CHAPTER 1

Survey of the Elementary Principles

The motion of material bodies formed the subject of some of the earliest research pursued by the pioneers of physics. From their efforts there has evolved a vast field known as analytical mechanics or dynamics, or simply, mechanics. In the present century the term "classical mechanics" has come into wide use to denote this branch of physics in contradistinction to the newer physical theories, especially quantum mechanics. We shall follow this usage, interpreting the name to include the type of mechanics arising out of the special theory of relativity. It is the purpose of this book to develop the structure of classical mechanics and to outline some of its applications of present-day interest in pure physics. Basic to any presentation of mechanics are a number of fundamental physical concepts, such as space, time, simultaneity, mass, and force. For the most part, however, these concepts will not be analyzed critically here; rather, they will be assumed as undefined terms whose meanings are familiar to the reader.

1.1. Mechanics of a Particle

Let r be the radius vector of a particle from some given origin and v its vector velocity:

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}. \tag{1.1}$$

The *linear momentum* $\boldsymbol{p}$ of the particle is defined as the product of the particle mass and its velocity:

$$\boldsymbol{p} = m\boldsymbol{v} \tag{1.2}$$

In consequence of interactions with external objects and fields, the particle may experience forces of various types, e.g., gravitational or electrodynamic; the vector sum of these forces exerted on the particle is the total force $\boldsymbol{F}$. The mechanics of the particle is contained in *Newton's second law of motion*, which states that there exist frames of reference in which the motion of the particle is described by the differential equation

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \equiv \dot{\boldsymbol{p}}, \tag{1.3}$$

or

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\boldsymbol{v}\right). \tag{1.4}$$

In most instances, the mass of the particle is constant and Eq. (1.4) reduces to

$$\boldsymbol{F} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\boldsymbol{a}, \tag{1.5}$$

where $\boldsymbol{a}$ is the vector acceleration of the particle defined by

$$\boldsymbol{a} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{r}. \tag{1.6}$$

The equation of motion is thus a differential equation of second order, assuming $\boldsymbol{F}$ does not depend on higher-order derivatives.

A reference frame in which Eq. (1.3) is valid is called an *inertial* or *Galilean system*. Even within classical mechanics the notion of an inertial system is something of an idealization. In practice, however, it is usually feasible to set up a coordinate system that comes as close to the desired properties as may be required. For many purposes, a reference frame fixed in Earth (the "laboratory system") is a sufficient approximation to an inertial system, while for some astronomical purposes it may be necessary to construct an inertial system (or inertial frame) by reference to distant galaxies.

Many of the important conclusions of mechanics can be expressed in the form of conservation theorems, which indicate under what conditions various mechanical quantities are constant in time. Equation (1.3) directly furnishes the first of these, the

THEOREM 1.1 (Conservation Theorem for the Linear Momentum of a Particle). *If the total force, $\boldsymbol{F}$, is zero, then $\dot{\boldsymbol{p}} = 0$ and the linear momentum, $\boldsymbol{p}$, is conserved.*

The angular momentum of the particle about point O, denoted by $\boldsymbol{L}$, is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}, \tag{1.7}$$

where $\boldsymbol{r}$ is the radius vector from O to the particle. Notice that the order of the factors is important. We now define the *moment of force* or *torque* about O as

$$\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F}. \tag{1.8}$$

The equation analogous to (1.3) for $\boldsymbol{N}$ is obtained by forming the cross product of $\boldsymbol{r}$ with Eq. (1.4):

$$\boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{N} = \boldsymbol{r} \times \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}). \tag{1.9}$$

Equation (1.9) can be written in a different form by using the vector identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times m\boldsymbol{v}) = \boldsymbol{v} \times m\boldsymbol{v} + \boldsymbol{r} \times \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}), \tag{1.10}$$

where the first term on the right obviously vanishes. In consequence of this identity, Eq. (1.9) takes the form

$$\boldsymbol{N} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times m\boldsymbol{v}) = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} \equiv \dot{\boldsymbol{L}}. \tag{1.11}$$

Note that both $\boldsymbol{N}$ and $\boldsymbol{L}$ depend on the point O, about which the moments are taken.

As was the case for Eq. (1.3), the torque equation, (1.11), also yields an immediate conservation theorem, this time the

THEOREM 1.2 (Conservation Theorem for the Angular Momentum of a Particle). *If the total torque, $\boldsymbol{N}$, is zero then $\dot{\boldsymbol{L}} = 0$, and the angular momentum $\boldsymbol{L}$ is conserved.*

Next consider the work done by the external force $\boldsymbol{F}$ upon the particle in going from point 1 to point 2. By definition, this work is

$$W_{12} = \int_1^2 \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}. \tag{1.12}$$

