Weighted Radon transforms on the plane

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1. Introduction

We consider the weighted ray transforms P_W defined by the formula

$$P_W f(s,\theta) = \int\limits_{\mathbb{R}} W(s\theta^\perp + t\theta, \theta) f(s\theta^\perp + t\theta) dt, \qquad (1)$$

$s \in \mathbb{R}$, $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1$, where $\theta^\perp = (-\theta_2, \theta_1)$, $W = W(x, \theta)$ is the weight, $f = f(x)$ is a test function , $x \in \mathbb{R}^2$, $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$. Up to change of variables, P_W is known also as the weighted Radon transform on the plane.

In definition (1) the product $\mathbb{R} \times \mathbb{S}^1$ is interpreted as the set of all oriented straight lines in $\mathbb{R}^2$. More precisely, if $\gamma = (s, \theta) \in \mathbb{R} \times \mathbb{S}^1$, then $\gamma = \{x \in \mathbb{R}^2 : \ x = s\theta^\perp + t\theta, \ t \in \mathbb{R}\}$ (modulo orientation) and θ gives the orientation of γ.

We assume that

$$W \text{ is complex} - \text{valued}, \; W \in C(\mathbb{R}^2 \times \mathbb{S}^1) \cap L^\infty(\mathbb{R}^2 \times \mathbb{S}^1), \quad (2)$$

$$w_0(x) = \frac{1}{2\pi} \int\limits_{\mathbb{S}^1} W(x, \theta) d\theta \neq 0, \quad x \in \mathbb{R}^2.$$

In addition, in important particular cases

$$W = \bar{W}, \quad W \geq c > 0. \quad (3)$$

If $W = 1$, then $P = P_W$ is known as classical ray (or Radon) transform on the plane. This transform arises, in particular, in the classical X- ray transmission tomography. In the X-ray tomography the quantity

$$S(\gamma) = \exp[-Pa(\gamma)],$$

$$Pa(\gamma) = \int_{\mathbb{R}} a(s\theta^{\perp} + t\theta)dt, \quad \gamma = (s, \theta) \in \mathbb{R} \times \mathbb{S}^1,$$

describes the X-ray photograph along γ, where $a = a(x)$ is the X-ray attenuation coefficient of the medium.

If

$$W(x,\theta) = W_a(x,\theta) = \exp(-Da(x,\theta)), \ \ Da(x,\theta) = \int_0^{+\infty} a(x+t\theta)dt, \tag{4}$$

where a is a complex-valued sufficiently regular function on $\mathbb{R}^2$ with sufficient decay at infinity, then P_W is known as the attenuated ray (or Radon) transform on the plane. This transform (at least, with $a \geq 0$) arises, in particular, in single-photon emission computed tomography (SPECT).

Transforms P_W with some other weight also arise in applications. For example, such transforms arise also in fluorescence tomography, optical tomography, positron emission tomography.

In single-photon emission computed tomography (SPECT) one considers a body containing radioactive isotopes emitting photons. The emission data p in SPECT consist in the radiation measured outside the body by a family of detectors during some fixed time. The basic problem of SPECT consists in finding the distribution f of these isotopes in the body from the emission data p and some a priori information concerning the body. Usually this a priori information consists in the photon attenuation coefficient a in the points of body, where this coefficient is found in advance by the methodes of the classical X-ray transmission tomography.
In SPECT the quantity $P_{W_a}f(\gamma)$, $\gamma = (s, \theta) \in \mathbb{R} \times \mathbb{S}^1$, describes the expected emission data along γ.

Exact and simultaneously explicit inversion formulas for the classical and attenuated Radon transforms on the plane were given for the first time in [Radon, 1917] and [R.Novikov, 2002], respectively. For some other weights W exact and simultaneously explicit inversion formulas are also known, see [Boman, Strömberg, 2004], [Gindikin, 2010], [R.Novikov, 2011].

