An effectivization of the global reconstruction in the Gel'fand-Calderon inverse problem in three dimensions

R. G. Novikov

18 October 2008

CNRS, Laboratoire de Mathématiques Jean Leray (UMR 6629), Université de Nantes, BP 92208, F-44322, Nantes cedex 03, France
e-mail: novikov@math.univ-nantes.fr

Abstract. By developing the $\bar{\partial}$-approach to global "inverse scattering" at zero energy we give a principal effectivization of the global reconstruction method for the Gel'fand-Calderon inverse boundary value problem in three dimensions. This work goes back to results published by the author in 1987, 1988 and proceeds from recent progress in the $\bar{\partial}$-approach to inverse scattering in 3D published by the author in 2005, 2006.

Mathematics Subject Classification (2000): 35R30 (primary); 81U40, 86A20 (secondary).

1. Introduction

The global reconstruction method for the Gel'fand-Calderon inverse boundary value problem in 3D was proposed for the first time by the author in 1987, 1988, see [HN] (Note added in proof) and [No1]. The sheme of this global reconstruction can be presented as follows:

$$\Phi \to h\big|_{\bar{\Theta}_\rho} \to v_\rho \to v, \qquad (*)$$

where Φ denotes boundary measurements, h is the Faddeev generalized scattering amplitude (at zero energy for simplicity) defined in complex domain Θ, $\bar{\Theta}_\rho$ is the subset of Θ with the imaginary part not greater than ρ, v_ρ is an approximate inverse scattering reconstruction from $h\big|_{\bar{\Theta}_\rho}$, $v_\rho \to v$ for $\rho \to +\infty$, where v is the unknown potential to be reconstructed from Φ. The global uniqueness in the Gel'fand- Calderon problem in 3D follows from this global reconstruction as a corollary, see [No1].

Slightly earlier with respect to [No1] the global uniqueness in the Calderon problem in 3D was proved in [SU]. Note that [SU] gives no reconstruction method.

Slightly later with respect to [No1] the global reconstruction from boundary measurements in 3D similar to the global reconstruction of [No1] was also published in [Na1]. Note that [Na1] contains a reference to the preprint of [No1].

The Gel'fand-Calderon problem is formulated as Problem 1.2 of Subsection 1.2 of this introduction.

Reconstruction problems from the Faddeev generalized scattering amplitude h in the complex domain at zero energy are formulated as Problems 1.1a, 1.1b and 1.1c of Subsection 1.1 of this introduction.

The reduction of Problem 1.2 to Problems 1.1 is given by formulas and equations (1.23)-(1.25) mentioned in Subsection 1.2 of this introduction, see also [No2] for more advanced version of these formulas and equations.

For a long time the main disadvantage of the global reconstruction (*) in 3D was related with the following two facts:

(1) The determination of $h\big|_{\bar\Theta_\rho}$ from Φ via formulas and equations of the type (1.23)-(1.25) is stable for relatively small ρ, but is very unstable for $\rho \to +\infty$ in the points of $\bar\Theta_\rho$ with sufficiently great imaginary part, see Subsection 1.2 of this introduction.

(2) The decay of the error $v - v_\rho$ for $\rho \to +\infty$ was very slow (not faster than $O(\rho^{-1})$ even for infinitely smooth compactly supported v) in existing global results for stable construction of v_ρ from $h\big|_{\bar\Theta_\rho}$ in 3D, see Remarks 1.1, 1.2, 1.3 of Subsection 1.1 of this introduction.

As a corollary, the global reconstruction (*) in 3D was not efficient with respect to its stability properties.

The key point is that in the present work we give a global and stable construction of v_ρ from $h\big|_{b\Theta_\rho}$ in 3D, where $b\Theta_\rho$ denotes the boundary of $\bar\Theta_\rho$, with rapid decay of the error $v - v_\rho$ for $\rho \to +\infty$ (in particular, with $v - v_\rho = O(\rho^{-\infty})$ for v of the Schwartz class). This gives a principal effectivization of the global reconstruction (*) with respect to its stability properties.

Our new results are presented in detail below in Subsections 1.1, 1.2, 1.3 (of the introduction) and in Sections 2 and 6. These results were obtained proceeding from [No3], [No4].

1.1. Inverse scattering at zero energy. Consider the equation

$$-\Delta\psi + v(x)\psi = 0, \quad x \in \mathbb{R}^d, \quad d \geq 2, \tag{1.1}$$

where

$$\begin{array}{l} v \ \text{ is a sufficiently regular function on } \ \mathbb{R}^d \\ \text{with sufficient decay at infinity} \end{array} \tag{1.2}$$

(precise assumptions on v are specified below in this introduction and in Section 2).

Equation (1.1) arises, in particular, in quantum mechanics, acoustics, electrodynamics. Formally, (1.1) looks as the Schrödinger equation with potential v at fixed energy $E = 0$.

For equation (1.1), under assumptions (1.2), we consider the Faddeev generalized scattering amplitude $h(k,l)$, where $(k,l) \in \Theta$,

$$\Theta = \{k \in \mathbb{C}^d,\ l \in \mathbb{C}^d :\ k^2 = l^2 = 0, \quad Im\, k = Im\, l\}. \tag{1.3}$$

Given v, to determine h on Θ one can use, in particular, the formula

$$h(k,l) = H(k, k-l), \quad (k,l) \in \Theta, \tag{1.4}$$

and the linear integral equation

$$H(k,p) = \hat v(p) - \int\limits_{\mathbb{R}^d} \frac{\hat v(p+\xi)H(k,-\xi)d\xi}{\xi^2 + 2k\xi}, \quad k \in \Sigma, \quad p \in \mathbb{R}^d, \tag{1.5}$$

where

$$\hat{v}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d, \tag{1.6}$$

$$\Sigma = \{k \in \mathbb{C}^d : \ k^2 = 0\}. \tag{1.7}$$

For more details concerning definitions of h, see [HN, Section 2.2], [No1, Section 2] and [No4, Sections 1 and 3].

Actually, h on Θ is a zero energy restriction of a function h introduced by Faddeev as an extension to the complex domain of the classical scattering amplitude for the Schrödinger equation at positive energies (see [F2], [HN]). Note that the restriction $h\big|_{\Theta}$ was not considered in Faddeev's works and that h in its zero energy restriction was considered for the first time in [BC] for $d = 3$ in the framework of Problem 1.1a formulated below. The Faddeev function h was, actually, rediscovered in [BC]. The fact that $\bar{\partial}$-scattering data of [BC] coincide with the Faddeev function h was observed, in particular, in [HN].

In the present work, in addition to h on Θ, we consider, in particular, $h\big|_{\bar{\Theta}_\rho}$ and $h\big|_{b\Theta_\rho}$, where

$$\begin{aligned} &\bar{\Theta}_\rho = \Theta_\rho \cup b\Theta_\rho, \\ &\Theta_\rho = \{(k,l) \in \Theta : \ |Im\, k| = |Im\, l| < \rho\}, \\ &b\Theta_\rho = \{(k,l) \in \Theta : \ |Im\, k| = |Im\, l| = \rho\}, \end{aligned} \tag{1.8}$$

where $\rho > 0$. Note that

$$dim\, \Theta = 3d - 4, \quad dim\, b\Theta_\rho = 3d - 5. \tag{1.9}$$

Using (1.4), (1.5) one can see that

$$h(k,l) \approx \hat{v}(p), \quad p = k - l, \quad (k,l) \in \Theta. \tag{1.10}$$

in the Born approximation (that is in the linear approximation near zero potential). In addition, one can see also that

$$(k,l) \in \bar{\Theta}_\rho \Longrightarrow \ p = k - l \in \bar{\mathcal{B}}_{2\rho}, \tag{1.11}$$

where

$$\begin{aligned} &\bar{\mathcal{B}}_r = \mathcal{B}_r \cup \partial\mathcal{B}_r, \\ &\mathcal{B}_r = \{p \in \mathbb{R}^d : \ |p| < r\}, \quad \partial\mathcal{B}_r = \{p \in \mathbb{R}^d : \ |p| = r\}, \ r > 0. \end{aligned} \tag{1.12}$$

In the present work we consider, in particular, the following inverse scattering problems for equation (1.1) under assumptions (1.2).

Problem 1.1.

(a) Given h on Θ, find v on $\mathbb{R}^d$;

(b) Given h on $\bar{\Theta}_\rho$ for some (sufficiently great) $\rho > 0$, find v on $\mathbb{R}^d$, at least, approximately;

(c) Given h on $b\Theta_\rho$ for some (sufficiently great) $\rho > 0$, find v on $\mathbb{R}^d$, at least, approximately.

Note that Problems 1.1a, 1.1b make sense for any $d \geq 2$, whereas Problem 1.1c is reasonable for $d \geq 3$ only: $dim\, b\Theta_\rho < dim\, \mathbb{R}^d$ for $d = 2$, see (1.9).

Note that: (1) any reconstruction method for Problem 1.1b with decaying error as $\rho \to +\infty$ gives also a reconstruction method for Problem 1.1a and (2) for $d \geq 3$, any reconstruction method for Problem 1.1c gives also a reconstruction method for Problem 1.1b.

Note that in the Born approximation (1.10): (a) Problem 1.1a is reduced to finding v on $\mathbb{R}^d$ from $\hat{v}$ on $\mathbb{R}^d$, (b) Problem 1.1b is reduced to (approximate) finding v on $\mathbb{R}^d$ from $\hat{v}$ on $\mathcal{B}_{2\rho}$, (c) Problem 1.1c for $d \geq 3$ is reduced to (approximate) finding v on $\mathbb{R}^d$ from $\hat{v}$ on $\mathcal{B}_{2\rho}$, where $\hat{v}$ is defined by (1.6). Thus, in the Born approximation, Problem 1.1c for $d \geq 3$ (as well as Problem 1.1b for $d \geq 2$) can be solved by the formula

$$\begin{gathered} v(x) = v^{lin}_{appr}(x,\rho) + v^{lin}_{err}(x,\rho), \\ v^{lin}_{appr}(x,\rho) = \int\limits_{\mathcal{B}_{2\rho}} e^{-ipx}\hat{v}(p)dp, \quad v^{lin}_{err}(x,\rho) = \int\limits_{\mathbb{R}^d\setminus\mathcal{B}_{2\rho}} e^{-ipx}\hat{v}(p)dp, \end{gathered} \tag{1.13}$$

where $x \in \mathbb{R}^d$. In addition, if, for example,

$$v \in W^{n,1}(\mathbb{R}^d) \ \text{ for some } \ n \in \mathbb{N}, \tag{1.14}$$

and $\|v\|^{n,1} \leq C$, where $W^{n,1}(\mathbb{R}^d)$ denotes the space of n-times smooth functions on $\mathbb{R}^d$ in L^1-sense and $\|\cdot\|^{n,1}$ denotes some fixed standard norm in $W^{n,1}(\mathbb{R}^d)$, then

$$|\hat{v}(p)| \leq c_1(n,d)C(1+|p|)^{-n}, \quad p \in \mathbb{R}^d, \tag{1.15}$$

and, therefore, for $n > d$,

$$|v^{lin}_{err}(x,\rho)| \leq c_2(n,d)C\rho^{-(n-d)}, \quad x \in \mathbb{R}^d, \quad \rho \geq 1, \tag{1.16}$$

where $c_1(n,d)$, $c_2(n,d)$ are some fixed positive constants and $v^{lin}_{err}(x,\rho)$ is the error term of (1.13).

Thus, in the Born approximation (1.10) (that is in the linear approximation near zero potential) we have that:

(1) h on $b\Theta_\rho$ for $d \geq 3$ (as well as h on $\bar{\Theta}_\rho$ for $d \geq 2$) stably determines $v^{lin}_{appr}(x,\rho)$ of (1.13) and

(2) the error $v^{lin}_{err}(x,\rho) = v(x) - v^{lin}_{appr}(x,\rho) = O(\rho^{-(n-d)})$ in the uniform norm as $\rho \to +\infty$ for n-times smooth v in the sense (1.14), where $n > d$. In particular, $v^{lin}_{err} = O(\rho^{-\infty})$ in the uniform norm as $\rho \to +\infty$ for v of the Schwartz class on $\mathbb{R}^d$.

The main results of the present work consist in global analogs for the nonlinearized case for $d = 3$ of the aforementioned Born-approximation results for Problem 1.1c, see Theorem 2.1 and Corollary 2.1 of Section 2. In particular, we give a stable approximate solution of nonlinearized Problem 1.1c for $d = 3$ and v satisfying (1.14), $n > d = 3$, with the error term decaying as $O(\rho^{-(n-d)}\ln\rho)$ in the uniform norm as $\rho\to+\infty$ (that is with almost the same decay rate of the error for $\rho\to+\infty$ as in the linearized case near zero potential, see (1.13), (1.16)). The results of the present work were obtained in the framework of a development of the $\bar\partial$-approach to inverse scattering at fixed energy in dimension $d\ge 3$ of [BC], [HN], [No3], [No4], with applications to the Gel'fand-Calderon inverse boundary value problem via the reduction going back to [No1]. See Subsections 1.2, 1.3 and Sections 2, 3, 4, 5, 6 for details.

