Elements of Digital Signal Processing in 1D

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These notes provides a concise account of the material covered in the first part of the DSIP course. Check the provided references if you care deepen your understanding.

Contents

1 Math minimalia

Signal processing requires some basic knowledge of linear algebra and calculus. Here is a very short account of the main concepts we are going to use throughout this course. In case you need more, any introductory book on complex numbers, linear vector spaces, and integration by part would do. Our favorite is the **Introduction to linear algebra** by *Gilbert Strang*.

1.1 Complex numbers

We provide a short introduction to the subject with special emphasis on the two basic representations of a complex number.

Imaginary unit Since $\forall x \in \mathbb{R}$ we have that $x^2 \geq 0$, the equation

$$x^2 + 1 = 0$$

has no solution in $\mathbb{R}$. However, if we introduce the *imaginary unit* i with $i^2 = -1$, we obtain the two solutions $x_+ = i$ and $x_- = -i$.

Same old algebra We can now compute the square root of an arbitrary real number and solve all second degree equations with real coefficients in the complex field $\mathbb{C}$. We write a complex number $z \in \mathbb{C}$ as

$$z = a + ib$$

with $a \in \mathbb{R}$ its *real part* and $b \in \mathbb{R}$ its *imaginary part*. A real number has zero imaginary part, while a complex number with zero real part is called *pure imaginary*.

By means of the usual algebra, if $z = a + ib$ and $w = c + id$ we find

$$\begin{aligned} z + w &= (a + ib) + (c + id) = (a + c) + i(b + d) \\ zw &= (a + ib)(c + id) = (ac - bd) + i(ad + bc). \end{aligned}$$

Observation 1. *Sum of two complex numbers*
The real and imaginary part of the sum of two complex numbers is the sum of their real and imaginary parts respectively. Instead, no easy interpretation for the product seems to be available. □

The number $z^* = a - ib$, obtained by changing the sign of the imaginary part of $z = a + ib$, is the *complex conjugate* of z. For all $z \in \mathbb{C}$,

$$zz^* = a^2 + b^2$$

is a real non-negative number. The quantity $|z| = \sqrt{a^2 + b^2}$ is the *module* of z.

Exercise 1. *Some simple facts on complex numbers*
Prove that

1. $\forall z \quad |z| = |z^*|$
2. $\forall z \quad (z^*)^* = z$
3. $|z| = 0$ iff $z = 0$.

Exercise 2. *A few minutes in the imaginary gym*
Let $z = a + ib$ and $w = c + id$. Computer the real and imaginary part of zw^* and z/w.

A more effective representation Let us represent $z = a + ib$ as the ordered pair of real numbers (a, b) in the complex plane $\mathbb{C}$ (see figure 1). The horizontal axis is the (usual) real axis. If $0 \leq \theta < 2\pi$ is the angle formed by the complex number z with the real axis we have

$$a = |z| \cos\theta \text{ and } b = |z| \sin\theta \quad \text{with } b/a = \tan\theta.$$

Observation 2. *Phase of a complex number*
The angle θ is called the *phase* of the complex number z. For all real positive numbers $\theta = 0$, while for real negative numbers $\theta = \pi$. For $z = 0$ the phase is not defined.

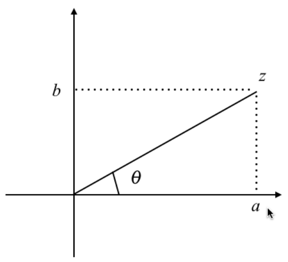

Figure 1: See text.

Observation 3. *Polar representation*
Since $z(\theta) = \cos\theta + i \sin\theta$ we have

$$\frac{\mathrm{d}z(\theta)}{\mathrm{d}\theta} = -\sin\theta + i\cos\theta = iz(\theta).$$

Integrating both sides of the equality

$$\frac{\mathrm{d}z}{z} = i\mathrm{d}\theta$$

we obtain

$$\ln z(\theta) = i\theta + C \tag{1}$$

for some constant C. Since $z(0) = 1$, we can conclude that $C = 0$. Applying the exponential to both sides of equation (1) we finally obtain

$$z(\theta) = \cos\theta + i\sin\theta = e^{i\theta}.$$

In general, for an arbitrary $z = a + ib \in \mathbb{C}$ we have $z = \rho\, e^{i\theta}$ with $\rho = |z|$.

Observation 4. *The exponential property*
The module of the product of two complex numbers $z_1 = \rho_1 e^{i\theta_1}$ and $z_2 = \rho_2 e^{i\theta_2}$ is the product of their modules, $\rho_1\rho_2$, while the phase is the sum of their phases, $\theta_1 + \theta_2$, since

$$zw = \rho_1 e^{i\theta_1} \rho_2\, e^{i\theta_2} = \rho_1\rho_2\, e^{i\theta_1} e^{i\theta_2} = \rho_1\rho_2\, e^{i(\theta_1+\theta_2)}.$$

Observation 5. *The most beautiful equation of all times (due to Euler)*
Since -1 can be written as $e^{i\pi}$, we have

$$e^{i\pi} + 1 = 0.$$

Five of the most recurrent numbers of mathematics - 0, 1, i, π and e - all in the same equality!

Exercise 3. *Some more time in the imaginary gym*
Find the polar representation of 2, -3, $4i$, $-5i$, $1+i$, and i^i.

Exercise 4. *Roots of unity*
Compute the n roots of the equation

$$z^n = 1$$

and show that they define a regular polygon of n sides inscribed in the unit circle in the complex plane with one of the vertices lying in $(1, 0)$.

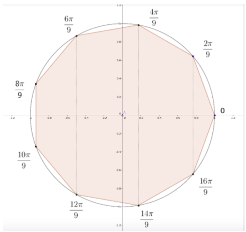

Figure 2: Roots of unity for $n = 9$.

1.2 Linear spaces

The concept of linear space is arguably one of the most important in mathematics. We review some basic facts, often without proof and restricting our attention to the finite dimensional case.

Generalities Let V be a collection of objects, or elements, on which we can define two operations. The first, called *vector sum*, is a map from $V \times V \to V$ which sends a pair of objects $\mathbf{u}$ and $\mathbf{v} \in V$ into another object $\mathbf{w} \in V$, or

$$\mathbf{w} = \mathbf{u} + \mathbf{v}.$$

The second, called *scalar multiplication*, is a map from $\mathbb{R} \times V \to V$ which sends a number $\alpha \in \mathbb{R}$ and an object $\mathbf{v} \in V$ into another object $\mathbf{w} \in V$, or

$$\mathbf{w} = \alpha \mathbf{v}.$$

Nothing changes if we replace $\mathbb{R}$ with some other field (like $\mathbb{C}$).

Definition 1. *Linear space*
The set V is a *linear space* iff

1. For all $\mathbf{u}$ and $\mathbf{v} \in V$ $\;\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;
2. For all $\mathbf{u}, \mathbf{v}$ and $\mathbf{w} \in V$ $\;\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{u}$;
3. There exists an element $\mathbf{0} \in V$, the *zero*, such that for all $\mathbf{v} \in V$ $\;\mathbf{v} + \mathbf{0} = \mathbf{v}$;
4. For all $\mathbf{v} \in V$ there exists an *inverse* element $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$;
5. For all α and $\beta \in \mathbb{R}$ and $\mathbf{v} \in V$ $\;\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$;
6. For all $\mathbf{v} \in V$ $\;1\mathbf{v} = \mathbf{v}$;
7. For all $\alpha \in \mathbb{R}$ and $\mathbf{u}$ and $\mathbf{v} \in V$ $\;\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$;

8. For all α and $\beta \in \mathbb{R}$ and $\mathbf{v} \in V$ $\;(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.

Observation 6. *A more compact version*
That V is a linear space can be verified by simply checking whether for all α and $\beta \in \mathbb{R}$ and $\mathbf{u}$ and $\mathbf{v} \in V$,

$$\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v} \;\in V,$$

giving for granted that the vector sum and the scalar multiplication satisfy the usual algebraic properties for sum and multiplication.

Observation 7. *One necessary element*
If $\mathbf{0} \notin V$, the set V cannot be a linear space!

Observation 8. *Vectors*
The objects of a linear space are often called *vectors*, without any implication about their true nature.

Example 1. $\mathbb{R}^3$
The set $\mathbb{R}^3$ which consists of all the ordered triplets of real numbers

$$\mathbf{v} = (v_1, v_2, v_3)$$

is a linear space with

1. the *vector sum* defined as $\mathbf{w} = \mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

2. the *scalar multiplication* as $\mathbf{w} = \alpha\mathbf{v} = (\alpha v_1, \alpha v_2, \alpha v_3)$.

Example 2. *Polynomials of degree n with real coefficients*
The set of polynomials of degree n with real coefficients written as

$$\mathbf{v} = v_0 + v_1 x + \ldots v_n x^n$$

is a linear space with

1. the *vector sum* defined as $\mathbf{w} = \mathbf{u} + \mathbf{v} = (u_0 + v_0) + (u_1 + v_1)x + \ldots (u_n + v_n)x^n$, and

2. the *scalar multiplication* as $\mathbf{w} = \alpha\mathbf{v} = \alpha(v_0 + v_1 x + \ldots v_n x^n)$.

Definition 2. *Linear independence*
Let V be a linear space: $\mathbf{v}_1, \ldots \mathbf{v}_k \in V$ are linearly independent if

$$\alpha_1\mathbf{v_1} + \ldots \alpha_k\mathbf{v}_k = 0 \quad \text{only if} \quad \alpha_1 = \cdots = \alpha_k = 0,$$

and linearly dependent otherwise.

Exercise 5. *Linear dependency and independency in* $\mathbb{R}^3$
Find the linearly independent pairs among $\mathbf{u} = (2, 3, 4)$, $\mathbf{v} = (3, 4, 5)$, and $\mathbf{w} = (-6, -8, -10)$.

Exercise 6. *Linear dependency*
Show that $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (0, 1, 0)$, $\mathbf{w} = (0, 0, 1)$, and $\mathbf{z} = (1, 1, 1) \in \mathbb{R}^3$ are pairwise linearly independent but linearly dependent as a set.

Exercise 7. *Linearly dependent and independent polynomials of degree 1*
Find the linearly independent pairs among $\mathbf{u} = 1 + x$, $\mathbf{v} = 2 + x$, and $\mathbf{w} = 4 + 2x$.

Definition 3. *Subspace*
A subset W of a linear space V over $\mathbb{R}$ is a *linear subspace* of V if W is a linear space over $\mathbb{R}$.

Observation 9. *Once again, the* **0** *vector is in*
Like for all good linear spaces, **0** must belong to any subspace!

Example 3. *A fake example*
The set W of all the triplets of the form $(1, t, t)$ with $t \in \mathbb{R}$ is not a subspace of $\mathbb{R}^3$, since $\mathbf{0} \notin W$.

Example 4. *A proper subspace of* $\mathbb{R}^3$
The set W of all the triplets of the form $(t, -2t, t)$ with $t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$. Indeed, for any two triplets $\mathbf{u} = (u, -2u, u)$ and $\mathbf{v} = (v, -2v, v)$ with u and $v \in \mathbb{R}$ we have that

$$\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v} = (\alpha u + \beta v, -2(\alpha u + \beta v), \alpha u + \beta v).$$

By setting $w = \alpha u + \beta v$, we see that $\mathbf{w} = (w, -2w, 0) \in W$. Notice that W is a proper subspace of $\mathbb{R}^3$ since $\bar{\mathbf{w}} = (0, 0, 1)$ is not in W.

Example 5. *Another proper subspace of* $\mathbb{R}^3$
The set W of all the triplets of the form $(s, t, 0)$ with $s, t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$. Indeed, for any two triplets $\mathbf{u} = (s_1, t_1, 0)$ and $\mathbf{v} = (s_2, t_2, 0)$ with s_1, t_1, s_2 and $t_2 \in \mathbb{R}$ we have that

$$\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v} = (\alpha s_1 + \beta s_2, \alpha t_1 + \beta t_2, 0).$$

By setting $s_3 = \alpha s_1 + \beta s_2$ and $t_3 = \alpha t_1 + \beta t_2$, we see that $\mathbf{w} = (s_3, t_3, 0) \in W$. Notice that $\bar{\mathbf{w}} = (1, 1, 1) \notin W$.

Example 6. *A proper subspace of the linear space of polynomials of degree 2*
The set W of all the polynomials of degree 2 of the form $\mathbf{v} = v_0 + v_2x^2$ is a proper subspace of the linear space of all polynomials of degree 2. Consider $\mathbf{u} = u_0 + u_2x^2$ and $\mathbf{v} = v_0 + v_2x^2$ with u_0, u_2, v_0 and $v_2 \in \mathbb{R}$. Since

$$\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v} = (\alpha u_0 + \beta v_0) + (\alpha u_2 + \beta v_2)x^2,$$

by setting $w_0 = \alpha u_0 + \beta v_0$ and $w_2 = \alpha u_2 + \beta v_2$, we see that $\mathbf{w} = w_0 + w_2x^2 \in W$. Notice that the polynomial $\bar{\mathbf{w}} = x \notin W$.

Definition 4. *Linear span*
The set W of all the linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$

$$W = Span\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k : t_1, \ldots, t_k \in \mathbb{R}\}$$

is the *linear span* generated by $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$.

Fact 1. *A linear span is a subspace*
For all possible choices of $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ the linear span W is a subspace of V. We do not provide a formal proof of this fact but we observe that the closure of W with respect to the vector sum and scalar multiplication appears to be guaranteed.

Fact 2. *One too many*
If $W = Span\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, then any set of $k + 1$ vectors in W cannot be linearly independent.

Observation 10. *Not a word about linear independency*
Nothing has been said about the linear independence of $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$. Intuitively the linear span W is going to be as rich as the elements $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ are diverse. If all the $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are one and the same, for example, the linear span W consists of all the multiples of the same vector, say $\mathbf{v}_1$. Consequently, if we take $k + 1$ vectors in W all of them, in this case, will be multiple of $\mathbf{w}_1$.

Fact 3. *Dimension of a subspace and basis*
If the k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, $W = Span\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ has a crucial minimal property: a proper subset of these k vectors cannot span W. In this case we say that the *dimension* of W equals k, or $dim(W) = k$, and that the set $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ is a *basis* for W.

Fact 4. *Unique expansion for any fixed basis*
If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a basis for W, any $\mathbf{w}$ of W can always be written as

$$\mathbf{w} = t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k$$

for exactly one choice of the real coefficients $t_1, \ldots, t_k \in \mathbb{R}$.

Exercise 8. *Dimension of* $\mathbb{R}^3$ *and possible bases*
Clearly, $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ span $\mathbb{R}^3$. The dimension of $\mathbb{R}^3$, therefore, is 3 and $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ form a basis for $\mathbb{R}^3$. Furthermore, for any $\mathbf{w} = (a, b, c) \in \mathbb{R}^3$ we have

$$\mathbf{w} = a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3.$$

Therefore, the unique choice of the coefficients t_1, t_2 and t_3 such that $\mathbf{w} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3$ are $t_1 = a$, $t_2 = b$, and $t_3 = c$.
Show that also $\mathbf{f}_1 = (1, 1, 0)$, $\mathbf{f}_2 = (1, 0, 1)$, and $\mathbf{f}_3 = (0, 1, 1)$ span $\mathbb{R}^3$ and form a basis.

Observation 11. *Always the same*
The same vector $\mathbf{w}$ is written as

$$\mathbf{w} = (a, b, c)_{\mathrm{e}}$$

in the $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e_3}$ basis and as

$$\mathbf{w} = \left(\frac{a+b-c}{2}, \frac{a+c-b}{2}, \frac{b+c-a}{2}\right)_{\mathrm{f}}$$

in the $\mathbf{f}_1, \mathbf{f}_2$, and $\mathbf{f_3}$ basis. Indeed, we have

$$\begin{aligned}
\mathbf{w} &= \frac{a+b-c}{2}\,\mathbf{f}_1 + \frac{a+c-b}{2}\,\mathbf{f}_2 + \frac{b+c-a}{2}\,\mathbf{f}_3 \\
&= \frac{a+b-c}{2}\,(1,1,0)_{\mathrm{e}} + \frac{a+c-b}{2}\,(1,0,1)_{\mathrm{e}} + \frac{b+c-a}{2}\,(0,1,1)_{\mathrm{e}} \\
&= \left(\frac{a}{2}+\frac{a}{2}, \frac{b}{2}+\frac{b}{2}, \frac{c}{2}+\frac{c}{2}\right)_{\mathrm{e}} = (a,b,c)_{\mathrm{e}} = a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3.
\end{aligned}$$

Therefore, it does not make sense to speak of the components of a vector without specifying the basis.