Inversion formula of [R.Novikov 2002]:

$$f = P_{w_a}^{-1} g, \quad \text{where} \quad g = P_{w_a} f, \tag{5}$$

$$P_{w_a}^{-1} g(x) = \frac{1}{4\pi} \int\limits_{\mathbb{S}} \theta^{\perp} \partial_x \big(\exp\left[-Da(x, -\theta)\right] \tilde{g}_{\theta}(\theta^{\perp} x) \big) \, d\theta,$$

$$\tilde{g}_{\theta}(s) = \exp\left(A_{\theta}(s)\right) \cos\left(B_{\theta}(s)\right) H(\exp\left(A_{\theta}\right) \cos\left(B_{\theta}\right) g_{\theta})(s) +$$

$$\exp\left(A_{\theta}(s)\right) \sin\left(B_{\theta}(s)\right) H(\exp\left(A_{\theta}\right) \sin\left(B_{\theta}\right) g_{\theta})(s),$$

$$A_{\theta}(s) = (1/2) P_0 a(s\theta^{\perp}, \theta), \quad B_{\theta}(s) = H A_{\theta}(s), \quad g_{\theta}(s) = g(s\theta^{\perp}, \theta),$$

$$Hu(s) = \frac{1}{\pi} p.v. \int\limits_{\mathbb{R}} \frac{u(t)}{s - t} dt,$$

$$x \in \mathbb{R}^2, \quad \theta^{\perp} = (-\theta_2, \theta_1) \quad \text{for} \quad \theta = (\theta_1, \theta_2) \in \mathbb{S}^1, \quad s \in \mathbb{R}.$$

For general P_W, under assumptions (2), (3), explicit and simultaneously exact (modulo $Ker\, P_W$) inversion formulas seem to be impossible. Nevertheless, due to [Boman, Quinto, 1987], P_W is injective on $C_0(\mathbb{R}^2)$ (where C_0 denotes continuous functions with compact support), i.e. $Ker\, P_W = 0$ in $C_0(\mathbb{R}^2)$, for real- analytic W satisfying (3). Besides, [Boman, 1993] gives an example of infinitely smooth W satisfying (2), (3) and such that $Ker\, P_W \neq 0$ in $C_0^\infty(\mathbb{R}^2)$ (where C_0^∞ denotes infinitely smooth functions with compact support).

On the other hand, the following Chang approximate inversion formula for P_W, where W is given by (4) with $a \geq 0$, has been used for a long time (see [Chang, 1978], [Kunyansky, 1992]):

$$f_{appr}(x) = F_0(x)/w_0(x), \tag{6}$$

$$F_0(x) = \frac{1}{4\pi} \int\limits_{\mathbb{S}^1} h'(x\theta^{\perp}, \theta) d\theta, \quad h'(s,\theta) = \frac{d}{ds} h(s,\theta),$$

$$h(s,\theta) = \frac{1}{\pi} p.v. \int\limits_{\mathbb{R}} \frac{P_W f(t,\theta)}{s-t} dt, \ s \in \mathbb{R}, \ \theta \in \mathbb{S}^1, \ x \in \mathbb{R}^2,$$

where w_0 is defined in (2). It is known that formula (5) is efficient as the first approximation in SPECT reconstructions and, in particular, is sufficiently stable with respect to strong Poisson noise in SPECT data.

2. Result of [R.Novikov, 2011]

Theorem 1. Let the assumptions (2) hold and let $f_{appr}(x)$ be given by (6). Then

$$f_{appr} = f \ \text{(in the sense of distributions) on } \mathbb{R}^2 \text{ for all } f \in C_0(\mathbb{R}^2), \tag{7}$$

if and only if

$$W(x,\theta) - w_0(x) \equiv w_0(x) - W(x,-\theta), \quad x \in \mathbb{R}^2, \quad \theta \in \mathbb{S}^1. \tag{8}$$

For the case when W is given by formulas (4) under the additional conditions that $a \geq 0$ and $supp\, a \subset D$, where D is some known bounded domain which is not too large, and for $f \in C(\mathbb{R}^2)$, $f \geq 0$, $supp\, f \subset D$, the transform $P_W f$ is relatively well approximated by $P_{W_{appr}} f$, where $W_{appr}(x,\theta) = w_0(x) + (W(x,\theta) - W(x,-\theta))/2$. In addition, this W_{appr} already satisfies (8). This explains the efficiency of formula (6) as the first approximation in SPECT reconstructions (at the level of integral geometry).

