Inverse problems in mathematical economics. Nonparametric method for analyses of products substitution

A. Shananin

Moscow Institute of Physics and Technology

Part 1

Generalized nonparametric method for computing positively homogenous indices of Konus-Divisia with applications to the analysis of commodity and stock markets

Economic indices construction

X – consumer bundle

p – price vector

$\langle p, X \rangle$ – value of consumer bundle

t – base period

τ – current period

$$\lambda = \frac{\langle p^t, X^\tau \rangle}{\langle p^t, X^t \rangle}$$ – Laspeyres demand index

$$\pi = \frac{\langle p^\tau, X^\tau \rangle}{\langle p^\tau, X^t \rangle}$$ – Paashe demand index

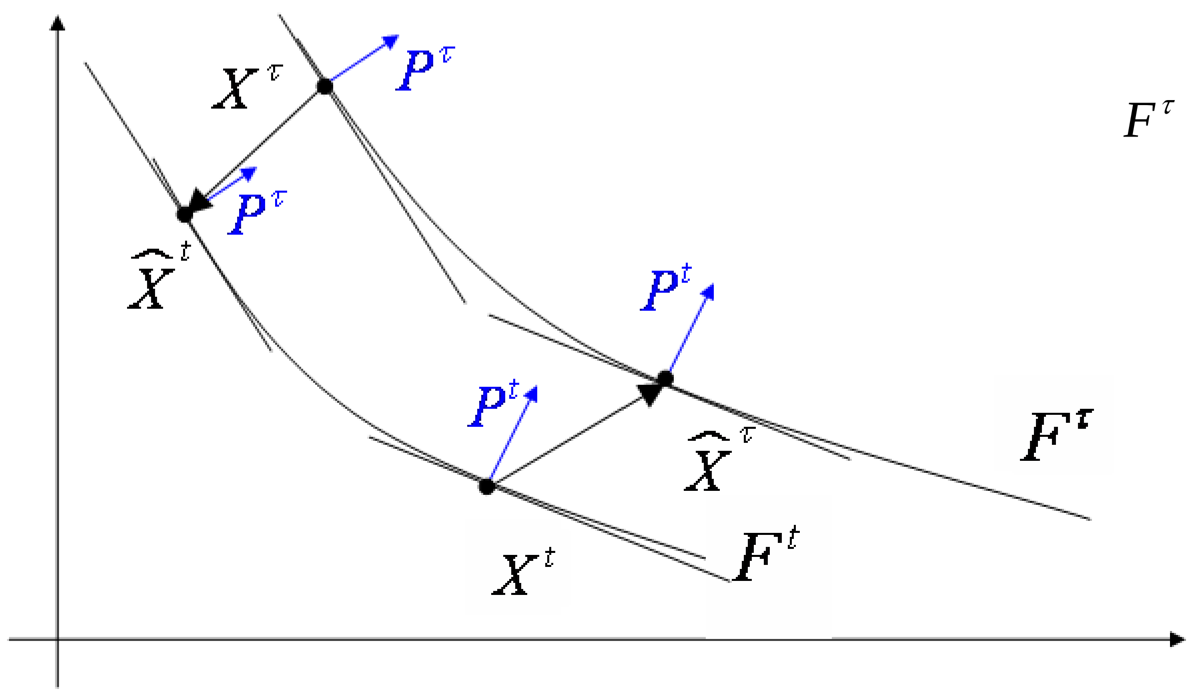
$\lambda > \pi$ – Gerschenkron effect: substitution of relatively expensive goods with relatively cheap ones

Konus indices

Consider the system of indifference surfaces. The levels of utility at X^{τ} and X^{t} are equal to F^{τ} and F^{t} respectively. Consider bundle X^{τ} which may be bought at prices P^{t} at period τ and has the utility level equal to F^{τ} and bundle x^{t} which may be bought at prices P^{τ} at period t and has the utility level equal to F^{t}.

$$\frac{\langle P^{t}, X^{\tau} \rangle}{\langle P^{t}, X^{t} \rangle}$$ Konus-Laspeyres demand index

$$\frac{\langle P^{\tau}, X^{\tau} \rangle}{\langle P^{\tau}, X^{t} \rangle}$$ Konus-Paashe demand index



Problem of rationalizability

$X=(X_1,\dots,X_m)$ – consumption levels

$P=(P_1,\dots,P_m)$ – prices

$P(X)=(P_1(X),\dots,P_m(X))$ – inverse demand functions

Φ_0 – set of continuous concave positively homogenous and positive in $\operatorname{int} R_+^m$ functions.

Definition. $P(X)$ *are rationalizable in* Φ_0 *if there exists a utility function* $F(X)\in\Phi_0$ *such that*

$$X\in \operatorname{Argmax}\left\{F(Y)\mid \left\langle P(X),Y\right\rangle \le \left\langle P(X),X\right\rangle,\ Y=(Y_1,\dots,Y_m)\ge 0\right\}$$

Formulations of the problem of rationalizability

Proposition 1.

Suppose that $\forall X > 0 \langle P(X), X \rangle > 0$. Then, the following statements are equivalent:

- There exist $F(X) \in \Phi_0, Q(P) \in \Phi_0,$ such that

$$\forall X > 0,\ \forall P \geq 0\quad Q(P)F(X) \leq \langle P, X \rangle;$$

$$\forall X > 0\qquad Q(P(X))F(X) = \langle P(X), X \rangle;$$

- There exist $F(X) \in \Phi_0,$ such that $\forall X > 0$

$$P(X) \in Q(P(X))\partial F(X)$$

where *Q(P)* – Yang transformation[1)] of function *F(X):*

$$Q(P) = \inf \left\{ \frac{\langle P, X \rangle}{F(X)} \,\middle|\, X \geq 0, F(X) > 0 \right\};$$

- There exists $F(X) \in \Phi_0$, which rationalizes the inverse demand functions $P(X)$.

[1)] Yang transformation is involutive in Φ_0, i.e. twice applied to a function it returns the function itself.

Konus indices

Proposition 2

Let the utility function $F(\cdot) \in \Phi_0$ *rationalizes inverse demand functions* $P(X)$. *Then Konus-Laspeyres index coincides with Konus-Paashe one.*

$$\frac{\left\langle P^t, X^\tau \right\rangle}{\left\langle P^t, X^t \right\rangle} = \frac{\left\langle P^\tau, X^\tau \right\rangle}{\left\langle P^\tau, X^t \right\rangle} = \frac{F(X^\tau)}{F(X^t)}$$

Proposition 3

Let the utility function $F(\cdot) \in \Phi_0$ *rationalizes inverse demand functions* $P(X)$. *Then Konus index is not greater than Laspeyres index and not less than Paashe index.*

$$\frac{\left\langle P^\tau, X^\tau \right\rangle}{\left\langle P^\tau, X^t \right\rangle} \le \frac{F(X^\tau)}{F(X^t)} \le \frac{\left\langle P^t, X^\tau \right\rangle}{\left\langle P^t, X^t \right\rangle}$$

Divisia indices

$$\frac{D(X(t))}{D(X(\tau))} = \exp\left(\int_{\tau}^{t} \frac{\sum_{i=1}^{m} P_i\left(X(\theta)\right)\frac{dX_i(\theta)}{d\theta}}{\sum_{i=1}^{m} P_i\left(X(\theta)\right)X_i(\theta)} d\theta \right)$$

- Trajectories (integration paths)

$$P(X(\theta)),\ X(\theta), \theta \in [\tau, t].$$

- Different trajectories – different formulas for practical computations;

Konus-Divisia indices

Proposition 6. (Balk, Halten) When the inverse demand functions are ratianalizable in set of differentiable functions from Φ_0 Konus index coincides with Divisia one.

$$\frac{D\left(X^t\right)}{D\left(X^\tau\right)} = \exp\left(\int_\tau^t \frac{\sum_{i=1}^m P_i\left(X(\theta)\right) \frac{dX_i(\theta)}{d\theta}}{\sum_{i=1}^m P_i\left(X(\theta)\right) X_i(\theta)} d\theta \right) = \exp\left(\int_\tau^t \frac{Q(P(X(\theta)))dF(X(\theta))}{Q(P(X(\theta)))F(X(\theta))} \right) =$$

$$= \exp\left(\int_\tau^t \frac{dF(X(\theta))}{F(X(\theta))} \right) = \exp\left(\int_\tau^t d\ln F(X(\theta)) \right) = \frac{F\left(X^t\right)}{F\left(X^\tau\right)}.$$

Critique: existence of functional relation between prices and volumes – demand functions of not general form. Whether such index exists?

Criterion of rationalizability – I

Definition. $F(X)$ Is a member of class U_m *if* $F(X) \in C(\mathbb{R}_+^m)$ *the following conditions are satisfied in* $\operatorname{int} \mathbb{R}_+^m$ *:*

1. $F(X) > 0 \quad \forall X > 0;$
2. $F(X) \in C^1(\operatorname{int} \mathbb{R}_+^m);$
3. $F(\lambda X) = \lambda F(X) \quad \forall \lambda > 0, X > 0;$
4. $F'(X) > 0 \quad \forall X > 0;$
5. $F(X)$ *strictly quasiconcave;*
6. $\forall Y > 0 \ \exists$ *at least one optimal over* $\operatorname{int} \mathbb{R}_+^m$ *solution of*

$$\frac{\langle X, Y \rangle}{F(X)} \to \inf_{X > 0}$$

Criterion of rationalizability – II

Statement 1.

Let $P(X) \in C^1(\mathbb{R}_+^m)$. *Denote* $M = \{1,...,m\}$. Then $P(X)$ rationalizable in U_m *if and only if:*

1. $P(X) > 0 \quad \forall X > 0$;

2. $\forall i, j \in \mathbb{N}, \quad \forall \lambda > 0, \ \forall X > 0 \qquad \dfrac{P_i(\lambda X)}{P_j(\lambda X)} = \dfrac{P_i(X)}{P_j(X)}$;

3. $\forall X_1, X_2 > 0: \ X_1 \neq \lambda X_2$ *for any* $\lambda > 0$

$$\langle P(X_1), X_2\rangle\langle P(X_2), X_1\rangle > \langle P(X_1), X_1\rangle\langle P(X_2), X_2\rangle;$$

4. $\forall$ *different* $i, j, k \in M, \quad \forall X > 0$

$$P_i(X)\left(\frac{\partial P_j}{\partial X_k}(X) - \frac{\partial P_k}{\partial X_j}(X)\right) + P_j(X)\left(\frac{\partial P_k}{\partial X_i}(X) - \frac{\partial P_i}{\partial X_k}(X)\right) + P_k(X)\left(\frac{\partial P_i}{\partial X_j}(X) - \frac{\partial P_j}{\partial X_i}(X)\right) = 0;$$

5. $\forall X \in \partial \mathbb{R}_+^m \quad (M \setminus \{i \in M \mid X_i = 0\}) \cap \{j \in M \mid P_j(X) = 0\} \neq \varnothing.$

Problem of integrability

Frobenius integrability conditions

$$P_i(X)\left(\frac{\partial P_j}{\partial X_k}(X)-\frac{\partial P_k}{\partial X_j}(X)\right)+P_j(X)\left(\frac{\partial P_k}{\partial X_i}(X)-\frac{\partial P_i}{\partial X_k}(X)\right)+P_k(X)\left(\frac{\partial P_i}{\partial X_j}(X)-\frac{\partial P_j}{\partial X_i}(X)\right)=0$$

- these are conditions in the form of equality

Consumption and price indices
Revealed preference theory

Utility function $F(X)$ – *consumption index*

$$Q(P) = \inf_{X>0, F(X)>0} \left\{ \frac{<P, X>}{F(X)} \right\} \quad - \textit{price index}$$

$$Q(P(X))F(X) = \langle P(X), X \rangle$$

Definition. $X^1 \in \mathbb{R}_+^m$ *is revealed preferred to* $X^2 \in \mathbb{R}_+^m$

(denoted as $X^1 \succ X^2$*), if and only if*

$$\langle P(X^1), X^1 \rangle \geq \langle P(X^1), X^2 \rangle, \quad X^1 \neq X^2.$$

Weak axiom. If $X^1, X^2 \in \mathbb{R}_+^m$, $\langle P(X^1), X^1 \rangle \geq \langle P(X^1), X^2 \rangle$,

$P(X^1) \neq P(X^2)$, *then* $\langle P(X^2), X^1 \rangle > \langle P(X^2), X^2 \rangle$.

