CHAPTER 1

The Real and Complex Number Systems

CHAPTER 2

Basic Topology

2.1. Finite, Countable, and Uncountable Sets

We begin this section with a definition of the function concept.

DEFINITION 2.1. Consider two sets A and B, whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B, which we denote by $f(x)$. Then f is said to be a *function* from A to B (or a *mapping* of A into B). The set A is called the *domain* of f (we also say f is defined on A), and the elements $f(x)$ are called the *values* of f. The set of all values of f is called the *range* of f.

DEFINITION 2.2. Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$. We call $f(E)$ the *image* of E under f. In this notation, $f(A)$ is the range of f. It is clear that $f(A) \subset B$. If $f(A) = B$, we say that f maps A *onto* B. (Note that, according to this usage, *onto* is more specific than *into*.)

If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the *inverse image* of E under f. If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$. If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A, then f is said to be a 1-1 (*one-to-one*) mapping of A into B. This may also be expressed as follows: f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1 \in A$, $x_2 \in A$.

(The notation $x_1 \neq x_2$ means that x_1 and x_2 are distinct elements; otherwise we write $x_1 = x_2$.)

DEFINITION 2.3. If there exists a 1-1 mapping of A *onto* B, we say that A and B, can be put in 1-1 *correspondence*, or that A and B have the same *cardinal number*, or briefly, that A and B are *equivalent*, and we write $A \sim B$. This relation clearly has the following properties:

- It is reflexive: $A \sim A$.
- It is symmetric: If $A \sim B$, then $B \sim A$.
- It is transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.

Any relation with these three properties is called an *equivalence relation*.

DEFINITION 2.4. For any positive integer n, let J_n be the set whose elements are the integers 1, 2, ..., n; let J be the set consisting of all positive integers. For any set A, we say:

(1) A is *finite* if $A \sim J_n$ for some n (the empty set is also considered to be finite).
(2) A is *infinite* if A is not finite.
(3) A is *countable* if $A \sim J$.
(4) A is *uncountable* if A is neither finite nor countable.
(5) A is *at most countable* if A is finite or countable.

Countable sets are sometimes called *enumerable*, or *denumerable*.

For two finite sets A and B, we evidently have $A \sim B$ if and only if A and B contain the same number of elements. For infinite sets, however, the idea of "having the same number of elements" becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

EXAMPLE 2.1. Let A be the set of all integers. Then A is countable. For, consider the following arrangement of the sets A and J:

$$\begin{array}{ll} A: & 0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3,\ \ldots \\ J: & 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots \end{array}$$

We can, in this example, even give an explicit formula for a function f from J to A which sets up a 1-1 correspondence:

$$f(n) = \begin{cases} \frac{n}{2} & (n \text{ even}), \\ -\frac{n-1}{2} & (n \text{ odd}). \end{cases}$$

REMARK 2.1. A finite set cannot be equivalent to one of its proper subsets. That this is, however, possible for infinite sets, is shown by Example 2.1, in which J is a proper subset of A.

In fact, we could replace Definition 2.4(2) by the statement: A is infinite if A is equivalent to one of its proper subsets.

DEFINITION 2.5. By a *sequence*, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by x_1, x_2, x_3, ... The values of f, that is, the elements x_n, are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a *sequence in* A, or a *sequence of elements of* A.

Note that the terms x_1, x_2, x_3, ... of a sequence need not be distinct.

Since every countable set is the range of a 1-1 function defined on J, we may regard every countable set as the range of a sequence of distinct terms. Speaking more loosely, we may say that the elements of any countable set can be "arranged in a sequence."

Sometimes it is convenient to replace J in this definition by the set of all nonnegative integers, i.e., to start with 0 rather than with 1.

