Phase-field approximations for direct and inverse problems

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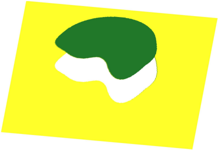
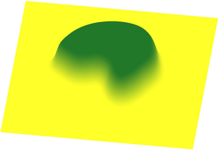
Phase-field method

A short introduction

Approximation of binary functions

set $E \subset \Omega$ $\longrightarrow$ binary function $\chi_E \in \{0, 1\}$

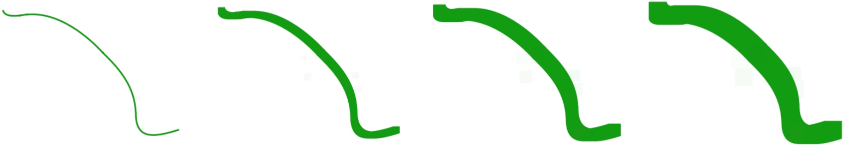




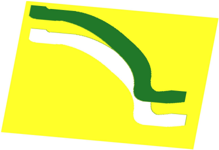
binary function $\chi_E \in \{0, 1\}$ $\longrightarrow$ phase-field function $\widetilde{v} \in H^1(\Omega, [0, 1])$

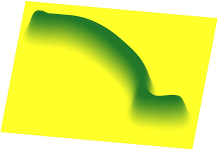
Approximation of jump sets

$u \in H^1(\Omega \setminus K)$, $K \subset \overline{\Omega}$ closed, $K = \overline{J(u)}$, $J(u)$ jump set of u



K $\longrightarrow$ $\overline{B_\delta(K)}$





binary function $\chi_{\overline{B_\delta(K)}} \in \{0, 1\}$ $\longrightarrow$ phase-field function $\tilde{v} \in H^1(\Omega, [0, 1])$

Different kinds of phase-field approximations

Remark: for any $\varepsilon > 0$ and any phase-field function $\widetilde{v} \in L^\infty(\Omega, [0,1])$, let

$$\widetilde{v} \longrightarrow v = 1 - \widetilde{v} \longrightarrow v_\varepsilon = (1 - \varepsilon^2)\psi(v) + \varepsilon^2$$

where $\psi(t) = -2t^3 + 3t^2$

Approximation of binary functions/jump sets

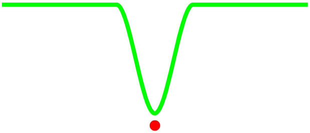
N-dimensional sets

$K = \partial E$

v_ε phase-field approximation

$(N-1)$-dimensional sets

$K = J(u)$



v_ε phase-field approximation

Different kinds of phase-field approximations

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Approximation of defects/scatterers

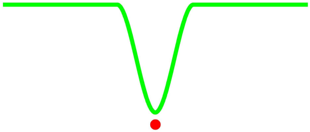
N-dimensional sets

$K = \partial\sigma$, σ cavity/obstacle

v_ε phase-field approximation

$(N-1)$-dimensional sets

K crack/screen



v_ε phase-field approximation

Different kinds of phase-field approximations

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Approximation of defects/scatterers

N-dimensional sets	$(N-1)$-dimensional sets
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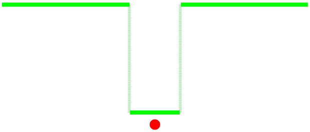
v_ε phase-field approximation



v_ε phase-field approximation

Example: approximation of perimeter functional

Perimeter functional: define $\mathcal{P} : L^1(\Omega) \to [0, +\infty]$ such that

$$\mathcal{P}(u) = \begin{cases} |Du|(\Omega) = TV(u, \Omega) & \text{if } u \in BV(\Omega),\ u \in \{0, 1\}, \\ +\infty & \text{otherwise} \end{cases}$$

Remarks:

- if $E \Subset \Omega$, $\mathcal{P}(\chi_E) = P(E)$ perimeter of E
- if $E \subset \Omega$ smooth, $\mathcal{P}(\chi_E) = \mathcal{H}^{N-1}(\partial E \cap \Omega)$

The Modica-Mortola functional: preparation

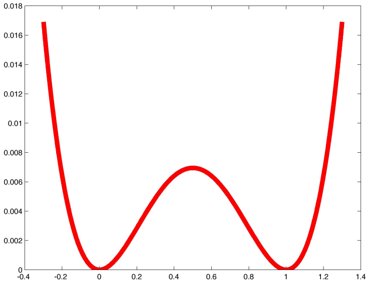
Remark: for any $\varepsilon > 0$ and any phase-field function $\widetilde{v} \in H^1(\Omega, [0, 1])$, let

$$\widetilde{v} \quad \longrightarrow \quad v = 1 - \widetilde{v} \quad \longrightarrow \quad v_\varepsilon = (1 - \varepsilon^2)\psi(v) + \varepsilon^2$$

where $\psi(t) = -2t^3 + 3t^2$

Let W be a double-well potential centered at 0 and 1.

