10 月 29 日作业

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题目 1. 求 $f(x) = x^2$ 在 $[a, b]$ 上的分段线性插值函数 $I_h(x)$, 并估计误差。

解答. 设插值节点为 $a = x_0 < x_1 < \cdots < x_n = b$, 则在 $[x_i, x_{i+1}]$ 上, 有

$$I_h(x) = f(x_i)\frac{x_{i+1} - x}{x_{i+1} - x_i} + f(x_{i+1})\frac{x - x_i}{x_{i+1} - x_i} = (x_i + x_{i+1})x - x_i x_{i+1}$$

设 $h_i = x_{i+1} - x_i, \quad h = \max_{0\le i\le n-1} h_i$, 则对任意 $x \in [a, b]$, 存在 i 使得 $x \in [x_i, x_{i+1}]$, 由插值误差公式, 存在 $\xi \in [x_i, x_{i+1}]$ 使得

$$f(x) - I_h(x) = \frac{f''(\xi)}{2!}(x - x_i)(x - x_{i+1}) = (x - x_i)(x - x_{i+1}) < \frac{h^2}{4}$$

其中 $f''(\xi) \equiv 2$.

题目 2. 给定数据如下表所示，试求三次样条插值 $S(x)$, 并满足条件

(1) $S'(0.25) = 1.0000, S'(0.53) = 0.6868$;

(2) $S''(0.25) = S''(0.53) = 0$.

表 1: 插值节点数据

x_j	0.25	0.30	0.39	0.45	0.53
y_j	0.5000	0.5477	0.6245	0.6708	0.7280

解答. (1) 由公式 $h_k = x_{k+1} - x_k$ 计算可得

$$h_0 = 0.05, h_1 = 0.09, h_2 = 0.06, h_3 = 0.08$$

再由 $\mu_k = \frac{h_{k-1}}{h_{k-1}+h_k}, \lambda_k = \frac{h_k}{h_{k-1}+h_k}$ 计算可得

$$\mu_1 = \frac{5}{14}, \lambda_1 = \frac{9}{14}; \quad \mu_2 = 0.6, \lambda_2 = 0.4; \quad \mu_3 = \frac{3}{7}, \lambda_3 = \frac{4}{7}$$

再由 $g_k = 3(\mu_k f[x_{k-1}, x_k] + \lambda_k f[x_k, x_{k+1}])$ 计算可得

$$g_1 = 3\left(\frac{5}{14} \times \frac{0.5477 - 0.5000}{0.30 - 0.25} + \frac{9}{14} \times \frac{0.6245 - 0.5477}{0.39 - 0.30}\right) = \frac{747}{280}$$

$$g_2 = 3\left(0.6 \times \frac{0.6245 - 0.5477}{0.39 - 0.30} + 0.4 \times \frac{0.6708 - 0.6245}{0.45 - 0.39}\right) = 2.462$$

$$g_3 = 3\left(\frac{3}{7} \times \frac{0.6708 - 0.6245}{0.45 - 0.39} + \frac{4}{7} \times \frac{0.7280 - 0.6708}{0.53 - 0.45}\right) = \frac{621}{280}$$

同时,$m_0 = 1.0000, m_4 = 0.6868$，因此可列线性方程组：

$$\begin{bmatrix} 2 & \frac{5}{14} & \\ 0.6 & 2 & 0.4 \\ & \frac{4}{7} & 2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} \frac{747}{280} - \frac{9}{14} \times 1.0000 \\ 2.462 \\ \frac{621}{280} - \frac{3}{7} \times 0.6868 \end{bmatrix}$$

解得

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0.8650 \\ 0.8263 \\ 0.7259 \end{bmatrix}$$

代入

$$\begin{aligned} s_k(x) = & \frac{(x - x_{k+1})^2[h_k + 2(x - x_k)]}{h_k^3} y_k + \frac{(x - x_k)^2[h_k + 2(x_{k+1} - x)]}{h_k^3} y_{k+1} \\ & + \frac{(x - x_{k+1})^2(x - x_k)}{h_k^2} m_k + \frac{(x - x_k)^2(x - x_{k+1})}{h_k^2} m_{k+1} \end{aligned}$$

可得分段三次样条插值函数 $S(x)$: