CHAPTER 1

Survey of the Elementary Principles

The motion of material bodies formed the subject of some of the earliest research pursued by the pioneers of physics. From their efforts there has evolved a vast field known as analytical mechanics or dynamics, or simply, mechanics. In the present century the term "classical mechanics" has come into wide use to denote this branch of physics in contradistinction to the newer physical theories, especially quantum mechanics. We shall follow this usage, interpreting the name to include the type of mechanics arising out of the special theory of relativity. It is the purpose of this book to develop the structure of classical mechanics and to outline some of its applications of present-day interest in pure physics. Basic to any presentation of mechanics are a number of fundamental physical concepts, such as space, time, simultaneity, mass, and force. For the most part, however, these concepts will not be analyzed critically here; rather, they will be assumed as undefined terms whose meanings are familiar to the reader.

1.1. Mechanics of a Particle

Let r be the radius vector of a particle from some given origin and v its vector velocity:

$$(1.1) \qquad\qquad v = \frac{\mathrm{d}r}{\mathrm{d}t}.$$

The *linear momentum* p of the particle is defined as the product of the particle mass and its velocity:

$$(1.2) \qquad\qquad p = mv$$

In consequence of interactions with external objects and fields, the particle may experience forces of various types, e.g., gravitational or electrodynamic; the vector sum of these forces exerted on the particle is the total force F. The mechanics of the particle is contained in *Newton's second law of motion*, which states that there exist frames of reference in which the motion of the particle is described by the differential equation

$$(1.3) \qquad\qquad F = \frac{\mathrm{d}p}{\mathrm{d}t} \equiv \dot{p},$$

or

$$(1.4) \qquad\qquad F = \frac{\mathrm{d}}{\mathrm{d}t}\left(mv\right).$$

In most instances, the mass of the particle is constant and Eq. (1.4) reduces to

$$(1.5) \qquad\qquad F = m\frac{\mathrm{d}v}{\mathrm{d}t} = ma,$$

where a is the vector acceleration of the particle defined by

$$(1.6) \qquad\qquad a = \frac{\mathrm{d}^2}{\mathrm{d}t^2}r.$$

The equation of motion is thus a differential equation of second order, assuming $\boldsymbol{F}$ does not depend on higher-order derivatives.

A reference frame in which Eq. (1.3) is valid is called an *inertial* or *Galilean system*. Even within classical mechanics the notion of an inertial system is something of an idealization. In practice, however, it is usually feasible to set up a coordinate system that comes as close to the desired properties as may be required. For many purposes, a reference frame fixed in Earth (the "laboratory system") is a sufficient approximation to an inertial system, while for some astronomical purposes it may be necessary to construct an inertial system (or inertial frame) by reference to distant galaxies.

Many of the important conclusions of mechanics can be expressed in the form of conservation theorems, which indicate under what conditions various mechanical quantities are constant in time. Equation (1.3) directly furnishes the first of these, the

THEOREM 1.1 (Conservation Theorem for the Linear Momentum of a Particle). *If the total force, $\boldsymbol{F}$, is zero, then $\dot{\boldsymbol{p}} = 0$ and the linear momentum, $\boldsymbol{p}$, is conserved.*

The angular momentum of the particle about point O, denoted by $\boldsymbol{L}$, is defined as

$$(1.7) \qquad \boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p},$$

where $\boldsymbol{r}$ is the radius vector from O to the particle. Notice that the order of the factors is important. We now define the *moment of force* or *torque* about O as

$$(1.8) \qquad \boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F}.$$

The equation analogous to (1.3) for $\boldsymbol{N}$ is obtained by forming the cross product of $\boldsymbol{r}$ with Eq. (1.4):

$$(1.9) \qquad \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{N} = \boldsymbol{r} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(m\boldsymbol{v}\right).$$

Equation (1.9) can be written in a different form by using the vector identity:

$$(1.10) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r} \times m\boldsymbol{v}\right) = \boldsymbol{v} \times m\boldsymbol{v} + \boldsymbol{r} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(m\boldsymbol{v}\right),$$

where the first term on the right obviously vanishes. In consequence of this identity, Eq. (1.9) takes the form

$$(1.11) \qquad \boldsymbol{N} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r} \times m\boldsymbol{v}\right) = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} \equiv \dot{\boldsymbol{L}}.$$

Note that both $\boldsymbol{N}$ and $\boldsymbol{L}$ depend on the point O, about which the moments are taken.

