CHAPTER 2

Basic Topology

2.1. Finite, Countable, and Uncountable Sets

We begin this section with a definition of the function concept.

Definition 2.1. Consider two sets A and B, whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B, which we denote by $f(x)$. Then f is said to be a *function* from A to B (or a *mapping* of A into B). The set A is called the *domain* of f (we also say f is defined on A), and the elements $f(x)$ are called the *values* of f. The set of all values of f is called the *range* of f.

Definition 2.2. Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$. We call $f(E)$ the *image* of E under f. In this notation, $f(A)$ is the range of f. It is clear that $f(A) \subset B$. If $f(A) = B$, we say that f maps A *onto* B. (Note that, according to this usage, *onto* is more specific than *into*.)

If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the *inverse image* of E under f. If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$. If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A, then f is said to be a 1-1 (*one-to-one*) mapping of A into B. This may also be expressed as follows: f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1 \in A$, $x_2 \in A$.

(The notation $x_1 \neq x_2$ means that x_1 and x_2 are distinct elements; otherwise we write $x_1 = x_2$.)

Definition 2.3. If there exists a 1-1 mapping of A *onto* B, we say that A and B, can be put in 1-1 *correspondence*, or that A and B have the same *cardinal number*, or briefly, that A and B are *equivalent*, and we write $A \sim B$. This relation clearly has the following properties:

- It is reflexive: $A \sim A$.
- It is symmetric: If $A \sim B$, then $B \sim A$.
- It is transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.

Any relation with these three properties is called an *equivalence relation*.

Definition 2.4. For any positive integer n, let J_n be the set whose elements are the integers 1, 2, ..., n; let J be the set consisting of all positive integers. For any set A, we say:

(1) A is *finite* if $A \sim J_n$ for some n (the empty set is also considered to be finite).
(2) A is *infinite* if A is not finite.
(3) A is *countable* if $A \sim J$.
(4) A is *uncountable* if A is neither finite nor countable.
(5) A is *at most countable* if A is finite or countable.

Countable sets are sometimes called *enumerable*, or *denumerable*.

For two finite sets A and B, we evidently have $A \sim B$ if and only if A and B contain the same number of elements. For infinite sets, however, the idea of "having the same number of elements" becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

EXAMPLE 2.1. Let A be the set of all integers. Then A is countable. For, consider the following arrangement of the sets A and J:

$$\begin{array}{ll} A: & 0, 1, -1, 2, -2, 3, -3, \ldots \\ J: & 1, 2, 3, 4, 5, 6, 7, \ldots \end{array}$$

We can, in this example, even give an explicit formula for a function f from J to A which sets up a 1-1 correspondence:

$$f(n) = \begin{cases} \frac{n}{2} & (n \text{ even}), \\ -\frac{n-1}{2} & (n \text{ odd}). \end{cases}$$

REMARK 2.1. A finite set cannot be equivalent to one of its proper subsets. That this is, however, possible for infinite sets, is shown by Example 2.1, in which J is a proper subset of A.

In fact, we could replace Definition 2.4(2) by the statement: A is infinite if A is equivalent to one of its proper subsets.

DEFINITION 2.5. By a *sequence*, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by x_1, x_2, x_3, ... The values of f, that is, the elements x_n, are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a *sequence in A*, or a *sequence of elements of A*.

Note that the terms x_1, x_2, x_3, ...of a sequence need not be distinct.

Since every countable set is the range of a 1-1 function defined on J, we may regard every countable set as the range of a sequence of distinct terms. Speaking more loosely, we may say that the elements of any countable set can be "arranged in a sequence."

Sometimes it is convenient to replace J in this definition by the set of all nonnegative integers, i.e., to start with 0 rather than with 1.

THEOREM 2.1. *Every infinite subset of a countable set A is countable.*

PROOF. Suppose $E \subset A$, and E is infinite. Arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements. Construct a sequence $\{n_k\}$ as follows:

Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen n_1, ..., n_{k-1} $(k = 2, 3, 4, \ldots)$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Putting $f(k) = x_{n_k}$ $(k = 1, 2, 3, \ldots)$, we obtain a 1-1 correspondence between E and J.

The theorem shows that, roughly speaking, countable sets represent the "smallest" infinity: No uncountable set can be a subset of a countable set. ☐

DEFINITION 2.6. Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α.

The set whose elements are the sets E_α will be denoted by $\{E_\alpha\}$. Instead of speaking of sets of sets, we shall sometimes speak of a collection of sets, or a family of sets.

The *union* of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

$$S = \bigcup_{\alpha \in A} E_\alpha. \tag{2.1}$$

If A consists of the integers 1, 2, ..., n, one usually writes

$$S = \bigcup_{m=1}^{n} E_m \tag{2.2}$$

or

$$S = E_1 \cup E_2 \cup \cdots \cup E_n. \tag{2.3}$$

If A is the set of all positive integers, the usual notation is

$$S = \bigcup_{m=1}^{\infty} E_m. \tag{2.4}$$

The symbol ∞ in (2.4) merely indicates that the union of a *countable* collection of sets is taken, and should not be confused with the symbols $+\infty$, $-\infty$, introduced in Definition **??**.

The *intersection* of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$. We use the notation

$$P = \bigcap_{\alpha \in A} E_\alpha, \tag{2.5}$$

or

$$P = \bigcap_{m=1}^{n} E_m = E_1 \cap E_2 \cap \cdots \cap E_n, \tag{2.6}$$

or

$$P = \bigcap_{m=1}^{\infty} E_m, \tag{2.7}$$

as for unions. If $A \cap B$ is not empty, we say that A and B *intersect*; otherwise they are *disjoint*.

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ & & \cdots & & \end{array} \tag{2.8}$$