Stability for nonlinear inverse problems with a finite number of measurements

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Outline

Motivations

Calderon's problem with a finite number of measurements:
global uniqueness and Lipschitz stability

Lipschitz stability: linear subspaces

Lipschitz stability: manifolds

X, Y Banach spaces, $F : X \to Y$ possibly nonlinear

$$\text{Given } y = F(x) \in Y, \text{ determine } x \in X.$$

X, Y Banach spaces, $F : X \to Y$ possibly nonlinear

Given $y = F(x) \in Y$, determine $x \in X$.

Stability estimate

$$\|x_1 - x_2\|_X \leqslant g(\|F(x_1) - F(x_2)\|_Y), \quad \text{where } g(t) \to 0, \text{ as } t \mapsto 0^+.$$

X, Y Banach spaces, $F : X \to Y$ possibly nonlinear

Given $y = F(x) \in Y$, determine $x \in X$.

Logarithmic stability estimate

$$\|x_1 - x_2\|_X \leqslant C \left|\log(\|F(x_1) - F(x_2)\|_Y^{-1})\right|^{-1}, \qquad C > 0$$

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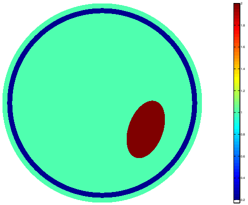
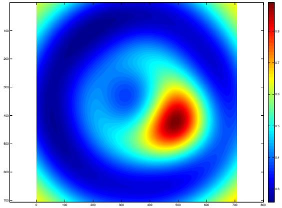
- target



- reconstruction



based on: A. Greenleaf, M. Lassas, M. Santacesaria, S. Siltanen, and G. Uhlmann. Analysis & PDE 11, no. 8 (2018).

Lipschitz stability estimate

$$\|x_1 - x_2\|_X \leqslant C\|F(x_1) - F(x_2)\|_Y, \qquad C > 0$$

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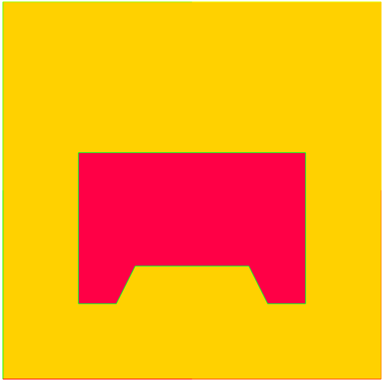

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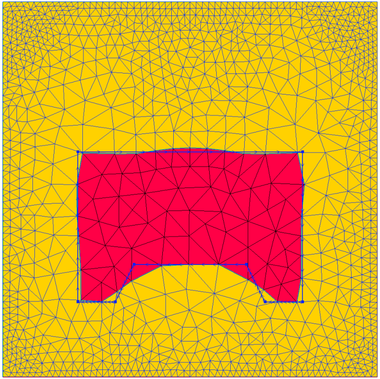

E. Beretta, S. Micheletti, S. Perotto, and M. Santacesaria. Journal of Computational Physics 353 (2018).

Other motivations

- Lipschitz stability can be used to prove a nonlinear RIP (restricted isometry property) in compressed sensing [Candès-Tao (2006), Blumensath(2013)].

- Lipschitz continuous mapping can be well approximated by neural networks [Yarotsky (2017)].

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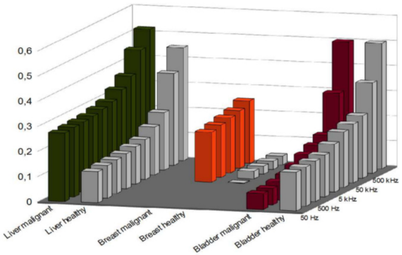
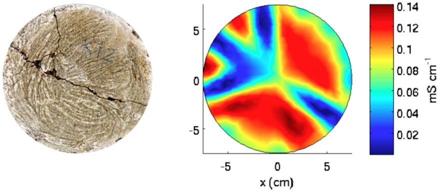
Conductivity imaging: motivations

- medical imaging



Low frequency contrast of σ

- nondestructive testing



[Karhunen, Seppänen, Lehikoinen, Monteiro & Kaipio 2010]
[Karhunen, Seppänen, Lehikoinen, Monteiro, Kaipio, Blunt, Hyvönen]

credits: Widlak, T., Scherzer, O. (2012). Inverse Problems, 28(8), 084008.

