tag{6.4}$$

and

$$\begin{aligned}\Delta_l([x_+(t)],t,d_l(t)) &= \sum_{k=[x_-(\tau)]}^{[x_+(t)]} (\Phi(f(k,t)-\Phi(\tilde f_l(k-c_lt-d_l(\tau))) \\ &= \sum_{[x_-(t)]+1}^{[x_+(t)]} (\Phi(f(k,t)-\Phi(\tilde f_l(k-c_lt-d_l(\tau)))+O(1/\sqrt{t}) \\ &= \sum_{[x_-(\tau)]}^{[x_-(t)]} (\Phi(f(k,t)-\Phi(\tilde f_l(k-c_lt-d_l(\tau)))+O(1/\sqrt{t}) \\ &\leq O_l(\delta)\end{aligned} \tag{6.5}$$

if $l=1,\dots,L-1$, $t\geq t_0$. From the definition (6.2) of $V_l^{\pm}(n,t)$, Lemma 15 and estimates (6.4), (6.5) we obtain

$$V_l^{\pm}([x_{\pm}(t)],t)\leq 0 \quad \text{if } t\in[t_\nu,t_{\nu+1}], \tag{6.6}$$

$\nu=0,1,\dots,$ where t_0 is large enough. Put

$$s_l = \frac{\max\{|\Delta_l(n,t_0,d_l(t_0))| : n\in E(t_0)\}}{\min\{|\omega_l(n,t_0,t_0)| : n\in E(t_0)\}}, \qquad l=0,\dots,L-1.$$

Relations (6.3), (6.6) and the maximum principle (Lemma 18) imply that $V_l^{\pm}(n,t)\leq 0$ for $n\in E(t_1)$, i.e.

$$|\Delta_l(n,t_1,d_l(t_1))|\leq s_l\omega_l(n,t_0,t_1)+O_l(\delta), \qquad l=0,\dots,L-1. \tag{6.7}$$

From (6.7), from the Corollaries to Propositions 4, 5 and from Lemmas 16, 17 we deduce

$$\begin{aligned}&\max\{|\Delta_l(n,t_1,d_l(t_1))| : n\in E(t_1)\} \\ &\qquad \leq e^{-\kappa_l}\max\{|\Delta_l(n,t_0,d_l(t_0))| : n\in E(t_1)\}+O_l(\delta), \qquad l=0,\dots,L-1,\end{aligned}$$

where

$$\kappa_0 = \frac{be^{-2\alpha\Gamma(\delta)}\sigma\delta\sqrt{t}}{c_0\alpha} = g_0\delta\sqrt{\tau} \tag{6.8}$$

with

$$g_0 := \frac{be^{-2\alpha\Gamma(\delta)}\sigma}{c_0\alpha} = \frac{b\sigma}{c_0\alpha}\delta^{2\alpha/\lambda_0},$$

and

$$\kappa_l = -\frac{\delta}{2} - b\sqrt{\rho/c_l}\,\delta + \frac{be^{-\alpha\Gamma}\hat\sigma\delta}{c_l\alpha} = g_l\delta \tag{6.9}$$

with

$$g_l = \frac{be^{-\alpha\Gamma}\hat\sigma}{c_l\alpha} - \frac{1}{2} - b\sqrt{\rho/c_l}.$$

Let a and α in Proposition 4 be such that $\alpha \leq \lambda_0/4$, i.e. $g_0 \geq \frac{b\sigma}{c_0\alpha}\sqrt{\delta}$. Fix also $\rho > 0$ so small and $b > 0$ so large that $g_l > 0$, $l = 1, \ldots, L-1$.

Repeating the construction above ν times we obtain

$$\begin{aligned} M_l(t_\nu) &:= \max\{|\Delta_l(n, t_\nu, d_l(t_\nu))| : n \in E(t_\nu)\} \\ &\leq e^{-\nu\kappa_l} \max\{|\Delta_l(n, t_0, d_l(t_0))| : n \in E(t_0)\} + (1 + e^{-\kappa_l} + \cdots + e^{-\kappa_l(\nu-1)})O_l(\delta) \\ &\leq e^{-\nu\kappa_l} M_l(t_0) + (1 - e^{-\kappa_l})^{-1} O_l(\delta). \end{aligned} \tag{6.10}$$