Exercise 9. *Dimension of the linear space of polynomials of degree 2 and possible bases*
Clearly, $\mathbf{e}_1 = 1$, $\mathbf{e}_2 = x$, and $\mathbf{e}_3 = x^2$ span the set. Therefore, the dimension is 3 and $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ form a basis. Show that also $\mathbf{f}_1 = 1 + x$, $\mathbf{f}_2 = 1 + x^2$, and $\mathbf{f}_3 = x + x^2$ span the set and form a basis.

Euclidean vector spaces We now turn our attention to linear spaces endowed with an additional structure.

Definition 5. *Scalar product*
A *scalar product* is a map $(\cdot, \cdot)\colon V \times V \to \mathbb{R}$ such that:

1. $\forall\, \mathbf{u}$ and $\mathbf{v} \in V$ $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$ (symmetry);

2. $\forall\, \mathbf{u} \in V$ $(\mathbf{u}, \mathbf{u}) \geq 0$ with $(\mathbf{u}, \mathbf{u}) = 0$ iff $\mathbf{u} = \mathbf{0}$ (positivity);

3. $\forall$ $\mathbf{u}, \mathbf{v}$ and $\mathbf{w} \in V$ and $\forall \alpha, \beta \in \mathbb{R}$ $(\alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}) = \alpha(\mathbf{u}, \mathbf{w}) + \beta(\mathbf{v}, \mathbf{w})$ (linearity).

Observation 12. *Bilinearity*
Combining symmetry with linearity in the *first* argument one can easily see that the scalar product is also linear in the *second* argument.

Observation 13. *Replacing $\mathbb{R}$ with $\mathbb{C}$*
If $\mathbb{C}$ is used instead, the definition must be modified by introducing the complex conjugate as follows

1. $\forall$ $\mathbf{u}$ and $\mathbf{v} \in V$ $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})^*$;
2. $\forall$ $\mathbf{u} \in V$ $(\mathbf{u}, \mathbf{u}) \geq 0$ with $(\mathbf{u}, \mathbf{u}) = 0$ iff $\mathbf{u} = \mathbf{0}$;
3. $\forall$ $\mathbf{u}, \mathbf{v}$ and $\mathbf{w} \in V$ and $\forall \alpha, \beta \in \mathbb{C}$ $(\alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}) = \alpha(\mathbf{u}, \mathbf{w}) + \beta(\mathbf{v}, \mathbf{w})$.

Observation 14. *Norm and distance induced by the scalar product*

- $\|\mathbf{v}\| = \sqrt{(\mathbf{v}, \mathbf{v})}$ is the *norm* of $\mathbf{v}$ (often called *length*)
- $\|\mathbf{u} - \mathbf{v}\|$ is the distance between $\mathbf{u}$ and $\mathbf{v}$, with $\|\mathbf{u} - \mathbf{v}\| = 0$ iff $\mathbf{u} = \mathbf{v}$.

Fact 5. *Three important properties*

1.	$\forall \alpha \in \mathbb{R}$ and $\mathbf{v} \in V$ $\|\alpha\mathbf{v}\| = \vert\alpha\vert\|\mathbf{v}\|$	homogeneity
2.	$\forall \mathbf{u}$ and $\mathbf{v} \in V$ $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$	triangular inequality
3.	$\forall \mathbf{u}$ and $\mathbf{v} \in V$ $\vert(\mathbf{u}, \mathbf{v})\vert \leq \|\mathbf{u}\|\|\mathbf{v}\|$	Cauchy-Schwartz inequality

□

Through the scalar product we can measure the *angle* between two non zero vectors and obtain several useful properties.

Fact 6. *Angle and orthogonality*
Through the Cauchy-Schwartz inequality, the angle θ between $\mathbf{u}$ and $\mathbf{v} \in V$, both different from $\mathbf{0}$ can be defined as

$$\theta = \arccos\left(\frac{(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|\|\mathbf{v}\|}\right).$$

If $(\mathbf{u}, \mathbf{v}) = 0$, $\theta = \pi/2$ and $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal*. Interestingly, the angle can be defined independently of the dimension of V.

Theorem 1. *Pythagora*
If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Proof

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}) = (\mathbf{u}, \mathbf{u})+(\mathbf{u}, \mathbf{v})+(\mathbf{v}, \mathbf{u})+(\mathbf{v}, \mathbf{v}) = (\mathbf{u}, \mathbf{u})+0+0+(\mathbf{v}, \mathbf{v}) = \|\mathbf{u}\|^2+\|\mathbf{v}\|^2$$

■
Notice that the same results hold true for an arbitrary number of mutually orthogonal vectors. We will see that Pythagora's theorem is true even for right triangles with infinite legs!

Definition 6. *Orthonormal basis*
An *orthonormal basis* is a basis which consists of mutually orthogonal vectors of unit length. □

The coefficients describing $\mathbf{v}$ in an orthonormal basis, called *components*, can be simply computed by taking the scalar product between $\mathbf{v}$ and each basis vector, operation which corresponds to the orthogonal projection of $\mathbf{v}$ on each basis vector. Let us obtain this result explicitly.

Observation 15. *A remarkable result*
Assume we are given an orthonormal basis for a space V with $dim(V) = n$. We know we can write any $\mathbf{v}$ as a linear combination of the n basis vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ each multiplied by an appropriate coefficient a_j with $j = 1, \dots, n$ or

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n.$$

The value of each coefficient can actually be computed very easily.
Taking the scalar product on both sides with $\mathbf{v}_j$, through the orthonormality of the basis vectors we obtain for all j

$$(\mathbf{v}, \mathbf{v}_j) = a_j.$$

Fact 7. *Scalar product in components*
If $\mathbf{u}$ and $\mathbf{v}$ are expressed through an orthonormal basis as $\mathbf{u} = u_1, \dots, u_n$ and $\mathbf{v} = v_1, \dots, v_n$, we have that

$$(\mathbf{u}, \mathbf{v}) = u_1v_1 + \dots u_nv_n$$

The norm of a vector can be written

$$\|\mathbf{u}\|^2 = (\mathbf{u}, \mathbf{u}) = u_1^2 + \dots u_n^2,$$

a further example of Pythagora's theorem for a right triangle with n legs!

1.3 Integration by parts

In what follows we assume that all the functions we considered can be differentiated. Assume you have to compute a definite integral of the form

$$\int_a^b f(x)g'(x)\mathrm{d}x$$

where $g'(x)$ denotes the derivative of $g(x)$. The formula of integration by parts tells you that this integral can also be computed as

$$\int_a^b f(x)g'(x)\mathrm{d}x = f(x)g(x)\,\Big|_a^b - \int_a^b g(x)f'(x)\mathrm{d}x.$$

Integration by parts is handy in case you realise that

1. the integrand can be conveniently viewed as the product of two functions, the second thought of as the derivative of a known function, and
2. you are better off computing the integral of the product of the known function with the derivative of the first function rather than the original integral.

Example 7. *Integral of $h(x) = \ln x$ between 1 and e*
We have to compute the integral

$$\int_1^e \ln x \,\mathrm{d}x.$$

If we let $f(x) = \ln x$ and $g(x) = x$, since $f'(x) = 1/x$ we can write

$$\int_1^e \ln x \,\mathrm{d}x = x\ln x\,\Big|_1^e - \int_1^e \mathrm{d}x = e - 0 - (e-1) = 1.$$

Exercise 10. *Prove the integration by parts formula.*

Exercise 11. *Three simple integrals*
Verify that

$$\int_{-\pi}^{\pi} x\cos x \,\mathrm{d}x = 0, \quad \int_{-\pi}^{\pi} \sin x\cos x \,\mathrm{d}x = 0, \quad \text{and} \quad \int_{-\pi}^{\pi} \cos^2 x \,\mathrm{d}x = \int_{-\pi}^{\pi} \sin^2 x \,\mathrm{d}x = \pi$$

2 Fourier series

Any textbook on Fourier Series can be used as a reference.

2.1 A linear space of functions

We know that a vector $\mathbf{v}$ in a Euclidean linear space V of dimension N can be represented by N real numbers $v_1, \ldots, v_N$, projections of $\mathbf{v}$ on each of the N mutually orthogonal unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$ which form an orthonormal basis for V. The vector $\mathbf{v}$ can be written as

$$\mathbf{v} = \sum_{i=n}^{N} v_n \mathbf{e}_n$$

We now consider a Euclidean linear space V in which a vector is a piece-wise continuous function over a fixed interval $[a, b]$. We will learn that a basis for this space can be constructed starting from an infinite number of mutually orthogonal periodic functions of period equal to $(b-a)/N$ for $N = 1, \ldots, +\infty$. Not surprisingly, each vector in V can be written as a linear combination of the basis elements. Neglecting the technical difficulties posed by the fact that the basis has infinite cardinality we obtain the remarkable result that any piece-wise continuous function can be written as a linear combination of a fixed and countable set of functions and thus represented by a discrete sequence of real numbers, components of the function in some fixed basis. Like in the finite dimensional case, we will be able to compute the components by simply taking the scalar product between the function and each basis element.

The fact that a and b are finite has a important consequence: the expansion we find of a function in the interval $[a, b]$ is a periodic function of period $b - a$ over the real line.

Definition 7. *Piece-wise continuitity*
A *piece-wise continuous* function $f : \mathbb{R} \to [a, b]$ has at most a finite number of points between a and b in which it is not continuous. $\square$

Let V be the set of all real valued *piece-wise continuous* functions over $[a, b]$ for $a < b \in \mathbb{R}$.

Fact 8. *V is a linear space*
Since any linear combination of piece-wise continuous functions over $[a, b]$ is piece-wise continuous, it is straightforward to realise that V is a linear space. For notational consistency we denote with $\mathbf{f}$ a piece-wise continuous function $f : \mathbb{R} \to [a, b]$ thought of as en element of V.

Definition 8. *A scalar product between functions*
We define the scalar product between two functions $\mathbf{f}$ and $\mathbf{g} \in V$ as

$$(\mathbf{f}, \mathbf{g}) = \int_a^b f(t) g(t) \mathrm{d}t. \tag{2}$$

We note that the integral is always well defined, since the product of piece-wise continuous functions is piece-wise continuous and, hence, Riemann integrable.

Exercise 12. *Prove that the product in Equation (2) satisfies the three properties of a scalar product.*

Observation 16. *Norm of a function*
We can measure the norm of a function $\mathbf{f}$ as

$$\|\mathbf{f}\| = \sqrt{\int_a^b f^2(t) \mathrm{d}t}. \tag{3}$$

Observation 17. *Orthogonality between functions*
Two functions $\mathbf{f}$ and $\mathbf{g}$ are *orthogonal* if

$$(\mathbf{f}, \mathbf{g}) = \int_a^b f(t)g(t)\mathrm{d}t = 0.$$

Observation 18. *Complex valued case*
If $\mathbf{f}$ and $\mathbf{g}$ are complex valued we define the scalar product as

$$(\mathbf{f}, \mathbf{g}) = \int_a^b f(t)g^*(t)\mathrm{d}t,$$

where $g^*(t)$ is the complex conjugate of $g(t)$ and $(\mathbf{f}, \mathbf{g}) = (\mathbf{g}, \mathbf{f})^*$.

Observation 19. *The intuition behind*
Let us assume, for simplicity, that the two functions $\mathbf{f}$ and $\mathbf{g}$ in Equation (2) are continuous. We divide the interval $[a, b]$ in N equal non-overlapping intervals of width $(b-a)/N$ and consider the N midpoints t_n with $n = 1, \ldots, N$. The integral in Equation (2) can then be rewritten as

$$(\mathbf{f}, \mathbf{g}) = \lim_{N \to +\infty} \frac{b-a}{N} \sum_{n=1}^{N} f(t_n)g(t_n).$$

Apart from the scaling factor $(b-a)/N$ and the limit, the analogy with the scalar product of two vectors $\mathbf{f}$ and $\mathbf{g}$ in a Euclidean vector space of dimension N,

$$(\mathbf{f}, \mathbf{g}) = \sum_{n=1}^{N} f_n g_n,$$

is immediate.

2.2 Bases

Definition 9. *Orthogonal and orthonormal system of functions*
If $(\boldsymbol{\phi}_n, \boldsymbol{\phi}_m) = 0 \ \forall n \neq m \in \mathbb{N}$, then $S_\phi = \{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, \ldots\}$ is an *orthogonal system*. Furthermore, if $(\boldsymbol{\phi}_n, \boldsymbol{\phi}_n) = 1 \ \forall n \in \mathbb{N}$, S_ϕ is *orthonormal*.

Exercise 13. *Orthonormality for* $[a, b] = [-\pi, \pi]$
Verify that $S_\phi = \{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, \ldots\}$ with

$$\boldsymbol{\phi}_0 = \phi_0(x) = \frac{1}{\sqrt{2\pi}}, \quad \boldsymbol{\phi}_{2n-1} = \phi_{2n-1}(x) = \frac{\sin nx}{\sqrt{\pi}}, \quad \boldsymbol{\phi}_{2n} = \phi_{2n}(x) = \frac{\cos nx}{\sqrt{\pi}} \tag{4}$$

with $n \in \mathbb{N}$ is orthonormal.

Exercise 14. *Easier with complex exponentials*
Verify that $S_\psi = \{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots\}$ with

$$\boldsymbol{\psi}_n = \psi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}} \quad \text{with} \quad n \in \mathbb{N}$$

is orthonormal.

We now extend the notion of linear independence to infinite sets. We recall that a finite set of N vectors $\{\boldsymbol{\phi}_1, \ldots \boldsymbol{\phi}_N\}$ are linearly independent if

$$\sum_{n=1}^{N} d_n \phi_n(x) = 0 \ \ \forall x \in [a,b] \text{ and } d_n \in \mathbb{R}$$

implies

$$d_n = 0 \quad \text{for} n = 1, \ldots, N$$

Observation 20. *Infinite case*
If S is infinite, the vectors in S are linearly independent if all finite subsets of S consist of linearly independent vectors.

Exercise 15. *Mutual orthogonality implies linearly independence*
Verify that the vectors of an orthogonal system S are linearly independent.

A natural question to ask is whether an infinite set of orthonormal functions is a basis for a certain vector space. For example, let us consider with the $\boldsymbol{\phi}_n$ as in equation (4) the *Fourier Series*

$$\sum_{n=0}^{+\infty} c_n \phi_n(x) \quad \text{with} \quad c_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t)\phi_n(t)\mathrm{d}t$$

For which values of x does the series converge? And if it does, does it converge to $f(x)$?

We first answer a simpler question. We let

$$\mathbf{c}_N = \sum_{n=1}^{N} c_n \boldsymbol{\phi}_n \ \text{ and } \ \mathbf{d}_N = \sum_{n=1}^{N} d_n \boldsymbol{\phi}_n.$$

While the coefficients c_n are computed by projecting the function $\mathbf{f}$ onto each of the $N < +\infty$ basis vectors considered, the d_n are arbitrary real numbers.

Theorem 2. *Best approximation property*

$$\forall N > 0 \ \ \|\mathbf{f} - \mathbf{d}_N\|^2 \geq \|\mathbf{f} - \mathbf{c}_N\|^2$$

Proof
Since

$$\|\mathbf{c}_N\|^2 = \sum_{n=1}^{N} c_n^2 \ \text{ and } \ (\mathbf{f}, \mathbf{c}_N) = \sum_{n=1}^{N} c_n^2,$$

we have

$$\|\mathbf{f} - \mathbf{c}_N\|^2 = \|\mathbf{f}\|^2 - \sum_{n=1}^{N} c_n^2 \geq 0. \tag{5}$$

Furthermore, since

$$\|\mathbf{d}_N\|^2 = \sum_{n=1}^{N} d_n^2 \ \text{ and } \ (\mathbf{f}, \mathbf{d}_N) = \sum_{n=1}^{N} d_n c_n,$$

adding and subtracting $\sum c_n^2$ and using inequality (5) we obtain

$$\begin{aligned}
\|\mathbf{f} - \mathbf{d}_N\|^2 &= \|\mathbf{f}\|^2 + \|\mathbf{d}_N\|^2 - 2(\mathbf{f}, \mathbf{d}_N) = \|\mathbf{f}\|^2 + \sum_{n=1}^{N} d_n^2 + \sum_{n=1}^{N} c_n^2 - \sum_{n=1}^{N} c_n^2 - 2\sum_{n=1}^{N} d_n c_n \\
&= \|\mathbf{f}\|^2 + \sum_{n=1}^{N} (d_n - c_n)^2 - \sum_{n=1}^{N} c_n^2 = \|\mathbf{f} - \mathbf{c}_N\|^2 + \sum_{n=1}^{N} (d_n - c_n)^2 \geq \|\mathbf{f} - \mathbf{c}_N\|^2
\end{aligned}$$

■

Observation 21. *A general result*
The result we proved holds true for any Euclidean space: no other linear combination leads to a better approximation of a vector $\mathbf{f}$ in a certain subspace than the one obtained by projecting $\mathbf{f}$ along the corresponding basis vectors of that subspace.

