Fokas Methods applied to a Boundary Valued Problem for Conjugate Conductivity Equations

Joint work with Slah Chaabi and Franck Wielonsky

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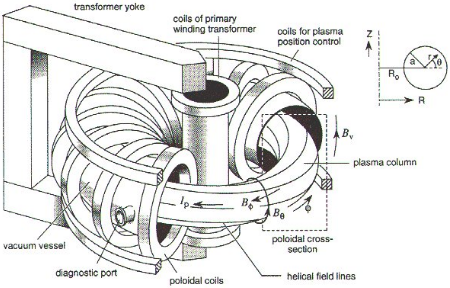
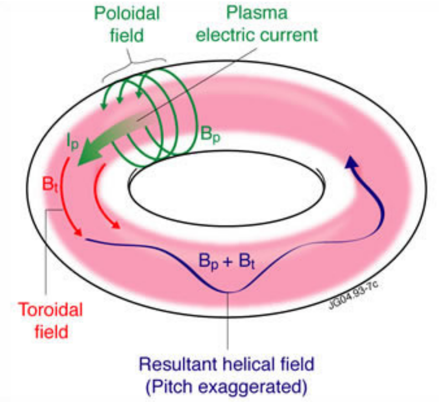
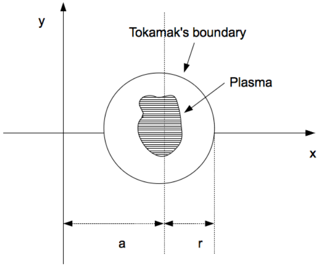
- Is it possible to obtain a relation between u and $\partial_{\vec{n}} u$ on $\partial\Omega$ without computing u in Ω ?

One other motivation : What is a Tokamak ?





Mathematical Formulation



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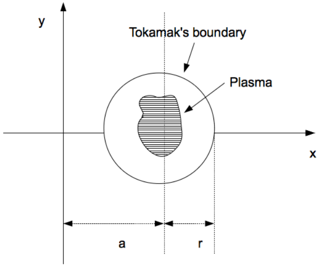

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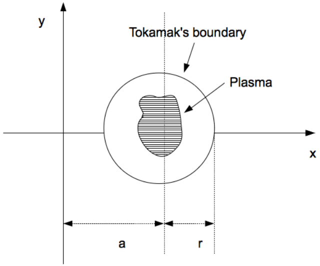

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- Generalized axisymetrical potential for $\alpha \in \mathbb{R}$:

$$\operatorname{div}(x^{\alpha}\nabla u) = 0 \quad \Leftrightarrow \quad \Delta u + \frac{\alpha}{x}\frac{\partial u}{\partial x} = 0.$$

The Fokas method Fokas (1997) : a very simple example.

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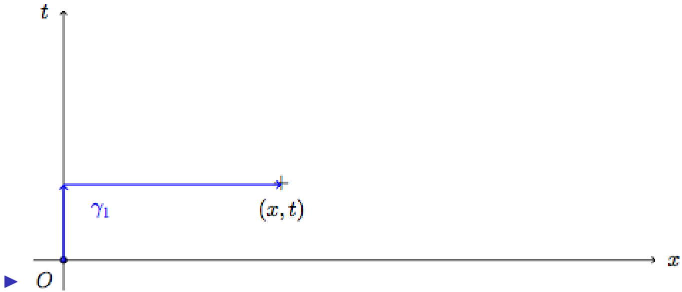
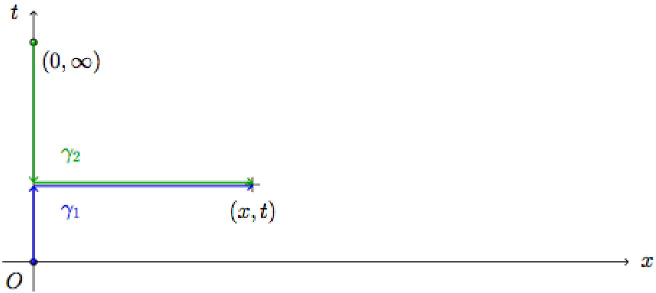
- Integration of ν between points of the boundary and (x,t).