Pros: high contrast, cheap, safe. **Cons:** low resolution.

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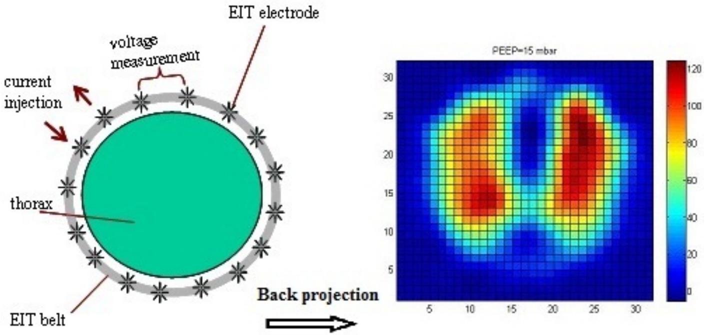
Electrical Impedance Tomography (EIT)



Example: monitoring lung ventilation distribution

credits: Zhao et al. Crit Care. 2010

Calderón's problem for EIT

- $D \subset \mathbb{R}^d$, $d \geqslant 2$: bounded Lipschitz domain
- $\sigma \in L^\infty(D)$, $\sigma(x) \geqslant \sigma_0 > 0$: unknown conductivity
- Conductivity equation:

$$\begin{cases} -\mathrm{div}(\sigma \nabla u) = 0 & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases} \tag{1}$$

- Dirichlet-to-Neumann (DN) map $\Lambda_\sigma : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$:

$$f \longmapsto \sigma \left.\frac{\partial u}{\partial \nu}\right|_{\partial D}$$

Calderón's problem

Given Λ_σ, determine σ in D.

Some known results

Basic questions:

- Uniqueness: injectivity of $\sigma \mapsto \Lambda_\sigma$
- stability estimates: continuity of $\Lambda_\sigma \mapsto \sigma$
- reconstruction algorithm

Theoretical contributions by: Calderón, Sylvester–Uhlmann, Nachman, Novikov, Alessandrini, Astala–Päivärinta, Haberman, Caro–Rogers and many others.

Usual reduction to the Gel'fand-Calderón inverse problem for the Schrödinger equation

$$(-\Delta + q)u = 0 \quad \text{in } D, \qquad \Lambda_q(u|_{\partial D}) = \left.\frac{\partial u}{\partial \nu}\right|_{\partial D},$$

which will be considered for the next few slides.

A finite number of measurements

$$\begin{cases} (-\Delta + q)u = 0 & \text{in } D, \\ u = f & \text{on } \partial D, \end{cases} \qquad \Lambda_q(f) = \left.\frac{\partial u}{\partial \nu}\right|_{\partial D}.$$

- Most results need an infinite number of measurement.

“Realistic” Calderón’s problem

$$\{(f_l, \Lambda_q(f_l))\}_{l=1,\dots,N} \qquad \rightsquigarrow \qquad q$$

A priori assumptions: $q \in \mathcal{W}_R$ if

- $q \in \mathcal{W}$: known finite dimensional subspace of $L^\infty(D)$;
- 0 is not a Dirichlet eigenvalue for $-\Delta + q$ in D;
- $\|q\|_{L^\infty(D)} \leqslant R$ for some $R > 0$.

Nonlinear prolem - global uniqueness

Theorem (G.S. Alberti, M.S. (2018)[1])

Take $d \geqslant 3$ and let $D \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $\mathcal{W} \subseteq L^\infty(D)$ be a finite dimensional subspace. There exists $N \in \mathbb{N}$ such that for any $R > 0$ and $q_1 \in \mathcal{W}_R$, the following is true.