If $l = 0$ then from (6.1) we deduce $\nu = \ln(t_\nu/t_0)/\ln(1+\delta)$. Using this equality together with (6.8), (6.10) we obtain

$$M_0(t) := \max\left\{|\Delta_0(n, t, d_0(t, A\sqrt{c_0}))| : d_0(t) - \frac{1}{\lambda_0}\ln(1/\delta) \leq n - c_0 t \leq A\sqrt{c_0 t}\right\}$$

$$\leq \left(\frac{t_0}{t}\right)^{\frac{g_0\delta\sqrt{t}}{\ln(1+\delta)}} + \frac{O_0(\delta)}{1 - e^{-g_0\delta\sqrt{t}}} \leq \left(\frac{t_0}{t}\right)^{\frac{b\sigma\sqrt{t}}{c_0\alpha}\delta^{2\alpha/\lambda_0}} + \frac{O_0(\delta)}{1 - e^{\frac{-b\sigma\sqrt{t}}{c_0\alpha}\delta^{2\alpha/\lambda_0+1}}}, \tag{6.11}$$

where $\delta \in (0,1)$, $A \in (0,1)$, $t \geq t_0$.

If $l = 1, \ldots, L-1$ then from (6.1) we deduce $\nu = 2\sqrt{\rho t_\nu/\delta} + o(1)$. Using this estimate together with (6.9), (6.10) we obtain

$$M_l(t) := \max\{|\Delta_l(n, t, d_l(t, \sqrt{\delta c_l}))| : -\sqrt{\delta c_l t} \leq n - c_l t \leq \sqrt{\delta c_l t}\}$$

$$\leq e^{-2\sqrt{\rho t/\delta} g_l \delta} M_0(t_0) + \frac{O_l(\delta)}{1 - e^{-g_l\delta}}, \quad \text{where} \quad A = \sqrt{\delta},\ \delta \in (0,1),\ t \geq t_0. \tag{6.12}$$

If $A = \sqrt{\delta}$, then the restriction $A^2 + Ab/\sqrt{c_l} < 1$ and the definition of g_l allow us to choose $g_l = O_+(b) = O_+(1/\sqrt{\delta})$ in (6.11), (6.12). Then (6.11), (6.12) imply the inequalities

$$M_l(t) \leq O(\delta) \qquad \text{for } l = 0, L;\ A = \sqrt{\delta}, \tag{6.13}$$

$$M_l(t) \leq O(\sqrt{\delta}) \qquad \text{for } l = 1, \ldots, L-1;\ A = \sqrt{\delta},\ t \geq t_0. \tag{6.14}$$

Putting $\delta = t^{-1/(2+4\alpha/\lambda_0)}$ in (6.11) we obtain for every $A \in (0,1)$ the following estimate for $l = 0$ and $t \geq t_0$:

$$M_0(t) \leq \left(\frac{t_0}{t}\right)^{\frac{b\sigma}{c_0\alpha}t^{1/[2+4\alpha/\lambda_0]}} + \frac{O(t^{-1/[2+4\alpha/\lambda_0]})}{1 - e^{-b\alpha/(c_0\alpha)}} = O(t^{-1/[2+4\alpha/\lambda_0]}) = O(t^{-1/3}). \tag{6.15}$$

Putting $g_l = O_+(1/\sqrt{\delta})$ in (6.12) with $\delta = t^{-1/2}$ we obtain for $A = \sqrt{\delta} = t^{-1/4}$, $l = 1, \ldots, L-1$ and $t \geq t_0$ the estimate

$$M_l(t) = O(t^{-1/4}). \tag{6.16}$$

From the definition of $\Delta_l(n, t, d_l(t))$ we deduce

$$|f(n,t) - \tilde{f}(n - c_l t - d_l(t))| \leq \frac{1}{b_0}|\Delta_l(n, t, d_l(t)) - \Delta_l(n-1, t, d_l(t))|, \tag{6.17}$$

$l = 0, \ldots, L-1$. From Lemma 16 and estimates (6.14)–(6.17) (or their immediate analogues for $l = L$) we obtain the following statement.

Proposition 6. *Let $f(n,t)$ be the solution of the Cauchy problem* (1b), (2) *where $\varepsilon = 1$, $L > 0$, $n \in \mathbb{Z}$, $t > 0$. Then under the assumptions and notations of Theorem* 1b *we have convergence of $f(n,t)$ to the shifted travelling waves $\tilde{f}_l(n - c_l t - d_l(t))$, $l = 0, \dots, L$, in the intervals*

$$\begin{cases} \{n : |n - c_l t| \le A\sqrt{c_l t}\} & \text{if } l = 0, L,\ A \in (0,1), \\ \{n : |n - c_l t| \le \sqrt{c_l} t^{1/3}\} & \text{if } l = 1, \dots, L-1,\ A = A(t) = t^{-1/4}, \end{cases}$$