Remark 1.1. Note that if

$$\begin{aligned} &v\in L^\infty(\mathbb{R}^d),\quad ess\ \sup_{x\in\mathbb{R}^d}(1+|x|)^{d+\varepsilon}|v(x)|\le C,\\ &\text{for some positive }\ \varepsilon\ \text{ and }\ C,\end{aligned}\tag{1.17}$$

then (see [HN], [No1], [Na1], [No4]):

$$\hat v(p)=\lim_{\rho\to+\infty,\ k-l=p} h(k,l)\ \text{ for any }\ p\in\mathbb{R}^d,\ d\ge 3,\tag{1.18}$$

$$|\hat v(k-l)-h(k,l)|\le c_3(\varepsilon,d)C^2\rho^{-1}\ \text{ as }\ \rho\to+\infty,\tag{1.19}$$

where $(k,l)\in b\Theta_\rho$, $c_3(\varepsilon,d)$ is some positive constant. Formulas (1.18), (1.19) show that the Born approximation (1.10) holds on $b\Theta_\rho$ (and on $\Theta\backslash\Theta_\rho$) for any sufficiently great ρ (actually, for any sufficiently great ρ in comparison with C of (1.17)). However, because of $O(\rho^{-1})$ in the right-hand side of (1.19), formulas (1.18), (1.19) give no method to reconstruct v on $\mathbb{R}^d$ from h on $b\Theta_\rho$ (or on $\bar\Theta_\rho$) with the error term decaying more rapidly than $O(\rho^{-1})$ in the uniform norm as $\rho\to+\infty$ (even for v of the Schwartz class on $\mathbb{R}^d$, $d\ge 3$).

Remark 1.2. On the other hand (in comparison with the result mentioned in Remark 1.1), for sufficiently small potentials v, in [No4] we succeeded, in particular, to give a stable method for solving Problem 1.1b for $d=3$ with the same type rapid decay of the error term for $\rho\to+\infty$ as in formulas (1.13), (1.16) for the linearized case near zero potential. Moreover, in this result of [No4], v is approximately reconstructed already from non-overdetermined restriction $h\big|_{\Theta_\rho\cap\Gamma}$, where $\Gamma\subset\Theta$, $dim\,\Gamma=d=3$. However, this result of [No4] is local: the smallness of v is used essentially.

Remark 1.3. We emphasize that before the present work no results were given, in general, in the literature on solving Problems 1.1c and 1.1b for $d\ge 3$ with the error term decaying more rapidly than $O(\rho^{-1})$ as $\rho\to+\infty$ even for v of the Schwartz class on $\mathbb{R}^d$ (and even for the infinitely smooth compactly supported case in the framework of sufficiently stable rigorous algorithms). In addition, rapid decay of this error term is a property of principal importance in the framework of applications of methods for solving Problems 1.1 to the Gel'fand-Calderon inverse boundary value problem (Problem 1.2) via the reduction of [No1], see the next part of introduction.

Note that Problem 1.1a was considered for the first time in [BC] for $d=3$ from pure mathematical point of view without any physical applications. No possibility to measure h on $\Theta\backslash\{(0)\}$ directly in some physical experiment is known at present (here

$\{0\} = \{(k,l) \in \Theta : \ |k| = |l| = 0\})$. However, as it was shown in [No1] (see also [HN] (Note added in proof), [Na1], [No2]), Problems 1.1 naturally arise in the electrical impedance tomography and, more generally, in connection with Problem 1.2 formulated in the next subsection.

1.2. The Gel'fand-Calderon problem. Consider the equation (1.1) in $D \subset \mathbb{R}^d$ only, where

$$\begin{aligned}&D \ \text{ is an open bounded domain in } \ \mathbb{R}^d, \quad d \geq 2,\\ &\text{with sufficiently regular boundary } \ \partial D,\\ &v \ \text{ is a sufficiently regular function on } \ \bar{D} = D \cup \partial D.\end{aligned} \tag{1.20}$$

For simplicity we assume also that

$$\begin{aligned}&0 \ \text{ is not a Dirichlet eigenvalue for}\\ &\text{the operator } \ -\Delta + v \ \text{ in } \ D.\end{aligned} \tag{1.21}$$

Consider the map Φ such that

$$\frac{\partial \psi}{\partial \nu}\big|_{\partial D} = \Phi(\psi\big|_{\partial D}) \tag{1.22}$$

for all sufficiently regular solutions ψ of (1.1) in $\bar{D}$, where ν is the outward normal to ∂D. The map Φ is called the Dirichlet-to-Neumann map for equation (1.1) in D. We consider the following inverse boundary value problem for equation (1.1) in D:

Problem 1.2. Given Φ, find v.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at zero energy (see [G], [No1]). This problem can be also considered as a generalization of the Calderon problem of the electrical impedance tomography (see [C], [SU], [No1]).

One can see that the Faddeev function h of Problems 1.1 does not appear in Problem 1.2. However, as it was shown in [No1] (see also [HN] (where this result of [No1] was announced in Note added in proof), [Na1], [No2]), if h corresponds to equation (1.1) on $\mathbb{R}^d$, where v is the potential of Problem 1.2 on D and $v \equiv 0$ on $\mathbb{R}^d \backslash \bar{D}$, then h on Θ can be determined from the Dirichlet-to-Neumann map Φ via the following formulas and equation:

$$h(k,l) = (2\pi)^{-d} \int\limits_{\partial D} \int\limits_{\partial D} e^{-ilx} (\Phi - \Phi_0)(x,y) \psi(y,k) dy dx \ \text{ for } \ (k,l) \in \Theta, \tag{1.23}$$

$$\psi(x,k) = e^{ikx} + \int\limits_{\partial D} A(x,y,k) \psi(y,k) dy, \quad x \in \partial D, \tag{1.24}$$

$$A(x,y,k) = \int\limits_{\partial D} G(x-z,k)(\Phi - \Phi_0)(z,y) dz, \quad x,y \in \partial D, \tag{1.25}$$

$$G(x,k) = -(2\pi)^{-d} e^{ikx} \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \quad x \in \mathbb{R}^d, \tag{1.26}$$

where $k \in \mathbb{C}^d$, $k^2 = 0$ in (1.24)-(1.26), Φ_0 denotes the Dirichlet-to-Neumann map for equation (1.1) in D with $v \equiv 0$, and $(\Phi - \Phi_0)(x, y)$ is the Schwartz kernel of the integral operator $\Phi - \Phi_0$. Note that (1.23), (1.25), (1.26) are explicit formulas, whereas (1.24) is a linear integral equation (with parameter k) for ψ on ∂D. In addition, G of (1.26) is the Faddeev's Green function of [F1] for the Laplacian Δ. Formulas and equation (1.23)-(1.26) reduce Problem 1.2 to Problems 1.1. In addition, from numerical point of view $h(k,l)$ for $(k,l) \in \bar{\Theta}_\rho$ can be relatively easily determined from Φ via (1.25), (1.24), (1.23) if ρ is sufficiently small. However, if $(k,l) \in \Theta \backslash \Theta_\rho$, where ρ is sufficiently great, then the determination of $h(k,l)$ from Φ via (1.25), (1.24), (1.23) is very unstable (especially on the step (1.24)); see, for example, [BRS], [No2], [No4]. This explains the principal importance (mentioned in Remark 1.3) of the error term rapid decay as $\rho \to +\infty$ in methods for solving Problems 1.1b and 1.1c.

1.3. Final remarks. In the present work we consider, mainly, Problems 1.1 and 1.2 for $d = 3$. The main results of the present work are presented in Sections 2 and 6. Some of these results were already mentioned above. Note that only restrictions in time prevent us from generalizing all main results of the present work to the case of the Schrödinger equation at arbitrary (not necessarily zero) fixed energy E for $d \geq 3$.

Note that results of the present work permit to complete (at least for $d = 3$) the proof of new stability estimates for Problem 1.2, $d \geq 3$, announced as Theorem 2.2 of [NN]. We plan to return to this proof in a separate article.

Our new global reconstruction for Problem 1.2 in 3D is summarized in schemes (6.1), (6.2) of Section 6. We expect that this reconstruction can be implemented numerically in a similar way with implementations developed in [ABR], where the parameter ρ of (6.1), (6.2) will play in some sense the role of the wave number k_0 of [ABR].

As regards results given in the literature on Problem 1.1, see [BC], [HN], [GN], [Na1], [Na2], [No4] and references therein.

As regards results given in the literature on Problem 1.2 (in its Calderon or Gel'fand form), see [SU], [No1], [Al], [Na1], [Na2], [Ma], [No2], [No4], [NN], [HM], [Am] and references therein.

2.Main new results

In the present work we consider, mainly, the three dimensional case $d = 3$. In addition, in the main considerations of the present work for $d = 3$ our basic assumption on v consists in the following condition on its Fourier transform:

$$\hat{v} \in L^\infty_\mu(\mathbb{R}^3) \cap \mathcal{C}(\mathbb{R}^3) \ \text{ for some real } \ \mu \geq 2, \tag{2.1}$$

where $\hat{v}$ is defined by (1.6),

$$\begin{aligned} &L^\infty_\mu(\mathbb{R}^d) = \{u \in L^\infty(\mathbb{R}^d) : \ \|u\|_\mu < +\infty\}, \\ &\|u\|_\mu = ess \sup_{p \in \mathbb{R}^d} (1 + |p|)^\mu |u(p)|, \quad \mu > 0, \end{aligned} \tag{2.2}$$

and $\mathcal{C}$ denotes the space of continuous functions. Actually, (2.1) is a specification of (1.2).

Note that

$$v \in W^{n,1}(\mathbb{R}^d) \Longrightarrow \hat{v} \in L^\infty_\mu(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d),$$
$$\|\hat{v}\|_\mu \le c_4(n,d)\|v\|^{n,1} \ \text{ for } \ \mu = n, \tag{2.3}$$

where $W^{n,1}$, L^∞_μ are the spaces of (1.14), (2.2).

Let

$$\Theta^\infty_{\rho,\tau} = \{(k,l) \in \Theta\backslash\bar{\Theta}_\rho : \ k-l \in \mathcal{B}_{2\rho\tau}\},$$
$$b\Theta_{\rho,\tau} = \{(k,l) \in b\Theta_\rho : \ k-l \in \mathcal{B}_{2\rho\tau}\}, \tag{2.4}$$

where Θ, $\bar{\Theta}_\rho$, $b\Theta_\rho$, $\mathcal{B}_r$ are defined by (1.3), (1.8), (1.12), $\rho > 0$, $0 < \tau < 1$. In (2.4) by symbol ∞ we emphasize that $\Theta^\infty_{\rho,\tau}$ is unbounded with respect to k, l. One can see also that by definition

$$b\Theta_{\rho,\tau} \subset b\Theta_\rho, \quad \rho > 0, \ 0 < \tau < 1. \tag{2.5}$$

Our main new results on Problem 1.1 are summarized as Theorem 2.1 and Corollary 2.1 below.

Theorem 2.1. *Let $\hat{v}$ satisfy* (2.1) *and $\|\hat{v}\|_\mu \le C$. Let $2 \le \mu_0 < \mu$, $0 < \delta < 1$,*

$$0 < \tau < \tau_1(\mu, \mu_0, C, \delta), \quad \rho \ge \rho_1(\mu, \mu_0, C, \delta), \tag{2.6}$$

where τ_1, ρ_1 are the constants of (4.36) *and, in particular, $0 < \tau_1 < 1$. Then h on $b\Theta_{\rho,\tau}$ stably determines $\hat{v}^\pm(\cdot,\tau,\rho)$ via* (6.2) (*where the nonlinear integral equation* (4.31) *is solved by successive approximations) and*

$$|\hat{v}(p) - \hat{v}^\pm(p,\tau,\rho)| \le c_5(\mu,\mu_0,\tau,\delta)C^2(1+|p|)^{-\mu_0}\rho^{-(\mu-\mu_0)} \ \ \textit{for} \ \ p \in \mathcal{B}_{2\tau\rho}, \tag{2.7}$$

where c_5 is the constant of (6.4), see Section 6.

Construction (6.2) is actually a definition of $\hat{v}^\pm(\cdot,\tau,\rho)$. We consider $\hat{v}^\pm(\cdot,\tau,\rho)$ as an approximation to $\hat{v}$ on $\mathcal{B}_{2\tau\rho}$. The error between $\hat{v}$ and $\hat{v}^\pm(\cdot,\tau,\rho)$ on $\mathcal{B}_{2\tau\rho}$ is estimated in (2.7). Estimate (2.7) is especially interesting for $\rho \to +\infty$ at fixed τ.

Theorem 2.1 follows from Proposition 4.2, Corollary 4.1 and formulas (5.2), (5.3), (5.6) (and related results of Sections 4 and 5). A more detailed version of (2.7) is given by (6.3). The stability mentioned in Theorem 2.1 follows from estimate (6.10). See Sections 4, 5, 6 for additional details.

An outline of the reconstruction formalized in Theorem 2.1 is given also at the end of the present section.

Let

$$v^\pm(x,\tau,\rho) = \int\limits_{\mathcal{B}_{2\tau\rho}} e^{-ipx}\hat{v}^\pm(p,\tau,\rho)dp, \ x \in \mathbb{R}^3, \tag{2.8}$$

where $\hat{v}^\pm(p,\tau,\rho)$ is the approximation of Theorem 2.1.