For constant mass (as will be assumed from now on unless otherwise specified), the integral in Eq. (1.12) reduces to

$$\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = m \int \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \boldsymbol{v}\, \mathrm{d}t = \frac{m}{2} \int \frac{\mathrm{d}}{\mathrm{d}t}(v^2)\, \mathrm{d}t,$$

and therefore

$$W_{12} = \frac{m}{2}\left(v_2^2 - v_1^2\right). \tag{1.13}$$

The scalar quantity $mv^2/2$ is called the kinetic energy of the particle and is denoted by T, so that the work done is equal to the change in the kinetic energy:

$$W_{12} = T_2 - T_1. \tag{1.14}$$

If the force field is such that the work W_{12} is the same for any physically possible path between points 1 and 2, then the force (and the system) is said to be *conservative*. An alternative description of a conservative system is obtained by imagining the particle being taken from point 1 to point 2 by one possible path and then being returned to point 1 by another path. The independence of W_{12} on the particular path implies that the work done around such a closed circuit is zero, i.e.:

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = 0. \tag{1.15}$$

Physically it is clear that a system cannot be conservative if friction or other dissipation forces are present, because $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$ due to friction is always positive and the integral cannot vanish.

By a well-known theorem of vector analysis, a necessary and sufficient condition that the work, W_{12}, be independent of the physical path taken by the particle is that $\boldsymbol{F}$ be the gradient of some scalar function of position:

$$\boldsymbol{F} = -\boldsymbol{\nabla} V(\boldsymbol{r}), \tag{1.16}$$

where V is called the *potential*, or *potential energy*. The existence of V can be inferred intuitively by a simple argument. If W_{12} is independent of the path of integration between the end points 1 and 2, it should be possible to express W_{12} as the change in a quantity that depends only upon the positions of the end points. This quantity may be designated by $-V$, so that for a differential path length we have the relation

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = -\,\mathrm{d}V$$

or

$$F_s = -\frac{\partial V}{\partial s},$$

which is equivalent to Eq. (1.16). Note that in Eq. (1.16) we can add to V any quantity constant in space, without affecting the results. Hence *the zero level of V is arbitrary.*

For a conservative system, the work done by the forces is

$$W_{12} = V_1 - V_2. \tag{1.17}$$

Combining Eq. (1.17) with Eq. (1.14), we have the result

$$T_1 + V_1 = T_2 + V_2, \tag{1.18}$$

which states in symbols the

THEOREM 1.3 (Energy Conservation Theorem for a Particle). *If the forces acting on a particle are conservative, then the total energy of the particle, $T + V$, is conserved.*

The force applied to a particle may in some circumstances be given by the gradient of a scalar function that depends explicitly on both the position of the particle and the time. However, the work done on the particle when it travels a distance $\mathrm{d}s$,

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = -\frac{\partial V}{\partial s}\,\mathrm{d}s,$$

is then no longer the total change in $-V$ during the displacement, since V also changes explicitly with time as the particle moves. Hence, the work done as the particle goes from point 1 to point 2 is no longer the difference in the function V between those points. While a total energy $T + V$ may still be defined, it is not conserved during the course of the particle's motion.

1.2. Mechanics of a System of Particles

In generalizing the ideas of the previous section to systems of many particles, we must distinguish between the *external forces* acting on the particles due to sources outside the system, and *internal forces* on, say, some particle i due to all other particles in the system. Thus, the equation of motion (Newton's second law) for the ith particle is written as

$$\sum_j \boldsymbol{F}_{ji} + \boldsymbol{F}_i^{(e)} = \dot{\boldsymbol{p}}_i, \tag{1.19}$$

where $\boldsymbol{F}_i^{(e)}$ stands for an external force, and $\boldsymbol{F}_{ji}$ is the internal force on the ith particle due to the jth particle ($\boldsymbol{F}_{ii}$ naturally, is zero). We shall assume that the $\boldsymbol{F}_{ij}$ (like the $\boldsymbol{F}_i^{(e)}$) obey Newton's third law of motion in its original form: that the forces two particles exert on each other are equal and opposite. This assumption (which does not hold for all types of forces) is sometimes referred to as the *weak law of action and reaction.*

Summed over all particles, Eq. (1.19) takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_i m_i \boldsymbol{r}_i = \sum_i \boldsymbol{F}_i^{(e)} + \sum_{\substack{i,\,j \\ i \neq j}} \boldsymbol{F}_{ji}. \tag{1.20}$$

The first sum on the right is simply the total external force $\boldsymbol{F}^{(e)}$, while the second term vanishes, since the law of action and reaction states that each pair $\boldsymbol{F}_{ij} + \boldsymbol{F}_{ji}$ is zero. To reduce the left-hand side, we define a vector $\boldsymbol{R}$ as the average of the radius vectors of the particles, weighted in proportion to their mass:

$$\boldsymbol{R} = \frac{\sum m_i \boldsymbol{r}_i}{\sum m_i} = \frac{\sum m_i \boldsymbol{r}_i}{M}. \tag{1.21}$$

The vector $\boldsymbol{R}$ defines a point known as the *center of mass*, or more loosely as the center of gravity, of the system (cf. Fig. 1.1). With this definition, (1.20) reduces to

$$M \frac{\mathrm{d}^2 \boldsymbol{R}}{\mathrm{d}t^2} = \sum_i \boldsymbol{F}_i^{(e)} \equiv \boldsymbol{F}^{(e)}, \tag{1.22}$$

which states that the center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. Purely internal forces, if they obey Newton's third law, therefore have no effect on the motion of the center of mass. An oft-quoted example is the motion of an exploding shell—the center of mass of the fragments traveling as if the shell were still in a single piece (neglecting air resistance). The same principle is involved in jet and rocket propulsion. In order that the motion of the center of mass be unaffected, the ejection of the exhaust gases at high velocity must be counterbalanced by the forward motion of the vehicle at a slower velocity.

