On Wiener type filters in SPECT

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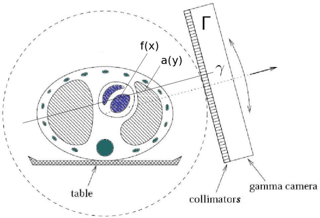
Abstract
For 2D data with Poisson noise we give explicit formulas for the optimal space-invariant Wiener type filter with some a priori geometric restrictions on the window function. We show that, under some natural geometric condition, this restrictedly optimal Wiener type filter admits a very efficient approximation by an approximately optimal filter with unknown object power spectrum. Generalizations to the case of some more general noise model are also given.
Proceeding from these results

1. we explain, in particular, an efficiency of some well-known "1D" approximately optimal space-invariant Wiener type filtering scheme in SPECT and PET imaging based on the FBP algorithms.
2. we propose efficient 2D approximately optimal space-invariant and space variant Wiener type filters for these tomography problems.

SPECT (Single Photon Emission Computed Tomography)



In the single-photon emission computed tomography (SPECT) one considers a body containing radioactive isotopes emitting photons. The emission data p in SPECT consist in the radiation measured outside the body by a family of detectors during some fixed time. The basic problem of SPECT consists in finding the distribution f of these isotopes in the body from the emission data p and some a priori information concerning the body. Usually this a priori information consists in the photon attenuation coefficient a in the points of body, where this coefficient is found in advance by the methods of the transmission computed tomography.

$f(x)$ - density of radioactive isotopes
$a(x)$ - photon attenuation coefficient
x - point of (the space containing the) body
$p(\gamma)$ - emission data
γ - point of detector set Γ
Γ - discrete subset of the set T of all oriented straight lines in the space containing the body
More precisely, $p(\gamma)$ is the number of photons coming from (the domain containing) the body along oriented straight line γ to the detector associated with γ.

In some approximation the emission data p are modeled as follows :

$$\begin{aligned}&p(\gamma) \text{ is a realization of a Poisson variate } \mathbf{p}(\gamma)\\&\text{with the mean } M\mathbf{p}(\gamma) = g(\gamma) = CP_a f(\gamma)\\&\text{for any } \gamma \in \Gamma \text{ and all } \mathbf{p}(\gamma),\ \gamma \in \Gamma,\ \text{are independent,}\end{aligned} \tag{1}$$

where

$$P_a f(\gamma) = \int\limits_{\gamma} \exp(-\mathcal{D}a(x, \hat{\gamma})) f(x) dx, \tag{2}$$

where $\hat{\gamma}$ is the direction of γ,

$$\mathcal{D}a(x, \theta) = \int\limits_{0}^{+\infty} a(x + t\theta) dt,\ x \in \mathbb{R}^d,\ \theta \in \mathbb{S}^{d-1}, \tag{3}$$

$C = C_1 t$, where t is the detection time.
$P_a f$ - attenuated ray transform of f.
The SPECT problem $p \to Cf$ can be restricted to each fixed 2D plane Ξ intersecting the body and identified with $\mathbb{R}^2$.

We remind that $T \approx \mathbb{R} \times \mathbb{S}^1$, where T is the set of all oriented straight lines in $\mathbb{R}^2$. If $\gamma = (s, \theta) \in \mathbb{R} \times \mathbb{S}^1$, then $\gamma = \{x \in \mathbb{R}^2 : \ x = s\theta + t\theta^{\perp}, \ t \in \mathbb{R}\}$ (modulo orientation) and θ gives the orientation of γ, where $\theta^{\perp} = (-\theta_2, \theta_1)$ for $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1$.
In practice, $d = 2$ (after restriction to 2D plane),
$a \geq 0$, $f \geq 0$, $supp\, a \subset \mathcal{B}_R$, $supp\, f \subset \mathcal{B}_R$,
$B_R = \{x \in \mathbb{R}^2 : \ |x| \leq R\}$,
R- radius of image support, Γ - is a uniform $n \times n$ sampling of

$$T_R = \{\gamma \in T : \ \gamma \cap B_R \neq \emptyset\} = \{(s, \theta) \in \mathbb{R} \times \mathbb{S}^1 : \ |s| \leq R\}.$$

In addition, the standard value for n is 128.