Formula (2.3) and Theorem 2.1 imply the following

Corollary 2.1. *Let v satisfy* (1.14), *$n > d = 3$, and $\|v\|^{n,1} \le D$. Let τ and ρ satisfy* (2.6) *for $\mu = n$, $\mu_0 = 3$, $C = c_4(n,3)D$ and fixed $\delta \in]0,1[$. Then h on $b\Theta_{\rho,\tau}$ stably determines $v^\pm(\cdot,\tau,\rho)$ via* (6.2), (2.8) *and*

$$|v(x) - v^\pm(x,\tau,\rho)| \le (c_6(n,\tau)D + c_7(n,\tau,\delta)D^2\ln(1+2\tau\rho))\rho^{-(n-3)} \ \ \textit{for} \ \ x \in \mathbb{R}^3, \tag{2.8}$$

where the constants c_6, c_7 are simply related with c_2 and c_5. In addition, in particular,

$$\begin{aligned}&\|v - v^{\pm}(\cdot,\tau,\rho)\|_{L^\infty(\mathbb{R}^3)} = O(\rho^{-(n-3)}\ln\rho) \ \ \textit{for} \ \ \rho\to+\infty \ \ \textit{for fixed} \ \ \tau,\\ &0<\tau<\tau_1(n,3,c_4(n,3)D,\delta).\end{aligned} \tag{2.9}$$

We consider Theorem 2.1 as a global nonlinear analog of the result that in the Born approximation $h\big|_{b\Theta_\rho}$ is reduced to a $\hat v$ on $\mathcal{B}_{2\rho}$ for $d\geq 3$. One can see, in particular, that in Theorem 2.1 we even do not try to reconstruct $\hat v$ on $\mathbb{R}^3\backslash\mathcal{B}_{2\rho}$ from h on $b\Theta_\rho$.

We consider Corollary 2.1 as a global nonlinear analog of the Born approximation result (for Problem 1.1c) consisting in formulas (1.13), (1.16).

In the derivations of the present work we rewrite h on Θ, $\bar\Theta_\rho$, $b\Theta_\rho$, $\Theta^\infty_{\rho,\tau}$ and $b\Theta_{\rho,\tau}$ as H on Ω, $\bar\Omega_\rho$, $\Omega^\infty_{\rho,\tau}$ and $b\Omega_{\rho,\tau}$ (respectively), where h is related with H by (1.4),

$$\begin{aligned}&\Omega = \{k\in\mathbb{C}^d,\ p\in\mathbb{R}^d:\ k^2=0,\ p^2=2kp\},\\ &\bar\Omega_\rho = \Omega_\rho\cup b\Omega_\rho,\\ &\Omega_\rho = \{(k,p)\in\Omega:\ |Im\,k|<\rho\},\\ &b\Omega_\rho = \{(k,p)\in\Omega:\ |Im\,k|=\rho\},\\ &\Omega^\infty_{\rho,\tau} = \{(k,p)\in\Omega\backslash\bar\Omega_\rho:\ p\in\mathcal{B}_{2\rho\tau}\},\\ &b\Omega_{\rho,\tau} = \{(k,p)\in b\Omega_\rho:\ p\in\mathcal{B}_{2\rho\tau}\},\end{aligned} \tag{2.10}$$

where $\rho>0$, $0<\tau<1$.

Note that

$$\begin{aligned}&\Omega\approx\Theta,\ \bar\Omega_\rho\approx\bar\Theta_\rho,\ b\Omega_\rho\approx b\Theta_\rho,\\ &\Omega^\infty_{\rho,\tau}\approx\Theta^\infty_{\rho,\tau},\ \ b\Omega_{\rho,\tau}\approx b\Theta_{\rho,\tau}.\end{aligned} \tag{2.11}$$

or more precisely

$$\begin{aligned}&(k,p)\in\Omega\Longrightarrow(k,k-p)\in\Theta,\ \ (k,l)\in\Theta\Longrightarrow(k,k-l)\in\Omega\\ &\text{and the same for }\ \bar\Omega_\rho,\ b\Omega_\rho,\ \Omega^\infty_{\rho,\tau},\ b\Omega_{\rho,\tau}\ \text{ and }\ \bar\Theta_\rho,\ b\Theta_\rho,\ \Theta^\infty_{\rho,\tau},\ b\Theta_{\rho,\tau},\\ &\text{respectively,}\ \ \text{in place of }\Omega\ \text{ and }\ \Theta.\end{aligned} \tag{2.12}$$

An outline of the reconstruction formalized in Theorem 2.1 consists in the following:

1. We rewrite h on $b\Theta_{\rho,\tau}\subset b\Theta_\rho$ as H on $b\Omega_{\rho,\tau}\subset b\Omega_\rho$ as mentioned above.
2. We consider H on $b\Omega_\rho$ as boundary data for H on $\Omega\backslash\Omega_\rho$, which solves the non-linear $\bar\partial$-equation (3.5) with estimates (3.2), (3.3).
3. Using this $\bar\partial$-equation and these estimates we obtain the non-linear integral equation (4.31) for finding $\tilde H_{\rho,\tau}$ on $\Omega^\infty_{\rho,\tau}$ from H on $b\Omega_{\rho,\tau}$, where $\tilde H_{\rho,\tau}$ approximates H on $\Omega_{\rho,\tau}$ (with estimate (4.31)).
4. The function $\tilde H_{\rho,\tau}$ determines $\hat v^{\pm}(\cdot,\tau,\rho)$ by formulas (4.32), where $\hat v^{\pm}(\cdot,\tau,\rho)$ approximates $\hat v$ on $\mathcal{B}_{2\tau\rho}$.

As it was already mentioned in introduction, the results of the present work were obtained in the framework of a development of the $\bar\partial$-approach to inverse scattering at fixed energy in dimension $d\geq 3$ of [BC], [HN], [No3], [No4]. In particular, there is a considerable similarity between the reconstruction scheme of the present work for the case of Problem 1.1c and the reconstruction scheme of [No3] for the case of inverse scattering at fixed positive energy E. Actually, in the present work the parameter ρ of Problem 1.1 plays the role of $E^{1/2}$ of [No3]. Some results of [BC], [HN], [No4] we use in the present work are recalled in the next section.

3. Background results

For simplicity always in this section we assume that $d = 3$.

3.1. Estimates for H on $\Omega\backslash\Omega_\rho$. Let v satisfy (2.1) and $\|\hat{v}\|_\mu \le C$. Let

$$\eta(C,\rho,\mu) \stackrel{\text{def}}{=} a(\mu)C(\ln\,\rho)^2\rho^{-1} < 1, \quad \ln\,\rho \ge 2, \tag{3.1}$$

where $a(\mu)$ is the constant $c_2(\mu)$ of [No4]. Then (according to [No4]):

$$H \in \mathcal{C}(\Omega\backslash\Omega_\rho), \tag{3.2}$$

$$|H(k,p)| \le \frac{C}{(1-\eta(C,\rho,\mu))(1+|p|)^\mu}, \quad (k,p) \in \Omega\backslash\Omega_\rho, \tag{3.3}$$

$$\hat{v}(p) = \lim_{|k|\to\infty,\ (k,p)\in\Omega} H(k,p), \quad p \in \mathbb{R}^3, \tag{3.4}$$

where $|k| = ((Re\,k)^2 + (Im\,k)^2)^{1/2}$. (These and some additional estimates on H are given in Proposition 3.2 of [No4].) Note that, for sufficiently regular v on $\mathbb{R}^3$ with sufficient decay at infinity, formula (3.4) was obtained for the first time in [HN].

3.2. The $\bar\partial$-equation for H on $\Omega\backslash\Omega_\rho$. Let v satisfy (2.1), $\|\hat{v}_\mu\| \le C$, and (3.1) hold. Then (see [No4]):

$$\begin{aligned}&\bar\partial_k H(k,p)\big|_{\Omega\backslash\Omega_\rho} = \\ &\sum_{j=1}^{3}\Big(-2\pi \int\limits_{\xi\in S_k} \xi_j H(k,-\xi)H(k+\xi,p+\xi)\frac{ds}{|Im\,k|^2}\Big)d\bar{k}_j\big|_{\Omega\backslash\Omega_\rho},\end{aligned} \tag{3.5}$$

where

$$S_k = \{\xi \in \mathbb{R}^3 :\ \xi^2 + 2k\xi = 0\}, \tag{3.6}$$

ds is arc-length measure on the circle S_k in $\mathbb{R}^3$. Actually, under some stronger assumptions on v than in the present subsection, the $\bar\partial$- equation (3.5) was obtained for the first time in [BC].

An important property of the $\bar\partial$-equation (3.5) is that (3.5) can be considered for H on $\Omega\backslash\Omega_\rho$ only, see, in particular, formulas (3.24) of Subsection 3.4.

3.3. Coordinates on $\Omega\backslash\Omega_\rho$. Let

$$\Omega_\nu = \{(k,p) \in \Omega :\ p \notin \mathcal{L}_\nu\}, \tag{3.7}$$

where

$$\mathcal{L}_\nu = \{p \in \mathbb{R}^3 :\ p = t\nu,\ t \in \mathbb{R}\}, \quad \nu \in \mathbb{S}^2. \tag{3.8}$$

For $p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu$ we consider $\theta(p)$ and $\omega(p)$ such that

$$\begin{aligned} &\theta(p), \omega(p) \ \text{ smoothly depend on } \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu, \\ &\text{take values in } \ \mathbb{S}^2, \ \text{ and} \\ &\theta(p)p = 0, \ \omega(p)p = 0, \ \theta(p)\omega(p) = 0. \end{aligned} \tag{3.9}$$

Assumptions (3.9) imply that

$$\omega(p) = \frac{p \times \theta(p)}{|p|} \ \text{ for } \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu \tag{3.10a}$$

or

$$\omega(p) = -\frac{p \times \theta(p)}{|p|} \ \text{ for } \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu, \tag{3.10b}$$

where $\times$ denotes vector product.

To satisfy (3.9), (3.10a) we can take

$$\theta(p) = \frac{\nu \times p}{|\nu \times p|}, \ \omega(p) = \frac{p \times \theta(p)}{|p|}, \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu. \tag{3.11}$$

Let θ, ω satisfy (3.9). Then (according to [No4]) the following formulas give a diffeomorphism between Ω_ν and $(\mathbb{C}\backslash 0) \times (\mathbb{R}^3 \backslash \mathcal{L}_\nu)$:

$$(k, p) \to (\lambda, p), \ \text{ where } \ \lambda = \lambda(k, p) = \frac{2k(\theta(p) + i\omega(p))}{i|p|}, \tag{3.12a}$$

$$\begin{aligned} &(\lambda, p) \to (k, p), \ \text{ where } \ k = k(\lambda, p) = \kappa_1(\lambda, p)\theta(p) + \kappa_2(\lambda, p)\omega(p) + \frac{p}{2}, \\ &\kappa_1(\lambda, p) = \frac{i|p|}{4}(\lambda + \frac{1}{\lambda}), \ \ \kappa_2(\lambda, p) = \frac{|p|}{4}(\lambda - \frac{1}{\lambda}), \end{aligned} \tag{3.12b}$$

where $(k, p) \in \Omega_\nu$, $(\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathbb{R}^3 \backslash \mathcal{L}_\nu)$. In addition, formulas (3.12a), (3.12b) for $\lambda(k)$ and $k(\lambda)$ at fixed $p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu$ give a diffeomorphism between $Z_p = \{k \in \mathbb{C}^3 : \ (k, p) \in \Omega\}$ for fixed p and $\mathbb{C}\backslash 0$.

In addition, for k and λ of (3.12) we have that

$$|Im\, k| = \frac{|p|}{4}\big(|\lambda| + \frac{1}{|\lambda|}\big), \ \ |Re\, k| = \frac{|p|}{4}\big(|\lambda| + \frac{1}{|\lambda|}\big), \tag{3.13}$$

where $(k, p) \in \Omega_\nu$, $(\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathbb{R}^3 \backslash \mathcal{L}_\nu)$.

Let

$$\Omega^\infty_{\rho,\tau,\nu} = \Omega^\infty_{\rho,\tau} \cap \Omega_\nu, \tag{3.14}$$

where $\Omega^\infty_{\rho,\tau}$, Ω_ν are defined as in (2.10), (3.7). Let

$$\Lambda_{\rho,\nu} = \{(\lambda, p) : \ \lambda \in \mathcal{D}_{\rho/|p|}, \ \ p \in \mathbb{R}^3 \backslash \mathcal{L}_\nu\}, \tag{3.15}$$

$$\Lambda_{\rho,\tau,\nu}=\{(\lambda,p):\ \lambda\in\mathcal{D}_{\rho/|p|},\ \ p\in\mathbb{R}^3\backslash\mathcal{L}_\nu,\ \ |p|<2\tau\rho\},$$
$$b\Lambda_{\rho,\tau,\nu}=\{(\lambda,p):\ \lambda\in\mathcal{T}_{\rho/|p|},\ \ p\in\mathbb{R}^3\backslash\mathcal{L}_\nu,\ \ |p|<2\tau\rho\},$$

where $\rho>0$, $0<\tau<1$, $\nu\in\mathbb{S}^2$,

$$\mathcal{D}_r=\{\lambda\in\mathbb{C}\backslash 0:\ \frac{1}{4}(|\lambda|+|\lambda|^{-1})>r\},\ r>0. \tag{3.16}$$

$$\mathcal{T}_r=\{\lambda\in\mathbb{C}:\ \frac{1}{4}(|\lambda|+|\lambda|^{-1})=r\},\ r\geq 1/2. \tag{3.17}$$

Using (3.13) one can see that formulas (3.12) give also the following diffeomorphisms

$$\begin{aligned}&\Omega_\nu\backslash\bar\Omega_\rho\approx\Lambda_{\rho,\nu},\quad \Omega^\infty_{\rho,\tau,\nu}\approx\Lambda_{\rho,\tau,\nu},\\ &b\Omega_{\rho,\tau}\cap\Omega_\nu\approx b\Lambda_{\rho,\tau,\nu},\\ &Z^\infty_{p,\rho}=\{k\in\mathbb{C}^3:\ (k,p)\in\Omega_\nu\backslash\bar\Omega_\rho\}\approx\mathcal{D}_{\rho/|p|}\ \text{ for fixed }\ p,\end{aligned} \tag{3.18}$$

where $\rho>0$, $0<\tau<1$, $\nu\in\mathbb{S}^2$.