Fact 9. *Parseval equality (Pythagora, once again)*
From Equation (5) we have that $\sum c_n^2 \leq \|\mathbf{f}\|^2$ and thus

$$\sum c_n^2 = \|\mathbf{f}\|^2 \Longleftrightarrow \text{for } N \to \infty \;\; \|\mathbf{f} - \mathbf{c}_N\| \to 0$$

Observation 22. *Sanity check*
Parseval formula can thus be used to control whether the representation of a vector in terms of linear combination of certain basis elements is complete. If the square of the vector norm equals the sum of the square of the coefficients we are done. If it is strictly larger, then something is missing. Furthermore, if the Parseval formula holds true for all $\mathbf{f}$, the orthonormal system S is a basis.

Fact 10. *Completeness*
The orthonormal systems of exercise (13) and (14) are complete.

2.3 Convergence and generalisation

Fact 11. *Uniform convergence*
Let $\mathbf{f}$ be piece-wise continuous and x a point of continuity for $\mathbf{f}$. For all $\epsilon > 0$ there exists N^* such that for all $N \geq N^*$

$$\left| \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_k \sin(nx) - f(x) \right| < \epsilon.$$

Observation 23. *Intuition behind uniform convergence*
Uniform convergence takes place when the convergence speed does not depend on x. For a fixed level of accuracy ϵ, it ensures the existence of a unique value of N^* such that the Fourier Series truncated after the first n^* terms is no more than ϵ away from $f(x)$ for all x.

Fact 12. *Gibbs phenomenon*
The Fourier Series converges also at a point x^* of discontinuity for $\mathbf{f}$, but it converges to the average of the limit of $f(x)$ for $x \to x^*_+$ and $x \to x^*_-$. For finite values of n the truncated series at points nearby x^* oscillates somewhat wildly, see Figure (3). This behaviour is known as Gibbs phenomenon.

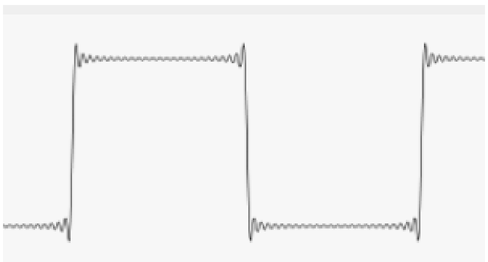

Figure 3: Gibbs phenomenon for a square wave.

Observation 24. *Arbitrary period*
There is nothing special in the period 2π. If the period is $T = b - a = 1/f$ the orthogonal systems vary accordingly. For example, in the case of complex exponentials, we have

$$\phi_n(x) = \frac{e^{i2\pi f n x}}{\sqrt{T}}$$

2.4 A simple example and a piece of advise

Once in a lifetime let us compute by hand the Fourier coefficients in a super simple example. We want to compute the Fourier series associated with the rectangular function $p_{\pi/2}(\cdot)$ where

$$p_{\pi/2}(x) = \begin{cases} 1 & -\pi/2 \leq x \leq \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

We first rewrite the Fourier series the way it is typically found in textbooks. Using the $\phi_n(x)$ defined by Equation (4) we have

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(nx) + b_n \sin(nx) \tag{6}$$

Observation 25. *Taking care of normalisation*
The coefficients of the linear expansion of Equation (6) are proportional but not equal to the projection of the original function on each of the basis element. There is a missing factor $1/\sqrt{\pi}$ in the projection and another $1/\sqrt{\pi}$ when the basis element is used in the expansion. When computing the square of the norm of a functions using the coefficients of Equation (6) this has to be taken into account! □

Since the rectangular function is even we only need to consider the projection of $p_{\pi/2}(\cdot)$ onto the cosine harmonics (including the constant). We thus find

$$\frac{a_0}{2} = \frac{1}{2}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} p_{\pi/2}(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\mathrm{d}t = \frac{1}{2}$$

and

$$a_n = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} p_{\pi/2}(t)\cos(nt)\mathrm{d}t = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(nt)\mathrm{d}t = \frac{2\sin(n\pi/2)}{\pi n} = \frac{2}{\pi n}(-1)^{(n-1)/2}$$

We see that $a_n > 0$ for $n = 1, 5, \ldots$ and negative otherwise. Therefore if we write $n = 2k - 1$ for $k = 1, \ldots$ and replace k with n we obtain

$$p_{\pi/2}(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{\cos((2n-1)x)}{2n-1}$$

Observation 26. *Beware of the equality sign*
The equality sign holds true at all points x for which $p_{\pi/2}$ is continuous. At the points $-\pi/2$ and $\pi/2$ the series converges to $1/2$.

Exercise 16. *The old good Pythagora (plus the usual magic from Euler)*
Check what Parseval tells you for $p_{\pi/2}(\cdot)$.

Observation 27. *Constants matter*
Normalisation factors are not the only constants to be checked. An additional source of confusion is related to M, the number of points used for sampling the considered interval. To be on the safe side M needs to be at least an order of magnitude greater than the highest harmonics. Speaking of M, do not forget that when computing an integral as a scalar product you need to enforce the normalisation factor $(b - a)/M$.

Furthermore, in the Parseval equality each coefficient is computed by taking the scalar product between the function and an element of an orthonormal basis. The sum of the square of the coefficients in Equation (7) and (8), therefore, equals the square of the norm of the function divided by the square of the normalisation factor (the basis element appears twice: in the scalar product computation and as one of the vectors in the linear combination).

Exercise 17. *Real or complex?*
In textbooks the Fourier series of a function $\mathbf{f}$, over the interval $[0, 2\pi]$ for simplicity, is typically presented as a series of sines and cosines

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos(nx) + b_n \sin(nx)) \tag{7}$$

or complex exponentials

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}. \tag{8}$$

Verify that for real valued functions if $c_0 = a_0/2$ and $c_n = (a_n - ib_n)/2$ for $n \neq 0$ and $c_{-n} = c_n^*$ the series in Equation (7) and (8) are the same.

3 Fourier Transform

This section is mainly taken from the **Fourier Integral and its Applications** by *Papoulis*.

3.1 A lengthy preamble

The Fourier Transform can be viewed as a generalisation of the Fourier Series for functions defined over the entire real line. Instead of an infinite set of discrete coefficients, the Fourier Transform is a complex valued function of the frequency. The representation provided by the Fourier Transform is thus much easier to interpret and use than the Fourier Series coefficients. Most importantly, while the Fourier Series of a signal in the interval $[a, b]$ represents and reconstructs the signal as a periodic function of period $b - a$, the Fourier Transform represents and reconstructs the signal for what it is.

A first model for the δ function Let $T > 0$ and consider the function

$$r_T(t) = \frac{1}{2T} p_T(t)$$

where, as usual,

$$p_T(t) = \begin{cases} 1 & -T \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

For all values of T, $r_T(\cdot)$ defines a *rectangle* of unit area since

$$\int_{-\infty}^{+\infty} r_T(t)\mathrm{d}t = \frac{1}{2T} \int_{-T}^{T} \mathrm{d}t = \frac{2T}{2T} = 1.$$

If $\phi : \mathbb{R} \to \mathbb{R}$ is a function continuous in $t = 0$, for small values of T we have

$$\int_{-\infty}^{+\infty} r_T(t)\phi(t)\mathrm{d}t = \frac{1}{2T} \int_{-T}^{T} \phi(t)\mathrm{d}t \approx \frac{1}{2T} 2T\phi(0) = \phi(0).$$

If we take the limit for $T \to 0$ we thus have

$$\lim_{T\to 0} \int_{-\infty}^{+\infty} r_T(t)\phi(t)\mathrm{d}t = \phi(0). \tag{9}$$

The *limit* in Equation (9) cannot be exchanged with the *integral*. However, if we set $\lim_{T\to 0} r_T(t) = \delta(t)$ we obtain

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ +\infty & t = 0 \end{cases}$$

with
$$\int_{-\infty}^{+\infty} \delta(t)\mathrm{d}t = 1.$$
Clearly, no ordinary function behaves like $\delta(t)$. Nevertheless, we write
$$\int_{-\infty}^{+\infty} \delta(t)\phi(t)\mathrm{d}t = \phi(0) \tag{10}$$
as a suggestive way to represent N_δ, the *distribution* which associates to a function $\phi(\cdot) : \mathbb{R} \to \mathbb{R}$, continuous in $t = 0$, its value $\phi(0)$, or
$$N_\delta[\phi] = \phi(0). \tag{11}$$

Observation 28. *Why bother*
Let alone the fact that integrals with a δ function inside are very easy to evaluate, the rationale for introducing $\delta(t)$, example of a *singularity* function or *generalised* function, is that Equation (10) makes it apparent that the distribution N_δ in Equation (11) enjoys all the properties applicable to an integral.

Observation 29. *Sampling*
In this course, the distribution N_δ is of paramount importance. It describes the process through which a 1D time-varying signal is sampled at the specific time stamp, or a 2D image at a specific pixel!

Observation 30. *Distributions*
We introduced the concept of distribution starting with the somewhat awkward example of a singularity function. Here are three examples you already came across in terms of ordinary functions. We invariably assume that for all the functions we use the integrals we write exist.

1. For all $T \neq 0$ we write
$$N_{p_T}[\phi] = \int_{-\infty}^{+\infty} p_T(t)\phi(t)\mathrm{d}t = \int_{-T}^{T} \phi(t)\mathrm{d}t.$$
The distribution $N_{p_T}[\phi]$ associates to ϕ the integral of ϕ between $-T$ and T.

2. If
$$U(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
is the *unit step* function, we have
$$N_U[\phi] = \int_{-\infty}^{+\infty} U(t)\phi(t)\mathrm{d}t = \int_{0}^{+\infty} \phi(t)\mathrm{d}t.$$
The distribution $N_U[\phi]$ associates to ϕ the integral of ϕ between 0 and $+\infty$.

3. If X is a random variable taking on values in the interval $[-\infty, \infty]$, $\phi(X)$ is function of X, and $f(x) \geq 0$ the probability density function of X with
$$\int_{-\infty}^{+\infty} f(x)\mathrm{d}x = 1,$$
the distribution
$$N_f[\phi] = \int_{-\infty}^{+\infty} \phi(x)f(x)\mathrm{d}x$$
is $\mathbb{E}[\phi(X)]$, the *expected value* of $\phi(X)$. Notice that the notation $N_f[\phi]$ unlike $\mathbb{E}[\phi(X)]$ makes it explicit the dependence of the expected value on the probability density function.

Properties In what follows we assume that any function ϕ can be differentiated as many times as we need and $\phi(t) \to 0$ faster than any polynomials for $t \to \pm\infty$.

Linearity For any distribution $g(t)$, as an immediate consequence of the integral notation, we have

$$\int_{-\infty}^{+\infty} g(t)\left(\alpha\phi_1(t) + \beta\phi_2(t)\right)\mathrm{d}t = \alpha\int_{-\infty}^{+\infty} g(t)\phi_1(t)\mathrm{d}t + \beta\int_{-\infty}^{+\infty} g(t)\phi_2(t)\mathrm{d}t.$$

Sifting Substituting $t - t_0 \to t$ gives for any distribution $g(t)$

$$\int_{-\infty}^{+\infty} g(t - t_0)\phi(t)\mathrm{d}t = \int_{-\infty}^{+\infty} g(t)\phi(t + t_0)\mathrm{d}t$$

For the δ function this reduces to

$$\int_{-\infty}^{+\infty} \delta(t - t_0)\phi(t)\mathrm{d}t = \phi(t_0).$$

Scale For all $a \neq 0$, substituting $at \to t$ gives that for any distribution $g(t)$

$$\int_{-\infty}^{+\infty} g(at)\phi(t)\mathrm{d}t = \frac{1}{|a|}\int_{-\infty}^{+\infty} g(t)\phi\left(\frac{t}{a}\right)\mathrm{d}t.$$

The absolute value is due to the need of interchanging the limits, when $a < 0$, to restore integration from $-\infty$ to $+\infty$.

Derivative Integrating by parts (since $\phi(t) \to 0$ for $t \to \pm\infty$) we have that for the derivative of a distribution $g(t)$ we have

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}g(t)}{\mathrm{d}t}\phi(t)\mathrm{d}t = -\int_{-\infty}^{+\infty} g(t)\frac{\mathrm{d}\phi(t)}{\mathrm{d}t}\mathrm{d}t.$$

Since

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}U(t)}{\mathrm{d}t}\phi(t)\mathrm{d}t = -\int_{-\infty}^{+\infty} U(t)\frac{\mathrm{d}\phi(t)}{\mathrm{d}t}\mathrm{d}t = -\int_{0}^{+\infty} \frac{\mathrm{d}\phi(t)}{\mathrm{d}t}\mathrm{d}t = \phi(0) - \phi(+\infty) = \phi(0),$$

we see that the $\delta(t)$ can be seen as the derivative of the unit step $U(t)$. This result is consistent with the intuition that the derivative of the unit step is 0 for all $t \neq 0$ and diverges in 0 because of the discontinuity.

Integral of the *sinc* We start by observing that

$$\begin{aligned}\int_0^{+\infty} e^{-xt}\sin x\mathrm{d}x &= \int_0^{+\infty} e^{-xt}\frac{e^{ix} - e^{-ix}}{2i}\mathrm{d}x = \frac{1}{2i}\left(\int_0^{+\infty} e^{x(i-t)}\mathrm{d}x - \int_0^{+\infty} e^{-x(i+t)}\mathrm{d}x\right) \\ &= \frac{1}{2i}\left.\frac{e^{x(i-t)}}{i-t}\right|_0^{+\infty} - \frac{1}{2i}\left.\frac{e^{-x(i+t)}}{-i-t}\right|_0^{+\infty} = \frac{1}{2i}\left(\frac{1}{t-i} - \frac{1}{t+i}\right) = \frac{1}{1+t^2} \qquad (12)\end{aligned}$$

Since

$$\int_0^{+\infty} e^{-xt}\mathrm{d}t = -\left.\frac{e^{-xt}}{x}\right|_0^{+\infty} = \frac{1}{x}$$

we have

$$\int_{-\infty}^{+\infty} \frac{\sin x}{\pi x}\,\mathrm{d}x = \frac{2}{\pi}\int_0^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x = \frac{2}{\pi}\int_{-\infty}^{+\infty} \sin x\left(\int_0^{+\infty} e^{-xt}\,\mathrm{d}t\right)\mathrm{d}x, \qquad (13)$$

and if we swap the order of integration in Equation (13) and use Equation (12) we finally obtain

$$\int_{-\infty}^{+\infty}\frac{\sin x}{\pi x}\,\mathrm{d}x = \frac{2}{\pi}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} e^{-xt}\sin x\,\mathrm{d}x\right)\mathrm{d}t = \frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{1}{1+t^2}\,\mathrm{d}t = \frac{2}{\pi}\arctan t\Big|_0^{+\infty} = 1 \qquad (14)$$

Observation 31. *Can it always be done?*
The conditions under which it is safe to swap the order of integration in a multiple integral are rather mild. In essence, we can always do it as far as the integral, as a whole, converges.

The Riemann-Lebesgue lemma In the case of Fourier Series we know that for a convergent series the coefficients c_k must approach 0 for $k \to \pm\infty$. The Riemann-Lebesgue lemma ensures that a similar result holds true in the continuous case or that

$$\lim_{\omega\to+\infty} \int_a^b f(t)\cos(t)\mathrm{d}t = 0 \text{ and } \lim_{\omega\to+\infty} \int_a^b f(t)\sin(t)\mathrm{d}t = 0$$

for all finite or infinite a and b. The proof is beyond the scope of this class. A hand-waving explanation is that for large values of ω, over each small interval around a point t, the functions $f(t)\cos(t)$ and $f(t)\sin(t)$ oscillate infinitely many times between $f(t)$ and $-f(t)$ thereby zeroing both integral values in the interval.