Problem 1. (problem of filtering) : Find (as well as possible) g from p, where g and p are the functions of (1).

Problem 2. (problem of reconstruction) : Find (as well as possible) Cf from p and a in the framework of the scheme

$$Cf \approx P_a^{-1} \mathcal{W} p, \tag{4}$$

where $\mathcal{W}$ is a filter for solving Problem 1 and P_a^{-1} is an inversion method for P_a for the noiseless case.
As P_a^{-1} we use the algorithm based on the explicit formula of [R.Novikov, An inversion formula for the attenuated x-ray transformation, Ark.Mat. **40** (2002), 145-167]
and on (stabilizing) iterations of
[T.Morosumi et al., Med.Imaging Technol. 2 (1984), 20-29].
However, the precise form of P_a^{-1} is not principal for our recent work. One can take P_a^{-1} just as a least square inversion method, for example.
Our main new results are related with the Wiener type filters for solving Problems 1 and 2.

Classical result (going back to N.Wiener)

Let $\hat{g}$, $\hat{\mathbf{p}}$, $\hat{p}$ on $\hat{\Gamma}$ denote the 2D discrete Fourier transforms of g, $\mathbf{p}$, p on Γ. Let $\mathcal{W}$ denote a space-invariant linear filter on Γ or, more precisely, $\mathcal{W}$ act in the frequency domain as follows :

$$\begin{aligned} &\hat{u}(j) \to \hat{W}(j)\hat{u}(j), \quad j \in \hat{\Gamma}, \\ &\hat{W} \ \text{ is real} - \text{valued}, \quad \hat{W}(j) = \hat{W}(-j), \end{aligned} \tag{5}$$

where $\hat{W}$ is the window function of $\mathcal{W}$.

Then the mean

$$\mu(\mathcal{W}, g) = M\|\mathcal{W}\mathbf{p} - g\|^2_{L^2(\Gamma)} \tag{6}$$

is minimal with respect to $\mathcal{W}$ iff

$$\begin{aligned} &\hat{W}(j) = \hat{W}^{opt}(j) \stackrel{\text{def}}{=} \tfrac{|\hat{g}(j)|^2}{|\hat{g}(j)|^2 + V}, \quad j \in \hat{\Gamma}, \\ &V = n^{-1}\hat{g}(0) = n^{-2}\textstyle\sum_{\gamma \in \Gamma} g(\gamma), \end{aligned} \tag{7}$$

where n is the sampling number.

An obvious obstacle for a direct use of $\mathcal{W}^{opt}$ for solving Problems 1 and 2 consists in the fact that $\mathcal{W}^{opt}$ of (6) is given in terms of g which is an unknown of Problems 1 and 2.

Regularization of $\mathcal{W}^{opt}$ (new result)

Let $S_1, \ldots, S_{n^*}$ be subsets of $\hat{\Gamma}$ such that

$$\hat{\Gamma} = \cup_{\alpha=1}^{n^*} S_\alpha, \quad S_\alpha \neq \emptyset, \quad S_\alpha \cap S_\beta = \emptyset \text{ if } \alpha \neq \beta, \quad -S_\alpha = S_{\beta[\alpha]}. \tag{8}$$

Let $\mathcal{W}$ act in the frequency domain as follows :

$$\begin{array}{l} \hat{u}(j) \to \hat{W}(j)u(j), \quad j \in \hat{\Gamma}, \\ \hat{W} \ \text{ is real} - \text{valued}, \quad \hat{W}(j) = \hat{W}(-j), \end{array} \tag{9}$$

$$\begin{array}{l} \hat{W} \ \text{ is constant on each fixed } \ S_\alpha, \ \alpha = 1, \ \ldots, n^* \\ \text{(regularization restrictions)}. \end{array} \tag{10}$$

