① L^2 and H^2

$f \in L^2 = L^2(\mathbb{T})$: $f(z) = \sum_{n=-\infty}^{\infty} f_n z^n$, $|z|=1$, $\sum |f_n|^2 < \infty$

$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$

$f \in H^2$: $f_n = 0$ for $n < 0$

$f(z) = \sum_{n=0}^{\infty} f_n z^n$, $|z| = 1$ or $|z| < 1$

H^2 is a Hilbert space,

$\|f\|^2 = \sum_{n=0}^{\infty} |f_n|^2 = \int_{\mathbb{T}} |f(z)|^2 dm(z)$

$P : L^2 \to H^2$ is the Szegö projection

$P : \sum_{n=-\infty}^{\infty} f_n z^n \mapsto \sum_{n=0}^{\infty} f_n z^n$.

Orthonormal basis in H^2 : $\{z^n\}_{n=0}^{\infty}$

$S : H^2 \to H^2$, $Sf(z) = z f(z)$ – shift operator

$$\sum_{n=0}^{\infty} f_n z^n \mapsto \sum_{n=0}^{\infty} f_n z^{n+1} = \sum_{m=1}^{\infty} f_{m-1} z^m$$

② Hankel operators

Ганкель

$$\Gamma_u = \begin{pmatrix} u_0 & u_1 & u_2 & \cdots \\ u_1 & u_2 & u_3 & \cdots \\ u_2 & u_3 & u_4 & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix} = \{u_{n+m}\}_{n,m=0}^{\infty}$$

$$\langle P(u\cdot\overline{z^n}), z^m\rangle = \langle u\cdot\overline{z^n}, \overline{Pz^m}\rangle$$

$$= \langle u\cdot\overline{z^n}, z^m\rangle = \langle u, z^n\cdot z^m\rangle$$

$$= \langle u, z^{n+m}\rangle = u_{n+m}$$

Symbol: $u(z) = \sum_{n=0}^{\infty} u_n z^n, \quad |z|=1$

$H_u f = P(u\bar f), \quad H^2 \to H^2$

$\langle H_u z^n, z^m\rangle = u_{m+n}, \quad m,n \geq 0$

$C\{x_n\} = \{\overline{x_n}\}$

H_u is anti-linear! $H_u \simeq \Gamma_u C$

$H_u^2 \simeq \Gamma_u (C\Gamma_u C) = \Gamma_u \Gamma_u^* \geq 0$

$H_u S = S^* H_u$

③ Motivation: the cubic Szegő equation

$u = u(z,t)$; $t \geq 0$, $z \in \mathbb{T}$

$$i \frac{\partial u}{\partial t} = P(u|u|^2)$$

Gérard-Grellier 2010-2017:

- has Lax pair
- explicit action-angle variables, associated with the spectral problem for H_u.

$$u(0) \xrightarrow[\text{problem}]{\text{direct}} \text{spectral data for } H_{u(0)} \xrightarrow[\text{evolution}]{\text{explicit}} \text{spectral data for } H_{u(t)} \longrightarrow$$

$$\xrightarrow[\text{problem}]{\text{inverse}} u(t)$$

(4) Basics of spectral theory of Hankel operators

$H_u f = P(u\bar{f})$, $H_u \sim \{u_{n+m}\}$

$$u(z) = \sum_{n=0}^{\infty} u_n z^n + \sum_{n=-\infty}^{-1} u_n z^n$$

$\|H_u\| \leq \|u\|_{L^\infty}$

$H_u = H_v$, if $u = Pv$

$u = Pv$: $\|H_u\| \leq \|v\|_{L^\infty}$

$\|H_u\| \leq \min\{\|v\|_{L^\infty} : u = Pv\}$

Z. Nehari (1957): "="

H_u is bdd iff $\exists v \in L^\infty$: $u = Pv$

L. Kronecker (1881):

H_u is finite rank iff $u = \frac{P}{Q}$

Corollary: $u \in C(\mathbb{T}) \Rightarrow H_u$ is compact

P. Hartman (1958):

H_u is compact iff $\exists v \in C(\mathbb{T})$: $u = Pv$

$$u(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+1} \; ; \quad \Gamma_u = \left\{\frac{1}{n+m+1}\right\}$$

u - unbounded

$$v(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+1} - \sum_{n=1}^{\infty} \frac{\bar{z}^n}{n+1}$$

$v \in L^\infty$, $\|v\|_{L^\infty} = \pi$

$$u(z) = \frac{1}{1-az}, \quad 0 < a < 1$$

$$= \sum_{n=0}^{\infty} a^n z^n, \quad \Gamma_u = \{a^{n+m}\} \text{ in } \ell^2$$

$$= \langle \cdot, \{a^n\} \rangle \{a^m\}$$

⑤ Some history of inverse problems for Hankel operators

Khrushchev-Peller (1982):

spectra of $|\Gamma_u| \simeq |H_u| = \sqrt{H_u^2}$

- $\dim \operatorname{Ker} H_u = 0$ or ∞:

  $S^* \underbrace{H_u f}_{} = H_u S f = 0 \Rightarrow Sf \in \operatorname{Ker} H_u$

  $S \operatorname{Ker} H_u \subset \operatorname{Ker} H_u$

- $0 \in \operatorname{spect}(|H_u|)$

Is every positive semi-definite operator, satisfying these conditions, unitarily equivalent to $|H_u|$?

