Approximate Global Convergence and Adaptivity for Inverse Problems With Simulated and Experimental Data

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Main results of the 2007-2011

- 1. L. Beilina and M.V. Klibanov, A globally convergent numerical method for a coefficient inverse problem, *SIAM J. Sci. Comp.*, 31, 478-509, 2008.
- 2. L. Beilina and M.V. Klibanov, Synthesis of global convergence and adaptivity for a hyperbolic coefficient inverse problem in 3D, *J. Inverse and Ill-posed Problems,* 18, 85-132, 2010.
- 3. L. Beilina and M.V. Klibanov, *A posteriori* error estimates for the adaptivity technique for the Tikhonov functional and global convergence for a coefficient inverse problem, *Inverse Problems,* 26, 045012, 2010.
- 4. M.V. Klibanov, M.A. Fiddy, L. Beilina, N. Pantong and J. Schenk, Picosecond scale experimental verification of a globally convergent numerical method for a coefficient inverse problem, *Inverse Problems*, 26, 045003, 2010.
- 5. L. Beilina and M.V.Klibanov, Reconstruction of dielectrics from experimental data via a hybrid globally convergent/adaptive inverse algorithm, *Inverse Problems*, 26, 125009, 2010.
- 6. L. Beilina, M.V. Klibanov and M.Yu. Kokurin, Adaptivity with relaxation for ill-posed problems and global convergence for a coefficient inverse problem, *Journal of Mathematical Sciences*, 167, 279-325, 2010.
- 7. M.V. Klibanov, A.B. Bakushinskii and L. Beilina, Why a minimizer of the Tikhonov functional is closer to the exact solution than the first guess? *J. Inverse and Ill-posed Problems,* accepted for publication, a preprint is available on-line at http://www.ma.utexas.edu/mp_arc/
- 8. A.V. Kuzhuget, L. Beilina, M.V. Klibanov, and V.G. Romanov, Global convergence and quasi-reversibility for a coefficient inverse problem with backscattering data, submitted for publication, a preprint is available on-line at http://www.ma.utexas.edu/mp_arc/
- 9. J. Xin, L.Beilina and M. V.Klibanov, Globally convergent numerical methods for coefficient inverse problems for imaging inhomogeneities, *Computing in Science and Engineering*, 12, 64-77, 2010.
- 10. M.Asadzadeh, L.Beilina, A posteriori error analysis in a globally convergent numerical method for a hyperbolic coefficient inverse problem, *Inverse Problems*, 26, 115007, 2010.
- 11. L.Beilina, M.V.Klibanov, A.Kuzhuget, New a posteriori error estimates for adaptivity technique and global convergence for a hyperbolic coefficient inverse problem, *Journal of Mathematical Sciences, JMS*, Springer, 2011.

Papers [4,5] are featured articles of *Inverse Problems* in 2010.

Preliminarities

- A Coefficient Inverse Problem (CIP) is a problem about the reconstruction of an unknown spatially dependent coefficient from boundary measurements.
- **Two Main Challenges:** nonlinearity and ill-posedness combined.
- The topic of the development of globally convergent numerical methods for CIPs is both one of most important and one of most challenging ones in the field of Inverse Problems.
- Least squares functionals feature multiple local minima and ravines.
- Thus, the vast majority of numerical methods for CIPs converge locally. Unsatisfactory.
- We have developed a new numerical method for some CIPs for the hyperbolic PDE $c(x) u_{tt} = \Delta u$.
- The key innovation is that this method does not need a good first guess for the solution.
- Only CIPs with single measurement data are considered: either a single source or a single incident plane wave.
- Non-local numerical methods were developed in the past only for the case of multiple measurements: Belishev, Kabanikhin, Isaacson, Mueller, Novikov, Siltanen.
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- Non-local numerical methods were developed in the past only for the case of multiple measurements: Belishev, Kabanikhin, Isaacson, Mueller, Novikov, Siltanen.

The First Central Question:

How to obtain a good approximation for the exact solution of a CIP without a priori knowledge of a small neighborhood of this solution?

The Second Central Question:

Given that first approximation, how to refine it?

THE TWO STAGE NUMERICAL PROCEDURE

Stage 1. Approximately globally convergent numerical method provides a good approximation for the exact solution
Stage 2. Adaptive Finite Element Method refines it

Our Definition of the Approximate Global Convergence Property

Definition (approximate global convergence)**.** Consider a nonlinear ill-posed problem. Suppose that a certain approximate mathematical model M_1 is proposed to solve this problem numerically. Assume that, within the framework of the model M_1, this problem has unique exact solution $x_1^* \in B$ for the noiseless data y^*. Here B is an appropriate Banach space with the norm $\|\cdot\|_B$. Consider an iterative numerical method for solving that problem. Suppose that this method produces a sequence of points $\{x_n\}_{n=1}^{N} \subset B, N \in [1, \infty)$. Let the number $\theta \in (0, 1)$. We call this numerical method *approximately globally convergent of the level* θ, or shortly *globally convergent*, if, within the framework of the approximate model M_1, a theorem is proven, which claims that, without any knowledge of a sufficiently small neighborhood of x^*, there exists a number $\overline{N} \in [1, N)$ such that the following inequality is valid

$$\frac{\|x_n - x_1^*\|_B}{\|x_n\|_B} \leq \theta, \forall n \geq \overline{N}.$$

Suppose that iterations are stopped at a certain number $k \geq \overline{N}$. Then the point x_k is denoted as $x_k := x_{glob}$ and is called "the approximate solution resulting from this method".

Two Additional Informal Conditions:

A. Numerical studies confirm that x_{glob} is indeed a sufficiently good approximation for the exact solution x^*.
B (optional). Testing of this numerical method on appropriate experimental data also demonstrates that iterative solutions provide a good approximation for the exact one.

$$c(x)\,u_{tt} = \Delta u, x \in \mathbb{R}^n, n = 2, 3. \qquad (1)$$

- Similarly for
$$c(x)\,v_t = \Delta v - a(x)\,v.$$
- 1. A globally convergent numerical method for an $n - D$ ($n = 2, 3$) CIP for the PDE (1) with single measurement data is developed.
- 2. This method is extended to the case of backscattering data.
- 3. The Adaptive Finite Element Method for Ill-Posed Problems is put in the framework of Functional Analysis.