Then $\mu(\mathcal{W}, g) = M\|\mathcal{W}\mathbf{p} - g\|^2_{L^2(\Gamma)}$ is minimal with respect to $\mathcal{W}$ iff

$$\hat{W}(j) = \hat{W}^{r.o.}(j) \stackrel{\text{def}}{=} \frac{\Sigma_{g,\alpha(j)}}{\Sigma_{g,\alpha(j)} + V}, \quad j \in \hat{\Gamma}, \tag{11}$$

$$\begin{array}{l} \Sigma_{g,\alpha} \stackrel{\text{def}}{=} \frac{1}{|S_\alpha|} \sum_{i \in S_\alpha} |\hat{g}(i)|^2, \quad \alpha = 1, \ldots, n^*, \\ V = n^{-1}\hat{g}(0) = n^{-2} \sum_{\gamma \in \Gamma} g(\gamma), \end{array} \tag{12}$$

where $|S_\alpha|$ denotes the number of elements in S_α and $\alpha(j)$ denotes α such that $j \in S_\alpha$.

If

$$S_{\alpha(j)} = \{j\} \ \text{ for any } \ j \in \hat{\Gamma}, \tag{13}$$

then $\mathcal{W}^{r.o.}$ is reduced to $\mathcal{W}^{opt}$.

One can take $S_1, \dots, S_{n^*}$ as a sequence of (boundaries of) squares centered at 0 in $\hat{\Gamma}$.

Approximation

Our next result is that the window

$$\begin{aligned} &\hat{A}(j) = \frac{\Sigma_{p,\alpha(j)} - V_p}{\Sigma_{p,\alpha(j)}} \text{ if } \Sigma_{p,\alpha(j)} - V_p > 0, \\ &\hat{A}(j) = 0 \text{ if } \Sigma_{p,\alpha(j)} - V_p \leq 0, \end{aligned} \tag{14}$$

$$\Sigma_{p,\alpha} = \frac{1}{|S_\alpha|} \sum_{i \in S_\alpha} |\hat{p}(i)|^2, \quad V_p = n^{-1}\hat{p}(0) = n^{-2} \sum_{\gamma \in \Gamma} p(\gamma), \tag{15}$$

where $|S_\alpha|$ is the number of elements in S_α, $\alpha(j)$ denotes α, such that $j \in S_\alpha$, $j \in \hat{\Gamma}$, is a very efficient approximation to $\mathcal{W}^{r.o.}(j)$, under the condition that

$$\begin{aligned} &|S_{\alpha(j)}| \text{ is great enough in comparison} \\ &\text{with } |j| \text{ for each fixed } j \in \hat{\Gamma}. \end{aligned} \tag{16}$$

Let $\mathcal{A}^{simp}$ denote $\mathcal{A}$ for the case (13). Then $|S_\alpha| = 1 \ \forall \alpha$ and condition (16) is not fulfilled for this case.

Let $\mathcal{A}^{1d}$ denote $\mathcal{A}$ for the case when

$$S_{\alpha(j)} = \{z = (z_1, z_2) \in \hat{\Gamma},\ z_1 = j_1\},\quad j = (j_1, j_2) \in \hat{\Gamma}.$$

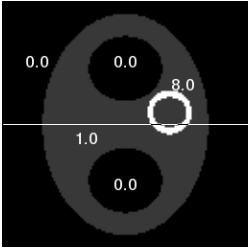
For this case (16) is fulfilled. The filter $\mathcal{A}^{1d}$ is actually used in [M.King et al., A Wiener filter for nuclear medicine images, Med. Phys. **10** (1983) 876-880] and in subsequent works without proper mathematical justification.

Let $\mathcal{A}^{sym}$ denote $\mathcal{A}$ for the case when $S_1 \dots S_{n^*}$ is an appropriate sequence (of boundaries) of squares centered at 0 in $\hat{\Gamma}$. For this case (16) is fulfilled.