For instance $W(t) = t^2(t-1)^2/9$



The Modica-Mortola functional

Modica-Mortola functional: for any $\varepsilon > 0$ define

$$MM_\varepsilon : L^1(\Omega) \to [0, +\infty] \quad \text{such that}$$

$$MM_\varepsilon(\widetilde{v}) = \begin{cases} \int_\Omega \left(\frac{c_2}{2\varepsilon} W(\widetilde{v}) + \frac{c_2 \varepsilon}{2} \|\nabla \widetilde{v}\|^2 \right) & \text{if } \widetilde{v} \in H^1(\Omega, [0,1]) \\ +\infty & \text{otherwise} \end{cases}$$

Modica & Mortola (1977)

As $\varepsilon \to 0^+$, MM_ε converges to the perimeter functional $\mathcal{P}$ in the sense fo Γ-convergence.

Approximation of the perimeter functional $\mathcal{P}$

Roughly speaking, given $E \subset \Omega$ and its characteristic function χ_E

the phase-field function $\widetilde{v}$ approximates χ_E
(or $v = 1 - \widetilde{v}$ and v_ε approximates $1 - \chi_E$)

$\Longrightarrow$

$MM_\varepsilon(\widetilde{v}) = MM_\varepsilon(v)$ approximates $\mathcal{P}(\chi_E)$

$$\mathcal{P}(\chi_E) \quad \rightsquigarrow \quad MM_\varepsilon(v) = \int_\Omega \left(\frac{c_2}{2\varepsilon} W(v) + \frac{c_2 \varepsilon}{2} \|\nabla v\|^2 \right)$$

Remark: for jump sets approximation replace the perimeter functional with the Mumford-Shah functional. Its corresponding phase-field approximation is the Ambrosio-Tortorelli functional.

Applications to inverse problems

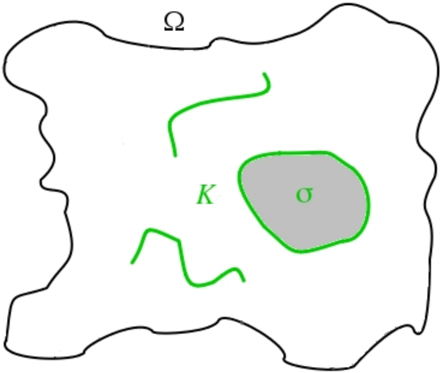
Electrical Impedance Tomography

Free-discontinuity methods applied to inverse problems

Inverse problems whose unknowns may be characterized by discontinuous functions:

1. discontinuous coefficients
   - inverse conductivity problem with discontinuous conductivity
     - phase-field method: Rondi & Santosa (2001)
     - similar approaches: Dobson & Santosa (1994); Chan & Tai (2003); Chung, Chan & Tai (2005)
     - justification: Rondi (2008)
2. shapes (domains) $\rightsquigarrow$ characteristic functions
   - inclusions determination: Rondi & Santosa (2001)
   - cavities determination: Rondi (2011) (theory) – Ring & Rondi (2012) (numerics)
3. discontinuous solutions of the forward problem
   - crack problems: Rondi (2006–2011) (theory) – Ring & Rondi (2012) (numerics)

The inverse crack or cavity problem



$\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$
bounded domain; $\partial\Omega$ Lipschitz
K defect in a (homogeneous and isotropic) conducting body Ω
In the case of cavities σ, $K = \partial\sigma$

We may also treat defects K that consist of (interior or surface-breaking) cracks, cavities, material losses at the boundary, and other kinds of defects, even simultaneously