By Eq. (1.21) the total linear momentum of the system,

$$\boldsymbol{P} = \sum m_i \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = M \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t}, \tag{1.23}$$

is the total mass of the system times the velocity of the center of mass. Consequently, the equation of motion for the center of mass, (1.23), can be restated as the

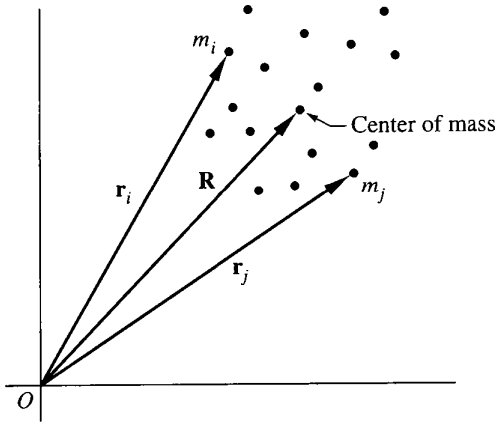

FIGURE 1.1. The center of mass of a system of particles.

THEOREM 1.4 (Conservation Theorem for the Linear Momentum of a System of Particles). *If the total external force is zero, the total linear momentum is conserved.*

We obtain the total angular momentum of the system by forming the cross product $\boldsymbol{r}_i \times \boldsymbol{p}_i$ and summing over i. If this operation is performed in Eq. (1.19), there results, with the aid of the identity, Eq. (1.10),

$$\sum_i (\boldsymbol{r}_i \times \dot{\boldsymbol{p}}_i) = \sum_i \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{r}_i \times \boldsymbol{p}_i) = \dot{\boldsymbol{L}} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \boldsymbol{r}_i \times \boldsymbol{F}_{ji}. \tag{1.24}$$

The last term on the right in (1.24) can be considered a sum of the pairs of the form

$$\boldsymbol{r}_i \times \boldsymbol{F}_{ji} + \boldsymbol{r}_j \times \boldsymbol{F}_{ij} = (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ji}, \tag{1.25}$$

using the equality of action and reaction. But $\boldsymbol{r}_i - \boldsymbol{r}_j$ is identical with the vector $\boldsymbol{r}_{ij}$ from j to i (cf. Fig. 1.2), so that the right-hand side of Eq. (1.25) can be written as

$$\boldsymbol{r}_{ij} \times \boldsymbol{F}_{ji}.$$

If the internal forces between two particles, in addition to being equal and opposite, also lie along the line joining the particles—a condition known as the *strong law of action and reaction*—then all of these cross products vanish. The sum over pairs is zero under this assumption and Eq. (1.24) may be written in the form

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{N}^{(e)}. \tag{1.26}$$

The time derivative of the total angular momentum is thus equal to the moment of the external force about the given point. Corresponding to Eq. (1.26) is the

THEOREM 1.5 (Conservation Theorem for Total Angular Momentum). $\boldsymbol{L}$ *is constant in time if the applied (external) torque is zero.*

(It is perhaps worthwhile to emphasize that this is a *vector* theorem; i.e., L_z will be conserved if $N_z^{(e)}$ is zero, even if $N_x^{(e)}$ and $N_y^{(e)}$ are not zero.)

Note that the conservation of linear momentum in the absence of applied forces assumes that the weak law of action and reaction is valid for the internal forces. The conservation of the total angular momentum of the system in the absence of applied torques requires the validity of the strong law of action and reaction—that the internal forces in addition be *central*. Many of the familiar physical forces, such as that of gravity, satisfy the strong form of the law. But it is possible to find forces for which action and reaction are equal even though the forces are not central

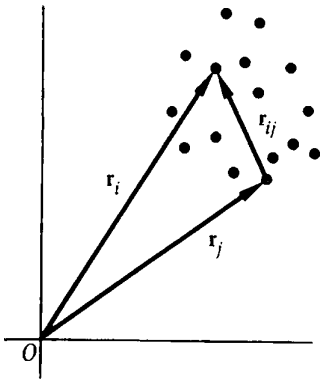

FIGURE 1.2. The vector $\boldsymbol{r}_{ij}$ between the ith and jth particles.

(see below). In a system involving moving charges, the forces between the charges predicted by the Biot-Savart law may indeed violate both forms of the action and reaction law.[1] Equations (1.23) and (1.26), and their corresponding conservation theorems, are not applicable in such cases, at least in the form given here. Usually it is then possible to find some generalization of $\boldsymbol{P}$ or $\boldsymbol{L}$ that is conserved. Thus, in an isolated system of moving charges it is the sum of the mechanical angular momentum and the electromagnetic "angular momentum" of the field that is conserved.