THEOREM 2.1. *Every infinite subset of a countable set A is countable.*

PROOF. Suppose $E \subset A$, and E is infinite. Arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements. Construct a sequence $\{n_k\}$ as follows:

Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen n_1, ..., n_{k-1} ($k = 2$, 3, 4, ...), let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Putting $f(k) = x_{n_k}$ ($k = 1$, 2, 3, ...), we obtain a 1-1 correspondence between E and J.

The theorem shows that, roughly speaking, countable sets represent the "smallest" infinity: No uncountable set can be a subset of a countable set. □

DEFINITION 2.6. Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α.

The set whose elements are the sets E_α will be denoted by $\{E_\alpha\}$. Instead of speaking of sets of sets, we shall sometimes speak of a collection of sets, or a family of sets.

The *union* of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

$$S = \bigcup_{\alpha \in A} E_\alpha. \tag{2.1}$$

If A consists of the integers 1, 2, ..., n, one usually writes

$$S = \bigcup_{m=1}^{n} E_m \tag{2.2}$$

or

$$S = E_1 \cup E_2 \cup \cdots \cup E_n. \tag{2.3}$$

If A is the set of all positive integers, the usual notation is

$$S = \bigcup_{m=1}^{\infty} E_m. \tag{2.4}$$

The symbol ∞ in (2.4) merely indicates that the union of a *countable* collection of sets is taken, and should not be confused with the symbols $+\infty$, $-\infty$, introduced in Definition **??**.

The *intersection* of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$. We use the notation

$$P = \bigcap_{\alpha \in A} E_\alpha, \tag{2.5}$$

or

$$P = \bigcap_{m=1}^{n} E_m = E_1 \cap E_2 \cap \cdots \cap E_n, \tag{2.6}$$

or

$$P = \bigcap_{m=1}^{\infty} E_m, \tag{2.7}$$

as for unions. If $A \cap B$ is not empty, we say that A and B *intersect*; otherwise they are *disjoint*.

Example 2.2.

(1) Suppose E consists of 1, 2, 3 and E_2 consists of 2, 3, 4. Then $E_1 \cup E_2$ consists of 1, 2, 3, 4, whereas $E_1 \cap E_2$ consists of 2, 3.
(2) Let A be the set of real numbers x such that $0 < x \leqslant 1$. For every $x \in A$, let E_x be the set of real numbers y such that $0 < y < x$. Then
   (a) $E_x \subset E_z$ if and only if $0 < x \leqslant z \leqslant 1$;
   (b) $\bigcup_{x \in A} E_x = E_1$;
   (c) $\bigcap_{x \in A} E_x$ is empty;
   (a) and (b) are clear. To prove (c), we note that for every $y > 0$, $y \notin E_x$ if $x < y$. Hence $y \notin \bigcap_{x \in A} E_x$.

Remarks 2.2. Many properties of unions and intersections are quite similar to those of sums and products, in fact, the words sum and product were sometimes used in this connection, and the symbols $\sum$ and $\prod$ were written in place of $\bigcup$ and $\bigcap$.

The commutative and associative laws are trivial:

$$A \cup B = B \cup A; \qquad A \cap B = B \cap A. \tag{2.8}$$

$$(A \cup B) \cup C = A \cup (B \cup C)\,; \qquad (A \cap B) \cap C = A \cap (B \cap C)\,. \tag{2.9}$$

Thus the omission of parentheses in (2.3) and (2.6) is justified.

The distributive law also holds:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\,. \tag{2.10}$$

To prove this, let the left and right members of (2.10) be denoted by E and F, respectively.

Suppose $x \in E$. Then $x \in A$ and $x \in B \cup C$, that is, $x \in B$ or $x \in C$ (possibly both). Hence $x \in A \cap B$ or $x \in A \cap C$, so that $x \in F$. Thus $E \subset F$.

Next, suppose $x \in F$. Then $x \in A \cap B$ or $x \in A \cap C$. That is, $x \in A$, and $x \in B \cup C$. Hence $x \in A \cap (B \cup C)$, so that $F \subset E$.

It follows that $E = F$.

