A function theory for discrete operators

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Complex analysis case

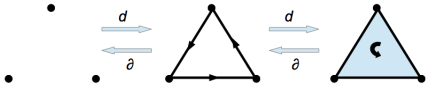
In the last two decades one sees an increasing interest in the analysis of discrete structures based on the simple fact that computers are restricted to work with discrete values.

Strong and broad theory behind $2D-$problems in Potential Theory, Boundary Value problems, Scattering problems, Ising Model, etc. (Ferrand, Lovasz, Kenyon, Bobenko, Mercat, Stephenson, Novikov, Smirnov, (with excuses to the many more left behind))

Current tools (e.g. FEM, calculation of fundamental solutions, numerical implementations of boundary operators) are based on the manipulation of the corresponding Fourier symbols as well as approximations (and subsequent study of their stability).

Higher dimensions

Several variables approach: Bobenko/Mercat/Suris, M. Desbrun (discrete holomorphic functions)



First approach in Clifford analysis: Gürlebeck/Sproessig, operator theory / potential theory, and based on Kirchhoff laws - one has the concept of a *potential function* satisfying to the *discrete star Laplacian*:

$$\Delta_h f^\#(x) := \frac{1}{h^2}\left[\sum_{i=1}^{n}(f^\#(x+he_i)+f^\#(x-he_i)) - 2n\, f^\#(x)\right].$$

This concept has been recently developed into a theory of discrete Dirac operators on bounded domains by C., Kähler, Faustino, de Ridder, Sommen, Legatiuk[2], which takes into consideration their Fourier symbols.

Forward and backward differences

Basic assumptions:

- forward and backward differences $\partial_h^{\pm j}$ are given by

$$\partial_h^{+j} f(hm) = \frac{1}{h}[f(hm + h\mathbf{e}_j) - f(hm)],$$

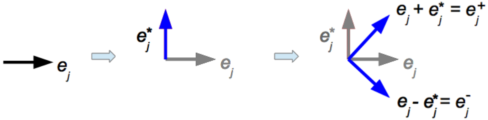
$$\partial_h^{-j} f(hm) = -\frac{1}{h}[f(hm) - f(hm - h\mathbf{e}_j)]$$

  whit lattice constant $h > 0$ and $m = m_1\mathbf{e}_1 + \cdots + m_n\mathbf{e}_n \in \mathbb{Z}^n$.

- Split the basis element $\mathbf{e}_j$ ($j = 1, \cdots, n$) into two basis elements

$$\mathbf{e}_j = \mathbf{e}_j^+ + \mathbf{e}_j^-$$

  corresponding to the forward and backward directions.



vector e_j $\Rightarrow$ copy of $\mathbb{C}$ $\Rightarrow$ Witt basis

Factorization of the star-Laplacian

- The elements $\mathbf{e}_j^+, \mathbf{e}_j^-$, $(j = 1, \cdots, n)$ form a Witt basis for the complexified Clifford algebra $\mathbb{C}_n$, satisfying to:

$$\begin{cases} \mathbf{e}_j^- \mathbf{e}_k^- + \mathbf{e}_k^- \mathbf{e}_j^- &= 0, \\ \mathbf{e}_j^+ \mathbf{e}_k^+ + \mathbf{e}_k^+ \mathbf{e}_j^+ &= 0, \\ \mathbf{e}_j^+ \mathbf{e}_k^- + \mathbf{e}_k^- \mathbf{e}_j^+ &= -\delta_{jk}, \end{cases}$$

  with δ_{jk} the Kronecker symbol.

- Discrete Dirac operator $D_h^{+-} = \sum_{j=1}^n \mathbf{e}_j^+ \partial_h^{+j} + \mathbf{e}_j^- \partial_h^{-j}$ which factorizes the star-Laplacian as

$$\Delta_h =: \sum_{j=1}^n \partial_h^{+j} \partial_h^{-j} = -(D_h^{+-})^2,$$

- Its adjoint Dirac operator

$$D_h^{-+} = \sum_{j=1}^n \mathbf{e}_j^+ \partial_h^{-j} + \mathbf{e}_j^- \partial_h^{+j}.$$

Discrete Fourier Transform

- ℓ^p spaces $(1 \leq p < +\infty)$

$$u \in \ell^p(\Omega) \quad iff \quad \|u\|_{\ell^p(\Omega)} := \left(\sum_{hm \in \Omega} |u(hm)|^p h^n \right)^{1/p} < \infty$$

- Discrete Fourier transform $\mathcal{F}_h : \ell^2(\mathbb{Z}^n) \to L_0^2\left(\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n\right)$,

$$u \mapsto \mathcal{F}_h u(\xi) = \begin{cases} \sum_{m \in \mathbb{Z}^n} e^{-ihm\cdot\xi} u(hm) h^n, & \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n \\ 0 & \text{other } \xi \end{cases}$$

- Inverse $\mathcal{F}_h^{-1} = R_h \mathcal{F}$ where R_h is the restriction to the lattice $h\mathbb{Z}^n$ and

$$\mathcal{F}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(\xi)\, \chi|_{\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n}(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

Discrete Fundamental Solutions

Discrete fundamental solution of D_h^{-+}

$$D_h^{-+}E_h^{-+}(hm) = \delta_h(hm) = \begin{cases} h^{-n} & m = 0 \\ 0 & m \neq 0 \end{cases}, \quad \forall m \in \mathbb{Z}^n$$

where δ_h is the discrete Dirac function.