S. Treil (1985),

S. Treil + V. Vasyunin (1989):

yes for compact operators

R. Ober (1987, 1990):

new approach

S. Treil (1990):

yes for all bounded operators

Megretskii-Peller-Treil (1995):

similar problem for self-adjoint Hankel operators.

(6) Inverse problem: positive compact Hankel operators (Gerard-Grellier 2010)

$\Gamma_u = \{u_{n+m}\}$, $\tilde{\Gamma}_u = \{u_{n+m+1}\}$

$$\begin{pmatrix} \cancel{u_0 \quad u_1 \quad u_2} \\ u_1 \quad u_2 \quad u_3 \\ u_2 \quad u_3 \end{pmatrix}$$

- Assume that Γ_u (and so $\tilde{\Gamma}_u$) is compact
- Assume $\Gamma_u \geq 0$, $\tilde{\Gamma}_u \geq 0$ (++)

$$\tilde{\Gamma}_u^2 = \Gamma_u^2 - \langle \cdot, u \rangle u$$

<u>Lemma</u> Eigenvalues of $\Gamma_u, \tilde{\Gamma}_u$ are simple and interlace:

$$s_1 > \tilde{s}_1 > s_2 > \tilde{s}_2 > \cdots \to 0 \qquad (*)$$

Spectral data: $\Lambda(u) = \left(\{s_n\}_{n=1}^{\infty}, \{\tilde{s}_n\}_{n=1}^{\infty} \right)$

<u>Theorem</u> The map $u \mapsto \Lambda(u)$ is a bijection between $\{u : (++)\}$ and $\{\Lambda(u) : (*)\}$

6a Explicit formula for $u(z)$

Assume $\operatorname{rank} \Gamma_u = N$:

$$\Lambda(u) = \left(\{ s_k \}_{k=1}^{N}, \{ \tilde{s}_k \}_{k=1}^{N} \right)$$

$$C(z) := \left\{ \frac{s_j - z\tilde{s}_k}{s_j^2 - \tilde{s}_k^2} \right\}_{j,k=1}^{N}, \quad |z| < 1$$

$$u(z) = \langle C(z)^{-1} \mathbb{1}, \mathbb{1} \rangle_{\mathbb{C}^N}, \quad |z| < 1$$

$$\mathbb{1} = (1, 1, \dots, 1).$$

7 Inverse problem: compact operators with simple spectrum (GG 2010)

$\tilde{H}_u = H_{\tilde{u}}$, $\tilde{u}(z) = \sum_{n=0}^{\infty} u_{n+1} z^n$

- Assume H_u (and so $\tilde{H}_u$) is compact
- Assume $H_u^2|_{(\operatorname{Ker} H_u)^\perp}$ and $\tilde{H}_u^2|_{(\operatorname{Ker}\tilde{H}_u)^\perp}$ have simple spectra

Lemma u is a cyclic element for both.

P_j - projection onto $\operatorname{Ker}(H_u^2 - s_j^2 I)$

$u_j = P_j u \neq 0$. Similarly $\tilde{u}_j = \tilde{P}_j u$.

$H_u u_j = s_j e^{i\varphi_j} u_j$

$\tilde{H}_u \tilde{u}_j = \tilde{s}_j e^{i\tilde{\varphi}_j} \tilde{u}_j$

$\Lambda(u) = (\{s_j\}, \{\tilde{s}_j\}, \{e^{i\varphi_j}\}, \{e^{i\tilde{\varphi}_j}\})$

$s_j, \tilde{s}_j$ interlace

Theorem The map $u \mapsto \Lambda(u)$ is a bijection.

Szegö dynamics:

$s_j, \tilde{s}_j$ – integrals of motion (action)

$\dot{\varphi}_j = s_j^2$, $\dot{\tilde{\varphi}}_j = \tilde{s}_j^2$ (angles)

Remark: (++) is destroyed by dynamics

8 Extensions

(a) Singular values with multiplicity:

$e^{i\varphi_j}$ are replaced by Blaschke products:

$$\boxed{\prod_k \frac{z_k - z}{1-\bar{z}_k z}}$$

(Gerard-Grellier 2014)

$\mathrm{Ker}\,(H_u^2 - s_j^2 I)$

(b) Non-compact case (++)

(Gerard-Pushnitski 2015)

$\Gamma_u \geq 0$, $\tilde{\Gamma}_u \geq 0$

spectral measure ρ of Γ_u on u

$$\int_0^\infty \frac{d\rho(x)}{x^2} \leq 1; \qquad u \longleftrightarrow \rho$$

(alternatively: via spectral shift function)

(c) Non-compact case, simple spectra of $|H_u|, |\tilde{H}_u|$

Work in progress (Gerard, Pushnitski, Treil)

Surjectivity breaks down!

(d) Hankel operators in $H^2_+(\mathbb{R})$

O. Pocovnicu, 2011

Gerard-Pushnitski (in progress)