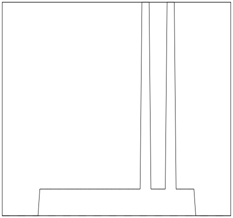
Let $\mathcal{A}^{sym}_{l_1,l_2}$ denote space-variant version of $\mathcal{A}^{sym}$, where $\mathcal{A}^{sym}_{l_1,l_2}$ uses space-invariant considerations in $l_1 \times l_2$ neighborhood of each detector $\gamma \in \Gamma$. In our numerical examples $l_1 = l_2 = 8$.

Numerical examples

Attenuation map a, (128×128)

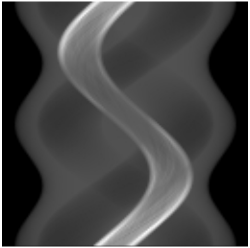


Emission activity f, (128×128)

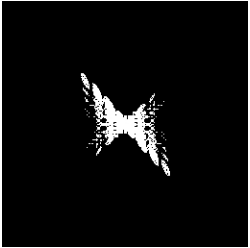


Emission profile

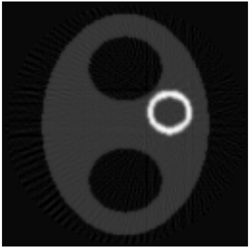
Noiseless projections and reconstruction



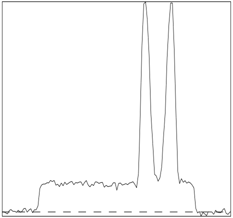
Projections g, (128×128)



Spectrum $|\hat{g}|$

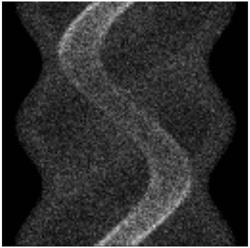


Reconstruction, $r_0 \approx f$

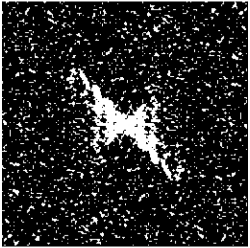


Profile

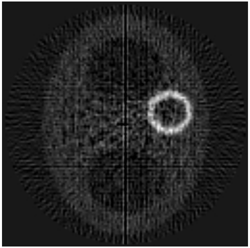
Noisy projections and reconstruction



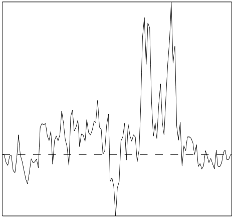
Projections $\|p - g\|_2 / \|g\|_2 = 30\%$
$b = \|Mean(p) - g\|_2 / \|g\|_2 = 0\%$



Spectrum

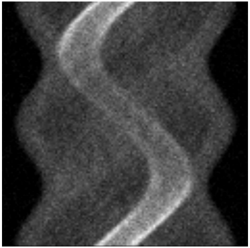


Reconstruction $\|r - r_0\|_2 / \|r_0\|_2 = 75\%$
$b = \|Mean(r) - r_0\|_2 / \|r_0\|_2 = 0\%$

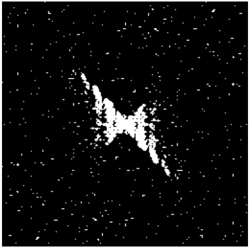


Profile

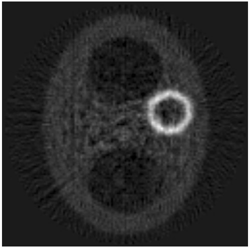
Simplest Wiener filter approximation A^{simp}



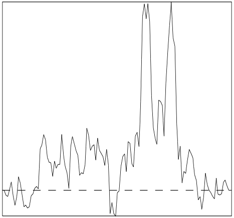
Projections $\|\tilde{p} - g\|_2 / \|g\|_2 = 16\%$
$b = \|Mean(\tilde{p}) - g\|_2 / \|g\|_2 = 4\%$