Another model for $\delta(t)$ We are now ready to show that

$$\lim_{\omega\to+\infty} \frac{\sin(\omega t)}{\pi t} = \delta(t)$$

Proof: We start by writing for some small $\epsilon > 0$

$$\lim_{\omega\to+\infty} \int_{-\infty}^{+\infty} \frac{\sin(\omega t)}{\pi t}\phi(t)\mathrm{d}t = \lim_{\omega\to+\infty}\left(\int_{-\infty}^{-\epsilon} \frac{\sin(\omega t)}{\pi t}\phi(t)\mathrm{d}t + \int_{-\epsilon}^{+\epsilon} \frac{\sin(\omega t)}{\pi t}\phi(t)\mathrm{d}t + \int_{+\epsilon}^{+\infty} \frac{\sin(\omega t)}{\pi t}\phi(t)\mathrm{d}t\right)$$

In virtue of the *Riemann-Lebesgue* lemma the first and the third integral in the limit vanish. If we set $x = \omega t$ and take into account Equation (14), the middle integral writes

$$\lim_{\omega\to+\infty} \int_{-\epsilon}^{+\epsilon} \frac{\sin(\omega t)}{\pi t}\phi(t)\mathrm{d}t = \phi(0) \lim_{\omega\to+\infty} \int_{-\epsilon\omega}^{+\epsilon\omega} \frac{\sin x}{\pi x}\mathrm{d}x = \phi(0) \int_{-\infty}^{+\infty} \frac{\sin x}{\pi x}\mathrm{d}x = \phi(0).$$

■

In summary, the functions $r_T(t)$ for $T \to 0$ and $\sin(\omega t)/(\pi t)$ for $\omega \to \infty$ behave very similarly: they both approximate an impulse of unit area at the origin.

And yet another (very important) model We are not finished. We also have another way to obtain the δ function. Namely

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \cos(\omega t)\mathrm{d}\omega = \lim_{\Omega\to\infty} \int_{-\Omega}^{+\Omega} \frac{\cos(\omega t)}{2\pi}\mathrm{d}\omega = \lim_{\Omega\to\infty} \left.\frac{\sin(\omega t)}{2\pi t}\right|_{-\Omega}^{+\Omega} = \lim_{\Omega\to\infty} \frac{\sin(\Omega t)}{\pi t} = \delta(t)$$

Notice that the above integral exists only as a distribution.

Observation 32. *One more step*
The parity of the *cosine* and *sine* functions tells us that we also have

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t}\mathrm{d}\omega = \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty} \cos\omega t\mathrm{d}\omega + i\int_{-\infty}^{+\infty} \sin\omega t\mathrm{d}\omega\right) = \delta(t) + 0 = \delta(t). \qquad (15)$$

Here again, the second integral exists and is equal to 0 only as a distribution.

3.2 A reversible transformation

We are now fully equipped to introduce the Fourier Transform.

Back and forth We define the Fourier Transform $F(\omega)$ of a function $f(t)$ as

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\mathrm{d}t.$$

The representation of $f(t)$ provided by $F(\omega)$ is *reversible* in the sense that we also have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\mathrm{d}\omega$$

Indeed we have

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(t')e^{-i\omega t'}\mathrm{d}t'\right)e^{i\omega t}\mathrm{d}\omega. \tag{16}$$

If we change the order of integration in Equation (16), and use Equation (15) and the sifting property, we finally obtain

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} e^{i\omega(t-t')}\mathrm{d}\omega\right)f(t')\mathrm{d}t' = \int_{-\infty}^{+\infty}\delta(t-t')f(t')\mathrm{d}t' = f(t)$$

In this definition we used the angular frequency $\omega = 2\pi\nu$ (where ν is the true frequency). This is the reason for which we have a factor $1/(2\pi)$ in the Inverse Transform.

Observation 33. *Existence*
If the function $f(t)$ is such that

$$\int_{-\infty}^{\infty} |f(t)|\mathrm{d}t < +\infty$$

then $F(\omega)$ exists as an ordinary function. Similarly to the Fourier Series case, the inversion formula converges point-wise at the points where f is continuous and to the average at the points of discontinuity. We will often consider cases in which the Fourier Transform and its inverse exist as distributions.

Observation 34. *Analogy with Fourier Series*
The analogy with the Fourier Series with complex exponentials is apparent. The Fourier Transform $F(\omega)$ corresponds to the coefficients c_n, while the Inverse Fourier Transform to the reconstruction of the signal as linear combination of the basis elements each multiplied by the corresponding coefficient. The sign change in the complex exponential between the Transform and the Inverse Transform completes the analogy.

Main properties

Linearity If $f(t) = ag(t) + bh(t)$, then $F(\omega) = aG(\omega) + bH(\omega)$. Immediate consequence of the definition of Fourier Transform as an integral

Conjugation If $f(t) \Leftrightarrow F(\omega)$, then $f^*(t) \Leftrightarrow F^*(-\omega)$. Since $zw^* = (z^*w)^*$, we have

$$\int_{-\infty}^{+\infty} f^*(t)e^{-i\omega t}\mathrm{d}t = \left(\int_{-\infty}^{+\infty} f(t)e^{i\omega t}\mathrm{d}t\right)^* = F^*(-\omega)$$

Duality If $f(t) \Leftrightarrow F(\omega)$, then $F(t) \Leftrightarrow 2\pi f(-\omega)$. Indeed viewing $F(t)$ has a Fourier Transform by setting $t = \Omega$ we obtain

$$\int_{-\infty}^{+\infty} F(t)e^{-i\omega t}\mathrm{d}t = \frac{2\pi}{2\pi}\int_{-\infty}^{+\infty} F(\Omega)e^{-i\omega\Omega}\mathrm{d}\Omega = 2\pi f(-\omega)$$

Time shift If $f(t) \Leftrightarrow F(\omega)$, then $f(t-t_0) \Leftrightarrow e^{-i\omega t_0}F(\omega)$. The substitution $t - t_0 \to t$ yields

$$\int_{-\infty}^{+\infty} f(t-t_0)e^{-i\omega t}\mathrm{d}t = \int_{-\infty}^{+\infty} f(t)e^{-i\omega(t+t_0)}\mathrm{d}t = e^{-i\omega t_0}\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\mathrm{d}t = e^{-i\omega t_0}F(\omega)$$

Frequency shift If $f(t) \Leftrightarrow F(\omega)$, then $e^{i\omega_0 t}f(t) \Leftrightarrow F(\omega-\omega_0)$. The substitution $\omega - \omega_0 \to \omega$ yields

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\omega-\omega_0)e^{i\omega t}\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\omega)e^{i(\omega+\omega_0)t}\mathrm{d}\omega = e^{i\omega_0 t}\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\omega)e^{i\omega t}\mathrm{d}\omega = e^{i\omega_0 t}f(t)$$

Derivative If $f(t) \Leftrightarrow F(\omega)$, then $\mathrm{d}f/\mathrm{d}t \Leftrightarrow i\omega F(\omega)$. If F', the Fourier Transform of the derivative, exists we have

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\mathrm{d}\omega\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} i\omega F(\omega)e^{i\omega t}\mathrm{d}\omega$$

and thus $F'(\omega) = i\omega F(\omega)$.

Integral If $f(t) \Leftrightarrow F(\omega)$ anf $F(0) = 0$, then $\int^t f(\tau)/\mathrm{d}\tau \Leftrightarrow F(\omega)/(i\omega)$. Indeed, interchanging the order of integration we find

$$\int_{-\infty}^{t}f(\tau)\mathrm{d}\tau=\frac{1}{2\pi}\int_{-\infty}^{t}\int_{-\infty}^{+\infty}F(\omega)e^{i\omega\tau}\mathrm{d}\omega\mathrm{d}\tau=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\omega)\int_{-\infty}^{t}e^{i\omega\tau}\mathrm{d}\tau\mathrm{d}\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{F(\omega)}{i\omega}e^{i\omega t}\mathrm{d}\omega.$$

Convolution If $h(t) = f * g$, then $H(\omega) = F(\omega)G(\omega)$. Since

$$h(t) = f * g = \int_{-\infty}^{+\infty} f(x)g(t-x)\mathrm{d}x,$$

interchanging the order of integration and using the time shift property we obtain

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(x)g(t-x)\mathrm{d}x\right)e^{-i\omega t}\,\mathrm{d}t = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-i\omega t}g(t-x)\,\mathrm{d}t\right)f(x)\,\mathrm{d}x \\ &= G(\omega)\int_{-\infty}^{+\infty} e^{-i\omega x}f(x)\,\mathrm{d}x = G(\omega)F(\omega) \end{aligned}$$

We also have that if $h(t) = f(t)g(t)$, then $2\pi H(\omega) = F * G$.

Parseval formula and Plancherel (Pythagora!) Theorem

$$\int_{-\infty}^{+\infty} f(t)g^*(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)G^*(\omega)\mathrm{d}\omega$$

Interchanging twice the order of integration, and using Equation (15) and the sifting property we find

$$
\begin{aligned}
\int_{-\infty}^{+\infty} f(t)g^*(t)\mathrm{d}t &= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\mathrm{d}\omega\right)\left(\int_{-\infty}^{+\infty} G^*(-\omega')e^{i\omega' t}\mathrm{d}\omega'\right)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\omega)G^*(-\omega')\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(\omega-\omega')t}\mathrm{d}t\right)\,\mathrm{d}\omega\,\mathrm{d}\omega' \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\omega)G^*(-\omega')\delta(\omega-\omega')\,\mathrm{d}\omega\,\mathrm{d}\omega' \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} G^*(-\omega')\delta(\omega-\omega')\,\mathrm{d}\omega'\right)F(\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)G^*(\omega)\mathrm{d}\omega
\end{aligned}
$$

Setting $g(t) = f^*(t)$ Parseval formula becomes Pythagora theorem (aka Plancherel theorem), or

$$
\int_{-\infty}^{+\infty} |f(t)|^2\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2\mathrm{d}\omega
$$

□

An important difference between Fourier series and Fourier Transform is that while the Fourier series of a real valued function is always real valued, the Fourier Transform of a real valued function f is, in general, complex. This can be readily seen applying the definition since we have $F(\omega) = R(\omega) - iX(\omega)$ with

$$
R(\omega) = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)\mathrm{d}t \quad \text{and} \quad X(\omega) = \int_{-\infty}^{+\infty} f(t)\sin(\omega t)\mathrm{d}t.
$$

Exercise 18. *Real valued and pure imaginary Fourier Transform*
Which functions have real and purely imaginary Fourier Transform?

Observation 35. *Module and phase*
Module and phase of a Fourier Transform follow the usual definition. We thus have

$$
F(\omega) = R(\omega) - iX(\omega) = A(\omega)e^{-i\Phi(\omega)}
$$

The module $A(\omega)$ tells you how much of the signal is captured at the angular frequency ω, the phase $\Phi(\omega)$ how the frequencies are lined up for reconstructing the signal. They are both essential. While the module is what you need to understand the weight of a certain frequency range in the representation of a function, without phase information no reconstruction can take place.

Observation 36. *Dependencies*
Unless some prior information on the signal (like causality, for example) is available, the real and the imaginary part, or module and phase of a Fourier Transform are independent.

Fact 13. *You cannot have both*
From the pairs we have seen, it appears that a function and its Fourier Transform exhibit a dual property. A function of finite support has a Fourier Transform with infinite support and the other way around.

3.3 Pairs

1. $p_T(t) \Leftrightarrow 2\sin(\omega T)/\omega$

   We first consider a translation in time of $p_T(t) \to p_T(t-t_0)$. Using the translation property we immediately have
   $$p_T(t-t_0) \Leftrightarrow \frac{2\sin(\omega T)}{\omega} e^{-i\omega t_0}$$
   From this pair we can appreciate the meaning of module and phase. The module is left unchanged while the phase change, not surprisingly, is determined by t_0.

   Furthermore, since
   $$p_T(t+2T) = \frac{2\sin(\omega T)}{\omega} e^{i2\omega T} \quad \text{and} \quad p_T(t-2T) = \frac{2\sin(\omega T)}{\omega} e^{-i2\omega T}$$
   we have
   $$p_T(t+2T) + p_T(t-2T) \Leftrightarrow \frac{4\sin(\omega T)}{\omega}\cos(2\omega T).$$
   Since
   $$p_{T/2}(t+T/2) = \frac{2\sin(\omega T/2)}{\omega} e^{i\omega T/2} \quad \text{and} \quad p_{T/2}(t-T/2) = \frac{2\sin(\omega T/2)}{\omega} e^{-i\omega T/2}$$
   we also have
   $$p_{T/2}(t+T/2) - p_{T/2}(t-T/2) \Leftrightarrow \frac{4i\sin^2(\omega T/2)}{\omega}.$$
   Since
   $$p_T(t)\cos(\omega_0 t) = p_T(t)\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2},$$
   using the frequency shift property we find
   $$p_T(t)\cos(\omega_0 t) \Leftrightarrow \frac{\sin[(\omega-\omega_0)t]}{(\omega-\omega_0)t} + \frac{\sin[(\omega+\omega_0)t]}{(\omega+\omega_0)t}.$$
   Finally, by interchanging the dependency on t and ω, and taking care of the factor $1/(2\pi)$ we have
   $$\frac{\sin(\omega_b t)}{\pi t} \Leftrightarrow p_{\omega_b}(\omega).$$
   This pair yields the first example of a Fourier Transform with *limited* support in the frequency domain. Functions with Fourier Transform of finite support are called *band limited*. We will see that the band limited functions can always be sampled safely.

2. $q_T(t) \Leftrightarrow 4\sin^2(\omega T/2)/(T\omega^2)$

   We consider the triangle function $q_T(t)$ defined as
   $$q_T(t) = \begin{cases} 1 + t/T & -T \le t \le 0 \\ 1 - t/T & 0 < t \le T \\ 0 & \text{otherwise.} \end{cases}$$
   Since
   $$q_T(t) = \frac{1}{T}\int_{-\infty}^{t} \left(p_{T/2}(\tau + T/2) - p_{T/2}(\tau - T/2)\right) \mathrm{d}\tau$$
   the pair $q_T(t) \Leftrightarrow 4\sin^2(\omega T/2)/(T\omega^2)$ follows from the integral property.

   Applying the symmetry property $F(t)/(2\pi) \Leftrightarrow f(-\omega)$ to this pair with $T = 2a$ we obtain
   $$\frac{\sin^2(at)}{\pi a t^2} \Leftrightarrow q_{2a}(\omega).$$

3. $e^{-\alpha t}U(t) \Leftrightarrow 1/(\alpha + i\omega)$

   From the definition of Fourier Transform we find

$$\int_{-\infty}^{+\infty} e^{-\alpha t}U(t)e^{-i\omega t}\mathrm{d}t = \int_{0}^{+\infty} e^{-(\alpha+i\omega)t}\mathrm{d}t = \left.\frac{-e^{-(\alpha+i\omega)t}}{\alpha+i\omega}\right|_{0}^{+\infty} = \frac{1}{\alpha+i\omega}.$$

   For the even part of $e^{-\alpha t}U(t)$ we have $f_e(t) = e^{-\alpha|t|}/2$. Since $f_e(t) \Leftrightarrow R(\omega)$, the Fourier Transform of $e^{-\alpha|t|}$ is simply twice the real part of $1/(\alpha + i\omega)$, or

$$2\mathrm{Re}\left(\frac{1}{\alpha+i\omega}\right) = \frac{2\alpha}{\alpha^2+\omega^2}.$$

4. $e^{-t^2/2\sigma^2} \Leftrightarrow \sqrt{2\pi}\sigma e^{-\omega^2\sigma^2/2}$

   Let $g(t) = e^{-t^2/2\sigma^2}$. For the Fourier Transform we have

$$G(\omega) = \int_{-\infty}^{+\infty} e^{-t^2/2\sigma^2}e^{-i\omega t}\mathrm{d}t.$$

   taking the derivative of $G(\omega)$ with respect to ω we obtain

$$\frac{\mathrm{d}G(\omega)}{\mathrm{d}\,\omega} = \int_{-\infty}^{+\infty} (-it)e^{-t^2/2\sigma^2}e^{-i\omega t}\mathrm{d}t.$$

   If we integrate by parts the integral in the right hand side with

$$u(t) = e^{-i\omega t} \quad \text{and} \quad v(t) = ie^{-t^2/(2\sigma^2)},$$

   since

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -i\omega e^{-i\omega t} \quad \text{and} \quad \frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{-it}{\sigma^2}e^{-t^2/(2\sigma^2)},$$

   and both $u(t), v(t) \to 0$ for $t \to \pm\infty$, we find

$$\frac{\mathrm{d}G(\omega)}{\mathrm{d}\,\omega} = \sigma^2\int_{-\infty}^{+\infty} \frac{\mathrm{d}v}{\mathrm{d}t}u\mathrm{d}t = uv\Big|_{-\infty}^{+\infty} - \sigma^2\int_{-\infty}^{+\infty} \frac{\mathrm{d}u}{\mathrm{d}t}v\mathrm{d}t = -\omega\sigma^2\int_{-\infty}^{+\infty} e^{-t^2/2\sigma^2}e^{-i\omega t}\mathrm{d}t = -\omega\sigma^2 G(\omega).$$

   The general solution to this differential equation, for some constant $K \in \mathbb{R}$, is

$$G(\omega) = Ke^{-\omega^2\sigma^2/2}.$$

   But we know that

$$G(0) = \int_{-\infty}^{+\infty} e^{-t^2/(2\sigma^2)}\mathrm{d}t = \sigma\sqrt{2\pi},$$

   therefore, $K = G(0) = \sigma\sqrt{2\pi}$ and we finally have

$$G(\omega) = \sigma\sqrt{2\pi}e^{-\omega^2\sigma^2/2}.$$

Observation 37. *The Gaussian is an eigenfunction of the Fourier operator*
The Fourier Transform of a Gaussian is a Gaussian! If σ^2 is the variance in time, $1/\sigma^2$ is the variance in frequency. Therefore the Fourier Transform of a Gaussian peaked around the origin in space is a shallow Gaussian in the frequency domain. *Viceversa*, the Fourier Transform of a shallow Gaussian in the spatial domain is a Gaussian peaked in the low frequency range.