We list a few more relations which are easily verified:

$$A \subset A \cup B, \tag{2.11}$$

$$A \cap B \subset A. \tag{2.12}$$

If 0 denotes the empty set, then

$$A \cup 0 = A, \qquad A \cap 0 = 0. \tag{2.13}$$

If $A \subset B$, then

$$A \cup B = B, \qquad A \cap B = A. \tag{2.14}$$

THEOREM 2.2. *Let $\{E_n\}$, $n = 1, 2, 3, \ldots$, be a sequence of countable sets, and put*

$$S = \bigcup_{n=1}^{\infty} E_n. \tag{2.15}$$

Then S is countable.

PROOF. Let every set E_n be arranged in a sequence $\{x_{nk}\}$, $k = 1, 2, 3, \ldots$, and consider the infinite array

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \end{array} \tag{2.16}$$

in which the elements of E_n form the nth row. The array contains all elements of S. As indicated by the arrows, these elements can be arranged in a sequence

$$x_{11};\ x_{21},\ x_{12};\ x_{31},\ x_{22},\ x_{13};\ x_{41},\ x_{32},\ x_{23},\ x_{14};\ \ldots \tag{2.17}$$

If any two of the sets E_n have elements in common, these will appear more than once in (2.17). Hence there is a subset T of the set of all positive integers such that $S \sim T$, which shows that S is at most countable (Theorem 2.1). Since $E_1 \subset S$, and E_1 is infinite, S is infinite, and thus countable. □

COROLLARY 2.1. *Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Put*

$$T = \bigcup_{\alpha \in A} B_\alpha.$$

Then T is at most countable.

For T is equivalent to a subset of (2.15).

THEOREM 2.3. *Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$, where $a_k \in A$ $(k = 1, \ldots, n)$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_n is countable.*

PROOF. That B_1 is countable is evident, since $B_1 = A$. Suppose B_{n-1}, is countable $(n = 2, 3, 4, \ldots)$. The elements of B_n are of the form

$$(b, a) \qquad (b \in B_{n-1},\ a \in A). \tag{2.18}$$

For every fixed b, the set of pairs (b, a) is equivalent to A, and hence countable. Thus B_n is the union of a countable set of countable sets. By Theorem 2.2, B_n is countable.

The theorem follows by induction. □

COROLLARY 2.2. *The set of all rational numbers is countable.*

PROOF. We apply Theorem 2.3, with $n = 2$, noting that every rational r is of the form b/a, where a and b are integers. The set of pairs (a, b), and therefore the set of fractions b/a, is countable. □

In fact, even the set of all algebraic numbers is countable (see Exercise (2)).

That not all infinite sets are, however, countable, is shown by the next theorem.

THEOREM 2.4. *Let A be the set of all sequences whose elements are the digits* 0 *and* 1. *This set A is uncountable.*

The elements of A are sequences like 1, 0, 0, 1, 0, 1, 1, 1, ...

PROOF. Let E be a countable subset of A, and let E consist of the sequences s_1, s_2, s_3, ... We construct a sequence s as follows. If the nth digit in s_n is 1, we let the nth digit of s be 0, and vice versa. Then the sequence s differs from every member of E in at least one place; hence $s \notin E$. But clearly $s \in A$, so that E is a proper subset of A.

We have shown that every countable subset of A is a proper subset of A. It follows that A is uncountable (for otherwise A would be a proper subset of A, which is absurd). □

The idea of the above proof was first used by Cantor, and is called Cantor's diagonal process; for, if the sequences s_1, s_2, s_3, ... are placed in an array like (2.16), it is the elements on the diagonal which are involved in the construction of the new sequence.

Readers who are familiar with the binary representation of the real numbers (base 2 instead of 10) will notice that Theorem 2.4 implies that the set of all real numbers is uncountable. We shall give a second proof of this fact in Theorem **??**.