Summary of results in 2007-2010

- 4. New *a posteriori* error estimates are obtained for the originating Tikhonov functional rather than for the previously used Lagrangian.
- 5. An adaptive globally convergent method for (1) is developed.
- 6. The two-stage numerical procedure is developed and its global convergence is rigorously established:
  - The first stage: the globally convergent method of item 1 provides a good approximation for the exact solution
  - The second stage: this approximation is refined via the adaptivity.
- 7. The first stage is verified on **blind** experimental data with a narrow observation angle.
- 8. The approximately globally convergent method is verified on **blind backscattering** experimental data measured in the field.
- Excellent accuracy of the reconstruction of target/background ratios of refractive indices.

$$c(x)\,u_{tt} = \Delta u \text{ in } \mathbb{R}^3 \times (0,\infty)\,,2 \tag{2}$$

$$u(x,0) = 0, u_t(x,0) = \delta(x - x_0)\,.3 \tag{3}$$

$$c(x) \in [1,d]\,, d = const. > 1, c(x) = 1 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega, 4 \tag{4}$$

$$c(x) \in C^2\left(\mathbb{R}^3\right).5 \tag{5}$$

- Applications: electromagnetic, acoustics

Inverse Problem 1 (complete data).

Suppose that the coefficient $c(x)$ satisfies (4) and (5), where the number $d > 1$ is given. Assume that the function $c(x)$ is unknown in the domain Ω. Determine the function $c(x)$ for $x \in \Omega$, assuming that the following function $g(x,t)$ is known for a single source position $x_0 \notin \overline{\Omega}$

$$u(x,t) = g(x,t), \forall (x,t) \in \partial\Omega \times (0,\infty). \tag{6}$$

- The function $g(x,t)$ in (6) is the result of measurement at the entire boundary.
- Uniqueness is known only if $\delta(x - x_0)$ is replaced with $f(x) \neq 0$ in $\overline{\Omega}$ (1981, the Bukhgeim-Klibanov method of Carleman estimates).
- Uniqueness theorem is a long standing open problem.
- We assume below that uniqueness holds true.

$$w(x,s) = \int\limits_0^\infty u(x,t)e^{-st}dt, \text{ for } s > \underline{s} = const. > 0,$$

$$\begin{aligned} \Delta w - s^2 c(x) w &= -\delta(x - x_0), x \in \mathbb{R}^3, \\ \lim_{|x| \to \infty} w(x,s) &= 0. \end{aligned}$$

Lemma 1 (follows from a result of V.G. Romanov, 1984)

Suppose that the function $c(x)$ satisfies conditions (4), (5). Assume that geodesic lines, generated by the eikonal equation corresponding to the function $c(x)$ are regular. Then the following asymptotic behavior of the function w and its derivatives takes place for $|\beta| \leq 2, \gamma = 0, 1, x \neq x_0$

$$D_x^\beta D_s^\gamma w(x,s) = D_x^\beta D_s^\gamma \left\{ \frac{\exp\left[-sl\left(x,x_0\right)\right]}{f\left(x,x_0\right)} \left[1 + O\left(\frac{1}{s}\right)\right] \right\}, s \to \infty,$$

where $f(x,x_0)$ is a certain function and $f(x,x_0) \neq 0$ for $x \neq x_0$.

- Conditions of regularity of geodesic lines cannot be effectively verified. However, the entire theory of CIPs for hyperbolic PDEs does not work without this condition.
- We verify the asymptotic behavior computationally.

$$w(x,s) > 0, x \in \overline{\Omega}, s >> 1.$$

$$\begin{aligned} v(x,s) &= \frac{\ln w(x,s)}{s^2}, \\ \Delta v + s^2 |\nabla v|^2 &= c(x), \\ q(x,s) &= \frac{\partial v(x,s)}{\partial s}. \end{aligned}$$

$$\Delta q - 2s^2 \nabla q \cdot \int_s^{\overline{s}} \nabla q(x,\tau)\, d\tau + 2s \left[\int_s^{\overline{s}} \nabla q(x,\tau)\, d\tau \right]^2$$

$$+ 2s^2 \nabla q \nabla V - 2s \nabla V \cdot \int_s^{\overline{s}} \nabla q(x,\tau)\, d\tau + 2s (\nabla V)^2 = 0,$$

$$q \mid_{\Omega} = \psi(x,s), \ \ (x,s) \in \partial\Omega \times [\underline{s}, \overline{s}].$$

- $V(x,\overline{s})$ is the "tail" function. It complements the rest of the integral, i.e.

$$\frac{\ln w(x,s)}{s^2} = -\int_s^{\overline{s}} q(x,\tau)\,d\tau + V(x,\overline{s}),$$

$$V(x,\overline{s}) = \frac{\ln w(x,\overline{s})}{\overline{s}^2}.$$

- Both functions q and V are unknown. We find q via inner iterations and V via outer iterations.

$$\begin{aligned} \underline{s} &= s_N < s_{N-1} < ... < s_1 = \overline{s}, h = s_{n-1} - s_n, \\ q(x,s) &= q_n(x) \text{ for } s \in (s_n, s_{n-1}]. \end{aligned}$$

- Multiply equation (7) by Carleman Weight Function (CWF) $e^{\mu(s-s_{n-1})}$, $\mu >> 1$.

