MAS316 Assignment 3: Chaos

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1 Bifurcations of 1-d maps

A population of lemmings can be modelled by the map

$$x_{t+1} = rx_t/(1 + x_t^2), \quad x, r \in [0, \infty)$$

We first want to find the fixed points (also known as critical points) of this map. Defining $x_{n+1} = f(x_n)$ a fixed point x_c is one that satisfies the equation$f(x_c) = x_c$, so in this case

$$\begin{aligned} x_c &= rx_c/(1 + x_c^2) \\ \implies x_c &= 0 \text{ or } x_c = \sqrt{r-1} \end{aligned}$$

At any given critical point x_c we have that a small perturbation of size δx_t from x_c is given by

$$\frac{f(x_c + \delta x_t) - (x_c)}{\delta x_t} = \frac{\delta x_{t+1}}{\delta x_t}$$

So if $|\delta x_{t+1}| < |\delta x_t|$ the fixed point is attracting and if $|\delta x_{t+1}| > |\delta x_t|$ then the fixed point must be repelling. Taking the limit $\delta x_t \to 0$ for a 1-D map we can deduce that:

$$\text{If } |f'(x_c)| < 1, \text{then } x_c \text{ is attracting, therefore stable}$$

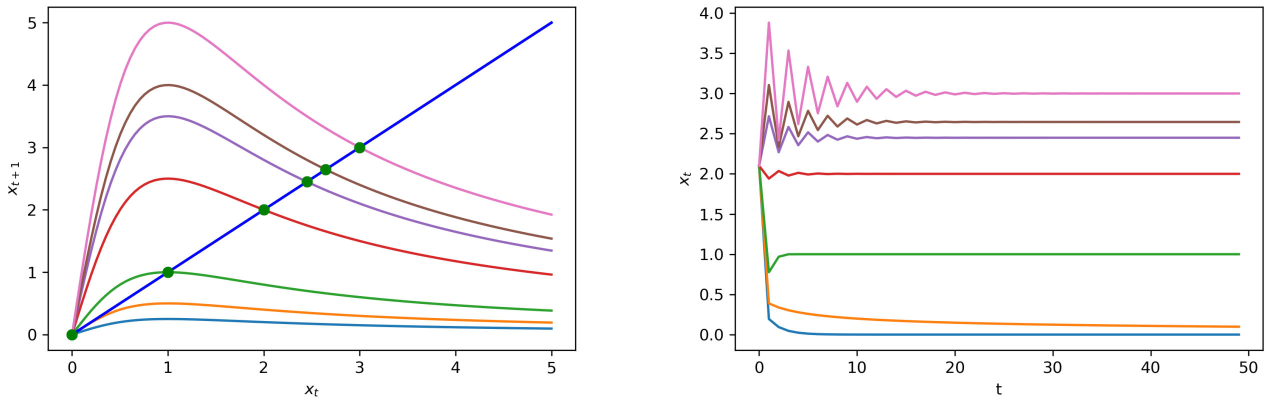
$$\text{If } |f'(x_c)| > 1, \text{then } x_c \text{ is repelling therefore unstable}$$

In the case of the lemming population we have that

$$|f'(x_c)| = -\frac{r(x_c^2 - 1)}{(x_c^2 + 1)^2}$$

So $|f'(0)| = r$ and the fixed point $x_c = 0$ is stable if $r < 1$. As $r \geq 1$ the fixed point $x_c = 0$ becomes unstable but $x_c = \sqrt{r-1}$ becomes stable, as long as $|f'(\sqrt{r-1})| = |-\frac{r((\sqrt{r-1})^2-1)}{((\sqrt{r-1})^2+1)^2}| = |-\frac{r-2}{r}| < 1 \implies r > 1$. Because the fixed points exchange stability at $r = 1$ this is called a transcritical bifurcation. Further to this the continued prevalence of stable fixed points over the whole range of r is an indicator chaos will not occur in this system. Instead the system reaches a steady state determined by the value of r. This means that the lemming population will always converge towards a steady level with very small or no fluctuations, the population level is determined by the value of r this is

Figure 1: (**left**) Lemming population map for $r = 0.5, 1, 2, 5, 7, 8, 10$ the critical points lie on the diagonal $x_t = x_{t+1}$ for each case, with $r > 1, x_c = \sqrt{r-1}$ and for $r < 1, x_c = 0$ (**right**) Time series graph showing how the value of x_t settles to a steady state, varying dependent on the initial value of r



likely to be the growth rate of the lemmings, the higher the growth rate the higher the potential population, this is a logical assumption to make if we were to ignore outside limiting factors to the population size, e.g food availability.

Now lets suppose the following two maps describe the flow of current around an electrical component. We want to explore the behaviour of these systems and how this might relate to the functionality of the electrical component.

