Blow up of smooth highly decreasing at infinity solutions to the compressible Navier-Stokes equations

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Abstract

We prove that the smooth solutions to the Cauchy problem for the Navier-Stokes equations with conserved total mass, finite total energy and finite momentum of inertia lose the initial smoothness within a finite time in the case of space of dimension 3 or greater even if the initial data are not compactly supported. The cases of isentropic and incompressible fluids are also considered.

Key words: compressible viscous fluid, the Cauchy problem, loss of smoothness
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1 System, known results and main problem

The motion of compressible viscous, heat-conductive, Newtonian polytropic fluid in $\mathbb{R}\times\mathbb{R}^n$, $n\ge 1$, is governed by the compressible Navier-Stokes (NS) equations

$$\partial_t\rho + \mathrm{div}_x(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \mathrm{div}_x(\rho u\otimes u) + \nabla_x p = \mathrm{Div} T, \tag{1.2}$$

$$\partial_t\left(\frac{1}{2}\rho|u|^2 + \rho e\right) + \mathrm{div}_x\left((\frac{1}{2}\rho|u|^2 + \rho e + p)u\right) = \mathrm{div}(Tu) + k\Delta_x\theta, \tag{1.3}$$

where ρ, $u=(u_1,...,u_n)$, p, e, θ denote the density, velocity, pressure, internal energy and absolute temperature, respectively, T is the stress tensor given by the Newton law

$$T = T_{ij} = \mu\,(\partial_i u_j + \partial_j u_i) + \lambda\,\mathrm{div} u\,\delta_{ij}, \tag{1.4}$$

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where the constants μ and λ are the coefficient of viscosity and the second coefficient of viscosity, $k \geq 0$ is the coefficient of heat conduction. We denote Div and div the divergency of tensor and vector, respectively. We assume that $\mu > 0$, $\lambda + \frac{2}{n}\mu > 0$.

The state equations have the forms

$$p = R\rho\theta, \quad e = c\theta, \quad p = A\exp\left(\frac{S}{c}\right)\rho^{\gamma}. \tag{1.5}$$

Here $A > 0$ is a constant, R is the universal gas constant, $S = \log e - (\gamma - 1)\log\rho$ is the specific entropy, $c = \frac{R}{\gamma-1}$, $\gamma > 1$ is the specific heat ratio,

The state equations (1.5) imply

$$p = (\gamma - 1)\rho e, \tag{1.6}$$

which allows us to consider (NS) as a system for the unknown ρ, u, p. Indeed, from (NS) and (1.6) it follows that

$$\partial_t p + (u, \nabla_x p) + \gamma p \,\mathrm{div} u = (\gamma - 1)\sum_{i,j=1}^{n} T_{ij}\partial_j u_i + \frac{k}{R}\Delta\frac{p}{\rho}. \tag{1.7}$$

Thus, therefore further we shall consider the system (1.1, 1.2, 1.7), denoted (NS*) for short.

(NS*) is supplemented with the initial data

$$(\rho, u, p)\Big|_{t=0} = (\rho_0(x), u_0(x), p_0(x)) \in H^m(\mathbb{R}^n),\ m > [n/2] + 2. \tag{1.8}$$

We also consider the isentropic case where the fluid obeys equations (1.1), (1.2) and $p = A\rho^{\gamma}$, we call this system (NSI) for short.

In the absence of vacuum, the local existence of classical solutions is known. Namely, in [1] it is proved that there exist classical solutions, having the Hölder continuous second derivatives with respect to space variables and the first ones with respect to time. In [2] the system of equations of viscous compressible fluid is considered as a particular case of combined systems of differential equations. The consideration is performed in the Sobolev spaces H^m with a sufficiently large m. The uniqueness of the solution was proved earlier in [3]. The existence and uniqueness of local strong solutions in the case where the initial density need not be positive and may vanish in an open set were proved recently in[4].

At the same time there exists a major open problem: to prove or disprove that a smooth solution to (NS) (in higher space dimensions) exists globally

in time. There are partial results concerning the Cauchy problem for (NS) away from a vacuum. In [5] it is proved that if there exists a constant $\bar\rho = const > 0$ such that $(\rho_0 - \bar\rho, u_0, S_0) \in H^m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),$ and the norm $\|\rho_0 - \bar\rho, u_0, S_0\|_{H^m},\ m > [n/2] + 2,$ is suitably small, then global solution to (NS) from $C^1([0,\infty), H^m(\mathbb{R}^n))$ exists. In [6] it is shown that for $n = 1$ the global existence takes place without assumptions on the smallness of the norm $\|\rho_0 - \bar\rho, u_0, S_0\|_{H^m}$.

However if the initial density ρ_0 is compact, then in arbitrary space dimensions no solution to (NS) from $C^1([0,\infty), H^m(\mathbb{R}^n))$ exists ([7]). This blowup result depends crucially on the assumption about compactness of support of the initial density. It does not seem to solve in a negative way the question of regularity for (NS). Indeed, (NS) is a model of non-dilute fluids where the density is bounded below away from zero, and therefore it is natural to expect the problem to be ill-posed when vacuum regions are present at the initial time. At the same time, the conservation of mass in the whole space requires a decrease of the density down to zero.

In [7] the author notes that the global smooth solution to (NS) seems to exist at least for small data in the case where initial vacuum appears only at infinity.

Nevertheless, in [8] a sufficient condition for the blow-up in case that the initial density is positive but has a decay at infinity was found. In this work it was expected a specific time decay of the velocity component that seems to be reasonable for a density away from zero. Further, in [9] it was proved that if the solution with finite moment of inertia to the (non-heat-conductive) Navier-Stokes system is smooth globally in time, then the solution components grow as $t \to \infty$ at least as a certain function specific for every component $\phi(t,x)$, where ϕ stands either for the density, pressure, velocity or gradient of velocity. Namely, we observe all trajectories of particles $x(t)$ that leave in a finite time a ball of finite radius R_0 and find that provided the solution is smooth, the inequality $|\phi(t, x(t))| \le M(t)|x(t)|^\lambda,\ t \to \infty,$ holds, with a continuous function $M(t)$ and a constant $\lambda,$ specific for every ϕ. Both in [8] and [9] the answer whether the solution blows up or not does not depend on the initial data, but only on the prescribed decay at infinity.

Thus, the question remains: is it true that in the case where the support of all components of initial data coincides with the whole space the global smooth solution exists for any smooth initial data?

Below we are going to show that, generally speaking, the answer is negative.

2 Integral functionals, solution with decreasing components and the statement of main theorem

The system (NS) is the differential form of conservation laws for the material volume $\Omega(t)$; it expresses conservation of mass

$$m = \int\limits_{\Omega(t)} \rho \, dx,$$

and balance of momentum

$$P = \int\limits_{\Omega(t)} \rho u \, dx,$$

and total energy

$$\mathcal{E} = \int\limits_{\Omega(t)} \left(\frac{1}{2}\rho|u|^2 + \rho e\right) \, dx \; = E_k(t) + E_i(t).$$

Here $E_k(t)$ and $E_i(t)$ are the kinetic and internal components of energy, respectively.

If we regard $\Omega(t) = \mathbb{R}^n$, the conservation of mass, momentum and energy takes place provided the components of the solution decrease at infinity sufficiently quickly.

Let us introduce the functionals

$$G(t) = \frac{1}{2} \int\limits_{\mathbb{R}^n} \rho(t,x)|x|^2 \, dx, \qquad F(t) = \int\limits_{\mathbb{R}^n} (u,x)\rho \, dx,$$

where the first one is the momentum of inertia, the scalar product of vectors is denoted as $(.,.)$.

For technical reasons we impose the following conditions of decay on the solution components to (NS*) as $|x| \to \infty$ at every fixed $t \in \mathbb{R}_+$:

$$\rho = O\left(\frac{1}{|x|^{n+2+\varepsilon}}\right), \qquad p = O\left(\frac{1}{|x|^{n+\varepsilon}}\right), \quad \varepsilon > 0, \tag{2.1}$$

$$|u| = o\left(\frac{1}{|x|^{n-1}}\right), \qquad |Du| = o\left(\frac{1}{|x|^{n}}\right). \tag{2.2}$$

If $k \neq 0$, we require additionally

$$|D\theta| = o\left(\frac{1}{|x|^{n-1}}\right), \quad |x| \to \infty, \quad t \in \mathbb{R}_+. \tag{2.3}$$

One can easily verify that these requirements guarantee the conservation of the mass m, energy $\mathcal{E}$, momentum P on solutions to (NS*) and ensure a convergence of the momentum of mass $G(t)$.

We impose no restriction on the solution support, however the decay of the solution as $|x| \to \infty$ in the class considered is greater than it is necessary for belonging to $C^1([0,T), H^m(\mathbb{R}^n))$.

Definition 1 *We will say that a solution (ρ, u, p) to the Cauchy problem (1.1, 1.2, 1.7), (1.8) belongs to the class $\mathfrak{K}$ if it has the following properties for all $t \geq 0$:*

(i) *the solution is classical;*

(ii) *the solution decays at infinity according to (2.1 – 2.3);*

(iii) $\rho(t,x) \geq 0$;

(iv) $\dfrac{dS(t,x)}{dt} = \sigma(t,x) \geq 0, \quad \|\sigma(t,x)\|_{L^\infty(\mathbb{R}^n)} = o(t^\alpha), \quad t \to \infty,$ *where* $\alpha = \frac{(\gamma-1)n^2+n-2}{n}$, *if* $\gamma \leq 1 + \frac{2}{n}$, *and* $\alpha = \frac{3n-2}{n}$, *otherwise.*

Remark 2.1 *In particular, (iv) results that $S(t,x) \geq S_0 = \inf\limits_{x\in\mathbb{R}^n} S(0,x)$. For (NSI) condition (iv) holds trivially.*

Remark 2.2 *From (1.7) we have*

$$\frac{p}{R}\left(\partial_t S + (u, \nabla_x S)\right) = \sum_{i,j=1}^n T_{ij}\partial_j u_i + k\Delta\theta.$$

Evidently, if the right-hand side is non-negative, then $S(t,x) \geq S_0$ for all $t > 0$. The first item on the right-hand side is non-negative. Indeed, according to [7]

$$\sum_{i,j=1}^n T_{ij}\partial_j u_i = \sum_{i,j=1}^n \partial_j(T_{ij}u_i) - \sum_{i,j=1}^n u_i\,\partial_j T_{ij} =$$

$$2\mu \sum_{i,j=1}^n (\partial_j u_j)^2 + \lambda(\operatorname{div} u)^2 + \mu\sum_{i\neq j}^n (\partial_j u_i)^2 + 2\mu\sum_{i>j}^n (\partial_j u_i)(\partial_i u_j). \tag{2.4}$$

For $\lambda \geq 0$ this implies the nonnegativity of the right-hand side of (2.4), for $\lambda < 0$ (2.4) can be estimated from below by

$$(2\mu + n\lambda) \sum_{i,j=1}^{n} (\partial_j u_j)^2 + \mu \sum_{i \neq j}^{n} (\partial_j u_i)^2 + 2\mu \sum_{i > j}^{n} (\partial_j u_i)(\partial_i u_j);$$

this expression is nonnegative by assumption on the viscosity coefficients.

Therefore we can guarantee the uniform boundedness of entropy from below in the case $k = 0$ and in the isothermal case $\theta = const.$

Theorem 2.1 *Let $n \geq 3$, $\gamma \geq \frac{2n}{n+2}$, the momentum $P \neq 0$. If $\inf\limits_{x \in \mathbb{R}^n} S(0, x) > -\infty$, then there exists no global in time solution to (NS*) from the class $\mathfrak{K}$.*

Analyzing condition (iv) we can see that the nondecreasing of entropy along trajectories seems natural, whereas the upper bound on the growth of entropy can be considered as unreasonable. So, we can re-formulate Theorem 2.1 as follows:

Theorem 2.2 *Let us assume $n \geq 3$, $\gamma \geq \frac{2n}{n+2}$ and $P \neq 0$. Then any solution to (NS*) with properties (i, ii, iii) such that the entropy does not decrease along the particles trajectories, blows up in a finite or infinite time. If the solution keeps smoothness for all $t > 0$, then $\|S\|_{L^\infty(\mathbb{R}^n)}$, $\|p\|_{L^\infty(\mathbb{R}^n)}$, $\|\mathrm{div} u\|_{L^\infty(\mathbb{R}^n)}$ rise at least as $O(t^{\alpha+1})$ as $t \to \infty$ (the constant $\alpha > 0$ is indicated in condition (iv).)*

For the isentropic case we have the following version of Theorem 2.1.

Theorem 2.3 *Let $n \geq 3$, $\gamma \geq \frac{2n}{n+2}$. If initial data $(\rho_0(x), u_0(x))$ satisfy that the momentum $P \neq 0$, then the solution to (NSI) from the class $\mathfrak{K}$ cannot exist for all $t > 0$.*

3 Proof of the theorems

We begin with the following extension of formula obtained in [10] to the viscous case.

Lemma 3.1 *For solutions to (1.1, 1.2, 1.7) with properties $(i), (ii)$*

$$G'(t) = F(t), \tag{3.1}$$

$$F'(t) = 2E_k(t) + n(\gamma - 1)E_i(t). \tag{3.2}$$

PROOF. The lemma can be proved by direct calculation using the general Stokes formula and taking into account the decay assumption (ii). For example, from (1.1) we get

$$G'(t) = \frac{1}{2}\int\limits_{\mathbb{R}^n} \rho'_t |x|^2\,dx =$$

$$= \int\limits_{\mathbb{R}^n} (u,x)\rho\,dx - \lim_{R\to\infty} \int\limits_{S_R} (u,x)\rho\frac{|x|}{2}\,dS_R =$$

$$= \int\limits_{\mathbb{R}^n} (u,x)\rho\,dx,$$

where S_R is a sphere of radius R with the center at the origin. This proves equality (3.1).

Then we get two-sided estimates of $G(t)$.

Lemma 3.2 *If $\gamma \le 1 + \frac{2}{n}$, then for solutions with properties $(i), (ii)$ the estimates*

$$\frac{n(\gamma-1)}{2}\mathcal{E}t^2 + F(0)t + G(0) \le G(t) \le \mathcal{E}t^2 + F(0)t + G(0) \tag{3.3}$$

(for (NS)) and*

$$\frac{P^2}{2m}t^2 + F(0)t + G(0) \le G(t) \le \mathcal{E}(0)t^2 + F(0)t + G(0) \tag{3.4}$$

(for (NSI)) hold.

If $\gamma > 1 + \frac{2}{n}$, then we have

$$\mathcal{E}t^2 + F(0)t + G(0) \le G(t) \le \frac{n(\gamma-1)}{2}\mathcal{E}t^2 + F(0)t + G(0) \tag{3.5}$$

for (NS) and*

$$\frac{P^2}{2m}t^2 + F(0)t + G(0) \le G(t) \le \frac{n(\gamma-1)}{2}\mathcal{E}(0)t^2 + F(0)t + G(0) \tag{3.6}$$

for (NSI).

PROOF. First of all (3.2) result

$$G''(t) = 2E_k(t) + n(\gamma-1)E_i(t) =$$

$$= 2\mathcal{E}(t) - (2 - n(\gamma - 1))E_i(t) = n(\gamma - 1)\mathcal{E}(t) + (2 - n(\gamma - 1))E_k(t). \quad (3.7)$$

In the case of (NS*) the total energy $\mathcal{E}$ is constant, therefore (3.3) and (3.5) follow from (3.7) after integration if we take into account nonnegativity of $E_k(t)$ and $E_i(t)$. Estimate (3.3) one can find in [10] for the zero viscosity.

In the case of (NSI) the total energy is only non-increasing, since (1.1) and (1.2) for a constant entropy result in

$$\mathcal{E}'(t) \leq -\nu \int_{\mathbb{R}^n} |Du|^2 \, dx \leq 0, \quad (3.8)$$

with some positive constant ν. Then to get the two-sided estimates of $G(t)$ from (3.7) we can use the non-increasing of total energy $\mathcal{E}(t) \leq \mathcal{E}(0)$, the nonnegativity of $E_i(t)$ and the estimate $E_k(t) \geq \frac{P^2}{2m}$, that follows from the Hölder inequality. This gives (3.4) and (3.6).

The next step is two-sides estimate of $E_i(t)$.

Lemma 3.3 *For solutions from the class $\mathfrak{K}$ to (NS*) for sufficiently large t we have*

$$\frac{C_1}{G^{(\gamma-1)n/2}(t)} \leq E_i(t) \leq \frac{C_2}{G^{(\gamma-1)n/2}(t)}, \quad (3.9)$$

for $\gamma \leq 1 + \frac{2}{n}$, and

$$\frac{C_1}{G^{(\gamma-1)n/2}} \leq E_i(t) \leq \frac{C_2}{G(t)}, \quad (3.10)$$

for $\gamma > 1 + \frac{2}{n}$, with constants $C_1, C_2\, (C_1 \leq C_2)$. The same estimate holds also for (NSI) provided $P \neq 0$.