- Next integrate the resulting equation with respect to $s \in (s_n, s_{n-1}]$. We obtain

$$L_n(q_n) := \Delta q_n - A_{1,n}\left(h\sum_{i=1}^{n-1}\nabla q_i\right)\nabla q_n + A_{1n}\nabla q_n \nabla V_n - \varkappa q_n$$

$$= B_n(\nabla q_n)^2 - A_{2,n}h^2\left(\sum_{i=1}^{n-1}\nabla q_i(x)\right)^2$$

$$+ 2A_{2,n}\nabla V_n\left(h\sum_{i=1}^{n-1}\nabla q_i\right) - A_{2,n}(\nabla V_n)^2,$$

$$q_n\mid_{\partial\Omega} = \psi_n(x), n = 1, ..., N.$$

- Important:

$$B_n = O\left(\frac{1}{\mu}\right), \mu \to \infty.$$

We solve Dirichlet boundary value problems (8) sequentially starting from q_1.

$$q_0 := 0, q_{1,1}^0 := 0, V_{1,1}(x) := V_{1,1}^0(x).$$

- The choice of the initial function $V_{1,1}(x)$ is described below
- **Step** $n_i, n \geq 1, i \geq 1$. Suppose that functions $q_1, ..., q_{n-1}, q_{n,0} := q_{n-1} \in C^{2+\alpha}(\overline{\Omega}), c_{n-1} \in C^{\alpha}(\overline{\Omega})$ and the tail function $V_{n,1}(x, \overline{s}) \in C^{2+\alpha}(\overline{\Omega})$ are constructed. We now construct the function $q_{n,i}$.

- Iterate with respect to the tails via solving the Dirichlet boundary value problems

$$
\begin{aligned}
L\left(q_{n,i}\right) &:= \Delta q_{n,i} - A_{1n}\left(h\sum_{j=0}^{n-1}\nabla q_j\right)\cdot\nabla q_{n,i} + A_{1n}\nabla q_{n,i}\cdot\nabla V_{n,i} \\
&= B_n\left(\nabla q_{n,i-1}\right)^2 - A_{2n}h^2\left(\sum_{j=0}^{n-1}\nabla q_j\left(x\right)\right)^2 \\
&+ 2A_{2n}\nabla V_{n,i}\cdot\left(h\sum_{j=0}^{n-1}\nabla q_j\left(x\right)\right) - A_{2n}\left(\nabla V_{n,i}\right)^2,
\end{aligned}
\tag{7}
$$

$$q_{n,i}\left(x\right) = \psi_n\left(x\right), x \in \partial\Omega.$$

- Given $q_{n,i}$, reconstruct the next approximation $c_{n,i} \in C^{\alpha}(\overline{\Omega})$ for the target coefficient

$$\begin{aligned} H_{n,i}(x) &= -hq_{n,i} - h\sum_{j=0}^{n-1} q_j(x) + V_{n,i}(x), x \in \Omega, \\ c_{n,i}(x) &= \Delta H_{n,i} + s_n^2 (\nabla H_{n,i})^2 . \end{aligned}$$

- Extend $c_{n,i}(x)$ in $\mathbb{R}^3 \diagdown \Omega$ as $c_{n,i}(x) := 1$ in $\mathbb{R}^3 \diagdown \Omega$. It is possible to arrange such an extension that $\widehat{c}_{n,i} \in C^{\alpha}(\mathbb{R}^3)$.
- Solve the original hyperbolic Cauchy problem (2), (3) with $c := \widehat{c}_{n,i}$.

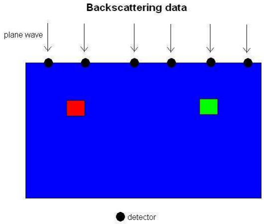
- Calculate the Laplace transform of that solution and update the tail as

$$V_{n,i+1}(x,\overline{s}) = \frac{1}{\overline{s}^2} \ln w_{n,i}(x,\overline{s}) \in C^{2+\alpha}\left(\overline{\Omega}\right).$$

- We iterate with respect to i until convergence occurs at the step $i := m_n$. Then we set

$$\begin{aligned} q_n &: = q_{n,m_n} \in C^{2+\alpha}\left(\overline{\Omega}\right), c_n := c_{n,m_n} \in C^{\alpha}\left(\overline{\Omega}\right), \\ V_{n+1,1}(x,\overline{s}) &= \frac{1}{\overline{s}^2} \ln w_{n,m_n}(x,\overline{s}) \in C^{2+\alpha}\left(\overline{\Omega}\right). \end{aligned}$$

Inverse Problem 2: the backscattering data



$$
\begin{aligned}
\Omega &= (-B, B) \times (0, 2B), \partial\Omega = \cup_{k=1}^{4} \Gamma_i, \\
\Gamma_1 &= \partial\Omega \cap \{z = 0\}, \Gamma_2 = \partial\Omega \cap \{x = B\}, \\
\Gamma_3 &= \partial\Omega \cap \{x = -B\}, \Gamma_4 = \partial\Omega \cap \{z = 2B\}.
\end{aligned}
$$

$$u \mid_{\Gamma} = g_0 (x, t), \partial_n u \mid_{\Gamma} = g_1 (x, t).$$

Hence,

$$w\mid_{\Gamma}=\widetilde{g}_0\left(x,s\right),\partial_n w\mid_{\Gamma}=\widetilde{g}_1\left(x,s\right).$$

Hence,

$$q_n\mid_{\Gamma}=\varphi_n\left(x\right),\partial_n q_n\mid_{\Gamma}=\eta_n\left(x\right). \tag{8}$$

The radiation condition gives

$$\partial_n q_n\mid_{\Gamma_k}=\frac{1}{s_{n-1}s_n},k=2,3,4. \tag{9}$$

- Complete data: the FEM. Backscattering data: The Quasi-Reversibility Method (QRM)
- Solve the problem (7)-(9).

- Let $L(q_{n,i})$ and $H_{n,i}$ be the left and right hand sides of (7) respectively. Minimize the Tikhonov functional J_α,

$$J_\alpha(q_{n,i}) = \|L(q_{n,i}) - H_{n,i}\|^2_{L_2(\Omega)} + \alpha \|q_{n,i}\|^2_{H^5(\Omega)}.$$

- The $H^5(\Omega)-$ norm is used for the theory only. In practice we use $H^2(\Omega)$.
- No local minima, since J_α is a quadratic functional.
- *A priori* estimate for the QRM is proven via a Carleman estimate.