with the estimates:

$$\sup_{l=0,L} \sup_{\{n:|n-c_l t| \le A\sqrt{c_l t}\}} |f(n,t) - \tilde{f}_l(n - c_l t - d_l(t, A\sqrt{c_l}))| = O(t^{-1/3}),$$

$$\sup_{l=1,\dots,L-1} \sup_{\{n:|n-c_l t| \le c_l^{1/2} t^{1/4}\}} |f(n,t) - \tilde{f}_l(n - c_l t - d_l(t, t^{-1/4}\sqrt{c_l}))| = O(t^{-1/4}), \qquad t \ge t_0,$$

$$\sup_{l=1,\dots,L-1} \sup_{\{n:|n-c_l t| \le \sqrt{c_l \delta t}\}} |f(n,t) - \tilde{f}_l(n - c_l t - d_l(t, \sqrt{\delta c_l}))| = O(\sqrt{\delta}), \qquad t \ge t_0(\delta).$$

The last two estimates of Proposition 6 and Proposition 3b imply that

$$d_l(t, t^{-1/4}\sqrt{c_l}) = d_l(t, \sqrt{\delta c_l}) + O(1) = \frac{c_l}{2}\gamma_l \ln t + o(\ln t) \quad \text{if } t \ge t_0,\ l = 1, \dots, L-1.$$

Hence, Propositions 1, 3, 6 for $L > 0$ and [HP2] for $L = 0$ imply Theorem 2.

Acknowledgments

I am grateful to Victor Polterovich, Alexandre Shananin and Alexander Tumanov for the fruitful collaboration on the Burgers type equations and their applications. Many constructions of this paper have their origin in our joint publications [HP1]–[HP4], [HS], [HST]. I am also grateful to V. Spivak and M. Rykova for the computational experiments [RS].

References

[B] H. Bateman, *Some recent researches on the motion of fluids.* Monthly Weather Rev. **43** (1915), 163–170.

[BP] P. Bauman and D. Phillips, *Large-time behavior of solutions to a scalar conservation law in several space dimensions.* Trans. Amer. Math. Soc. **298** (1986), 401–419.

[Be] V. Belenky, *Diagram of growth of a monotonic function and a problem of its reconstruction from the diagram.* Preprint, Central Economics and Mathematics Institute, Acad. Sci. USSR, Moscow, 1990, 1–44 (in Russian).

[BHR] S. Benzoni-Gavage, P. Huot and F. Rousset, *Nonlinear stability of semidiscrete shock waves.* SIAM J. Math. Anal. **35** (2003), 639–707.

[BL] S. E. Buckley and M. C. Leverett, *Mechanism of fluid displacement in sands.* Trans. A.I.M.E. **146** (1942), 107–116.

[Bu] J. M. Burgers, *Application of a model system to illustrate some points of the statistical theory of free turbulence.* Proc. Acad. Sci. Amsterdam **43** (1940), 2–12.

[C] K. S. Cheng, *Asymptotic behavior of solutions of a conservation law without convexity conditions*. J. Differential Equations **40** (1981), 343–376.

[Co] J. D. Cole, *On a quasi-linear parabolic equation occurring in aerodynamics*. Quart. Appl. Math. **9** (1951), 225–236.

[EO] B. Engquist and S. Osher, *One-sided difference approximations for nonlinear conservation laws*. Math. Comp. **36** (1981), 321–351.

[F] V. A. Florin, *Some of the simplest nonlinear problems arising in the consolidation of wet soil*. Izv. Akad. Nauk SSSR Otdel. Tekhn. Nauk **9** (1948), 1389–1397 (in Russian).

[G] I. M. Gelfand, *Some problems in the theory of quasilinear equations*. Uspekhi Mat. Nauk **14** (1959), no. 2, 87–158 (in Russian); English transl.: Amer. Math. Soc. Transl. **29** (1963), 295–381.

[HHL] A. Harten, J. M. Hyman and P. D. Lax, *On finite-difference approximations and entropy conditions for shocks*. Comm. Pure Appl. Math. **29** (1976), 297–322.

[H] G. M. Henkin, *Cauchy problem for Burgers type equations*. In: Encyclopedia of Math. Physics, J.-P. Françoise et al. (eds.), Vol. 1, Elsevier, 2006, 446–454.

[HP1] G. M. Henkin and V. M. Polterovich [= V. M. Polterovich and G. M. Khenkin], *An evolutionnary model with interaction between development and adoption of new technologies*. Econom. Math. Methods **24** (1988), 1071–1083 (in Russian); English transl.: Matecon **24** (1988), no. 6, 3–19.