The forward problem

K is a perfectly insulating defect in Ω

$f \in L^s(\partial\Omega)$, $s > N-1$, is the prescribed current density on $\partial\Omega$ such that

$$\int_{\partial\Omega} f = 0$$

$u = u(f, K)$ is the electrostatic potential in Ω, (unique) solution to

(1)
$$\begin{cases} \Delta u = 0 & \text{in } \Omega \backslash K \\ \nabla u \cdot \nu = 0 & \text{on (either sides of) } K \\ \nabla u \cdot \nu = f & \text{on } \partial\Omega \\ \int_{\partial\Omega} u = 0 & \end{cases}$$

with the following normalization in the case of cavities σ

(2)
$$u = 0 \quad \text{in } \sigma$$

The inverse crack or cavity problem

K unknown perfectly insulating defect in Ω

f prescribed current density on $\partial\Omega$

u electrostatic potential in Ω, solution to (1) with the normalization (2)

Measurement: $g = u|_{\partial\Omega}$ is the measured voltage on $\partial\Omega$

Remark: $g \in L^2(\partial\Omega)$ such that

$$\int_{\partial\Omega} g = \int_{\partial\Omega} u = 0$$

Aim of the inverse problem

From one or more voltage and current measurements (prescribing the current f on $\partial\Omega$ and measuring the corresponding voltage $g = u|_{\partial\Omega}$ on $\partial\Omega$), reconstruct the unknown defect K

The least-squares problem for phase-field functions

Given a phase-field function v, the corresponding potential $u = u(f_\varepsilon, v_\varepsilon)$ solves

$$\begin{cases} \mathrm{div}(v_\varepsilon \nabla u) = 0 & \text{in } \Omega \\ \nabla u \cdot \nu = f_\varepsilon & \text{on } \partial\Omega \\ \int_{\partial\Omega} u = 0 & \end{cases}$$

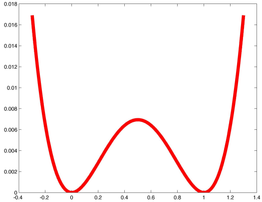
We look for a phase-field function $v \in H^1(\Omega, [0,1])$ solving

$$\min_{v} \int_{\partial\Omega} |u(f_\varepsilon, v_\varepsilon) - g_\varepsilon|^2 + \text{regularization}$$

Proposed phase-field method for cavities

Let W be a double-well potential centered at 0 and 1.

For instance $W(t) = t^2(t-1)^2/9$



Find $v \in H^1(\Omega, [0,1])$ solving

$$\min_{v} \mathcal{G}_\varepsilon(v)$$

$$\mathcal{G}_\varepsilon(v) = \frac{1}{\varepsilon^{\tilde{q}}} \int_{\partial\Omega} |u(f_\varepsilon, v_\varepsilon) - g_\varepsilon|^2 + \int_\Omega \left(v_\varepsilon |\nabla u(f_\varepsilon, v_\varepsilon)|^2 + \frac{c_2}{2\varepsilon} W(v) + \frac{c_2 \varepsilon}{2} |\nabla v|^2 \right)$$

Proposed phase-field method for cracks

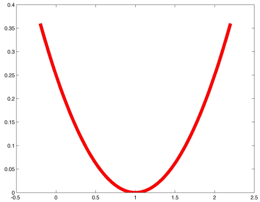
Find $v \in H^1(\Omega, [0,1])$ solving

$$\min_{v} \mathcal{F}_\varepsilon(v)$$

$$\mathcal{F}_\varepsilon(v) = \frac{1}{\varepsilon^{\tilde{q}}} \int_{\partial\Omega} |u(f_\varepsilon, v_\varepsilon) - g_\varepsilon|^2 + \int_\Omega \left(v_\varepsilon |\nabla u(f_\varepsilon, v_\varepsilon)|^2 + \frac{c_2}{2\varepsilon} V(v) + \frac{c_2 \varepsilon}{2} |\nabla v|^2 \right)$$

where V is a single-well potential centered at 1.