THE APPROXIMATE GLOBAL CONVERGENCE THEOREM

- The exact solution $c^*(x)$ satisfies (4), (5).
- The corresponding function $w^*(x,s)$ is the Laplace transform of the exact solution of the forward problem (2), (3).

$$
\begin{aligned}
v^*(x,s) &= \frac{\ln w^*(x,s)}{s^2}, q^*(x,s) = \frac{\partial v^*(x,s)}{\partial s}, \\
q^*(x,s) &\mid_{\partial\Omega} = \psi^*(x,s), \\
q_n^*(x) &= \frac{1}{h}\int_{s_n}^{s_n} q^*(x,s)\,ds, \psi_n^*(x) = \frac{1}{h}\int_{s_n}^{s_n} \psi^*(x,s)\,ds,
\end{aligned}
$$

$$
\|q_n^*(x) - q^*(x,s)\|_{C^{2+\alpha}(\overline{\Omega})\times C[s_n,s_{n-1}]} + \|\psi_n^*(x) - \psi^*(x,s)\|_{C^{2+\alpha}(\partial\Omega)\times C[s_n,s_{n-1}
$$

- σ is the level of the error in the data

The asymptotic behavior is

$$\frac{\ln w\left(x,s\right)}{s^2} := v\left(x,s\right) = \frac{p\left(x\right)}{s} + O\left(\frac{1}{s^2}\right), s \to \infty.$$

- $\overline{s}$ is the truncation pseudo frequency

Assumption 1.

$$V^*(x,\bar{s}) = \frac{p^*(x)}{\bar{s}} = \frac{\ln w^*(x,\bar{s})}{\bar{s}^2}.$$

Assumption 2. Consider $q^*(x,\bar{s}) = \partial_s \left(V^*(x,s)\right)|_{s=\bar{s}}$. Then

$$q^*(x,\bar{s}) = -\frac{p^*(x)}{\bar{s}^2}.$$

The first guess for tail

$$V_{1,1}(x) = \frac{p(x)}{\overline{s}}.$$

Here

$$\Delta p = 0,$$

$$p \mid_{\partial\Omega} = -\overline{s}^2 \psi(x, \overline{s}),$$

$$\Delta p^* = 0,$$

$$p^* \mid_{\partial\Omega} = -\overline{s}^2 \psi^*(x, \overline{s}).$$

Conclusion: The error in the first tail is determined only by the error in the data ψ, since

$$\|p - p^*\|_{C^{2+\alpha}(\overline{\Omega})} \leq C\overline{s}^2 \|\psi(x, \overline{s}) - \psi^*(x, \overline{s})\|_{C^{2+\alpha}(\partial\Omega)}.$$

- This truncation is similar with, e.g. geometrical optics,

$$\Delta u + k^2 a(x) u = 0, a \geq const. > 0$$

- The truncation in the WKB method is

$$u = e^{ikS(x)} \left(1 + O\left(\frac{1}{k}\right)\right) \approx e^{ikS(x)}, k \to \infty.$$

- The entire X-ray Computer Tomography is based on geometrical optics and works very well in hospitals.
- The diffraction optics is based on the Huygens-Fresnel theory which cannot be derived from the Maxwell equations rigorously.
  Introduce the class of functions $P(d, d^*)$,

$$P(d, d^*) = \left\{ c \in C^{\alpha}\left(\overline{\Omega}\right) : |c|_{\alpha} \leq d^* + \frac{1}{2}, c \in \left[1, d + \frac{1}{2}\right] \right\}.$$

Approximate Global Convergence Theorem

1 (rough formulation) *Let* $\Omega, \Omega_1 \subset \mathbb{R}^3$ *be a bounded domain with* $\partial\Omega \in C^3$. *Let the source* $x_0 \notin \overline{\Omega}$. *Let* σ *be the error in the boundary data,*

$$\|\psi_n - \psi_n^*\|_{C^{2+\alpha}(\partial\Omega)} \leq C^*\sigma.$$

Consider the error parameter

$$\eta = 2\left(h + \sigma\right).$$

Let m *be the maximal number of functions* $q_{n,i}$ *for each* n *and* N *be the maximal number of functions* q_n :

$$\left\{q_{n,i}\right\}_{(i,n)=(1,1)}^{(m,N)}.$$

Let the exact solution

$$c^* \in P\left(d, d^*\right), c^* \in C^2\left(\mathbb{R}^3\right)$$

$$c^* \geq 1 \text{ in } \mathbb{R}^3, c^* = 1 \text{ in } \mathbb{R}^3 \diagdown \Omega.$$

Assume that all reconstructed functions $c_{n,i}(x) \geq 1$ *in* Ω. *Let* $\omega \in (0,1)$ *be a number.*

Then there exists a constant $B = B(\Omega, \overline{s}, d, d^*) > 2$ *such that if the error* η *is so small that*

$$\eta \in \left(0, \frac{1}{B^{3Nm/\omega}}\right),$$

then all functions $c_{n,i} \in P(d, d^*) \cap C^{\alpha}\left(\overline{\Omega}\right)$ *and*

$$\frac{\left\| c_{n,i} - c^* \right\|_{C^{\alpha}\left(\overline{\Omega}\right)}}{\left\| c_{n,i} \right\|_{C^{\alpha}\left(\overline{\Omega}\right)}} \leq \eta^{1-\omega} := \theta \in (0,1).$$

- The **DECISIVE ADVANTAGE** of this theorem is that it guarantees to provide a good approximation for the exact solution c^* without an a priori knowledge of a small neighborhood of c^*.
- Of course, this guarantee is valid only within the framework of the above approximate mathematical model.

THE REFINEMENT STAGE: THE ADAPTIVITY

QUESTION: Why a regularized solution of the Tikhonov functional is usually more accurate than the first guess for the practical case of a single value of the regularization parameter? Indeed, the theory guarantees this only in the limiting case.