As was the case for Eq. (1.3), the torque equation, (1.11), also yields an immediate conservation theorem, this time the

THEOREM 1.2 (Conservation Theorem for the Angular Momentum of a Particle). *If the total torque, $\boldsymbol{N}$, is zero then $\dot{\boldsymbol{L}} = 0$, and the angular momentum $\boldsymbol{L}$ is conserved.*

Next consider the work done by the external force $\boldsymbol{F}$ upon the particle in going from point 1 to point 2. By definition, this work is

$$(1.12) \qquad W_{12} = \int_{1}^{2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}.$$

For constant mass (as will be assumed from now on unless otherwise specified), the integral in Eq. (1.12) reduces to

$$\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = m \int \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \boldsymbol{v}\,\mathrm{d}t = \frac{m}{2} \int \frac{\mathrm{d}}{\mathrm{d}t}\left(v^2\right)\mathrm{d}t,$$

and therefore

$$(1.13) \qquad W_{12} = \frac{m}{2}\left(v_2^2 - v_1^2\right).$$

The scalar quantity $mv^2/2$ is called the kinetic energy of the particle and is denoted by T, so that the work done is equal to the change in the kinetic energy:

$$(1.14) \qquad W_{12} = T_2 - T_1.$$

If the force field is such that the work W_{12} is the same for any physically possible path between points 1 and 2, then the force (and the system) is said to be *conservative.* An alternative description of a conservative system is obtained by imagining the particle being taken from point 1 to point 2 by one possible path and then being returned to point 1 by another path. The independence of W_{12} on the particular path implies that the work done around such a closed circuit is zero, i.e.:

$$(1.15) \qquad \oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = 0.$$

Physically it is clear that a system cannot be conservative if friction or other dissipation forces are present, because $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$ due to friction is always positive and the integral cannot vanish.

By a well-known theorem of vector analysis, a necessary and sufficient condition that the work, W_{12}, be independent of the physical path taken by the particle is that $\boldsymbol{F}$ be the gradient of some scalar function of position:

$$(1.16) \qquad \boldsymbol{F} = -\nabla V(\boldsymbol{r}),$$

where V is called the *potential*, or *potential energy.* The existence of V can be inferred intuitively by a simple argument. If W_{12} is independent of the path of integration between the end points 1 and 2, it should be possible to express W_{12} as the change in a quantity that depends only upon the positions of the end points. This quantity may be designated by $-V$, so that for a differential path length we have the relation

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = -\mathrm{d}V$$

or

$$F_s = -\frac{\partial V}{\partial s},$$

which is equivalent to Eq. (1.16). Note that in Eq. (1.16) we can add to V any quantity constant in space, without affecting the results. Hence *the zero level of V is arbitrary.*

For a conservative system, the work done by the forces is

$$(1.17) \qquad W_{12} = V_1 - V_2.$$

Combining Eq. (1.17) with Eq. (1.14), we have the result

$$(1.18) \qquad T_1 + V_1 = T_2 + V_2,$$

which states in symbols the

THEOREM 1.3 (Energy Conservation Theorem for a Particle). *If the forces acting on a particle are conservative, then the total energy of the particle, $T + V$, is conserved.*

The force applied to a particle may in some circumstances be given by the gradient of a scalar function that depends explicitly on both the position of the particle and the time. However, the work done on the particle when it travels a distance $\mathrm{d}s$,

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = -\frac{\partial V}{\partial s}\,\mathrm{d}s,$$

is then no longer the total change in $-V$ during the displacement, since V also changes explicitly with time as the particle moves. Hence, the work done as the particle goes from point 1 to point 2 is no longer the difference in the function V between those points. While a total energy $T + V$ may still be defined, it is not conserved during the course of the particle's motion.

1.2. Mechanics of a System of Particles

In generalizing the ideas of the previous section to systems of many particles, we must distinguish between the *external forces* acting on the particles due to sources outside the system, and *internal forces* on, say, some particle i due to all other particles in the system. Thus, the equation of motion (Newton's second law) for the ith particle is written as

$$(1.19) \qquad \sum_j \boldsymbol{F}_{ji} + \boldsymbol{F}_i^{(e)} = \dot{\boldsymbol{p}}_i,$$

where $\boldsymbol{F}_i^{(e)}$ stands for an external force, and $\boldsymbol{F}_{ji}$ is the internal force on the ith particle due to the jth particle ($\boldsymbol{F}_{ii}$ naturally, is zero). We shall assume that the $\boldsymbol{F}_{ij}$ (like the $\boldsymbol{F}_i^{(e)}$) obey Newton's third law of motion in its original form: that the forces two particles exert on each other are equal and opposite. This assumption (which does not hold for all types of forces) is sometimes referred to as the *weak law of action and reaction.*