Strong axiom and homogenous strong axiom of revealed preference theory

Definition. $X \geq 0$ *indirectly revealed preferred to* $Y \geq 0$ *(denoted as* $X\ R\,Y$*)* *if and only if* $\exists\, X^1 \geq 0, \ldots, X^k \geq 0$, *such that*

$$X = X^1 \succ X^2, X^2 \succ X^3, \ldots, X^{k-1} \succ X^k = \mathrm{Y}.$$

Strong axiom. *If* $X \geq 0, Y \geq 0,\ X\ R\,Y$, *then* $\langle \mathrm{P(Y), X - Y} \rangle \geq 0$

Definition. $P(X)$ *satisfy homogenous strong axiom of revealed preference theory (HAS) if* $\forall \left\{X^1, \ldots, X^T\right\} \in \mathbb{R}^m_+$

$$\left\langle P(X^1), X^2 \right\rangle \left\langle P(X^2), X^3 \right\rangle \ldots \left\langle P(X^T), X^1 \right\rangle \geq$$

$$\geq \left\langle P(X^1), X^1 \right\rangle \left\langle P(X^2), X^2 \right\rangle \ldots \left\langle P(X^T), X^T \right\rangle$$

Rationalizability of reverse demand functions

Statement 2

Let $P(X)\geq 0,\ \ P(X)\in C(\mathbb{R}_{+}^{m}),\langle P(X),X\rangle>0\quad \forall X\in \mathbb{R}_{+}^{m}\setminus\{0\}$

Then the following statements are equivalent.

1. $P(X)$ *are rationalizable in* Φ_0.

2. $\exists$ *solution* $\lambda(X)>0,\ \lambda(X)\in C(\operatorname{int}\ \mathbb{R}_{+}^{m})$ *of system*

$$\lambda(Y)\langle P(Y),X\rangle\geq\lambda(X)\langle P(X),X\rangle,\qquad \forall X,Y\in\mathbb{R}_{+}^{m}$$

3. $P(X)$ *satisfy HSA.*

Rationalizability of trade statistic

$\{P^t, X^t\}_{t=0}^{T}$ – trade statistic

$X^t = (X_1^t, \ldots, X_m^t)$ – consumption levels

$P^t = (P_1^t, \ldots, P_m^t)$ – prices

Trade statistic – values of inverse demand functions at

$$X^t = (X_1^t, \ldots, X_m^t)$$

Definition. *Trade statistic is rationalizable if it can be extended to inverse demand functions rationalizable in* Φ_0.

Afriat-Varian theorem

The following statements are equivalent:

1) $\exists$ *utility function of the form* $\mathrm{F(X)} = \min \lambda_{\mathrm{t}} \left\langle \mathrm{P}^{\mathrm{t}}, \mathrm{X} \right\rangle$

which rationalizes trade statistic $\left\{P^t, X^t\right\}_{t=0}^{T}$*, i.e.*

$$\mathrm{X}^{\mathrm{t}} \in \mathrm{Argmax}\left\{ \mathrm{F}(\mathrm{X}) \,|\, \left\langle \mathrm{P}^{\mathrm{t}}, \mathrm{X} \right\rangle \le \left\langle \mathrm{P}^{\mathrm{t}}, \mathrm{X}^{\mathrm{t}} \right\rangle,\ \mathrm{X} \ge 0 \right\}, \quad \mathrm{t} = \overline{0,\mathrm{T}}$$

2) $\exists$ *solution* $(\lambda_0, \ldots, \lambda_T)$ *of the system of linear inequalities*

$$\lambda_{\tau} \left\langle \mathrm{P}^{\tau}, \mathrm{X}^{\mathrm{t}} \right\rangle \ge \lambda_{\mathrm{t}} \left\langle \mathrm{P}^{\mathrm{t}}, \mathrm{X}^{\mathrm{t}} \right\rangle, \lambda_{\mathrm{t}} > 0, \quad \tau, \mathrm{t} = \overline{0,\mathrm{T}} \qquad \text{(I)}$$

3) $\left\{P^t, X^t\right\}_{t=0}^{T}$ *satisfies homogenous strong axiom of revealed preference theory (HSA):* $\forall \left\{\mathrm{t}_1, \ldots, \mathrm{t}_k\right\} \subset \left\{\overline{0,\mathrm{T}}\right\}$

$$\left\langle \mathrm{P}^{\mathrm{t}_1}, \mathrm{X}^{\mathrm{t}_2} \right\rangle \left\langle \mathrm{P}^{\mathrm{t}_2}, \mathrm{X}^{\mathrm{t}_3} \right\rangle \ldots \left\langle \mathrm{P}^{\mathrm{t}_k}, \mathrm{X}^{\mathrm{t}_1} \right\rangle \ge \left\langle \mathrm{P}^{\mathrm{t}_1}, \mathrm{X}^{\mathrm{t}_1} \right\rangle \left\langle \mathrm{P}^{\mathrm{t}_2}, \mathrm{X}^{\mathrm{t}_2} \right\rangle \ldots \left\langle \mathrm{P}^{\mathrm{t}_k}, \mathrm{X}^{\mathrm{t}_k} \right\rangle$$

Nonparametric method of Konus-Divisia index construction

Proposition 7

Let $F(X) = \min_{\tau=\overline{0,T}} \lambda_\tau \left\langle P^\tau, X \right\rangle$, where $\lambda_0 > 0, \ldots, \lambda_T > 0$ satisfy (I), and

$$Q(P) = \inf \left\{ \frac{\langle P, Y \rangle}{F(Y)} \;\middle|\; Y \geq 0, F(Y) > 0 \right\}$$

Then

$$Q(P^t) = 1/\lambda_t \qquad\qquad F(X^t) = \lambda_t \left\langle P^t, X^t \right\rangle$$

Such method for computation of indices is called **nonparametric**.

Floyd–Warshall algorithm

$$C_{\tau t} = \frac{\langle P^t, X^t \rangle}{\langle P^\tau, X^t \rangle}$$ – matrix of Paashe price indices

2) $\Leftrightarrow \ \exists \lambda_t > 0,\ t = \overline{0,T}$, such that $\lambda_t C_{\tau t} \le \lambda_\tau,\ \forall \tau, t = \overline{0,T}$

$$C^*_{\tau t} = \max\left\{ C_{\tau t_1} C_{t_1 t_2} \ldots C_{t_k t} \,\middle|\, \{t_1, t_2, \ldots, t_k\} \subset \mathrm{T}, k \in \mathbb{N} \right\}$$

(I) is solvable $\Leftrightarrow\ C^*_{tt} \le 1,\ t = \overline{0,T}$, and $\lambda_t = \max\limits_{\beta = \overline{0,T}} C^*_{t\beta},\ \ t = \overline{0,T}$

Consider idempotent semi ring with operations

$a \oplus b = \max(a, b)$ and $a \otimes b = ab$

Then

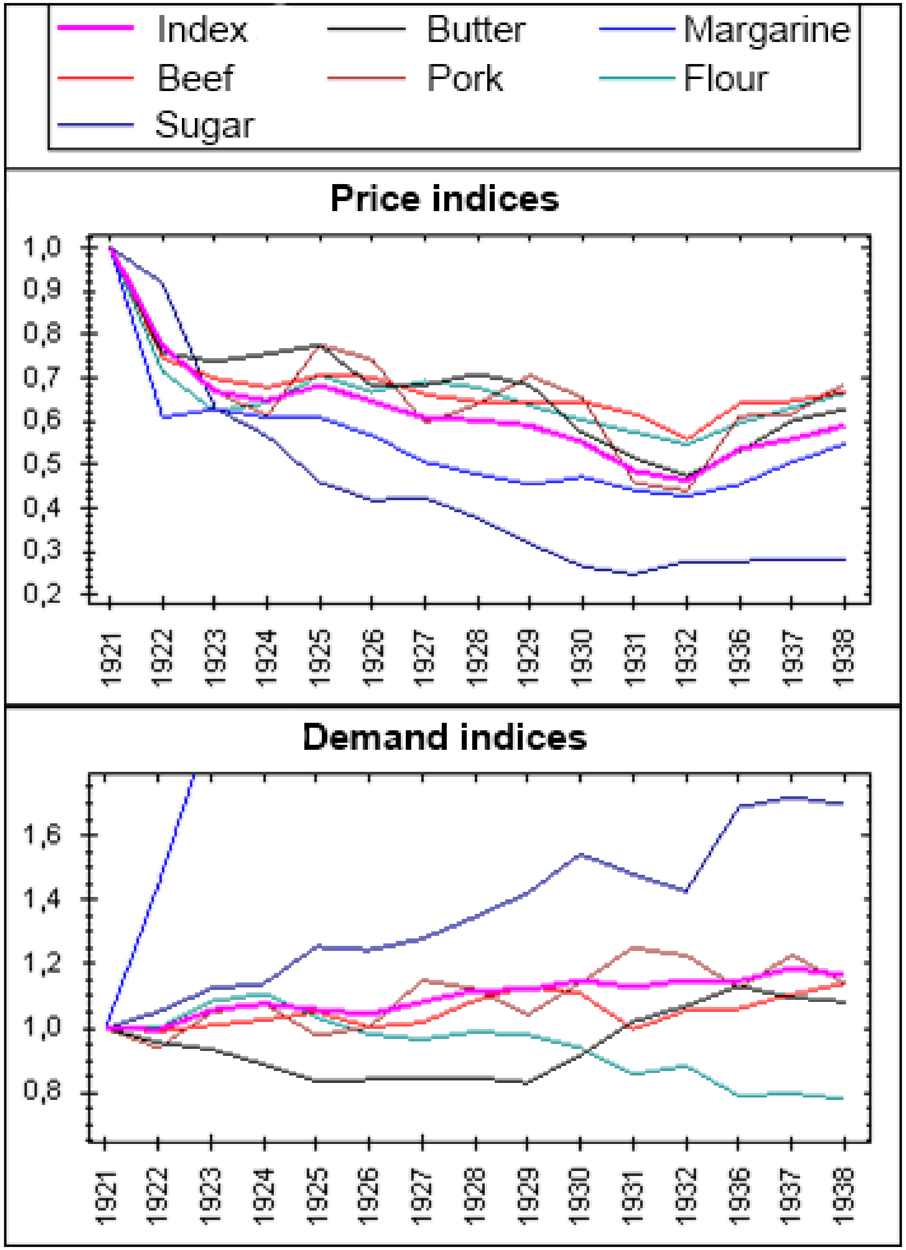
$$C^* = C \oplus C^{\bullet 2} \oplus \ldots \oplus C^{\bullet k} \oplus \ldots,$$