Remark that the symbol d^2 of the discrete star Laplacian is known

$$\mathcal{F}_h(-\Delta_h u)(\xi) = \frac{4}{h^2}\sum_{j=1}^{n}\sin^2\left(\frac{\xi_j h}{2}\right)\mathcal{F}_h u(\xi) := d^2\mathcal{F}_h u(\xi)$$

and that

$$\mathcal{F}_h(D_h^{-+}u)(\xi) := \tilde{\xi}_- \mathcal{F}_h u(\xi) = \sum_{j=1}^{n}\left(\mathbf{e}_j^+ \xi_{-j}^D + \mathbf{e}_j^- \xi_{+j}^D\right)\mathcal{F}_h u(\xi),$$

with $\xi_{\pm j}^D = \mp h^{-1}\left(1 - e^{\mp i h \xi_j}\right)$.

Computation in Fourier Domain - 1

These leads to

$$E_h^{-+} = R_h \mathcal{F}\left(\frac{\widetilde{\xi_-}}{d^2}\right) = \sum_{j=1}^{n} e_j^+ R_h \mathcal{F}\left(\frac{\xi_{-j}^D}{d^2}\right) + e_j^- R_h \mathcal{F}\left(\frac{\xi_{+j}^D}{d^2}\right),$$

with

- $D_h^{-+} E_h^{-+}(hm) = \delta_h(hm), \quad hm \in \ h\mathbb{Z}^n,$
- $\lim_{h\to 0} \frac{\widetilde{\xi_-}}{d^2} = \frac{-i\xi}{|\xi|^2};$
- $\left| \int_{[-\frac{\pi}{h}, \frac{\pi}{h}]^n} \frac{\xi_{\pm j}^D}{d^2} e^{-i\langle hm, \xi\rangle} d\xi \right| \approx \mathcal{O}(\frac{1}{|hm|^{n-1}}),$

so that $E_h^{-+} \in \ell_p(\mathbb{Z}^n, \mathbb{C}_n)$ for $p > \frac{n}{n-1}$.

Computation in Fourier Domain - 2

Lemma

For each fix point $m \in h\mathbb{Z}^n$ of the lattice, there exists a constant $C > 0$ such that

$$\left|E_h^{-+}(m) - E(m)\right| \leq C\frac{h}{|m|^n}, \quad m \neq \mathbf{0},$$

where E is the fundamental solution of the Dirac operator in $\mathbb{R}^n$.

Similarly for $E_h^{+-} = R_h\mathcal{F}\left(\frac{\widetilde{\xi}_+}{d^2}\right) = \sum_{j=1}^n e_j^+ R_h\mathcal{F}\left(\frac{\xi_{+j}^D}{d^2}\right) + e_j^- R_h\mathcal{F}\left(\frac{\xi_{-j}^D}{d^2}\right)$.

Discrete upper/lower Hilbert Transform

Theorem (C., Kähler, Ku, Sommen, JFAA (2014))

Let $f \in \ell_p(\mathbb{Z}^{n-1}, \mathbb{C}_n)$ be a boundary value of a discrete monogenic function in the upper half space. Then its $(n-1)$D-Fourier transform $F = \mathcal{F}_h f$ satisfies

$$\pm \frac{\widetilde{\underline{\xi}_-}}{\underline{d}} \left(\mathbf{e}_n^{\mp} \frac{2}{h\underline{d} - \sqrt{4 + h^2 \underline{d}^2}} + \mathbf{e}_n^{\pm} \frac{h\underline{d} - \sqrt{4 + h^2 \underline{d}^2}}{2} \right) F = F.$$

Definition (Discrete upper/lower Hilbert transform)

$$H_{\pm} f = \mathcal{F}_h^{-1} \left[\pm \frac{\widetilde{\underline{\xi}_-}}{\underline{d}} \left(\mathbf{e}_n^{\mp} \frac{2}{h\underline{d} - \sqrt{4 + h^2 \underline{d}^2}} + \mathbf{e}_n^{\pm} \frac{h\underline{d} - \sqrt{4 + h^2 \underline{d}^2}}{2} \right) \right] \mathcal{F}_h f.$$

Moreover,

$$(H_+)^2 = id = (H_-)^2.$$

Symbol calculus of band-dominated operators - V. Rabinovich, S. Roch, B. Silbermann (1998)

- Alternative approach - uses limit operators

Limit operator

A_h is called limit operator of a linear bounded operator A if

$$\begin{aligned} \lim_{m\to\infty} \|(V_{-h_m} A V_{h_m} - A_h)\chi I\| &= 0 \\ \lim_{m\to\infty} \|\chi(V_{-h_m} A V_{h_m} - A_h)\| &= 0 \end{aligned}$$

for every function $\chi \in \ell^\infty$ with finite support. V_h denotes the shift operator and $(h_m)_{m=0}^{\infty}$ is a sequence tending to infinity.

- A is band-dominated if $A = \sum_\alpha a_\alpha V_\alpha$ where the sum is finite (finite combination of shifts and multiplication operators); obviously this definition includes both star-Laplacian and discrete Dirac operators.

Symbol calculus of band-dominated operators - V. Rabinovich, S. Roch, B. Silbermann

- Symbol calculus for such operators;
- Boundedness result is stronger than in our case, i.e. allows also the case of $p = 1$ and $p = \infty$;
- With our calculus of discrete pseudo-differential operators we can prove that if for a band-dominated operator A all limit operators are invertible then A is Fredholm.

