ANALYTIC EXTENSIONS OF NORMAL FORMS

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Abstract. This is a summary of a conference given by the author at the seminar ””Quasilinear equations and inverse problems” organized by Roman Novikov at the Ecole Polytechnique on February 1, 2013.

1. The Jacobian conjecture

The Jacobian conjecture deals with a polynomial map $F = (F_1, ..., F_n) : k^n \to k^n$, k is any algebraically closed field of characteristic zero. By Lefschetz principle we can assume $k = \mathbb{C}$. Assume that the Jacobian of F: $Jac_{ij}F := \frac{\partial F_i}{\partial x_j}$ is so that $J(F) = \det Jac(F) = 1$. Then the (local analytic) inverse of F should be a polynomial map. For more history and known results on the Jacobian conjecture, see [1], [6] and the references therein. There has been recently many interesting developments about this conjecture. In particular, the problem was recently reduced to the case of symmetric jacobians ([3, 9]):

$$F : x \mapsto x - \nabla P, \tag{1}$$

where P denotes a polynomial with terms of degree strictly larger than 2.

W. Zhao proved ([10]) the following:

Theorem 1. *Let t be a small parameter, consider the deformation $F_t(z) = z - t\nabla P$. The inverse map of $z \mapsto F_t(z)$ can be written:*

$$G_t(z) = z + t\nabla Q_t(z), \tag{2}$$

where $Q_t(z)$ is the unique solution of the Cauchy problem for the Hamilton-Jacobi equation:

$$\begin{gathered} \frac{\partial Q_t(z)}{\partial t} = \frac{1}{2}\langle \nabla Q_t, \nabla Q_t\rangle, \\ Q_{t=0}(z) = P(z) \end{gathered} \tag{3}$$

2. Measures defined on the complement of an arrangement of hyperplanes

Let A be a finite arrangement of hyperplanes in the n-dimensional complex affine space $\mathbb{C}^n$. Let $N(A)$ be the union of hyperplanes of A in $\mathbb{C}^n$ and $M(A)$ be its complement. It is further assumed that A is real, meaning that the defining function of every hyperplane $f_h(z) = u_{h,0} + \sum_{i=1}^n u_{h,i}z_i$ has real coefficients. We consider the family of functions:

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$$V = \frac{1}{2}\sum_{i=1}^{n}\left(x_i - \sum_{h\in A} t\frac{\lambda_h u_{h,i}}{f_h(x)}\right)^2, t \in [0,1], \lambda_h > 0, \tag{4}$$

and the associated family of measures defined by densities:

$$\rho_t = (\frac{\beta}{2\pi})^{n/2}\exp(-\beta V). \tag{5}$$

In the case when the arrangement of hyperplanes is given by the walls of the Weyl chambers of a root system, the function V represents the potential of the generalized Calogero hamiltonian system (where all masses must be equal). In that case, it was possible to prove that all the measures $\rho_t dx$ are indeed probability measure because the system is associated to a symplectic action of the torus (cf [7], [4]).

In ([2]), Aomoto and Forrester proved that, in general, for any finite arrangement of hyperplanes, we have:

$$\int_{M(A)} \rho_t dx = 1. \tag{6}$$

Although the Jacobian conjecture stands for complex polynomials, theorem 1 above makes perfectly sense for real polynomials. With the help of this theorem 1, it is possible to give a new interpretation of the Aomoto-Forrester result.

Theorem 2. *The mapping*

$$F_t : x \mapsto w, \quad w_i = x_i - t\frac{\partial P}{\partial x_i}, \quad P(x) = \log[\Pi_{h\in A} < f_h(x) >^{\lambda_h}] \tag{7}$$

displays a local inverse for t small, solution of the Hamilton-Jacobi equation. This local inverse admits an analytic prolongation to $t \in [0,1]$ on each connected components $\Delta_j, j = 1, ..., k$. This inverse mapping of F_t achieves an optimal transport of $\rho_0 dx = (\frac{\beta}{2\pi})^{n/2}e^{-\frac{1}{2}\beta|w|^2}dx$ to $\rho_1 dx = (\frac{\beta}{2\pi})^{n/2}e^{-\frac{1}{2}\beta|x-\nabla P|^2}dx$:

$$\rho_1(T(w))\mathrm{Det}(DT(w)) = \rho_0(w), \tag{8}$$

$$T_j(w) = x_j = w_j + t\frac{\partial Q_t}{\partial w_j}, \tag{9}$$

where $\Sigma_j \frac{1}{2}w_j^2 + tQ_t$ is convex and Q_t solves Hamilton-Jacobi equation:

$$\begin{gathered}\frac{\partial Q_t(z)}{\partial t} = \frac{1}{2}\langle\nabla Q_t, \nabla Q_t\rangle, \\ Q_{t=0}(z) = P(z)\end{gathered} \tag{10}$$

Proof. Consider the Aomoto-Forrester mapping:

$$F : x \mapsto w, \quad w_i = x_i - \sum_{h\in A}\frac{\lambda_h u_{h,i}}{f_h(x)}. \tag{11}$$

A connected component $\Delta_j, j = 1, ..., k$ of $M(A) \cap \mathbb{R}^n$ is called a chamber. Aomoto and Forrester proved that the mapping F defines a bijection of Δ_j on $\mathbb{R}^n$. Let us denote $T_j : \mathbb{R}^n \to \Delta_j$ the inverses of the mapping F. Given a $w \in \mathbb{R}^n$, there exists a unique $x \in \Delta_j$ so that $w = F(x), x = T_j(w)$ and:

$$\int_{M(A)} \rho_1 = \Sigma_{j=1}^k \int_{\Delta_j} \exp(-\beta V) d^n x = \Sigma_{j=1}^k \int_{\mathbb{R}^n} \exp(-\beta \Sigma_i w_i^2) \mathrm{Jac}(T_j(w)) dw. \tag{12}$$

Aomoto and Forrester proved (by a convexity argument) that for all $w \in \mathbb{R}^n$, $\mathrm{Jac}(T_j(w)) > 0$. Furthermore they showed using Griffiths-Harris residue techniques that:

$$\Sigma_{j=1}^k \mathrm{Jac}(T_j(w)) = 1. \tag{13}$$

This implies that:

$$\int_{M(A)} \rho_1 dx = \int_{\mathbb{R}^n} \exp(-\beta \Sigma_i w_i^2) [\Sigma_{j=1}^k \mathrm{Jac}(T_j(w))] dw = 1. \tag{14}$$

□

Several consequences of Theorem 1 have been analyzed from the viewpoint of perturbation analysis of Hamiltonian systems and symplectic geometry in an article to appear ([5]).

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