$$\mu_{46}(x,t,k) = e^{ikx-k^2t}\int_{\gamma_1}\nu$$



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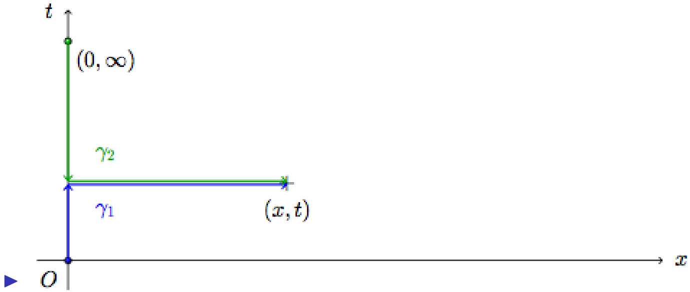

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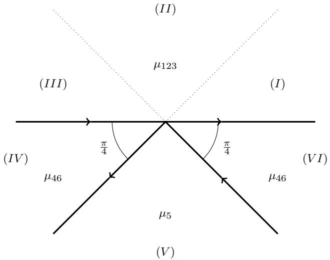
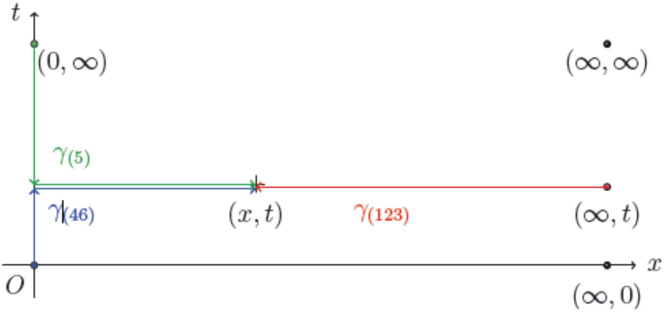


- 

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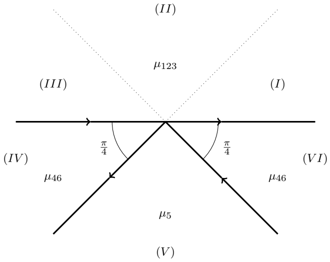
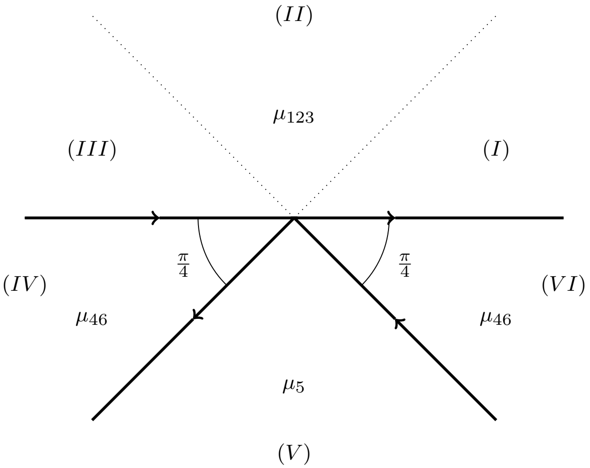
$$\mu_{123}(x,t,k) = e^{ikx-k^2t}\int_{\gamma_{123}} \nu$$