The torus case $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$

Agranovich 1979, Ruzhansky/Turunen/Wirth since 2006

- For $\hat{f} : \mathbb{Z}^n \to \mathbb{C}$ we define the difference operator $\partial_{\xi_j}^{+j}$ as

$$\partial_{\xi_j}^{+j}\hat{f} = \hat{f}(\xi + \mathbf{e}_j) - \hat{f}(\xi), \quad j = 1, \cdots, n,$$

  and

$$\partial_{\xi}^{+\alpha}\hat{f} = \partial_{\xi_1}^{+\alpha_1} \cdots \partial_{\xi_n}^{+\alpha_n}\hat{f}, \quad \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n.$$

- For periodic $f : \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{C}$ we have

$$\hat{f}(\xi) = \int_{\mathbb{T}^n} e^{-2\pi i x\cdot\xi} f(x)dx, \quad \xi \in \mathbb{Z}^n.$$

  Then,

$$Af(x) = \sum_{\xi\in\mathbb{Z}^n} \sigma(x, \xi) e^{2\pi i x\cdot\xi}\, \hat{f}(\xi),$$

  with symbol class $S_{1,0}^m(\mathbb{R}^n \times \mathbb{Z}^n)$ defined as

$$\left|\partial_{\xi}^{+\alpha}\partial_x^{\beta}\sigma(x, \xi)\right| \leq C_{\alpha,\beta}\left(1 + |\xi|^2\right)^{(m-|\alpha|)/2}$$

Difference equations on $\mathbb{Z}^n$ (*Botchway / Kibiti / Ruzhansky, 2017*)

Consider now difference operators (e.g. discrete star Laplacian):

$$\Delta^{\#} f(x) := \sum_{i=1}^{n} [f(x+\mathbf{e}_i) + f(x-\mathbf{e}_i)] - 2n\, f(x), \quad x \in \mathbb{Z}^n.$$

Given $g \in \ell^2(\mathbb{Z}^n)$ we have

- $\hat{g}(\xi) = \sum_{x\in\mathbb{Z}^n} e^{-2\pi i \xi\cdot x} g(x), \quad \xi \in \mathbb{T}^n;$
- the difference equation $\Delta^{\#} f(x) = g(x), \quad x \in \mathbb{Z}^n$ has solution

$$f(x) = \int_{\mathbb{T}^n} e^{2\pi i \xi\cdot x} \frac{1}{2\,\sum_{j=1}^{n} \cos(2\pi\xi_j) - 2n}\, \hat{g}(\xi) d\xi;$$

- for $1 < p \leq 2 \leq q < \infty$ it holds
  - $g \in \ell^p(\mathbb{Z}^n) \Rightarrow f \in \ell^q(\mathbb{Z}^n);$
  - $\sum_{k\in\mathbb{Z}^n} (1+|x|^2)^{s/2} |g(x)|^2 < \infty \Rightarrow \sum_{x\in\mathbb{Z}^n} (1+|x|^2)^{s/2} |f(x)|^2 < \infty.$

Discrete symbols

Let $\sigma : \mathbb{Z}^n \times \mathbb{T}^n \to \mathbb{C}$ be measurable.

- We define the pseudodifferential operator acting on $\mathbb{Z}^n$ and associated to σ as
$$Op(\sigma)f(x) := \int_{\mathbb{T}^n} e^{2\pi i x\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi, \quad x \in \mathbb{Z}^n.$$

- The discrete Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ (space of rapidly decreasing function) is the space of $\varphi : \mathbb{Z}^n \to \mathbb{C}$ if for any $M > 0$ there exists $C_{\varphi,M} > 0$ such that
$$|\varphi(x)| \leq C_{\varphi,M}(1+|x|^2)^{-M/2}, \quad \forall x \in \mathbb{Z}^n,$$
endowed with the topology induced by the seminorms
$$p_j(\varphi) := \sup_{x\in\mathbb{Z}^n}(1+|x|^2)^{j/2}|\varphi(x)|.$$

- The space of tempered distributions $\mathcal{S}'(\mathbb{Z}^n)$ is the topological dual to $\mathcal{S}(\mathbb{Z}^n)$.

Symbol class

Definition (symbol class $S^m_{\rho,\delta}(\mathbb{Z}^n \times \mathbb{T}^n)$)

We say that $\sigma : \mathbb{Z}^n \times \mathbb{T}^n \to \mathbb{C}$ belongs to the class $S^m_{\rho,\delta}(\mathbb{Z}^n \times \mathbb{T}^n)$ if

- $\sigma(x, \cdot) \in C^\infty(\mathbb{T}^n)$ for all $x \in \mathbb{Z}^n$;
- for all $\alpha, \beta \in \mathbb{N}_0^n$ there exists $C_{\alpha,\beta} > 0$ s. t.

$$\left|\partial_\xi^\beta \partial_x^{+\alpha} \sigma(x,\xi)\right| \leq C_{\alpha,\beta}(1 + |x|^2)^{\frac{1}{2}(m-\rho|\alpha|+\delta|\beta|)},$$

  for all $x \in \mathbb{Z}^n, \xi \in \mathbb{T}^n$.

Of particular interest is the class $S^m_{1,0}$

$$\left|\partial_\xi^\beta \partial_x^{+\alpha} \sigma(x,\xi)\right| \leq C_{\alpha,\beta}(1 + |x|^2)^{\frac{1}{2}(m-|\alpha|)},$$

thus corresponding to controlled decay in "space"-variable.

Convolutional kernel representations

For a measurable symbol $\sigma : \mathbb{Z}^n \times \mathbb{T}^n \to \mathbb{C}$ we define the pseudo-difference operator

$$f \in \ell^2(\mathbb{Z}^n) \to Op(\sigma)f(x) := \int_{\mathbb{T}^n} e^{2\pi i x\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi,$$

and the quantization $\sigma \mapsto Op(\sigma)$ is called the lattice quantization.