- Let B_1 and B_2 be two Banah spaces.
- Let Q be another space, $Q \subset B_1$ as a set and $\overline{Q} = B_1$, Also assume that Q is compactly embedded in B_1.
- Let $G \subseteq Q$ be a convex set with interior points, closed in the norm of B_1.
- A continuous one-to-one operator $F : G \to B_2$. The continuity means here in terms of the pair of spaces B_1 and B_2.

$$F(x) = y, x \in G. \tag{6}$$

- Let y^* be the ideal noiseless data corresponding to the ideal solution x^*,

- The Tikhonov regularization functional $J_\alpha : G \to \mathbb{R}$,

$$J_\alpha(x) = \frac{1}{2}\|F(x) - y\|_{B_2}^2 + \frac{\alpha}{2}\|x - x_0\|_Q^2, x_0 \in G, \tag{12}$$

$$\alpha = \alpha(\delta) = \delta^{2\mu}, \mu \in (0,1). \tag{13}$$

Theorem 2. *Let $B_1, Q \subset B_1$ and B_2 be above Banach spaces. Let $G \subseteq Q$ be a set which is closed in the B_1-norm and $F : G \rightarrow B_2$ be a one-to-one operator, continuous in terms of norms $\|\cdot\|_{B_1}, \|\cdot\|_{B_2}$. Consider the problem of solution of equation (10). Let y^* be the ideal noiseless right hand side of (18) and x^* be the corresponding exact solution of equation (11). Consider the Tikhonov functional (12) and assume that (13) holds. Let*

$$m(\delta) = \inf_G J_\alpha(x)$$

and $\{x_n\}_{n=1}^{\infty} \subset G$ be a minimizing sequence,
$\lim_{n \to \infty} J_\alpha(x_n) = m(\delta)$.
Then there exists a sufficiently small number $\delta_0 = \delta_0(\xi) \in (0, 1)$ such that for all $\delta \in (0, \delta_0)$ the following inequality holds

$$\|x_n - x^*\|_{B_1} \leq \xi \|x_0 - x^*\|_Q, \forall n. \tag{14}$$

In particular, if $\dim B_1 < \infty$, *then all norms in* B_1 *are equivalent and there exists a regularized solution* $x_{\alpha(\delta)}$ *such that* $m(\delta) = J_{\alpha(\delta)}\left(x_{\alpha(\delta)}\right)$. *In this case we set* $Q = B_1$. *Then (14) becomes*

$$\left\|x_{\alpha(\delta)} - x^*\right\|_{B_1} \leq \xi \left\|x_0 - x^*\right\|_{B_1}.$$

- **Conclusion:** For a given pair $(\delta, \alpha(\delta))$, the accuracy of the regularized solution is completely defined by the accuracy of the first guess x_0. Also, a regularized solution is more accurate than the first guess.
- The local strong convexity of the Tikhonov functional. Let $V_1(x^*) = \{x : \|x - x^*\| < 1\}$.

Theorem 3. *Let $B_1 := H_1$ and $B_1 := H_2$ be two Hilbert spaces, $\dim H_1 < \infty, G \subseteq H_1$ be a convex compact set and $F : G \to H_2$ be a continuous one-to-one operator. Let $x^* \in G$ be the exact solution of equation (12), $\delta \in (0,1)$ be the error in the data, let (11) be satisfied and $V_1(x^*) \subset G$. Assume that for every $x \in V_1(x^*)$ the operator F has the Frechét derivative $F'(x) \in \mathcal{L}(H_1, H_2)$ this derivative is uniformly bounded in $V_1(x^*)$,*

$$\|F'(x)\| \leq N_1, \forall x \in V_1(x^*)$$

and Lipschitz continuous, i.e.

$$\|F'(x) - F'(z)\| \leq N_2 \|x - z\|, \forall x, z \in V_1(x^*),$$

where $N_1, N_2 = const. > 0$. Assume that

$$\alpha = \alpha(\delta) = \delta^{2\mu}, \forall \delta \in (0,1),$$

$$\mu = const. \in \left(0, \frac{1}{4}\right).$$

Then there exists a sufficiently small number $\delta_0 = \delta_0(N_1, N_2, \beta, \mu) \in (0,1)$ such that for all $\delta \in (0, \delta_0)$ the functional $J_{\alpha(\delta)}(x)$ is strongly convex in the neighborhood $V_{\alpha(\delta)}(x^*)$ of the exact solution x^* with the strong convexity constant $\alpha/2$.

Furthermore, let the first guess x_0 for the exact solution x^* be so accurate that

$$\|x^* - x_0\| < \frac{\delta^{3\mu}}{3}.$$

Then there exists the unique regularized solution $x_{\alpha(\delta)}$ of equation (10) and $x_{\alpha(\delta)} \in V_{\delta^{3\mu}/3}(x^*)$. In addition, the gradient method of the minimization of the functional $J_{\alpha(\delta)}(x)$, which starts at x_0, converges to $x_{\alpha(\delta)}$.
Furthermore, let $\xi \in (0,1)$ be an arbitrary number. Then there exists a number $\delta_1 = \delta_1(\xi, \delta_0) \in (0, \delta_0]$ such that for any $\delta \in (0, \delta_1)$

$$\left\|x_{\alpha(\delta)} - x^*\right\| \leq \xi \|x_0 - x^*\|, \ \forall \delta \in (0, \delta_1).$$

THE FRAMEWORK OF THE FUNCTIONAL ANALYSIS FOR THE ADAPTIVITY TECHNIQUE FOR ILL-POSED PROBLEMS

- Only standard piecewise linear finite elements are considered
- Let a bounded domain $\Omega \subset \mathbb{R}^n, n = 2, 3.$
- Let T_0 be a triangulation of Ω. Let $\Phi \subseteq \Omega$ be the corresponding polygonal subdomain
- All other triangulations are obtained via the standard mesh refinements with

$$a_1 \leq h_{\min} \leq r_{\max} a_2, a_1, a_2 = const. > 0.$$

Here $h_{\min}$ is the minimal grid step size of triangulations and $r_{\max}$ is the maximal radius of a circle/sphere contained in a corresponding triangle/tetrahedron. In other words, triangulations are regular ones.