PROOF. The lower estimate is due to [10]. It follows from the inequality

$$\|f\|_{L^1(\mathbb{R}^n;\, dx)} \;\leq\; C_{\gamma,n} \|f\|^{\frac{2\gamma}{(n+2)\gamma - n}}_{L^\gamma(\mathbb{R}^n;\, dx)} \, \|f\|^{\frac{n(\gamma-1)}{(n+2)\gamma - n}}_{L^1(\mathbb{R}^n;\, |x|^2\, dx)},$$

together with the lower estimate of internal energy

$$E_i(t) \geq \frac{A e^{S_0/c}}{\gamma - 1} \int_{\mathbb{R}^n} \rho^\gamma(t,x) \, dx.$$

The latter inequality follows from the state equations in (1.5). The constant

$$C_1 = A \frac{e^{S_0/c}}{\gamma - 1} (m C_{\gamma,n}^{-1})^{\frac{\gamma(n+2)-n}{2}}$$

with

$$C_{\gamma,n} = \left(\frac{2\gamma}{n(\gamma-1)}\right)^{\frac{n(\gamma-1)}{(n+2)\gamma-n}} + \left(\frac{2\gamma}{n(\gamma-1)}\right)^{\frac{-2\gamma}{(n+2)\gamma-n}}.$$

The method of the upper estimate of $E_i(t)$ is also similar to [10]. Namely, let us consider the function $Q(t) = 4G(t)\mathcal{E}(t) - F^2(t)$. The Hölder inequality gives $F^2 \leq 4G(t)E_k(t)$, therefore $\mathcal{E}(t) = E_k(t) + E_i(t) \geq E_i(t) + \frac{F^2(t)}{4G(t)}$ and

$$E_i(t) \leq \frac{Q(t)}{4G(t)}. \tag{3.11}$$

We notice also that $Q(t) > 0$ provided the pressure does not equal to zero identically. Then taking into account (3.1), (3.2) and (3.7) we have

$$Q'(t) = 4G'(t)\mathcal{E}(t) - 2G'(t)G''(t) + 4G(t)\mathcal{E}'(t) =$$

$$= 2(2 - n(\gamma-1))\, G'(t)E_i(t) + 4G(t)\mathcal{E}'(t). \tag{3.12}$$

Further, one can see from (3.2) that in the (NS*) case $G'(t) > 0$ beginning from a positive t_0 for all initial data, whereas for (NSI) equality (3.2) result in

$$G''(t) \geq \frac{P^2}{m},$$

and we can guarantee the positivity of $G'(t)$ for sufficiently large t only for $P \neq 0$. Thus, for $\gamma \leq 1 + \frac{2}{n}$ (3.11), (3.12) (and (3.8) for (NSI)) result

$$\frac{Q'(t)}{Q(t)} \leq \frac{2 - n(\gamma-1)}{2} \frac{G'(t)}{G(t)}. \tag{3.13}$$

Then (3.13) and (3.11) give

$$E_i(t) \leq \frac{C_2}{G^{(\gamma-1)n/2}(t)}, \quad C_2 = \frac{Q(0)G^{(\gamma-1)n/2}(0)}{4}. \tag{3.14}$$

If $\gamma > 1 + \frac{2}{n}$, then from (3.12) taking into account the lower estimate of $E_i(t)$ we get

$$Q'(t) \leq -2((\gamma-1)n - 2)\, \frac{C_1}{G^{(\gamma-1)n/2}(t)},$$

$$Q(t) \leq \tilde{C} + 4\, C_1\, G^{1-(\gamma-1)n/2}(t), \quad \tilde{C} = Q(0) - 4\, C_1\, G^{1-(\gamma-1)n/2}(0),$$

and, at last, from (3.11)

$$E_i(t) \leq \frac{\tilde{C}}{4\, G(t)} + C_1\, G^{-(\gamma-1)n/2}(t).$$

Thus, since the leading term in the right hand side of the latter inequality is the first one, beginning from a moment $t > 0$ we get

$$E_i(t) \leq \frac{C_2}{G(t)}, \quad C_2 = \frac{\tilde{C}}{4} + C_1.$$

The proof is over.

The next lemma is a key point of the theorem's proof.