Equation (1.23) states that the total linear momentum of the system 1s the same as if the entire mass were concentrated at the center of mass and moving with it. The analogous theorem for angular momentum is more complicated. With the origin O as reference point, the total angular momentum of the system is

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i.$$

Let $\boldsymbol{R}$ be the radius vector from O to the center of mass, and let $\boldsymbol{r}_i'$ be the radius vector from the center of mass to the ith particle. Then we have (cf. Fig. 1.3)

$$\boldsymbol{r}_i = \boldsymbol{r}_i' + \boldsymbol{R} \tag{1.27}$$

and

$$\boldsymbol{v}_i = \boldsymbol{v}_i' + \boldsymbol{v}$$

where

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t}$$

is the velocity of the center of mass relative to O, and

$$\boldsymbol{v}_i' = \frac{\mathrm{d}\boldsymbol{r}_i'}{\mathrm{d}t}$$

is the velocity of the ith particle relative to the center of mass of the system. Using Eq. (1.27), the total angular momentum takes on the form

$$\boldsymbol{L} = \sum_i \boldsymbol{R} \times m_i \boldsymbol{v} + \sum_i \boldsymbol{r}_i' \times m_i \boldsymbol{v}_i' + \left(\sum_i m_i \boldsymbol{r}_i' \right) \times \boldsymbol{v} + \boldsymbol{R} \times \frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i \boldsymbol{r}_i'.$$

[1] If two charges are moving uniformly with parallel velocity vectors that are not perpendicular to the line joining the charges, then the net mutual forces are equal and opposite but do not lie along the vector between the charges. Consider, further, two charges moving (instantaneously) so as to "cross the T," i.e., one charge moving directly at the other, which in turn is moving at right angles to the first. Then the second charge exerts a nonvanishing magnetic force on the first, without experiencing any magnetic reaction force at that instant.

The last two terms in this expression vanish, for both contain the factor $\sum m_i \boldsymbol{r}'_i$, which, it will be recognized, defines the radius vector of the center of mass in the very coordinate system whose origin is the center of mass and is therefore a null vector. Rewriting the remaining terms, the total angular momentum about O is

$$\boldsymbol{L} = \boldsymbol{R} \times M\boldsymbol{v} + \sum_i \boldsymbol{r}'_i \times \boldsymbol{p}'_i. \tag{1.28}$$

In words, Eq. (1.28) says that the total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass, plus the angular momentum of motion about the center of mass. The form of Eq. (1.28) emphasizes that in general $\boldsymbol{L}$ depends on the origin O, through the vector $\boldsymbol{R}$. Only if the center of mass is at rest with respect to O will the angular momentum be independent of the point of reference. In this case, the first term in (1.28) vanishes, and $\boldsymbol{L}$ always reduces to the angular momentum taken about the center of mass.

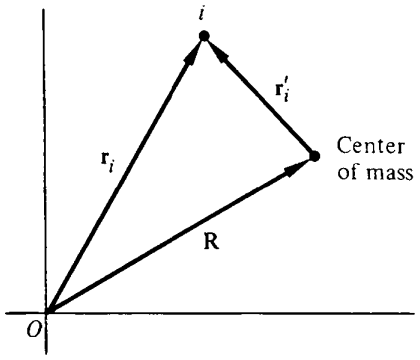

FIGURE 1.3. The vectors involved in the shift of reference point for the angular momentum.

Finally, let us consider the energy equation. As in the case of a single particle, we calculate the work done by all forces in moving the system from an initial configuration 1, to a final configuration 2:

$$W_{12} = \sum_i \int_1^2 \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{s}_i = \sum_i \int_1^2 \boldsymbol{F}_i^{(e)} \cdot \mathrm{d}\boldsymbol{s}_i + \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \boldsymbol{F}_{ji} \cdot \mathrm{d}\boldsymbol{s}_i. \tag{1.29}$$

Again, the equations of motion can be used to reduce the integrals to

$$\sum_i \int_1^2 \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{s}_i = \sum_i \int_1^2 m_i \dot{\boldsymbol{v}}_i \cdot \boldsymbol{v}_i \, \mathrm{d}t = \sum_i \int_1^2 \mathrm{d}\left(\frac{1}{2} m_i v_i^2\right).$$

Hence, the work done can still be written as the difference of the final and initial kinetic energies:

$$W_{12} = T_2 - T_1,$$

where T, the total kinetic energy of the system, is

$$T = \frac{1}{2} \sum_i m_i v_i^2. \tag{1.30}$$

Making use of the transformations to center-of-mass coordinates, given in Eq. (1.27), we may also write T as

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i\,(\boldsymbol{v}+\boldsymbol{v}_i')\cdot(\boldsymbol{v}+\boldsymbol{v}_i') \\ &= \frac{1}{2}\sum_i m_i v^2 + \frac{1}{2}\sum_i m_i v_i'^2 + \boldsymbol{v}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i m_i \boldsymbol{r}_i'\right), \end{aligned}$$

and by the reasoning already employed in calculating the angular momentum, the last term vanishes, leaving

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}\sum_i m_i v_i'^2. \tag{1.31}$$

The kinetic energy, like the angular momentum, thus also consists of two parts: the kinetic energy obtained if all the mass were concentrated at the center of mass, plus the kinetic energy of motion about the center of mass.