$\cdot B(T)$ is the basis in the linear space of finite elements corresponding to the triangulation T,

$$\begin{aligned} H_1 &= \cup_T Span(B(T)), \\ \dim H_1 &< \infty, \\ H_1 &\subset H^1(\Phi), f_{x_i} \in L_\infty(\Omega), \forall f \in H_1. \\ \text{Set } \|\cdot\|_{H_1} &: = \|\cdot\|_{L_2(\Phi)} := \|\cdot\|. \end{aligned}$$

- We can construct a finite sequence of subspaces $\{M_n\}_{n=1}^N \subset H_1, M_n \subset M_{n+1}$. M_{n+1} is obtained from M_n via a mesh refinement.
- Let $G \subseteq H_1$ be a closed convex set.

- A continuous one-to-one operator $F : G \to H_2$. Consider the equation

$$F(x) = y, x \in G. \tag{15}$$

- Let y^* be the ideal noiseless data corresponding to the ideal solution x^*,

$$F(x^*) = y^*, \|y - y^*\|_{H_2} < \delta. \tag{16}$$

- The Tikhonov regularization functional $J_\alpha : G \to \mathbb{R}$,

$$J_\alpha(x) = \frac{1}{2}\|F(x) - y\|_{H_2}^2 + \frac{\alpha}{2}\|x - x_0\|^2, x_0 \in G. \tag{17}$$

• Let conditions of Theorem 3 (about local strong convexity) be satisfied. Then there exists unique minimizer $x_{\alpha(\delta)} \in V_{\alpha(\delta)}(x^*)$ of the functional (17).

THE ADAPTIVITY PROBLEM

Approximate the regularized solution on a sequence of subspaces $\{M_n\}_{n=1}^{N} \subset H_1, M_n \subset M_{n+1}$ *obtained one from another one via a local mesh refinement.*

- Minimize the Tikhonov functional on a sequence of subspaces

$$\min_{V_{\alpha(\delta)}(x^*)\cap M_n} J_\alpha(x).$$

- Does a minimizer on $V_{\xi\varepsilon(\delta)}(x^*) \cap M_n$ exist? Yes!
- Relaxation property of mesh refinements

Theorem 4. *Let conditions of theorem 3 be satisfied. Let $P_n : H_1 \to M_n$ be the orthogonal projection operator on the subspace M_n. Assume that*

$$x_{\alpha(\delta)} \neq P_n x_{\alpha(\delta)}$$

(otherwise, the regularized solution $x_{\alpha(\delta)}$ is found and belongs to the subspace M_n). Then there

$$\|x_n - x_\alpha\| \leq \theta_n \|x_{n-1} - x_\alpha\|, \theta_n \in (0,1).$$

$$
\begin{aligned}
c(x)\, u_{tt} &= \Delta u \text{ in } \mathbb{R}^3 \times (0, \infty), \\
u(x, 0) &= 0, u_t(x, 0) = \delta(x - x_0).
\end{aligned}
$$

$$
\begin{aligned}
c(x) &\in [1, d], d = const. > 1, c(x) = 1 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega, \\
c(x) &\in C^2\left(\mathbb{R}^3\right).
\end{aligned}
$$

$$
u(x, t) = g(x, t), \forall (x, t) \in \partial\Omega \times (0, \infty).
$$

- Uniquely solve the initial boundary value problem (18)-(22) in $\left(\mathbb{R}^3 \diagdown \Omega\right) \times (0, T)$.

$$Q_T = \Omega \times (0, T), S_T = \partial\Omega \times (0, T).$$

- We obtain that the following function $p(x, t)$ is known

$$\partial_n u \mid_{S_T} = p(x, t).$$

- A simplification is necessary because of an insufficient smoothness of the function u.

- Replace $\delta\left(x-x_0\right)$ with

$$\delta_\varepsilon\left(x-x_0\right)=\left\{\begin{array}{c} C_\varepsilon \exp\left(\frac{1}{\left|x-x_0\right|^2-\varepsilon^2}\right), \left|x-x_0\right|<\varepsilon, \\ 0, \left|x-x_0\right|\geq\varepsilon. \end{array}\right., \quad \int\limits_{\mathbb{R}^3}\delta_\varepsilon\left(x-x_0\right)dx=1$$

- Solutions of state and adjoint problems $u(x,t,c), \lambda(x,t,c) \in H^4(Q_T)$, where

$$\begin{aligned} c(x)\, u_{tt} &= \Delta u \text{ in } Q_T, \\ u(x,0) &= u_t(x,0) = 0, \\ \partial_n u &\mid_{S_T} = p(x,t). \end{aligned}$$

$$\begin{aligned} c(x)\, \lambda_{tt} &= \Delta \lambda \text{ in } Q_T, \\ \lambda(x,T) &= \lambda_t(x,T) = 0, \\ \partial_n \lambda &\mid_{S_T} = (g-u)\mid_{S_T}. \end{aligned}$$

- The operator $F: Y \to L_2(S_T)$,

$$F(c) := [g(x,t) - u(x,t,c)]\mid_{S_T}.$$

- Thus, we need to solve the equation

$$\begin{aligned} F(c) &= 0, c \in Y. \\ \|F(c^*)\|_{L_2(S_T)} &< \delta. \end{aligned}$$

- This operator has the Frechét derivative $F'(c)$ and this derivative is Lipschitz continuous.
- Therefore, Theorems 2,3 are applicable to the Tikhonov functional $E : Y \to \mathbb{R}, Y \subset L_2(\Omega)$,

$$E(c) = \frac{1}{2} \int\limits_{S_T} (u\mid_{S_T} - g)^2 \, dS + \frac{\alpha}{2} \int\limits_{\Omega} (c - c_{glob})^2 \, dx.$$

Theorem 5 (rough formulation). *The functional $E(c)$ has the Frechet derivative $E'(c) \in C\left(\overline{\Omega}\right)$ for each $c \in Y$ and*

$$E'(c)\,(x) = \alpha\,(c - c_{glob})\,(x) - \int\limits_0^T (u_t \lambda_t)\,(x,t)\,dt, x \in \Omega. \quad (18)$$

- *A posteriori* error estimates help us to understand how we can refine solutions.