These operators have a (convolutional) kernel representation

$$Op(\sigma) = \int_{\mathbb{T}^n} e^{2\pi i x\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi = \int_{\mathbb{T}^n} e^{2\pi i x\cdot\xi}\sigma(x,\xi)\sum_{s\in\mathbb{Z}^n} e^{-2\pi i s\cdot\xi}f(s)d\xi$$

$$= \sum_{m\in\mathbb{Z}^n}\left(\int_{\mathbb{T}^n} e^{2\pi i m\cdot\xi}\sigma(x,\xi)d\xi\right)f(x-m) = \sum_{m\in\mathbb{Z}^n}\kappa(x,m)f(x-m).$$

Ellipticity

Definition

We say $\sigma \in S^m_{1,0}(\mathbb{Z}^n \times \mathbb{T}^n)$ is elliptic of order m if it exits constants $C, M > 0$ such that

$$|\sigma(x,\xi)| \geq C(1+|x|^2)^{m/2},$$

for all $\xi \in \mathbb{T}^n$ and $x \in \mathbb{Z}^n : |x| \geq M$.

But the discrete star Laplacian

$$\Delta^{\#} f := \sum_{i=1}^{n} [f(\cdot + \mathbf{e}_i) + f(\cdot - \mathbf{e}_i)] - 2n\, f$$

has symbol

$$\sigma(x,\xi) = 2\sum_{j=1}^{n} \cos(2\pi\xi_j) - 2n = 4\sum_{j=1}^{n} \sin^2(\pi\xi_j)$$

and, hence it is **not** a discrete elliptic operator.

Composition and adjoint

Composition formula

Let $0 \leq \delta < \rho \leq 1$. Given symbols $\sigma_A \in S^{m_1}_{\rho,\delta}(\mathbb{Z}^n \times \mathbb{T}^n)$ and $\sigma_B \in S^{m_2}_{\rho,\delta}(\mathbb{Z}^n \times \mathbb{T}^n)$ we have that its composition $Op(\sigma_A) \circ Op(\sigma_B)$ is a pseudo-differential operator with symbol $\sigma \in S^{m_1+m_2}_{\rho,\delta}(\mathbb{Z}^n \times \mathbb{T}^n)$, given as an asymptotic sum

$$\sigma(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_A(x,\xi) \partial_x^{+\alpha} \sigma_B(x,\xi).$$

Adjoint formula

Let $0 \leq \delta < \rho \leq 1$. For each $\sigma \in S^m_{\rho,\delta}(\mathbb{Z}^n \times \mathbb{T}^n)$ there exists a symbol $\sigma^* \in S^m_{\rho,\delta}(\mathbb{Z}^n \times \mathbb{T}^n)$ such that the adjoint operator $Op(\sigma)^*$ is a pseudo-difference operator with symbol σ^*, and

$$\sigma^*(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha \partial_x^{+\alpha} \overline{\sigma(x,\xi)}.$$

In particular, for $\sigma(x,\xi) = \sigma(\xi)$ we have $\sigma^*(x,\xi) \sim \overline{\sigma(x,\xi)}$.

Boundedness on $\ell^2(\mathbb{Z}^n)$

Recall: a linear bounded operator A over $\ell_2(\mathbb{Z}^n)$ has finite Hilbert-Schmidt norm

$$\|A\|_{HS}^2 = \sum_{k\in\mathbb{Z}^n} \|A\delta_k(\cdot)\|^2,$$

where $(\delta_k)_{k\in\mathbb{N}_0^n}$ (Kronecker delta) is the canonical basis in $\ell_2(\mathbb{Z}^n)$.

Lemma

The discrete pseudo-difference operator $Op(\sigma) : \ell_2(\mathbb{Z}^n) \mapsto \ell_2(\mathbb{Z}^n)$ is a Hilbert-Schmidt operator if and only if $\sigma \in L_2(\mathbb{Z}^n \times \mathbb{T}^n)$, where it holds

$$\|Op(\sigma)\|_{HS} = \|\sigma\|_{L_2(\mathbb{Z}^n\times\mathbb{T}^n)} = \left(\sum_{x\in\mathbb{Z}^n\times\mathbb{T}^n} \int_{[-\frac{\pi}{h},\frac{\pi}{h}]^n} |\sigma(x,\xi)|^2 d\xi \right)^{1/2},$$

Additionally, we have in the case that $\sigma \in L_2(\mathbb{Z}^n \times \mathbb{T}^n)$ then $Op(\sigma) : \ell_p(\mathbb{Z}^n) \mapsto \ell_q(\mathbb{Z}^n)$ is bounded for all $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$ with

$$\|Op(\sigma)\|_{\ell_p(\mathbb{Z}^n)\mapsto\ell_q(\mathbb{Z}^n)} \leq 2^{n/2}\|\sigma\|_{L_2(\mathbb{Z}^n\times\mathbb{T}^n)}.$$

Boundedness on $\ell^2(\mathbb{Z}^n)$

Theorem (Mikhlin type of boundedness on $\ell^2(\mathbb{Z}^n)$)

Let $N \in \mathbb{N}$ such that $N > n/2$. Suppose that the symbol $\sigma : \mathbb{Z}^n \times \mathbb{T}^n \mapsto \mathbb{C}_n$ satisfies

$$|\partial_\xi^\beta \sigma(x,\xi)| \leq C, \qquad \text{for all } (x,\xi) \in \mathbb{Z}^n \times \mathbb{T}^n, \text{ and all } |\beta| \leq N.$$

Then $Op(\sigma)$ extends to a bounded operator on $\ell^2(\mathbb{Z}^n)$.