Riemann-Hilbert Problems



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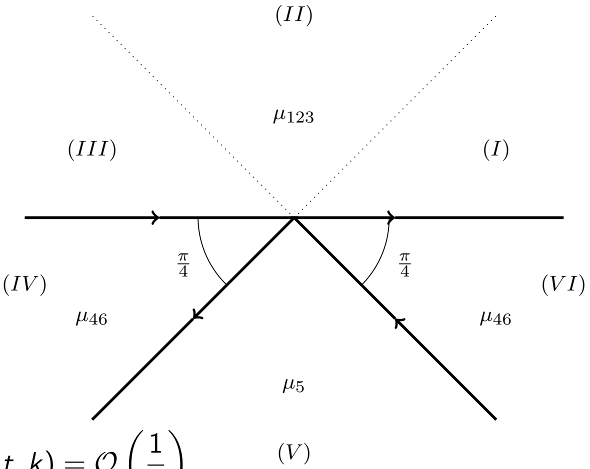

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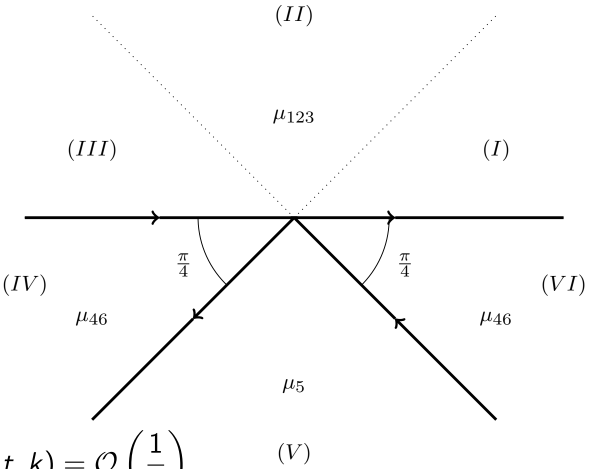

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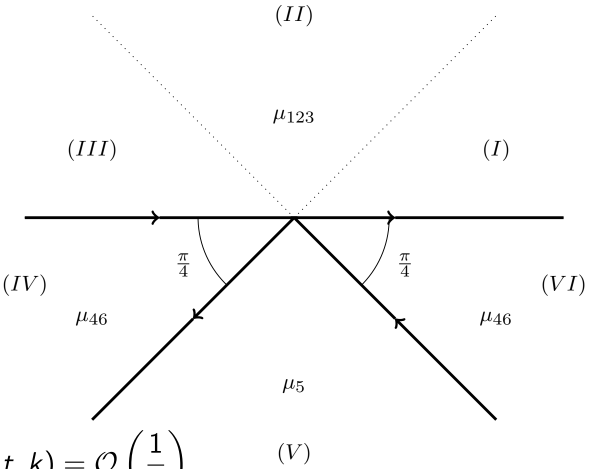

- $\mu(x, t, k) = \mathcal{O}\left(\frac{1}{k}\right)$
- $\mu^+ - \mu^- = \phi$
- $\mu = \frac{1}{2\pi i} \int_L \frac{\phi(k')dk'}{k' - k}$ (Plemelj Formula)

Lax Pairs and closed differential form for the GASP equation.

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  Note that, when $\alpha \in 2\mathbb{N}^*$, the differential form has no singularity in Ω and k may be any complex number. Otherwise, for $\alpha \in \mathbb{R} \setminus 2\mathbb{N}^*$, $W(z,k)$ has a pole or a branching point in k or $-\bar k$ if one of this point is in Ω.

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$\alpha = -2m,\ m \in \mathbb{N}$.

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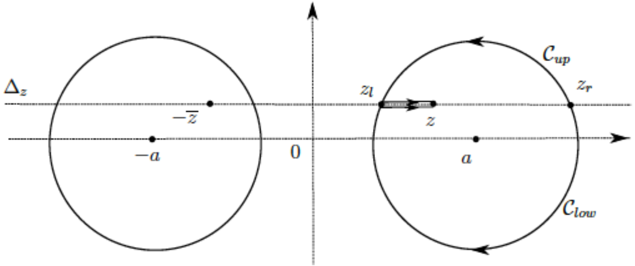