Consider now the right-hand side of Eq. (1.29). In the special case that the external forces are derivable in terms of the gradient of a potential, the first term can be written as

$$\sum_i \int_1^2 \boldsymbol{F}_i^{(e)}\cdot\mathrm{d}\boldsymbol{s}_i = -\sum_i \int_1^2 \boldsymbol{\nabla}_i V_i\cdot\mathrm{d}\boldsymbol{s}_i = -\sum_i V_i\Big|_1^2,$$

where the subscript i on the del operator indicates that the derivatives are with respect to the components of $\boldsymbol{r}_i$. If the internal forces are also conservative, then the mutual forces between the ith and jth particles, $\boldsymbol{F}_{ij}$ and $\boldsymbol{F}_{ji}$, can be obtained from a potential function V_{ij}. To satisfy the strong law of action and reaction, V_{ij} can be a function only of the distance between the particles:

$$V_{ij} = V_{ij}\left(|\boldsymbol{r}_i - \boldsymbol{r}_j|\right). \tag{1.32}$$

The two forces are then automatically equal and opposite,

$$\boldsymbol{F}_{ji} = -\boldsymbol{\nabla}_i V_{ij} = +\boldsymbol{\nabla}_j V_{ij} = -\boldsymbol{F}_{ij}, \tag{1.33}$$

and lie along the line joining the two particles,

$$\boldsymbol{\nabla} V_{ij}\left(|\boldsymbol{r}_i - \boldsymbol{r}_j|\right) = (\boldsymbol{r}_i - \boldsymbol{r}_j)\,f, \tag{1.34}$$

where f is some scalar function. If V_{ij} were also a function of the difference of some other pair of vectors associated with the particles, such as their velocities or (to step into the domain of modern physics) their intrinsic "spin" angular momentum, then the forces would still be equal and opposite, but would not necessarily lie along the direction between the particles.

When the forces are all conservative, the second term in Eq. (1.29) can be rewritten as a sum over *pairs* of particles, the terms for each pair being of the form

$$-\int_1^2 \left(\boldsymbol{\nabla}_i V_{ij}\cdot\mathrm{d}\boldsymbol{s}_i + \boldsymbol{\nabla}_j V_{ij}\cdot\mathrm{d}\boldsymbol{s}_j\right).$$

If the difference vector $\boldsymbol{r}_i - \boldsymbol{r}_j$ is denoted by $\boldsymbol{r}_{ij}$, and if $\boldsymbol{\nabla}_{ij}$ stands for the gradient with respect to $\boldsymbol{r}_{ij}$, then

$$\boldsymbol{\nabla}_i V_{ij} = \boldsymbol{\nabla}_{ij} V_{ij} = -\boldsymbol{\nabla}_j V_{ij},$$

and

$$\mathrm{d}\boldsymbol{s}_i - \mathrm{d}\boldsymbol{s}_j = \mathrm{d}\boldsymbol{r}_i - \mathrm{d}\boldsymbol{r}_j = \mathrm{d}\boldsymbol{r}_{ij},$$

so that the term for the ij pair has the form

$$-\int \boldsymbol{\nabla}_{ij} V_{ij}\cdot\mathrm{d}\boldsymbol{r}_{ij}.$$

The total work arising from internal forces then reduces to

$$-\frac{1}{2}\sum_{\substack{i,\,j\\ i\neq j}}\int_1^2 \boldsymbol{\nabla}_{ij}V_{ij}\cdot \mathrm{d}\boldsymbol{r}_{ij} = -\frac{1}{2}\sum_{\substack{i,\,j\\ i\neq j}} V_{ij}\Bigg|_1^2. \tag{1.35}$$

The factor $\frac{1}{2}$ appears in Eq. (1.35) because in summing over *both* i and j each member of a given pair is included twice, first in the i summation and then in the j summation.

From these considerations, it is clear that if the external and internal forces are both derivable from potentials it is possible to define a *total potential energy*, V, of the system,

$$V = \sum_i V_i + \frac{1}{2}\sum_{\substack{i,\,j\\ i\neq j}} V_{ij}, \tag{1.36}$$

such that the total energy $T+V$ is conserved, the analog of the conservation theorem (1.18) for a single particle.

The second term on the right in Eq. (1.36) will be called the internal potential energy of the system. In general, it need not be zero and, more important, it may vary as the system changes with time. Only for the particular class of systems known as *rigid bodies* will the internal potential always be constant. Formally, a rigid body can be defined as a system of particles in which the distances r_{ij} are fixed and cannot vary with time. In such case, the vectors $\mathrm{d}\boldsymbol{r}_{ij}$ can only be perpendicular to the corresponding $\boldsymbol{r}_{ij}$, and therefore to the $\boldsymbol{F}_{ij}$. Therefore, in a rigid body the *internal forces do no work*, and the internal potential must remain constant. Since the total potential is in any case uncertain to within an additive constant, an unvarying internal potential can be completely disregarded in discussing the motion of the system.