Idea of the proof (Ruzhansky, Turunen (2010)): usage of duality between our discrete symbols and toroidal symbols, i.e.

$$Op_{\mathbb{Z}^n}(\sigma) = R_h\mathcal{F} \circ Op_{\mathbb{T}^n}(\tau) \circ \mathcal{F}_h;$$

The discrete Fourier transform $\mathcal{F}_h$ is an isometry from $\ell_2(\mathbb{Z}^n)$ to $L_2(\mathbb{T}^n)$, so the discrete $Op(\sigma) \equiv Op_{\mathbb{Z}^n}(\sigma)$ is bounded on $\ell_2(\mathbb{Z}^n)$ iff the pseudo-differential operator $Op_{\mathbb{T}^n}(\tau)$ is bounded on $L_2(\mathbb{T}^n)$ for the toroidal symbol with values in the Clifford algebra $\mathbb{C}_n$

Essencial spectrum

For a linear (closed) operator A on a Clifford-Hilbert module its essential spectrum is defined as

$$\Sigma_{ess}(A) = \{\lambda : A - M_\lambda \text{ is Fredholm and } i(A - M_\lambda) = 0\}$$

where $M_\lambda f(x) := f(x)\lambda$.

Another characterization of the essential spectrum is so-called S-spectrum Colombo/Sabadini (originally for quaternions, extended later to Clifford algebras)

$$\Sigma_{ess}(A) = \mathbb{C}_n \setminus \{s : A^2 - 2s_0 A + |s|^2 I \text{ is Fredholm and its index is } 0\}.$$

Gohberg Lemma

Gohberg Lemma

Let $\sigma \in S^0_{1,0}(\mathbb{Z}^n \times \mathbb{T}^n)$. Define

$$d_{max} := \limsup_{|x|\to\infty} \sup_{\xi\in\mathbb{T}^n} |\sigma(x,\xi)|.$$

Then for all compact operators K on $\ell_2(\mathbb{Z}^n)$ we have

$$\|Op(\sigma) - K\| \geq d_{\max}.$$

Theorem (Compactness)

Let $\sigma \in S^0_{1,0}(\mathbb{Z}^n \times \mathbb{T}^n)$. Define $d_{\max} := \limsup_{|x|\to\infty} \sup_{\xi\in[-\frac{\pi}{h},\frac{\pi}{h}]^n} |\sigma(x,\xi)|$.
Then $Op(\sigma)$ is compact on $\ell_2(\mathbb{Z}^n)$ if and only if

$$d_{\max} = 0.$$

Localization of the essential spectrum

Theorem

Let $\sigma \in S^0_{1,0}(\mathbb{Z}^n \times \mathbb{T}^n)$ be of the type

$$\sigma = \sigma_0 + \sum_{j=0}^{n} (e_j^+ \sigma_j^+ + e_j^- \sigma_j^-),$$

with $\sigma_j^{\pm}$ complex-valued for $j = 0, \ldots, n$, and

$$d_{\max} := \limsup_{|x| \to \infty} \sup_{\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]^n} |\sigma(x, \xi)|.$$

Then for $Op(\sigma) : \ell_2(\mathbb{Z}^n) \mapsto \ell_2(\mathbb{Z}^n)$ we have

$$\Sigma_{ess}(Op(\sigma)) \subset \{s \in \mathbb{C}_n : |s| \leq d_{\max}\}.$$

Weighted boundedness on $\ell_2(h\mathbb{Z}^n)$

Theorem

Let $m \in \mathbb{R}$ and $\sigma \in S^m_{0,0}(\mathbb{Z}^n \times \mathbb{T}^n)$.
Then $Op(\sigma)$ is bounded from $\ell^s_2(\mathbb{Z}^n)$ to $\ell^{s-m}_2(\mathbb{Z}^n)$ for all $s \in \mathbb{R}$.

Overview of the classical continuous case

Operator	symbol	class
Laplace operator Δ	$\sigma_\Delta(x,\xi) = -\lvert\xi\rvert^2$	$S^2_{1,0}$
Dirac operator D	$\sigma_D(x,\xi) = -i\xi$	$S^1_{1,0}$
Teodorescu operator T	$\sigma_T(x,\xi) = \frac{i\xi}{\lvert\xi\rvert^2}$	$S^{-1}_{1,0}$
Hilbert transform H	$\sigma_H(x,\xi) = \frac{i\xi}{\lvert\xi\rvert}$	$S^0_{1,0}$

. . . versus the discrete case

Operator	symbol $\sigma(x,\xi)$, $\quad x \in h\mathbb{Z}^n, \xi \in \mathbb{T}^n$	class
star-Laplace operator Δ_h	$d^2 := \frac{4}{h^2} \sum_{j=1}^n \sin^2\left(\frac{h\xi_j}{2}\right)$	$S^0_{1,0}$
Dirac operator $D_h^{\mp}$	$\sum_{j=1}^n \left(\mathbf{e}_j^+ \xi^D_{-j} + \mathbf{e}_j^- \xi^D_{+j}\right)$	$S^0_{1,0}$
Teodorescu operator T_h	$\frac{\sum_{j=1}^n \left(\mathbf{e}_j^+ \xi^D_{-j} + \mathbf{e}_j^- \xi^D_{+j}\right)}{\frac{4}{h^2} \sum_{j=1}^n \sin^2\left(\frac{h\xi_j}{2}\right)}$	$S^0_{1,0}$
Hilbert transforms $H_\pm$	$\pm \frac{\widetilde{\xi}_-}{\underline{d}} \left(\mathbf{e}_n^{\mp} \frac{2}{h\underline{d} - \sqrt{4+h^2\underline{d}^2}} + \mathbf{e}_n^{\pm} \frac{h\underline{d} - \sqrt{4+h^2\underline{d}^2}}{2}\right)$	$S^0_{1,0}$

What about boundedness and compactness?