Summed over all particles, Eq. (1.19) takes the form

$$(1.20) \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_i m_i \boldsymbol{r}_i = \sum_i \boldsymbol{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \boldsymbol{F}_{ji}.$$

The first sum on the right is simply the total external force $\boldsymbol{F}^{(e)}$, while the second term vanishes, since the law of action and reaction states that each pair $\boldsymbol{F}_{ij} + \boldsymbol{F}_{ji}$ is zero. To reduce the left-hand side, we define a vector $\boldsymbol{R}$ as the average of the radius vectors of the particles, weighted in proportion to their mass:

$$(1.21) \qquad \boldsymbol{R} = \frac{\sum m_i \boldsymbol{r}_i}{\sum m_i} = \frac{\sum m_i \boldsymbol{r}_i}{M}.$$

The vector $\boldsymbol{R}$ defines a point known as the *center of mass*, or more loosely as the center of gravity, of the system (cf. Fig. 1.1). With this definition, (1.20) reduces to

$$(1.22) \qquad M\frac{\mathrm{d}^2 \boldsymbol{R}}{\mathrm{d}t^2} = \sum_i \boldsymbol{F}_i^{(e)} \equiv \boldsymbol{F}^{(e)},$$

which states that the center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. Purely internal forces, if they obey Newton's third law, therefore have no effect on the motion of the center of mass. An oft-quoted example is the motion of an exploding shell—the center of mass of the fragments traveling as if the shell were still in a single piece (neglecting air resistance). The same principle is involved in jet and rocket propulsion. In order that the motion of the center of mass be unaffected, the ejection of the exhaust gases at high velocity must be counterbalanced by the forward motion of the vehicle at a slower velocity.

FIGURE 1.1. The center of mass of a system of particles.

By Eq. (1.21) the total linear momentum of the system,

$$(1.23) \qquad \boldsymbol{P} = \sum m_i \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = M\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t},$$

is the total mass of the system times the velocity of the center of mass. Consequently, the equation of motion for the center of mass, (1.23), can be restated as the

THEOREM 1.4 (Conservation Theorem for the Linear Momentum of a System of Particles). *If the total external force is zero, the total linear momentum is conserved.*

We obtain the total angular momentum of the system by forming the cross product $\boldsymbol{r}_i \times \boldsymbol{p}_i$ and summing over i. If this operation is performed in Eq. (1.19), there results, with the aid of the identity, Eq. (1.10),

$$(1.24) \qquad \sum_i \left(\boldsymbol{r}_i \times \dot{\boldsymbol{p}}_i\right) = \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_i \times \boldsymbol{p}_i\right) = \dot{\boldsymbol{L}} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \boldsymbol{r}_i \times \boldsymbol{F}_{ji}.$$

The last term on the right in (1.24) can be considered a sum of the pairs of the form

$$(1.25) \qquad \boldsymbol{r}_i \times \boldsymbol{F}_{ji} + \boldsymbol{r}_j \times \boldsymbol{F}_{ij} = \left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) \times \boldsymbol{F}_{ji},$$

using the equality of action and reaction. But $\boldsymbol{r}_i - \boldsymbol{r}_j$ is identical with the vector $\boldsymbol{r}_{ij}$ from j to i (cf. Fig. 1.2), so that the right-hand side of Eq. (1.25) can be written as

$$\boldsymbol{r}_{ij} \times \boldsymbol{F}_{ji}.$$

If the internal forces between two particles, in addition to being equal and opposite, also lie along the line joining the particles—a condition known as the *strong law of action and reaction*—then all of these cross products vanish. The sum over pairs is zero under this assumption and Eq. (1.24) may be written in the form

$$(1.26) \qquad \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{N}^{(e)}.$$

The time derivative of the total angular momentum is thus equal to the moment of the external force about the given point. Corresponding to Eq. (1.26) is the

THEOREM 1.5 (Conservation Theorem for Total Angular Momentum). *$\boldsymbol{L}$ is constant in time if the applied (external) torque is zero.*

FIGURE 1.2. The vector $\boldsymbol{r}_{ij}$ between the ith and jth particles.