For instance $V(t) = (t-1)^2/4$



Results

Convergence results

Crack case:

- Phase-field model: there is evidence that there may be no convergence
- there is convergence for a strictly related functional (Rondi (2008))

Cavity case:

- Phase-field model: there is convergence (Rondi (2011))

Numerical results

Crack and cavity cases:

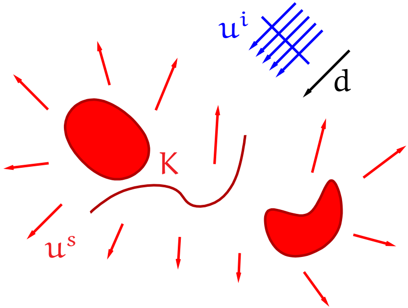
Ring & Rondi (2012)

Direct scattering problem

Sound-hard scatterers

Acoustic scattering

K sound-hard scatterer including obstacles and screens



k wavenumber

d direction of propagation

$u^i(x) = e^{ikx \cdot d}$, $x \in \mathbb{R}^N$, incident field

u^s scattered field

u total field

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^N \setminus K \\ u = u^i + u^s & \text{in } \mathbb{R}^N \setminus K \\ \nabla u \cdot \nu = 0 & \text{on } \partial K \\ \lim_{r \to \infty} r^{(N-1)/2} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 & r = \|x\| \end{cases}$$

Stability with respect to scatterers

K_n, $n \in \mathbb{N}$, scatterer contained in $\overline{B_R}$, $R > 0$
u_n solution to the scattering problem with scatterer K_n

K scatterer contained in $\overline{B_R}$
u solution to the scattering problem with scatterer K

Question

$$K_n \to K \qquad \Longrightarrow \qquad u_n \to u\,?$$

Convergences used:

- $K_n \to K$ with respect to the Hausdorff distance
- $u_n \to u$ strongly in $L^2(B_r)$ for any $r > 0$

Remark: u_n extended to zero in K_n, u extended to zero in K

Key ingredient: Mosco convergence

$A_n = H^1(B_{R+1} \backslash K_n)$, $A = H^1(B_{R+1} \backslash K)$ subspaces of $L^2(B_{R+1}, \mathbb{R}^{N+1})$

Remark: $v \in H^1(B_{R+1} \backslash K)$ is identified with $(v, \nabla v) \in L^2(B_{R+1}, \mathbb{R}^{N+1})$, where v and ∇v are extended to zero in K.

Definition: Mosco convergence

$A_n \to A$ in the sense of Mosco if

- for any $(v, V) \in L^2(B_{R+1}, \mathbb{R}^{N+1})$ if $\exists$ $(u_k, \nabla u_k) \in A_{n_k}$ such that $(u_k, \nabla u_k) \rightharpoonup (v, V)$ weakly in $L^2(B_{R+1}, \mathbb{R}^{N+1})$, then $(v, V) \in A$
- for any $(u, \nabla u) \in A$ $\exists$ $(u_n, \nabla u_n) \in A_n$ such that $(u_n, \nabla u_n) \to (u, \nabla u)$ strongly in $L^2(B_{R+1}, \mathbb{R}^{N+1})$

Remark: $(v, V) \in A$ means v and V are zero in K and $V = \nabla v$ in $B_{R+1} \backslash K$

Conditions for Mosco convergence

Assume $K_n \to K$ in the Hausdorff distance

$N = 2$

- Bucur & Varchon (2000)
  Assume $\#\{$connected components of $K_n\} \leqslant M$. Then
  $$A_n \to A \text{ in the sense of Mosco} \quad \Longleftrightarrow \quad |K_n| \to |K|$$
- Chambolle & Doveri (1997)
  Assume $\#\{$connected components of $\partial K_n\} \leqslant M$ and $\mathcal{H}^1(\partial K_n) \leqslant C$.
  Then $A_n \to A$ in the sense of Mosco

$N \geqslant 3$

- Giacomini (2004) – Menegatti & Rondi (2013)
  Uniform Lipschitz type regularity assumptions on K_n.
  Then $A_n \to A$ in the sense of Mosco

Stability result

Stability theorem (Menegatti & Rondi (2013))

- **Convergence in the Hausdorff distance**
  $K_n \to K$ in the Hausdorff distance
- **Convergence in the sense of Mosco**
  $A_n = H^1(B_{R+1} \backslash K_n) \to A = H^1(B_{R+1} \backslash K)$ in the sense of Mosco
- **Uniform Sobolev inequality**
  $\exists\, p > 2$ and $C > 0$ such that

$$\|v\|_{L^p(B_{R+1} \backslash K_n)} \leqslant C \|v\|_{H^1(B_{R+1} \backslash K_n)} \quad \text{for any } v \in H^1(B_{R+1} \backslash K_n)$$

Then $u_n \to u$ strongly in $L^2(B_r)$ for any $r > 0$

Consequence: uniform decay estimates for scattered fields as $\|x\| \to \infty$ for a large class of admissible scatterers

Approximation of screens by thin obstacles: assumptions

K scatterer formed by Lipschitz screens.