Necessity of numerical experiments

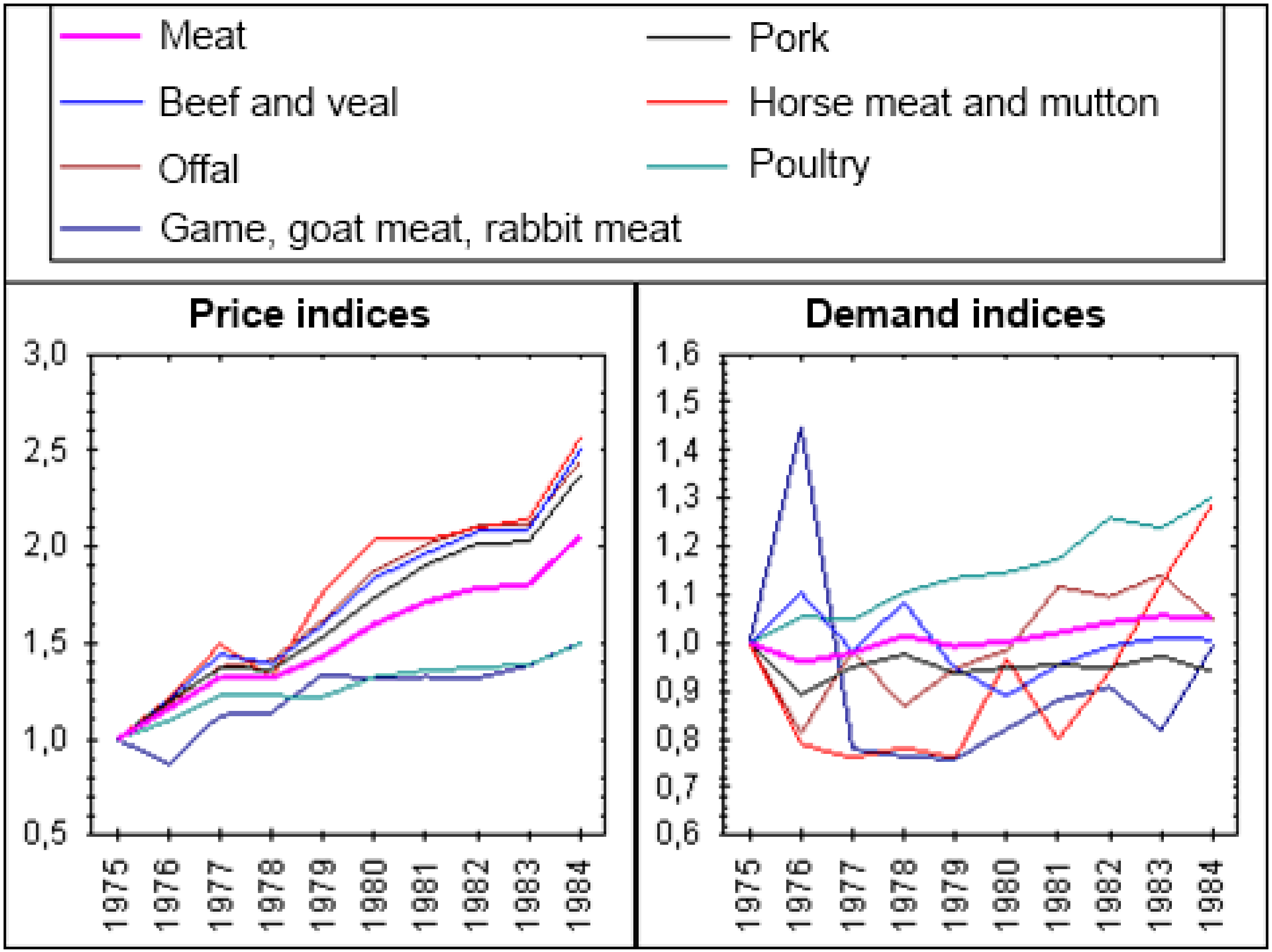
- What stands behind the violation of conditions for existence of positively homogenous Konus-Divisia indices?
- How the conditions for existence are affected by preprocessing of statistics and the choice of goods group?
- How to use Konus-Divisia indices for analysis of markets segmentation and demand structure?

Statistics of Sweden 1921-1938.

- Fluctuations of Konus-Divisia index are smoothed (compared to raw data);
- 1933-1935: violation of conditions for rationalizability;
- The consequences of Great Economic Depression: new needs and commodities appear (refrigerators e.g.);
- Relation between system restructuring of economy and violation of conditions for rationalizability are revealed by nonparametric method.



Price and consumption indices. Example: Hungary 1975-1984.



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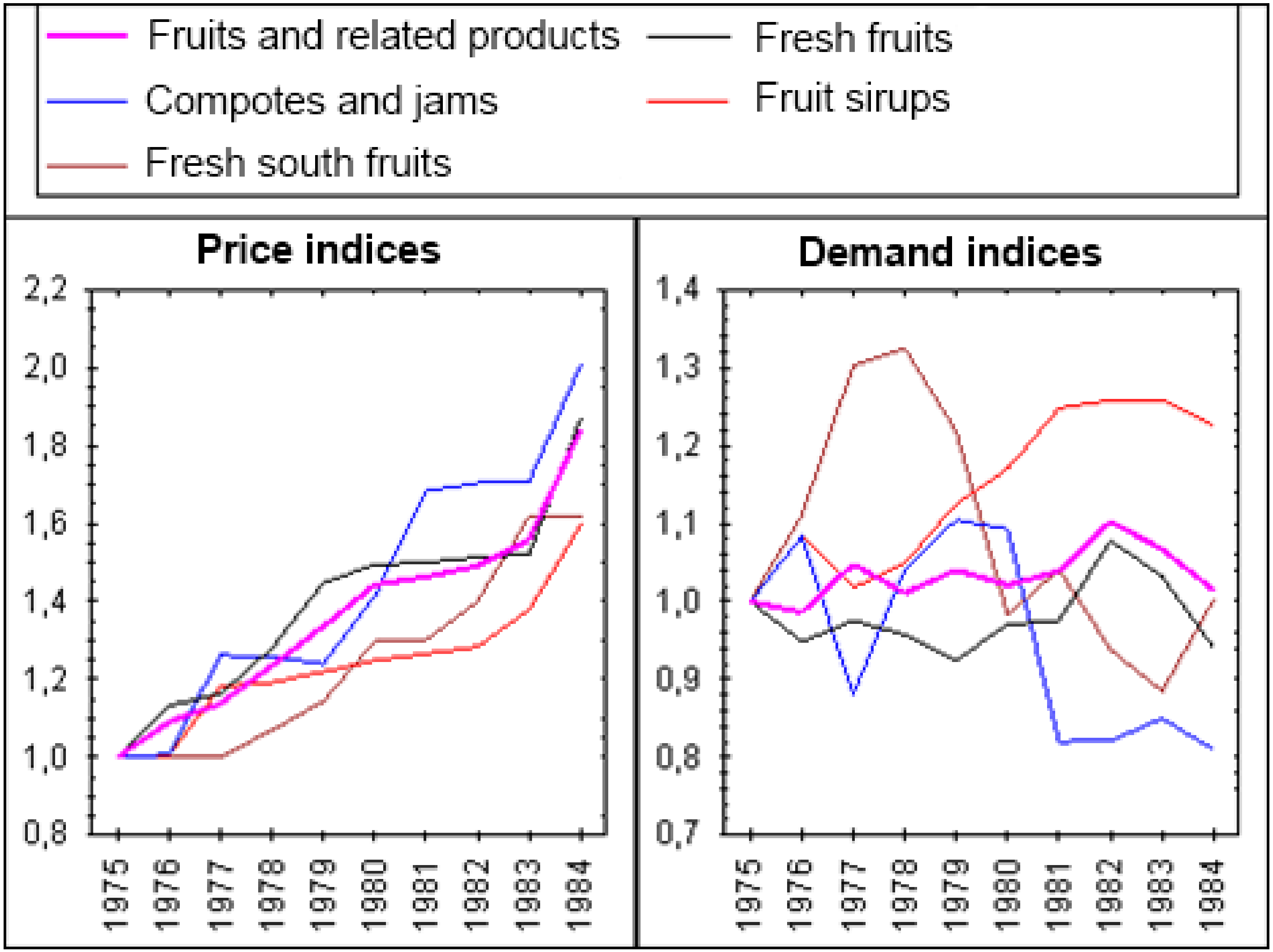


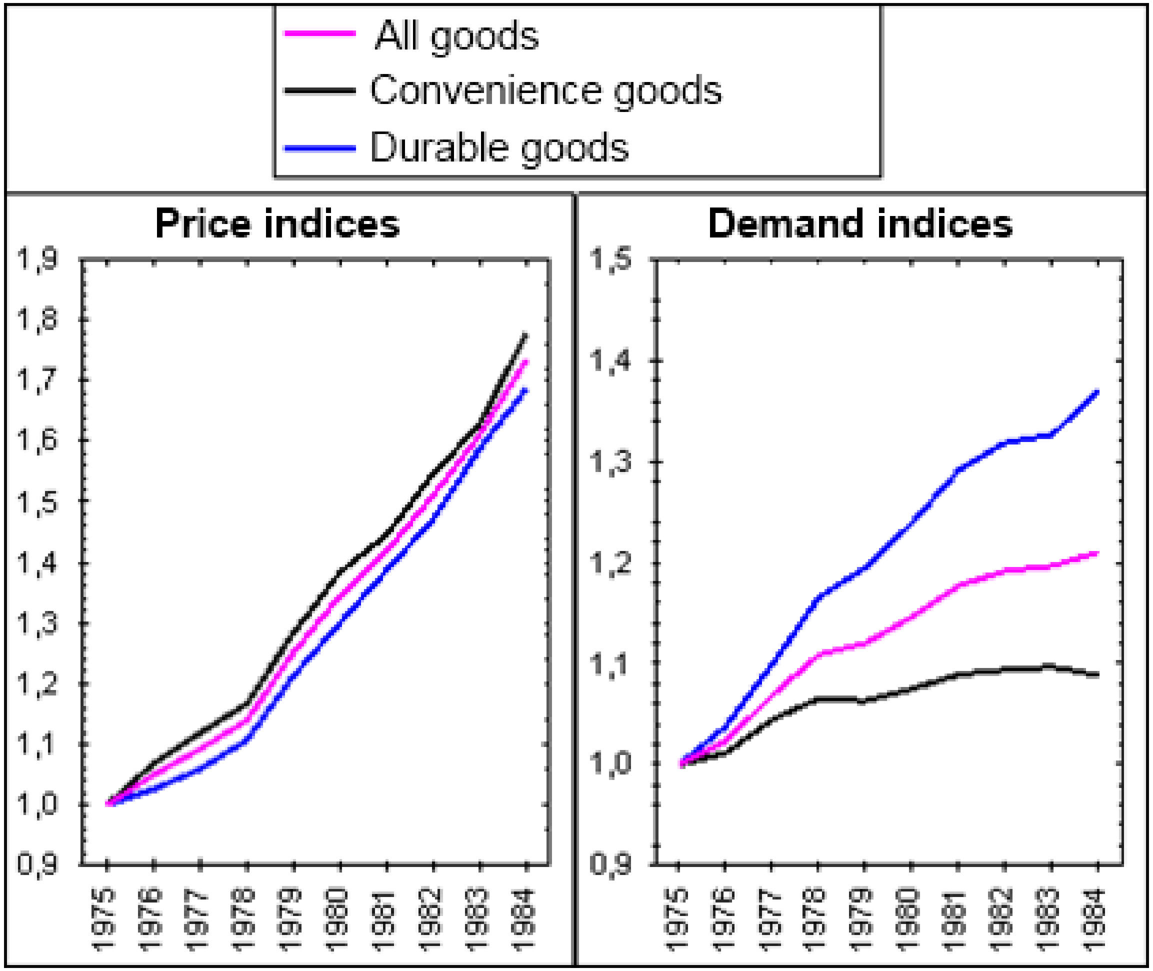
Hungary: commodities classification

Number	Group	Number of goods
1	Consumption goods	49
2	Beverages	15
3	Tobacco goods	3
4	Clothes	31
5	Housing service	5
6	Heating, energy in household use	12
7	Home equipment	30
8	Health care, hygiene	7
9	Transport, information	11
10	Education, culture, sport, leisure	23
11	Other items of consumption	10

Commodity groups differ in duration of consumption of goods. The first three groups represent convenience goods with duration of consumption of one month. For clothes the duration is about a year. The rest goods have the duration of 5-10 years (durable goods and services)