In [No4] λ,p of (3.12) were used as coordinates on Ω. In the present work we use them also as coordinates on $\Omega\backslash\Omega_\rho$ (or more precisely on $\Omega_\nu\backslash\Omega_\rho$).

3.4. $\bar\partial$-equation for H in the λ,p coordinates and some related estimate. Let λ,p be the coordinates of Subsection 3.3, where θ, ω satisfy (3.9), (3.10). Then (see Lemma 5.1 of [No4]) in these coordinates the $\bar\partial$-equation (3.5) for $p\neq 0$ takes the form:

$$\begin{aligned}&\frac{\partial}{\partial\bar\lambda}H(k(\lambda,p),p)=-\frac{\pi}{4}\int_{-\pi}^{\pi}\left(\frac{|p|}{2}\frac{(|\lambda|^2-1)}{\bar\lambda|\lambda|}(\cos\varphi-1)-|p|\frac{1}{\bar\lambda}\sin\varphi\right)\times\\ &H(k(\lambda,p),-\xi(\lambda,p,\varphi))H(k(\lambda,p,\varphi),p+\xi(\lambda,p,\varphi))d\varphi\end{aligned} \tag{3.19}$$

for $(\lambda,p)\in\Lambda_{\rho,\nu}$, where $k(\lambda,p)$ is defined in (3.12b) (and also depends on ν, θ, ω), $\Lambda_{\rho,\nu}$ is defined in (3.15),

$$\xi(\lambda,p,\varphi)=Re\,k(\lambda,p)(\cos\varphi-1)+k^\perp(\lambda,p)\sin\varphi, \tag{3.20}$$

$$k^\perp(\lambda,p)=\frac{Im\,k(\lambda,p)\times Re\,k(\lambda,p)}{|Im\,k(\lambda,p)|}, \tag{3.21}$$

where $\times$ in (3.21) denotes vector product.

Note that (3.19) can be written as

$$\frac{\partial}{\partial\bar\lambda}H(k(\lambda,p),p)=\{H,H\}(\lambda,p),\quad (\lambda,p)\in\Lambda_{\rho,\nu}, \tag{3.22}$$

where

$$\begin{aligned}&\{U_1,U_2\}(\lambda,p)=-\frac{\pi}{4}\int_{-\pi}^{\pi}\left(\frac{|p|}{2}\frac{|\lambda|^2-1}{\bar\lambda|\lambda|}(\cos\varphi-1)-\frac{|p|}{\bar\lambda}\sin\varphi\right)\times\\ &U_1(k(\lambda,p),-\xi(\lambda,p,\varphi))U_2(k(\lambda,p)+\xi(\lambda,p,\varphi),p+\xi(\lambda,p,\varphi))d\varphi,\end{aligned} \tag{3.23}$$

where U_1, U_2 are test functions on $\Omega\backslash\bar{\Omega}_\rho$, $k(\lambda,p)$, $\xi(\lambda,p,\varphi)$ are defined by (3.12b), (3.20), $(\lambda,p)\in\Lambda_{\rho,\nu}$. Note that in the left-hand side of (3.19), (3.22)

$$(k(\lambda,p),p)\in\Omega_\nu\backslash\bar{\Omega}_\rho \tag{3.24a}$$

and in the right-hand side of (3.19), (3.23)

$$\begin{aligned}&(k(\lambda,p),-\xi(\lambda,p,\varphi))\in\Omega\backslash\bar{\Omega}_\rho,\\&(k(\lambda,p)+\xi(\lambda,p,\varphi),p+\xi(\lambda,p,\varphi))\in\Omega\backslash\bar{\Omega}_\rho,\end{aligned} \tag{3.24b}$$

where $(\lambda,p)\in\Lambda_{\rho,\nu}$, $\varphi\in[-\pi,\pi]$.

Let $U_1,U_2\in L^\infty_\mu(\Omega\backslash\bar{\Omega}_\rho)$, $\mu\geq 2$, where

$$\begin{aligned}&L^\infty_\mu(\Omega\backslash\bar{\Omega}_\rho)=\{U\in L^\infty(\Omega\backslash\bar{\Omega}_\rho):\ |||U|||_{\rho,\mu}<+\infty\},\\&|||U|||_{\rho,\mu}=ess\sup_{(k,p)\in\Omega\backslash\bar{\Omega}_\rho}(1+|p|)^\mu|U(k,p)|,\ \mu>0.\end{aligned} \tag{3.25}$$

Let $\{U_1,U_2\}$ be defined by (3.23). Then (as a corollary of Lemma 5.2 of [No4]):

$$\{U_1,U_2\}\in L^\infty_{local}(\Lambda_{\rho,\nu}) \tag{3.26}$$

and

$$\begin{aligned}&|\{U_1,U_2\}(\lambda,p)|\leq\frac{|||U_1|||_{\rho,\mu}|||U_2|||_{\rho,\mu}}{(1+|p|)^\mu}b(\mu,|\lambda|,|p|)\\&\text{for almost all }\ (\lambda,p)\in\Lambda_{\rho,\nu},\quad b(\mu,|\lambda|,|p|)=\\&\left(\frac{b_1(\mu)|\lambda|}{(|\lambda|^2+1)^2}+\frac{b_2(\mu)|p|||\lambda|^2-1|}{|\lambda|^2(1+|p|(|\lambda|+|\lambda|^{-1}))^2}+\frac{b_3(\mu)|p|}{|\lambda|(1+|p|(|\lambda|+|\lambda|^{-1}))}\right),\end{aligned} \tag{3.27}$$

where $\Lambda_{\rho,\nu}$ is defined in (3.15), $b_1(\mu)$, $b_2(\mu)$, $b_3(\mu)$ are the constants $c_3(\mu)$, $c_4(\mu)$, $c_5(\mu)$ of [No4].

4. Approximate finding H on $\Omega^\infty_{\rho,\tau}$ from H on $b\Omega_{\rho,\tau}$

We recall that $\Omega^\infty_{\rho,\tau}$ and $b\Omega_{\rho,\tau}$ were defined in Section 2, see formulas (2.10). We assume that $d=3$.

Consider $\chi_r H$, where χ_r denotes the multiplication operator by the function

$$\chi_r(p)=1\ \text{ for }\ |p|<r,\ \chi_r(p)=0\ \text{ for }\ |p|\geq r,\ \text{ where }\ p\in\mathbb{R}^3,\ r>0. \tag{4.1}$$

Note that

$$\begin{aligned}&\chi_{2\tau\rho}H(k,p)=H(k,p)\ \text{ for }\ (k,p)\in\Omega^\infty_{\rho,\tau}\\&\chi_{2\tau\rho}H(k,p)=0\ \text{ for }\ (k,p)\in(\Omega\backslash\bar{\Omega}_\rho)\backslash\Omega^\infty_{\rho,\tau},\end{aligned} \tag{4.2}$$

where $\rho>0$, $\tau\in]0,1[$.

As a corollary of (3.5), (3.22), (3.23) we have that

$$\frac{\partial}{\partial\bar\lambda}\chi_{2\tau\rho}H(k(\lambda,p),p)=\{\chi_{2\tau\rho}H,\chi_{2\tau\rho}H\}(\lambda,p)+R_{\rho,\tau}(\lambda,p), \tag{4.3}$$

$$\begin{aligned}&R_{\rho,\tau}(\lambda,p)=\\&\{(1-\chi_{2\tau\rho})H,\chi_{2\tau\rho}H\}(\lambda,p)+\{\chi_{2\tau\rho}H,(1-\chi_{2\tau\rho})H\}(\lambda,p)+\\&\{(1-\chi_{2\tau\rho})H,(1-\chi_{2\tau\rho})H\}(\lambda,p)\end{aligned} \tag{4.4}$$

for $(\lambda,p)\in\Lambda_{\rho,\tau,\nu}$ of (3.15).

Because of the remainder $R_{\rho,\tau}$ of (4.3), (4.4), the $\bar\partial$-equation (3.5), (4.3) is only an approximate $\bar\partial$- equation for $\chi_{2\tau\rho}H=H$ on $\Omega^\infty_{\rho,\tau}$ or on $\Lambda_{\rho,\tau,\nu}$ in the coordinates λ, p. However, $R_{\rho,\tau}$ rapidly vanishes when ρ increases for fixed $\tau\in]0,1[$; see Lemma 4.1.

For approximate finding H on $\Omega^\infty_{\rho,\tau}$ from H on $\Omega^\infty_{\rho,\tau}$ we proceed from (3.2), (3.3), (4.3), (4.4), (3.26), (3.27) and the following formulas

$$u_+(\lambda)=\frac{1}{2\pi i}\int\limits_{\mathcal{T}_r^+}u_+(\zeta)\frac{d\zeta}{\zeta-\lambda}-\frac{1}{\pi}\iint\limits_{\mathcal{D}_r^+}\frac{\partial u_+(\zeta)}{\partial\bar\zeta}\frac{d\,Re\,\zeta\;d\,Im\,\zeta}{\zeta-\lambda},\quad\lambda\in\mathcal{D}_r^+, \tag{4.5a}$$

$$u_-(\lambda)=-\frac{1}{2\pi i}\int\limits_{\mathcal{T}_r^-}u_-(\zeta)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda)}-\frac{1}{\pi}\iint\limits_{\mathcal{D}_r^-}\frac{\partial u_-(\zeta)}{\partial\bar\zeta}\frac{\lambda d\,Re\,\zeta\;d\,Im\,\zeta}{\zeta(\zeta-\lambda)},\quad\lambda\in\mathcal{D}_r^-, \tag{4.5b}$$

where

$$\begin{aligned}&\mathcal{D}_r^\pm=\{\lambda\in\mathbb{C}\backslash 0:\ \tfrac{1}{4}(|\lambda|+|\lambda|^{-1})>r,\ \ |\lambda|^{\pm1}<1\},\\&\mathcal{T}_r^\pm=\{\lambda\in\mathbb{C}:\ \tfrac{1}{4}(|\lambda|+|\lambda|^{-1})=r,\ \ |\lambda|^{\pm1}\le 1\},\quad r>1/2,\end{aligned} \tag{4.6}$$

$u_+(\lambda)$ is continuous and bounded on $\mathcal{D}_r^+\cup\mathcal{T}_r^+$, $\partial u_+(\lambda)/\partial\bar\lambda$ is bounded on $\mathcal{D}_r^+$, $u_-(\lambda)$ is continuous and bounded on $\mathcal{D}_r^-\cup\mathcal{T}_r^-$, $\partial u_-(\lambda)/\partial\bar\lambda$ is bounded on $\mathcal{D}_r^-$, and $\partial u_-(\lambda)/\partial\bar\lambda=O(|\lambda|^{-2})$ as $|\lambda|\to\infty$ (and where the integrals along $\mathcal{T}_r^\pm$ are taken in the counter-clockwise direction). The aforementioned assumptions on $u_\pm$ in (4.5) can be somewhat weakened. Formulas (4.5) follow from the well-known Cauchy-Green formula

$$u(\lambda)=\frac{1}{2\pi i}\int\limits_{\partial\mathcal{D}}u(\zeta)\frac{d\zeta}{\zeta-\lambda}-\frac{1}{\pi}\iint\limits_{\mathcal{D}}\frac{\partial u(\zeta)}{\partial\bar\zeta}\frac{d\,Re\,\zeta\;d\,Im\,\zeta}{\zeta-\lambda},\quad\lambda\in\mathcal{D}, \tag{4.7}$$

where $\mathcal{D}$ is a bounded open domain in $\mathbb{C}$ with sufficiently regular boundary and u is a sufficiently regular function on $\bar{\mathcal{D}}=\mathcal{D}\cup\partial\mathcal{D}$.

Let

$$H(\lambda,p)=H(k(\lambda,p),p),\quad(\lambda,p)\in(\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu), \tag{4.8}$$

where λ,p are the coordinates of Subsection 3.3 under assumption (3.10a).

Let

$$\begin{aligned}&\Lambda^\pm_{\rho,\tau,\nu}=\{(\lambda,p):\ \lambda\in\mathcal{D}^\pm_{\rho/|p|},\ p\in\mathcal{B}_{2\tau\rho}\backslash\mathcal{L}_\nu\},\\&b\Lambda^\pm_{\rho,\tau,\nu}=\{(\lambda,p):\ \lambda\in\mathcal{T}^\pm_{\rho/|p|},\ p\in\mathcal{B}_{2\tau\rho}\backslash\mathcal{L}_\nu\},\end{aligned} \tag{4.9}$$

where $\mathcal{B}_r$, $\mathcal{L}_\nu$, $\mathcal{D}_r^\pm$, $\mathcal{T}_r^\pm$ are defined by (1.12) for $d=3$, (3.8), (4.6), $\rho>0$, $\tau\in]0,1[$, $\nu\in\mathbb{S}^2$.