$\exists\ \widetilde{d} : \mathbb{R}^N \to [0, +\infty)$ Lipschitz such that

- $\exists\ 0 < a \leqslant 1 \leqslant b$ such that

$$a\,\mathrm{dist}(x, K) \leqslant \widetilde{d}(x) \leqslant b\,\mathrm{dist}(x, K) \quad \text{for any } x \in \mathbb{R}^N.$$

- Let $K_\varepsilon = \{x \in \mathbb{R}^N :\ \widetilde{d}(x) \leqslant \varepsilon\}$. For any $0 < \varepsilon \leqslant \widetilde{\varepsilon}$

$$K_\varepsilon \subset B_{R+1/2} \ \text{ and } \ \mathbb{R}^N \backslash K_\varepsilon \text{ is connected.}$$

- $\exists\ p > 2$ and$C > 0$ such that for any $0 < \varepsilon \leqslant \widetilde{\varepsilon}$

$$\|v\|_{L^p(B_{R+1}\backslash K_\varepsilon)} \leqslant C\|v\|_{H^1(B_{R+1}\backslash K_\varepsilon)} \quad \text{for any } v \in H^1(B_{R+1}\backslash K_\varepsilon).$$

Approximation of screens by thin obstacles: result

K scatterer formed by Lipschitz screens satisfying the previous assumptions.

Let $0 < \varepsilon_n \leqslant \widetilde{\varepsilon}$ such that $\lim_n \varepsilon_n = 0$. Let $K_n = K_{\varepsilon_n}$. Then

- Convergence in the Hausdorff distance
  $K_n \to K$ in the Hausdorff distance
- Convergence in the sense of Mosco
  $\mathcal{A}_n = H^1(B_{R+1} \backslash K_n) \to \mathcal{A} = H^1(B_{R+1} \backslash K)$ in the sense of Mosco
- Uniform Sobolev inequality
  $\exists\, p > 2$ and $C > 0$ such that

$$\|v\|_{L^p(B_{R+1} \backslash K_n)} \leqslant C \|v\|_{H^1(B_{R+1} \backslash K_n)} \quad \text{for any } v \in H^1(B_{R+1} \backslash K_n)$$

Therefore $u_n \to u$ strongly in $L^2(B_r)$ for any $r > 0$

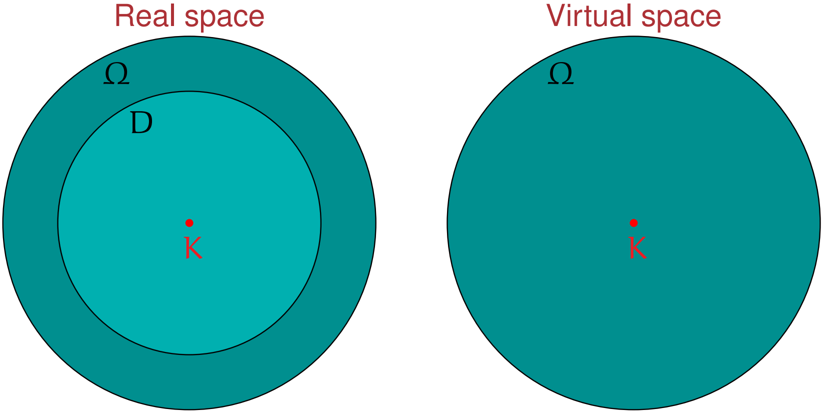
Full and partial approximate cloaking

Phase-field methods and transformation optics

Full cloack: real and virtual space

Greenleaf, Lassas & Uhlmann (2003)

$K = \{0\}$, D cloaked region, $\Omega \backslash D$ cloaking region



$F : \mathbb{R}^N \backslash K \to \mathbb{R}^N \backslash D$ bijective such that

$$F|_{\mathbb{R}^N \backslash \Omega} = \mathrm{Id} \quad \text{and} \quad F|_{\Omega \backslash K} = F^{(1)} : \Omega \backslash K \to \Omega \backslash D$$

Transformation optics

Virtual space

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^N \backslash K \qquad (\sigma \equiv \mathrm{Id},\ q \equiv 1)$$