(It is perhaps worthwhile to emphasize that this is a *vector* theorem; i.e., L_z will be conserved if $N_z^{(e)}$ is zero, even if $N_x^{(e)}$ and $N_y^{(e)}$ are not zero.)

Note that the conservation of linear momentum in the absence of applied forces assumes that the weak law of action and reaction is valid for the internal forces. The conservation of the total angular momentum of the system in the absence of applied torques requires the validity of the strong law of action and reaction—that the internal forces in addition be *central.* Many of the familiar physical forces, such as that of gravity, satisfy the strong form of the law. But it is possible to find forces for which action and reaction are equal even though the forces are not central (see below). In a system involving moving charges, the forces between the charges predicted by the Biot-Savart law may indeed violate both forms of the action and reaction law.[1] Equations (1.23) and (1.26), and their corresponding conservation theorems, are not applicable in such cases, at least in the form given here. Usually it

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[1]If two charges are moving uniformly with parallel velocity vectors that are not perpendicular to the line joining the charges, then the net mutual forces are equal and opposite but do not lie along the vector between the charges. Consider, further, two charges moving (instantaneously) so as to "cross the T," i.e., one charge moving directly at the other, which in turn is moving at right angles to the first. Then the second charge exerts a nonvanishing magnetic force on the first, without experiencing any magnetic reaction force at that instant.

is then possible to find some generalization of $\boldsymbol{P}$ or $\boldsymbol{L}$ that is conserved. Thus, in an isolated system of moving charges it is the sum of the mechanical angular momentum and the electromagnetic "angular momentum" of the field that is conserved.

Equation (1.23) states that the total linear momentum of the system 1s the same as if the entire mass were concentrated at the center of mass and moving with it. The analogous theorem for angular momentum is more complicated. With the origin O as reference point, the total angular momentum of the system is

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i.$$

Let $\boldsymbol{R}$ be the radius vector from O to the center of mass, and let $\boldsymbol{r}'_i$ be the radius vector from the center of mass to the ith particle. Then we have (cf. Fig. 1.3)

$$\text{(1.27)} \qquad\qquad\qquad \boldsymbol{r}_i = \boldsymbol{r}'_i + \boldsymbol{R}$$

and

$$\boldsymbol{v}_i = \boldsymbol{v}'_i + \boldsymbol{v}$$

where

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t}$$

is the velocity of the center of mass relative to O, and

$$\boldsymbol{v}'_i = \frac{\mathrm{d}\boldsymbol{r}'_i}{\mathrm{d}t}$$

is the velocity of the ith particle relative to the center of mass of the system. Using Eq. (1.27), the total angular momentum takes on the form

$$\boldsymbol{L} = \sum_i \boldsymbol{R} \times m_i \boldsymbol{v} + \sum_i \boldsymbol{r}'_i \times m_i \boldsymbol{v}'_i + \left(\sum_i m_i \boldsymbol{r}'_i \right) \times \boldsymbol{v} + \boldsymbol{R} \times \frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i \boldsymbol{r}'_i.$$

The last two terms in this expression vanish, for both contain the factor $\sum m_i \boldsymbol{r}'_i$, which, it will be recognized, defines the radius vector of the center of mass in the very coordinate system whose origin is the center of mass and is therefore a null vector. Rewriting the remaining terms, the total angular momentum about O is

$$\text{(1.28)} \qquad\qquad\qquad \boldsymbol{L} = \boldsymbol{R} \times M\boldsymbol{v} + \sum_i \boldsymbol{r}'_i \times \boldsymbol{p}'_i.$$

In words, Eq. (1.28) says that the total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass, plus the angular momentum of motion about the center of mass. The form of Eq. (1.28) emphasizes that in general $\boldsymbol{L}$ depends on the origin O, through the vector $\boldsymbol{R}$. Only if the center of mass is at rest with respect to O will the angular momentum be independent of the point of reference. In this case, the first term in (1.28) vanishes, and $\boldsymbol{L}$ always reduces to the angular momentum taken about the center of mass.

FIGURE 1.3. The vectors involved in the shift of reference point for the angular momentum.