One has to remark that, as all these operators are in the class $S^0_{1,0}$, all satisfy the Mikhlin type of boundedness condition.

In difference to the continuous case, the star-Laplacian and the discrete Dirac are now bounded operators.

Furthermore, none of the operators satisfies the compactness condition (in particular, the discrete Teodorescu operator).

We can modify the discrete Dirac operator into an elliptic operator i.e.

$$D^{\pm}_{h,m} = D^{\pm}_{h} + mI,$$

which has symbol

$$\sum_{j=1}^{n} \left(\mathbf{e}_j^{+} \xi^D_{-j} + \mathbf{e}_j^{-} \xi^D_{+j} \right) + mI$$

which for $\Re(m) \neq 0$ satisfy the ellipticity condition. In physical terms it represents a discretization of the Dirac operator with rest mass m.

Remarks on calculus of pseudodifference operators by S. Roch & B. Silbermann

Roch & Silbermann 1998

Given a function $a : \mathbb{Z}^n \times \mathbb{T}^n \mapsto \mathbb{C}$ we can define the symbol class *OPS*

$$\sup_{(x,t)\in\mathbb{Z}^n\times\mathbb{T}^n,\beta\in\mathbb{N}^n:|\beta|_\infty\leq k} |\partial_t^\beta a(x,t)| \leq \infty$$

for each non-negative integer k.

- Similar quantization - in particular, same symbols for difference operators
- Different symbol classes (Mellin-type symbol classes)
- Study based on spectral resolution - not easy in Clifford analysis

Discrete half Dirichlet problems

Problem I

Given $g \in \ell^p(\mathbb{Z}^{n-1})(1 \leq p < \infty)$ we want to find $f : \mathbb{Z}^{n-1} \times \mathbb{Z}_0^+ \to \mathbb{C}_n$ such that

$$\begin{cases} D_h^{-+} f(hn) = 0, & n \in \mathbb{Z}^{n-1} \times \mathbb{Z}^+ \\ f(h(\underline{n}, 1)) = g(h\underline{n}), & \underline{n} \in \mathbb{Z}^{n-1}. \end{cases} \tag{1}$$

Discrete Cauchy Transforms

Definition (Upper Cauchy transform)

For a discrete ℓ^p-function f, $1 \leq p < +\infty$, defined on the boundary layers $(\underline{\eta}, 0), (\underline{\eta}, 1), \underline{\eta} \in \mathbb{Z}^{n-1}$, we define the upper Cauchy transform for $m \in \mathbb{Z}^{n-1} \times \mathbb{Z}_+$ as

$$C^+[f](m) = -\sum_{\underline{\eta} \in \mathbb{Z}^{n-1}} \left[E_h^{-+}(\underline{\eta} - \underline{m}, -m_n)\mathbf{e}_n^+ f(\underline{\eta}, 1) \right. \\ \left. + E_h^{-+}(\underline{\eta} - \underline{m}, 1 - m_n)\mathbf{e}_n^- f(\underline{\eta}, 0) \right] h^{n-1}.$$

Definition (Lower Cauchy Transform)

For a discrete ℓ^p-function f, $1 \leq p < +\infty$, defined on the boundary layers $(\underline{\eta}, -1), (\underline{\eta}, 0), \underline{\eta} \in \mathbb{Z}^{n-1}$, we define the lower Cauchy transform for $m \in \mathbb{Z}^{n-1} \times \mathbb{Z}_-$ as

$$C^-[f](m) = \sum_{\underline{\eta} \in \mathbb{Z}^{n-1}} \left[E_h^{-+}(\underline{\eta} - \underline{m}, -1 - m_n)\mathbf{e}_n^+ f(\underline{\eta}, 0) + \right. \\ \left. + E_h^{-+}(\underline{\eta} - \underline{m}, -m_n)\mathbf{e}_n^- f(\underline{\eta}, -1) \right] h^{n-1}.$$

Plemelj-Sokhotzki Formulae

Due to the properties of H_+ and H_- we can introduce the projectors into the respective Hardy spaces

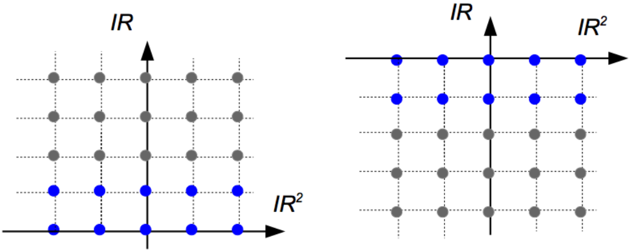
Definition

$$P_+ = \frac{1}{2}(I + H_+), \quad Q_+ = \frac{1}{2}(I - H_+).$$
$$P_- = \frac{1}{2}(I + H_-), \quad Q_- = \frac{1}{2}(I - H_-).$$

Lemma

$$f \in h_p^+ \quad \text{iff} \quad P_+ f = f; \qquad f \in h_p^- \quad \text{iff} \quad P_- f = f.$$

Boundary Data



Visualization of the upper / lower boundary data.