2.2. Metric Spaces

2.3. Compact Sets

2.4. Perfect Sets

2.5. Connected Sets

2.6. Exercises

(1) Prove that the empty set is a subset of every set.
(2) A complex number z is said to be *algebraic* if there are integers a_0, ..., a_n, not all zero, such that
$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$
Prove that the set of all algebraic numbers is countable. *Hint*: For every positive integer N there are only finitely many equations with
$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

CHAPTER 3

Numerical Sequences and Series

CHAPTER 4

Continuity

CHAPTER 5

Differentiation

CHAPTER 6

The Riemann–Stieltjes Integral

The present chapter is based on a definition of the Riemann integral which depends very explicitly on the order structure of the real line. Accordingly, we begin by discussing integration of real-valued functions on intervals. Extensions to complex- and vector-valued functions on intervals follow in later sections. Integration over sets other than intervals is discussed in Chaps. ?? and ??.

6.1. Definition and Existence of the Integral

DEFINITION 6.1. Let $[a, b]$ be a given interval. By a *partition* P of $[a, b]$ we mean a finite set of points x_0, x_1, ..., x_n, where

$$a = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{n-1} \leqslant x_n = b.$$

We write

$$\Delta x_i = x_i - x_{i-1} \qquad (i = 1, \ldots, n).$$

Now suppose f is a bounded real function defined on $[a, b]$. Corresponding to each partition P of $[a, b]$ we put

$$\begin{aligned} M_i &= \sup f(x) \qquad (x_{i-1} \leqslant x \leqslant x_i), \\ m_i &= \inf f(x) \qquad (x_{i-1} \leqslant x \leqslant x_i), \\ U(P, f) &= \sum_{i=1}^{n} M_i \Delta x_i, \\ L(P, f) &= \sum_{i=1}^{n} m_i \Delta x_i, \end{aligned}$$

and finally

$$\overline{\int_a^b} f \, \mathrm{d}x = \inf U(P, f), \tag{6.1}$$

$$\underline{\int_a^b} f \, \mathrm{d}x = \sup L(P, f), \tag{6.2}$$

where the inf and the sup are taken over all partitions P of $[a, b]$. The left members of (6.1) and (6.2) are called the *upper* and *lower Riemann integrals* of f over $[a, b]$, respectively.

If the upper and lower integrals are equal, we say that f is *Riemann-integrable* on $[a, b]$, we write $f \in \mathscr{R}$ (that is, $\mathscr{R}$ denotes the set of Riemann-integrable functions), and we denote the common value of (6.1) and (6.2) by

$$\int_a^b f \, \mathrm{d}x \tag{6.3}$$

or by

$$\int_a^b f(x) \, \mathrm{d}x. \tag{6.4}$$

This is the *Riemann integral* of f over $[a, b]$. Since f is bounded, there exist two numbers, m and M, such that

$$m \leqslant f(x) \leqslant M \qquad (a \leqslant x \leqslant b).$$

Hence, for every P

$$m(b-a) \leqslant L(P, f) \leqslant U(P, f) \leqslant M(b-a),$$

so that the numbers $L(P, f)$ and $U(P, f)$ form a bounded set. This shows that *the upper and lower integrals are defined for every* bounded function f. The question of their equality, and hence the question of the integrability of f, is a more delicate one. Instead of investigating it separately for the Riemann integral, we shall immediately consider a more general situation.

Definition 6.2. Let α be a monotonically increasing function on $[a, b]$ (since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bounded on $[a, b]$). Corresponding to each partition P of $[a, b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$

It is clear that $\Delta\alpha_i \geqslant 0$. For any real function f which is bounded on $[a, b]$ we put

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i,$$
$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i,$$

where M_i, m_i have the same meaning as in Definition 6.1, and we define

$$\overline{\int_a^b} f\,\mathrm{d}\alpha = \inf U(P, f, \alpha), \tag{6.5}$$

$$\underline{\int_a^b} f\,\mathrm{d}\alpha = \sup L(P, f, \alpha), \tag{6.6}$$

the inf and sup again being taken over all partitions.