Finally, let us consider the energy equation. As in the case of a single particle, we calculate the work done by all forces in moving the system from an initial configuration 1, to a final configuration 2:

$$\text{(1.29)} \qquad W_{12} = \sum_i \int_1^2 \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{s}_i = \sum_i \int_1^2 \boldsymbol{F}_i^{(e)} \cdot \mathrm{d}\boldsymbol{s}_i + \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \boldsymbol{F}_{ji} \cdot \mathrm{d}\boldsymbol{s}_i.$$

Again, the equations of motion can be used to reduce the integrals to

$$\sum_i \int_1^2 \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{s}_i = \sum_i \int_1^2 m_i \dot{\boldsymbol{v}}_i \cdot \boldsymbol{v}_i \, \mathrm{d}t = \sum_i \int_1^2 \mathrm{d}\left(\frac{1}{2} m_i v_i^2\right).$$

Hence, the work done can still be written as the difference of the final and initial kinetic energies:

$$W_{12} = T_2 - T_1,$$

where T, the total kinetic energy of the system, is

$$(1.30) \qquad\qquad T = \frac{1}{2}\sum_i m_i v_i^2.$$

Making use of the transformations to center-of-mass coordinates, given in Eq. (1.27), we may also write T as

$$T = \frac{1}{2}\sum_i m_i \left(\boldsymbol{v} + \boldsymbol{v}_i'\right) \cdot \left(\boldsymbol{v} + \boldsymbol{v}_i'\right)$$

$$= \frac{1}{2}\sum_i m_i v^2 + \frac{1}{2}\sum_i m_i v_i'^2 + \boldsymbol{v} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i m_i \boldsymbol{r}_i'\right),$$

and by the reasoning already employed in calculating the angular momentum, the last term vanishes, leaving

$$(1.31) \qquad\qquad T = \frac{1}{2}M v^2 + \frac{1}{2}\sum_i m_i v_i'^2.$$

The kinetic energy, like the angular momentum, thus also consists of two parts: the kinetic energy obtained if all the mass were concentrated at the center of mass, plus the kinetic energy of motion about the center of mass.

Consider now the right-hand side of Eq. (1.29). In the special case that the external forces are derivable in terms of the gradient of a potential, the first term can be written as

$$\sum_i \int_1^2 \boldsymbol{F}_i^{(e)} \cdot \mathrm{d}\boldsymbol{s}_i = -\sum_i \int_1^2 \nabla_i V_i \cdot \mathrm{d}\boldsymbol{s}_i = -\sum_i V_i \bigg|_1^2,$$

where the subscript i on the del operator indicates that the derivatives are with respect to the components of $\boldsymbol{r}_i$. If the internal forces are also conservative, then the mutual forces between the ith and jth particles, $\boldsymbol{F}_{ij}$ and $\boldsymbol{F}_{ji}$, can be obtained from a potential function V_{ij}. To satisfy the strong law of action and reaction, V_{ij} can be a function only of the distance between the particles:

$$(1.32) \qquad\qquad V_{ij} = V_{ij}\left(\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|\right).$$

The two forces are then automatically equal and opposite,

$$(1.33) \qquad\qquad \boldsymbol{F}_{ji} = -\nabla_i V_{ij} = +\nabla_j V_{ij} = -\boldsymbol{F}_{ij},$$

and lie along the line joining the two particles,

$$(1.34) \qquad\qquad \nabla V_{ij}\left(\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|\right) = \left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) f,$$

where f is some scalar function. If V_{ij} were also a function of the difference of some other pair of vectors associated with the particles, such as their velocities or (to step into the domain of modern physics) their intrinsic "spin" angular momentum, then the forces would still be equal and opposite, but would not necessarily lie along the direction between the particles.

When the forces are all conservative, the second term in Eq. (1.29) can be rewritten as a sum over *pairs* of particles, the terms for each pair being of the form

$$-\int_{1}^{2}\left(\nabla_i V_{ij}\cdot \mathrm{d}\boldsymbol{s}_i + \nabla_j V_{ij}\cdot \mathrm{d}\boldsymbol{s}_j\right).$$