Note that

$$\Lambda_{\rho,\tau,\nu}=\Lambda^+_{\rho,\tau,\nu}\cup\Lambda^-_{\rho,\tau,\nu},\quad \Lambda^+_{\rho,\tau,\nu}\cap\Lambda^-_{\rho,\tau,\nu}=\emptyset,\quad b\Lambda_{\rho,\tau,\nu}=b\Lambda^+_{\rho,\tau,\nu}\cup\Lambda^-_{\rho,\tau,\nu},\tag{4.10}$$

where $\Lambda_{\rho,\tau,\nu}$, $b\Lambda_{\rho,\tau,\nu}$ were defined in (3.15), $\rho>0$, $\tau\in]0,1[$, $\nu\in\mathbb{S}^2$.

As a corollary of (3.2), (3.3), (4.3), (4.4), (3.26), (3.27), (4.5), we obtain the following

Proposition 4.1. *Let v and ρ satisfy the same assumptions that in Subsection 3.1. Let $H(\lambda,p)$ be defined by (4.8). Then $H=H(\lambda,p)$ as a function of $(\lambda,p)\in\Lambda_{\rho,\tau,\nu}$ of (4.10), where $\tau\in]0,1[$, satisfies the following nonlinear integral equation*

$$H=H^0+M_{\rho,\tau}(H)+Q_{\rho,\tau},\quad \tau\in]0,1[,\tag{4.11}$$

where

$$H^0(\lambda,p)=\frac{1}{2\pi i}\int\limits_{\mathcal{T}^+_{\rho/|p|}}H(\zeta,p)\frac{d\zeta}{\zeta-\lambda},\ (\lambda,p)\in\Lambda^+_{\rho,\tau,\nu},\tag{4.12a}$$

$$H^0(\lambda,p)=-\frac{1}{2\pi i}\int\limits_{\mathcal{T}^-_{\rho/|p|}}H(\zeta,p)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda)},\ (\lambda,p)\in\Lambda^-_{\rho,\tau,\nu},\tag{4.12b}$$

where $\mathcal{T}_r^\pm$ are defined by (4.6);

$$\begin{aligned}&M_{\rho,\tau}(U)(\lambda,p)=M^+_{\rho,\tau}(U)(\lambda,p)=\\ &-\frac{1}{\pi}\iint\limits_{\mathcal{D}^+_{\rho/|p|}}(U,U)_{\rho,\tau}(\zeta,p)\frac{d\,Re\,\zeta\;d\,Im\,\zeta}{\zeta-\lambda},\quad(\lambda,p)\in\Lambda^+_{\rho,\tau,\nu},\end{aligned}\tag{4.13a}$$

$$\begin{aligned}&M_{\rho,\tau}(U)(\lambda,p)=M^-_{\rho,\tau}(U)(\lambda,p)=\\ &-\frac{1}{\pi}\iint\limits_{\mathcal{D}^-_{\rho/|p|}}(U,U)_{\rho,\tau}(\zeta,p)\frac{\lambda d\,Re\,\zeta\;d\,Im\,\zeta}{\zeta(\zeta-\lambda)},\quad(\lambda,p)\in\Lambda^-_{\rho,\tau,\nu},\end{aligned}\tag{4.13b}$$

$$\begin{aligned}&(U_1,U_2)_{\rho,\tau}(\zeta,p)=\{\chi_{2\tau\rho}U_1',\chi_{2\tau\rho}U_2'\}(\zeta,p),\ (\zeta,p)\in\Lambda_{\rho,\tau,\nu},\\ &\chi_{2\tau\rho}U_j'(k,p)=U_j(\lambda(k,p),p),\ (k,p)\in\Omega^\infty_{\rho,\tau,\nu},\\ &\chi_{2\tau\rho}U_j'(k,p)=0,\ |p|\ge 2\tau\rho,\ j=1,2,\end{aligned}\tag{4.14}$$

where U,U_1,U_2 are test functions on $\Lambda_{\rho,\tau,\nu}$, $\{\cdot,\cdot\}$ is defined by (3.23), $\lambda(k,p)$ is defined in (3.12a);

$$Q_{\rho,\tau}(\lambda,p)=-\frac{1}{\pi}\iint\limits_{\mathcal{D}^+_{\rho/|p|}}R_{\rho,\tau}(\zeta,p)\frac{d\,Re\,\zeta\;d\,Im\,\zeta}{\zeta-\lambda},\quad(\lambda,p)\in\Lambda^+_{\rho,\tau,\nu},\tag{4.15a}$$

$$Q_{\rho,\tau}(\lambda,p)=-\frac{1}{\pi}\iint\limits_{\mathcal{D}^-_{\rho/|p|}}R_{\rho,\tau}(\zeta,p)\frac{\lambda d\,Re\,\zeta\;d\,Im\,\zeta}{\zeta(\zeta-\lambda)},\quad(\lambda,p)\in\Lambda^-_{\rho,\tau,\nu},\tag{4.15b}$$

where $R_{\rho,\tau}$ *is defined by* (4.4).

Remark 4.1. In addition to (4.14), note that the definition of $(U_1,U_2)_{\rho,\tau}$ can be also written as

$$\begin{aligned}(U_1,U_2)_{\rho,\tau}(\lambda,p)=-\frac{\pi}{4}\int_{-\pi}^{\pi}\Big(\frac{|p|}{2}\frac{|\lambda|^2-1}{\bar\lambda|\lambda|}(\cos\varphi-1)-\frac{|p|}{\bar\lambda}\sin\varphi\Big)\times\\ U_1(z_1(\lambda,p,\varphi),-\xi(\lambda,p,\varphi))U_2(z_2(\lambda,p,\varphi),p+\xi(\lambda,p,\varphi))\times\\ \chi_{2\tau\rho}(\xi(\lambda,p,\varphi))\chi_{2\tau\rho}(p+\xi(\lambda,p,\varphi))d\varphi,\end{aligned}\tag{4.16}$$

where

$$\begin{aligned}z_1(\lambda,p,\varphi)&=\frac{2k(\lambda,p)(\theta(-\xi(\lambda,p,\varphi))+i\omega(-\xi(\lambda,p,\varphi)))}{i|p|},\\ z_2(\lambda,p,\varphi)&=\frac{2(k(\lambda,p)+\xi(\lambda,p,\varphi))(\theta(p+\xi(\lambda,p,\varphi))+i\omega(p+\xi(\lambda,p,\varphi)))}{i|p|},\end{aligned}\tag{4.17}$$

$(\lambda,p)\in\Lambda_{\rho,\tau,\nu}$, $\varphi\in[-\pi,\pi]$, $k(\lambda,p)$ is defined in (3.12b), $\xi(\lambda,p,\varphi)$ is defined by (3.20), θ, ω are the vector functions of (3.9), (3.10a).

We consider (4.11) as an integral equation for finding H from H^0 with unknown remainder $Q_{\rho,\tau}$, where H, H^0, $Q_{\rho,\tau}$ are considered on $\Lambda_{\rho,\tau,\nu}$. Thus, actually, we consider (4.11) as an approximate equation for finding H on $\Lambda_{\rho,\tau,\nu}$ from H^0 on $\Lambda_{\rho,\tau,\nu}$. To deal with (4.11) we use Lemmas 4.1-4.5 given below.

Let

$$|||U|||_{\rho,\tau,\mu}=ess\sup_{(\lambda,p)\in\Lambda_{\rho,\tau,\nu}}(1+|p|)^{\mu}|U(\lambda,p)|\tag{4.18}$$

for $U\in L^\infty(\Lambda_{\rho,\tau,\nu})$, where $\rho>0$, $\tau\in]0,1[$, $\nu\in\mathbb{S}^2$, $\mu>0$.

Lemma 4.1. *Let v and ρ satisfy the same assumptions that in Subsection 3.1. Let $R_{\rho,\tau}$, $Q_{\rho,\tau}$ be defined by* (4.4), (4.15), $\tau\in]0,1[$. *Then*

$$R_{\rho,\tau}\in L^\infty_{local}(\Lambda_{\rho,\tau,\nu}),\tag{4.19a}$$

$$|R_{\rho,\tau}(\lambda,p)|\le\frac{3b(\mu_0,|\lambda|,|p|)C^2}{(1-\eta)^2(1+2\tau\rho)^{\mu-\mu_0}(1+|p|)^{\mu_0}},\ (\lambda,p)\in\Lambda_{\rho,\tau,\nu},\tag{4.19b}$$

$$Q_{\rho,\tau}\in L^\infty(\Lambda_{\rho,\tau,\nu}),\tag{4.20a}$$

$$|||Q_{\rho,\tau}|||_{\rho,\tau,\mu_0}\le\frac{3b_4(\mu_0)C^2}{(1-\eta)^2(1+2\tau\rho)^{\mu-\mu_0}},\tag{4.20b}$$

where $2\le\mu_0\le\mu$, $b(\mu,|\lambda|,|p|)$ *is defined in* (3.27), $\eta=\eta(C,\rho,\mu)$ *is defined by* (3.1),

$$b_4(\mu)=\frac{1}{\pi}(b_1(\mu)n_1+b_2(\mu)n_2+b_3(\mu)n_3),\tag{4.21}$$

where b_1,b_2,b_3 *are the constants of* (3.27) (*the constants* c_3,c_4,c_5 *of Lemma* 5.2 *of* [No4]), n_1,n_2,n_3 *are the constants of Lemma* 11.1 *of* [No4].

Lemma 4.1 is proved in Section 7.

Lemma 4.2. *Let v and ρ satisfy the same assumptions that in Subsection 3.1. Let H^0 be defined by* (4.8), (4.12), $\tau \in]0,1[$. *Then*

$$H^0 \in L^\infty(\Lambda_{\rho,\tau,\nu}), \tag{4.22a}$$

$$|||H^0|||_{\rho,\tau,\mu_0} \le \frac{C}{1-\eta}\left(1+\frac{b_4(\mu_0)C}{1-\eta}\right), \tag{4.22b}$$

where $2 \le \mu_0 \le \mu$, $\eta = \eta(C,\rho,\mu)$ *is defined by* (3.1), b_4 *is defined by* (4.21).

Lemma 4.2 is proved in Section 7.

Lemma 4.3. *Let* $\rho > 0$, $\nu \in \mathbb{S}^2$, $\tau \in]0,1[$, $\mu \ge 2$. *Let* $M_{\rho,\tau}$ *be defined by* (4.13), (4.14) (*where* λ, p *the coordinates of Subsection* 3.3 *under assumption* (3.10a)). *Let* $U_1, U_2 \in L^\infty(\Lambda_{\rho,\tau,\nu})$, $|||U_1|||_{\rho,\tau,\mu} < +\infty$, $|||U_2|||_{\rho,\tau,\mu} < +\infty$. *Then*

$$M_{\rho,\tau}(U_j) \in L^\infty(\Lambda_{\rho,\tau,\nu}),\ j = 1,2, \tag{4.23}$$

$$|||M_{\rho,\tau}(U_j)|||_{\rho,\tau,\mu} \le c_8(\mu,\tau,\rho)(|||U_j|||_{\rho,\tau,\mu})^2,\ j = 1,2, \tag{4.24}$$

$$\begin{aligned}&|||M_{\rho,\tau}(U_1) - M_{\rho,\tau}(U_2)|||_{\rho,\tau,\mu} \le \\ &c_8(\mu,\tau,\rho)(|||U_1|||_{\rho,\tau,\mu} + |||U_2|||_{\rho,\tau,\mu})|||U_1 - U_2|||_{\rho,\tau,\mu},\end{aligned} \tag{4.25}$$

where

$$c_8(\mu,\tau,\rho) = 3b_1(\mu)\tau^2 + 4b_2(\mu)\rho^{-1} + 4b_3(\mu)\tau, \tag{4.26}$$

where b_1, b_2, b_3 *are the constants of* (3.27).

Lemma 4.3 is proved in Section 7.

Lemmas 4.1, 4.2, 4.3 show that, under the assumptions of Proposition 4.1, the nonlinear integral equation (4.11) for unknown H can be analysed for H^0, $Q_{\rho,\tau}$, $H \in L^\infty(\Lambda_{\rho,\tau,\nu})$ using the norm $|||\cdot|||_{\rho,\tau,\mu_0}$, where $2 \le \mu_0 \le \mu$.

Consider the equation

$$U = U^0 + M_{\rho,\tau}(U), \quad \rho > 0, \quad \tau \in]0,1[, \tag{4.27}$$

for unknown U (where U^0, U are functions on $\Lambda_{\rho,\tau,\nu}$). Actually, under the assumptions of Proposition 4.1, we suppose that $U^0 = H^0 + Q_{\rho,\tau}$ or consider U^0 as an approximation to $H^0 + Q_{\rho,\tau}$.

Lemma 4.4. *Let* $\rho > 0$, $\nu \in \mathbb{S}^2$, $\tau \in]0,1[$, $\mu \ge 2$ *and* $0 < r < (2c_8(\mu,\tau,\rho))^{-1}$. *Let* $M_{\rho,\tau}$ *be defined by* (4.13), (4.14) (*where* λ, p *are the coordinates of Subsection* 3.3 *under assumption* (3.10a)). *Let* $U^0 \in L^\infty(\Lambda_{\rho,\tau,\nu})$ *and* $|||U^0|||_{\rho,\tau,\mu} \le r/2$. *Then equation* (4.27) *is uniquely solvable for* $U \in L^\infty(\Lambda_{\rho,\tau,\nu})$, $|||U|||_{\rho,\tau,\mu} \le r$, *and* U *can be found by the method of successive approximations, in addition,*

$$|||U - (M_{\rho,\tau,U^0})^n(0)|||_{\rho,\tau,\mu} \le \frac{r(2c_8(\mu,\tau,\rho)r)^n}{2(1-2c_8(\mu,\tau,\rho)r)},\ n \in \mathbb{N}, \tag{4.28}$$

where M_{ρ,τ,U^0} *denotes the map* $U \to U^0 + M_{\rho,\tau}(U)$.