If the left members of (6.5) and (6.6) are equal, we denote their common value by

$$\int_a^b f\,\mathrm{d}\alpha \tag{6.7}$$

or sometimes by

$$\int_a^b f(x)\,\mathrm{d}\alpha(x). \tag{6.8}$$

This is the *Riemann–Stieltjes integral* (or simply the *Stieltjes integral*) of f with respect to α, over $[a, b]$.

If (6.7) exists, i.e., if (6.5) and (6.6) are equal, we say that f is integrable with respect to α, in the Riemann sense, and write $f \in \mathscr{R}(\alpha)$.

By taking $\alpha(x) = x$, the Riemann integral is seen to be a special case of the Riemann–Stieltjes integral. Let us mention explicitly, however, that in the general case α need not even be continuous.

A few words should be said about the notation. We prefer (6.7) to (6.8), since the letter x which appears in (6.8) adds nothing to the content of (6.7). It is immaterial which letter we use to represent the so-called "variable of integration." For instance, (6.8) is the same as

$$\int_a^b f(y)\,\mathrm{d}\alpha(y)\,.$$

The integral depends on f, α, a and b, but not on the variable of integration, which may as well be omitted.

The role played by the variable of integration is quite analogous to that of the index of summation: The two symbols

$$\sum_{i=1}^{n} c_i, \qquad \sum_{k=1}^{n} c_k$$

are the same, since each means $c_1 + c_2 + \cdots + c_n$.

Of course, no harm is done by inserting the variable of integration, and in many cases it is actually convenient to do so.

We shall now investigate the existence of the integral (6.7). Without saying so every time, f will be assumed real and bounded, and α monotonically increasing on $[a, b]$; and, when there can be no misunderstanding, we shall write $\int$ in place of $\int_a^b$.

Definition 6.3. We say that the partition P^* is a *refinement* of P if $P^* \supset P$ (that is, if every point of P is a point of P^*). Given two partitions, P_1 and P_2, we say that P^* is their *common refinement* if $P^* = P_1 \cup P_2$.

Theorem 6.1. *If P^* is a refinement of P, then*

$$L(P, f, \alpha) \leqslant L(P^*, f, \alpha) \tag{6.9}$$

and

$$U(P^*, f, \alpha) \leqslant U(P, f, \alpha)\,. \tag{6.10}$$

Proof. To prove (6.9), suppose first that P^* contains just one point more than P. Let this extra point be x^*, and suppose $x_{i-1} < x^* < x_i$, where x_{i-1} and x_i are two consecutive points of P. Put

$$\begin{aligned} w_1 &= \inf f(x) \qquad (x_{i-1} \leqslant x \leqslant x^*), \\ w_2 &= \inf f(x) \qquad (x^* \leqslant x \leqslant x_i). \end{aligned}$$

Clearly $w_1 > m_i$ and $w_2 > m_i$, where, as before,

$$m_i = \inf f(x) \qquad (x_{i-1} \leqslant x \leqslant x_i).$$

Hence

$$\begin{aligned} &L(P^*, f, \alpha) - L(P, f, \alpha) \\ &= w_1[\alpha(x^*) - \alpha(x_{i-1})] + w_2[\alpha(x_i) - \alpha(x^*)] - m_i[\alpha(x_i) - \alpha(x_{i-1})] \\ &= (w_1 - m_i)[\alpha(x^*) - \alpha(x_{i-1})] + (w_2 - m_i)[\alpha(x_i) - \alpha(x^*)] \geqslant 0. \end{aligned}$$

If P^* contains k points more than P, we repeat this reasoning k times, and arrive at (6.9). The proof of (6.10) is analogous. □

Theorem 6.2. $\underline{\int_a^b} f\,\mathrm{d}\alpha \leqslant \overline{\int_a^b} f\,\mathrm{d}\alpha$.