1.3. Constraints

From the previous sections one might obtain the impression that all problems in mechanics have been reduced to solving the set of differential equations (1.19):

$$m_i\ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i^{(e)} + \sum_j \boldsymbol{F}_{ji}.$$

One merely substitutes the various forces acting upon the particles of the system, turns the mathematical crank, and grinds out the answers! Even from a purely physical standpoint, however, this view is oversimplified. For example, it may be necessary to take into account the *constraints* that limit the motion of the system. We have already met one type of system involving constraints, namely rigid bodies, where the constraints on the motions of the particles keep the distances r_{ij} unchanged. Other examples of constrained systems can easily be furnished. The beads of an abacus are constrained to one-dimensional motion by the supporting wires. Gas molecules within a container are constrained by the walls of the vessel to move only *inside* the container. A particle placed on the surface of a solid sphere is subject to the constraint that it can move only on the surface or in the region exterior to the sphere.

Constraints may be classified in various ways, and we shall use the following system. If the conditions of constraint can be expressed as equations connecting the coordinates of the particles (and possibly the time) having the form

$$f(\boldsymbol{r}_1,\, \boldsymbol{r}_2,\, \boldsymbol{r}_3,\, \ldots,\, t) = 0, \tag{1.37}$$

then the constraints are said to be *holonomic*. Perhaps the simplest example of holonomic constraints is the rigid body, where the constraints are expressed by equations of the form

$$\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)^2 - c_{ij}^2 = 0.$$

A particle constrained to move along any curve or on a given surface is another obvious example of a holonomic constraint, with the equations defining the curve or surface acting as the equations of a constraint.

Constraints not expressible in this fashion are called nonholonomic. The walls of a gas container constitute a nonholonomic constraint. The constraint involved in the example of a particle placed on the surface of a sphere is also nonholonomic, for it can be expressed as an inequality

$$r^2 - a^2 \geqslant 0$$

(where a is the radius of the sphere), which is not in the form of (1.37). Thus, in a gravitational field a particle placed on the top of the sphere will slide down the surface part of the way but will eventually fall off.

Constraints are further classified according to whether the equations of constraint contain the time as an explicit variable (rheonomous) or are not explicitly dependent on time (scleronomous). A bead sliding on a rigid curved wire fixed in space is obviously subject to a scleronomous constraint; if the wire is moving in some prescribed fashion, the constraint is rheonomous. Note that if the wire moves, say, as a reaction to the bead's motion, then the time dependence of the constraint enters in the equation of the constraint only through the coordinates of the curved wire (which are now part of the system coordinates). The overall constraint is then scleronomous.

Constraints introduce two types of difficulties in the solution of mechanical problems. First, the coordinates r_i are no longer all independent, since they are connected by the equations of constraint; hence the equations of motion (1.19) are not all independent. Second, the forces of constraint, e.g., the force that the wire exerts on the bead (or the wall on the gas particle), is not furnished a priori. They are among the unknowns of the problem and must be obtained from the solution we seek. Indeed, imposing constraints on the system is simply another method of stating that there are forces present in the problem that cannot be specified directly but are known rather in terms of their effect on the motion of the system.

In the case of holonomic constraints, the first difficulty is solved by the introduction of *generalized coordinates*. So far we have been thinking implicitly in terms of Cartesian coordinates. A system of N particles, free from constraints, has $3N$ independent coordinates or *degrees of freedom*. If there exist holonomic constraints, expressed in k equations in the form (1.37), then we may use these equations to eliminate k of the $3N$ coordinates, and we are left with $3N - k$ independent coordinates, and the system is said to have $3N - k$ degrees of freedom. This elimination of the dependent coordinates can be expressed in another way, by the introduction of new, $3N - k$, independent variables $q_1, q_2, \ldots, q_{3N-k}$ in terms of which the old coordinates $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N$ are expressed by equations of the form

$$\begin{aligned} \boldsymbol{r}_1 &= \boldsymbol{r}_1\left(q_1, q_2, \ldots, q_{3N-k}, t\right), \\ &\vdots \\ \boldsymbol{r}_N &= \boldsymbol{r}_N\left(q_1, q_2, \ldots, q_{3N-k}, t\right) \end{aligned} \tag{1.38}$$

containing the constraints in them implicitly. These are *transformation* equations from the set of $(\boldsymbol{r}_l)$ variables to the (q_l) set, or alternatively Eqs. (1.38) can be considered as parametric representations of the $(\boldsymbol{r}_l)$ variables. It is always assumed that we can also transform back from the (q_l) to the $(\boldsymbol{r}_l)$ set, i.e., that Eqs. (1.38)

combined with the k equations of constraint can be inverted to obtain any q_i as a function of the $(\boldsymbol{r}_l)$ variable and time.