3.4 Sampling

Infinite Impulse Train An infinite impulse train with period T_s can be written as

$$i(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \tag{17}$$

Setting $\omega_s = 2\pi/T_s$ we can expand $i(t)$ in its Fourier Series in the interval $[0, T_s]$ and obtain

$$i(t) = \sum_{k=-\infty}^{+\infty} c_k e^{ikt\omega_s}$$

Using Equation (17) and observing that the only relevant δ in $[0, T_s]$ is obtained for $n = 0$, for the coefficient c_k we find

$$c_k = \frac{1}{T_s}\int_0^{T_s} i(t)e^{-ikt\omega_s}\mathrm{d}t = \frac{1}{T_s}\int_0^{T_s} \delta(t)e^{-ikt\omega_s}\mathrm{d}t = \frac{1}{T_s}$$

and thus we have

$$i(t) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} e^{ikt\omega_s}$$

If we now take the Fourier Transform of $i(t)$ and interchange the integral with the series we obtain

$$I(\omega) = \frac{1}{T_s}\int_{-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} e^{ikt\omega_s}e^{-i\omega t}\mathrm{d}t = \frac{2\pi}{T_s}\sum_{k=-\infty}^{+\infty}\delta(\omega - k\omega_s) = \omega_s\sum_{k=-\infty}^{+\infty}\delta(\omega - k\omega_s)$$

Observation 38. *Duality*
The Fourier Transform of a periodic impulse train is a periodic impulse train. Interestingly, if the impulses are T_s far apart in time, the transformed impulses are $1/T_s$ far apart in frequency.

What really happens We are now ready to analyse what happens when a signal is sampled over time. Let $s_{smp}(t)$ be the sampling of $s(t)$ at a discrete infinite sequence of equidistant times t_n with $n \in (-\infty, \infty)$. By definition of sampling we have

$$s_{smp}(t) = \begin{cases} s(t) & t = t_n \\ 0 & \text{otherwise} \end{cases}$$

We set $t_n = nT_s$ for some fixed time interval T_s and we model the sampled signal s_{smp} through a train of δ functions

$$s_{smp}(t) = s(t)i(t) = \sum_{n=-\infty}^{+\infty} s(t)\delta(t - nT_s) = \sum_{n=-\infty}^{+\infty} s(nT_s)\delta(t - nT_s).$$

Via the convolution property we have

$$S_{smp}(\omega) = \frac{1}{2\pi}S * I = \frac{1}{2\pi}\omega_s\sum_{k=-\infty}^{+\infty}\int_{-\infty}^{+\infty} S(\bar{\omega})\delta(\omega - k\omega_s - \bar{\omega})\mathrm{d}\bar{\omega} = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} S(\omega - k\omega_s) \tag{18}$$

Observation 39. *Aliasing*
From Equation (18) we see that $S_{smp}(\omega)$ is the sum of equidistant copies of $S(\omega)$, ω_s far apart. Consequently, if $\mathcal{L}(S)$ - length of the smallest interval over which $S(\omega) \neq 0$ - is larger than ω_s, $S_{smp}(\omega)$ and $S(\omega)$ over the interval $[-\omega_s/2, \omega_s/2]$ are different. This phenomenon is known as *aliasing* because higher frequencies of $S(\omega)$ are disguised as lower frequencies of $S_{smp}(\omega)$, see Figure (4). □

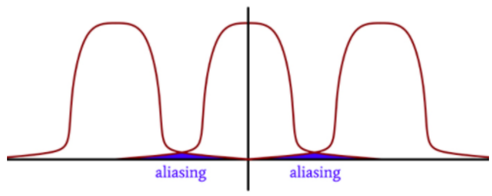

Figure 4: See text.

We are now ready to state and prove the Sampling Theorem, the cornerstone of digital signal processing. The Theorem establishes the condition under which the sampling of a signal does not induce aliasing.

Theorem 3. *Sampling theorem (Shannon)*
Let $S(\omega)$ be the Fourier Transform of a signal $s(t)$ with $S(\omega) = 0$ for $|\omega| \geq \omega_b$ (we thus know that $\mathcal{L}(S) \leq 2\omega_b$). If $t_n = n\pi/\omega_b$ with $n \in (-\infty, \infty)$ are equidistant times and $s_{smp}(t)$ the sampling of $s(t)$ at the times t_n, then $s(t)$ can be reconstructed *exactly* for all t from $s_{smp}(t)$ through the formula

$$s(t) = \sum_{n=-\infty}^{+\infty} s_{smp}(t_n) \frac{\sin(\omega_b(t - t_n))}{\omega_b(t - t_n)}.$$

As usual whenever we write $(\sin x)/x$ we assume that for $x = 0$ $(\sin x)/x = 1$.
Proof
From the hypothesis we have that for the Inverse Fourier Transform of $S(\omega)$ we have

$$s(t) = \frac{1}{2\pi} \int_{-\omega_b}^{\omega_b} S(\omega) e^{i\omega t} \mathrm{d}\omega \tag{19}$$

For $t_n = n\pi/\omega_b$, therefore, Equation (19) writes

$$s_{smp}(t_n) = s(t_n) = s\left(n\frac{\pi}{\omega_b}\right) = \frac{1}{2\pi} \int_{-\omega_b}^{\omega_b} S(\omega) e^{in\pi\omega/\omega_b} \mathrm{d}\omega \tag{20}$$

On the other hand, the Fourier Series expansion of $S(\omega)$ in the interval $[-\omega_b, \omega_b]$ is

$$S(\omega) = \sum_{n=-\infty}^{+\infty} c_n e^{in\pi\omega/\omega_b}$$

with

$$c_n = \frac{1}{2\omega_b} \int_{-\omega_b}^{\omega_b} S(\omega) e^{-in\pi\omega/\omega_b} \mathrm{d}\omega \tag{21}$$

Comparing Equation (20) and (21) we find

$$c_n = \frac{\pi}{\omega_b} s\left(-n\frac{\pi}{\omega_b}\right) = \frac{\pi}{\omega_b} s(t_{-n}) = \frac{\pi}{\omega_b} s_{smp}(t_{-n})$$

Absorbing the change of sign we write

$$S_{smp}(\omega) = \sum_{n=-\infty}^{\infty} \frac{\pi}{\omega_b} s(t_{-n}) e^{in\pi\omega/\omega_b} = \frac{\pi}{\omega_b} \sum_{n=-\infty}^{\infty} s_{smp}(t_n) e^{-in\pi\omega/\omega_b}$$

The function $S_{smp}(\omega)$ is periodic of period $2\omega_b$ with $S_{smp}(\omega) = S(\omega)$ for $\omega \in [-\omega_b, \omega_b]$. Therefore, we can write

$$S(\omega) = p_{\omega_b} S_{smp}(\omega) = \frac{\pi}{\omega_b} \sum_{n=-\infty}^{\infty} s_{smp}(t_n) p_{\omega_b}(\omega) e^{-in\pi\omega/\omega_b}.$$

Finally, from the pair

$$\frac{\omega_b}{\pi} \frac{\sin(\omega_b t - n\pi)}{\omega_b t - n\pi} \Leftrightarrow p_{\omega_b}(\omega) e^{-in\pi\omega/\omega_b}$$

and the convolution property we find

$$s(t) = \sum_{n=-\infty}^{+\infty} s_{smp}(t_n) \frac{\sin(\omega_b(t - t_n))}{\omega_b(t - t_n)} \tag{22}$$

■

Observation 40. *Nyquist frequency (beware of a factor 2...)*
The frequency ω_b/π is the celebrated Nyquist frequency. Its inverse, π/ω_b is time lag between consecutive samples. Sampling at the inverse of the Nyquist frequency or higher allows for perfect signal reconstruction. Even if you forget the proof of Shannon theorem, all what you need to remember is the duality between an infinite impulse train in time with time lag equal to T_s and its Fourier Transform, an infinite impulse train in frequency with frequency lag equal to $1/T_s$. Aliasing is avoided if

$$T_s \geq \frac{\pi}{\omega_b} = \frac{2\pi}{\mathcal{L}(S)}$$

Notice that $\omega_b = \mathcal{L}(S)/2$!

Exercise 19. *Which sampling rate should we use for* $\cos t$*?*

4 Linear systems

Here again, we took inspiration from the *Papoulis*.

4.1 Definition and properties

A physical system L is a transducer which for a signal $f_{in}(t)$ in input produces a signal $f_{out}(t)$ in output. The output signal $f_{out}(t)$, or *effect*, is entirely determined by the input signal $f_{in}(t)$, or *cause*. The system L can be thought of as a transformation with

$$L\{f_{in}(t)\} = f_{out}(t)$$

We immediately restrict our attention to linear systems which satisfy two constraints.

Linearity A system L is *linear* if from

$$L\{f_{in}(t)\} = f_{out}(t) \quad \text{and} \quad L\{f'_{in}(t)\} = f'_{out}(t)$$

it follows that

$$L\{af_{in}(t) + bf'_{in}(t)\} = af_{out}(t) + bf'_{out}(t)$$

for all possible input signals $f_{in}(t)$ and $f'_{in}(t)$.

Time-invariance A system L is *time-invariant* if $\forall s \in \mathbb{R}$ and input signal $f_{in}(t)$

$$L\{f_{in}(t)\} = f_{out}(t) \quad \Rightarrow \quad L\{f_{in}(t-s)\} = f_{out}(t-s).$$

In words, the parameters of a time-invariant system do not change over time.

Impulse response Let $h(t)$ be the response of a *linear time-invariant* (LTI) system to an impulse $\delta(t)$. We have

$$L\{\delta(t-s)\} = h(t-s).$$

Characterisation of a LTI system If we write a signal $f_{in}(t)$ as an infinite sum of impulses,

$$f_{in}(t) = \int_{-\infty}^{\infty} f_{in}(s)\delta(t-s)\mathrm{d}s,$$

and use the linearity of the integral and the time-invariance property we have

$$f_{out}(t) = L\{f_{in}(t)\} = \int_{-\infty}^{\infty} f_{in}(s)L\{\delta(t-s)\}\mathrm{d}s = \int_{-\infty}^{\infty} f_{in}(s)h(t-s)\mathrm{d}s \tag{23}$$

Equation (23) provides a powerful characterisation of an LTI system. The output of an LTI system can be obtained as the *convolution of the input with the impulse response*. Remarkably, all what is needed to compute the output $f_{out}(t)$ of a linear time-invariant system for a given input $f_{in}(t)$ is the knowledge of just one function of time: the impulse response $h(t)$.

Stability A system is *stable* if its response to a bounded input is bounded, or if

$$|f_{in}(t)| < M \quad \Rightarrow \quad |f_{out}(t)| < MI$$

where I is a constant independent of the input.

Exercise 20. *Prove that the integrability of the absolute value of the impulse response implies stability.*

Causality A system is *causal* if when $f_{in}(t) = 0$ for $t < t_0$ we also have that $f_{out}(t) = 0$ for $t < t_0$. For a causal LTI system we thus find

$$f_{out}(t) = L\{f_{in}(t)\} = 0 \quad \text{for } \ t < t_0.$$

Exercise 21. *Prove that the impulse response of a causal LTI system is causal and viceversa.*

Eigenfunction and eigenvalue Let us consider the output of LTI system for an exponential input $f_{in}(t) = e^{i\omega_0 t}$. We have

$$L\{e^{i\omega_0 t}\} = f_{out}(t)$$

for some function $f_{out}(t)$. For the time-invariance we can write

$$L\{e^{i\omega_0 (t+s)}\} = f_{out}(t+s).$$

For the linearity and the fact that $e^{i\omega_0(t+s)} = e^{i\omega_0 t}e^{i\omega_0 s}$ we have

$$L\{e^{i\omega_0 (t+s)}\} = L\{e^{i\omega_0 t}e^{i\omega_0 s}\} = e^{i\omega_0 s}L\{e^{i\omega_0 t}\} = e^{i\omega_0 s}f_{out}(t).$$

Therefore we conclude that

$$f_{out}(t+s) = e^{i\omega_0 s}f_{out}(t)$$

If we set $t = 0$, $f_{out}(0) = k$, and observe that s is arbitrary we obtain (with $s \to t$)

$$f_{out}(t) = ke^{i\omega_0 t}. \tag{24}$$

We thus find that the complex exponential is an eigenfunction of L, since the output to a complex exponential input is the same complex exponential. We now express the eigenvalue k in terms of the Fourier Transform of the impulse response.

System function Through the convolution property of the Fourier Transform we have that taking the Fourier Transform of Equation (23) we find

$$F_{out}(\omega) = F_{in}(\omega)H(\omega).$$

The Fourier Transform $H(\omega)$ of the impulse response $h(t)$ is known as *system function*.

Exercise 22. *Eigenvalue characterisation*
Show that the eigenvalue k for a complex exponential input of angular frequency ω_0 is the system function evaluated at ω_0.

Observation 41. *Consequences*
We already knew that the output of a sinusoidal input is a sinusoid with the same angular frequency ω_0. From Equation (24) we can also compute the attenuation factor $A(\omega_0) = |H(\omega_0)|$ and the phase Φ as in $H(\omega_0) = A(\omega_0)e^{i\Phi(\omega_0)}$. For $f_{in}(t) = \cos(\omega_0 t)$, for example, we have

$$f_{out}(t) = L\{\cos(\omega_0 t)\} = A(\omega_0)\cos\left(\omega_0 t + \Phi(\omega_0)\right)$$

All input sinusoids maintain the same frequency on output but are *attenuated* by a frequency dependent factor and *delayed* by a frequency dependent phase!

Observation 42. *Constant input*
If $\omega_0 = 0$, we find that the output signal of a constant input signal, $f_{in}(t) = \bar{f}_{in}$ is constant with

$$f_{out}(t) = \bar{f}_{out} = H(0)\bar{f}_{in} \tag{25}$$

A second characterisation Since $U(t) = \int_{-\infty}^{t} \delta(s)\mathrm{d}s$ and $h(t) = 0$ for $t < 0$, we have that

$$a(t) = \int_0^t h(s)\mathrm{d}s.$$

For a causal system, it is sometimes easier to compute the response $a(t)$ to the unit step $U(t)$,

$$L\{U(t)\} = a(t).$$

We start by writing the input signal $f_{in}(t)$ as

$$f_{in}(t) = f_{in}(-\infty) + \int_{-\infty}^{t} \frac{df_{in}(s)}{\mathrm{d}s}\mathrm{d}s = f_{in}(-\infty) + \int_{-\infty}^{\infty} \frac{df_{in}(s)}{\mathrm{d}s}U(t-s)\mathrm{d}s$$

Thus, by the linearity of L and using Equation (25), we obtain

$$\begin{aligned} f_{out}(t) = L\{f_{in}(t)\} &= L\{f_{in}(-\infty)\} + \int_{-\infty}^{\infty} \frac{df_{in}(s)}{\mathrm{d}s}L\{U(t-s)\}\mathrm{d}s \\ &= H(0)f_{in}(-\infty) + \int_{-\infty}^{\infty} \frac{df_{in}(s)}{\mathrm{d}s}a(t-s)\}\mathrm{d}s \\ &= H(0)f_{in}(-\infty) + \int_{-\infty}^{t} \frac{df_{in}(s)}{\mathrm{d}s}a(t-s)\}\mathrm{d}s \end{aligned}$$

since, clearly, $a(t)$ is a causal function.