Proof. Let P^* be the common refinement of two partitions P_1, and P_2. By Theorem 6.1,

$$L(P_1, f, \alpha) \leqslant L(P^*, f, \alpha) \leqslant U(P^*, f, \alpha) \leqslant U(P_2, f, \alpha)$$

Hence

$$L(P_1, f, \alpha) \leqslant U(P_2, f, \alpha)\,. \tag{6.11}$$

If P_2 is fixed and the sup is taken over all P_1, (6.11) gives

$$\underline{\int} f\,\mathrm{d}\alpha \leqslant U(P_2, f, \alpha)\,. \tag{6.12}$$

The theorem follows by taking the inf over all P_2 in (6.12). □

THEOREM 6.3. *$f \in \mathscr{R}(a)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that*

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon. \tag{6.13}$$

PROOF. For every P we have

$$L(P, f, \alpha) \leqslant \underline{\int} f\,\mathrm{d}\alpha \leqslant \overline{\int} f\,\mathrm{d}\alpha \leqslant U(P, f, \alpha)\,.$$

Thus (6.13) implies

$$0 \leqslant \overline{\int} f\,\mathrm{d}\alpha - \underline{\int} f\,\mathrm{d}\alpha < \varepsilon.$$

Hence, if (6.13) can be satisfied for every $\varepsilon > 0$, we have

$$\overline{\int} f\,\mathrm{d}\alpha = \underline{\int} f\,\mathrm{d}\alpha,$$

that is, $f \in \mathscr{R}(\alpha)$.

Conversely, suppose $f \in \mathscr{R}(\alpha)$, and let $\varepsilon > 0$ be given. Then there exist partitions P_1 and P_2 such that

$$U(P_2, f, \alpha) - \int f\,\mathrm{d}\alpha < \frac{\varepsilon}{2}, \tag{6.14}$$

$$\int f\,\mathrm{d}\alpha - L(P_1, f, \alpha) < \frac{\varepsilon}{2}. \tag{6.15}$$

We choose P to be the common refinement of P_1 and P_2. Then Theorem 6.1, together with (6.14) and (6.15), shows that

$$U(P, f, \alpha) \leqslant U(P_2, f, \alpha) < \int f\,\mathrm{d}\alpha + \frac{\varepsilon}{2} < L(P_1, f, \alpha) + \varepsilon \leqslant L(P, f, \alpha) + \varepsilon,$$

so that (6.13) holds for this partition P. □

Theorem 6.3 furnishes a convenient criterion for integrability. Before we apply it, we state some closely related facts.

THEOREM 6.4.

(1) If (6.13) *holds for some P and some ε, then* (6.13) *holds (with the same ε) for every refinement of P.*

(2) If (6.13) *holds for $P = \{x_0, \dots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then*

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i < \varepsilon.$$

(3) If $f \in \mathscr{R}(\alpha)$ and the hypotheses of (2) hold, then

$$\left|\sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_a^b f\,\mathrm{d}\alpha\right| < \varepsilon.$$

PROOF. Theorem 6.1 implies (1). Under the assumptions made in (2), both $f(s_i)$ and $f(t_i)$ lie in $[m_i, M_i]$, so that $|f(s_i) - f(t_i)| \leqslant M_i - m_i$. Thus

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i \leqslant U(P, f, \alpha) - L(P, f, \alpha),$$

which proves (2). The obvious inequalities

$$L(P, f, \alpha) \leqslant \sum f(t_i) \Delta\alpha_i \leqslant U(P, f, \alpha)$$

and

$$L(P, f, \alpha) \leqslant \int f \mathrm{d}\alpha \leqslant U(P, f, \alpha)$$

prove (3). □

THEOREM 6.5. *If f is continuous on $[a, b]$ then $f \in \mathscr{R}(\alpha)$ on $[a, b]$.*

PROOF. Let $\varepsilon > 0$ be given. Choose $\eta > 0$ so that

$$[\alpha(b) - \alpha(a)] \eta < \varepsilon.$$

Since f is uniformly continuous on $[a, b]$ (Theorem ??), there exists a $\delta > 0$ such that

$$|f(x) - f(t)| < \eta \tag{6.16}$$

if $x \in [a, b]$, $t \in [a, b]$, and $|x - t| < \delta$.