Spectrum

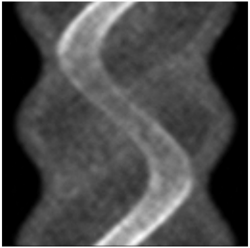


Reconstruction $\|r - r_0\|_2 / \|r_0\|_2 = 40\%$
$b = \|Mean(r) - r_0\|_2 / \|r_0\|_2 = 15\%$

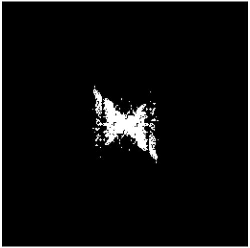


Profile

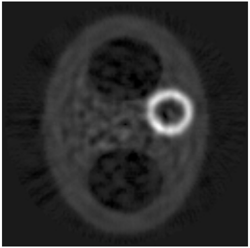
Symmetric 2D Wiener filter approximation A^{sym}



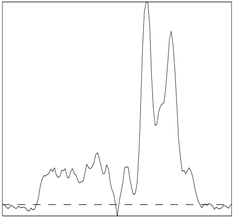
Projections $\|\tilde{p} - g\|_2 / \|g\|_2 = 9\%$
$b = \|Mean(\tilde{p}) - g\|_2 / \|g\|_2 = 6\%$



Spectrum

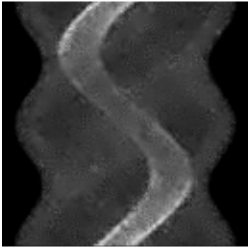


Reconstruction $\|r - r_0\|_2 / \|r_0\|_2 = 27\%$
$b = \|Mean(r) - r_0\|_2 / \|r_0\|_2 = 22\%$

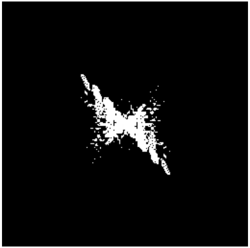


Profile

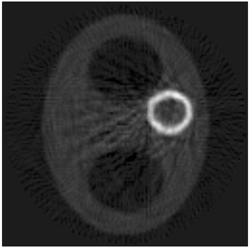
Space-variant Wiener filter $A^{sym}_{8,8}$



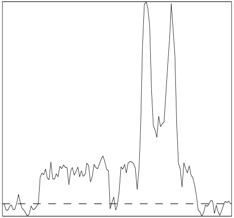
Projections $\|\tilde{p} - g\|_2 / \|g\|_2 = 11\%$
$b = \|Mean(\tilde{p}) - g\|_2 / \|g\|_2 = 3\%$



Spectrum



Reconstruction $\|r - r_0\|_2 / \|r_0\|_2 = 27\%$
$b = \|Mean(r) - r_0\|_2 / \|r_0\|_2 = 12\%$



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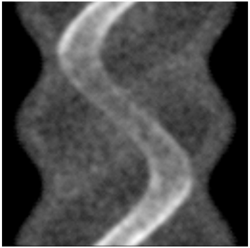
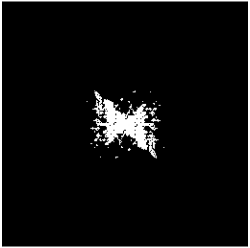
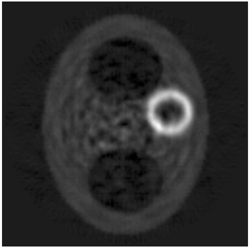
Penalized-likelihood sinogram smoothing



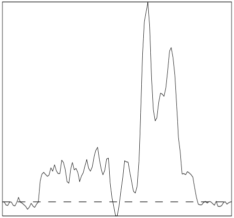
Projections $\|\tilde{p} - g\|_2 / \|g\|_2 = 11\%$
$b = \|Mean(\tilde{p}) - g\|_2 / \|g\|_2 = 7\%$



Spectrum



Reconstruction $\|r - r_0\|_2 / \|r_0\|_2 = 30\%$
$b = \|Mean(r) - r_0\|_2 / \|r_0\|_2 = 23\%$



Profile

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