Let us know discuss an example of a stable and causal LTI system.

4.2 RC circuit

We consider the circuit in Figure (5). All what you are supposed to remember about the physics of a circuit (unless you want to risk your life every time you interact with a socket) are two facts: *(a)* when you apply a voltage V_{in}, a current will start to flow across the circuit, and *(b)* the same current, with no leakage, flows across the entire circuit.

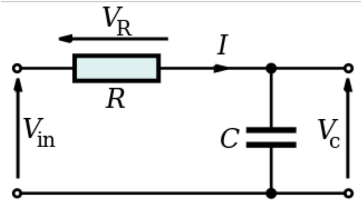

Figure 5: See text.

An equation involving the derivative of a function We have that

$$V_R + V_C = V_{in}$$

From Ohm's law, we know that for the voltage drop between the resistor ends we have $V_R = R\,\mathrm{d}q/\mathrm{d}t$, while the voltage at the plates of a capacitor is inversely proportional to the capacitance $V_C = q/C$. We thus have

$$R\,\frac{\mathrm{d}q(t)}{\mathrm{d}t} + \frac{1}{C}\,q(t) = V_{in}$$

If we set $V_C = q/C = V_{out}$ we find

$$\frac{\mathrm{d}V_{out}(t)}{\mathrm{d}t} + \frac{1}{\tau}\,V_{out} = \frac{1}{\tau}\,V_{in} \tag{26}$$

where we introduce the time constant $\tau = RC$. In order to write $V_{out}(t)$ in terms of $V_{in}(t)$ we need to solve the differential Equation (26) for $V_{out}(t)$.

The trick is to multiply each term by a differentiable function $\mu(t)$

$$\frac{\mathrm{d}V_{out}(t)}{\mathrm{d}t}\mu(t) + \frac{1}{\tau}V_{out}(t)\mu(t) = \frac{1}{\tau}V_{in}\mu(t) \tag{27}$$

and look for $\mu(t)$ such that the left hand side of Equation (27) can be written as a derivative. Indeed, if we choose $\mu(t) = e^{t/\tau}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(V_{out}(t)\mu(t)\right) = \frac{\mathrm{d}V_{out}(t)}{\mathrm{d}t}e^{t/\tau} + \frac{1}{\tau}V_{out}(t)e^{t/\tau}.$$

A causal, stable LTI system Therefore, integrating both sides of Equation (27) we find

$$V_{out}(t)e^{t/\tau} = \frac{1}{\tau}\int_{-\infty}^{t} e^{s/\tau}V_{in}(s)\mathrm{d}s \quad \to \quad V_{out} = \frac{1}{\tau}\int_{-\infty}^{t} e^{-(t-s)/\tau}V_{in}(s)\mathrm{d}s \quad \text{for } t \geq s$$

We thus obtain for the impulse response

$$h(t) = \frac{1}{\tau}e^{-t/\tau}U(t)$$

where $U(t)$ ensures that $h(t) = 0$ for $t < 0$. Intuitively, the capacitor discharges at an exponential rate depending on the characteristic time $\tau = RC$.

The system is clearly linear, time-invariant, and causal. Stability is guaranteed by its physical nature.

Unit step response We have

$$a(t) = \frac{e^{-t/\tau}}{\tau} \int_0^t e^{s/\tau} \mathrm{d}s = \frac{e^{-t/\tau}}{\tau} \tau e^{s/\tau} \Big|_{s=0}^{s=t} = \left(1 - e^{-t/\tau}\right) \quad \text{for } t \geq 0$$

The constant τ is the characteristic time needed by the system, see Figure (6), to respond to the unit step input.

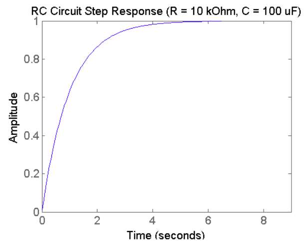

Figure 6: See text.

Impulse response The impulse response can also be found by taking the derivative of the unit step response. Indeed we have

$$h(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left(1 - e^{-t/\tau}\right) = \frac{1}{\tau} e^{-t/\tau} U(t)$$

System function For the Fourier transform of $h(t)$ we have

$$H(\omega) = \frac{1}{\tau} \frac{1}{1/\tau + i\omega} = \frac{1}{1 + i\tau\omega} = A(\omega) e^{i\Phi(\omega)} = \frac{1}{\sqrt{1 + \tau^2\omega^2}} e^{i\tau\omega}$$

This is a first example of filter. The shape of both the unit step response and the impulse response can be understood in terms of the action of the system function on the Fourier Transform of $U(t)$ and $\delta(t)$. The higher the frequency, the stronger the attenuation in the module $A(\omega)$. Here again, the constant τ is the characteristic time. The larger τ the more visible is the filter effect (for the same frequency). The phase $\Phi(\omega)$, instead, introduces distortion in the reconstruction.

5 Kalman filtering

The filters we discussed so far do not take into account the fact that, in time-varying signal, future samples are often correlated with the current sample. We now present a scheme, Kalman flltering, in which this correlation is exploited to design optimal filters which can also be implemented efficiently. As usual the topic is vast and in class we only cover the fundamentals. We invariably assume the discrete time setting. We are partially indebted to the **Tutorial: The Kalman Filter** by *Thacker and Lacey*.

5.1 Computing the average revisited

The scenario Let us assume we want to estimate the height of a building. We refer to s as the unknown quantity to be estimated, the true height, and view it as the *state* of a certain system, the building (in the far-fetched assumption that all what we are interested in is the building height). We do not have direct access to the state but at each time $k = 1, \ldots$ we acquire a measurement, m_k. The problem is to obtain the optimal estimate $\hat{s}_k$ of s when the k-th measurement is acquired. In this simple example the state s does not change over time.

Known answer We already know the solution we should find: the best we can do is to compute the empirical average of the measurements acquired up to time k,

$$\hat{s}_k = \frac{1}{k}\sum_{i=1}^{k} m_i.$$

The ˆ sign makes it clear that $\hat{s}_k$ is an *estimate*, the subscript k that this estimate is obtained *after* the k-th measurement is acquired.

Problem setting Let us derive this result in the *Kalman filtering* setting. We write

$$m_k = s + r_k \tag{28}$$

where r_k is the noise affecting the measurement m_k. We assume that the noise is zero mean, $\mathbb{E}[r_k] = 0$, with variance $\mathbb{E}[r_k^2] = R > 0$.

We let $\hat{s}_k^-$ denote the state estimate we predict to obtain *before* the acquisition of the k-th measurement. We write the **state update** equation as

$$\hat{s}_k = \hat{s}_k^- + K_k(m_k - \hat{s}_k^-) \tag{29}$$

and aim at finding K_k, the *Kalman gain*, which allows us to recursively obtain the optimal estimate $\hat{s}_k$ as the weighted sum of $\hat{s}_k^-$, and the difference between the new measurement m_k and $\hat{s}_k^-$, called *innovation* or *measurement residual*. If $e_k = s - \hat{s}_k$ is the *error*, let us then look for the Kalman gain minimising the *mean squared error*, or variance of the state estimate *after* the k-th measurement is acquired,

$$\mathbb{E}[e_k^2] = \mathbb{E}[(s - \hat{s}_k)^2]$$

Solution Using Equation (28) and (29) we have

$$e_k = s - \hat{s}_k = s - \hat{s}_k^- - K_k s - K_k r_k + K_k \hat{s}_k^- = (1 - K_k)(s - \hat{s}_k^-) - K_k r_k$$

and thus

$$e_k^2 = (1 - K_k)^2 (s - \hat{s}_k^-)^2 + K_k^2 r_k^2 - 2K_k(1 - K_k) r_k (s - \hat{s}_k^-).$$

Taking into account that the error in the state estimate and the noise measurement are uncorrelated, the expected value of the last term in the right hand side of the previous equality vanishes and we are left with

$$\mathbb{E}[e_k^2] = (1 - K_k)^2 \mathbb{E}[(s - \hat{s}_k^-)^2] + K_k^2 \mathbb{E}[r_k^2]. \tag{30}$$

If we denote with P_k the variance $\mathbb{E}[(s - \hat{s}_k)^2]$ and let $P_k^- = \mathbb{E}[(s - \hat{s}_k^-)^2]$ for the variance of the state estimate predicted *before* the k-th measurement has been acquired, since $\mathbb{E}[r_k^2] = R$, we rewrite Equation (30) as

$$P_k = P_k^- + K_k^2(P_k^- + R) - 2K_k P_k^- \tag{31}$$

Setting the derivative of P_k in Equation (31) with respect to K_k equal to 0

$$\frac{\mathrm{d}P_k}{\mathrm{d}K_k} = -(1 - K_k)P_k^- + K_k R = 0$$

we obtain the **Kalman gain** minimising the mean squared error P_k as

$$K_k = \frac{P_k^-}{P_k^- + R}.$$

By substituting this expression for K_k in Equation (29) we are now in a position to compute the state update following the acquisition of the k-th measurement. In order to complete the derivation we still need to update the variance, and project the state and variance estimates from k to $k+1$. For the **variance update**, by plugging the expression for K_k into Equation (31), we obtain

$$P_k = P_k^- + \frac{(P_k^-)^2}{P_k^- + R} - 2\frac{(P_k^-)^2}{P_k^- + R} = P_k^-(1 - K_k)$$

For the **state projection**, since we know that x is not changing over time, we simply have

$$\hat{s}_{k+1}^- = \hat{s}_k. \tag{32}$$

For the **variance projection**, using Equations (32), we obtain

$$P_{k+1}^- = \mathbb{E}[(s - \hat{s}_{k+1}^-)^2] = \mathbb{E}[(s - \hat{s}_k)^2] = P_k.$$

In summary, the solution to our problem - relying on the prior knowledge of the measurement noise variance R - consists of 7 steps.

Algorithm 1. *FixedStateKalman*
Acquire the measurement z_1 and set

$$\hat{s}_1 = m_1, \quad \hat{s}_2^- = \hat{s}_1, \quad \text{and} \quad P_2^- = R$$

Set $k = 2$ and repeat

1.	Acquire the measurement m_k	
2.	Set the Kalman gain:	$K_k = 1/k$
3.	State update:	$\hat{s}_k = \hat{s}_k^- + K_k(m_k - \hat{s}_k^-) = \hat{s}_k^- + (m_k - \hat{s}_k^-)/k$
4.	Variance update:	$P_k = P_k^-(1 - K_k) = P_k^-(k-1)/k = R/k$
5.	State estimate projection:	$\hat{s}_{k+1}^- = \hat{s}_k$
6.	Variance estimate projection:	$P_{k+1}^- = P_k = R/k$
7.	$k \leftarrow k + 1$	

Exercise 23. *Checking the obtained result*
Go through a few iterations of the *FixedStateKalman* algorithm and check that the state estimate is a recursive computation of the empirical average.

Exercise 24. *Average grade*
Mike is not sure whether to accept or reject the 18/30 he got in his last exam. If all the grades have the same weight and Mike's average after 9 exams is 28, compute Mike's new average assuming he is going to accept the grade.

5.2 General case

We now generalise what we have seen so far in several directions. First, we consider a system described by an N-dimensional state vector $\mathbf{s}_k = [s_{1,k}\ s_{2,k}\ \dots\ s_{N,k}]^\top$ where each component changes, or might change, with k. In a tracking scenario, for example, the system is a ball moving on a billiard table. The state consists of two components for the ball position, and two for its velocity, and is thus a 4-dimensional vector. Second, we do have some knowledge about the way the state evolves over time.

Definition 10. *Linear (stationary) process model*
A *linear process model* expresses $\mathbf{s}_{k+1}$ in terms of $\mathbf{s}_k$ through an $N \times N$ *transition* matrix Φ

$$\mathbf{s}_{k+1} = \Phi \mathbf{s}_k + \mathbf{q}_k \tag{33}$$

where $\mathbf{q}_k = [q_{1,k}\ q_{2,k}\ \dots\ q_{N,k}]^\top$ is the *process noise.*

Definition 11. *Process noise expected value and covariance*
We assume that $\mathbb{E}[\mathbf{q}_k] = 0$ and denote the $N \times N$ *process noise covariance* matrix with

$$Q = \mathbb{E}[\mathbf{q}_k \mathbf{q}_k^\top]$$

Example 8. *Back to the average*
In our first, very simple example the state $\mathbf{s}_k$, the process noise $\mathbf{q}_k$, and the covariance matrix Q are scalar with no dependence on k. Since Equation (33) reads $s_{k+1} = s_k$, we have $\Phi = 1$, $q_k = 0$ for all k, and $Q = 0$. The process model is *exact.*

Example 9. *Tracking*
We want to track a ball moving with constant velocity on the billiard table. In this case $\mathbf{s}_k = [x_k\ y_k\ \dot{x}_k\ \dot{y}_k]^\top$ is the 4 dimensional state vector: x_k and y_k the two components for the ball position, and $\dot{x}_k$ and $\dot{y}_k$ the two components for its constant velocity along the X and Y direction at time k. According to the linear model of Equation (33) the velocity does not depend on k, and thus

$$\dot{x}_{k+1} = \dot{x}_k, \quad \text{and} \quad \dot{y}_{k+1} = \dot{y}_k,$$

while the ball position at time $k + 1$ can be computed from the ball position and velocity at time k as

$$x_{k+1} = x_k + \dot{x}_k \Delta t, \quad \text{and} \quad y_{k+1} = y_k + \dot{y}_k \Delta t.$$

Since $\Delta t = (k + 1) - k = 1$, if we set

$$\Phi = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Equation (33) reads

$$\begin{bmatrix} x_{k+1} \\ \dot{x}_{k+1} \\ y_{k+1} \\ \dot{y}_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ \dot{x}_k \\ y_k \\ \dot{y}_k \end{bmatrix} + \begin{bmatrix} q_{x_k} \\ q_{\dot{x}_k} \\ q_{y_k} \\ q_{\dot{y}_k} \end{bmatrix}$$

with the process noise vector $\mathbf{q}_k = [q_{x_k}\ q_{y_k}\ q_{\dot{x}_k}\ q_{\dot{y}_k}]^\top = 0$ for all k. The role of the process noise is to allow some variability in the model. In this case, for example, $\mathbf{q}_k \neq 0$ models a process in which the ball velocity is only piece-wise constant and includes the tracking of a ball rebounding from the cushion. $\square$

As before, we do not have direct access to the state $\mathbf{s}_k$ but, at each given time k, we acquire measurements, M instead of just 1, which we are going to use to build an optimal estimate $\hat{\mathbf{s}}_k$ of the true state $\mathbf{s}_k$. Again we assume a linear relationship between measurements and the system state at time k.

Definition 12. *Linear (stationary) observation model*
Let $\mathbf{m}_k = [m_{1,k}\ m_{2,k}\ \dots\ m_{M,k}]^\top$ be the *measurement* vector and $\mathbf{r}_k = [r_{1,k}\ r_{2,k}\ \dots\ r_{M,k}]^\top$ the *measurement noise* vector. If H is an $M \times N$ *measurement* matrix, the *observation model* expresses $\mathbf{m}_k$ in terms of $\mathbf{s}_k$ and $\mathbf{r}_k$, or

$$\mathbf{m}_k = H\mathbf{s}_k + \mathbf{r}_k \tag{34}$$

Definition 13. *Measurement noise expected value and covariance*
We assume that $\mathbb{E}[\mathbf{r}_k] = 0$ and denote the $M \times M$ *measurement noise covariance* matrix with

$$R = \mathbb{E}[\mathbf{r}_k \mathbf{r}_k^\top]$$

Example 10. *Back to averaging, once again*
In our first example both $\mathbf{m}_k$ and $\mathbf{r}_k$ are scalar. Since $m_k = s + r_k$, we have $H = 1$ and the covariance matrix R reduces to the noise variance.

Example 11. *Tracking a tennis ball*
At each time k through some computer vision algorithm we estimate the ball position $\mathbf{m}_k = [m_{x_k} \; m_{y_k}]^\top$ through an image captured by a camera calibrated with respect to the billiard table. We thus have

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and Equation (34) reads

$$\begin{bmatrix} m_{x_k} \\ m_{y_k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \dot{x}_k \\ y_k \\ \dot{y}_k \end{bmatrix} + \begin{bmatrix} r_{x_k} \\ r_{y_k} \end{bmatrix}$$

where $\mathbf{r}_k = [r_{x_k} \; r_{y_k}]$ is the measurement noise vector.