If P is any partition of $[a, b]$ such that $\Delta x_i < \delta$ for all i, then (6.16) implies that

$$M_i - m_i \leqslant \eta \qquad (i = 1, \dots, n) \tag{6.17}$$

and therefore

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i \\ &\leqslant \eta \sum_{i=1}^{n} \Delta\alpha_i = \eta [\alpha(b) - \alpha(a)] < \varepsilon. \end{aligned}$$

By Theorem 6.3, $f \in \mathscr{R}(\alpha)$. □

THEOREM 6.6. *If f is monotonic on $[a, b]$, and if α is continuous on $[a, b]$, then $f \in \mathscr{R}(\alpha)$. (We still assume, of course, that α is monotonic.)*

PROOF. Let $\varepsilon > 0$ be given. For any positive integer n, choose a partition such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \qquad (i = 1, \dots, n).$$

This is possible since α is continuous (Theorem ??).

We suppose that f is monotonically increasing (the proof is analogous in the other case). Then

$$M_i = f(x_i), \qquad m_i = f(x_{i-1}) \qquad (i = 1, \dots, n),$$

so that

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \\ &= \frac{\alpha(b) - \alpha(a)}{n} \cdot [f(b) - f(a)] < \varepsilon \end{aligned}$$

if n is taken large enough. By Theorem 6.3, $f \in \mathscr{R}(\alpha)$. □

THEOREM 6.7. *Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $F \in \mathscr{R}(\alpha)$.*

Proof. Let $\varepsilon > 0$ be given. Put $M = \sup|f(x)|$, let E be the set of points at which f is discontinuous. Since E is finite and α is continuous at every point of E, we can cover E by finitely many disjoint intervals $[u_j, v_j] \subset [a, b]$ such that the sum of the corresponding differences $\alpha(v_j) - \alpha(u_j)$ is less than ε. Furthermore, we can place these intervals in such a way that every point of $E \cap (a, b)$ lies in the interior of some $[u_j, v_j]$.

Remove the segments (u_j, v_j) from $[a, b]$. The remaining set K is compact. Hence f is uniformly continuous on K, and there exists $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ if $s \in K$, $t \in K$, $|s - t| < \delta$.

Now form a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, as follows: Each u_j occurs in P. Each v_j occurs in P. No point of any segment (u_j, v_j) occurs in P. If x_{i-1} is not one of the u_j, then $\Delta x_i < \delta$.

Note that $M_i - m_i \leqslant 2M$ for every i, and that $M_i - m_i \leqslant \varepsilon$ unless x_{i-1} is one of the u_j. Hence, as in the proof of Theorem 6.5,

$$U(P, f, \alpha) - L(P, f, \alpha) \leqslant [\alpha(b) - \alpha(a)]\,\varepsilon + 2M\varepsilon.$$

Since ε is arbitrary, Theorem 6.3 shows that $f \in \mathscr{R}(\alpha)$. □

Note. If f and α have a common point of discontinuity, then f need not be in $\mathscr{R}(\alpha)$. Exercise **??** shows this.

Theorem 6.8. *Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \leqslant f \leqslant M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a, b]$.*

Proof. Choose $\varepsilon > 0$. Since ϕ is uniformly continuous on $[m, M]$, there exists $\delta > 0$ such that $\delta < \varepsilon$ and $|\phi(s) - \phi(t)| < \varepsilon$ if $|s - t| \leqslant \delta$ and $s, t \in [m, M]$.

Since $f \in \mathscr{R}(\alpha)$, there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2. \tag{6.18}$$

Let M_i, m_i have the same meaning as in Definition 6.1, and let M_i^*, m_i^* be the analogous numbers for h. Divide the numbers 1, ..., n into two classes: $i \in A$ if $M_i - m_i < \delta$, $i \in B$ if $M_i - m_i \geqslant \delta$.