Lemma 4.4 is proved in Section 8 (using Lemma 4.3 and the lemma about contraction maps).

Lemma 4.5. *Let the assumptions of Lemma* 4.4 *be fulfilled. Let also* $\tilde U^0 \in L^\infty(\Lambda_{\rho,\tau,\nu})$, $|||\tilde U^0|||_{\rho,\tau,\mu} \le r/2$, *and* $\tilde U$ *denote the solution of* (4.27) *with* U^0 *replaced by* $\tilde U^0$, *where* $\tilde U \in L^\infty(\Lambda_{\rho,\tau,\nu})$, $|||\tilde U|||_{\rho,\tau,\mu} \le r$. *Then*

$$|||U - \tilde U|||_{\rho,\tau,\mu} \le (1 - 2c_8(\mu,\tau,\rho)r)^{-1}|||U^0 - \tilde U^0|||_{\rho,\tau,\mu}. \tag{4.29}$$

Lemma 4.5 is proved in Section 8.

Estimates (3.2), (3.3), Proposition 4.1 and Lemmas 4.1, 4.2, 4.3, 4.4, 4.5 imply, in particular, the following result.

Proposition 4.2. *Let* v *and* ρ *satisfy the same assumptions that in Subsection 3.1. Let* $\nu \in \mathbb{S}^2$, $\tau \in]0,1[$, $2 \le \mu_0 < \mu$. *Let* H, H^0 *be defined on* $\Lambda_{\rho,\tau,\nu}$ *by* (4.8), (4.12) *and* $M_{\rho,\tau}$ *be defined by* (4.13). *Let*

$$\begin{aligned} &r_{min}(\mu,\mu_0,\tau,\rho,C) \le r < (2c_8(\mu_0,\tau,\rho))^{-1},\\ &r_{min} \stackrel{\text{def}}{=} \frac{2C}{1-\eta(C,\rho,\mu)} + \frac{2b_4(\mu_0)C^2}{(1-\eta(C,\rho,\mu))^2}\left(1 + \frac{3}{(1+2\tau\rho)^{\mu-\mu_0}}\right), \end{aligned} \tag{4.30}$$

where η *is defined in* (3.1), c_8 *is defined by* (4.26). *Then the equation*

$$\tilde H_{\rho,\tau} = H^0 + M_{\rho,\tau}(\tilde H_{\rho,\tau}) \tag{4.31}$$

is uniquely solvable for $\tilde H_{\rho,\tau} \in L^\infty(\Lambda_{\rho,\tau,\nu})$, $|||\tilde H_{\rho,\tau}|||_{\rho,\tau,\mu_0} \le r$, *by the method of successive approximations and*

$$|||H - \tilde H_{\rho,\tau}|||_{\rho,\tau,\mu_0} \le \frac{3b_4(\mu_0)C^2}{(1-2c_8(\mu_0,\tau,\rho)r)(1-\eta(C,\rho,\mu))^2(1+2\tau\rho)^{\mu-\mu_0}}. \tag{4.32}$$

Note that (4.30) can be fulfilled if and only if

$$r_{min}(\mu,\mu_0,\tau,\rho,C) < (2c_8(\mu_0,\tau,\rho))^{-1}. \tag{4.33}$$

Using the definitions of η of (3.1) and c_8 of (4.26) one can see that:

$$\begin{aligned} &\text{conditions (3.1) and (4.33) are fulfilled,}\\ &\text{if } \ C \le c_9(\mu,\mu_0,\rho,\tau) \ \text{ for appropriate positive } c_9, \end{aligned} \tag{4.34}$$

where $2 \le \mu_0 < \mu$, $\ln\rho \ge 2$, $0 < \tau < 1$. Using (4.34) one can see that Proposition 4.2 gives a method for approximate finding H on $\Omega^\infty_{\rho,\tau}$ from H on $b\Omega_{\rho,\tau}$ with estimate (4.32), at least, for sufficiently small potentials v in the sense $\|\hat v\|_\mu < C$, $C \le c_9(\mu,\mu_0,\rho,\tau)$. However, the main point is that Proposition 4.2 also contains a global result, see considerations given below in this section.

Due to (4.26) we have that

$$c_8(\mu_0,\tau,\rho) \le \varepsilon \ \text{ if } \ 0<\tau\le\tau(\varepsilon,\mu_0), \quad \rho \ge \rho(\varepsilon,\mu_0) \tag{4.35}$$

for any arbitrary small $\varepsilon > 0$ and appropriate sufficiently small $\tau(\varepsilon,\mu_0) \in]0,1[$ and sufficiently great $\rho(\varepsilon,\mu_0)$. Using (4.35) and the definition of η of (3.1) we obtain that:

$$\begin{aligned} &\text{conditions } \ (3.1),(4.33) \ \text{ are fulfilled and} \\ &0 \le \eta(C,\rho,\mu) < \delta, \quad 0 \le 2c_8(\mu_0,\tau,\rho)\, r_{min}(\mu,\mu_0,\tau,\rho,C) < \delta, \\ &\text{if } \ 0<\tau\le\tau_1(\mu,\mu_0,C,\delta), \quad \rho\ge\rho_1(\mu,\mu_0,C,\delta), \end{aligned} \tag{4.36}$$

where τ_1 and ρ_1 are appropriate constants such that $\tau_1 \in]0,1[$ is sufficiently small and ρ_1 is sufficiently great, $2 \le \mu_0 < \mu$, $0 < \delta < 1$.

As a corollary of Proposition 4.2 and property (4.36), we obtain the following result.

Corollary 4.1. *Let v satisfy* (2.1) *and* $\|\hat v\|_\mu \le C$. *Let*

$$0<\tau\le\tau_1(\mu,\mu_0,C,\delta), \quad \rho\ge\rho_1(\mu,\mu_0,C,\delta), \tag{4.37}$$

where $2 \le \mu_0 \le \mu$, $0 < \delta < 1$. *Then H on $b\Omega_{\rho,\tau}$ determines via* (4.12), (4.31) *the approximation $\tilde H_{\rho,\tau}$ to H on $\Omega^\infty_{\rho,\tau}$ with the error estimate* (4.32) (*where r can be taken, for example, as $r = r_{min}$ of* (4.30)) *and, in particular, with*

$$|||H - \tilde H_{\rho,\tau}|||_{\rho,\tau,\mu_0} = O(\rho^{-(\mu-\mu_0)}) \ \ \textit{as} \ \ \rho\to+\infty. \tag{4.38}$$

The constant C can be arbitrary great in Corollary 4.1 and, therefore, the result of Corollary 4.1 is global.

5. Approximate finding $\hat v$ on $\mathcal{B}_{2\tau\rho}$ from $\tilde H_{\rho,\tau}$ on $\Omega^\infty_{\rho,\tau}$

Consider, first, H on $\Omega^\infty_{\rho,\tau,\nu}$ in the coordinates k, p as H on $\Lambda_{\rho,\tau,\nu}$ in the coordinates λ, p according to (4.8). If $\hat v$ satisfies (2.1), then formulas (3.4), (4.8), (3.12b), (3.13) imply that

$$\begin{aligned} &H(\lambda,p) \to \hat v(p) \ \text{ as } \ \lambda\to 0, \\ &H(\lambda,p) \to \hat v(p) \ \text{ as } \ \lambda\to \infty, \end{aligned} \tag{5.1}$$

where $p \in \mathcal{B}_{2\tau\rho}\backslash\mathcal{L}_\nu$, $\tau \in]0,1[$.

Consider now $\tilde H_{\rho,\tau}$ defined in Proposition 4.2. Under the assumptions of Proposition 4.2, the following formulas hold:

$$\tilde H_{\rho,\tau}(\lambda,p) \to \hat v^+(p,\rho,\tau) \ \text{ as } \ \lambda\to 0, \tag{5.2a}$$

$$\tilde H_{\rho,\tau}(\lambda,p) \to \hat v^-(p,\rho,\tau) \ \text{ as } \ \lambda\to \infty, \tag{5.2b}$$

where

$$\begin{aligned} &\hat v^+(p,\rho,\tau) = \frac{1}{2\pi i}\int\limits_{\mathcal{T}^+_{\rho/|p|}} H(\zeta,p)\frac{d\zeta}{\zeta} - \\ &\frac{1}{\pi}\int\int\limits_{\mathcal{D}^+_{\rho/|p|}} (\tilde H_{\rho,\tau},\tilde H_{\rho,\tau})_{\rho,\tau}(\zeta,p)\frac{d\,Re\,\zeta d\,Im\,\zeta}{\zeta}, \end{aligned} \tag{5.3a}$$

$$\begin{aligned}\hat v^-(p,\rho,\tau)=\frac{1}{2\pi i}\int\limits_{\mathcal{T}^-_{\rho/|p|}} H(\zeta,p)\frac{d\zeta}{\zeta}+\\ \frac{1}{\pi}\iint\limits_{\mathcal{D}^-_{\rho/|p|}}(\tilde H_{\rho,\tau},\tilde H_{\rho,\tau})_{\rho,\tau}(\zeta,p)\frac{d\,Re\,\zeta\,d\,Im\,\zeta}{\zeta},\end{aligned}\qquad(5.3b)$$

where $p\in\mathcal{B}_{2\tau\rho}\backslash\mathcal{L}_\nu$, $\tau\in]0,1[$, $(\tilde H_{\rho,\tau},\tilde H_{\rho,\tau})_{\rho,\tau}$ is defined by means of (4.14), (4.16), (4.17).

Formulas (5.2), (5.3) follow from (4.31), where $|||\tilde H_{\rho,\tau}|||_{\rho,\tau,\mu_0}<\infty$, $\mu_0\ge 2$, formulas (4.12), (4.13) and estimate (3.27).

Formulas (5.1), (5.2), (4.18) imply that

$$\|\hat v-\hat v^\pm(\cdot,\rho,\tau)\|_{2\tau\rho,\mu_0}\le|||H-\tilde H_{\rho,\tau}|||_{\rho,\tau,\mu_0},\qquad(5.4)$$

where

$$\|w\|_{r,\mu}=ess\sup_{p\in\mathcal{B}_r\backslash\mathcal{L}_\nu}(1+|p|)^\mu|w(p)|,\quad \mu>0,\ r>0,\qquad(5.5)$$

and ρ,τ,μ_0 are the same that in Proposition 4.2. Thus, under the assumptions of Proposition 4.2 (or under the assumptions of Corollary 4.1), formulas (5.2), (5.3), (5.4), (4.32) imply that $\hat v$ on $\mathcal{B}_{2\tau\rho}$ can be approximately determined from $\tilde H_{\rho,\tau}$ on $\Omega^\infty_{\rho,\tau}$ as $\hat v_\pm(\cdot,\rho,\tau)$ of (5.2), (5.3) and

$$\begin{aligned}&\|\hat v-\hat v^\pm(\cdot,\rho,\tau)\|_{2\tau\rho,\mu_0}\ \text{ is smaller or equal than the right}-\text{hand side of }\ (4.32)\\ &\text{and, in particular, is }\ O(\rho^{-(\mu-\mu_0)})\ \text{ as }\ \rho\to+\infty.\end{aligned}\qquad(5.6)$$

6. Reconstruction of v from Φ

In this section we summarize our global 3D reconstruction

$$\Phi\xrightarrow{1}h\big|_{b\Theta_{\rho,\tau}}\xrightarrow{2}\hat v\big|_{\mathcal{B}_{2\tau\rho}}\xrightarrow{3}v\qquad(6.1)$$

developed in [No1], [No2] and in Sections 4, 5 of the present work. See formulas (1.22), (1.3)-(1.12), (2.4) for notations used in (6.1). In (6.1) the numbers $\rho>0$ and $\tau\in]0,1[$ are parameters. The reconstruction (6.1) for fixed ρ and τ is approximate on the steps 2 and 3. The steps 1, 2, 3 of (6.1) consist in the following:

(1) To find $h\big|_{b\Theta_{\rho,\tau}}$ from Φ we use formulas and equations (1.23)-(1.25). In addition, if v is sufficiently close to some known non-zero background potential v_0, then instead of (1.23)-(1.25) one can use their advanced version of [No2] for improving the reconstruction stability.

(2) To find $\hat v$ on $\mathcal{B}_{2\tau\rho}$ from h on $b\Theta_{\rho,\tau}$ (approximately but stably and with minimal approximation error) we proceed as follows

$$\begin{aligned}&h\big|_{b\Theta_{\rho,\tau}}\underset{(1.4)}{\longrightarrow}H\big|_{b\Omega_{\rho,\tau}}\underset{(3.12)}{\longrightarrow}H\big|_{b\Lambda_{\rho,\tau,\nu}}\\ &\underset{(4.12)}{\longrightarrow}H^0\big|_{\Lambda_{\rho,\tau,\nu}}\underset{(4.31)}{\longrightarrow}\tilde H_{\rho,\tau}\ \text{ on }\ \Lambda_{\rho,\tau,\nu}\\ &\underset{(5.3)}{\longrightarrow}\hat v^\pm(\cdot,\tau,\rho)\ \text{ on }\ \mathcal{B}_{2\tau\rho},\end{aligned}\qquad(6.2)$$

where $\hat{v}^{\pm}(\cdot,\tau,\rho)$ approximates $\hat{v}$ on $\mathcal{B}_{2\tau\rho}$. See also formulas (2.4), (2.10), (2.11), (3.15), (3.18) concerning the sets $b\Omega_{\rho,\tau}$, $b\Lambda_{\rho,\tau,\nu}$, $\Lambda_{\rho,\tau,\nu}$ mentioned in (6.2). In (6.2) the substep from H^0 to $\tilde{H}_{\rho,\tau}$ consists in solving the nonlinear integral equation (4.31), whereas all other substeps are given by explicit formulas.