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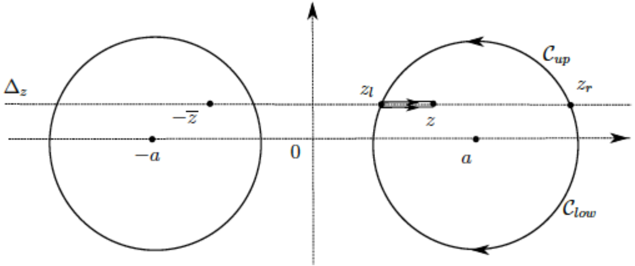

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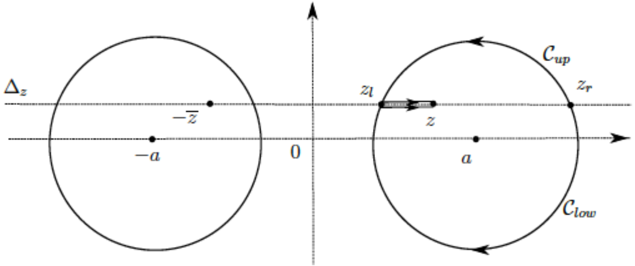

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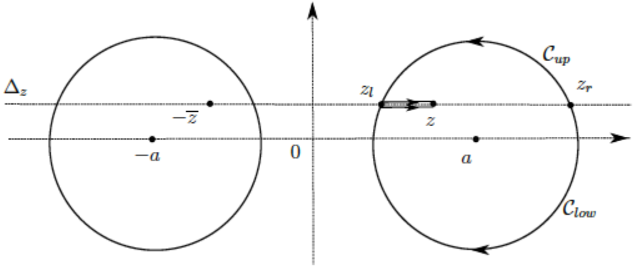

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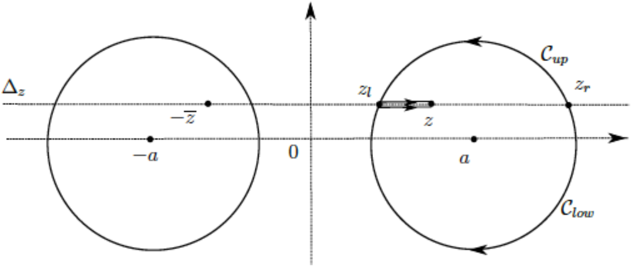

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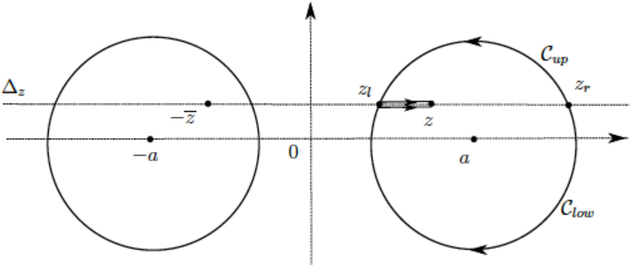
- 

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- $\phi \sim_{k \to +\infty} \frac{u(z)-u(z_r)}{k^{2m-1}}$

- Renormalization : $\widetilde{\phi}(z,k) = ((k-z)(k+\overline{z}))^m \phi(z,k)$

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- $\widetilde{\phi}$ regular in z and $-\overline{z}$, polar singularities in z_r et $-\overline{z}_r$.

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Computation of the residue a_r of $\widetilde{\phi}$ in z_r

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- m integrations by parts give $\widetilde{\phi}_{z_r, -\overline{z}_r}$, then a_r.