Usually the generalized coordinates, q_l, unlike the Cartesian coordinates, will not divide into convenient groups of three that can be associated together to form vectors. Thus, in the case of a particle constrained to move *on* the surface of a sphere, the two angles expressing position on the sphere, say latitude and longtude, are obvious possible generalized coordinates. Or, in the example of a double pendulum moving in a plane (two particles connected by an inextensible light rod and suspended by a similar rod fastened to one of the particles), satisfactory generalized coordinates are the two angles θ_1, θ_2. (Cf. Fig. 1.4.) Generalized coordinates, in the sense of coordinates other than Cartesian, are often useful in systems without constraints. Thus, in the problem of a particle moving in an external central force field ($V = V(r)$), there is no constraint involved, but it is clearly more convenient to use spherical polar coordinates than Cartesian coordinates. Do not, however, think of generalized coordinates in terms of conventional orthogonal position coordinates. All sorts of quantities may be invoked to serve as generalized coordinates. Thus, the amplitudes in a Fourier expansion of $\boldsymbol{r}_j$ may be used as generalized coordinates, or we may find it convenient to employ quantities with the dimensions of energy or angular momentum.

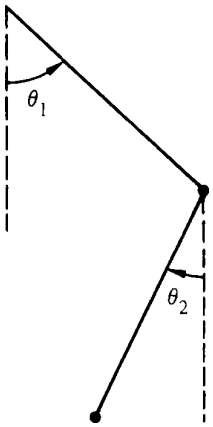

FIGURE 1.4. Double pendulum.

If the constraint is nonholonomic, the equations expressing the constraint cannot be used to eliminate the dependent coordinates. An oft-quoted example of a nonholonomic constraint is that of an object rolling on a rough surface without slipping. The coordinates used to describe the system will generally involve angular coordinates to specify the orientation of the body, plus a set of coordinates describing the location of the point of contact on the surface. The constraint of "rolling" connects these two sets of coordinates; they are not independent. A change in the position of the point of contact inevitably means a change in its orientation. Yet we cannot reduce the number of coordinates, for the "rolling" condition is not expressible as a equation between the coordinates, in the manner of (1.37). Rather, it is a condition on the *velocities* (i.e., the point of contact is stationary), a differential condition that can be given in an integrated form only *after* the problem is solved.

A simple case will illustrate the point. Consider a disk rolling on the horizontal xy plane constrained to move so that the plane of the disk is always vertical. The coordinates used to describe the motion might be the x, y coordinates of the center of the disk, an angle of rotation ϕ about the axis of the disk, and an angle θ between

the axis of the disk and say, the x axis (cf. Fig 1.5). As a result of the constraint the velocity of the center of the disk, $\boldsymbol{v}$, has a magnitude proportional to $\dot{\phi}$,

$$v = a\dot{\phi},$$

where a is the radius of the disk, and its direction is perpendicular to the axis of the disk:

$$\dot{x} = v\sin\theta,$$
$$\dot{y} = -v\cos\theta.$$

Combining these conditions, we have two *differential* equations of constraint:

$$\begin{aligned} \mathrm{d}x - a\sin\theta\,\mathrm{d}\phi &= 0, \\ \mathrm{d}y + a\cos\theta\,\mathrm{d}\phi &= 0. \end{aligned} \tag{1.39}$$

Neither of Eqs. (1.39) can be integrated without in fact solving the problem; i.e., we cannot find an integrating factor $f(x, y, \theta, \phi)$ that will turn either of the equations into perfect differentials (cf. Derivation 4).[2] Hence, the constraints cannot be reduced to the form of Eq. (1.37) and are therefore nonholonomic. Physically we can see that there can be no direct functional relation between ϕ and the other coordinates x, y, and θ by noting that at any point on its path the disk can be made to roll around in a circle tangent to the path and of arbitrary radius. At the end of the process, x, y, and θ have been returned to their original values, but ϕ has changed by an amount depending on the radius of the circle.

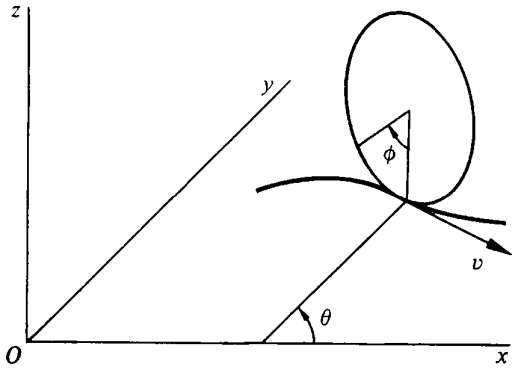

FIGURE 1.5. Vertical disk rolling on a horizontal plane.

Nonintegrable *differential* constraints of the form of Eqs. (1.39) are of course not the only type of nonholonomic constraints. The constraint conditions may involve higher-order derivatives, or may appear in the form of inequalities, as we have seen.

Partly because the dependent coordinates can be eliminated, problems involying holonomic constraints are always amenable to a formal solution. But there is no general way to attack nonholonomic examples. True, if the constraint is nonintegrable, the differential equations of constraint can be introduced into the problem along with the differential equations of motion, and the dependent equations eliminated, in effect, by the method of Lagrange multipliers.