Change of consumption structure

- New market relations appear on consumer market;
- Consumption shifts to durable goods;



Tree of economic indices. Markets segmentation

$$\mathbf{X} = (\chi_1, \ldots, \chi_k, \varsigma) \geq 0$$

$$\chi_i = (X_{i_1}, \ldots, X_{i_{k_i}}) : \exists\ F_i(\chi_i) \in \Phi_0$$

$$\varsigma = (X_{j_1}, \ldots, X_{j_z})$$ – all other goods

$$\mathbf{F(X)} = \mathbf{F}\left(F_1(\chi_1), \ldots, F_k(\chi_k), \varsigma\right)$$

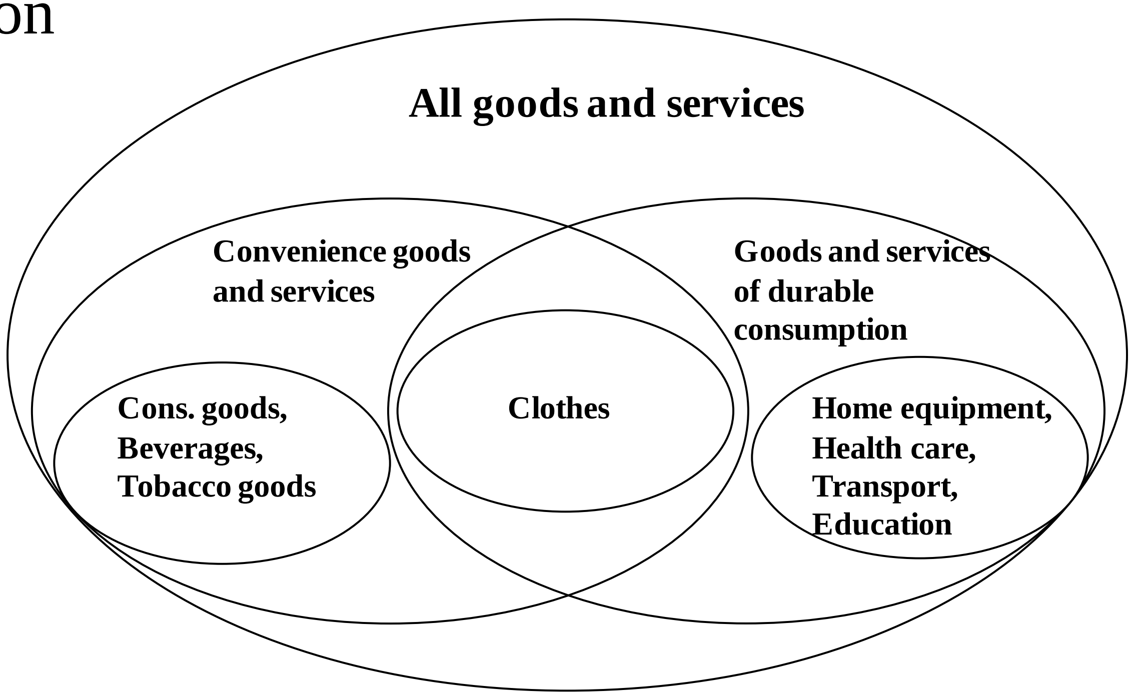
$$// \; F_i(\chi_i) = F_i\left(F_{i1}(\chi_{i1}), \ldots, F_{il}(\chi_{il}), \varsigma_i\right)$$

- All goods
  - Convenience goods
    - Consum. goods
      - Fruits and related productsия
      - Meat
    - Beverages
  - Durable goods

Statistics of Hungary. Separability.

- Classification used by commodity experts proved to be inadequate; what corresponded to the processes happening in the country is the classification based on characteristic duration of consumption;

- The group “Clothes" does not satisfy HSA. However, when combined with another aggregated good “Consumption goods“ it satisfies HSA.



Statistics of Netherlands

- Preprocessing of statistic (classification by groups) in general reflects its properties inadequately since it is based on heuristic experience and linguistic features. How preprocessing distorts the information about substitution among goods?
- Statistics of Netherlands: among groups chosen by commodity experts none is rationalizable. Nevertheless, the statistics itself is rationalizable.

Generalized nonparametric method

Generalization of nonparametric method:

$$\omega\lambda_\tau < P^\tau, X^t > \ \geq \lambda_t < P^t, X^t >, \quad \lambda_t > 0, \ \ \tau, t = \overline{0, T} \qquad \text{(I)}$$

The minimum $\omega \geq 1$, with which the system (I) is solvable, is called irrationality index of a trade statistics.

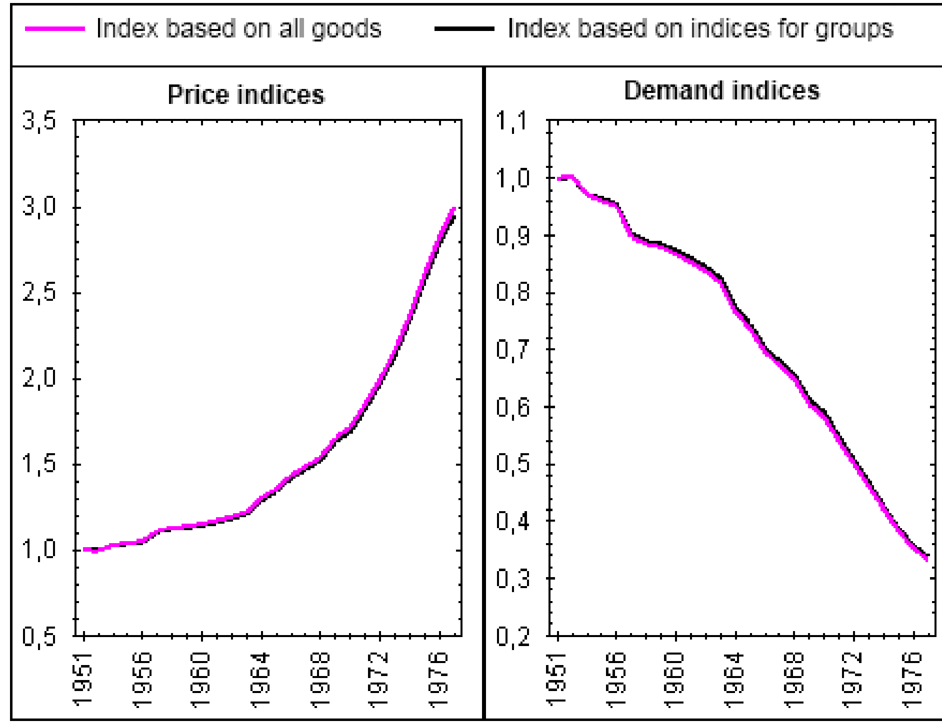
Mathematical meaning of irrationality index: relation to idempotent analog of Frobenius-Perron number of the matrix of Paashe indices.

Statistics of Netherlands. GNM.

- Commodity groups selected by office of statistics are not rationalizable;
- By means of GNM one may compute the following indices:
  - Index based on indices for groups;
  - Index based on all goods;
- Maximum deviation between the two indices is 1.76%.

$$\mathbf{F}(\mathbf{X}) \approx \mathbf{F}\left(F_1(\chi_1),\ldots,\ F_k(\chi_k)\right)$$

- GNM allows studying the demand structure



- One obtains the opportunity to analyze the influence of preprocessing of trade statistics.

Analysis of index tree by means of GNM

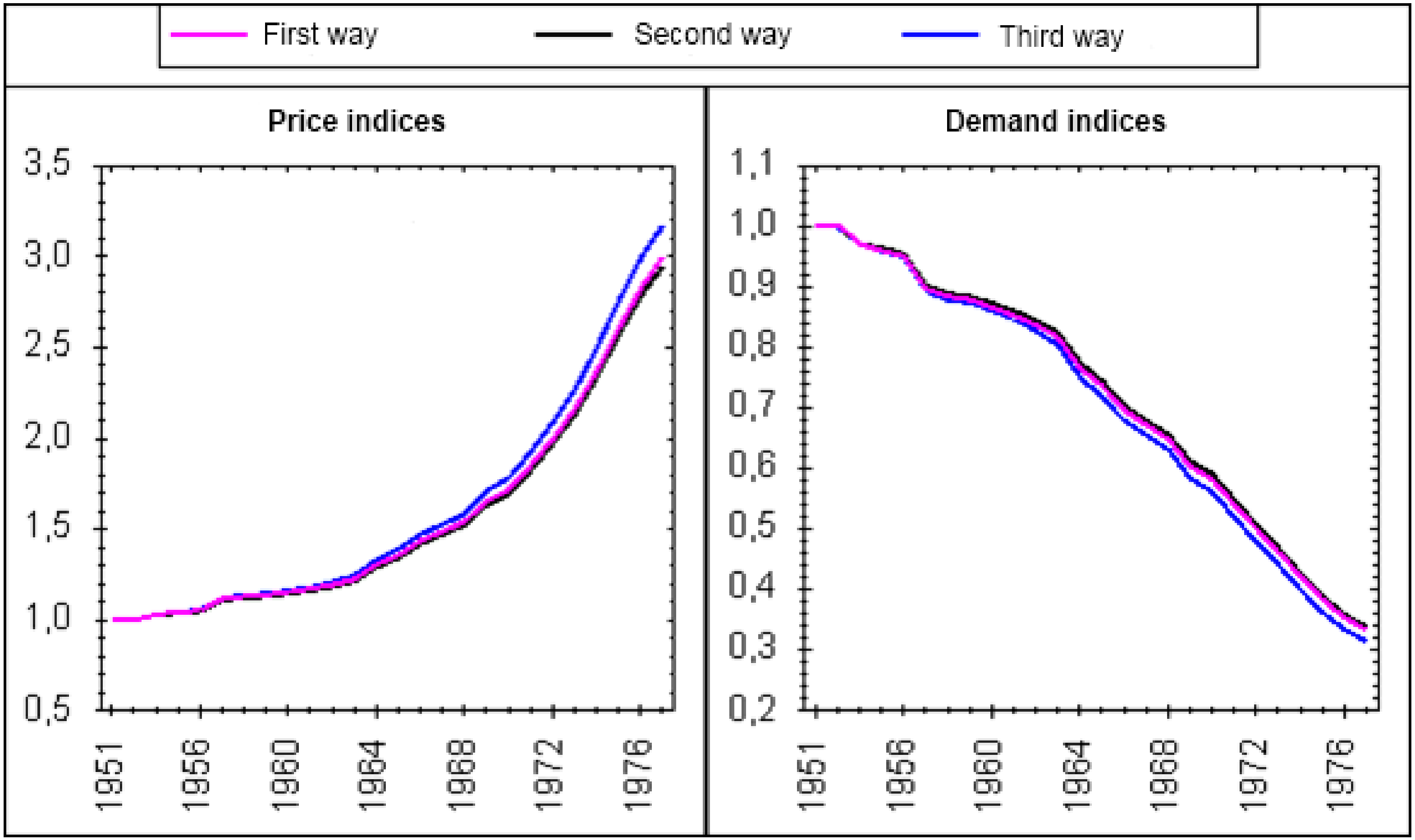
When using GNM we have opportunity to check how the preprocessing including computation of Laspeyres price and Paashe demand indices influences rationalizability. Indeed, we may compute the index in three ways:

First way: directly over all goods in trade statistics.

Second way: with preliminary aggregation by groups, i.e. compute first Konus-Divisia indices for groups from classifier and then compute the index of all statistics.