3. Finite Fourier series weights

We consider weights of the form

$$W(x, \theta(\varphi)) = \sum_{n=-N}^{N} e^{in\varphi} w_n(x), \tag{9}$$

$x \in \mathbb{R}^2$, $\theta(\varphi) = (\cos\varphi, \sin\varphi)$, $\varphi \in [-\pi, \pi]$, $N \in \mathbb{N} \cup 0$,
where $w_n \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and, for example, $|w_0| > c_0 > 0$.
Such weights approximate, in particular, W satisfying (2), (3).
Due to the formula

$$\frac{1}{2}(P_W f(s, \theta) + P_W f(-s, -\theta)) = P_{W_{sym}} f(s, \theta),\ (s, \theta) \in \mathbb{R} \times \mathbb{S}^1, \tag{10}$$

$$W_{sym}(x, \theta) = \frac{1}{2}(W(x, \theta) + W(x, -\theta)),\ x \in \mathbb{R}^2,\ \theta \in \mathbb{S}^1,$$

inversion of P_W can be reduced to inversion of $P_{W_{sym}}$. Therefore, we consider in more detail the case when W is given by (9) and

$$W(x, \theta) = W(x, -\theta), \quad x \in \mathbb{R}^2, \quad \theta \in \mathbb{S}^1. \tag{11}$$

One can see that $(9), (11) \Rightarrow$

$$W(x, \theta(\varphi)) = \sum_{l=-m}^{m} e^{2il\varphi} w_{2l}(x), \qquad (12)$$

$x \in \mathbb{R}^2,\ \theta(\varphi) = (\cos\varphi, \sin\varphi),\ \varphi \in [-\pi, \pi]$, where $w_{2l} \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $|w_0| \geq c_0 > 0$, $m = [N/2]$ (i.e. m is the integer part of $N/2$).

If general W satisfying (2) is approximated as $W(x, \theta) \approx\ w_0(x)$, i.e. via (12) with $m = 0$, then $f_{appr} = P_{w_0}^{-1}(P_W f)$ yelds the Chang approximate inversion formula (6).

Further, we consider inversion of P_W for W of the form (12), where $m > 0$, in general.

Consider

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2, \tag{13}$$

$$\partial_z = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big), \; \partial_{\bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big),$$

where $x = (x_1, x_2) \in \mathbb{R}^2$.
We will write $f = f(z)$, $w_n = w_n(z)$.
Let Π, $\bar{\Pi}$ denote the linear integral operators on the complex plane such that

$$\Pi u(z) = -\frac{1}{\pi}\int\limits_{\mathbb{C}} \frac{u(\zeta)}{(\zeta - z)^2} dRe\zeta \, dIm\zeta, \tag{14}$$

$$\bar{\Pi} u(z) = -\frac{1}{\pi}\int\limits_{\mathbb{C}} \frac{u(\zeta)}{(\bar{\zeta} - \bar{z})^2} dRe\zeta \, dIm\zeta,$$

where u is a test function, $z \in \mathbb{C}$; see, for example, [Vekua, 1962] for detail properties of these operators.

Theorem 2 ([R.Novikov, 2012]). Let W be given by (12), where $m \geq 1$. Let $f \in C(\mathbb{C})$, *supp* $f \subset D$, where D is an open bounded domain in $\mathbb{C}$. Let $F = w_0 f$ and let F_0 be defined as in (6). Then

$$F + \sum_{l=1}^{m} \big((-\bar{\Pi})^l \frac{w_{2l}}{w_0} + (-\Pi)^l \frac{w_{-2l}}{w_0}\big)\chi_D F = F_0, \qquad (15)$$

where χ_D is the characteristic function of D, Π, $\bar{\Pi}$ are defined by (14). In addition, equation (15) is uniquely solvable for F in $L^2(\mathbb{C})$ (by the method of successive approximations) if

$$\sum_{l=1}^{m} \big(\sup_{z \in D} |\frac{w_{2l}(z)}{w_0(z)}| + \sup_{z \in D} |\frac{w_{-2l}(z)}{w_0(z)}|\big) < 1. \qquad (16)$$