Lemma 3.4 *Let $n \geq 3, \gamma \geq \frac{2n}{n+2}$. If $|P| \neq 0$, then there exists a positive constant K such that for the solutions of the class $\mathfrak{K}$ the following inequality holds:*

$$\int\limits_{\mathbb{R}^n} |Du|^2\,dx \geq \; K\, E_i^{-\frac{n-2}{n(\gamma-1)}}(t). \tag{3.15}$$

PROOF. First of all we use the Hölder inequality to get

$$|P| = \Big|\int\limits_{\mathbb{R}^n} \rho u\,dx\Big| \leq \left(\int\limits_{\mathbb{R}^n} \rho^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{2n}} \left(\int\limits_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{2n}} \leq$$

$$\leq e^{-\frac{n-2}{2n(\gamma-1)}\frac{S_0}{c}} \left(\int\limits_{\mathbb{R}^n} e^{\frac{n-2}{(n+2)(\gamma-1)}\frac{S}{c}} \rho^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{2n}} \left(\int\limits_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{2n}}. \tag{3.16}$$

Further, using the Jensen inequality we have for $\frac{(\gamma-1)(n+2)}{n-2} \geq 1$ (or $\gamma \geq \frac{2n}{n+2}$)

$$\left(\frac{1}{m}\int\limits_{\mathbb{R}^n} e^{\frac{n-2}{(n+2)(\gamma-1)}\frac{S}{c}}\,\rho^{\frac{2n}{n+2}}\,dx\right)^{\frac{(\gamma-1)(n+2)}{n-2}} \leq \frac{\int\limits_{\mathbb{R}^n} e^{\frac{S}{c}}\rho^{\gamma}\,dx}{m} = \frac{(\gamma-1)E_i(t)}{mA}.$$

Thus, the latter inequality and (3.16) give

$$|P| \leq K_1\,(E_i(t))^{\frac{n-2}{2n(\gamma-1)}} \left(\int\limits_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{2n}}, \tag{3.17}$$

with the positive constant K_1 that depends on γ, n, m, S_0. Further, we take into account of the inequality

$$\left(\int\limits_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \leq K_2 \int\limits_{\mathbb{R}^n} |Du|^2\,dx, \tag{3.18}$$

where the constant $K_2 > 0$ depends on $n,\ n \geq 3$. The latter inequality holds for $u \in H^1(\mathbb{R}^n)$ ([11], p.22) and follows from the Sobolev embedding.

Thus, the lemma statement, the inequality (3.15), follows from (3.17), (3.18), with the constant $K = \frac{|P|^2}{K_1^2 K_2}$.

We begin from the isentropic case (NSI).

Proof of theorem 2.3. From (3.8), (3.15) we have

$$\mathcal{E}'(t) \le -\nu\, K\, (E_i(t))^{-\frac{n-2}{n(\gamma-1)}}. \tag{3.19}$$

Together with (3.4), (3.6) and Lemma 3.3 inequality (3.19) implies

$$\mathcal{E}'(t) \le -\nu\, K\, C_2^{\frac{2-n}{n(\gamma-1)}}\, G^{\frac{n-2}{2n}}(t) \le -L\, t^{\frac{n-2}{n}},$$

with a positive constant L, for $\gamma \le 1 + \frac{2}{n}$, and

$$\mathcal{E}'(t) \le -\nu\, K\, C_2^{\frac{2-n}{n(\gamma-1)}}\, G^{\frac{n-2}{n(\gamma-1)}}(t) \le -L\, t^{\frac{2(n-2)}{n(\gamma-1)}},$$

for $\gamma > 1 + \frac{2}{n}$. In both cases this contradicts to the non-negativity of $\mathcal{E}(t)$. Thus, theorem 2.3 is proved.

Proof of theorem 2.1. Let us remind that in the case of (NS*) the total energy $\mathcal{E}$ is constant for the solutions of the class $\mathfrak{K}$. However, the derivatives of both kinetic and internal components of the total energy can be estimated. Namely, taking into account the state equation in (1.5) we have

$$\frac{d\, E_k(t)}{dt} = \int\limits_{\mathbb{R}^n} (u, \mathrm{Div} T)\, dx - \int\limits_{\mathbb{R}^n} (u, \nabla p)\, dx \le$$

$$\le -\nu \int\limits_{\mathbb{R}^n} |Du|^2\, dx - \frac{A}{c} \int\limits_{\mathbb{R}^n} e^{\frac{S}{c}}\, \rho^\gamma\, (u, \nabla S)\, dx - A \int\limits_{\mathbb{R}^n} e^{\frac{S}{c}}\, (u, \nabla \rho^\gamma)\, dx, \tag{3.20}$$

with a positive constant ν. Further, together with (1.1) we obtain

$$\frac{d\, E_i(t)}{dt} = \frac{d}{dt}\left(\int\limits_{\mathbb{R}^n} \frac{A\, e^{\frac{S}{c}}\, \rho^\gamma}{\gamma - 1}\, dx\right) =$$

$$= \frac{A}{c(\gamma-1)} \int\limits_{\mathbb{R}^n} e^{\frac{S}{c}}\, \rho^\gamma \partial_t S\, dx + \frac{A\gamma}{\gamma-1} \int\limits_{\mathbb{R}^n} e^{\frac{S}{c}}\, \rho^{\gamma-1} \partial_t \rho\, dx =$$

$$= \frac{A}{c(\gamma-1)} \int\limits_{\mathbb{R}^n} e^{\frac{S}{c}}\, \rho^\gamma \frac{dS}{dt}\, dx - \frac{A\gamma}{\gamma-1} \int\limits_{\mathbb{R}^n} e^{\frac{S}{c}}\, \rho^{\gamma-1} \mathrm{div}(\rho u)\, dx -$$

$$-\frac{A}{c(\gamma-1)}\int\limits_{\mathbb{R}^n} e^{\frac{S}{c}}\,\rho^\gamma(u,\nabla S)\,dx \le \tag{3.21}$$

$$\le \frac{\|\sigma(t,x)\|_{L^\infty(\mathbb{R}^n)}}{c}\,E_i(t) + \frac{A\gamma}{\gamma-1}\int\limits_{\mathbb{R}^n} \rho u\,\nabla(e^{\frac{S}{c}}\,\rho^{\gamma-1})\,dx -$$

$$-\frac{A}{c(\gamma-1)}\int\limits_{\mathbb{R}^n} e^{\frac{S}{c}}\,\rho^\gamma(u,\nabla S)\,dx =$$

$$= \frac{\|\sigma(t,x)\|_{L^\infty(\mathbb{R}^n)}}{c}\,E_i(t) + \frac{A}{c}\int\limits_{\mathbb{R}^n} e^{\frac{S}{c}}\,\rho^\gamma\,(u,\nabla S)\,dx + A\int\limits_{\mathbb{R}^n} e^{\frac{S}{c}}\,(u,\nabla\rho^\gamma)\,dx.$$

At last, from (3.20), (3.21), the condition (iv), and estimates (3.9), (3.10), (3.3), (3.5) one can get for $\gamma \le 1+\frac{2}{n}$

$$0 = \frac{d}{dt}(E_k(t)+E_i(t)) \le$$

$$\le -\nu\int\limits_{\mathbb{R}^n} |Du|^2\,dx + \frac{\|\sigma(t,x)\|_{L^\infty(\mathbb{R}^n)}}{c}\,E_i(t) \le$$

$$\le -\nu\,K\,C_2^{\frac{2-n}{n(\gamma-1)}}\,G^{\frac{n-2}{2n}}(t) + o(t^\alpha)\,G^{-\frac{n(\gamma-1)}{2}}(t) \le$$

$$\le -L\,t^{\frac{n-2}{n}} + o(t^{\alpha-n(\gamma-1)}),$$

with some positive constant L. Analogously, for $\gamma > 1+\frac{2}{n}$,

$$0 = \frac{d}{dt}(E_k(t)+E_i(t)) \le -\nu\,K\,C_2^{\frac{2-n}{n(\gamma-1)}}\,G^{\frac{n-2}{2n}}(t) + o(t^\alpha)\,G^{-1}(t) \le -L\,t^{\frac{n-2}{n}} + o(t^{\alpha-2}).$$

If we take α from the condition (iv), we get a contradiction that proves the theorem.

Proof of theorem 2.2. It remains to prove the upper estimates for $\|p\|_{L^\infty(\mathbb{R}^n)}$ and $\|\mathrm{div} u\|_{L^\infty(\mathbb{R}^n)}$. It is easy to compute that

$$E_i'(t) = -\int\limits_{\mathbb{R}^n} p\ \mathrm{div}u\,dx + (\mu+\lambda)\int\limits_{\mathbb{R}^n} |\mathrm{div}u|^2\,dx + \mu\int\limits_{\mathbb{R}^n} |Du|^2\,dx. \tag{3.22}$$