Theorem 6 (rough formulation). *Let H_1 be the above space of finite elements with $\dim H_1 < \infty$. Let c_α be the regularized solution. Let $c_n \in V_{\alpha(\delta)}(c^*)$ be the minimizer of the functional $E(c)$ on a subspace M_n with its maximal grid step size h_n. Then the following approximate a posteriori error estimate holds*

$$\|c_n - c_\alpha\|_{L_2(\sigma)} \leq \frac{2}{\delta^{2\mu}} \left\|E'(c_n)\right\|_{L_2(\sigma)} \leq \frac{C_1}{\delta^{2\mu}} m_n^2 \exp\left(CM_n T\right),$$

where $m_n = \|c_n\|_{C(\overline{\sigma})}, M_n = \|\nabla c_n\|_{L_\infty(\sigma)}.$

- *A posteriori* error estimate of Theorem 6 implies the following two mesh refinement recommendations:

  The First Mesh Refinement Recommendation. *Refine the mesh in neighborhoods of those points* $x \in \sigma \subseteq \Omega$ *where the function* $|E'(c_n)(x)|$ *defined in (18) attains its maximal values. More precisely, let* $\varkappa \in (0,1)$ *be the tolerance number which should be chosen in computational experiments. Refine the mesh in such subdomains of* Ω *where*

$$\left|E'(c_n)(x)\right| \geq \varkappa \max_{\overline{\sigma}} \left|E'(c_n)(x)\right|.$$

The Second Mesh Refinement Recommendation. *Refine the mesh in neighborhoods of those points* $x \in \sigma \subseteq \Omega$ *where the function* $|c_n(x)|$ *attains its maximal values. More, precisely in such subdomains of* Ω *where*

$$|c_n(x)| \geq \widetilde{\varkappa} \max_{\overline{\sigma}} |c_n(x)| .$$

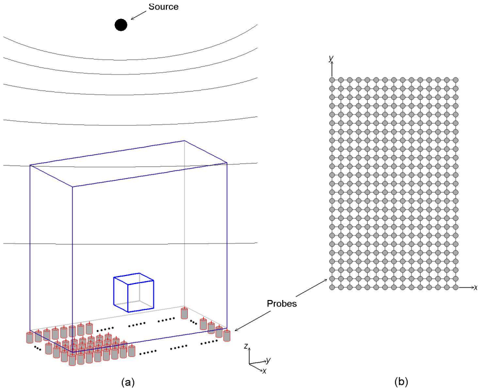
VERIFICATIONS OF OUR METHOD FOR BLIND EXPERIMENTAL DATA

- The main challenge of working with time dependent experimental data is a huge discrepancy between these data and computationally simulated ones.
- Conventional data denoising techniques, such as, e.g. Fourier transform, Hilbert transform, spline interpolation and others provide only a very insignificant help.
- Therefore, radically new data pre-processing procedures were invented.
- The main idea is to **immerse** the experimental data in computationally simulated data.
- These procedures cannot be justified neither by physics nor by mathematics. They are based on the intuition only.
- The single justification for these procedures is a surprisingly excellent accuracy of our reconstruction results

$$\varepsilon_r(x)\, u_{tt} = \Delta u.$$

- $\varepsilon_r(x)$ is the relative dielectric permittivity
- $n(x) = \sqrt{\varepsilon_r(x)}$ is the refractive index
- It is $n(x)$ which can be directly measured in experiments. $\varepsilon_r(x)$ cannot be measured.

The experimental setup

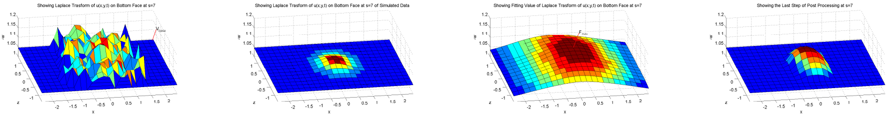


a) The rectangular prism depicts our computational domain Ω. Only a single source location outside of this prism was used. Tomographic measurements of the scattered time resolved EM wave were conducted on the bottom side of this prism. The signal was measured with the time interval 20 picoseconds with total time 12.3 nanoseconds. b) Schematic diagram of locations of detectors on the bottom side of the prism Ω. The distance between neighboring detectors was 10 mm.

Table 1. Accuracy of blindly computed refractive indices $n = \sqrt{\varepsilon_r}$ of dielectric abnormalities from experimental data

Case	Comput. n	Measured n, error	Comput. Error
1	2.16	2.17, 6%	0.5%
2	2	2.17, 6%	7.8%
3	2.16	2.17, 6%	0.5%
4	2.19	2.17, 6%	1%
5	1.73	1.78, 6%	2.8%
6	1.79	1.78, 6%	0.56%

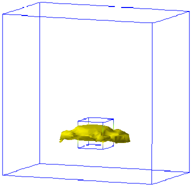
Data immersing procedure in the Laplace Domain for the globally convergent algorithm



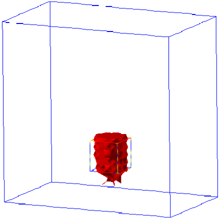
a) b) c) d)

Let $w_{incl}(x,s)$ be the Laplace transform of the pre-processed data in time domain with inclusion present and $\widetilde{w}_{incl}(x,s) = -(\ln w_{incl}(x,s))/s^2$. a) The function $\widetilde{w}_{incl}(x,\overline{s})$, $\overline{s} = 7.5$. b) The function $-(\ln w_{sim}(x,\overline{s}))/\overline{s}^2$ is depicted, where $w_{sim}(x,\overline{s})$ is the Laplace transform of the function $u_{sim}(x,t)$ for a computationally simulated data. Figure b) is given only for the sake of comparison with Figure a). c) The function $\widetilde{w}_{smooth}(x,\overline{s})$ resulting from fitting of a) by the Lowess Fitting procedure in the 2-D case, see MATLABR 2009a. d) The final function $\widetilde{w}_{immers}(x,\overline{s})$. Values of $\widetilde{w}_{immers}(x,s)$ are used to produce the Dirichlet boundary conditions $\overline{\psi}_n(x)$ for our elliptic PDEs of the globally convergent algorithm.