Theorem 3 ([R.Novikov, 2012]). Let W be given by (12) for $m = 1$ and condition (3) be fulfilled, and let $f \in C(\mathbb{C})$, $supp\, f \subset D$, where D is an open bounded domain in $\mathbb{C}$. Then f is uniquely determined by $P_W f$ and W via the linear integral equation (15) for $m = 1$ (solvable by the method of successive approximations in $L^2(\mathbb{C})$).

It is interesting to point out that finding $f \in C(\mathbb{C})$, *supp* $f \subset D$, from $P_W f$, where W satisfies (3) and is given by (12) for $m = 1$ is reduced to finding u such that

$$\partial_z u - 2\chi_D \frac{w_2}{w_0} Re\, \partial_{\bar z} u = \chi_D \frac{w_2}{w_0} F_0, \qquad (17)$$

$$u \in C(\mathbb{C}), \quad \partial_z u \in L^2(\mathbb{C}), \quad u(z) \to 0, \quad \text{as} \quad |z| \to \infty,$$

$$\text{(Beltrami type equation)}$$

where F_0 is the function of (6). In addition,

$$f = \frac{F_0 + 2Re\, \partial_{\bar z} u}{w_0}.$$

For general $m \geq 1$ such a reduction yields a system of first order differential equations on the plane, generalizing (17). For details see [R.Novikov, 2012].

Exact inversion of Theorem 2 for P_W, where W is given by (12) and satisfies (16) leads to an approximate inversion method for P_W with general W satisfying (2). This research direction is developed in detail in [Guillement, R.Novikov, 2013].

References

J.Boman, An example of non-uniqueness for a generalized Radon transform, J.Anal.Math. **61** (1993), 395-401

J.Boman and E.T.Quinto, Support theorems for real-analytic Radon transforms, Duke Math. J. **55** (1987), 943-948

J.Boman and J.O.Strömberg, Novikov's inversion formula for the attenuated Radon transform - a new approach, *J.Geom.Anal.* **14** (2004), 185-198

L.T.Chang, A method for attenuation correction in radionuclide computed tomography, IEEE Trans. Nucl. Sci. NS-25 (1978), 638-643

S.Gindikin, A remark on the weighted Radon transform on the plane, Inverse Problems and Imaging **4** (2010), 649-653

J.-P.Guillement and R.G.Novikov, Optimized analytic reconstruction for SPECT, J.Inv. Ill-Posed Problems **20** (2012), 489-500

References

J.-P.Guillement and R.G.Novikov, Inversion of weighted Radon transforms via finite Fourier series weight approximations, Inverse Problems in Sciences and Engineering, http://dx.doi.org/10.1080/17415977.2013.823417, e-print: http://hal.archives-ouvertes.fr/hal-00794083,

L.A.Kunyansky, Generalized and attenuated Radon transforms: restorative approach to the numerical inversion, Inverse Problems **8** (1992), 809-819

R.G.Novikov, An inversion formula for the attenuated X-ray transformation, Ark. Mat. 40 (2002), 145-167

R.G.Novikov, Weighted Radon transforms for which Chang's approximate inversion formula is exact, Uspekhi Mat. Nauk **66** (2) (2011), 237-238

R.G.Novikov, Weighted Radon transforms and first order differential systems on the plane, e-print: http://hal.archives-ouvertes.fr/hal-00714524

References

J.Radon, Uber die Bestimmung von Funktionen durch ihre Integralwerte langs gewisser Mannigfaltigkeiten, Ber. Verh. Sachs. Akad. Wiss. Leipzig, Math-Nat., **K**1 **69** (1917), 262-267

I.N.Vekua, Generalized Analytic Functions, Pergamon Press Ltd. 1962