For $i \in A$, our choice of δ shows that $M_i^* - m_i^* \leqslant \varepsilon$.

For $i \in B$, $M_i^* - m_i^* \leqslant 2K$, where $K = \sup|\phi(t)|$, $m \leqslant t \leqslant M$. By (6.18), we have

$$\delta \sum_{i \in B} \Delta\alpha_i \leqslant \sum_{i \in B} (M_i - m_i)\,\Delta\alpha_i < \delta^2 \tag{6.19}$$

so that $\sum_{i \in B} \Delta\alpha_i < \delta$. It follows that

$$\begin{aligned} U(P, h, \alpha) - L(P, h, \alpha) &= \sum_{i \in A} (M_i^* - m_i^*)\,\Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*)\,\Delta\alpha_i \\ &\leqslant \varepsilon\,[\alpha(b) - \alpha(a)] + 2K\delta < \varepsilon\,[\alpha(b) - \alpha(a) + 2K]. \end{aligned}$$

Since ε was arbitrary, Theorem 6.3 implies that $h \in \mathscr{R}(\alpha)$. □

Remark. This theorem suggests the question: Just what functions are Riemann-integrable? The answer is given by Theorem **????**.

6.2. Properties of the Integral

Theorem 6.9.

(1) If $f_1 \in \mathscr{R}(\alpha)$ and $f_2 \in \mathscr{R}(\alpha)$ on $[a, b]$, then

$$f_1 + f_2 \in \mathscr{R}(\alpha),$$

$cf \in \mathscr{R}(\alpha)$ for every constant c, and

$$\int_a^b (f_1 + f_2)\,\mathrm{d}\alpha = \int_a^b f_1\,\mathrm{d}\alpha + \int_a^b f_2\,\mathrm{d}\alpha,$$

$$\int_a^b cf\,\mathrm{d}\alpha = c\int_a^b f\,\mathrm{d}\alpha.$$

(2) If $f_1(x) \leqslant f_2(x)$ on $[a, b]$ then

$$\int_a^b f_1\,\mathrm{d}\alpha \leqslant \int_a^b f_2\,\mathrm{d}\alpha.$$

(3) If $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and if $a < c < b$, then $f \in \mathscr{R}(\alpha)$ on $[a, c]$ and on $[c, b]$, and

$$\int_a^c f\,\mathrm{d}\alpha + \int_c^b f\,\mathrm{d}\alpha = \int_a^b f\,\mathrm{d}\alpha.$$

(4) If $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and if $|f(x)| \leqslant M$ on $[a, b]$, then

$$\left|\int_a^b f\,\mathrm{d}\alpha\right| \leqslant M[\alpha(b) - \alpha(a)].$$

(5) If $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$, then $f \in \mathscr{R}(\alpha_1 + \alpha_2)$ and

$$\int_a^b f\,\mathrm{d}(\alpha_1 + \alpha_2) = \int_a^b f\,\mathrm{d}\alpha_1 + \int_a^b f\,\mathrm{d}\alpha_2;$$

if $f \in \mathscr{R}(\alpha)$ and c is a positive constant, then $f \in \mathscr{R}(c\alpha)$ and

$$\int_a^b f\,\mathrm{d}(c\alpha) = c\int_a^b f\,\mathrm{d}\alpha.$$

PROOF. If $f = f_1 + f_2$ and P is any partition of $[a, b]$, we have

$$L(P, f_1, \alpha) + L(P, f_2, \alpha) \leqslant L(P, f, \alpha) \leqslant U(P, f, \alpha) \leqslant U(P, f_1, \alpha) + U(P, f_2, \alpha)\,. \tag{6.20}$$

□

6.3. Integration and Differentiation

6.4. Integration of Vector-valued Functions

6.5. Rectifiable Curves