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题目 1. 求 $f(x)=x^2$ 在 $[a,b]$ 上的分段线性插值函数 $I_h(x)$, 并估计误差。

解答. 设插值节点为 $a=x_0<x_1<\cdots<x_n=b$, 则在 $[x_i,x_{i+1}]$ 上, 有

$$I_h(x)=f(x_i)\frac{x_{i+1}-x}{x_{i+1}-x_i}+f(x_{i+1})\frac{x-x_i}{x_{i+1}-x_i}=(x_i+x_{i+1})x-x_ix_{i+1}$$

设 $h_i=x_{i+1}-x_i,\quad h=\max_{0\leq i\leq n-1}h_i$, 则对任意 $x\in[a,b]$, 存在 i 使得 $x\in[x_i,x_{i+1}]$, 由插值误差公式, 存在 $\xi\in[x_i,x_{i+1}]$ 使得

$$f(x)-I_h(x)=\frac{f''(\xi)}{2!}(x-x_i)(x-x_{i+1})=(x-x_i)(x-x_{i+1})<\frac{h^2}{4}$$

其中 $f''(\xi)\equiv 2$.

题目 2. 给定数据如下表所示，试求三次样条插值 $S(x)$, 并满足条件

(1) $S'(0.25)=1.0000, S'(0.53)=0.6868$;

(2) $S''(0.25)=S''(0.53)=0$.

表 1: 插值节点数据

x_j	0.25	0.30	0.39	0.45	0.53
y_j	0.5000	0.5477	0.6245	0.6708	0.7280

解答. (1) 由公式 $h_k = x_{k+1} - x_k$ 计算可得

$$h_0 = 0.05, h_1 = 0.09, h_2 = 0.06, h_3 = 0.08$$

再由 $\mu_k = \frac{h_{k-1}}{h_{k-1}+h_k}, \lambda_k = \frac{h_k}{h_{k-1}+h_k}$ 计算可得

$$\mu_1 = \frac{5}{14}, \lambda_1 = \frac{9}{14}; \quad \mu_2 = 0.6, \lambda_2 = 0.4; \quad \mu_3 = \frac{3}{7}, \lambda_3 = \frac{4}{7}$$

再由 $g_k = 3(\mu_k f[x_{k-1}, x_k] + \lambda_k f[x_k, x_{k+1}])$ 计算可得

$$g_1 = 3\left(\frac{9}{14} \times \frac{0.5477 - 0.5000}{0.30 - 0.25} + \frac{5}{14} \times \frac{0.6245 - 0.5477}{0.39 - 0.30}\right) = 2.7541$$

$$g_2 = 3\left(0.4 \times \frac{0.6245 - 0.5477}{0.39 - 0.30} + 0.6 \times \frac{0.6708 - 0.6245}{0.45 - 0.39}\right) = 2.413$$

$$g_3 = 3\left(\frac{4}{7} \times \frac{0.6708 - 0.6245}{0.45 - 0.39} + \frac{3}{7} \times \frac{0.7280 - 0.6708}{0.53 - 0.45}\right) = 2.2421$$

同时,$m_0 = 1.0000, m_4 = 0.6868$，因此基于边界条件 1，可列线性方程组:

$$\begin{bmatrix} 2 & \frac{5}{14} & \\ 0.4 & 2 & 0.6 \\ & \frac{4}{7} & 2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 2.7541 - \frac{9}{14} \times 1.0000 \\ 2.413 \\ 2.2421 - \frac{3}{7} \times 0.6868 \end{bmatrix}$$

解得

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0.9127 \\ 0.8004 \\ 0.7452 \end{bmatrix}$$

代入

$$\begin{aligned} s_k(x) = &\frac{(x - x_{k+1})^2[h_k + 2(x - x_k)]}{h_k^3} y_k + \frac{(x - x_k)^2[h_k + 2(x_{k+1} - x)]}{h_k^3} y_{k+1} \\ &+ \frac{(x - x_{k+1})^2(x - x_k)}{h_k^2} m_k + \frac{(x - x_k)^2(x - x_{k+1})}{h_k^2} m_{k+1} \end{aligned}$$

可得分段三次样条插值函数:

$$S(x) = \begin{cases} 1.88x^3 - 2.424x^2 + 1.8595x + 0.15725, & x \in [0.25, 0.30], \\ 0.794239x^3 - 1.44593x^2 + 1.56581x + 0.186646, & x \in [0.30, 0.39], \\ 0.62963x^3 - 1.25333x^2 + 1.4907x + 0.19641, & x \in [0.39, 0.45], \\ 0.3125x^3 - 0.824375x^2 + 1.29729x + 0.225477, & x \in [0.45, 0.53]. \end{cases}$$

(2)$g_0=3\frac{y_1-y_0}{h_0}-0=2.862,\quad g_4=3\frac{y_4-y_3}{h_3}+0=2.145$，因此基于边界条件 2，可列线性方程组:

$$\begin{bmatrix} 2 & 1 & & & \\ \frac{9}{14} & 2 & \frac{5}{14} & & \\ & 0.4 & 2 & 0.6 & \\ & & \frac{4}{7} & 2 & \frac{3}{7} \\ & & & 1 & 2 \end{bmatrix}\begin{bmatrix} m_0\\ m_1\\ m_2\\ m_3\\ m_4 \end{bmatrix}=\begin{bmatrix} 2.862\\ 2.7541\\ 2.413\\ 2.2421\\ 2.145 \end{bmatrix}$$

解得

$$\begin{bmatrix} m_0\\ m_1\\ m_2\\ m_3\\ m_4 \end{bmatrix}=\begin{bmatrix} 0.9697\\ 0.9226\\ 0.7992\\ 0.7424\\ 0.7013 \end{bmatrix}$$

代入

$$\begin{aligned} s_k(x)=&\frac{(x-x_{k+1})^2[h_k+2(x-x_k)]}{h_k^3}y_k+\frac{(x-x_k)^2[h_k+2(x_{k+1}-x)]}{h_k^3}y_{k+1}\\ &+\frac{(x-x_{k+1})^2(x-x_k)}{h_k^2}m_k+\frac{(x-x_k)^2(x-x_{k+1})}{h_k^2}m_{k+1} \end{aligned}$$

可得分段三次样条插值函数:

$$S(x)=\begin{cases} -6.28x^3+4.71x^2-0.2078x+0.3557, & x\in[0.25,0.30],\\ 1.8683x^3-2.6193x^2+1.9897x+0.1360, & x\in[0.30,0.39],\\ -0.4815x^3+0.1333x^2+0.9149x+0.2760, & x\in[0.39,0.45],\\ 2.1406x^3-3.4036x^2+2.5052x+0.03762, & x\in[0.45,0.53]. \end{cases}$$