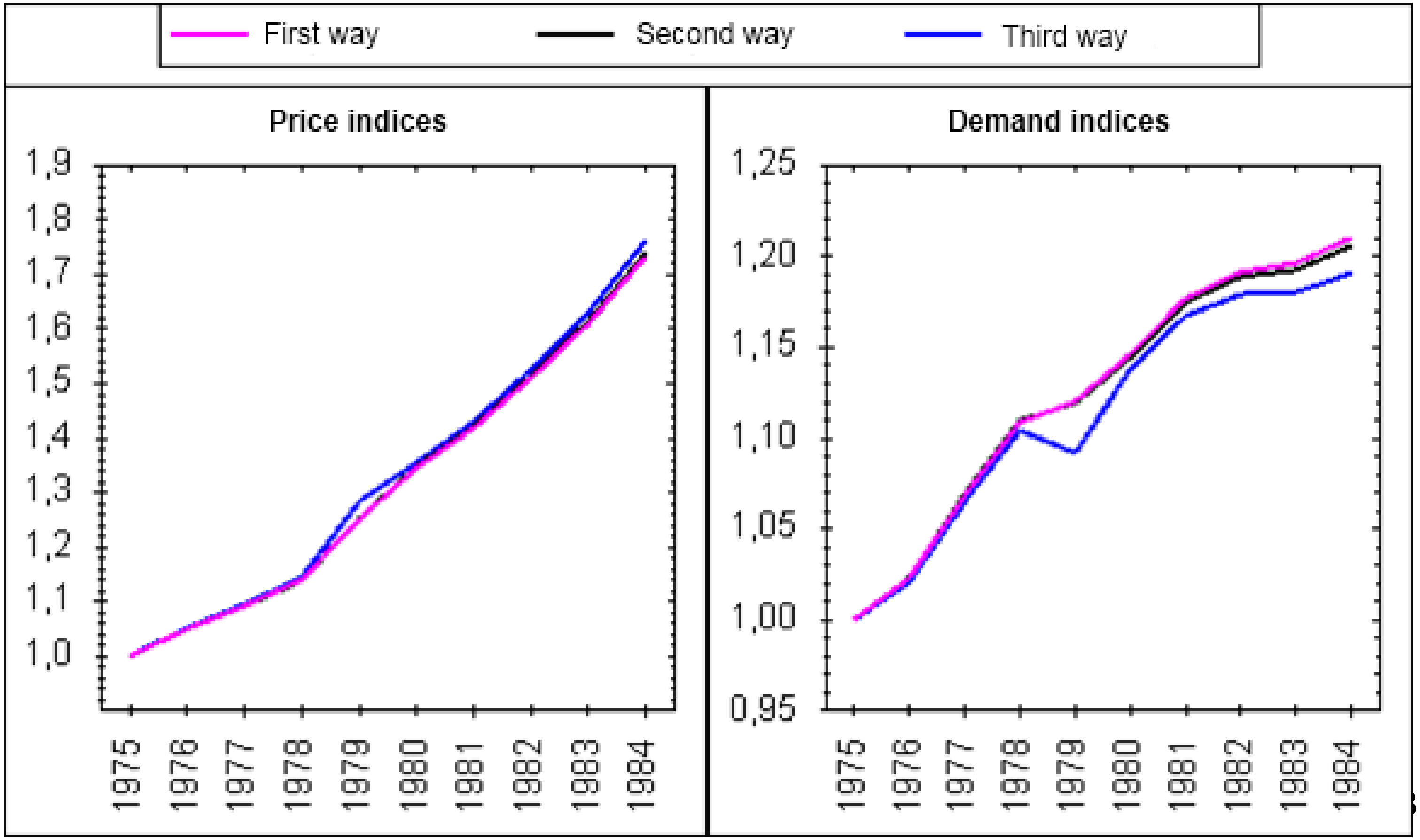
Third way: with preliminary aggregation by groups, but with computation of Laspeyres price and Paashe demand indices; this imitates the processing which the office of statistics usually does.

Analysis of index tree by means of GNM. Netherlands.



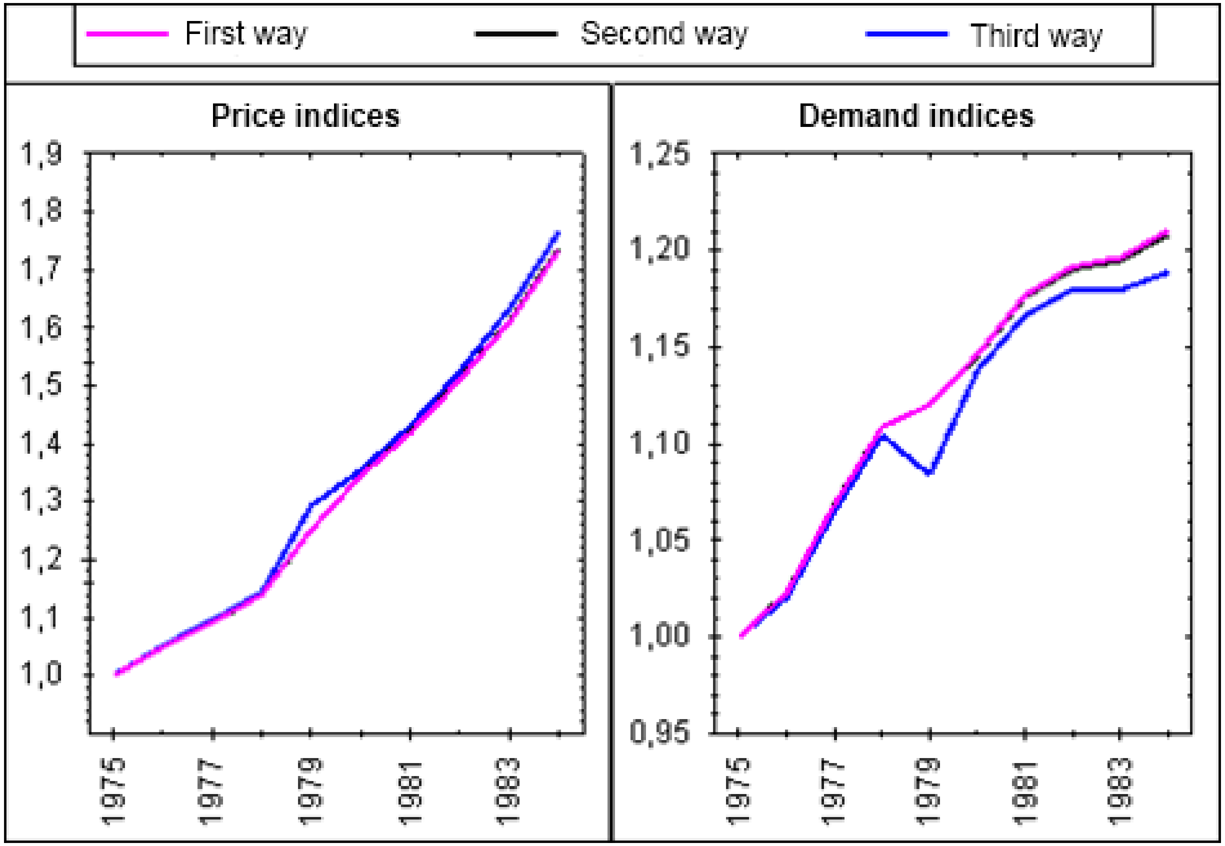
Analysis of index tree by means of GNM. Hungary. Classification by office of statistics.

For computations the office of statistics classification was used: Cons. goods, Beverages, Housing service, Health etc.



Analysis of index tree by means of GNM. Hungary. Classification based on consumption duration.

For computations the classification based on consumption duration was used: convenience goods and durable goods



Analysis of index tree by means of GNM. Conclusion.

All figures show that the deviation between the curves plotted in the first and the second ways is less than that between the curves plotted in the first and the third ways. Therefore, one may argue that the aggregation with computation of Konus-Divisia indices reflects the structure of consumer demand to a greater extent.

Initial statistics. Alcohol-free beverages.

- The statistics contains monthly data (January 2007 - December 2007) on sales of alcohol-free beverages (14 various commodity names) from 643 shops in Moscow;
- A commodity – a certain item in a certain shop;
- We have pairs of complementary goods: the trade service and the commodity itself;
- **Question:** does the segmentation depends on trade service or item name (brand)?

Initial statistics. Alcohol-free beverages.

- Almost all groups including all goods of a particular brand are rationalizable;
- More than 2/3 "shop-classes" have the irrationality indices greater than that of the unique non-rationalizable "brand-class";
- Spread of prices inside one brand is pretty large (up to 50%);

N	Class name	ω
	All goods	1.0
1	shop 12	1.0
2	shop 30	1.0
…	…	…
161	shop 350	1.0
162	shop 36	1.00003
163	shop 21	1.00006
…	…	….
197	shop 258	1.00048
198	shop 227	1.00051
199	shop 114	1.00054
…	…	…
218	shop 519	1.00095
219	shop 327	1.00101
…	…	…
643	shop 22	1.07729

N	Class name	$\max_t\left(\frac{\sqrt{D_t}}{E_t}\right)$	ω
	All goods	0.2984	1.0
1	Coca-Cola 0.5 l.	0.3564	1.0
2	Coca-Cola 0.33 l.	0.5158	1.0
3	Coca-Cola 1 l.	0.2363	1.0
4	Coca-Cola 2 l.	0.1327	1.0005
5	Coca-Cola Light 2 l.	0.1108	1.0
6	Coca-Cola Light 1 l.	0.2114	1.0
7	Coca-Cola Light 0.5 l.	0.32	1.0
8	Pepsi-Cola 1.25 l.	0.2392	1.0
9	Pepsi-Cola 0.33 l.	0.5347	1.0
10	Pepsi-Cola 0.6 l.	0.369	1.0
11	Pepsi-Cola Light 0.6 l.	0.3416	1.0
12	Pepsi-Cola Light 2 l.	0.1913	1.0
13	Pepsi-Cola Light 1.25 l.	0.1956	1.0
14	Pepsi-Cola 2.5 l.	0.1942	1.0

- Conclusion: segmentation by brands is revealed.

Initial statistics. Computer equipment

The group "All goods" has the smallest irrationality index. Apparently, it accounts for substitution and complementarity to a greater extent.

Group name	$\omega_{\min}$	**Group name**	$\omega_{\min}$
All goods	1,0082	Diskettes and disks	1,0314
Consumables	1,0089	Scanners	1,0322
Network equipment	1,0109	Memory	1,0337
Printers	1,0115	Speakers	1,0363
Processors	1,0163	Monitors	1,0373
Sound cards	1,0181	Video cards	1,0527
Other	1,0183	Controllers	1,0889
Mice	1,0202	Motherboards	1,1041
Keyboards	1,0223	Hard disks	1,1730
Office equipment	1,0242	Coolers	1,2575
CD-ROM/ DVD/ Disk drives	1,0262		

Computer equipment. Join of classes.

Joined groups have much smaller irrationality index then the groups themselves. This is so because the joined groups accounts for substitution and complementarity to a greater extent. A good example is the rationalizability of the group "Memory" joined with the groups "Processors" and "Network equipment". This illustrates the behavior of consumers: they usually choose memory cards together with processors such that their characteristics correspond to each other, rather than choose them separately.

Наименование	ω_{min}
Memory, Processors, Network equipment	1
Speakers, Memory, Processors	1,00007
Sound cards, Memory, Processors	1,00018
Controllers, Memory, Processors	1,00047
Memory, Processors	1,00058
Coolers, Memory, Processors	1,00074
Video cards, Processors, Network equipment	1,00096
Memory, Processors, Other	1,00083
Memory, Processors, Consumables	1,001

Statistics of stock market

- P_t - stock prices, X_t - trading volumes (in pieces)
- 21 largest stock exchanges:
  - New-York stock exchange, London stock exchange,
  - Tokyo stock exchange,
  - Frankfort stock exchange,
  - Hong-Kong stock exchange,
  - Shanghai stock exchange...

Problems:

- Different numbers of huge stock packs resales, the degree of speculators' activity influences rationalizability;
- Stocks are traded in different currencies.

Foreign exchange market analysis. Arbitrage sequences.

Let a_{ij} be the amount of *j* currency which may be obtained for a unit of *i* currency. The resulting matrix *A* is called cross-rate matrix. We say that the cross-rate matrix *A* allows for an arbitrage sequence $(i_1, i_2, \ldots, i_k)$, if

$$a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{k-1} i_k} a_{i_k i_1} > 1$$

Theorem (Afriat, Varian) Let *A* be a positive matrix. Then the following statements are equivalent:

1. The matrix *A* does not have arbitrage sequences;
2. The system of linear inequalities $a_{ij}\lambda_j \le \lambda_i$ has positive solution

The existence of solution may be related to the productivity of matrix *A* in idempotent sense

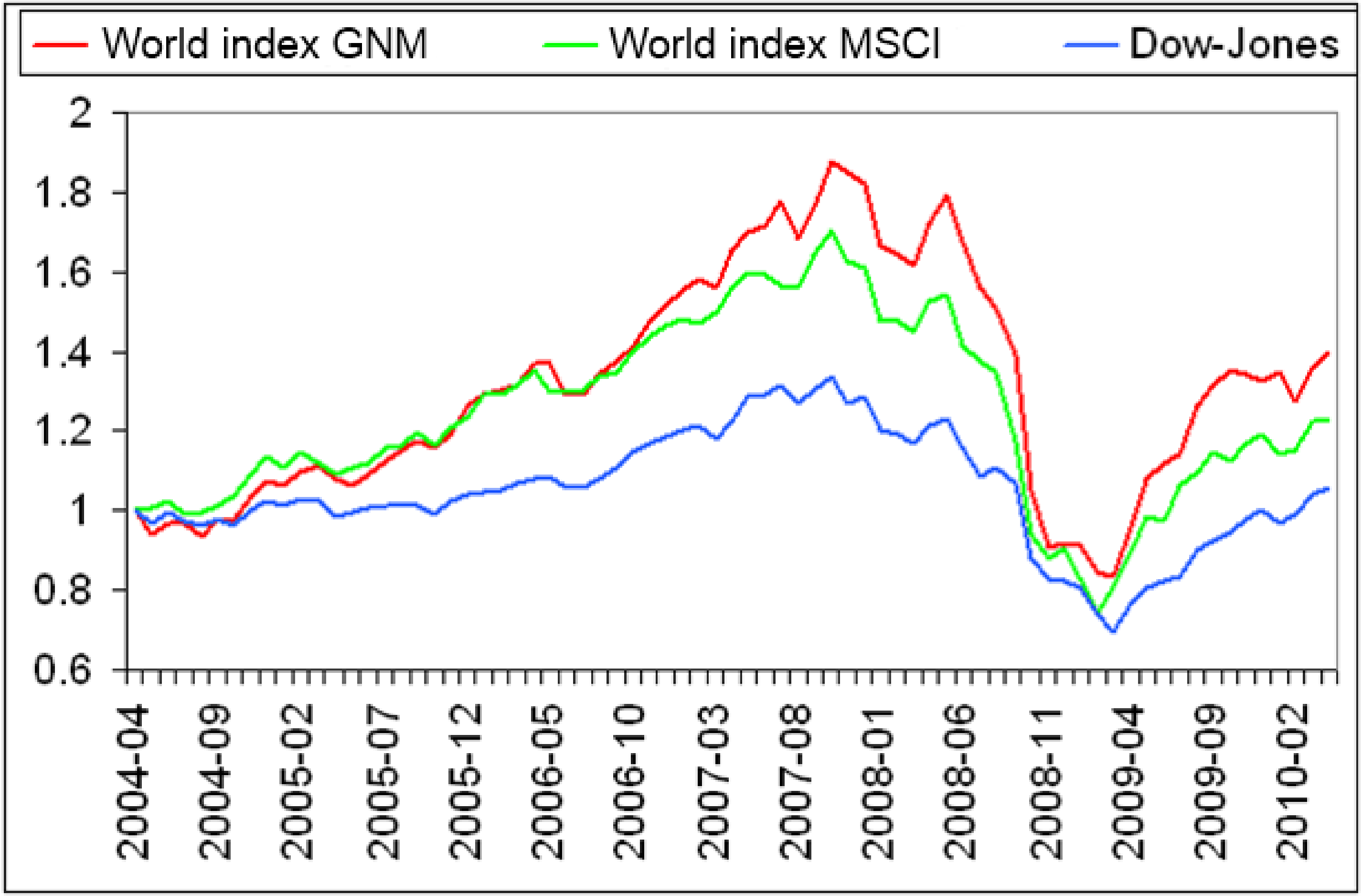
Foreign exchange market analysis. Reduction to a common currency.

Let us solve the system of inequalities

$$a_{ij}\lambda_j \le \lambda_i, \ \lambda_i > 0$$

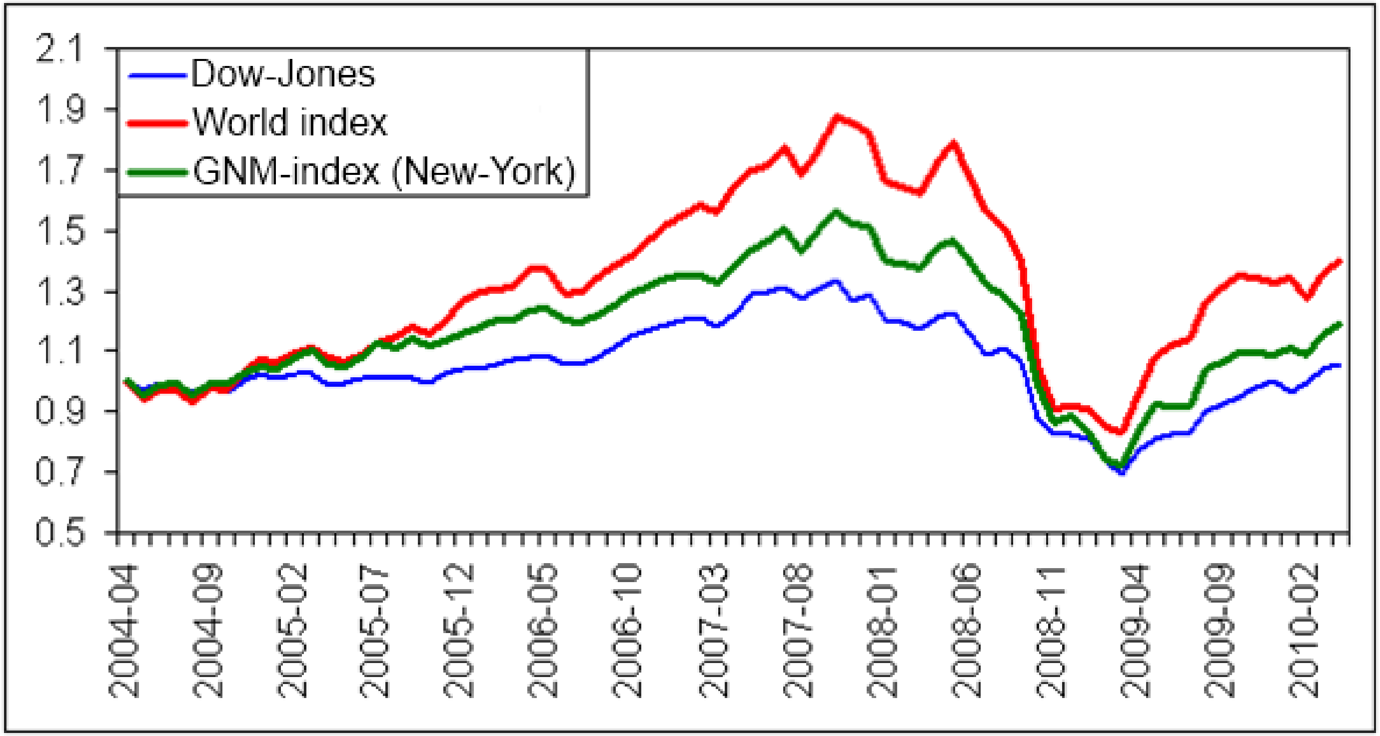
by means of Floyd-Warshall algorithm. We obtain the "weights" of currencies λ_i. Multiplying the solution by a any positive number we also obtain a solution. Let the "weight" of the USA dollar be equal to 1.

World index


World index GNM
World index MSCI
Dow-Jones
2
1.8
1.6
1.4
1.2
1
0.8
0.6
2004-04
2004-09
2005-02
2005-07
2005-12
2006-05
2006-10
2007-03
2007-08
2008-01
2008-06
2008-11
2009-04
2009-09
2010-02

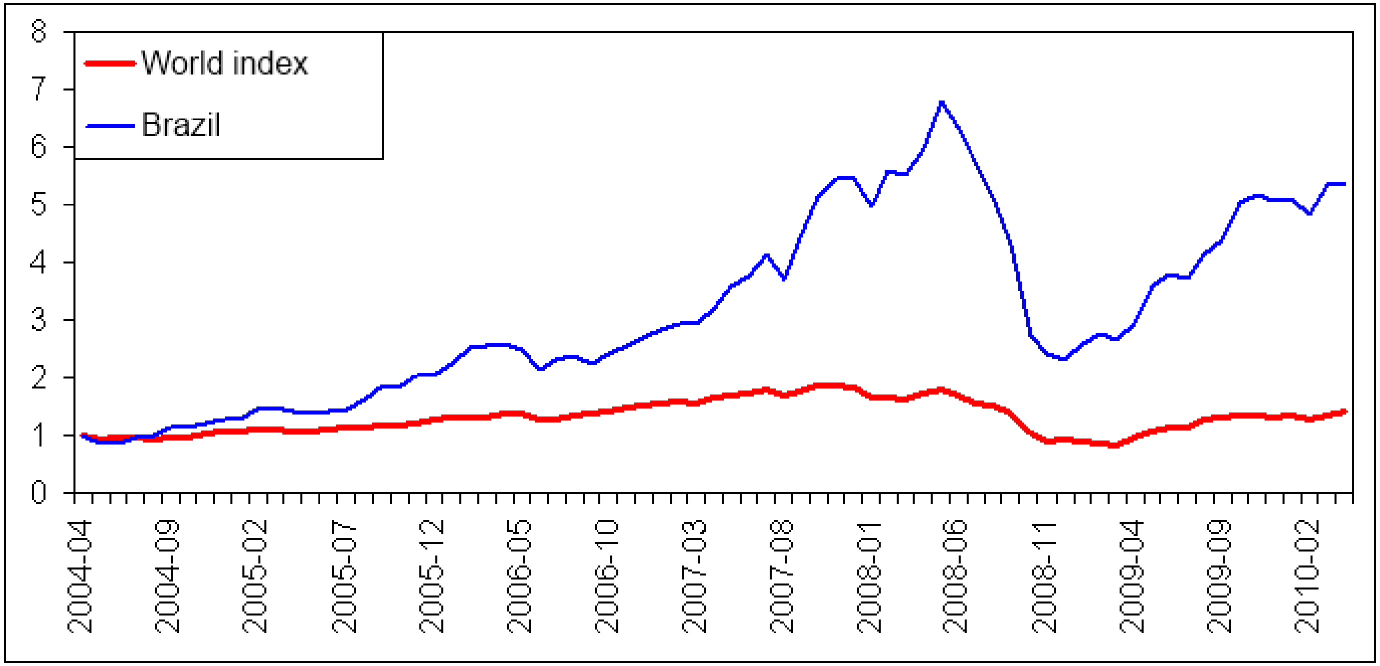
World index and New-York

World index is more volatile the index for NYSE, which is more volatile than the Dow-Jones index.



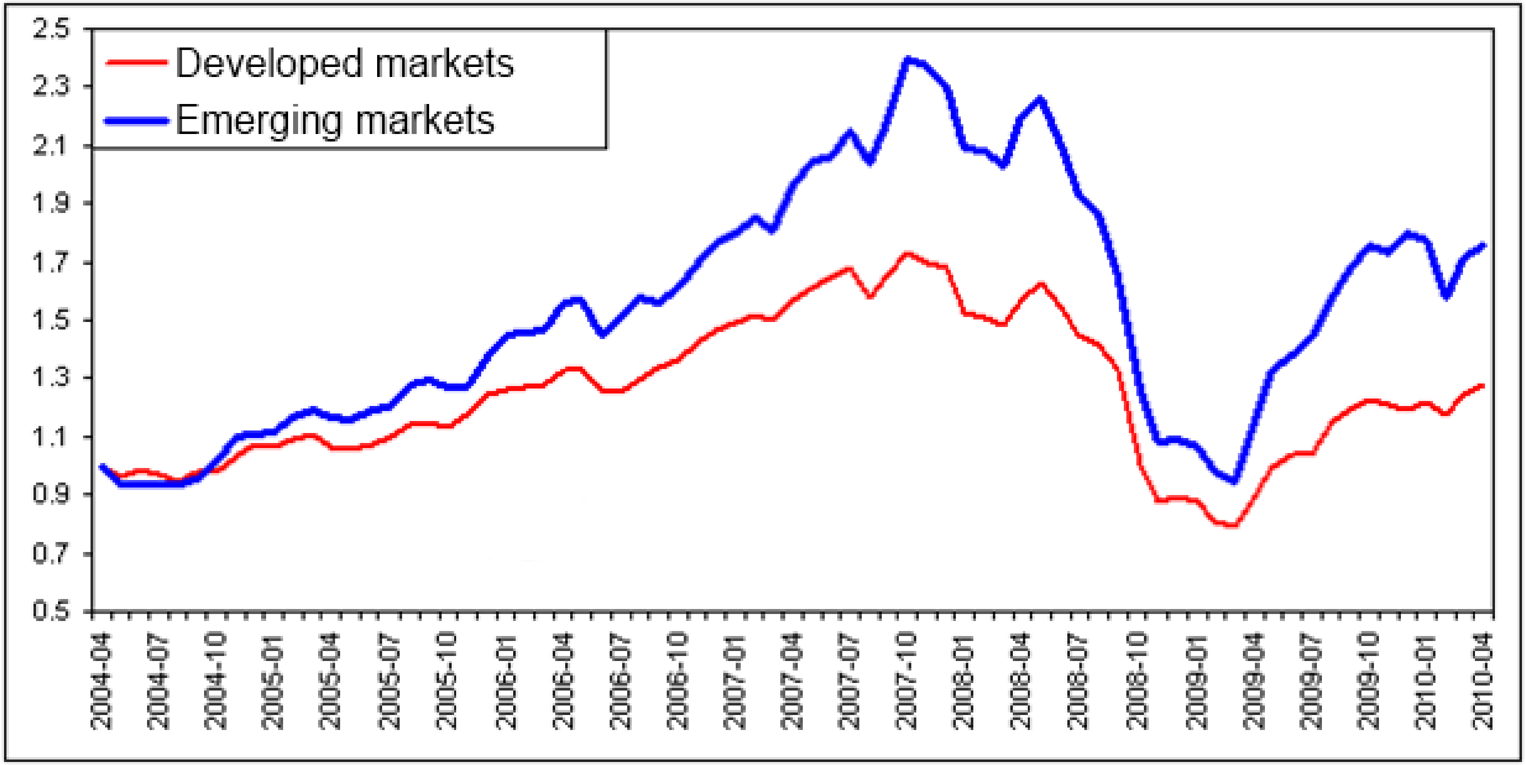
Brazil

Brazil market has very large volatility compared for example to world index.



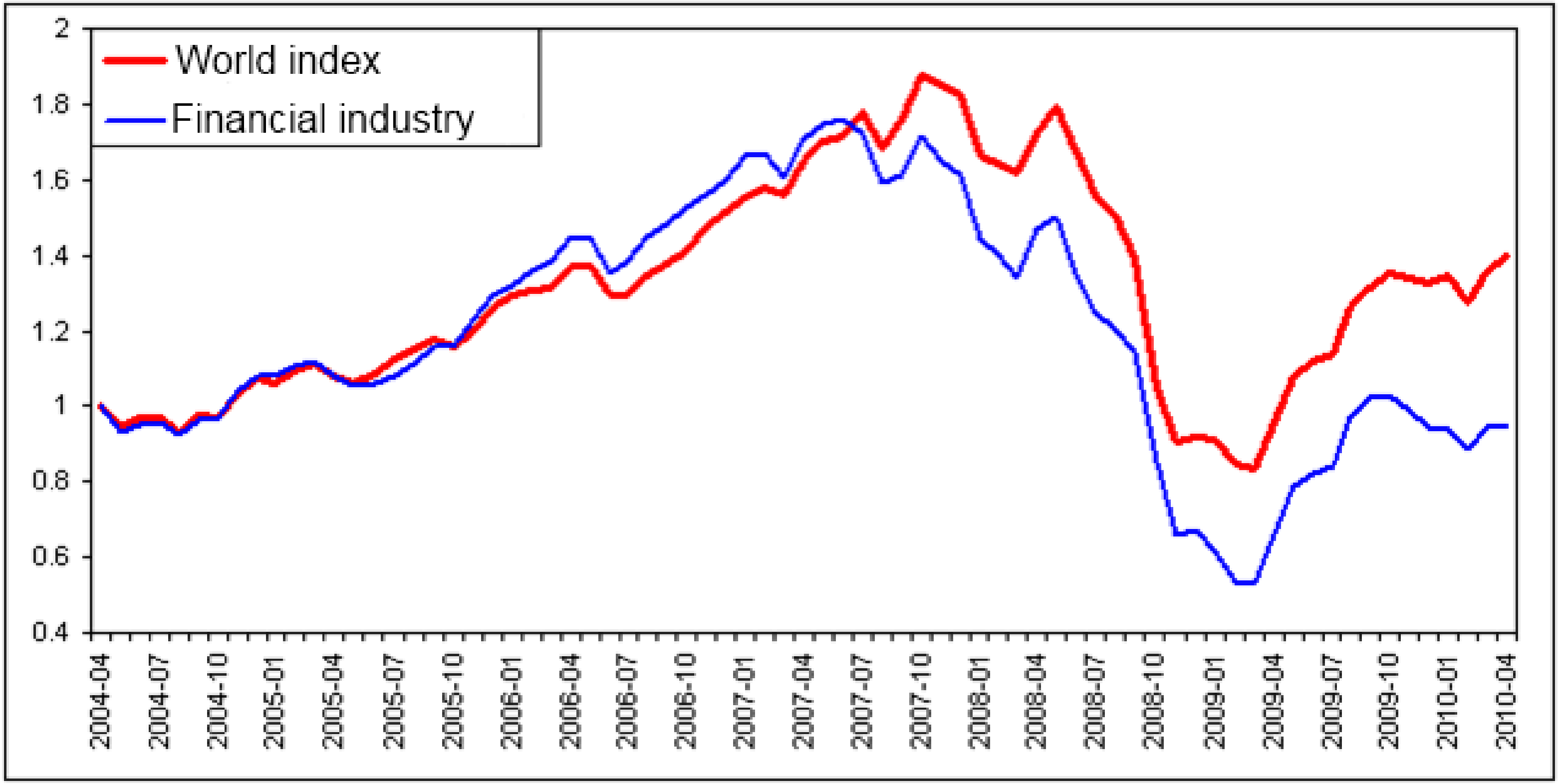
Developed and emerging markets

On the whole emerging markets are more volatile.



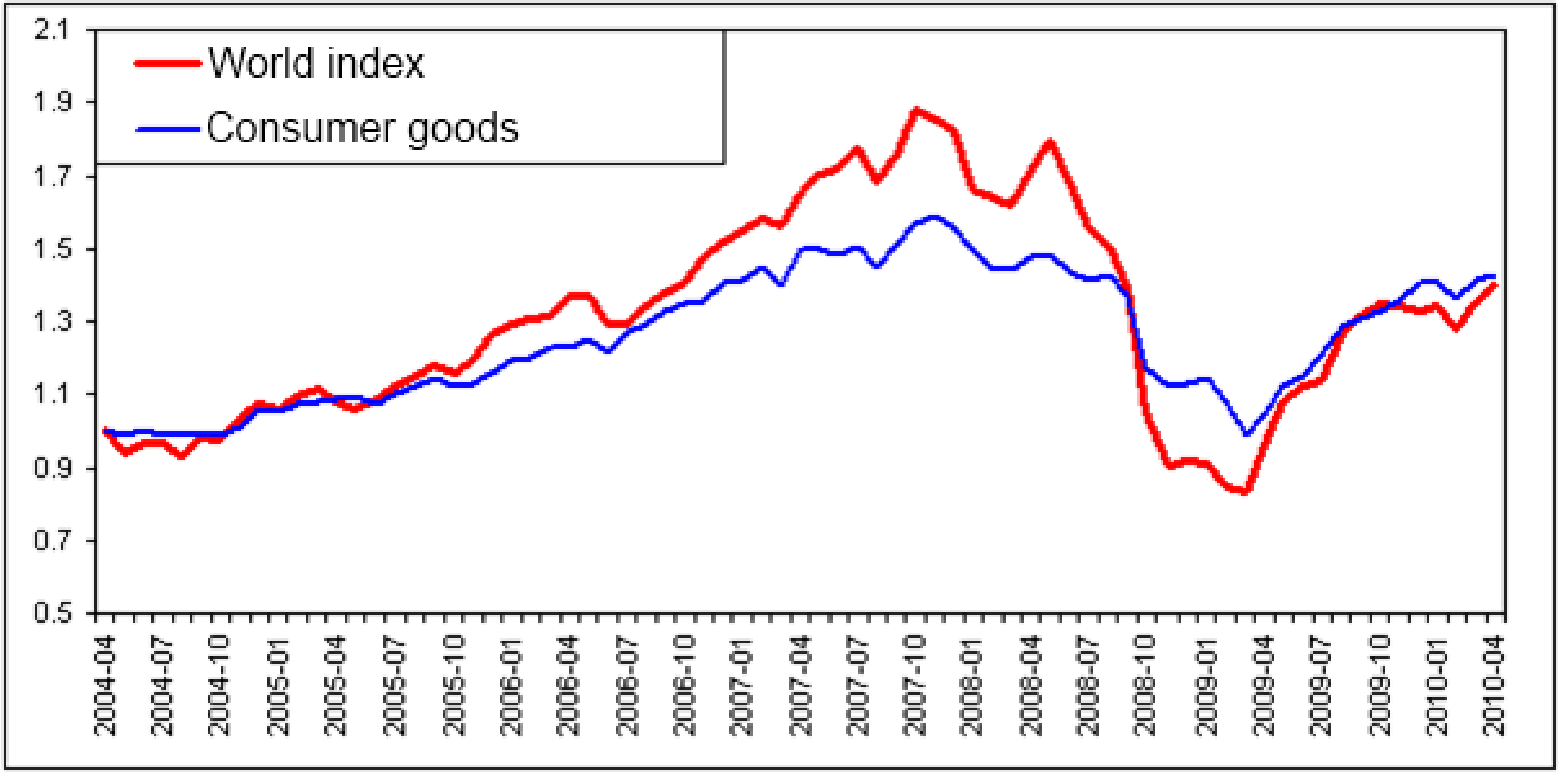
Financial industry

GNM allows one to compute indices for various sectors. The analysis showed that financial industry, suffered from the 2008 crisis the most.



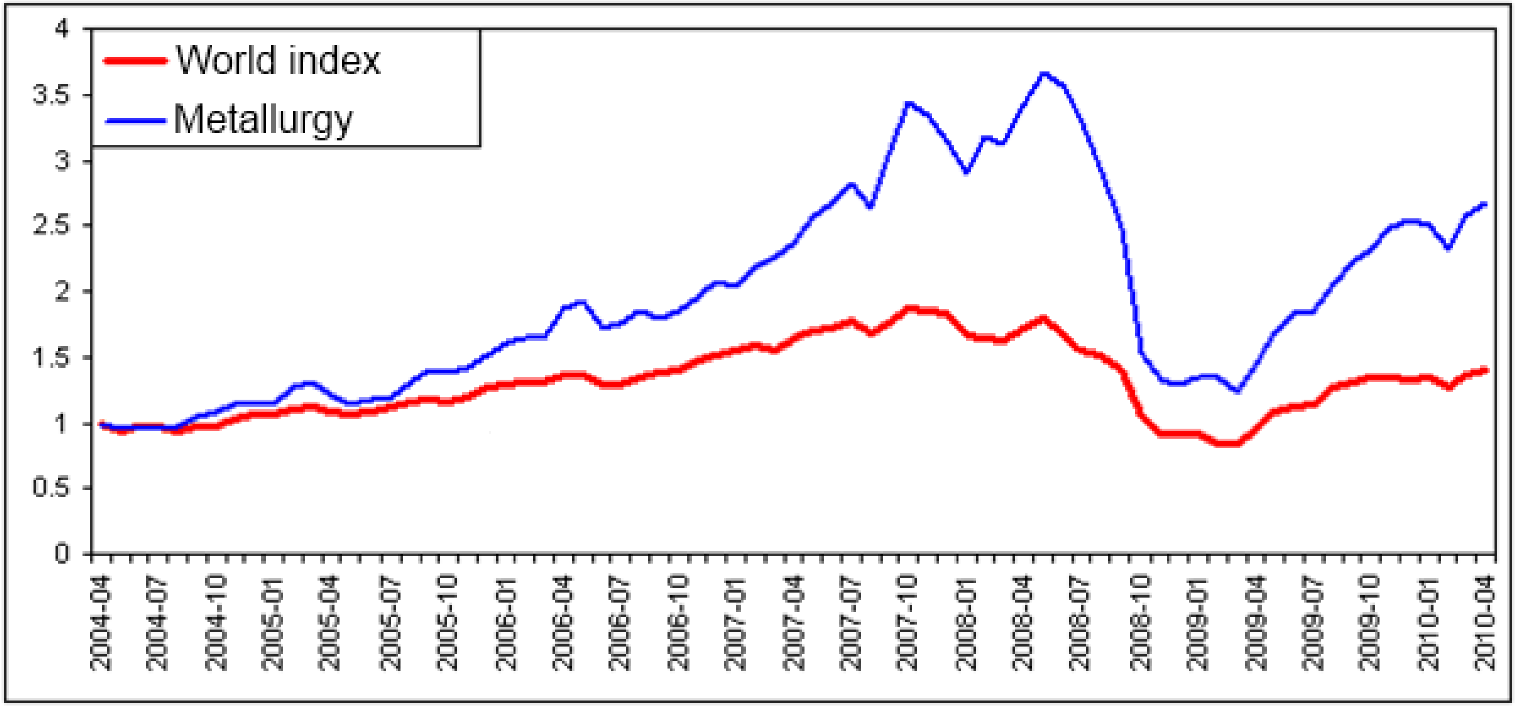
Consumer goods companies

Consumer goods companies had a stabilizing effect on the market.



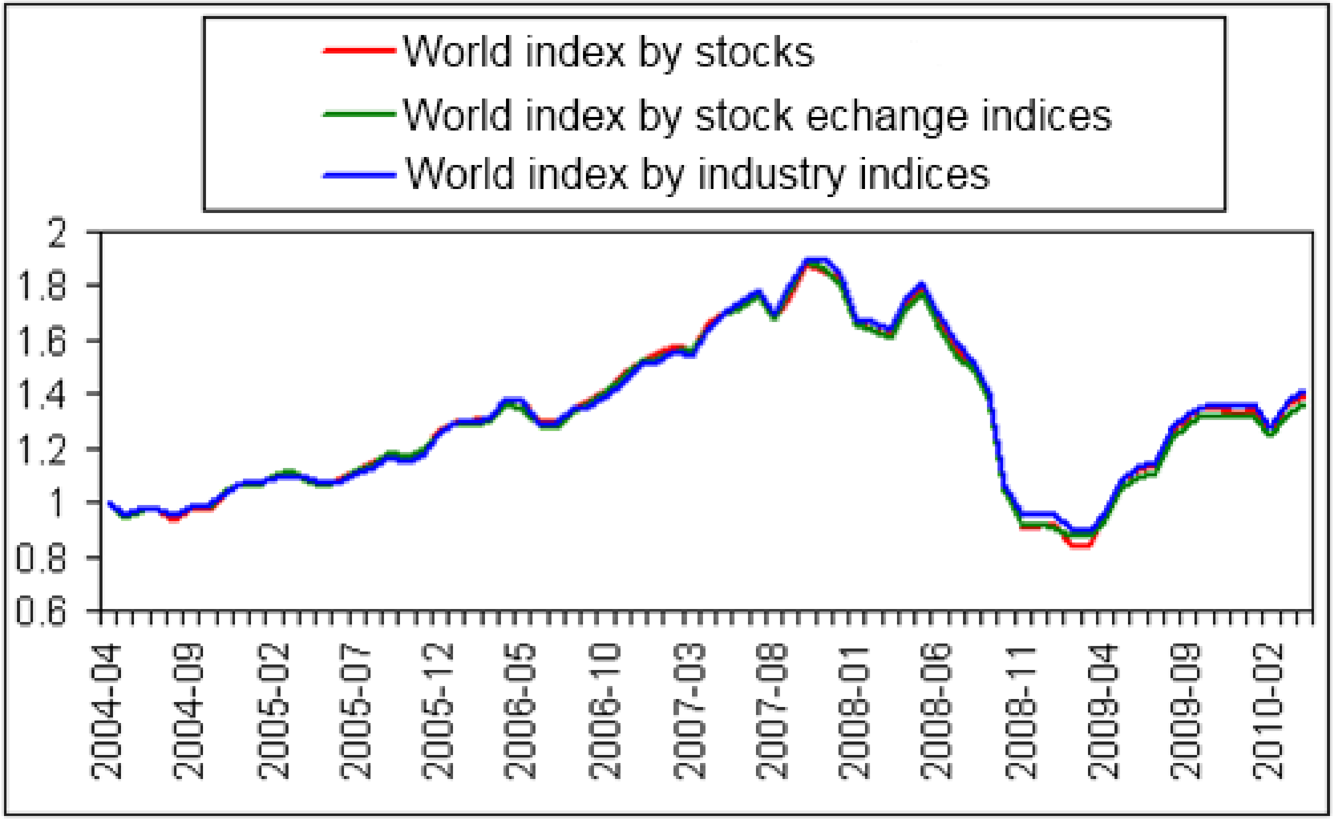
Metallurgy

Metallurgy companies demonstrated the largest growth before the crisis.



Stock market segmentation

Indices computed by industry indices, stock exchange indices and stocks almost coincides.



Rationalizability

Irrationality index is the smallest for the world market.

World stock market with aggregation by industries	1.0108
World stock market with aggregation by stock exchanges	1.0208
World stock market	1.0212
USA (NYSE)	1.0232
Financial	1.0264
Metallurgy	1.0727
Brazil	1.0966
Consumer goods companies	1.1229

Neoclassical model of consumer choice

M – *number of social groups*

$u_\alpha(X) \in U_m,\ \alpha = \overline{1, M}$ – *utility functions of the groups*

$\varphi_\alpha(P)$ – *incomes of the groups,* $I(P) = \sum_{\alpha=1}^{M} \varphi_\alpha(P)$

Statement 3. *Let* $X^\alpha(P) = (X_1^\alpha(P), \ldots, X_m^\alpha(P))$

$$X_i^\alpha(P) = \operatorname{Arg\,max}\left(u_\alpha(X) : \langle P, X \rangle \le \varphi_\alpha(P), X \ge 0\right)$$

Then

$$X_i^\alpha(P) = \frac{\varphi_\alpha(P)}{q_\alpha(P)} \frac{\partial q_\alpha(P)}{\partial P_i} \left(i = \overline{1, m}\right),$$

where $q_\alpha(P) = \inf\limits_{\{X \ge 0 | u_\alpha(X) > 0\}} \dfrac{\langle P, X \rangle}{u_\alpha(X)}$ – *price index from the point of view of group* α.

Integrability and distribution of incomes

$$X(P) = \sum_{\alpha} X^{\alpha}(P)$$

$$\omega = \sum_{i} X_i\left(P\right) dP_i = \sum_{\alpha=1}^{M} \frac{\varphi_{\alpha}\left(P\right)}{q_{\alpha}\left(P\right)} dq_{\alpha}\left(P\right)$$

Statement 4. *if* $\omega = F(X(P))dQ(P)$, *where* $F(X) \in U_m, Q(P) \in U_m$

$$Q(P) = \inf_{\{X \geq 0 | F(X) > 0\}} \frac{\langle P, X \rangle}{F(X)},$$

then there exists $\Phi(q) \in U_M$, *such that* $Q(P) = \Phi(q(P))$

$$\varphi_{\alpha}(P) = I(P) \frac{q_{\alpha}(P)}{\Phi(q(P))} \frac{\partial \Phi(q(P))}{\partial q_{\alpha}}.$$

Bergson function of social welfare

$$W\left(u_1,\ldots,u_M\right)=\inf_{\left\{q\geq 0 \middle| \Phi\left(q\right)>0\right\}}\frac{\sum_{\alpha=1}^{M} q_\alpha u_\alpha}{\Phi(q)}$$

Link between Bergson welfare function and product index

Statement 5. *Let* $\Phi(q) \in U_M, \varphi_\alpha(P) = I(P)\psi_\alpha(q(P)),$
where

$$\psi_\alpha(P) = \frac{q_\alpha(P)}{\Phi(q)} \frac{\partial \Phi(q)}{\partial q_\alpha}, \ \alpha = \overline{1, M}.$$

Then the product index $F(X)$ *is not less than the optimal value of the functional in the problem*

$$\begin{cases} W\big(u_1(X^1), \dots, u_M(X^M)\big) \to \max, \\ X^1 + \dots + X^M = X, \quad X^\alpha \geq 0 \quad (\alpha = \overline{1, M}) \end{cases} \qquad (\#)$$

If $X_i = \sum_\alpha \frac{\psi_\alpha(q)}{q_\alpha} \frac{\partial q_\alpha}{\partial p_i} \ (i = 1, \dots, m)$ *and* $X^1(X), \dots, X^M(X)$ *form the solution of (#), then*

$$F(X) = W(u_1(X^1(X)), \dots, u_M(X^M(X))) \quad \text{and} \quad u_\alpha = \frac{1}{\Phi(q)} \frac{\partial \Phi}{\partial q_\alpha} = \frac{\psi_\alpha(q)}{q_\alpha}, \ q_\alpha = \Phi(q) \frac{\partial W(u(q))}{\partial u_\alpha}$$

$$\gamma = \sum_{\alpha=1}^{M} u_\alpha(q) dq_\alpha = W(u(q)) d\Phi(q), \ \varepsilon = \sum_{\alpha=1}^{M} q_\alpha(u) du_\alpha = \Phi(q(u)) dW(u)$$

Violation of integrability conditions and the social structure of the society

$$\gamma = \sum_{\alpha=1}^{M} u_\alpha(q)dq_\alpha = \sum_{\beta=1}^{\kappa} W_\beta(u(q))dV_\beta(q),$$

$$u_\alpha(q) = \sum_{\beta=1}^{\kappa} W_\beta(u(q)) \frac{\partial V_\beta(q)}{\partial q_\alpha}$$

$$\varepsilon = \sum_{\alpha=1}^{M} q_\alpha(u)du_\alpha = \sum_{\beta=1}^{\kappa} V_\beta(q(u))dW_\beta(u)$$

$$q_\alpha(u) = \sum_{\beta=1}^{\kappa} V_\beta(q) \frac{\partial W_\beta(u(q))}{\partial u_\alpha}$$

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Thank you for your attention!