If the difference vector $\boldsymbol{r}_i - \boldsymbol{r}_j$ is denoted by $\boldsymbol{r}_{ij}$, and if ∇_{ij} stands for the gradient with respect to $\boldsymbol{r}_{ij}$, then

$$\nabla_i V_{ij} = \nabla_{ij} V_{ij} = -\nabla_j V_{ij},$$

and

$$\mathrm{d}\boldsymbol{s}_i - \mathrm{d}\boldsymbol{s}_j = \mathrm{d}\boldsymbol{r}_i - \mathrm{d}\boldsymbol{r}_j = \mathrm{d}\boldsymbol{r}_{ij},$$

so that the term for the ij pair has the form

$$-\int \nabla_{ij} V_{ij}\cdot \mathrm{d}\boldsymbol{r}_{ij}.$$

The total work arising from internal forces then reduces to

$$(1.35) \qquad -\frac{1}{2}\sum_{\substack{i,j \\ i\neq j}}\int_{1}^{2}\nabla_{ij}V_{ij}\cdot \mathrm{d}\boldsymbol{r}_{ij} = -\frac{1}{2}\sum_{\substack{i,j \\ i\neq j}} V_{ij}\Big|_{1}^{2}.$$

The factor $\frac{1}{2}$ appears in Eq. (1.35) because in summing over *both* i and j each member of a given pair is included twice, first in the i summation and then in the j summation.

From these considerations, it is clear that if the external and internal forces are both derivable from potentials it is possible to define a *total potential energy*, V, of the system,

$$(1.36) \qquad V = \sum_i V_i + \frac{1}{2}\sum_{\substack{i,j \\ i\neq j}} V_{ij},$$

such that the total energy $T+V$ is conserved, the analog of the conservation theorem (1.18) for a single particle.

The second term on the right in Eq. (1.36) will be called the internal potential energy of the system. In general, it need not be zero and, more important, it may vary as the system changes with time. Only for the particular class of systems known as *rigid bodies* will the internal potential always be constant. Formally, a rigid body can be defined as a system of particles in which the distances r_{ij} are fixed and cannot vary with time. In such case, the vectors $\mathrm{d}\boldsymbol{r}_{ij}$ can only be perpendicular to the corresponding $\boldsymbol{r}_{ij}$, and therefore to the $\boldsymbol{F}_{ij}$. Therefore, in a rigid body the *internal forces do no work*, and the internal potential must remain constant. Since the total potential is in any case uncertain to within an additive constant, an unvarying internal potential can be completely disregarded in discussing the motion of the system.

1.3. Constraints

From the previous sections one might obtain the impression that all problems in mechanics have been reduced to solving the set of differential equations (1.19):

$$m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i^{(e)} + \sum_j \boldsymbol{F}_{ji}.$$

One merely substitutes the various forces acting upon the particles of the system, turns the mathematical crank, and grinds out the answers! Even from a purely physical standpoint, however, this view is oversimplified. For example, it may be necessary to take into account the *constraints* that limit the motion of the

system. We have already met one type of system involving constraints, namely rigid bodies, where the constraints on the motions of the particles keep the distances r_{ij} unchanged. Other examples of constrained systems can easily be furnished. The beads of an abacus are constrained to one-dimensional motion by the supporting wires. Gas molecules within a container are constrained by the walls of the vessel to move only *inside* the container. A particle placed on the surface of a solid sphere is subject to the constraint that it can move only on the surface or in the region exterior to the sphere.

Constraints may be classified in various ways, and we shall use the following system. If the conditions of constraint can be expressed as equations connecting the coordinates of the particles (and possibly the time) having the form

(1.37) $$f(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \dots, t) = 0,$$

then the constraints are said to be *holonomic*. Perhaps the simplest example of holonomic constraints is the rigid body, where the constraints are expressed by equations of the form

$$\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)^2 - c_{ij}^2 = 0.$$

A particle constrained to move along any curve or on a given surface is another obvious example of a holonomic constraint, with the equations defining the curve or surface acting as the equations of a constraint.

Constraints not expressible in this fashion are called nonholonomic. The walls of a gas container constitute a nonholonomic constraint. The constraint involved in the example of a particle placed on the surface of a sphere is also nonholonomic, for it can be expressed as an inequality

$$r^2 - a^2 \geqslant 0$$

(where a is the radius of the sphere), which is not in the form of (1.37). Thus, in a gravitational field a particle placed on the top of the sphere will slide down the surface part of the way but will eventually fall off.

Constraints are further classified according to whether the equations of constraint contain the time as an explicit variable (rheonomous) or are not explicitly dependent on time (scleronomous). A bead sliding on a rigid curved wire fixed in space is obviously subject to a scleronomous constraint; if the wire is moving in some prescribed fashion, the constraint is rheonomous. Note that if the wire moves, say, as a reaction to the bead's motion, then the time dependence of the constraint enters in the equation of the constraint only through the coordinates of the curved wire (which are now part of the system coordinates). The overall constraint is then scleronomous.