There exist $\{f_l\}_{l=1}^N \subseteq H^{1/2}(\partial D)$ such that for any $q_2 \in \mathcal{W}_R$, if

$$\Lambda_{q_1} f_l = \Lambda_{q_2} f_l, \qquad l = 1, \ldots, N,$$

then

$$q_1 = q_2.$$

Similar result for Calderón's problem as well.

[1] Calderón's inverse problem with a finite number of measurements, Forum of Mathematics, Sigma (2019)

Ideas of the proof

- Alessandrini's identity to go from the boundary to the interior.

$$\langle g, (\Lambda_q - \Lambda_0) f\rangle_{H^{\frac{1}{2}}(\partial D)\times H^{-\frac{1}{2}}(\partial D)} = \int_D q\, u_g^0 u_f^q \, dx$$

- CGO solutions (Faddeev, Sylvester-Uhlmann): the complex parameters belong to a countable subset of $\mathbb{C}^d$. For $k \in \mathbb{Z}^d$, take $u^0(x) = e^{\zeta_2^k \cdot x}$ and CGO solution $u^q(x) = e^{\zeta_1^k \cdot x}(1 + r^k(x))$, with $\zeta_1^k, \zeta_2^k \in \mathbb{C}^d$ such that

$$\zeta_j^k \cdot \zeta_j^k = 0, \qquad \zeta_1^k + \zeta_2^k = -2\pi i k, \qquad \|r^k\|_{L^2(\mathbb{T}^d)} \leqslant c/t_k$$

- Order the frequencies: $\rho \colon l \in \mathbb{N} \mapsto k_l \in \mathbb{Z}^d$ (bijection)
- Define the nonlinear measurement operator $U \colon L^\infty([0,1]^d) \to \ell^\infty$ by

$$(U(q))_l = \int_D q(x) e^{-2\pi i k_l \cdot x}(1 + r^{k_l}(x))\, dx$$

- $U = F + B$, where, F Fourier transform, B is a contraction (t_k large)

Sketch of the proof

- Define the nonlinear operator $U\colon L^\infty(D) \to \ell^\infty$ by

$$(U(q))_l = \int_D q(x) e^{-2\pi i k_l \cdot x}(1 + r^{k_l}(x))\, dx, \qquad U = F + B$$

- Assume that $\Lambda_{q_1} f_l = \Lambda_{q_2} f_l$ for $l = 1, \ldots, N$
- Then $(P_N U)(q_1) = (P_N U)(q_2)$
- Using that B is a contraction we obtain $q_1 = q_2$, since

$$\begin{aligned} \|q_1 - q_2\|_{L^2} &= \|F(q_1 - q_2)\|_{\ell^2} \\ &\leqslant \|P_N^\perp F(q_1 - q_2)\|_{\ell^2} + \|P_N(B(q_2) - B(q_1))\|_{\ell^2} \\ &\leqslant \|P_N^\perp F P_{\mathcal{W}}(q_1 - q_2)\|_{\ell^2} + \frac{1}{2}\|q_1 - q_2\|_{L^2}, \end{aligned}$$

provided that N is chosen so that

$$\|P_N^\perp F P_{\mathcal{W}}\|_{L^2([0,1]^d) \to \ell^2} \leqslant \frac{1}{4}.$$

On the number of measurements N

- The number of measurements N depends only on $\mathcal{W}$ through

$$\|(I - P_N)FP_{\mathcal{W}}\|_{\mathcal{H}\to\ell^2} \leqslant 1/4.$$

- Relation with sampling theory: how many Fourier measurements does one need to reconstruct a function in $\mathcal{W}$?
- It allows for an explicit calculation of N:
  - bandlimited potentials
  $$N = \dim \mathcal{W}$$
  - piecewise constant potentials
  $$N = O((\dim \mathcal{W})^4)$$
  (up to log factors, and possibly not optimal)
  - low-scale wavelets
  $$N = O(\dim \mathcal{W})$$
  (up to log factors, proven only in 1D, but easy generalization)
- The ordering of $\mathbb{Z}^d$ is crucial