Let us denote $f_1(t) = \|\mathrm{div}u\|_{L^\infty(\mathbb{R}^n)}$ and $f_2(t) = \|p\|_{L^\infty(\mathbb{R}^n)}$. As in the proof of Theorem 2.1 taking into account Lemma 3.3 we get from (3.22)

$$E_i'(t) \ge -(\gamma-1)\,E_i(t)\,f_1(t) + \mu\int\limits_{\mathbb{R}^n} |Du|^2\,dx \ge$$

$$\ge -(\gamma-1)\,E_i(t)\,f_1(t) + K\,(E_i(t))^{-\frac{n-2}{n(\gamma-1)}} \ge -L_1\,t^\beta\,f_1(t) + L_2\,t^{\frac{n-2}{n}},$$

$$E_i'(t) \ge -\frac{\gamma-1}{\mu+\lambda} E_i(t) f_2(t) + \mu \int\limits_{\mathbb{R}^n} |\nabla u|^2\,dx \ge$$

$$\ge -\frac{\gamma-1}{\mu+\lambda} E_i(t) f_2(t) + K(E_i(t))^{-\frac{n-2}{n(\gamma-1)}} \ge -L_3\, t^{\beta}\, f_2(t) + L_2\, t^{\frac{n-2}{n}},$$

with positive constants L_1, L_2, L_3 where $\beta = -n(\gamma-1)$, for $\gamma \le 1 + \frac{2}{n}$, and $\beta = -2$, otherwise. If the growth rate of $f_1(t)$ and $f_2(t)$ is less then prescribed in Theorem 3.1 statement, we get a contradiction. Thus, the theorem is proved.

Remark 3.1 *As follows from (3.22) and Lemmas 3.2, 3.3, 3.4, the requirement of incompressibility* $\mathrm{div}_x u = 0$ *signifies that in conditions of Theorem 2.2 the solution to (NS*) loses its initial smoothness within a finite time.*

3.1 Long-time behavior of solution and the blow up

In [8] it was found that there exists no global smooth solution to (NS) such that

$$\limsup_{t\to\infty} \left\| t\,\frac{(u,x)}{1+|x|^2} \right\|_{L^\infty(\mathbb{R}^n)} < 1. \tag{3.23}$$

One can also derive that there exists no global smooth solution to (NS) such that

$$\limsup_{t\to\infty}\ \|u\|_{L^\infty(\mathbb{R}^n)} < \frac{|P|}{m} = const \tag{3.24}$$

or

$$\limsup_{t\to\infty} \left(\int\limits_{t_0}^{t} \|\mathrm{div} u\|_{L^\infty(\mathbb{R}^n)}(\tau)\,d\tau - n\ln t \right) < const, \quad t_0 \ge 0. \tag{3.25}$$

Condition (3.24) follows from the Hölder inequality:

$$|P| = \Big| \int\limits_{\mathbb{R}^n} \rho u\,dx \Big| \le m\ \|u\|_{L^\infty(\mathbb{R}^n)}.$$

To prove (3.25) (e.g. for $\gamma \le 1 + \frac{2}{n}$) we note that (3.22) results in

$$E_i'(t) \ge -\int\limits_{\mathbb{R}^n} p\ \mathrm{div} u\,dx \ge -(\gamma-1)\,E_i(t)\ \|\mathrm{div} u\|_{L^\infty(\mathbb{R}^n)}.$$

Further, integrating from any $t_0 \ge 0$ gives

$$\ln E_i(t) \ge -(\gamma-1) \int\limits_{t_0}^{t} \|\mathrm{div} u\|_{L^\infty(\mathbb{R}^n)}(\tau)\,d\tau + \ln E_i(t_0). \tag{3.26}$$

Further, inequalities (3.3), (3.9) give the following estimate for sufficiently large t:

$$E_i(t) \le C_2 \,(\mathcal{E}t^2 + F(0)t + G(0))^{-(\gamma-1)n/2}. \tag{3.27}$$

Estimate (3.25) follows from (3.26) and (3.27) immediately.

However, condition (3.23) just as conditions (3.24) and (3.25) does not use the fact that the velocity belongs to the space $H^1(\mathbb{R}^n)$. These conditions can be applied for the case of zero coefficients of viscosity, i.e. for the gas dynamics equations. Moreover, it is possible to construct global in time exact solutions to the gas dynamics equations with the velocity that increases by modulus as $|x| \to \infty$ and has the form $u = A(t)\,x, \quad A(t) \sim t^{-1},\ t \to \infty$ (see e.g. [12],[9] for details) such that conditions (3.23), (3.24), (3.25) become equalities. These solutions satisfy (NS) as well. As follows from (3.25), for smooth solutions to (NS) the function $\int\limits_{t_0}^{t} \|\mathrm{div} u\|_{L^\infty(\mathbb{R}^n)}\,(\tau)\,d\tau \ \ge\ n\ln t + const,\ t_0 \ \ge\ 0,$ and the comparison with with the statement of Theorem 2.2 shows that for solutions of class $H^1(\mathbb{R}^n)$ this estimate is very far to be exact.

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