(3) Finally, from $\hat{v}^{\pm}(\cdot,\tau,\rho)$ of (6.2) we find $v^{\pm}(\cdot,\tau,\rho)$ by formula (2.8), where $v^{\pm}(\cdot,\tau,\rho)$ approximates v on $\mathbb{R}^3$.

One can see that on its steps 1 and 3 reconstruction (6.1) is reduced to results of [No1], [No2] and to the inverse Fourier transform, whereas (6.2) is developed in the present work. Some rigorous results concerning (2.8), (6.2), were already summarized as Theorem 2.1 and Corollary 2.1 of Section 2.

In addition, under the assumptions of Theorem 2.1, a more detailed version of (2.7) is given by

$$\begin{aligned} &|\hat{v}(p)-\hat{v}^{\pm}(p,\tau,\rho)| \le \frac{q(\mu,\mu_0,\tau,\rho,C)C^2}{(1+2\tau\rho)^{\mu-\mu_0}},\\ &q(\mu,\mu_0,\tau,\rho,C)=\frac{3b_4(\mu_0)}{(1-2c_8(\mu_0,\tau,\rho)r_{min}(\mu,\mu_0,\tau,\rho,C))(1-\eta(C,\rho,\mu))^2}, \end{aligned} \tag{6.3}$$

where η, b_4, c_8, r_{min} are defined in (3.1), (4.21), (4.26), (4.30). Estimate (2.7) with

$$c_5(\mu,\mu_0,\tau,\delta)=\frac{3b_4(\mu_0)}{(1-\delta)^3(2\tau)^{\mu-\mu_0}} \tag{6.4}$$

follows from (6.3) and (4.36).

Note that $h\big|_{b\Theta_{\rho,\tau}}$, $H\big|_{b\Omega_{\rho,\tau}}$ and $H\big|_{b\Lambda_{\rho,\tau,\nu}}$ represent the same function in different coordinates. For stability analysis of (6.2) it is convenient to fix this function as $H\big|_{b\Lambda_{\rho,\tau,\nu}}$. Under the assumptions of Theorem 2.1, this function has, in particular, the following properties (see (3.2), (3.3) and the proof of (4.22b)):

$$H\in\mathcal{C}(b\Lambda_{\rho,\tau,\nu}), \tag{6.5}$$

$$\|H\|_{\rho,\tau,\mu_0}\le\frac{C}{1-\eta(C,\rho,\mu)}, \tag{6.6}$$

$$\|T_bH\|_{\rho,\tau,\mu_0}\le\frac{C}{1-\eta(C,\rho,\mu)}\Big(1+\frac{b_4(\mu_0)C}{1-\eta(C,\rho,\mu)}\Big), \tag{6.7}$$

where $\mathcal{C}$ denotes the space of continuous functions, η is defined by (3.1),

$$\begin{aligned} &(T_bU)(\lambda,p)=\frac{1}{2\pi i}\int\limits_{\mathcal{T}^{+}_{\rho/|p|}} U(\zeta,p)\frac{d\zeta}{\zeta-\lambda(1-0)},\quad (\lambda,p)\in b\Lambda^{+}_{\rho,\tau,\nu},\\ &(T_bU)(\lambda,p)=-\frac{1}{2\pi i}\int\limits_{\mathcal{T}^{-}_{\rho/|p|}} U(\zeta,p)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda(1+0))},\quad (\lambda,p)\in b\Lambda^{-}_{\rho,\tau,\nu}, \end{aligned} \tag{6.8}$$

$$\|U\|_{\rho,\tau,\mu_0} = \sup_{(\lambda,p)\in b\Lambda_{\rho,\tau,\nu}} (1+|p|)^{\mu_0}|U(\lambda,p)|, \tag{6.9}$$

where U is a test function on $b\Lambda_{\rho,\tau,\nu}$, $b\Lambda^{\pm}_{\rho,\tau,\nu}$ are defined in (4.9)
(and $b\Lambda_{\rho,\tau,\nu} = b\Lambda^{+}_{\rho,\tau,\nu} \cap b\Lambda^{-}_{\rho,\tau,\nu}$, see (4.10)).

Properties (6.5)-(6.7) are necessary properties of $H\big|_{b\Lambda_{\rho,\tau,\nu}}$ under the assumptions of Theorem 2.1. In addition, if two functions H_1, H_2 satisfy (6.5)-(6.7), where τ, ρ satisfy (2.6), $C>0$, $2\le\mu_0<\mu$, $0<\delta<1$, then $\hat v_i^{\pm}(\cdot,\tau,\rho)$ on $\mathcal{B}_{2\tau\rho}$ can be constructed from H_i via (4.12), (4.31), (5.2), (5.3), $i=1,2$ (in the same way as $\hat v^{\pm}(\cdot,\tau,\rho)$ is constructed from $H\big|_{b\Lambda_{\rho,\tau,\nu}}$ in the framework of Theorem 2.1), and

$$\|\hat v_1^{\pm}(\cdot,\tau,\rho) - \hat v_2^{\pm}(\cdot,\tau,\rho)\|_{2\tau\rho,\mu_0} \le (1-\delta)^{-1}\|T_b(H_1-H_2)\|_{\rho,\tau,\mu_0}, \tag{6.10}$$

where $\|\cdot\|_{2\tau\rho,\mu_0}$ in the left-hand side of (6.10) is defined as in (5.5), $\|\cdot\|_{\rho,\tau,\mu_0}$ in the right-hand side of (6.10) is defined by (6.9), T_b is defined by (6.8).

The stability estimate (6.10) follows from:

(a) the maximum principle in λ for H_1^0, H_2^0, $H_1^0-H_2^0$, where H_n^0 is constructed from H_n via (4.12), $n=1,2$,

(b) Lemma 4.5 and statement of (4.36),

(c) arguments similar with the arguments used for (5.4).

In the present work, restrictions in time prevent us from discussing the stability of (6.1), (6.2) in more detail.

7. Proofs of Lemmas 4.1, 4.2, 4.3

Proof of Lemma 4.1. Under the assumptions of Subsection 3.1, due to (3.2), (3.3), (3.25), (4.2), we have that

$$\begin{aligned}
&H,\ \chi_{2\tau\rho}H,\ (1-\chi_{2\tau\rho})H \in L^\infty_{\mu_0}(\Omega\backslash\bar\Omega_\rho),\\
&|||H|||_{\rho,\mu_0} \le (1-\eta)^{-1}C,\ |||\chi_{2\tau\rho}H|||_{\rho,\mu_0} \le (1-\eta)^{-1}C,\\
&|||(1-\chi_{2\tau\rho})H|||_{\rho,\mu_0} \le (1-\eta)^{-1}(1+2\tau\rho)^{-(\mu-\mu_0)}C,
\end{aligned} \tag{7.1}$$

where η is given by (3.1), $\tau\in]0,1[$, $0\le\mu_0\le\mu$.

Formulas (4.19) follow from (4.4), (7.1), (3.26), (3.27).

Formulas (4.20) follow from (4.15), (4.19), Lemma 11.1 of [No4] and the following formulas

$$\begin{aligned}
&\int\limits_{\mathcal{D}_r^-} u_j(\zeta,s)\frac{|\lambda|}{|\zeta|}\frac{d\,Re\,\zeta\ d\,Im\,\zeta}{|\zeta-\lambda|} = \int\limits_{\mathcal{D}_r^+} u_j(\zeta,s)\frac{d\,Re\,\zeta\ d\,Im\,\zeta}{|\zeta-\lambda^{-1}|},\\
&j=1,2,3,\quad r>1/2,\quad \lambda\in\mathcal{D}_r^-,\quad s>0,
\end{aligned} \tag{7.2}$$

$$\begin{aligned}
&u_1(\zeta,s) = \frac{|\zeta|}{(|\zeta|^2+1)^2},\quad u_2(\zeta,s) = \frac{(|\zeta|^2+1)s}{|\zeta|^2(1+s(|\zeta|+|\zeta|^{-1}))^2},\\
&u_3(\zeta,s) = \frac{s}{|\zeta|(1+s(|\zeta|+|\zeta|^{-1}))}.
\end{aligned} \tag{7.3}$$

Lemma 4.1 is proved.

Proof of Lemma 4.2. Using (4.5), (4.12) and (4.8), (3.22) we obtain that

$$H^0(\lambda,p) = H(\lambda,p) + \frac{1}{\pi}\int\!\!\int\limits_{\mathcal{D}^+_{\rho/|p|}} \{H,H\}(\zeta,p)\frac{d\,Re\,\zeta\; d\,Im\,\zeta}{\zeta-\lambda}, \quad (\lambda,p)\in\Lambda^+_{\rho,\tau,\nu}, \tag{7.4a}$$

$$H^0(\lambda,p) = H(\lambda,p) + \frac{1}{\pi}\int\!\!\int\limits_{\mathcal{D}^-_{\rho/|p|}} \{H,H\}(\zeta,p)\frac{\lambda\, d\,Re\,\zeta\; d\,Im\,\zeta}{\zeta(\zeta-\lambda)}, \quad (\lambda,p)\in\Lambda^-_{\rho,\tau,\nu}. \tag{7.4b}$$

Formulas (4.22) follow from (7.4), (3.2), (3.3), (3.26), (3.27), Lemma 11.1 of [No4] and formulas (7.2), (7.3).

Lemma 4.2 is proved.

Proof of Lemma 4.3. Consider

$$\begin{aligned} &I_{\rho,\tau}(U,V)(\lambda,p) = I^+_{\rho,\tau}(U,V)(\lambda,p) = \\ &-\frac{1}{\pi}\int\!\!\int\limits_{\mathcal{D}^+_{\rho/|p|}} (U,V)_{\rho,\tau}(\zeta,p)\frac{d\,Re\,\zeta\; d\,Im\,\zeta}{\zeta-\lambda}, \quad (\lambda,p)\in\Lambda^+_{\rho,\tau,\nu}, \end{aligned} \tag{7.5a}$$

$$\begin{aligned} &I_{\rho,\tau}(U,V)(\lambda,p) = I^-_{\rho,\tau}(U,V)(\lambda,p) = \\ &-\frac{1}{\pi}\int\!\!\int\limits_{\mathcal{D}^-_{\rho/|p|}} (U,V)_{\rho,\tau}(\zeta,p)\frac{\lambda\, d\,Re\,\zeta\; d\,Im\,\zeta}{\zeta(\zeta-\lambda)}, \quad (\lambda,p)\in\Lambda^-_{\rho,\tau,\nu}, \end{aligned} \tag{7.5b}$$

where $U,V\in L^\infty(\Lambda_{\rho,\tau,\nu})$, $|||U|||_{\rho,\tau,\mu}<+\infty$, $|||V|||_{\rho,\tau,\mu}<+\infty$, $(U,V)_{\rho,\tau}$ is defined by (4.14), (4.16), ρ, ν, τ, μ are the same that in Lemma 4.3.

Note that

$$M_{\rho,\tau}(U_j) = I_{\rho,\tau}(U_j,U_j), \quad j=1,2, \tag{7.6}$$

$$M_{\rho,\tau}(U_1) - M_{\rho,\tau}(U_2) = I_{\rho,\tau}(U_1-U_2,U_1) + I_{\rho,\tau}(U_2,U_1-U_2), \tag{7.7}$$

where U_1, U_2 are the functions of Lemma 4.3.

Using (7.6), (7.7) one can see that in order to prove Lemma 4.3 it is sufficient to prove that

$$I_{\rho,\tau}(U,V)\in L^\infty(\Lambda_{\rho,\tau,\nu}), \tag{7.8}$$

$$|||I_{\rho,\tau}(U,V)|||_{\rho,\tau,\mu} \le c_8(\mu,\tau,\rho)|||U|||_{\rho,\tau,\mu}|||V|||_{\rho,\tau,\mu} \tag{7.9}$$

under the same assumptions that in (7.5).

Formulas (7.8), (7.9) follow from (7.5), (4.14), (4.16), (3.26), (3.27), (7.2), (7.3) and the following estimates:

$$\int\limits_{\mathcal{D}^+_{\rho/|p|}} u_j(\zeta,|p|)\frac{d\,Re\,\zeta\; d\,Im\,\zeta}{|\zeta-\lambda|} \le \begin{cases} (3/4)\pi(|p|/\rho)^2 & \text{for } j=1 \\ 4\pi/\rho & \text{for } j=2 \\ 2\pi|p|/\rho & \text{for } j=3, \end{cases} \tag{7.10}$$

where u_1, u_2, u_3 are defined by (7.3), $0 < |p| < 2\tau\rho$, $0 < \tau < 1$, $\lambda \in \mathcal{D}^+_{\rho/|p|}$.