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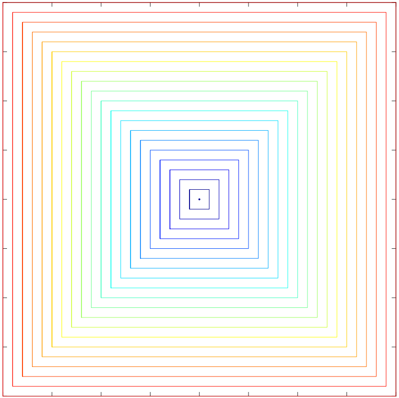
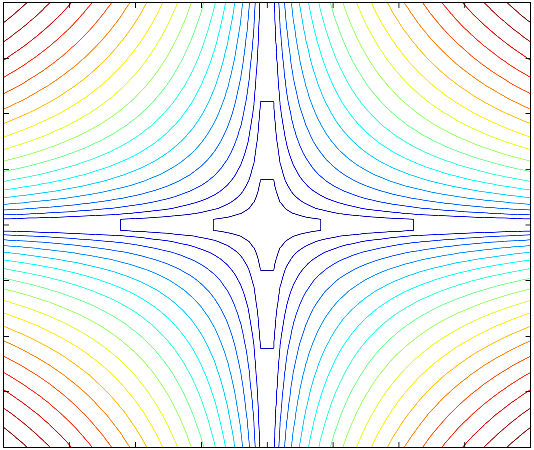
Possible orderings of $\mathbb{Z}^d$



(a) Linear ordering



(b) Hyperbolic ordering (Jones, Adcock, Hansen, 2017)

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Lipschitz stability

Theorem (G.S. Alberti, M.S. (2018)[2])

Under the same assumptions, there exist $\{f_l\}_{l=1}^N \subseteq H^{1/2}(\partial D)$ such that for every $q_2 \in \mathcal{W}_R$, we have

$$\|q_2 - q_1\|_{L^2(D)} \leqslant e^{CN^{\frac{1}{2}+\alpha}} \left\| \left(\Lambda_{q_2} f_l - \Lambda_{q_1} f_l\right)_{l=1}^N \right\|_{H^{-1/2}(\partial D)^N}$$

for some $C > 0$ depending only on D, R and α.

- Many Lipschitz stability estimates with the full DN map (Alessandrini, Beretta, Francini, Gaburro, de Hoop, Scherzer, Sincich, Vessella...).
- The exponential $e^{CN^{\frac{1}{2}+\alpha}}$ is consistent with the severe ill-posedness of this IP.
- Nonlinear reconstruction algorithm based on Banach fixed point theorem.

[2] Calderón's inverse problem with a finite number of measurements, Forum of Mathematics, Sigma (2019)

Local stability with a priori known boundary measurements

Theorem (G.S. Alberti, M.S. (2019)[3])

Let $d \in \{3, 4\}$, let N be as in the previous Theorem and take $q_0 \in \mathcal{W}_R$. There exist $\delta, C > 0$ and $L \in \mathbb{N}$ depending only on Ω, C_ρ, R, $\mathcal{W}$ and $\|(-\Delta + q_0)^{-1}\|_{H^{-1}(\Omega)\to H^1_0(\Omega)}$ such that for every $q_1, q_2 \in \mathcal{W}_R$, if

$$\|q_0 - q_j\|_{L^2(\Omega)} \leqslant \delta \qquad j = 1, 2, \tag{2}$$

then

$$\|q_2 - q_1\|_{L^2(\Omega)} \leqslant C \left\| \left(f^L_{n,1} - f^L_{n,2}\right)_{n=1}^N \right\|_{H^{1/2}(\partial\Omega)^N},$$

where

$$f^L_{n,j} = \sum_{l=1}^{L} \left(\left(S^{q_0}_{\zeta_n} (\Lambda_{q_j} - \Lambda_{q_0}) \right)^l (f^n_{q_0}) \right), \qquad j = 1, 2, \tag{3}$$

and $S^{q_0}_{\zeta_n}$ is the generalized single layer operator corresponding to the Faddeev-Green function and to the potential q_0 and $f^n_{q_0}$ are Faddeev-CGO functions associated to q_0.