[HP2] G. M. Henkin and V. M. Polterovich, *Schumpeterian dynamics as a nonlinear wave theory*. J. Math. Econom. **20** (1991), 551–590.

[HP3] G. M. Henkin and V. M. Polterovich, *A difference-differential analogue of the Burgers equation: stability of the two-wave behavior*. J. Nonlinear Sci. **4** (1994), 497–517.

[HP4] G. M. Henkin and V. M. Polterovich, *A difference-differential analogue of the Burgers equation and some models of economic development*. Discrete Contin. Dyn. Syst. **4** (1999), 697–728.

[HS] G. M. Henkin and A. A. Shananin, *Asymptotic behavior of solutions of the Cauchy problem for Burgers type equations*. J. Math. Pures Appl. **83** (2004), 1457–1500.

[HST] G. M. Henkin, A. A. Shananin and A. E. Tumanov, *Estimates for solutions of Burgers type equations and some applications*. J. Math. Pures Appl. **84** (2005), 717–752.

[HZ] D. Hoff and K. Zumbrun, *Asymptotic behavior of multidimensional viscous shock fronts*. Indiana Univ. Math. J. **49** (2000), 427–474.

[Ho] E. Hopf, *The partial differential equation* $u_t + uu_x = \mu u_{xx}$. Comm. Pure Appl. Math. **3** (1950), 201–230.

[IO] A. M. Il'in and O. A. Oleĭnik, *Asymptotic long-time behaviour of the Cauchy problem for some quasilinear equation*. Mat. Sb. **51** (1960), 191–216 (in Russian).

[KP] S. N. Kruzhkov and N. S. Petrosyan, *Asymptotic behavior of the solutions of the Cauchy problem for the non-linear first order equations*. Russian Math. Surveys **42** (1987), 1–47.

[LSU] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*. Nauka, Moscow, 1967 (in Russian); English transl.: Amer. Math. Soc., Providence, RI, 1968.

[LL] L. D. Landau and E. M. Lifschitz, *Fluid Mechanics*. Addison-Wesley, 1953.

[La] P. D. Lax, *Hyperbolic systems of conservation laws, II*. Comm. Pure Appl. Math. **10** (1957), 537–566.

[L1] J. Leray, *Essai sur les mouvements plans d'un liquide visqueux que limite des parois*. J. Math. Pures Appl. **13** (1934), 331–418.

[L2] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*. Acta Math. **63** (1934), 193–248.

[Li] T.-P. Liu, *Admissible solutions of hyperbolic conservation laws*. Mem. Amer. Math. Soc. **30** (1981), no. 240.

[LMN] T.-P. Liu, A. Matsumura and K. Nishihara, *Behaviors of solutions for the Burgers equation with boundary corresponding to rarefaction waves*. SIAM J. Math. Anal. **29** (1998), 293–308.

[LY] T.-P. Liu and S.-H. Yu, *Propagation of stationary viscous Burgers shock under the effet of boundary*. Arch. Ration. Mech. Anal. **139** (1997), 57–92.

[O] O. A. Oleĭnik, *Uniqueness and stability of the generalized solution of the Cauchy problem for a quasi-linear equation*. Uspekhi Mat. Nauk **14** (1959), 167–170 (in Russian); English transl.: Amer. Math. Soc. Transl. **33** (1963), 285–290.

[PW] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*. Springer, 1986.

[RS] M. I. Rykova and V. A. Spivak, *A numerical study of the Polterovich–Henkin model for propagation of new technologies*. Preprint, 2004 (in Russian).

[S] J. A. Schumpeter, *Business Cycles: A Theoretical, Historical and Statistical Analysis of the Capitalist Process*. McGraw-Hill, New York, 1939.

[S1] D. Serre, *Systems of Conservation Laws. I*. Cambridge Univ. Press, 1999.

[S2] D. Serre, *L^1-stability of nonlinear waves in scalar conservation laws*. In: C. Dafermos and E. Feireisl (eds.), Handbook of Differential Equations, 2004, 473–553.

[W] H. F. Weinberger, *Long-time behavior for a regularized scalar conservation law in the absence of genuine nonlinearity*. Ann. Inst. H. Poincaré Anal. Non Linéaire **7** (1990), 407–425.

G. M. Henkin
Université Pierre et Marie Curie
case 247, 75252 Paris, France
and
EMI, Academy of Science
117418, Moscow, Russia
e-mail: `henkin@math.jussieu.fr`