Space vs. Fourier domain: (-1)- and 1-layers

The boundary values equation (at the layer $m_n = 1$ (resp. $m_n = -1$) of a function which is discrete monogenic in the upper (resp., lower) half plane

$$\begin{cases} \sum_{\underline{\eta}\in\mathbb{Z}^{n-1}} \left[E_h^{-+}(\underline{\eta}-\underline{m},-1)e_n^+ f(\underline{\eta},1) + E_h^{-+}(\underline{\eta}-\underline{m},0)e_n^- f(\underline{\eta},0)\right] h^{n-1} = -f(\underline{m},1) \\ \sum_{\underline{\eta}\in\mathbb{Z}^{n-1}} \left[E_h^{-+}(\underline{\eta}-\underline{m},0)e_n^+ f(\underline{\eta},0) + E_h^{-+}(\underline{\eta}-\underline{m},1)e_n^- f(\underline{\eta},-1)\right] h^{n-1} = f(\underline{m},-1) \end{cases}$$

becomes

$$\begin{cases} \mathcal{F}_h E^{-+}(\underline{\xi},-1)e_n^+ \mathcal{F}_h f^+(\underline{\xi},1) + \mathcal{F}_h E^{-+}(\underline{\xi},0)e_n^- \mathcal{F}_h f^+(\underline{\xi},0) = -\mathcal{F}_h f^+(\underline{\xi},1) \\ \mathcal{F}_h E^{-+}(\underline{\xi},1)e_n^- \mathcal{F}_h f^-(\underline{\xi},-1) + \mathcal{F}_h E^{-+}(\underline{\xi},0)e_n^+ \mathcal{F}_h f^-(\underline{\xi},0) = \mathcal{F}_h f^-(\underline{\xi},-1) \end{cases}$$

Upper and lower boundary generators

Upper and lower boundary generators

Given $g \in \ell^p(h\mathbb{Z}^{n-1})$, we define the

(i) upper boundary generator $\mathcal{G}_+ : \ell^p(h\mathbb{Z}^{n-1}) \to \ell^p(h\mathbb{Z}^{n-1}) \times \ell^p(h\mathbb{Z}^{n-1})$ as

$$\mathcal{G}_+[g] := \left(\mathbf{e}_n^- \mathcal{A}_+[-\mathbf{e}_n^+ g], \mathbf{e}_n^+ g\right).$$

(ii) lower boundary generator $\mathcal{G}_- : \ell^p(h\mathbb{Z}^{n-1}) \to \ell^p(h\mathbb{Z}^{n-1}) \times \ell^p(h\mathbb{Z}^{n-1})$ as

$$\mathcal{G}_-[g] := \left(\mathbf{e}_n^+ \mathcal{A}_- g, \mathbf{e}_n^- g\right).$$

Upper and lower trace operators

Upper and lower trace operator

Given $f \in \ell_p(h\mathbb{Z}^n)$, we define the

i) upper trace operator $\mathrm{tr}_+ : \ell_p(h\mathbb{Z}^n) \to \ell_p(h\mathbb{Z}^{n-1}) \times \ell_p(h\mathbb{Z}^{n-1})$ as

$$\mathrm{tr}_+[f] := \left(e_n^- \mathcal{A}_+[-e_n^+ f^1], e_n^+ f^1 \right),$$

with $f^1(h\underline{m}) := f(h(\underline{m}, 1))$.

ii) lower trace operator $\mathrm{tr}_- : \ell_p(h\mathbb{Z}^n) \to \ell_p(h\mathbb{Z}^{n-1}) \times \ell_p(h\mathbb{Z}^{n-1})$ as

$$\mathrm{tr}_-[f] := \left(e_n^+ \mathcal{A}_-[f^{-1}], e_n^- f^{-1} \right),$$

with $f^{-1}(h\underline{m}) := f(h(\underline{m}, -1))$.

Lemma

Given $f \in \ell_p(h\mathbb{Z}^n)$, then

$$C^+\mathrm{tr}_+[C^+\mathrm{tr}_+[f]] = C^+\mathrm{tr}_+[f]; \qquad C^-\mathrm{tr}_-[C^-\mathrm{tr}_-[f]] = C^-\mathrm{tr}_-[f].$$

Solution of Problem I

Theorem

Problem I is uniquely solvable if and only if $g \in h_p^+$.
Moreover, its solution is given by

$$f = C^+[\mathcal{G}_+ g], \quad \text{in } \mathbb{Z}_+^n.$$

Idea of proof: $g \notin h_p^+$ implies it doesn't exist a discrete monogenic f such that its $1-$ layer fulfils the given boundary condition. So, $g \in h_p^+$.

Now, $f = C^+[\mathcal{G}_+ g]$ is a discrete monogenic function on the upper half lattice with $f(h(\underline{m}, 1)) = g(\underline{m})$ and uniqueness ensured by the maximum principle.

Jump problem

Problem II

Given $g \in \ell^p(h\mathbb{Z}^{n-1}), (1 \leq p < \infty)$ determine a discrete function $f : h\mathbb{Z}^n \to \mathbb{C}_n$ subjected to a jump condition,

$$\left\{ \begin{array}{ll} D_h^{-+} f(hm) = 0, & m \in \mathbb{Z}^n \setminus \{m_n = 0\} \\ \mathbf{e}_n^- f_+(h\underline{m}) - \mathbf{e}_n^+ f_-(h\underline{m}) = \mathbf{e}_n g(h\underline{m}), & \underline{m} \in \mathbb{Z}^{n-1} \end{array} \right. . \qquad (2)$$

Theorem

The Riemann boundary value problem (2) is uniquely solvable, and its solution is explicitly given by

$$f(hm) = \left\{ \begin{array}{ll} C^+ \mathcal{G}_+[g_+](hm) & , m_n \geq +1 \\ C^- \mathcal{G}_-[-g_-](hm) & , m_n \leq -1 \end{array} \right. ,$$

where $g_- := g^1 + \mathbf{e}_n^- g^3, \; g_+ := g^1 - g^4 + \mathbf{e}_n^+ g^2$.