[3] Calderón's inverse problem with a finite number of measurements II: independent data. Applicable Analysis (2020)

Other results

- (Rüland-Sincich 2019) fractional Calderón problem,
- (Harrach 2019) complete electrode model for EIT.

(Partially) Open questions

- Two-dimensional case.
- Extensions to other infinite dimensional IP, e.g. inverse scattering, elasticity.
- General Lipschitz stability result for a class of ill-posed inverse problems.

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Lipschitz stability: linear subspaces

Lipschitz stability: manifolds

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Lipschitz stability: basic result

- X, Y Banach spaces, $A \subseteq X$ open subset
- **(priors)** $W \subseteq X$ finite-dimensional, $K \subseteq W \cap A$ compact and convex.

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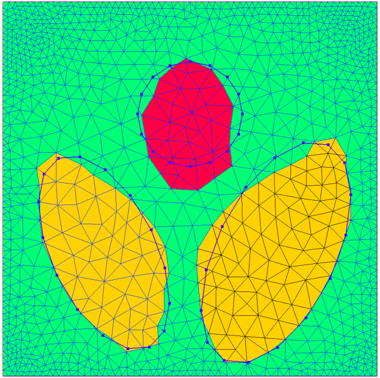
Lipschitz stability: basic result



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Lipschitz stability: basic result

- X, Y Banach spaces, $A \subseteq X$ open subset
- **(priors)** $W \subseteq X$ finite-dimensional, $K \subseteq W \cap A$ compact and convex.

Theorem

Let $F \in C^1(A, Y)$ be such that $F|_{W \cap A}$ and $F'(x)|_W$, $x \in W \cap A$, are injective.

Then there exists a constant $C > 0$ such that

$$\|x_1 - x_2\|_X \leqslant C\|F(x_1) - F(x_2)\|_Y, \qquad x_1, x_2 \in K.$$

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Many Lipschitz stability estimates for EIT with **full** DN map: Alessandrini, Beretta, Francini, Gaburro, de Hoop, Meftahi, Scherzer, Sincich, Vessella and many others.

Lipschitz stability with finite measurements: setting

Discretization: $Q_N : Y \to Y$ bounded linear map such that there exists a subspace $\widetilde{Y} \subseteq Y$ satisfying:

1. $\|Q_N|_{\widetilde{Y}}\|_{\mathcal{L}(\widetilde{Y},Y)} \leqslant D$ for every $N \in \mathbb{N}$ and some $D > 0$;
2. $Q_N|_{\widetilde{Y}} \to I_{\widetilde{Y}}$ as $N \to \infty$ with respect to the strong operator topology, i.e.

$$\lim_{N\to\infty} \|y - Q_N y\|_Y = 0$$

for every $y \in \widetilde{Y}$.

Examples

- Y Hilbert space, $\{G_j\}_{j\in\mathbb{N}}$ exhaustive sequence of finite dimensional and nested subspaces.

  $$Q_N = P_{G_N} \text{ orthogonal projection onto } G_N.$$

  $\widetilde{Y} = Y$.
- $Y = \mathcal{L}_c(Y^1, Y^2)$ with Y^1, Y^2 Banach spaces. $P^2_N \to I_{Y^2}$ and $(P^1_N)^* \to I_{Y^1}$ strongly.

  $$Q_N(y) = P^2_N y P^1_N.$$

  $\widetilde{Y} = \{T \in Y : T \text{ is compact}\}$.

Lipschitz stability with finite measurements: main result

Theorem (G.S. Alberti, M.S. (2019) [4])

Let $K \subseteq A$ be convex. Suppose there exists $C > 0$ such that

$$\|x_1 - x_2\|_X \leqslant C\|F(x_1) - F(x_2)\|_Y, \qquad \text{for } x_1, x_2 \in K.$$

(i) If $K \subseteq W \cap A$ is compact, where W is a finite dimensional subset of X and for every $\xi \in K$, $\operatorname{ran}\left(F'(\xi)|_W\right) \subseteq \tilde{Y}$, then

$$\lim_{N \to +\infty} s_N = 0, \qquad s_N = \sup_{\xi \in K} \|(I - Q_N)F'(\xi)\|_{W \to Y}.$$

(ii) If $s_N \leqslant \frac{1}{2C}$, then

$$\|x_1 - x_2\|_X \leqslant 2C\|Q_N(F(x_1)) - Q_N(F(x_2))\|_Y, \qquad x_1, x_2 \in K.$$

[4] Infinite-dimensional inverse problems with finite measurements, arXiv:1906.10028.