In turn, estimates (7.10) follow from the estimates

$$\zeta \in \mathcal{D}^+_r \Rightarrow |\zeta| \le (2r)^{-1}, \quad r \ge 1/2, \tag{7.11}$$

$$\int\limits_{|\zeta|\le\varepsilon} u_j(\zeta, s)\frac{d\,Re\,\zeta\; d\,Im\,\zeta}{|\zeta-\lambda|} \le \begin{cases} 3\pi\varepsilon^2 & \text{for } \; j=1 \\ 8\pi\varepsilon/(\varepsilon+s) & \text{for } \; j=2 \\ 4\pi\varepsilon & \text{for } \; j=3, \end{cases} \tag{7.12}$$

where $0 \le \varepsilon \le 1$, $s > 0$, $|\lambda| \le \varepsilon$. (To obtain (7.10) we use (7.11), (7.12) for $r = \rho/|p|$, $\varepsilon = (2r)^{-1}$, $s = |p|$.)

The proof of (7.11). One can see that

$$\begin{aligned} &\zeta \in \mathcal{D}^+_r \overset{(4.6)}{\Rightarrow} |\zeta|^2 - 4|\zeta|r + 1 > 0, \quad |\zeta| < 1 \Rightarrow \\ &|\zeta| < 2r(1-\sqrt{1-1/(2r)^2}) \le (2r)^{-1}, \end{aligned} \tag{7.13}$$

where $r \ge 1/2$.

The proof of (7.12). We have that

$$\begin{aligned} &\int\limits_{|\zeta|\le\varepsilon} u_j(\zeta, s)\frac{d\,Re\,\zeta\; d\,Im\,\zeta}{|\zeta-\lambda|} \le \\ &\Bigg(\int\limits_{|\zeta|\le\varepsilon,\ |\zeta|\le|\zeta-\lambda|} + \int\limits_{|\zeta|\le\varepsilon,\ |\zeta|\ge|\zeta-\lambda|}\Bigg) u_j(\zeta, s)\frac{d\,Re\,\zeta\; d\,Im\,\zeta}{|\zeta-\lambda|} \le A_j + B_j, \\ &A_j = \int\limits_{|\zeta|\le\varepsilon} u_j(\zeta, s)\frac{d\,Re\,\zeta\; d\,Im\,\zeta}{|\zeta|}, \\ &B_j = \int\limits_{|\zeta-\lambda|\le|\zeta|\le\varepsilon} u_j(\zeta, s)\frac{d\,Re\,\zeta\; d\,Im\,\zeta}{|\zeta-\lambda|}, \end{aligned} \tag{7.14}$$

where $j = 1, 2, 3$. Further,

$$\begin{aligned} &A_1 \le \int\limits_{|\zeta|\le\varepsilon} d\,Re\,\zeta\; d\,Im\,\zeta = \pi\varepsilon^2, \\ &B_1 \le \varepsilon \int\limits_{|\zeta-\lambda|\le\varepsilon} \frac{d\,Re\,\zeta\; d\,Im\,\zeta}{|\zeta-\lambda|} = 2\pi\varepsilon^2, \end{aligned} \tag{7.15}$$

$$
\begin{aligned}
&A_2 \le \int\limits_{|\zeta|\le\varepsilon} \frac{(1+\varepsilon^2)s\,d\,Re\,\zeta\,d\,Im\,\zeta}{(|\zeta|+s(|\zeta|^2+1))^2|\zeta|},\\
&B_2 \le \int\limits_{|\zeta-\lambda|\le\varepsilon} \frac{(1+\varepsilon^2)s\,d\,Re\,\zeta\,d\,Im\,\zeta}{(|\zeta-\lambda|+s(|\zeta-\lambda|^2+1))|\zeta-\lambda|},\\
&A_2+B_2 \le 4s\int\limits_{|\zeta|\le\varepsilon} \frac{d\,Re\,\zeta\,d\,Im\,\zeta}{(|\zeta|+s)^2|\zeta|} = 8\pi s\int\limits_0^\varepsilon \frac{dr}{(r+s)^2} = 8\pi\varepsilon/(\varepsilon+s),
\end{aligned}
\tag{7.16}
$$

$$
\begin{aligned}
&A_3 \le \int\limits_{|\zeta|\le\varepsilon} \frac{s\,d\,Re\,\zeta\,d\,Im\,\zeta}{(|\zeta|+s(|\zeta|^2+1))|\zeta|},\\
&B_3 \le \int\limits_{|\zeta-\lambda|\le\varepsilon} \frac{s\,d\,Re\,\zeta\,d\,Im\,\zeta}{(|\zeta-\lambda|+s(|\zeta-\lambda|^2+1))|\zeta-\lambda|},\\
&A_3+B_3 \le 2\int\limits_{|\zeta|\le\varepsilon} \frac{s\,d\,Re\,\zeta\,d\,Im\,\zeta}{(|\zeta|+s)|\zeta|} = 4\pi s\int\limits_0^\varepsilon \frac{dr}{r+s} = 4\pi s\ln\bigl(1+\frac{\varepsilon}{s}\bigr) \le 4\pi\varepsilon.
\end{aligned}
\tag{7.17}
$$

Estimates (7.12) follow from (7.14)-(7.17).

Lemma 4.3 is proved.

8. Proof of Lemmas 4.4 and 4.5

Proof of Lemma 4.4. For

$$
\begin{aligned}
&U^0, U, U_1, U_2 \in L^\infty(\Lambda_{\rho,\tau,\nu}),\\
&|||U^0|||_{\rho,\tau,\mu} \le r/2,\quad |||U|||_{\rho,\tau,\mu} \le r,\quad |||U_1|||_{\rho,\tau,\mu} \le r,\quad |||U_2|||_{\rho,\tau,\mu} \le r,
\end{aligned}
\tag{8.1}
$$

using Lemma 4.3 and the assumptions of Lemma 4.4 we obtain that

$$
\begin{aligned}
&M_{\rho,\tau,U^0}(U) \in L^\infty(\Lambda_{\rho,\tau,\nu}),\\
&|||M_{\rho,\tau,U^0}(U)|||_{\rho,\tau,\mu} \le |||U^0|||_{\rho,\tau,\mu} + |||M_{\rho,\tau}(U)|||_{\rho,\tau,\mu} \le\\
&r/2 + c_8(\mu,\tau,\rho)r^2 < r,
\end{aligned}
\tag{8.2}
$$

$$
\begin{aligned}
&|||M_{\rho,\tau,U^0}(U_1) - M_{\rho,\tau,U^0}(U_2)|||_{\rho,\tau,\mu} \le 2c_8(\mu,\tau,\rho)r|||U_1-U_2|||_{\rho,\tau,\mu},\\
&2c_8(\mu,\tau,\rho)r < 1,
\end{aligned}
\tag{8.3}
$$

where

$$M_{\rho,\tau,U^0}(U) = U^0 + M_{\rho,\tau}(U). \tag{8.4}$$

Due to (8.1)-(8.4), M_{ρ,τ,U^0} is a contraction map of the ball $U \in L^\infty(\Lambda_{\rho,\tau,\nu})$, $|||U|||_{\rho,\tau,\mu} \le r$. Using now the lemma about contraction maps and using the formulas

$$|||U - M^n_{\rho,\tau,U^0}(0)|||_{\rho,\tau,\mu} \le \sum_{j=n}^{\infty} |||M^{j+1}_{\rho,\tau,U^0}(0) - M^j_{\rho,\tau,U^0}(0)|||_{\rho,\tau,\mu}, \tag{8.5}$$

$$|||M_{\rho,\tau,U^0}(0) - 0|||_{\rho,\tau,\mu} = |||U^0|||_{\rho,\tau,\mu} \le r/2, \tag{8.6}$$

$$|||M^{j+1}_{\rho,\tau,U^0}(0) - M^{j}_{\rho,\tau,U^0}(0)|||_{\rho,\tau,\mu} \overset{(8.3)}{\leq} 2c_8(\rho,\tau,\mu)r\times \\ |||M^{j}_{\rho,\tau,U^0}(0) - M^{j-1}_{\rho,\tau,U^0}(0)|||_{\rho,\tau,\mu}, \quad j = 1,2,3,\ldots, \tag{8.7}$$

where U is the fixed point of M_{ρ,τ,U_0} in the aforementioned ball, $M^0_{\rho,\tau,U^0}(0) = 0$, we obtain Lemma 4.4.

Proof of Lemma 4.5. We have that

$$U - \tilde{U} = U^0 - \tilde{U}^0 + M_{\rho,\tau}(U) - M_{\rho,\tau}(\tilde{U}), \tag{8.8a}$$

$$M_{\rho,\tau}(U) - M_{\rho,\tau}(\tilde{U}) \overset{(7.7)}{=} I_{\rho,\tau}(U - \tilde{U}, U) + I_{\rho,\tau}(\tilde{U}, U - \tilde{U}), \tag{8.8b}$$

where $I_{\rho,\tau}(U,V)$ is defined by (7.5).

In view of (8.8b) we can consider (8.8a) as a linear integral equation for "unknown" $U - \tilde{U}$ with given $U^0 - \tilde{U}^0$, U, $\tilde{U}$. Using (7.9) and the properties $|||U|||_{\rho,\tau,\mu} \leq r$, $|||\tilde{U}|||_{\rho,\tau,\mu} \leq r$ we obtain that

$$|||I_{\rho,\tau}(U - \tilde{U}, U) - I_{\rho,\tau}(\tilde{U}, U - \tilde{U})|||_{\rho,\tau,\mu} \leq 2c_8(\mu,\tau,\rho)r|||U - \tilde{U}|||_{\rho,\tau,\mu}. \tag{8.9}$$

Using (8.8b), (8.9) and solving (8.8a) by the method of successive approximations we obtain (4.29). Lemma 4.5 is proved.

References

[Al] G.Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal. **27** (1988), 153-172.

[ABR] N.V.Alexeenko, V.A.Burov and O.D.Rumyantseva, *Solution of three-dimensional acoustical inverse scattering problem,II: modified Novikov algorithm*, Acoust. J. **54**(3) (2008) (in Russian), English transl.: Acoust. Phys. **54**(3) (2008).

[Am] H.Ammari, *An Introduction to Mathematics of Emerging Biomedical Imaging*, Springer, Berlin, 2007.

[BC] R.Beals and R.R.Coifman, *Multidimensional inverse scattering and nonlinear partial differential equations*, Proc. Symp. Pure Math. **43** (1985), 45-70.

[BRS] V.A.Burov, O.D.Rumyantseva and T.V.Suchkova, *Practical application possibilities of the functional approach to solving inverse scattering problems*, Moscow: Physical Society, **3** (1990), 275-278 (in Russian).

[C] A.-P.Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), pp.65-73, Soc. Brasil. Mat. Rio de Janeiro, 1980.

[F1] L.D.Faddeev, *Growing solutions of the Schrödinger equation*, Dokl. Akad. Nauk SSSR **165** (1965), 514-517 (in Russian); English Transl.: Sov. Phys. Dokl. **10** (1966), 1033-1035.

[F2] L.D.Faddeev, *Inverse problem of quantum scattering theory II*, Itogi Nauki i Tekhniki, Sovr. Prob. Math. **3** (1974), 93-180 (in Russian); English Transl.: J.Sov. Math. **5** (1976), 334-396.

[G] I.M.Gelfand, *Some problems of functional analysis and algebra*, Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, pp.253-276.

[GN] P.G.Grinevich, S.P.Novikov, *Two-dimensional "inverse scattering problem" for negative energies and generalized-analytic functions. I. Energies below the ground state*, Funkt. Anal. i Pril. **22**(1) (1988), 23-33 (In Russian); English Transl.: Funkt. Anal. and Appl. **22** (1988), 19-27.

[HM] G.Henkin and V.Michel, *Inverse Conductivity Problem on Riemann Surfaces*, J.Geom.Anal. **18** (2008), 1033-1052.

[HN] G.M.Henkin and R.G.Novikov, *The $\bar{\partial}$- equation in the multidimensional inverse scattering problem*, Uspekhi Mat. Nauk **42(3)** (1987), 93-152 (in Russian); English Transl.: Russ. Math. Surv. **42(3)** (1987), 109-180.

[M] N.Mandache, *Exponential instability in an inverse problem for the Schrödinger equation*, Inverse Problems **17** (2001), 1435-1444.

[Na1] A.I.Nachman, *Reconstructions from boundary measurements*, Ann. Math. **128** (1988), 531-576.

[Na2] A.I.Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem*, Ann, Math. **142** (1995), 71-96.

[No1] R.G.Novikov, *Multidimensional inverse spectral problem for the equation* $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, Funkt. Anal. i Pril. **22(4)** (1988), 11-22 (in Russian); English Transl.: Funct. Anal. and Appl. **22** (1988), 263-272.

[No2] R.G.Novikov, *Formulae and equations for finding scattering data from the Dirichlet-to-Neumann map with nonzero background potential*, Inverse Problems **21** (2005), 257-270.

[No3] R.G.Novikov, *The $\bar{\partial}$- approach to approximate inverse scattering at fixed energy in three dimensions*, International Mathematics Research Papers, **2005:6**, (2005), 287-349.

[No4] R.G.Novikov, *On non-overdetermined inverse scattering at zero energy in three dimensions*, Ann., Scuola Norm. Sup. Pisa Cl. Sci. **5** (2006), 279-328.

[NN] R.G.Novikov and N.N.Novikova, *On stable determination of potential by boundary measurements*, ESAIM: Proceedings, to appear 2008.

[SU] J.Sylvester and G.Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. Math. **125** (1987), 153-169