Real space

$$\mathrm{div}(\widetilde{\sigma}\nabla\widetilde{u}) + k^2\widetilde{q}\widetilde{u} = 0 \quad \text{in } \mathbb{R}^N \backslash D$$

where

$$(\widetilde{\sigma}, \widetilde{q}) = F_*(\sigma, q) \quad \text{and} \quad \widetilde{u} = u \circ F^{-1}$$

that is

$$\widetilde{\sigma}(\widetilde{x}) = F_*\sigma(\widetilde{x}) := \left.\frac{DF(x)\cdot\sigma(x)\cdot DF(x)^T}{|\det(DF(x))|}\right|_{x=F^{-1}(\widetilde{x})}$$

$$\widetilde{q}(\widetilde{x}) = F_*q(\widetilde{x}) := \left.\frac{q(x)}{|\det(DF(x))|}\right|_{x=F^{-1}(\widetilde{x})}$$

Cloaking by transformation optics

$u^i(x) = e^{ikx\cdot d}$, $x \in \mathbb{R}^N$, incident field

u solution to the scattering problem

Cloaking

$\tilde{u} = u = u^i$ in $\mathbb{R}^N \setminus \Omega$ no matter which medium is contained in D

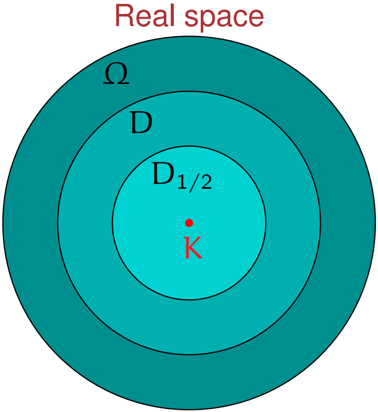
that is

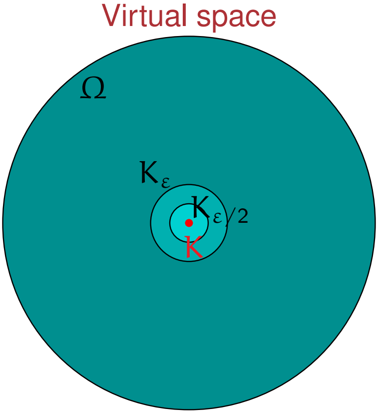
the cloaking medium $F_*^{(1)}(Id, 1)$ in $\Omega \setminus D$ fully cloaks the medium in the cloaked region D

Drawback: $F^{(1)}$ is not bi-Lipschitz, therefore the cloaking medium $F_*^{(1)}(Id, 1)$ is singular!

Approximate full cloak: real and virtual space

$K = \{0\}$, $D_{1/2}$ cloaked region, $\Omega \backslash D_{1/2}$ cloaking region
$D \backslash D_{1/2}$ lossy layer, $\varepsilon > 0$



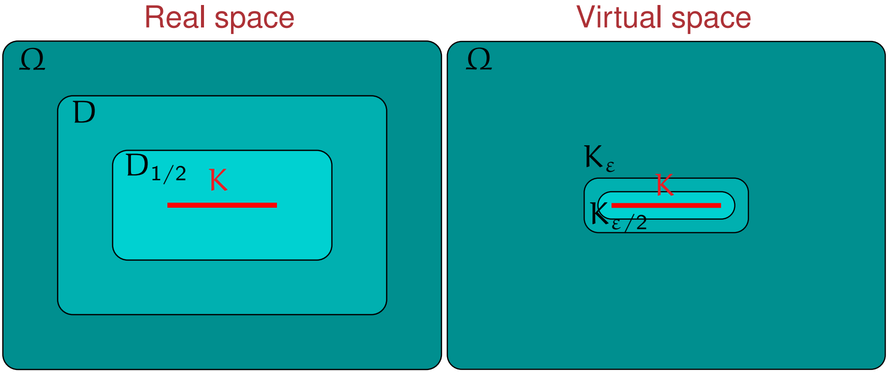


$F_\varepsilon : \mathbb{R}^N \backslash K \to \mathbb{R}^N \backslash K$ bijective such that $F_\varepsilon|_{\mathbb{R}^N \backslash \Omega} = \mathrm{Id}$ and

$$F_\varepsilon|_{\Omega \backslash K_\varepsilon} = F_\varepsilon^{(1)} : \Omega \backslash K_\varepsilon \to \Omega \backslash D \quad \text{and} \quad F_\varepsilon|_{K_\varepsilon \backslash K} = F_\varepsilon^{(2)} : K_\varepsilon \backslash K \to D \backslash K$$