The smoothing condition: $\operatorname{ran}\left(F'(\xi)|_W\right) \subseteq \tilde{Y}$

- Y Hilbert space, $\{G_j\}_{j\in\mathbb{N}}$ exhaustive sequence of finite dimensional and nested subspaces. $Q_N = P_{G_N}$ orthogonal projection onto G_N.

  Since $Q_N \to I_Y$ strongly and $\widetilde{Y} = Y$, the condition is satisfied.

- $Y = \mathcal{L}_c(Y^1, Y^2)$ with Y^1, Y^2 Banach spaces. $P_N^2 \to I_{Y^2}$ and $(P_N^1)^* \to I_{Y^1}$ strongly.

$$Q_N(y) = P_N^2 y P_N^1.$$

  Assuming that $F'(\xi)\tau : Y^1 \to Y^2$ is compact, i.e. $F'(\xi)\tau \in \widetilde{Y}$, for every $\xi \in K, \tau \in W$ then the condition is satisfied.

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On the number of measurements N

N depends on the Lipschitz constant C for the full data and on the subspace W:

$$\sup_{\xi \in K} \|(I - Q_N)F'(\xi)\|_{W \to Y} \leqslant \frac{1}{2C}$$

which can be explicitly computed in several cases.

Example I: electrical impedance tomography

Let $\mathcal{N}_\sigma$ be the Neumann-to-Dirichlet map and assume

$$\|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \leqslant C\|\mathcal{N}_{\sigma_1} - \mathcal{N}_{\sigma_2}\|_{L^2_\diamond(\partial\Omega)\to L^2_\diamond(\partial\Omega)}, \qquad \sigma_1, \sigma_2 \in K,$$

where K is a compact subset of a finite dimensional subspace of L^∞ conductivities ($L^2_\diamond(\partial\Omega) = \{f \in L^2(\partial\Omega) : \int_{\partial\Omega} f\, ds = 0\}$). Then there exists $N \in \mathbb{N}$ such that

$$\|\sigma_1 - \sigma_2\|_\infty \leqslant 2C\, \|P_N \mathcal{N}_{\sigma_1} P_N - P_N \mathcal{N}_{\sigma_2} P_N\|_{L^2_\diamond(\partial\Omega)\to L^2_\diamond(\partial\Omega)}, \qquad \sigma_1, \sigma_2 \in K.$$

$\Omega \subseteq \mathbb{R}^2$ unit disk. Let P_N be the projection on the trigonometric current patterns

$\sin(n\theta), \cos(n\theta)$, for $n \leqslant N$, $\theta \in \partial\Omega$.

Then we have $N = O(C^2)$ (recall that for EIT $C = O(\exp(\dim W))$).

Example II: inverse scattering

$$\begin{cases} \Delta u + k^2 n(x) u = 0 & \text{in } \mathbb{R}^3, \\ u = e^{ikx\cdot d} + u^s & \text{in } \mathbb{R}^3, \\ \text{radiation condition for } u^s & \end{cases}$$

- $k > 0$ is the (fixed) wavenumber, $d \in S^2$,
- $n \in L^\infty(\mathbb{R}^3; \mathbb{C})$ is the refractive index with $\mathrm{Im}(n) \geqslant 0$ in $\mathbb{R}^3$ and $\mathrm{supp}\,(1-n) \subseteq B$ for some open ball B.

Problem. Given the far field $u_n^\infty(\hat{x}, d) \in L^2(S^2 \times S^2)$ at fixed $k > 0$, find n in B.