Let u be a solution to the equation $\Delta u + \alpha x^{-1}\partial_x u = 0$, $\alpha = -2m$, $m \in \mathbb{N}$, in the domain $\mathcal{D}$ with smooth tangential derivatives u_t and normal derivatives u_n on the boundary $\mathcal{C}$.

$$u(z) = -\frac{1}{\pi}\mathrm{Im}\int_{(z,z_r)} \big((k-z)(k+\bar{z})\big)^m J(z,k)dk + 2\mathrm{Re}\, a_r + u(z_r), \tag{1}$$

where a_r can be explicitly computed in terms of the tangential derivative along $\mathcal{C}$ of u_t and u_n, up to the order $m-1$, in z_r. Function $J(z,k)$ is given by

$$J(z,k) = -\int_{\mathcal{C}} W(z',k),$$

where $W(z,k)$ is the differential form

$$W(z,k) = \big((k-z)(k+\bar{z})\big)^{-m-1}\left((k+\bar{z})u_z(z)dz + (k-z)u_{\bar{z}}(z)d\bar{z}\right) \tag{2}$$

$$= \big((k-z)(k+\bar{z})\big)^{-m-1}\left((k-iy)u_t(z) + ixu_n(z)\right)ds, \tag{3}$$

with $z = x+iy$ and ds the unit length element on $\mathcal{C}$.

The case $\alpha = -1$.

- $W(z,k) = \dfrac{(k+\overline{z})u_z dz + (k-z)u_{\overline{z}}d\overline{z}}{\sqrt{(k-z)(k+\overline{z})}}$

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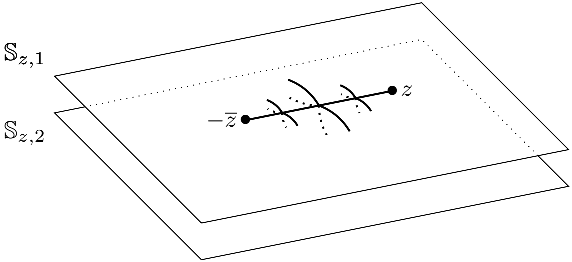

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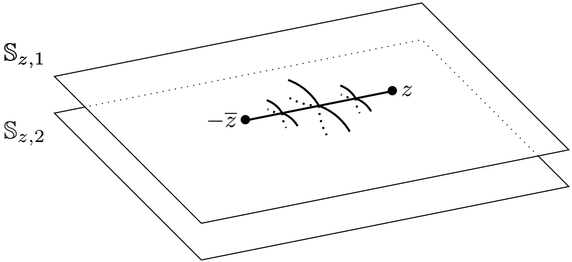

-
- $\lambda_1(z,k) \sim k$ when $k \to \infty_1$ on the sheet above $\mathbb{S}_{z,1}$
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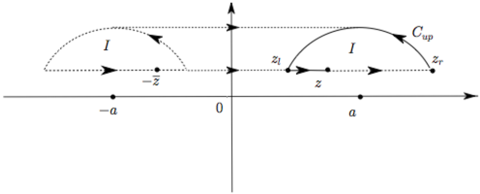

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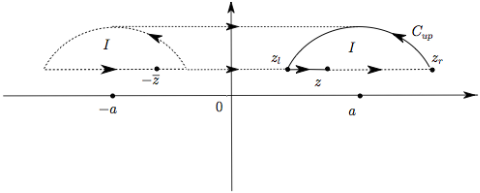
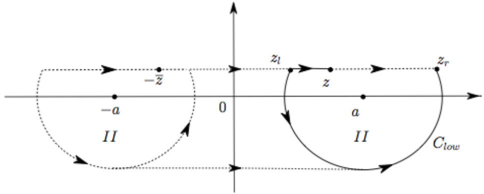


- Sheet 2



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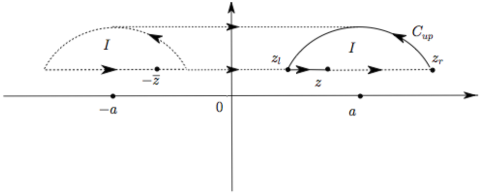