Convergence Result - Discrete to Continuous

- $g \in L_p(\mathbb{R}^{n-1}, \mathbb{C}_n) \cap C^\alpha(\mathbb{R}^{n-1}, \mathbb{C}_n)$ $(0 < \alpha < 1)$, and $1 < p < \infty$, then we have

$$\|R_h g\|_{\ell^{p+\frac{n-1}{\alpha}}} \leq C\|g\|_{L_p};$$

- The continuous Cauchy transform

$$C_\Gamma f(y) = \int_{\mathbb{R}^{n-1}} E(x-y)(-\mathbf{e}_n)f(x)d\Gamma_x, \quad y = (\underline{y}, y_n) \in \mathbb{R}^n, y_n > 0,$$

with $E(x)$ being the fundamental solution to the Dirac operator $D = \sum_{j=1}^n \mathbf{e}_j \frac{\partial}{\partial_{x_j}}$

Theorem

Let $g \in H_p^+ \cap C^\alpha\left(\mathbb{R}^{n-1}, \mathbb{C}_n\right) \cap W_p^1\left(\mathbb{R}^{n-1}, \mathbb{C}_n\right)$, for $0 < \alpha \leq 1$, and $1 < p < n$. Then, the following estimate for the point-wise error between the discrete solution of (1) and its continuous counterpart holds:

$$\left|C_\Gamma[g](mh) - C^+\mathcal{G}_+[g](mh)\right| \leq \left(Ah^{n-1} + Bh\right)\|g\|_{L^p}, \tag{3}$$

for all $m \in \mathbb{Z}_+^n$, with $A, B > 0$ constants independent of h and g.

Remarks on the case of bounded domain

joint work with U. Kähler, A. Legatiuk, D. Legatiuk

Definition (convolution kernels)

$$H_i^+(mh) := R_h \mathcal{F}^{-1}_{\xi\mapsto(mh)_i}\left[\frac{\widetilde{\xi}_{-,i}}{\underline{d}}\left(\mathbf{e}_i^+ \frac{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}{2} + \mathbf{e}_i^- \frac{2}{h\underline{d}-\sqrt{4+h^2\underline{d}^2}}\right)\right]$$

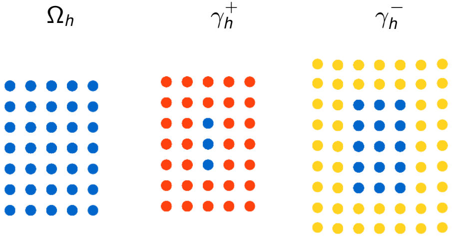
$$H_i^-(mh) := R_h \mathcal{F}^{-1}_{\xi\mapsto(mh)_i}\left[\frac{\widetilde{\xi}_{-,i}}{\underline{d}}\left(\mathbf{e}_i^+ \frac{2}{h\underline{d}-\sqrt{4+h^2\underline{d}^2}} + \mathbf{e}_i^- \frac{h\underline{d}-\sqrt{4-h^2\underline{d}^2}}{2}\right)\right]$$

Definition (Hilbert transforms)

$$H_+ f(mh) := \sum_{i=1}^3 \sum_{nh\in\gamma_{h,i}^+} H_i^+(nh-mh)f(nh)h^2, \qquad mh \in \gamma_h^+$$

$$H_- f(mh) = \sum_{i=1}^3 \sum_{nh\in\gamma_{h,i}^-} H_i^-(nh-mh)f(nh)h^2, \qquad mh \in \gamma_h^-.$$

Inner and outer boundaries of a cuboid domain



Original rectangular domain, its inner and outer boundaries

Practical application: Lena

Gray scale image “Lena”; (image 652×582 pixel, with computing time: ca. $0,130656$ sec)

Lena: discrete Riesz transforms

Riesz transforms ($x-$ and $y-$directions)

Lena: absolute value and phase angle

Absolute value and phase of the original image

References

- Rabinovich, Roch, Silbermann, Fredholm theory and finite section method for band-dominated operators, *Integ. Eq. and Op. Theory*, **30(4)** (1998), 452–495.
- Faustino, Kähler, Sommen, Discrete Dirac operators in Clifford analysis, *Adv. Appl. Cliff. Alg.* **17(3)** (2007), 451–467.
- C., Faustino, Vieira, Numerical Clifford analysis for nonlinear Schrödinger problem, *Numer. Methods Partial Differ. Equations* **24(4)** (2008), 1181–1202.
- C., Kähler, Ku, Sommen, Discrete Hardy spaces, *J. Fourier Anal. Appl.*, **20 (4)**, 2014, 715–750.
- C., Kähler, Ku, Discrete Riemann-Hilbert boundary value problems on half lattice, *J. Difference Equ. Appl.*, **21 (12)**, 2015, 1277–1304.
- L. N. A. Botchway, P. G. Kibiti, M. Ruzhansky, Difference equations and pseudo-differential operators on $\mathbb{Z}^n$, *ArXiv 2017*.

References

- J. Bory-Reyes, PC, A. Guzman-Adan, U. Kähler, *A Short Note on the Local Solvability of the Quaternionic Beltrami Equation*, Advances in Applied Clifford Algebras, **24** (2014), 945–053.
- M. Ruzhansky, V. Turunen, Pseudo-differential operators and symmetries: background analysis and advanced topics, Birkhäuser, Basel, 2010.
- V. Thomée, *Discrete interior Schauder estimates for elliptic difference operators*, SIAM J. Numer. Anal. **5** (1968), 626–645.
- PC, U Kähler, A. Legatiuk, D. Legatiuk, *Discrete Hardy Spaces for Bounded Domains in* $\mathbb{R}^n$, Complex Analysis and Operator Theory, **15 (4)**, (2021).

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