Approximate partial cloak: real and virtual space

$K = [-1/2, 1/2]^{N-1} \times \{0\}$, $D_{1/2}$ cloaked region, $\Omega \backslash D_{1/2}$ cloaking region
$D \backslash D_{1/2}$ lossy layer, $\varepsilon > 0$

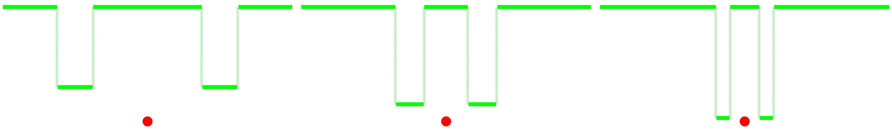


$F_\varepsilon : \mathbb{R}^N \backslash K \to \mathbb{R}^N \backslash K$ bijective such that $F_\varepsilon|_{\mathbb{R}^N \backslash \Omega} = \mathrm{Id}$ and

$$F_\varepsilon|_{\Omega \backslash K_\varepsilon} = F_\varepsilon^{(1)} : \Omega \backslash K_\varepsilon \to \Omega \backslash D \quad \text{and} \quad F_\varepsilon|_{K_\varepsilon \backslash K} = F_\varepsilon^{(2)} : K_\varepsilon \backslash K \to D \backslash K$$

Approximation by phase-field functions

$\mathrm{K} = \{0\}$ or K scatterer formed by Lipschitz screens satisfying the previous assumptions



v_ε phase-field approximations as $\varepsilon \to 0^+$

Aim

Approximate in the virtual space the sound-hard scatterer K by a thin lossy layer using phase-field functions

Approximation of a screen by a thin lossy layer

σ^ε and q^ε coefficients of the reduced wave equation in the virtual space

Assumptions on σ^ε and q^ε

- in $\mathbb{R}^N \setminus \Omega$ and $\Omega \setminus K_\varepsilon$: $\quad \sigma^\varepsilon \equiv \mathrm{Id} \quad$ and $\quad q^\varepsilon \equiv 1$
- in the thin lossy layer $K_\varepsilon \setminus K_{\varepsilon/2}$:

$$\frac{1}{\varepsilon^2} \sigma^\varepsilon \xi \cdot \xi \leqslant \Lambda \|\xi\|^2 \quad \text{for any } \xi \in \mathbb{R}^N$$

$$\lim_{\varepsilon \to 0^+} \int_{K_\varepsilon \setminus K_{\varepsilon/2}} |q^\varepsilon| = 0$$

$$0 < \Re q^\varepsilon \leqslant E_1(\omega_1(\varepsilon))^{-1} \quad \text{and} \quad 0 < E_2(\omega_1(\varepsilon))^{-1} \leqslant \Im q^\varepsilon$$

ω_1 positive continuous nondecreasing function s.t. $\lim_{s \to 0^+} \omega_1(s) = 0$

- in the cloaked region $K_{\varepsilon/2}$: $\quad$ NO assumptions on σ^ε and q^ε

Approximation result

$u^i(x) = e^{ikx\cdot d}$, $x \in \mathbb{R}^N$, incident field

u_ε solution to the scattering problem

$$\begin{cases} \mathrm{div}(\sigma^\varepsilon \nabla u_\varepsilon) + k^2 q^\varepsilon u_\varepsilon = 0 & \text{in } \mathbb{R}^N \\ u_\varepsilon = u^i + u_\varepsilon^s & \text{in } \mathbb{R}^N \\ \lim\limits_{r\to\infty} r^{(N-1)/2}\left(\dfrac{\partial u_\varepsilon^s}{\partial r} - iku_\varepsilon^s\right) = 0 & r = \|x\| \end{cases}$$

Then, independently of σ^ε and q^ε in the cloaked region $K_{\varepsilon/2}$, as $\varepsilon \to 0^+$ $(1-\chi_{K_{\varepsilon/2}})u_\varepsilon$ converges to u strongly in $L^2(B_r)$ for any $r > 0$, where u solves