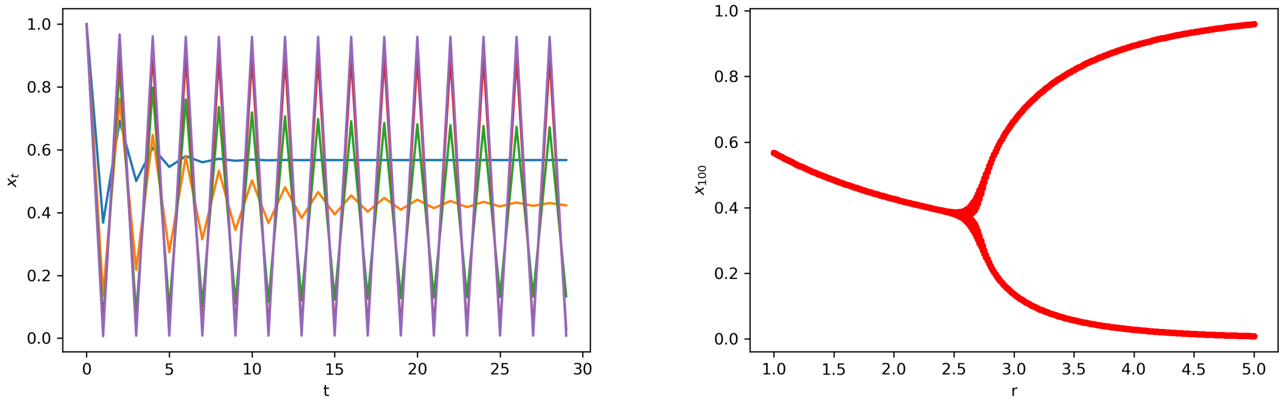
$$x_{t+1} = e^{-rtx_t} \tag{1}$$

$$x_{t+1} = r \cos x_t \tag{2}$$

Taking map (1) and plotting a time series for a few different values of r (Figure 2) we can see that for some values of r it appears x_t finds a steady state and for other r there seems there is oscillation around a steady state forming a cycle of period 2. Changing the initial value x_0 has no effect of the finishing position of these steady states and cycles. Plotting the bifurcation diagram for this map shows how as the value of r increases through $r \approx 2.6$ there is a period doubling bifurcation, in which a small change in the parameter r causes a change in the state space, in this case a change in the electrical current. That means for $r > 2.6$ the electrical current running through the circuit changes from a steady current to one that oscillates between two values.

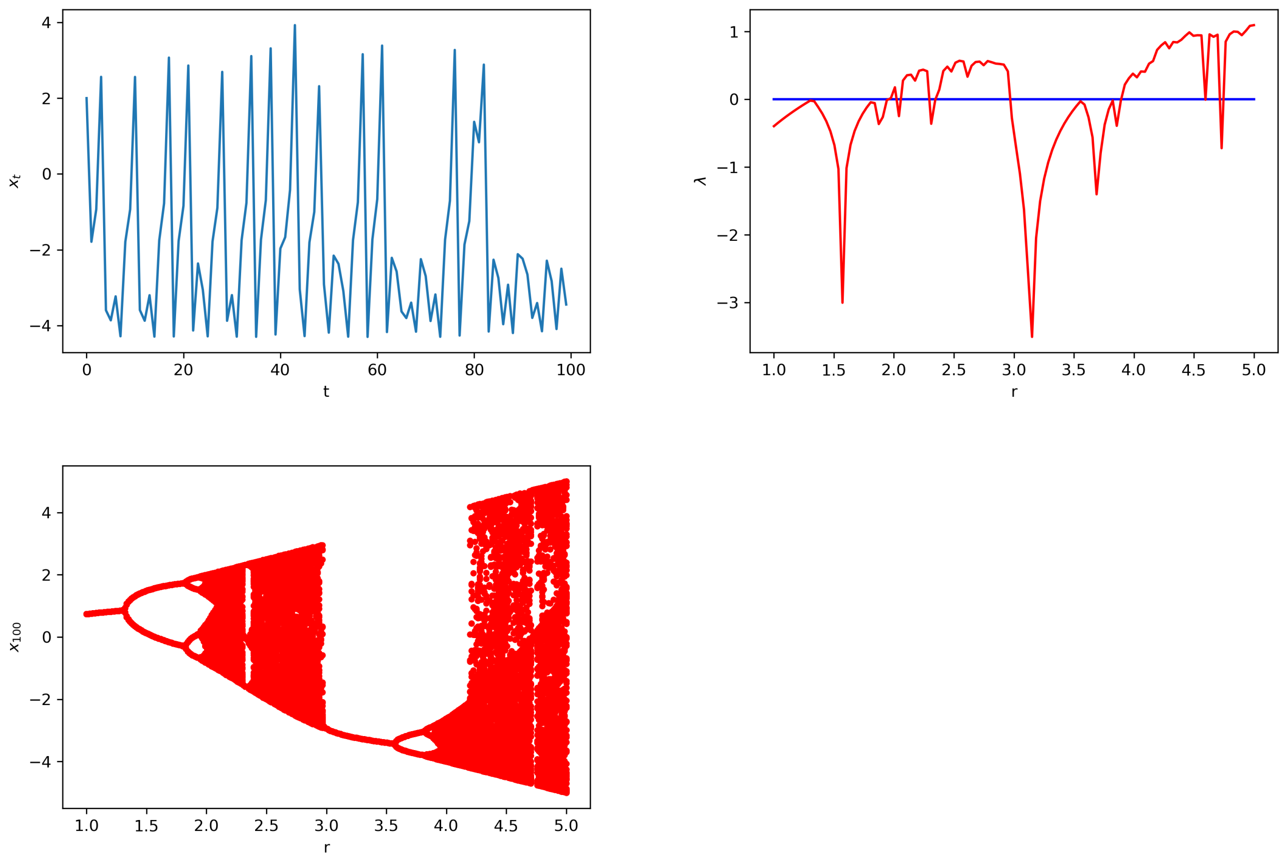
Taking map (2) it can be seen for some values of r e.g. 4.2 (Figure 3) the value of x_t never settles down to any fixed value or any periodic cycle. It looks to be chaotic. We can examine this further by looking at the Lyapunov exponent (λ) for the system, $\lambda > 0$ is a necessary condition for chaos. Looking at the plot in figure 3 it's clear there are many values for which $\lambda > 0$, comparing this to the bifurcation diagram for the map there is direct relationship between the values of r for which $\lambda < 0$ and those for which there is a steady state or periodic cycle. Descending into chaos for the values of r that give $\lambda > 0$. For different values of r then we would get either a fixed current with the component receiving a consistent level of power, a current that is not fixed but periodic in nature