Definition 14. *Error*
Let $\hat{\mathbf{s}}_k$ be the *estimate* of $\mathbf{s}_k$ obtained *after* the acquisition of the measurement vector $\mathbf{m}_k$. The difference between the true state and the *posterior state estimate* is expressed by the N-dimensional *error* vector

$$\mathbf{e}_k = \mathbf{s}_k - \hat{\mathbf{s}}_k$$

Definition 15. *Posterior covariance*
This time P_k is an $N \times N$ matrix, the *posterior covariance*, or

$$P_k = \mathbb{E}[\mathbf{e}_k \mathbf{e}_k^\top] \tag{35}$$

Same objective As before, we use the "$^-$" superscript to denote the projection of a given quantity from time k to time $k+1$ *before* the acquisition of the measurement at time $k+1$.

The plan is the same. We want to determine the matrix K_k, or *Kalman gain*, which after the acquisition of the measurement $\mathbf{m}_k$ gives the **state update** equation at time k as

$$\hat{\mathbf{s}}_k = \hat{\mathbf{s}}_k^- + K_k(\mathbf{m}_k - H\hat{\mathbf{s}}_k^-). \tag{36}$$

The state estimate $\hat{\mathbf{s}}_k$, again, is a weighted average between $\hat{\mathbf{s}}_k^-$, the state estimate at time k we predict to obtain before the k-th measurement is acquired, and the innovation $\mathbf{m}_k - H\hat{\mathbf{s}}_k^-$. The optimal weight, the Kalman gain matrix, is obtained by minimising the trace of the covariance matrix. There is only one technical difference with respect to the starting example: we have to deal with matrices every step of the way.

Observation 43. *To the moon (and back) in 1969!*
This scenario is much richer than before. Starting from a linear model of how a state evolves over time and a linear model of how measurements are obtained, for each k our goal is to obtain the Kalman gain, the state and covariance estimate updates, and project our estimates at time $k+1$. The recursive nature of the solution, combined with the predictive property, ensures that this program can be carried out effectively and with limited computational resources. Since many decades Kalman filtering has been, and still is, the basis for tracking and control tasks in many different application domains.

5.3 Linear Kalman Filter

As before, the program is to find the *Kalman gain*, this time an $N \times M$ matrix K_k.

By plugging Equation (33) into Equation (36) we rewrite Equation (36) as

$$\hat{\mathbf{s}}_k = \hat{\mathbf{s}}_k^- + K_k(H\mathbf{s}_k + \mathbf{r}_k) - K_k H \hat{\mathbf{s}}_k^-. \tag{37}$$

Using Equation (37), Equation (35) becomes

$$P_k = \mathbb{E}\left[[(I - K_k H)(\mathbf{s}_k - \hat{\mathbf{s}}_k^-) - K_k \mathbf{r}_k][(I - K_k H)(\mathbf{s}_k - \hat{\mathbf{s}}_k^-) - K_k \mathbf{r}_k]^\top\right]$$

which, since the error of the prior state estimate and the measurement noise are uncorrelated, can be written

$$P_k = (I - K_k H)\mathbb{E}[(\mathbf{s}_k - \hat{\mathbf{s}}_k^-)(\mathbf{s}_k - \hat{\mathbf{s}}_k^-)^\top](I - K_k H)^\top + K_k \mathbb{E}[\mathbf{r}_k \mathbf{r}_k^\top] K_k^\top. \tag{38}$$

If we set $P_k^- = \mathbb{E}[(\mathbf{s}_k - \hat{\mathbf{s}}_k^-)(\mathbf{s}_k - \hat{\mathbf{s}}_k^-)^\top]$, since $\mathbb{E}[\mathbf{r}_k \mathbf{r}_k^\top] = R$, we rewrite Equation (38) as

$$\begin{aligned} P_k &= (I - K_k H)P_k^-(I - K_k H)^\top + K_k R K_k^\top \\ &= P_k^- - K_k H P_k^- - P_k^- H^\top K_k^\top + K_k(H P_k^- H^\top + R)K_k^\top \end{aligned}$$

We know that the trace of P_k, $Tr[P_k]$, is the sum of the mean squared errors. To minimise this sum we simply have to solve the equation

$$\frac{\mathrm{d}tr(P_k)}{\mathrm{d}K_k} = 0. \tag{39}$$

The trace of a square is the sum of its diagonal entries. Equation (39) defines an $N \times N$ matrix A with A_{ij} computed as the derivative of $tr(P_k)$ with respect to $K_{ij,k}$.

For the trace we have

$$tr(P_k) = tr(P_k^-) - 2tr(K_k H P_k^-) + tr(K_k(H P_k^- H^\top + R)K_k^\top).$$

Notice that the matrix $K_k(HP_k^- H^\top + R)K_k^\top$ is symmetric. Thus Equation (39), see the last section for details, becomes

$$0 = -2(HP_k^-)^\top + 2K_k(HP_k^- H^\top + R)$$

which can be solved for K_k to give the **Kalman gain**

$$K_k = P_k^- H^\top (HP_k^- H^\top + R)^{-1} \tag{40}$$

From Equation (40) we have that

$$\hat{K}_k(HP_k^- H^\top + R) = P_k^- H^\top (HP_k^- H^\top + R)^{-1}(HP_k^- H^\top + R) = P_k^- H^\top.$$

Therefore, the third and fourth term in the sum of the right hand side of Equation (39) cancel out and we finally obtain the **update covariance** equation

$$P_k = (I - K_k H)P_k^-.$$

To complete the program, as before, we use the process model to obtain the **state projection** equation, or the estimate of the state $\mathbf{s}_k$ *before* the acquisition of the measurement vector $\mathbf{m}_k$

$$\hat{\mathbf{s}}_{k+1}^- = \Phi \hat{\mathbf{s}}_k. \tag{41}$$

Using Equation (33) and (41) we obtain the **covariance projection** equation

$$\begin{aligned} P_{k+1}^- &= \mathbb{E}[\mathbf{e}_{k+1}^- {\mathbf{e}_{k+1}^-}^\top] = \mathbb{E}[(\mathbf{s}_{k+1} - \hat{\mathbf{s}}_{k+1}^-)(\mathbf{s}_{k+1} - \hat{\mathbf{s}}_{k+1}^-)^\top] \\ &= \mathbb{E}[(\Phi\mathbf{s}_k + \mathbf{q}_k - \Phi\hat{\mathbf{s}}_k)(\Phi\mathbf{s}_k + \mathbf{q}_k - \Phi\hat{\mathbf{s}}_k)^\top] = \mathbb{E}[(\Phi\mathbf{e}_k + \mathbf{q}_k)(\Phi\mathbf{e}_k + \mathbf{q}_k)^\top] \\ &= \mathbb{E}[\Phi\mathbf{e}_k \Phi\mathbf{e}_k^\top] + \mathbb{E}[\mathbf{q}_k \mathbf{q}_k^\top] = \Phi P_k \Phi^\top + Q \end{aligned}$$

Similarly to the static case, the solution to our problem - this time relying on the prior knowledge of the model noise and measurement noise covariance matrices Q and R - consists of 7 steps.

Algorithm 2. *LinearKalmanFilter*
Guess the state $\mathbf{s}_1$ and the matrix P_1^- (random initialisation is an option). Set $k = 1$ and repeat

1.	Acquire the measurement $\mathbf{m}_k$	
2.	Set the Kalman gain:	$K_k = P_k^- H^\top (H P_k^- H^\top + R)^{-1}$
3.	State update:	$\hat{\mathbf{s}}_k = \hat{\mathbf{s}}_k^- + K_k(\mathbf{m}_k - H\hat{\mathbf{s}}_k^-)$
4.	Covariance update:	$P_k = (I - K_k H)P_k^-$
5.	State estimate projection:	$\hat{\mathbf{s}}_{k+1}^- = \Phi\hat{\mathbf{s}}_k$
6.	Covariance estimate projection:	$P_{k+1}^- = \Phi P_k \Phi^\top + Q$
7.	$k \leftarrow k + 1$	

5.4 Computing the derivative of a function affected by noise

We conclude with a non-trivial example which can be implemented without matrix inversion. We assume to consider an unspecified system described by a time-varying function f and its derivative f' with respect to time. For the true state we thus write $s_k = [f_k \; f_k']^\top$ where f_k and f_k' are the true value of the function f and of its temporal derivative both evaluated at time k. We do not have access to the true function f but at each time k we obtain a sample m_k and we want to use Kalman filtering to obtain the optimal state estimate $\hat{s}_k = [\hat{f}_k \; \hat{f}_k']^\top$. Let us proceed by writing down what we have in this case.

As for Φ we could use

$$\Phi = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Assuming again a unit time interval $\Delta t = 1$ between samples, we model f_{k+1} as

$$f_{k+1} = f_k + f_k' \, \Delta t = f_k + f_k'$$

and the temporal derivative as a constant. We let Q be the 2×2 diagonal matrix

$$Q = \begin{bmatrix} q_1^2 & 0 \\ 0 & q_2^2 \end{bmatrix}$$

with q_1^2 and q_2^2 the variances of the noise affecting the piece-wise linear model we are adopting and thus allowing for changes in the derivative value.

The measurement model is immediate. We have $H = [1 \; 0]$ and

$$m_k = [1 \; 0] \begin{bmatrix} f_k \\ f_k' \end{bmatrix} + r_k = f_k + r_k$$

with R, as usual, for the variance of the noise r_k. If

$$P_k = \begin{bmatrix} p_{11,k} & p_{12,k} \\ p_{21,k} & p_{22,k} \end{bmatrix} \quad \text{and} \quad P_k^- = \begin{bmatrix} p_{11,k}^- & p_{12,k}^- \\ p_{21,k}^- & p_{22,k}^- \end{bmatrix},$$

since

$$P_k^- H^\top = \begin{bmatrix} p_{11,k}^- & p_{12,k}^- \\ p_{21,k}^- & p_{22,k}^- \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11,k}^- \\ p_{21,k}^- \end{bmatrix}$$

and

$$H P_k^- H^\top = [1 \; 0] \begin{bmatrix} p_{11,k}^- & p_{12,k}^- \\ p_{21,k}^- & p_{22,k}^- \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [1 \; 0] \begin{bmatrix} p_{11,k}^- \\ p_{21,k}^- \end{bmatrix} = p_{11,k}^-,$$

we have that

$$(HP_k^- H^\top + R)^{-1} = \frac{1}{p_{11,k}^- + R},$$

so that the **Kalman gain**, $P_k^- H^\top (HP_k^- H^\top + R)^{-1}$, is given by

$$K_k = \frac{1}{p_{11,k}^- + R} \begin{bmatrix} p_{11,k}^- \\ p_{21,k}^- \end{bmatrix}$$

For the **state update**, $\hat{\mathbf{s}}_k = \hat{\mathbf{s}}_k^- + K_k(\mathbf{m}_k - H\hat{\mathbf{s}}_k^-)$, we thus have

$$\begin{bmatrix} \hat{f}_k \\ \hat{f}'_k \end{bmatrix} = \begin{bmatrix} \hat{f}_k^- \\ \hat{f}_k^{\prime -} \end{bmatrix} + \frac{m_k - \hat{f}_k^-}{p_{11,k}^- + R} \begin{bmatrix} p_{11,k}^- \\ p_{21,k}^- \end{bmatrix}$$

Since

$$I - K_k H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{p_{11,k}^- + R} \begin{bmatrix} p_{11,k}^- \\ p_{21,k}^- \end{bmatrix} [1\ 0] = \frac{1}{p_{11,k}^- + R} \begin{bmatrix} R & 0 \\ -p_{21,k}^- & p_{11,k}^- + R \end{bmatrix}$$

the **covariance update** reads

$$\begin{aligned} \begin{bmatrix} p_{11,k} & p_{12,k} \\ p_{21,k} & p_{22,k} \end{bmatrix} &= \frac{1}{p_{11,k}^- + R} \begin{bmatrix} R & 0 \\ -p_{21,k}^- & p_{11,k}^- + R \end{bmatrix} \begin{bmatrix} p_{11,k}^- & p_{12,k}^- \\ p_{21,k}^- & p_{22,k}^- \end{bmatrix} \\ &= \frac{1}{p_{11,k}^- + R} \begin{bmatrix} Rp_{11,k}^- & Rp_{12,k}^- \\ Rp_{21,k}^- & -p_{21,k}^- p_{12,k}^- + p_{11,k}^- p_{22,k}^- + p_{22,k}^- R \end{bmatrix} \\ &= \frac{R}{p_{11,k}^- + R} \begin{bmatrix} p_{11,k}^- & p_{12,k}^- \\ p_{21,k}^- & p_{22,k}^- + |P_k^-|/R \end{bmatrix} \end{aligned}$$

where $|P_k^-|$ denotes the determinant of P_k^-. For the **state projection**, instead, we have

$$\begin{bmatrix} \hat{f}_{k+1}^- \\ \hat{f}_{k+1}^{\prime -} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{f}_k \\ \hat{f}'_k \end{bmatrix} = \begin{bmatrix} \hat{f}_k + \hat{f}'_k \\ \hat{f}'_k \end{bmatrix}$$

Finally, since

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11,k} & p_{12,k} \\ p_{21,k} & p_{22,k} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11,k} + p_{12,k} & p_{12,k} \\ p_{21,k} + p_{22,k} & p_{22,k} \end{bmatrix} \\ &= \begin{bmatrix} p_{11,k} + p_{12,k} + p_{21,k} + p_{22,k} & p_{21,k} + p_{22,k} \\ p_{21,k} + p_{22,k} & p_{22,k} \end{bmatrix} \end{aligned}$$

we write the **covariance projection** as

$$\begin{bmatrix} p_{11,k}^- & p_{12,k}^- \\ p_{21,k}^- & p_{22,k}^- \end{bmatrix} = \begin{bmatrix} p_{11,k} + p_{12,k} + p_{21,k} + p_{22,k} & p_{12,k} + p_{22,k} \\ p_{21,k} + p_{22,k} & p_{22,k} \end{bmatrix} + \begin{bmatrix} q_1^2 & 0 \\ 0 & q_2^2 \end{bmatrix}$$

The derivative of a trace with respect to a matrix

For the sake of simplicity we consider 2×2 matrices, leaving the generalisation as an exercise. A and B are arbitrary matrices while C is symmetric. We thus have

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix}$$

We first compute $tr(-2AB + ACA^\top)$ and obtain

$$\begin{aligned} tr(-2AB + ACA^\top) &= -2(a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}c_{22}) \\ &+ a_{11}^2 c_{11} + 2a_{11}a_{12}c_{12} + a_{12}^2 c_{22} + a_{21}^2 c_{11} + 2a_{21}a_{22}c_{12} + a_{22}^2 c_{22}. \end{aligned}$$

For the derivative of $tr(-2AB + ACA^\top)$ with respect to the entries of A we obtain

$$\begin{aligned} \frac{tr(-2AB + ACA^\top)}{\mathrm{d}a_{11}} &= -2b_{11} + 2a_{11}c_{11} + 2a_{12}c_{12} \\ \frac{tr(-2AB + ACA^\top)}{\mathrm{d}a_{12}} &= -2b_{21} + 2a_{11}c_{12} + 2a_{12}c_{22} \\ \frac{tr(-2AB + ACA^\top)}{\mathrm{d}a_{21}} &= -2b_{12} + 2a_{21}c_{11} + 2a_{22}c_{12} \\ \frac{tr(-2AB + ACA^\top)}{\mathrm{d}a_{22}} &= -2b_{22} + 2a_{21}c_{12} + 2a_{22}c_{22} \end{aligned}$$

which in matrix form writes

$$-2B^\top + 2AC.$$

This is precisely the result we got for $\mathrm{d}tr(P_k)/\mathrm{d}K_k$ with $K_k = A$, $HP_k^- = B$, and $HP_k^- H^\top + R = C$.

6 1D Haar wavelets

This class is adapted and sometimes taken word by word from the excellent **Wavelets for Computer Graphics: A Primer** by *Stollnitz*, *DeRose*, and *Salesin*.