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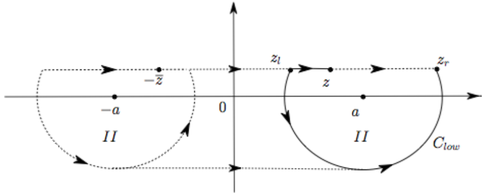
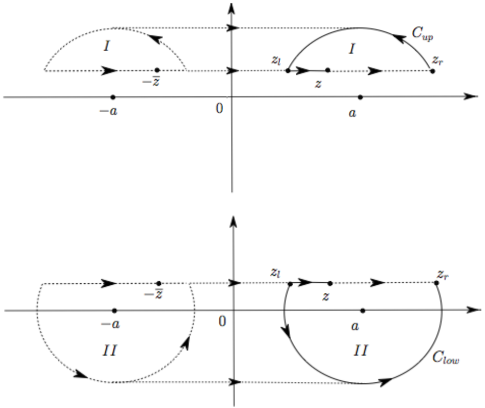
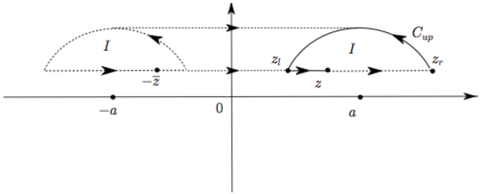
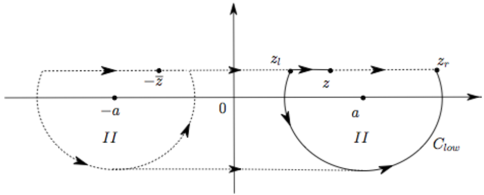


- $\phi(z,\infty_1) = -\phi(z,\infty_2)$.



C_{up}
I
I
z_l
z_r
z
$-\overline{z}$
$-a$
0
a
z_l
z_r
z
$-\overline{z}$
$-a$
0
a
II
II
C_{low}

- 

- 

$$\phi(z,k) = \frac{1}{4i\pi}\int_{\mathcal{C}_{up}\cup -\overline{\mathcal{C}}_{up}} J(z,k')\left(\frac{\lambda(z,k)}{\lambda_1(z,k')}+1\right)\frac{dk'}{k'-k}$$

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- Since f is real valued, on has for $z \in \mathbb{T}$

$$f(z) = g(z) + \overline{g}(1/z)$$

with $g \in H(\mathbb{D})$ and $\overline{g}(1/z) \in H(\mathbb{C} \setminus \overline{\mathbb{D}})$.

- $$\int_{\mathbb{T}} \frac{z^{m-1}(g(z) + \overline{g}(1/z))dz}{(z-(k-a))^m \left(z + \frac{1}{k+a}\right)^m} = 0.$$

- $$\int_{\mathbb{T}} \frac{z^{m-1}(g(z)+\overline{g}(1/z))dz}{(z-(k-a))^m\left(z+\frac{1}{k+a}\right)^m} = 0.$$

- $$\left(\frac{-1}{k+a}\right)^m \left[\frac{z^{m-1}g(z)}{(z-(k-a))^m}\right]^{(m-1)} \left(\frac{-1}{k+a}\right) +$$
$$\left(\frac{1}{k-a}\right)^m \left[\frac{z^{m-1}\bar{g}(z)}{(z+(k+a))^m}\right]^{(m-1)} \left(\frac{1}{k-a}\right) = 0.$$

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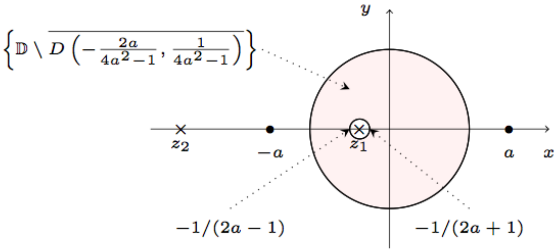