We shall return to this method at a later point. However, the more vicious cases of nonholonomic constraint must be tackled individually, and consequently in the development of the more formal aspects of classical mechanics, it is almost invariably assumed that any constraint, if present, is holonomic. This restriction

[2]In principle, an integrating factor can always be found for a first-order differential equation of constraint in systems involving only two coordinates and such constraints are therefore holonomic. A familiar example is the two-dimensional motion of a circle rolling on an inclined plane.

does not greatly limit the applicability of the theory, despite the fact that many of the constraints encountered in everyday life are nonholonomic. The reason is that the entire concept of constraints imposed in the system through the medium of wires or surfaces or walls is particularly appropriate only in macroscopic or large-scale problems. But today physicists are more interested in atomic and nuclear problems. On this scale all objects, both in and out of the system, consist alike of molecules, atoms, or smaller particles, exerting definite forces, and the notion of constraint becomes artificial and rarely appears. Constraints are then used only as mathematical idealizations to the actual physical case or as classical approximations to a quantum-mechanical property, e.g., rigid body rotations for "spin." Such constraints are always holonomic and fit smoothly into the framework of the theory.

To surmount the second difficulty, namely, that the forces of constraint are unknown a priori, we should like to so formulate the mechanics that the forces of constraint disappear. We need then deal only with the known applied forces. A hint as to the procedure to be followed is provided by the fact that in a particular system with constraints, i.e., a rigid body, the work done by internal forces (which are here the forces of constraint) vanishes. We shall follow up this clue in the ensuing sections and generalize the ideas contained in it.

Derivations

(1) Show that for a single particle with constant mass the equation of motion implies the following differential equation for the kinetic energy:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \boldsymbol{F} \cdot \boldsymbol{v},$$

while if the mass varies with time the corresponding equation is

$$\frac{\mathrm{d}\,(mT)}{\mathrm{d}t} = \boldsymbol{F} \cdot \boldsymbol{p}.$$

(2) Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,\,j} m_i m_j r_{ij}^2.$$

(3) Suppose a system of two particles is known to obey the equations of motion, Eqs. (1.22) and (1.26). From the equations of the motion of the individual particles show that the internal forces between particles satisfy both the weak and the strong laws of action and reaction. The argument may be generalized to a system with arbitrary number of particles, thus proving the converse of the arguments leading to Eqs. (1.22) and (1.26).

(4) The equations of constraint for the rolling disk, Eqs. (1.39), are special cases of gen- eral linear differential equations of constraint of the form

$$\sum_{i=1}^{n} g_i\,(x_1,\,\ldots,\,x_n)\,\mathrm{d}x_i = 0.$$

A constraint condition of this type is holonomic only if an integrating function $f\,(x_1,\,\ldots,\,x_n)$ can be found that turns it into an exact differential. Clearly the function must be such that

$$\frac{\partial\,(fg_i)}{\partial x_j} = \frac{\partial\,(fg_j)}{\partial x_i}$$

for all $i \neq j$. Show that no such integrating factor can be found for either of Eqs. (1.39).

(5) Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane. Show that there are two nonholonomic equations of constraint,

$$\cos\theta\,\mathrm{d}x + \sin\theta\,\mathrm{d}y = 0,$$
$$\sin\theta\,\mathrm{d}x - \cos\theta\,\mathrm{d}y = \frac{1}{2}a\left(\mathrm{d}\phi + \mathrm{d}\phi'\right),$$

(where θ, ϕ, and ϕ' have meanings similar to those in the problem of a single vertical disk, and $(x,\, y)$ are the coordinates of a point on the axle midway between the two wheels) and one holonomic equation of constraint,

$$\theta = C - \frac{a}{b}\left(\phi - \phi'\right).$$

where C is a constant.

(6) A particle moves in the xy plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is nonholonomic.

(7) Show that Lagrange's equations in the form of Eqs. (**??**) can also be written as

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j.$$

These are sometimes known as the *Nielsen* form of the Lagrange equations.

(8) If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{\mathrm{d}F(q_1,\, \ldots,\, q_n,\, t)}{\mathrm{d}t}$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments.

(9) The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\psi\left(\boldsymbol{r},\, \boldsymbol{t}\right),$$
$$\phi \to \phi - \frac{1}{c}\frac{\partial \psi}{\partial t},$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected?

(10) Let $q_1,\, \ldots,\, q_n$ be a set of independent generalized coordinates for a system of n degrees of freedom, with a Lagrangian $L\left(q,\, \dot{q},\, t\right)$. Suppose we transform to another set of independent coordinates $s_1,\, \ldots,\, s_n$ by means of transformation equations

$$q_i = q_i\left(s_1,\, \ldots,\, s_n,\, t\right), \qquad i = 1,\, \ldots,\, n.$$

(Such a transformation is called a *point transformation.*) Show that if the Lagrangian function is expressed as a function of s_j, $\dot{s}_j$, and t through the equations of transformation, then L satisfies Lagrange's equations with respect to the s coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \frac{\partial L}{\partial s_j} = 0.$$

In other words, the form of Lagrange's equations is invariant under a point transformation.