Results of the globally convergent algorithm: cube nr.1 (small)



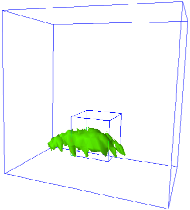
a) $\varepsilon_r^{(5,2)} = 3.9$, $n^{(5,2)} = 1.97$



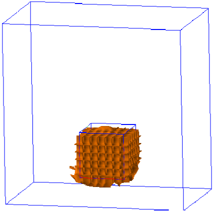
b) $\varepsilon_{r,h} \approx 4.2$, $n_{glob} = \sqrt{\varepsilon_{r,h}} \approx 2.05$

a) A sample of the reconstruction result of dielectric cube No. 1 (4 cm side) via the first stage of our two-stage numericalprocedure. b) The final reconstruction result after applying the adaptive stage (2nd stage). Both the refractive index and the shape are reconstructed with a good accuracy. The side 4 cm=1.33 wavelength.

Results of the globally convergent algorithm: cube nr.2 (big)



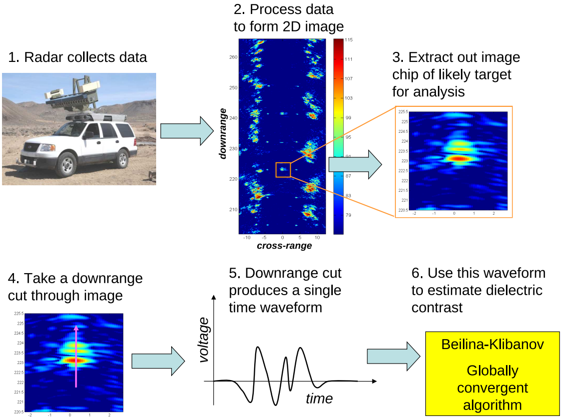
a) $\varepsilon_r(5,5) = 3.19$, $n^{(5,5)} = 1.79$



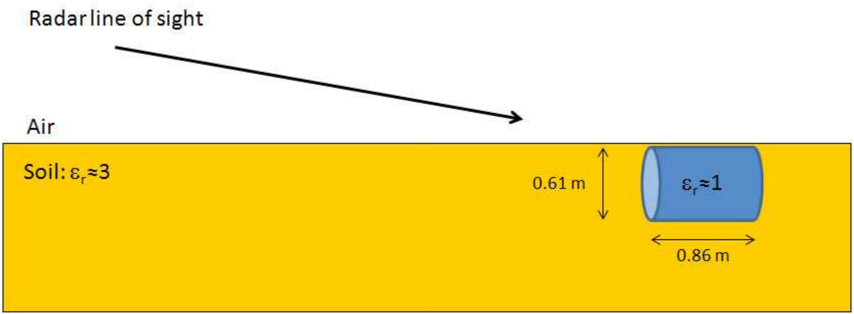
b) $\varepsilon_{r,h} \approx 3.0$, $n_{glob} = \sqrt{\varepsilon_{r,h}} \approx 1.73$

a) A sample of the reconstruction result of the dielectric cube No. 2 (6 cm side) via the first stage of our two-stage numericalprocedure. b) The final reconstruction result after applying the adaptive stage (2nd stage). Both the refractive index and the shape are reconstructed with an excellent accuracy. The side 6 cm=2 wavelength.

The experimental setup for back-scattered data collected in the field by a radar of the US Army Research Laboratory

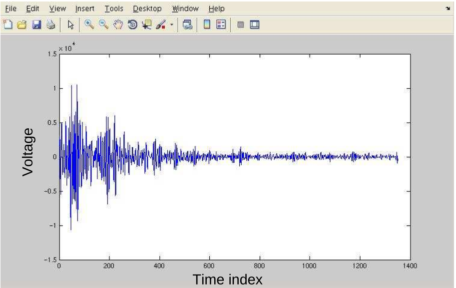


The simplified model



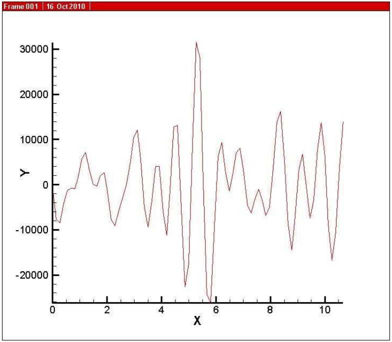
Target is flush buried empty (air-filled) plastic cylindrical container.
Target-to-background dielectric contrast is around 1/3=0.333

Sample Radar Record

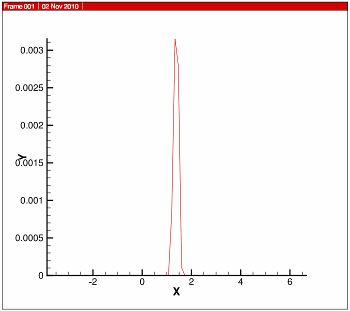


This is a typical radar record collected by one of the receive channels. To form the image (step 2 from previous email), we literally integrate hundreds of these radar records. Note that the target is embedded in this signal somewhere. It is hard to know where it is until we do the image formation process.

A sample of the truncated signal

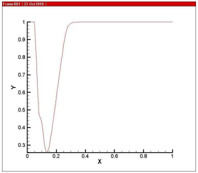


Preprocessed signal



Preprocessed signal. Only the largest peak is left. The globally convergent method works only with this signal.

Result of the blind reconstruction using the backscattering data collected in the field by a radar of the US Army Research Laboratory.



The number ε_r (target) $/\varepsilon_r$ (background) $= 0.26$. The correct ratio ≈ 0.33.

Real ratio	Blindly computed ratio
≈ 0.33	0.26

Table 2. Summary Of Results Of Blind Imaging Of The Data Collected By The Forward Looking Radar

Test Number	Computed ε_r of the target	Real ε_r
1	0.84	≈ 1
2	14.4	≥ 14 (metal)

Acknowledgments

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