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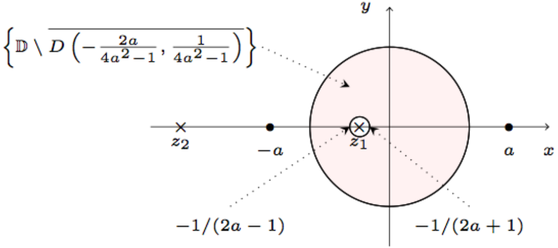
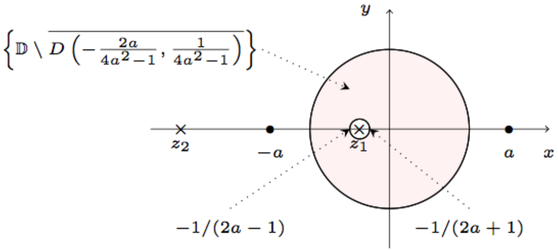


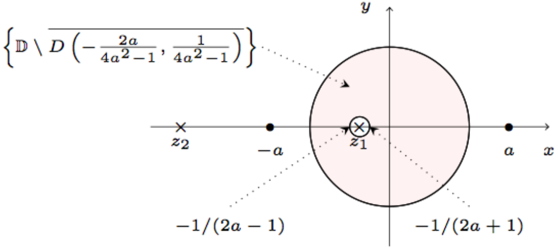
- For $\mu \in \mathbb{A}$

$$\Phi(\mu) = \int_{\mathbb{T}} \frac{z^{m-1} g(z) dz}{(1-\varphi(\mu)z)^m (z-\mu)^m} = \frac{2\pi i}{(m-1)!} \left(\frac{h(z)}{(1-\varphi(\mu)z)^m}\right)^{(m-1)} (\mu)$$

with $h(z) = z^{m-1} g(z)$.



- $\varphi : \mathbb{A} \to \mathbb{A}$, and $D(-\frac{2a}{4a^2-1}, \frac{1}{4a^2-1}) \to \mathbb{C} \setminus \overline{\mathbb{D}}$ and $-1/(2a) \to \infty$.



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- z_1 and z_2 the roots of $z^2 + 2az + 1$.

$$z_1 = -a + \sqrt{a^2 - 1} \in D(-\frac{2a}{4a^2 - 1}, \frac{1}{4a^2 - 1})$$

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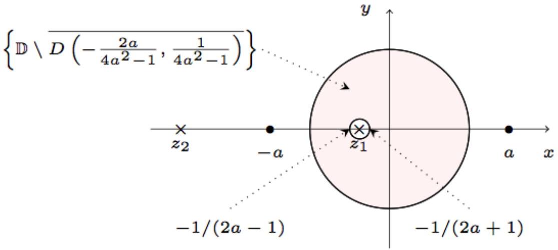

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- $1 - \varphi(\mu)\mu = \frac{\mu^2+2a\mu+1}{1+2a\mu} \Rightarrow \Phi$ has zero of order at least m at $-1/(2a)$.

- Multiplying $\Phi(z) + \bar{\Phi}(\varphi(z)) = 0$ by

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- $$\int_{\mathbb{T}} \frac{z^{m-1} f(z)}{(z-(k-a))^m (z+(k+a)^{-1})^m} = 0, \qquad \forall k \in \mathbb{C} \backslash \{\overline{\mathcal{D}_a} \cup \overline{-\mathcal{D}_a}\},$$

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- We put $\xi = -(k+a)^{-1}$, we get

$$\int_{\mathbb{T}} \frac{z^{m-1} f(z)}{(\xi z + 2a\xi + 1)^m (z - \xi)^m} = 0$$

$$\forall \xi \in \mathbb{D} \setminus \overline{D\left(-\frac{2a}{4a^2-1}, \frac{1}{4a^2-1}\right)}.$$

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$$\sum_{k=0}^{m-1} \frac{(-1)^k (m+k-1)!}{k!(m-1-k)!} \frac{\xi^k}{(\xi^2 + 2a\xi + 1)^{m+k}} F^{(m-1-k)}(\xi) = 0.$$

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- Thank you very much for your attention !