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^N \backslash K \\ u = u^i + u^s & \text{in } \mathbb{R}^N \backslash K \\ \nabla u \cdot \nu = 0 & \text{on } \partial K \\ \lim\limits_{r\to\infty} r^{(N-1)/2}\left(\dfrac{\partial u^s}{\partial r} - iku^s\right) = 0 & r = \|x\| \end{cases}$$

Main theorem

Sommerfeld radiation condition $\Longrightarrow$

$$u^s(x) = \frac{e^{ik\|x\|}}{\|x\|^{\frac{N-1}{2}}}\left(u^\infty(x/\|x\|) + \mathcal{O}(1/\|x\|)\right) \quad \text{as } \|x\| \to +\infty$$

u^∞ far-field pattern

Theorem (Li, Liu, Rondi & Uhlmann (2013))

There exists a positive function ω satisfying $\lim_{s\to 0^+}\omega(s) = 0$, such that for any $0 < \varepsilon \leqslant \widetilde{\varepsilon}$

$$\|u_\varepsilon - u\|_{L^2(B_{R+2}\setminus\overline{B_{R+1}})} \leqslant \omega(\varepsilon)$$

and

$$\|u_\varepsilon^\infty - u^\infty\|_{L^\infty(\mathbb{S}^{N-1})} \leqslant C\omega(\varepsilon)$$

independently of σ^ε and q^ε in the cloaked region $K_{\varepsilon/2}$ and of the direction of propagation d

Approximate cloaking by transformation optics

Let

$$(\widetilde{\sigma}^{\varepsilon}, \widetilde{q}^{\varepsilon}) = (F_{\varepsilon})_{*}(\sigma^{\varepsilon}, q^{\varepsilon}) \quad \text{and} \quad \widetilde{u}_{\varepsilon} = u_{\varepsilon} \circ F_{\varepsilon}^{-1}$$

Then $\widetilde{u}_{\varepsilon}$ solves in the real space

$$\begin{cases} \mathrm{div}(\widetilde{\sigma}^{\varepsilon} \nabla \widetilde{u}_{\varepsilon}) + k^2 \widetilde{q}^{\varepsilon} \widetilde{u}_{\varepsilon} = 0 & \text{in } \mathbb{R}^N \\ \widetilde{u}_{\varepsilon} = u^{i} + \widetilde{u}^{s}_{\varepsilon} & \text{in } \mathbb{R}^N \\ \lim\limits_{r \to \infty} r^{(N-1)/2} \left(\dfrac{\partial \widetilde{u}^{s}_{\varepsilon}}{\partial r} - ik \widetilde{u}^{s}_{\varepsilon} \right) = 0 & r = \|x\| \end{cases}$$

and

$$\widetilde{u}^{\infty}_{\varepsilon} = u^{\infty}_{\varepsilon}$$

Approximate full cloak

$K = \{0\}$, $D_{1/2}$ cloaked region, $\Omega \backslash D_{1/2}$ cloaking region

Then for any direction of propagation d we have $u^\infty \equiv 0$, therefore

Theorem (Li, Liu, Rondi & Uhlmann (2013))

For any direction of propagation d

$$\|\tilde{u}_\varepsilon^\infty\|_{L^\infty(\mathbb{S}^{N-1})} \leqslant C\omega(\varepsilon)$$

independently of $\tilde{\sigma}^\varepsilon$ and $\tilde{q}^\varepsilon$ in the cloaked region $D_{1/2}$

Remark: in the cloaked region we may also have the presence of scatterers and, under certain conditions, of sources

Approximate partial cloak

$K = [-1/2, 1/2]^{N-1} \times \{0\}$, $D_{1/2}$ cloaked region, $\Omega \backslash D_{1/2}$ cloaking region

Then for any direction of propagation d such that $|d \cdot e_N| \leqslant \tau$ we have

$$\|u^\infty\|_{L^\infty(\mathbb{S}^{N-1})} \leqslant C\tau,$$

therefore

Theorem (Li, Liu, Rondi & Uhlmann (2013))

For any direction of propagation d such that $|d \cdot e_N| \leqslant \tau$

$$\|\widetilde{u}_\varepsilon^\infty\|_{L^\infty(\mathbb{S}^{N-1})} \leqslant C(\omega(\varepsilon) + \tau)$$

independently of $\widetilde{\sigma}^\varepsilon$ and $\widetilde{q}^\varepsilon$ in the cloaked region $D_{1/2}$

Remark: in the cloaked region we may also have the presence of scatterers and, under certain conditions, of sources