GLOBAL UNIQUENESS AND RECONSTRUCTION FOR THE MULTI-CHANNEL GEL'FAND-CALDERÓN INVERSE PROBLEM IN TWO DIMENSIONS

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Abstract. We study the multi-channel Gel'fand-Calderón inverse problem in two dimensions, i.e. the inverse boundary value problem for the equation $-\Delta\psi + v(x)\psi = 0$, $x \in D$, where v is a smooth matrix-valued potential defined on a bounded planar domain D. We give an exact global reconstruction method for finding v from the associated Dirichlet-to-Neumann operator. This also yields a global uniqueness results: if two smooth matrix-valued potentials defined on a bounded planar domain have the same Dirichlet-to-Neumann operator then they coincide.

1. Introduction

Let D be an open bounded domain in $\mathbb{R}^2$ with with C^2 boundary and let $v \in C^1(\bar{D}, M_n(\mathbb{C}))$, where $M_n(\mathbb{C})$ is the set of the $n \times n$ complex-valued matrices. The Dirichlet-to-Neumann map associated to v is the operator $\Phi : C^1(\partial D, M_n(\mathbb{C})) \to L^p(\partial D, M_n(\mathbb{C}))$, $p < \infty$ defined by:

$$\Phi(f) = \left.\frac{\partial \psi}{\partial \nu}\right|_{\partial D} \tag{1.1}$$

where $f \in C^1(\partial D, M_n(\mathbb{C}))$, ν is the outer normal of ∂D and ψ is the $H^1(\bar{D}, M_n(\mathbb{C}))$-solution of the Dirichlet problem

$$-\Delta\psi + v(x)\psi = 0 \ \text{ on } D, \ \ \psi|_{\partial D} = f; \tag{1.2}$$

here we assume that

$$0 \text{ is not a Dirichlet eigenvalue for the operator } -\Delta + v \text{ in } D. \tag{1.3}$$

Equation (1.2) arises, in particular, in quantum mechanics, acoustics, electrodynamics; formally, it looks like the Schrödinger equation with potential v at zero energy.

In addition, (1.2) comes up as a 2D-approximation for the 3D equation (see section 2).

The following inverse boundary value problem arises from this construction.

Problem 1. Given Φ, find v.

This problem can be considered as the Gel'fand inverse boundary value problem for the multi-channel 2D Schrödinger equation at zero energy (see [11], [13]) and can also be seen as a generalization of the 2D Calderón problem for the electrical impedance tomography (see [8], [13]). In addition, the history of inverse problems for the two-dimensional Schrödinger equation at fixed energy goes back to [9] (see also [14], [12] and references therein). Note also that Problem 1 can be considered as a model problem for the monochromatic ocean tomography (e.g. see [3] for similar problems arising in this tomography).

In the case of complex-valued potentials the global injectivity of the map $v \to \Phi$ was firstly proved in [13] for $D \subset \mathbb{R}^d$ with $d \geq 3$ and in [6] for $d = 2$ with $v \in L^p$: in particular, these results were obtained by the use of global reconstructions developed in the same papers.

This is the first paper which gives global (uniqueness and reconstruction) results for Problem 1 with $M_n(\mathbb{C})$-valued potentials with $n \geq 2$. Results in this direction were only known for potentials with many restrictions (e.g. see [19]).

We emphasize that Problem 1 is not overdetermined, in the sense that we consider the reconstruction of a $M_n(\mathbb{C})$-valued function $v(x)$ of two variables, $x \in D \subset \mathbb{R}^2$, from a $M_n(\mathbb{C})$-valued function $\Phi(\theta, \theta')$ of two variables, $(\theta, \theta') \in \partial D \times \partial D$, where $\Phi(\theta, \theta')$ is the Schwartz kernel of the Dirichlet-to-Neumann operator Φ: this is one of the principal differences between Problem 1 and its analogue for $D \subset \mathbb{R}^d$ with $d \geq 3$. At present, very few global results are proved for non-overdetermined inverse problems for the Schrödinger equation on $D \subset \mathbb{R}^d$ with $d \geq 2$. Concerning these results, our paper develops the two-dimensional works [6], [17] and indicates 3D applications of the method. The non-overdetermined inverse problems, including multi-channel ones, are much more developed for the Schrödinger equation in dimension $d = 1$ (e.g. see [1], [20]).

We recall that in global results one does not assume that the potential v is small in some sense or is (piecewise) real analytic or is subject to some other serious restrictions.

Our global reconstruction procedure for Problem 1 follows the same scheme as in the scalar case given in [13], with some fundamental modifications inspired by [6].

Let us identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, where $(x_1, x_2) \in \mathbb{R}^2$. We define a special family of solutions of equation (1.2), which we call the Buckhgeim analogues of the Faddeev solutions: $\psi_{z_0}(z, \lambda)$, for $z, z_0 \in \bar{D}, \lambda \in \mathbb{C}$, such that $-\Delta\psi + v(x)\psi = 0$ over D, where in particular $\psi_{z_0}(z,\lambda) \to e^{\lambda(z-z_0)^2} I$ for $\lambda \to \infty$ (i.e. for $|\lambda| \to +\infty$) and I is the identity matrix.