6.1 A different representation for a 4-pixel image

We start off by discussing a very simple example. We are given a *4-pixel* image

$$\mathcal{I}_4 = [\, 8\ 4\ 1\ 3 \,]$$

and we want to construct a multi-resolution representation for $\mathcal{I}$. If we average the first and the second pixel, and the third and the fourth, we obtain a lower resolution *2-pixel* image

$$\mathcal{I}_2 = \left[\ \frac{8+4}{2}\ \frac{1+3}{2}\ \right] = [\, 6\ 2 \,]$$

The information lost in the averaging can be recovered if along with the averages, 6 and 2, we also save the *detailed coefficients*: **2** for the first average and **-1** for the second. Combining the averages with the detailed coefficients we obtain back the four original pixel values of $\mathcal{I}_4$,

$$6 + \mathbf{2} = 8,\ 6 - \mathbf{2} = 4,\ 2 + (-\mathbf{1}) = 1,\ \text{and } 2 - (-\mathbf{1}) = 3.$$

If we repeat the process by averaging the two pixels of the lower resolution *2-pixel* image, we obtain the *1-pixel* image

$$\mathcal{I}_1 = \left[\ \frac{6+2}{2}\ \right] = [\, 4 \,]$$

and the detail coefficient **2** needed to recover the two pixel values of $\mathcal{I}_2$. The full process is summarised in figure 7.

Nothing is gained and nothing is lost if, instead of saving $\mathcal{I}_4$, we save $\mathcal{I}_1$ and *three* detail coefficients: *one* for the reconstruction of $\mathcal{I}_2$ from $\mathcal{I}_1$, and *two* for the reconstruction of $\mathcal{I}_4$ from $\mathcal{I}_2$. It is conceivable

Resolution	Averages	Detail coefficients
4	$\begin{bmatrix} 8 & 4 & 1 & 3 \end{bmatrix}$	
2	$\begin{bmatrix} 6 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 \end{bmatrix}$
1	$\begin{bmatrix} 4 \end{bmatrix}$	$\begin{bmatrix} 2 \end{bmatrix}$

Figure 7: From Stollnitz, DeRose, and Salesin.

that in this new representation the truncation of small detail coefficients at high resolution could lead to a lossy image compression of higher quality than straightforward downsampling.

The representation of images like $\mathcal{I}_4$ in terms of a single average value and three detailed coefficients is a very simple example of wavelet transform, known as *Haar wavelet transform*.

Let us go through the same example this time using wavelet notation and concepts along the way,

6.2 Haar wavelet transform

The set V^0 of all constant functions over the interval $[0, 1)$ allows us to describe *1-pixel* images. We increase the resolution by introducing the set V^1 of all the functions made by two constant pieces: the first over the interval $[0, 1/2)$ and the second over the interval $[1/2, 1)$. V^1 is what we need to describe *2-pixel* images. If we keep going, the set V^j consists of all piece-wise constant functions made by 2^j pieces (or $2^j - pixel$ images), with the interval $[0, 1)$ divided in 2^j disjoint $1/2^j$ wide intervals. As shown in Figure (8) the accuracy of the approximation of a given function increases with j.

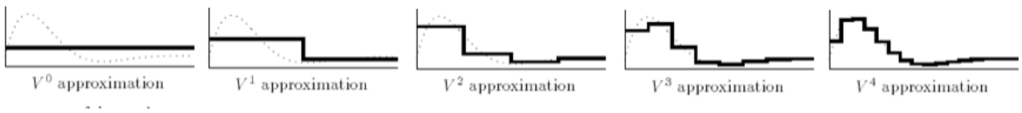

Figure 8: From Stollnitz, DeRose, and Salesin.

Fact 14. *Multiresolution analysis*
For all j, the set V^j of all 2^j-*pixel* images is clearly a vector space. In addition, any 2^j-*pixel* image in V^j is also a 2^{j+1}-*pixel* image in V^{j+1}.

Exercise 25. *Nested structure of vector spaces*
Prove that

$$V^0 \subset V^1 \subset V^2 \subset \ldots$$

□

As basis functions for each space V^j we introduce the set of *scaled* and *translated* functions

$$\phi_i^j = \phi(2^j x - i), \quad \text{with } i = 0, \ldots, 2^j - 1$$

where the **box** function

$$\phi(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is the *scaling* function.

Observation 44. *Scale and translation*
The 2^j functions ϕ_i^j of V^j for $i = 0, \ldots, 2^j - 1$ are non-overlapping translated copies of a scaled box function of width $1/2^j$ in the interval $[0, 1)$. Figure (9) shows the four functions of V^2.

Figure 9: From Stollnitz, DeRose, and Salesin.

Let us adopt the standard definition for computing the scalar product between two vectors in V^j. If $\mathbf{u} = u(x)$ and $\mathbf{v} = v(x)$ are two vectors in V^j, or two 2^j*-pixel* images we have

$$(\mathbf{u}, \mathbf{v}) = \int_0^1 u(x)v(x)\mathrm{d}x.$$

Fact 15. *An orthogonal basis for* V^j
The 2^j vectors ϕ_i^j for $i = 0, \ldots, 2^j - 1$ are mutually orthogonal and thus a basis for the 2^j dimensional vector space V^j. Indeed no two functions ϕ_i^j and ϕ_k^j for $i \neq k$ are both different from zero on each of the 2^j subinterval (see again Figure (9) for $j = 2$). Therefore, we have

$$\begin{aligned}
(\phi_i^j, \phi_k^j) &= \int_0^1 \phi(2^j x - i)\phi(2^j x - k)\mathrm{d}x \\
&= \int_0^{1/2^j} \phi(2^j x - i)\phi(2^j x - k)\mathrm{d}x + \cdots + \int_{(2^j-1)/2^j}^1 \phi(2^j x - i)\phi(2^j x - k)\mathrm{d}x \\
&= \underbrace{0 + \cdots + 0}_{2^k} = 0
\end{aligned}$$

Observation 45. *Easy step: from orthogonal to orthonormal*
The ϕ_i^j are not orthonormal since for $i = k$ we have

$$(\phi_k^j, \phi_k^j) = \int_0^1 \phi(2^j x - k)\phi(2^j x - k)\mathrm{d}x = \int_{k/2^j}^{(k+1)/2^j} 1\mathrm{d}x = \frac{1}{2^j}.$$

To obtain an orthonormal basis we multiply each ϕ_i^j by the square root of 2^j and define $\hat{\phi}_i^j$ as

$$\hat{\phi}_i^j = 2^{j/2}\phi(2^j x - i), \quad \text{with } i = 0, \ldots, 2^j - 1. \ \Box$$

Before completing our first wavelet construction, we review the notion of orthogonal complement of a vector subspace.

Fact 16. *Orthogonal complement as a subspace*
Let V be a vector space. The orthogonal complement $W^\perp$ of a vector subspace U in V, or the set of all vectors $w \in V$ orthogonal to all vectors in U, is a vector subspace.

Example 12. *The XY-plane, the Z-axis, and the XYZ-space*
The set U of vectors which can be written as $(s_1, s_2, 0)$ for all s_1 and $s_2 \in \mathbb{R}$, the points in the *XY-plane*, is a subspace of $V = \mathbb{R}^3$, the *XYZ-space*. The orthogonal complement of U is the set $W^\perp$ of all vectors which can be written as $(0, 0, s_3)$ for all $s_3 \in \mathbb{R}$, the points in the *Z-axis*. It is trivial to verify that $W^\perp$ is a subspace and that the scalar product between any vector in U with any vector in $W^\perp$ is always 0.

Fact 17. *Sum of the dimension of a subspace and of its orthogonal complement*
For finite dimensional spaces it is always true that

$$dim(V) = dim(U) + dim(W^{\perp}).$$

Moreover a basis for V can be obtained as the union of a basis for U with a basis for $W^{\perp}$.

Observation 46. *Orthogonal complement of V^j in V^{j+1}*
An orthogonal basis for V^{j+1} can be obtained by adding to the 2^j mutually orthogonal ϕ_i^j, basis of V^j, the 2^j mutually orthogonal vectors ϕ_i^j basis for the vector space W^j, the orthogonal complement of V^j in V^{j+1}. □

The main idea leading to the wavelet construction is that the basis for the orthogonal complement of V^j are the basis functions needed to recover the details lost in the averaging from V^{j+1} to V^j.

In the case of the *box* scaling function the basis functions are defined as

$$\psi_i^j = \psi(2^j x - i), \quad \text{with } i = 0, \ldots, 2^j - 1$$

where

$$\psi(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1/2 \\ -1 & \text{for } 1/2 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is the *wavelet* function.

Figure (9) shows ψ_0^1 and ψ_1^1, the two wavelets of W^1.

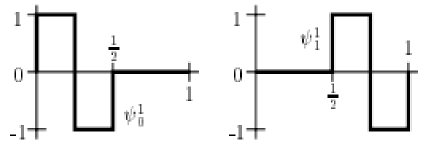

Figure 10: From Stollnitz, DeRose, and Salesin.

Exercise 26. *Constructing an orthogonal basis for V^{j+1}*
Prove that the ϕ_i^j and the ψ_i^j for $i = 0, \ldots, 2^j - 1$ are a basis for V^{j+1}.

Observation 47. *Orthonormality*
Like in the case of the scaling functions ϕ_i^j, to obtain unit norm vectors we need to set $\hat{\psi}_i^j = 2^{j/2}\psi_i^j$.

In V^2 we write

$$\mathcal{I}_4(x) = c_0^2\phi_0^2(x) + c_1^2\phi_1^2(x) + c_2^2\phi_2^2(x) + c_3^2\phi_3^2(x) = 8\phi_0^2(x) + 4\phi_1^2(x) + 1\phi_2^2(x) + 3\phi_3^2(x).$$

The weight of each of the four scaled and translated boxes, ϕ_i^2 $i = 1, \ldots, 4$, is simply the value of the corresponding pixel i.

In V^1 and W^1, instead,

$$\mathcal{I}_4(x) = c_0^1\phi_0^1(x) + c_1^1\phi_1^1(x) + d_0^1\psi_0^1(x) + d_1^1\psi_1^1(x) = 6\phi_0^1(x) + 2\phi_1^1(x) + 2\psi_0^1(x) - 1\psi_1^1(x).$$

The weights of the two scaled and translated boxes ϕ_0^1 and ϕ_1^1 are the two averages, 6 and 2, while the weights of two wavelets ψ_0^1 and ψ_1^1 the corresponding detailed coefficients, 2 and -1.

Finally, in V^0, W^0, and W^1, we have

$$\mathcal{I}_4(x) = c_0^0\phi_0^0(x) + d_0^0\psi_0^0(x) + d_0^1\psi_0^1(x) + d_1^1\psi_1^1(x) = 4\phi_0^0(x) + 2\psi_0^0(x) + 2\psi_0^1(x) - 1\psi_1^1(x).$$

The weights of the only box at the lowest resolution, ϕ_0^0, is the average, 4, while the rest of the basis functions are wavelets at different resolution each weighted by the corresponding detailed coefficient.

6.3 Multiresolution analysis

We now go through the same steps once more and show that both the *analysis*, in which average and detail coefficients are computed at all scales, and *synthesis*, in which the original function is reconstructed from the computed coefficients, can be described in terms of linear filters and, therefore, matrices. Our analysis is limited to Haar wavelets. Wherever appropriate we mention what happens in the general case. For simplicity we stick to the *power of 2* scenario.

Synthesis fitlers

If we introduce the 2^j-*D row* vector

$$\Phi^j(x) = \begin{bmatrix} \phi_0^j & \dots & \phi_{2^j-1}^j \end{bmatrix}$$

and the $2^j \times 2^{j-1}$ constant matrix

$$P^j = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ & & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

we can obtain the 2^{j-1}-*D row* vector $\Phi^{j-1}(x)$ of scaling functions at the coarser level as

$$\Phi^{j-1}(x) = \Phi^j(x)P^j.$$

Therefore, each scaling function at the level $j-1$ can be written as a linear combination of the scaling functions at the finer scale j.

Similarly, we introduce the 2^j-*D row* vector

$$\Psi^j(x) = \begin{bmatrix} \psi_0^j & \dots & \psi_{2^j-1}^j \end{bmatrix}$$

and the $2^j \times 2^{j-1}$ constant matrix

$$Q^j = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ & & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \end{bmatrix} \tag{42}$$

and write

$$\Psi^{j-1}(x) = \Phi^j(x)Q^j.$$

Here again, each wavelet function at the level $j-1$ can be written as a linear combination of the scaling functions at the finer scale j.

Example 13. *Back to* $j = 2$
Since

$$P^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q^2 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

we have

$$\left[\phi_0^1(x)\ \ \phi_1^1(x)\right] = \left[\phi_0^2(x)\ \ \phi_1^2(x)\ \ \phi_2^2(x)\ \ \phi_3^2(x)\right] \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$\left[\psi_0^1(x)\ \ \psi_1^1(x)\right] = \left[\phi_0^2(x)\ \ \phi_1^2(x)\ \ \phi_2^2(x)\ \ \phi_3^2(x)\right] \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

Observation 48. *Is Q^j uniquely determined by the choice of the ϕ_i^j and P^j?*
The answer is in the negative. The columns of Q^j must form a basis for the orthogonal complement of V^{j-1} in V^j. Of the many possible choices available once we choose the scaling functions Φ^j and the synthesis filters P^j, the Q^j matrix of the Haar wavelet in Equation (42) is uniquely determined by the additional constraint of having the least number of non-zero entries in each column. □

We now introduce the 2^j-*D column* vectors C^j and D^j with

$$C^j = \left[c_0^j\ \ldots\ c_{2^j-1}^j\right]^\top \ \text{ and } \ D^j = \left[d_0^j\ \ldots\ d_{2^j-1}^j\right]^\top$$

where the c_i^j are the coefficients of the scaling functions at the level j needed to approximate a given function $f(x)$, and d_i^j, the detail coefficients for $i = 0, \ldots, 2^j - 1$ needed to reconstruct the function $f(x)$ at the finer scale $j + 1$.

The P^j and Q^j are called *synthesis* filters since we have

$$C^j = P^j C^{j-1} + Q^j D^{j-1} = \left[P^j | Q^j\right] \left[\frac{C^{j-1}}{D^{j-1}}\right]. \tag{43}$$

Exercise 27. *Verify that the dimension of all the vectors and matrices in Equation (43) are consistent.*

Example 14. *Again the case of $j = 2$*
If

$$C^2 = [\,8\,4\,1\,3\,]^\top \quad C^1 = [\,6\,2\,]^\top \quad D^1 = [\,2\ -1\,]^\top \quad C^0 = [\,4\,] \text{ and } D^0 = [\,2\,],$$

consistently with what we just derived with

$$P^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad Q^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad P^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } Q^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

we have

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 1 \\ 3 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} [\,4\,] + \begin{bmatrix} 1 \\ -1 \end{bmatrix} [\,2\,] = \begin{bmatrix} 4 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

Analysis filters

We now want to find the $2^{j-1} \times 2j$ matrices A^j and B^j needed to obtain the lower resolution coefficients

$$C^{j-1} = A^j C^j$$

$$D^{j-1} = B^j C^j$$

For the unnormalised Haar basis we have

$$A^j = \frac{1}{2}(P^j)^\top \text{ and } B^j = \frac{1}{2}(Q^j)^\top. \tag{44}$$

Exercise 28. *Verify Equation (44) in the case of Example (14)*

Observation 49. *Analysis filters*
The matrices A^j and B^j are *analysis* filters since they are used to obtain the coefficients at level $j-1$ from the finer resolution at level j.

Observation 50. *Relationship between analysis and synthesis filters*
In the general case the analysis filters and the synthesis filters are not the transpose of one another though it is always true that, stacking A^j and B^j one on top of the other and C^j and D^j next to each other, it holds true that

$$\left[\frac{A^j}{B^j}\right] = \left[C^j | D^j\right]^{-1}$$

7 Projects

Fourier Series Study theoretically and empirically the convergence of the Fourier Series at the points of continuity as a function of the number of terms. Quantify the Gibbs phenomenon at points of discontinuity. Check the quality of the approximation you obtain on real signals as a function of the number of terms in the series.

Fourier Transform Compute by hand all the pairs listed in Subsection 3.3 and verify the obtained results using your favorire FFT package.

Low-Pass Filtering Design and implement an ideal low-pass filter and a Gaussian filter for a 1D signal. Build a smooth signal and add some high frequency noise to it. Compare the result you obtain after filtering the noisy signal with the ideal low-pass and the Gaussian filter as a function of the filter width.

Kalman Filter Design and implement a Kalman filter for tracking a billiard ball. Assume the ball moves at constant speed and rebounds from the cushion without losing its kinetic energy.

Wavelet vs Fourier transform Using the packages you prefer compare the Wavelet and the Fourier Transform on a few smooth signals, signals with discontinuities, and signals varying first slowly and then fast.

Lossy image compression Study the 2-D Haar wavelets and its application to image compression.

Frames Study the theoretical properties of frames and get to know the beauty and the power of over-completeness.