Example II: inverse scattering

- $X = L^\infty(B;\mathbb{C})$, $Y = L^2(S^2 \times S^2)$;
- $A = L^\infty_+(B) = \{f \in L^\infty(B;\mathbb{C}) : \mathrm{Im}(n) \geqslant \lambda$ in B for some $\lambda > 0\}$;
- W finite-dimensional subspace of $L^\infty(B;\mathbb{C})$
- K convex and compact subset of $W \cap A$;

[Bourgeois 2013] proved Lipschitz stability in this case:

$$\|n_1 - n_2\|_{L^\infty(B)} \leqslant C \left\|u^\infty_{n_1} - u^\infty_{n_2}\right\|_{L^2(S^2\times S^2)}, \qquad n_1, n_2 \in K.$$

- $Q_N : L^2(S^2 \times S^2) \to L^2(S^2 \times S^2)$ bounded linear maps, $Q_N \to I_{L^2(S^2\times S^2)}$ strongly.

$$\|n_1 - n_2\|_{L^\infty(B)} \leqslant 2C \left\|Q_N(u^\infty_{n_1}) - Q_N(u^\infty_{n_2})\right\|_{L^2(S^2\times S^2)}, \qquad n_1, n_2 \in K.$$

Example: Q_N projections onto the span of the first N spherical harmonics.

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Low-dimensional manifolds

$M \subseteq X$ (Banach space) a n-dimensional C^1 manifold with atlas $\{(U_i, \varphi_i)\}_{i \in I}$:

- topologically embedded in X (i.e. not a C^1 submanifold),
- α-Hölder for $\alpha \in (0, 1]$, that is $\varphi_i^{-1} : \varphi_i(U_i) \to M$ is α-Hölder, for $i \in I$.

Examples:

- Indicator functions on balls with variable centres, radii and intensities,
  - Hölder in $L^p(\mathbb{R}^n)$, $p > 1$, Lipschitz in $L^1(\mathbb{R}^n)$, not C^1 embedded.
- Indicator functions on simplexes.
- A manifold generated by a neural network (e.g. autoencoder, GAN, etc.)

Hölder-Lipschitz stability from an infinite amount of measurements

Theorem (G.S. Alberti, A. Arroyo, M.S. (2020) [5])

Let X and Y be Banach spaces, $\alpha \in (0,1]$, $M \subseteq X$ be an n-dimensional differentiable manifold α-Hölder in X and $K \subseteq M$ be a compact set. Consider $F \in C^1(M,Y)$ with:

1. F is injective,
2. the differential $dF_x \colon T_x M \to Y$ is injective for every $x \in M$.

Then there exists a constant $C > 0$ such that

$$\|x - y\|_X \leqslant C \|F(x) - F(y)\|_Y^\alpha, \qquad x, y \in K.$$

Key ideas:

- short distance and long distance cases;
- workaround for the lack of convexity of K.

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[5] Inverse problems on low-dimensional manifolds, arXiv:2009.00574.

Finite number of measurements

Let $Q_N : Y \to Y$ and $\widetilde{Y} \subseteq Y$ be as in the linear case.

Theorem (G.S. Alberti, A. Arroyo, M.S. (2020)[6])

Let X and Y be Banach spaces, $M \subseteq X$ an n-dimensional C^1 manifold Lipschitz in X, $K \subseteq M$ a compact set. Consider $F \in C^1(M, Y)$ satisfying:

1. F is injective;
2. the differential $dF_x : T_x M \to Y$ is injective for every $x \in M$;
3. $\mathrm{ran}(F|_K) \subseteq \widetilde{Y}$;
4. $\mathrm{ran}(dF_x) \subseteq \widetilde{Y}$ for every $x \in M$.

Then

$$\|x - y\|_X \leqslant C \|Q_N F(x) - Q_N F(y)\|_Y, \qquad x, y \in K,$$

for some $C > 0$ and every sufficiently large $N \in \mathbb{N}$.

[6] Inverse problems on low-dimensional manifolds, arXiv:2009.00574.

Ongoing work and conclusions

- Global reconstruction algorithms for both linear and manifold cases.
- Applications of the results on manifolds on classical inverse problems.
- Add sparsity and random subsampling: non-linear compressed sensing.
- Approximate the inverse mapping with a deep neural network.

Thank you!