More precisely, for a matrix valued potential v of size n, we define $\psi_{z_0}(z,\lambda)$ as

$$\psi_{z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}\mu_{z_0}(z,\lambda), \tag{1.4}$$

where $\mu_{z_0}(\cdot,\lambda)$ solves the integral equation

$$\mu_{z_0}(z,\lambda) = I + \int_D g_{z_0}(z,\zeta,\lambda)v(\zeta)\mu_{z_0}(\zeta,\lambda) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{1.5}$$

I is the identity matrix of size $n \in \mathbb{N}$, $z, z_0 \in \bar{D}$, $\lambda \in \mathbb{C}$ and

$$g_{z_0}(z,\zeta,\lambda) = \frac{e^{\lambda(\zeta-z_0)^2-\bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}}{4\pi^2}\int_D \frac{e^{-\lambda(\eta-z_0)^2+\bar{\lambda}(\bar{\eta}-\bar{z}_0)^2}}{(z-\eta)(\bar{\eta}-\bar{\zeta})} d\mathrm{Re}\eta\, d\mathrm{Im}\eta \tag{1.6}$$

is a Green function of the operator $4\left(\frac{\partial}{\partial z} + 2\lambda(z-z_0)\right)\frac{\partial}{\partial \bar{z}}$ in D, for $z_0 \in D$. We consider equation (1.5), at fixed z_0 and λ, as a linear integral equation for $\mu_{z_0}(\cdot,\lambda) \in C^1_{\bar{z}}(\bar{D})$: we will see that it is uniquely solvable for $|\lambda| > \rho_1(D, N_1, n)$, where $\|v\|_{C^1_{\bar{z}}(\bar{D}, M_n(\mathbb{C}))} < N_1$ (see Proposition 1.3).

In order to state the reconstruction method we also define the Bukhgeim analogue of the Faddeev generalized scattering amplitude

$$h_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{z}-\bar{z}_0)^2} v(z)\mu_{z_0}(z,\lambda) d\mathrm{Re}z\, d\mathrm{Im}z, \tag{1.7}$$

for $z_0 \in \bar{D}$, $\lambda \in \mathbb{C}$.

Theorem 1.1. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with C^2 boundary and let $v \in C^1(\bar{D}, M_n(\mathbb{C}))$ be a matrix-valued potential which satisfies* (1.3) *and $v|_{\partial D} = 0$. Consider, for $z_0 \in D$, the functions h_{z_0}, ψ_{z_0}, g_{z_0} defined above and Φ, Φ_0 the Dirichlet-to-Neumann maps associated to the potentials v and* 0*, respectively. Then the following reconstruction formulas and equation hold:*

$$v(z_0) = \lim_{\lambda\to\infty} \frac{2}{\pi}|\lambda| h_{z_0}(\lambda), \tag{1.8}$$

$$h_{z_0}(\lambda) = \int_{\partial D} e^{-\bar{\lambda}(\bar{z}-\bar{z}_0)}(\Phi - \Phi_0)\psi_{z_0}(z,\lambda)|dz|, \tag{1.9}$$

$$\psi_{z_0}(z,\lambda)|_{\partial D} = e^{\lambda(z-z_0)^2} I + \int_{\partial D} G_{z_0}(z,\zeta,\lambda)(\Phi-\Phi_0)\psi_{z_0}(\zeta,\lambda)|d\zeta|, \tag{1.10}$$

where

$$G_{z_0}(z,\zeta,\lambda) = e^{\lambda(z-z_0)^2} g_{z_0}(z,\zeta,\lambda) e^{-\lambda(\zeta-z_0)^2}, \tag{1.11}$$

$z_0 \in D$, $z, \zeta \in \partial D$, $\lambda \in \mathbb{C}$, $|\lambda| > \rho_1(D, N_1, n)$, *where* $\|v\|_{C^1_{\bar{z}}(\bar{D}, M_n(\mathbb{C}))} < N_1$.

In addition, if $v \in C^2(\bar{D}, M_n(\mathbb{C}))$ with $\|v\|_{C^2(\bar{D}, M_n(\mathbb{C}))} < N_2$ and $\frac{\partial v}{\partial \nu}|_{\partial D} = v|_{\partial D} = 0$ then the following estimates hold:

$$\left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}(\lambda) \right| \le a(D,n) \frac{\log(3|\lambda|)}{|\lambda|^{1/2}} N_2(N_2+1), \tag{1.12a}$$

$$\left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}(\lambda) \right| \le b(D,n) \frac{(\log(3|\lambda|))^2}{|\lambda|^{3/4}} N_2(N_2^2+1), \tag{1.12b}$$

for $|\lambda| > \rho_2(D, N_1, n)$, $z_0 \in D$.

Remark 1. Note that in Theorem 1.1, $\rho_j = \rho_j(D, N_1, n)$, $j = 1, 2$ (where $\|v\|_{C^1_{\bar{z}}(\bar{D}, M_n(\mathbb{C}))} < N_1$), are arbitrary fixed positive constants such that

$$\begin{aligned} &2n \frac{c_2(D)}{|\lambda|^{\frac{1}{2}}} \|v\|_{C^1_{\bar{z}}(\bar{D})} < 1, \ |\lambda| \ge 1, \ \text{if } |\lambda| > \rho_1, \\ &2n \frac{c_2(D)}{|\lambda|^{\frac{1}{2}}} \|v\|_{C^1_{\bar{z}}(\bar{D})} \le \frac{1}{2}, \ |\lambda| \ge 1, \ \text{if } |\lambda| > \rho_2, \end{aligned} \tag{1.13}$$

where c_2 is the constant in Lemma 3.1.

Remark 2. Note that estimate (1.12b) is not strictly stronger than (1.12a) because of the presence of the N_2^3 factor.

In order to make use of the reconstruction given by Theorem 1.1, the following two propositions are necessary:

Proposition 1.2. *Under the assumptions of Theorem 1.1 (without the additional assumptions used for* (1.12)*), equation* (1.10) *is a Fredholm linear integral equation of the second kind for $\psi_{z_0} \in C(\partial D)$.*

Proposition 1.3. *Under the assumptions of Theorem 1.1 (without the additional assumptions used for* (1.12)*), for $|\lambda| > \rho_1(D, N_1, n)$, where $\|v\|_{C^1_{\bar{z}}(\bar{D}, M_n(\mathbb{C}))} < N_1$, equations* (1.5) *and* (1.10) *are uniquely solvable in the spaces of continuous functions on $\bar{D}$ and ∂D, respectively.*

Remark 3. Note that the assumption that $v|_{\partial D} = 0$ is unnecessary for formula (1.9), equation (1.10) and Propositions 1.2, 1.3. In addition, formula (1.8) also holds without this assumption if

$$\int_{\partial D} e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} w(z) |dz| \to 0 \qquad \text{as } |\lambda| \to \infty, \tag{1.14}$$

for fixed $z_0 \in D$ and each $w \in C^1(\partial D)$. The class of domains D for which (1.14) holds for each $z_0 \in D$ is large and includes, for example, all ellipses.

Note also that if $v|_{\partial D} \neq 0$ but $v \equiv \Lambda \in M_n(\mathbb{C})$ on some open neighborhood of ∂D in $\bar{D}$, then estimates (1.12) hold with $h_{z_0}(\lambda)$ replaced by

$$h^+_{z_0}(\lambda) = h_{z_0}(\lambda) + \int_{\mathbb{R}^2 \setminus D} e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z}_0)^2} \Lambda \chi(z) d\mathrm{Re} z \, d\mathrm{Im} z, \tag{1.15}$$

where $\chi \in C^2(\mathbb{R}^2, \mathbb{R})$, $\chi \equiv 1$ on D, suppχ is compact, and with the constants a, b depending also on χ. The aforementioned matrix Λ, for example, can be related with a diagonal matrix composed by the eigenvalues $\{\lambda_i\}_{1\le i\le n}$ arising in section 2.

Theorem 1.1 and Propositions 1.2, 1.3 yield the following corollary:

Corollary 1.4. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with C^2 boundary, let $v_1, v_2 \in C^1(\bar{D}, M_n(\mathbb{C}))$ be two matrix-valued potentials which satisfy* (1.3) *and Φ_1, Φ_2 the corresponding Dirichlet-to-Neumann operators. If $\Phi_1 = \Phi_2$ then $v_1 = v_2$.*

Theorem 1.1, Propositions 1.2, 1.3 and Corollary 1.4 are proved in section 4.

The global reconstruction of Theorem 1.1 is fine in the sense that is consists in solving Fredholm linear integral equations of the second type and using explicit formulas; nevertheless this reconstruction is not optimal with respect to its stability properties: see [7], [16], [5] for discussions and numerical implementations of the aforementioned similar (but overdetermined) reconstruction of [13] for $d = 3$ and $n = 1$. An approximate but more stable reconstruction method for Problem 1 will be published in another paper.

The present paper is focused on global uniqueness and reconstruction for Problem 1 for $n \geq 2$. In addition, using the techniques developed in the present work and following the scheme of [17] it is also possible to obtain a global logarithmic stability estimate for Problem 1 in the multi-channel case. Following inverse problem traditions (e.g. see [2], [16], [17]) this result will be published in another paper.

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2. Approximation of the 3D equation

In this section we recall how the multi-channel two-dimensional Schrödinger equation can be seen as an approximation of the scalar 3D equation in a cylindrical domain; in this framework, three-dimensional inverse problems can be approximated by two-dimensional ones.

Let $L = [a, b]$ for some $a, b \in \mathbb{R}$ and consider the complex-valued potential $v(x, z)$ defined on the set $D \times L$, where $x = (x_1, x_2) \in D \subset \mathbb{R}^2$, $z \in L$. We consider the equation

$$-\Delta\psi(x, z) + v(x, z)\psi(x, z) = 0 \quad \text{in } D \times L. \tag{2.1}$$

Now, for every $x \in D$ we can write $\psi(x, z) = \sum_{j=1}^{\infty} \psi_j(x)\phi_j(z)$, where $\{\phi_j\}$ is the orthonormal basis of $L^2(L)$ given by the eigenfunctions of $-\frac{d^2}{dz^2}$: more precisely

$$-\frac{d^2}{dz^2}\phi_j(z) = \lambda_j \phi_j(z) \text{ for } z \in L, \tag{2.2}$$

$$\phi_j|_{\partial L} = 0 \quad \text{(for example)} \tag{2.3}$$

$$\int_L \bar{\phi}_i(z)\phi_j(z)dz = \delta_{ij}$$

and $\psi_j(x) = \int_L \psi(x,z)\bar{\phi}_j(z)dz$. Now equation (2.1) reads

$$\sum_{j=1}^{\infty}(-\Delta_x \psi_j(x)\phi_j(z) - \psi_j(x)\Delta_z\phi_j(z)) + v(x,z)\sum_{j=1}^{\infty}\psi_j(x)\phi_j(z) = 0. \tag{2.4}$$

Using (2.2)-(2.4) and the properties of $\{\phi_j(z)\}$, we obtain that equation (2.1) is equivalent to the following infinite-dimensional system

$$-\Delta_x\psi_i(x) + \lambda_i\psi_i(x) + \sum_{j=1}^{\infty} V_{ij}(x)\psi_j(x) = 0, \text{ for } i = 1, \dots, \tag{2.5}$$

where

$$V_{ij}(x) = \int_L \bar{\phi}_i(z)v(x,z)\phi_j(z)dz.$$

Notice that if $\bar{v} = v$ then $V^* = V$. Now, if we impose $1 \leq i, j \leq n$ for some $n \in \mathbb{N}$, we find equation (1.2).

We also give here the relation between the Dirichlet-to-Neumann (D-t-N) operators of the 3D equation and that of the 2D multi-channel equation. If $\Phi(\theta, z, \theta', z')$ is the Schwartz kernel of the D-t-N operator of the 3D problem, and $(\Phi_{ij}(\theta,\theta'))_{i,j\geq 1}$ that of the 2D infinity-channel problem, we have

$$\Phi_{ij}(\theta,\theta') = \int_{L\times L}\Phi(\theta,z,\theta',z')\bar{\phi}_i(z)\phi_j(z')dz\,dz', \tag{2.6}$$

where $\theta, \theta' \in \partial D$, $z, z' \in L$. This follows from

(2.7)

$$\int_{\partial D\times L}\Phi(\theta,z,\theta',z')f(\theta',z')d\theta'\,dz' = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\int_{\partial D}\Phi_{ij}(\theta,\theta')f_j(\theta')d\theta'\right)\phi_i(z),$$

for every $f \in C^1(\partial(D\times L))$ such that $f|_{D\times\partial L} = 0$ and $f(\theta,z) = \sum_{j=1}^{\infty} f_j(\theta)\phi_j(z)$.

Let us remark that reductions of 3D direct and inverse problems to multi-channel 2D problems are well known in the physical literature for a long time (e.g. see [3]). Nevertheless, we do not know a reference containing formula (2.6) in its precise form.

3. Preliminaries

In this section we introduce and give details about the above-mentioned family of solutions of equation (1.2), which will be used throughout all the paper.

Let us define the function spaces $C^1_{\bar z}(\bar D) = \{u : u, \frac{\partial u}{\partial \bar z} \in C(\bar D, M_n(\mathbb{C}))\}$ with the norm $\|u\|_{C^1_{\bar z}(\bar D)} = \max(\|u\|_{C(\bar D)}, \|\frac{\partial u}{\partial \bar z}\|_{C(\bar D)})$, $\|u\|_{C(\bar D)} = \sup_{z\in \bar D} |u|$ and $|u| = \max_{1\le i,j\le n} |u_{i,j}|$; we define also $C^1_z(\bar D) = \{u : u, \frac{\partial u}{\partial z} \in C(\bar D, M_n(\mathbb{C}))\}$ with an analogous norm.

The functions $G_{z_0}(z,\zeta,\lambda),\ g_{z_0}(z,\zeta,\lambda),\ \psi_{z_0}(z,\lambda),\ \mu_{z_0}(z,\lambda)$ defined in Section 1, satisfy

$$4\frac{\partial^2}{\partial z\partial \bar z} G_{z_0}(z,\zeta,\lambda) = \delta(z-\zeta), \tag{3.1}$$

$$4\frac{\partial^2}{\partial \zeta\partial \bar \zeta} G_{z_0}(z,\zeta,\lambda) = \delta(\zeta-z), \tag{3.2}$$

$$4\left(\frac{\partial}{\partial z} + 2\lambda(z-z_0)\right)\frac{\partial}{\partial \bar z} g_{z_0}(z,\zeta,\lambda) = \delta(z-\zeta), \tag{3.3}$$

$$4\frac{\partial}{\partial \bar\zeta}\left(\frac{\partial}{\partial \zeta} - 2\lambda(\zeta-z_0)\right) g_{z_0}(z,\zeta,\lambda) = \delta(\zeta-z), \tag{3.4}$$

$$-4\frac{\partial^2}{\partial z\partial \bar z}\psi_{z_0}(z,\lambda) + v(z)\psi_{z_0}(z,\lambda) = 0, \tag{3.5}$$

$$-4\left(\frac{\partial}{\partial z} + 2\lambda(z-z_0)\right)\frac{\partial}{\partial \bar z}\mu_{z_0}(z,\lambda) + v(z)\mu_{z_0}(z,\lambda) = 0, \tag{3.6}$$

where $z, z_0, \zeta \in D$, $\lambda \in \mathbb{C}$, δ is the Dirac's delta. (In addition, it is assumed that (1.5) is uniquely solvable for $\mu_{z_0}(\cdot,\lambda) \in C^1_{\bar z}(\bar D)$ at fixed z_0 and λ.) Formulas (3.1)-(3.6) follow from (1.5), (1.6), (1.11) and from

$$\frac{\partial}{\partial \bar z}\frac{1}{\pi(z-\zeta)} = \delta(z-\zeta),$$

$$\left(\frac{\partial}{\partial z} + 2\lambda(z-z_0)\right)\frac{e^{-\lambda(z-z_0)^2+\bar\lambda(\bar z-\bar z_0)^2}}{\pi(\bar z-\bar\zeta)} e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2} = \delta(z-\zeta),$$

where $z, \zeta, z_0, \lambda \in \mathbb{C}$.

We say that the functions G_{z_0}, g_{z_0}, ψ_{z_0}, μ_{z_0}, h_{z_0} are the Bukhgeim-type analogues of the Faddeev functions (see [17]). We recall that the history of these functions goes back to [10] and [4].

Now we state some fundamental lemmata. Let

$$g_{z_0,\lambda}u(z) = \int_D g_{z_0}(z,\zeta,\lambda)u(\zeta)d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta,\ z\in\bar D,\ z_0,\lambda\in\mathbb{C}, \tag{3.7}$$

where $g_{z_0}(z,\zeta,\lambda)$ is defined by (1.6) and u is a test function.

Lemma 3.1 ([17]). *Let $g_{z_0,\lambda}u$ be defined by* (3.7)*. Then, for $z_0, \lambda \in \mathbb{C}$, the following estimates hold:*

$$g_{z_0,\lambda}u \in C^1_{\bar z}(\bar D), \quad \text{for } u \in C(\bar D), \tag{3.8}$$

$$\|g_{z_0,\lambda}u\|_{C^1(\bar D)} \le c_1(D,\lambda)\|u\|_{C(\bar D)}, \quad \text{for } u \in C(\bar D), \tag{3.9}$$

$$\|g_{z_0,\lambda}u\|_{C^1_{\bar z}(\bar D)} \le \frac{c_2(D)}{|\lambda|^{\frac{1}{2}}}\|u\|_{C^1_{\bar z}(\bar D)}, \quad \text{for } u \in C^1_{\bar z}(\bar D),\ |\lambda| \ge 1. \tag{3.10}$$

Given a potential $v \in C^1_{\bar z}(\bar D)$ we define the operator $g_{z_0,\lambda}v$ simply as $(g_{z_0,\lambda}v)u(z) = g_{z_0,\lambda}w(z),\ w = vu$, for a test function u. If $u \in C^1_{\bar z}(\bar D)$, by Lemma 3.1 we have that $g_{z_0,\lambda}v : C^1_{\bar z}(\bar D) \to C^1_{\bar z}(\bar D)$,

$$\|g_{z_0,\lambda}v\|^{op}_{C^1_{\bar z}(\bar D)} \le 2n\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar z}(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}, \tag{3.11}$$

where $\|\cdot\|^{op}_{C^1_{\bar z}(\bar D)}$ denotes the operator norm in $C^1_{\bar z}(\bar D)$, $z_0, \lambda \in \mathbb{C}$. In addition, $\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar z}(\bar D)}$ is estimated in Lemma 3.1. Inequality (3.11) and Lemma 3.1 implies existence and uniqueness of $\mu_{z_0}(z,\lambda)$ (and thus also $\psi_{z_0}(z,\lambda)$) for $|\lambda| > \rho_1(D, N_1, n)$.

Let

$$\mu^{(k)}_{z_0}(z,\lambda) = \sum_{j=0}^{k}(g_{z_0,\lambda}v)^j I,$$

$$h^{(k)}_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}v(z)\mu^{(k)}_{z_0}(z,\lambda)d\mathrm{Re}z\, d\mathrm{Im}z,$$

where $z, z_0 \in D$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}\cup\{0\}$.

Lemma 3.2 ([17]). *For $v \in C^1_{\bar z}(\bar D)$ such that $v|_{\partial D} = 0$ the following formula holds:*

$$v(z_0) = \frac{2}{\pi}\lim_{\lambda\to\infty}|\lambda|h^{(0)}_{z_0}(\lambda), \qquad z_0 \in D. \tag{3.12}$$

In addition, if $v \in C^2(\bar D)$, $v|_{\partial D} = 0$ and $\frac{\partial v}{\partial \nu}|_{\partial D} = 0$ then

$$\left|v(z_0) - \frac{2}{\pi}|\lambda|h^{(0)}_{z_0}(\lambda)\right| \le c_3(D,n)\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar D)}, \tag{3.13}$$

for $z_0 \in D$, $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$.

Following the proof of [17, Lemma 6.2] and assuming (1.14), we have that limit (3.12) is valid without the assumption that $v|_{\partial D} = 0$. In addition, if $v|_{\partial D} \neq 0$ but $v \equiv \Lambda \in M_n(\mathbb{C})$ on some open neighborhood of ∂D in $\bar D$, then estimate (3.13) holds with $h^{(0)}_{z_0}(\lambda)$ replaced by

$$h^{(0),+}_{z_0}(\lambda) = h^{(0)}_{z_0}(\lambda) + \int_{\mathbb{R}^2\setminus D} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\Lambda\chi(z)d\mathrm{Re}z\, d\mathrm{Im}z, \tag{3.14}$$

where $\chi \in C^2(\mathbb{R}^2, \mathbb{R})$, $\chi \equiv 1$ on D, $\mathrm{supp}\chi$ is compact, and the constant c_3 depending also on χ.

Let

$$W_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} w(z) d\mathrm{Re}\, z d\mathrm{Im}\, z, \tag{3.15}$$

where $z_0 \in \bar D$, $\lambda \in \mathbb{C}$ and w is some $M_n(\mathbb{C})$-valued function on $\bar D$. (One can see that $W_{z_0} = h_{z_0}^{(0)}$ for $w = v$.)

Lemma 3.3 ([17]). *For $w \in C^1_{\bar z}(\bar D)$ the following estimate holds:*

$$|W_{z_0}(\lambda)| \leq c_4(D) \frac{\log(3|\lambda|)}{|\lambda|} \|w\|_{C^1_{\bar z}(\bar D)}, \; z_0 \in \bar D, \; |\lambda| \geq 1. \tag{3.16}$$

Lemma 3.4. *For $v \in C^1_{\bar z}(\bar D)$ and for $\|g_{z_0,\lambda} v\|^{op}_{C^1_{\bar z}(\bar D)} \leq \delta < 1$ we have that*

$$\|\mu_{z_0}(\cdot, \lambda) - \mu_{z_0}^{(k)}(\cdot, \lambda)\|_{C^1_{\bar z}(\bar D)} \leq \frac{\delta^{k+1}}{1-\delta}, \tag{3.17}$$

$$|h_{z_0}(\lambda) - h_{z_0}^{(k)}(\lambda)| \leq c_5(D, n) \frac{\log(3|\lambda|)}{|\lambda|} \frac{\delta^{k+1}}{1-\delta} \|v\|_{C^1_{\bar z}(\bar D)}, \tag{3.18}$$

where $z_0 \in D, \; \lambda \in \mathbb{C}, \; |\lambda| \geq 1, \; k \in \mathbb{N} \cup \{0\}$.

The proof of Lemma 3.4 in the scalar case can be found in [17]: the generalization to the matrix-valued case is straightforward.

Lemma 3.5. *The function $g_{z_0}(z, \zeta, \lambda)$ satisfies the following properties:*

$$g_{z_0}(z, \zeta, \lambda) \quad \text{is continuous for } z, \zeta \in \bar D, \; z \neq \zeta, \; z_0 \in D, \tag{3.19}$$

$$|g_{z_0}(z, \zeta, \lambda)| \leq c_6(D) |\log|z - \zeta||, \qquad z, \zeta \in \bar D, \; z_0 \in D, \tag{3.20}$$

where $\lambda \in \mathbb{C}$.

These properties follow from the definition (1.6) and from classical estimates (see [18]).

Lemma 3.6. *Under the assumptions of Proposition 1.2, the Schwartz kernel $(\Phi - \Phi_0)(z, \zeta)$ of the operator $\Phi - \Phi_0$ satisfies the following properties:*

$$(\Phi - \Phi_0)(z, \zeta) \quad \text{is continuous for } z, \zeta \in \partial D, \; z \neq \zeta, \tag{3.21}$$

$$|(\Phi - \Phi_0)(z, \zeta)| \leq c_7(D, v, n) |\log|z - \zeta||, \qquad z, \zeta \in \partial D. \tag{3.22}$$

For a proof of this Lemma in the scalar case we refer to [13, 15]: the generalization to the matrix-valued case is straightforward.

4. Proofs of Theorem 1.1, Propositions 1.2, 1.3 and Corollary 1.4

We begin with a matrix version of Alessandrini's identity (see [2] for the scalar case):

$$\int_{\partial D} u_0(z)(\Phi-\Phi_0)u(z)|dz| = \int_D u_0(z)v(z)u(z)d\mathrm{Re}z\, d\mathrm{Im}z \tag{4.1}$$

for any sufficiently regular $M_n(\mathbb{C})$-valued function u (resp. u_0) such that $\Delta u_0 = 0$ (resp. $(-\Delta+v)u=0$) in D. This follows from Stokes's theorem, exactly as in the scalar case.

The general matrix version of Alessandrini's identity (that will not be used)

$$\int_{\partial D} u_1(z)(\Phi_2-\Phi_1)u_2(z)|dz| = \int_D u_1(z)(v_2(z)-v_1(z))u_2(z)d\mathrm{Re}z\, d\mathrm{Im}z \tag{4.2}$$

for $u_1, u_2 \in C^2(\bar D, M_n(\mathbb{C}))$ such that $(-\Delta+v_j)u_j = 0$ in D, works if u_1 and v_1 commute each other (but does not work in general).

Proof of Theorem 1.1. Let us begin with the proof of formulas (1.8) and (1.12): we have indeed

(4.3)
$$\left|v(z_0)-\frac{2}{\pi}|\lambda|h_{z_0}(\lambda)\right| \le \left|v(z_0)-\frac{2}{\pi}|\lambda|h^{(0)}_{z_0}(\lambda)\right| + \frac{2}{\pi}|\lambda||h_{z_0}(\lambda)-h^{(0)}_{z_0}(\lambda)|.$$

The first term in the right side goes to zero as $|\lambda|\to\infty$ by Lemma 3.2, while the other by Lemmata 3.1 and 3.4. In addition, for $v\in C^2(\bar D, M_n(\mathbb{C}))$ with $\|v\|_{C^2(\bar D)} < N_2$ and $\frac{\partial v}{\partial \nu}|_{\partial D} = 0$, using (3.10), (3.11), (3.13) and (3.18) we obtain, from (4.3):

$$\begin{aligned}\left|v(z_0)-\frac{2}{\pi}|\lambda|h_{z_0}(\lambda)\right| &\le c_3(D,n)\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar D)}\\ &\quad + c_5(D,n)\frac{\log(3|\lambda|)}{|\lambda|^{1/2}}\|v\|^2_{C^1_{\bar z}(\bar D)}\\ &\le c_8(D,n)\frac{\log(3|\lambda|)}{|\lambda|^{1/2}}(\|v\|_{C^2(\bar D)}+\|v\|^2_{C^1_{\bar z}(\bar D)}),\end{aligned}$$

for λ such that

$$2n\frac{c_2(D)}{|\lambda|^{\frac12}}\|v\|_{C^1_{\bar z}(\bar D)} \le \frac12,\ \ |\lambda|\ge 1,$$

which implies (1.12a). In order to prove (1.12b) we will need the following lemma:

Lemma 4.1. *Let $g_{z_0,\lambda}u$ be defined by (3.7), where $u \in C^1_{\bar z}(\bar D)$, $z_0, \lambda \in \mathbb{C}$. Then the following estimate holds:*

$$\|g_{z_0,\lambda}u\|_{C(\bar D)} \le \eta(D)\frac{\log(3|\lambda|)}{|\lambda|^{\frac{3}{4}}}\|u\|_{C^1_{\bar z}(\bar D)},\ |\lambda| \ge 1. \tag{4.4}$$

Proof of Lemma 4.1. As in the proof of [17, Lemma 3.1], we can write $g_{z_0,\lambda} = \frac{1}{4}T\bar T_{z_0,\lambda}$, for $z_0, \lambda \in \mathbb{C}$, where

$$Tu(z) = -\frac{1}{\pi}\int_D \frac{u(\zeta)}{\zeta - z} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta,$$

$$\bar T_{z_0,\lambda}u(z) = -\frac{e^{-\lambda(z-z_0)^2+\bar\lambda(\bar z-\bar z_0)^2}}{\pi}\int_D \frac{e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}}{\bar\zeta-\bar z}u(\zeta)d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta,$$

for $z \in \bar D$ and u a test function. We have that (see [17]):

$$Tw \in C^1_{\bar z}(\bar D), \tag{4.5}$$

$$\|Tw\|_{C^1_{\bar z}(\bar D)} \le \eta_1(D)\|w\|_{C(\bar D)},\ \text{where } w \in C(D), \tag{4.6}$$

$$\bar T_{z_0,\lambda}u \in C(\bar D), \tag{4.7}$$

$$\|\bar T_{z_0,\lambda}u\|_{C(\bar D)} \le \frac{\eta_2(D)}{|\lambda|^{\frac{1}{2}}}\|u\|_{C^1_{\bar z}(\bar D)},\ |\lambda| \ge 1, \tag{4.8}$$

$$\|\bar T_{z_0,\lambda}u\|_{C(\bar D)} \le \frac{\log(3|\lambda|)(1+|z-z_0|)\eta_3(D)}{|\lambda||z-z_0|^2}\|u\|_{C^1_{\bar z}(\bar D)},\ |\lambda| \ge 1, \tag{4.9}$$

where $u \in C^1_{\bar z}(\bar D)$, $z_0, \lambda \in \mathbb{C}$.

Let $z_0 \in D$, $0 < \delta < \frac{1}{2}$ and $B_{z_0,\delta} = \{z \in \mathbb{C} : |z - z_0| < \delta\}$. We have

(4.10)

$$\begin{aligned}
|4\pi g_{z_0,\lambda}u(z)| &= \left|\int_D \frac{\bar T_{z_0,\lambda}u(\zeta)}{\zeta - z} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta\right| \\
&\le \int_{B_{z_0,\delta}\cap D} \frac{|\bar T_{z_0,\lambda}u(\zeta)|}{|\zeta - z|} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \int_{D\setminus B_{z_0,\delta}} \frac{|\bar T_{z_0,\lambda}u(\zeta)|}{|\zeta - z|} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \\
&\le 2\pi\delta\frac{\eta_2(D)}{|\lambda|^{\frac{1}{2}}}\|u\|_{C^1_{\bar z}(\bar D)} + \frac{\log(3|\lambda|)\eta_4(D)}{|\lambda|\delta}\|u\|_{C^1_{\bar z}(\bar D)},
\end{aligned}$$

where we used the following estimate:

$$\begin{aligned}
&\int_{D\setminus B_{z_0,\delta}} \frac{1}{|\zeta - z||\zeta - z_0|^2} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \\
&\quad = \int_{B_{z,\delta}\cap(D\setminus B_{z_0,\delta})} \frac{1}{|\zeta - z||\zeta - z_0|^2} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \\
&\qquad + \int_{D\setminus(B_{z,\delta}\cup B_{z_0,\delta})} \frac{1}{|\zeta - z||\zeta - z_0|^2} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \\
&\quad \le \frac{2\pi}{\delta} + \int_{D\setminus(B_{z,\delta}\cup B_{z_0,\delta})} \frac{1}{|\zeta - z|^3} + \frac{1}{|\zeta - z_0|^3} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta
\end{aligned}$$

$$\leq \frac{\eta_5(D)}{\delta}.$$

Putting $\delta = \frac{1}{2}|\lambda|^{-\frac{1}{4}}$ in (4.10) we obtain the result. Thus Lemma 4.1 is proved.

We now come back to the proof of (1.12b). Proceeding from (4.3) and Lemma 3.2 we obtain:

(4.11)
$$\left| v(z_0) - \frac{2}{\pi}|\lambda| h_{z_0}(\lambda) \right| \leq c_3(D,n) \frac{\log(3|\lambda|)}{|\lambda|} \|v\|_{C^2(\bar D)} + \frac{2}{\pi}|\lambda||h_{z_0}(\lambda) - h^{(0)}_{z_0}(\lambda)|,$$

for $|\lambda| \geq 1$. In addition, from the definitions of $h^{(k)}$, $\mu^{(k)}$, Lemmata 3.1 and 3.4, we have

$$\begin{aligned} &|h_{z_0}(\lambda) - h^{(0)}_{z_0}(\lambda)| \\ &\quad \leq \left| \int_D e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} v(z) g_{z_0,\lambda} v(z) d\mathrm{Re} z\, d\mathrm{Im} z \right| + O\left(\frac{\log(3|\lambda|)}{|\lambda|^2} \right) n^2 \|v\|^3_{C^1_{\bar z}(\bar D)}, \end{aligned}$$

for λ such that $2n \frac{c_2(D)}{|\lambda|^{1/2}} \|v\|_{C^1_{\bar z}(\bar D)} \leq \frac{1}{2}$, $|\lambda| \geq 1$.

Repeating the proof of [17, Lemma 3.3] and using also Lemma 4.1, we have, for $0 < \varepsilon \leq 1$,

(4.12)
$$\begin{aligned} &\left| \int_D e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} v(z) g_{z_0,\lambda} v(z) d\mathrm{Re} z\, d\mathrm{Im} z \right| \\ &\leq \int_{D \cap B_{z_0,\varepsilon}} \|v(z) g_{z_0,\lambda} v(z)\|_{C(\bar D)} d\mathrm{Re} z\, d\mathrm{Im} z + \frac{1}{4|\lambda|} \int_{\partial(D \setminus B_{z_0,\varepsilon})} \frac{\|v(z) g_{z_0,\lambda} v(z)\|_{C(\bar D)}}{|\bar z - \bar z_0|} |dz| \\ &\quad + \frac{1}{2|\lambda|} \int_{D \setminus B_{z_0,\varepsilon}} \left| \frac{\partial}{\partial \bar z} \left(\frac{v(z) g_{z_0,\lambda} v(z)}{\bar z - \bar z_0} \right) \right| d\mathrm{Re} z\, d\mathrm{Im} z \\ &\leq \sigma_1(D,n) \|v\|_{C(\bar D)} \|v\|_{C^1_{\bar z}(\bar D)} \frac{\varepsilon^2 \log(3|\lambda|)}{|\lambda|^{3/4}} \\ &\quad + \sigma_2(D,n) \|v\|_{C(\bar D)} \|v\|_{C^1_{\bar z}(\bar D)} \frac{\log(3\varepsilon^{-1}) \log(3|\lambda|)}{|\lambda|^{1+3/4}} \\ &\quad + \frac{1}{8|\lambda|} \left| \int_{D \setminus B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} v(z) \frac{\bar T_{z_0,\lambda} v(z)}{\bar z - \bar z_0} d\mathrm{Re} z\, d\mathrm{Im} z \right|, \quad |\lambda| \geq 1, \end{aligned}$$

where we also used integration by parts and the fact that $\frac{\partial}{\partial \bar z} g_{\lambda,z_0} u(z) = \frac{1}{4} \bar T_{z_0,\lambda} u(z)$. The last term in (4.12) can be estimated independently on ε by

$$\sigma_3(D,n) \frac{\log(3|\lambda|)}{|\lambda|^{1+3/4}} \|v\|_{C(\bar D)} \|v\|_{C^1_{\bar z}(\bar D)} \tag{4.13}$$

using the same argument as in the proof of Lemma 4.1 (see estimate (4.10)). Now putting $\varepsilon = |\lambda|^{-1/2}$ in (4.12) we obtain

$$|\lambda||h_{z_0}(\lambda) - h_{z_0}^{(0)}(\lambda)| \leq \sigma_4(D,n)\frac{(\log(3|\lambda|))^2}{|\lambda|^{3/4}}\|v\|^2_{C^1_{\bar{z}}(\bar{D})}(\|v\|_{C^1_{\bar{z}}(\bar{D})}+1),$$

for $|\lambda| > \rho_2(D, N_1, n)$, which, together with (4.11), gives us (1.12b).

The proofs of the other formulas of Theorem 1.1 are based on identity (4.1). As $\mu_{z_0}(z,\lambda) = e^{-\lambda(z-z_0)^2}\psi_{z_0}(z,\lambda)$, we can write the generalized scattering amplitude as

$$h_{z_0}(\lambda) = \int_D e^{-\bar{\lambda}(\bar{z}-\bar{z}_0)^2} v(z)\psi_{z_0}(z,\lambda) d\mathrm{Re} z\, d\mathrm{Im} z.$$

Now identity (4.1) with $u_0(z) = e^{-\bar{\lambda}(\bar{z}-\bar{z}_0)^2} I$ and $u(z) = \psi_{z_0}(z,\lambda)$ reads

$$\int_{\partial D} e^{-\bar{\lambda}(\bar{z}-\bar{z}_0)^2}(\Phi - \Phi_0)\psi_{z_0}(z,\lambda)|dz| = \int_D e^{-\bar{\lambda}(\bar{z}-\bar{z}_0)^2} v(z)\psi_{z_0}(z,\lambda) d\mathrm{Re} z\, d\mathrm{Im} z$$

which gives formula (1.9).

Since μ_{z_0} is a solution of equation (1.5), $\psi_{z_0}(z,\lambda)$ satisfies the equation

$$\psi_{z_0}(z,\lambda) = e^{\lambda(z-z_0)^2} I + \int_D G_{z_0}(z,\zeta,\lambda) v(\zeta)\psi_{z_0}(\zeta,\lambda) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{4.14}$$

for $z_0, z \in \bar{D}$, $\lambda \in \mathbb{C}$, $|\lambda| > \rho_1(D, N_1, n)$. Thus again by identity (4.1), with $u_0 = G_{z_0}(z,\zeta,\lambda) I$ and $u(z) = \psi_{z_0}(\zeta,\lambda)$, by (3.2) and (4.14) we obtain, for $z \in \partial D$,

$$\begin{aligned}\int_{\partial D} G_{z_0}(z,\zeta,\lambda)(\Phi - \Phi_0)\psi_{z_0}(\zeta,\lambda)|d\zeta| &= \int_D G_{z_0}(z,\zeta,\lambda) v(\zeta)\psi_{z_0}(\zeta,\lambda) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \\ &= \psi_{z_0}(z,\lambda) - e^{\lambda(z-z_0)^2} I.\end{aligned}$$

This finish the proof of Theorem 1.1. □

Proof of Proposition 1.2. By (1.11) we have that $G_{z_0}(z,\zeta,\lambda)$ satisfies the same properties as $g_{z_0}(z,\zeta,\lambda)$ in Lemma 3.5, with the difference that the constant in (3.20) depends also on λ. This observation, along with Lemma 3.6, implies that the operator $A(\lambda)$ defined as

$$A(\lambda)u(z) = \int_{\partial D} G_{z_0}(z,\zeta,\lambda)(\Phi - \Phi_0) u(\zeta)|d\zeta|, \qquad z \in \partial D,$$

for a test function u, is compact on the space of continuous functions on ∂D. Thus equation (1.10) is a Fredholm linear integral equation of the second kind in the space of continuous functions on ∂D. □

Proof of Proposition 1.3. First we have that equations (1.5) and (1.10) are well defined (i.e. Fredholm linear integral equations of the second type) on the spaces of continuous functions on $\bar{D}$ and ∂D respectively. This follows from (3.9) for the first equation and from Proposition 1.2 for the second one.

Now if (1.5) admits a solution $\mu_{z_0}(z,\lambda) \in C(\bar{D})$, then by (3.8) and (1.5) one readily obtains $\mu_{z_0}(z,\lambda) \in C^1_{\bar{z}}(\bar{D})$. This solution is unique by Lemma 3.1 for $|\lambda| > \rho_1(D, N_1, n)$ and by the same arguments as in the proof of Theorem 1.1 one has that $\psi_{z_0}(z,\lambda)|_{z\in\partial D}$ satisfies equation (1.10).

Conversely, suppose that $\psi_{z_0}(z,\lambda) \in C(\partial D)$ satisfies equation (1.10): we have to show that $\psi_{z_0}(z,\lambda)$, defined on $\bar{D}$ as the solution of the Dirichlet problem $(-\Delta + v)\psi_{z_0}(z,\lambda) = 0$ with boundary values given by a solution of equation (1.10), satisfies (4.14).

By identity (4.1), $\psi_{z_0}(z,\lambda)$ satisfies already equation (4.14) with $z \in \partial D$. Now, the function

$$\varphi(z) = \psi_{z_0}(z,\lambda) - e^{\lambda(z-z_0)^2} I - \int_D G_{z_0}(z,\zeta,\lambda) v(\zeta) \psi_{z_0}(\zeta,\lambda) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \tag{4.15}$$

satisfies $\Delta\varphi = 0$ in D and $\varphi|_{\partial D} = 0$, so $\varphi \equiv 0$ in D. Proposition 1.3 is proved. □

Proof of Corollary 1.4. If $v_j|_{\partial D} = 0$, for $j = 1, 2$, then we can apply Theorem 1.1 and Propositions 1.2, 1.3. As $\Phi_1 = \Phi_2$, then $\psi^1_{z_0}(\cdot,\lambda)|_{\partial D} = \psi^2_{z_0}(\cdot,\lambda)|_{\partial D}$ for $|\lambda| > \rho_1(D, N_1, n)$ (where we called $\psi^j_{z_0}(z,\lambda)$ the Bukhgeim analogues of the Faddeev solutions corresponding to v_j, for $j = 1, 2$). Thus we also have equality between the corresponding generalized scattering amplitudes, $h^1_{z_0}(\lambda) = h^2_{z_0}(\lambda)$ for $|\lambda| > \rho_1(D, N_1, n)$, which yields $v_1(z_0) = v_2(z_0)$ for $z_0 \in D$.

If $v_j|_{\partial D} \neq 0$, for $j = 1, 2$, and D is such that (1.14) holds, then by Remark 3 we can apply Theorem 1.1 and argue as above.

The general case follows from stability estimates which will be published in another paper, following the scheme of [17]. □

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