Figure 2: **left** A time series plot for map (1) with $r = 1, 2, 3, 4, 5$ **right** Bifurcation diagram for map (1)



thus predictable giving the component a cyclic range of power input or a completely random and unpredictable current offering anything from no power input to extremely high current in short period of time, this is likely to be far from optimal and would likely cause damage to any electrical component.

Figure 3: (**left**) A time series plot for map (2) with $r = 4.2$ showing chaotic pattern. (**right**) Lyapunov exponent value λ varying with r, values above the line $\lambda = 0$ suggest chaotic behaviour is possible.(**below**) Bifurcation diagram for map (2) over $1 < r < 5$



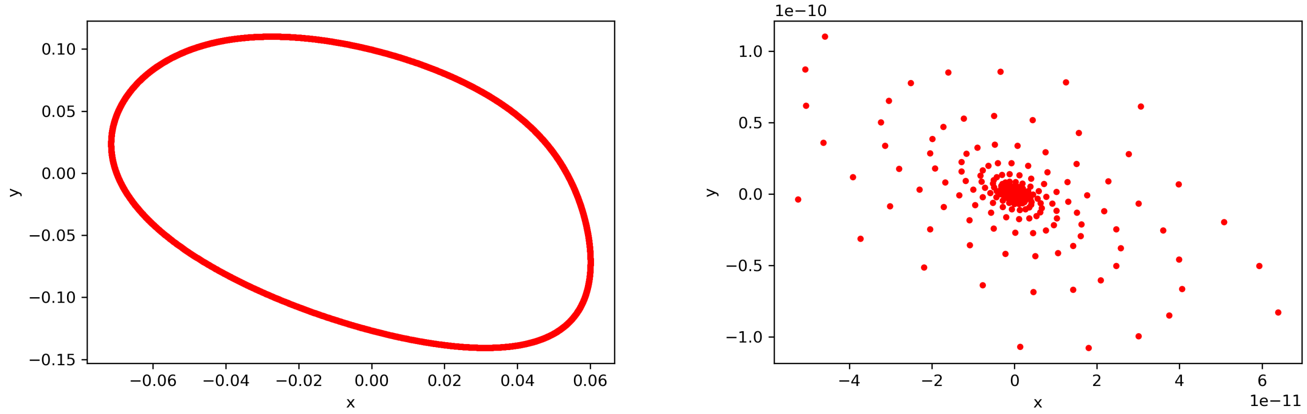
2 The Tinkerbell map

We will define the Tinkerbell map to be

$$\boldsymbol{f}(x, y) = (x^2 - y^2 + c_1x + c_2y, 2xy + c_3x + c_4y)$$

We see in Figure 4 that if we plot the phase portrait after 1000 iterations the system with

Figure 4: Phase portrait for Tinkerbell map with (**left**) $c_4 = 0.5$ a periodic cycle and (**right**) $c_4 = 0.8$ a fixed point



parameters $c_1 = -0.3, c_2 = -0.6, c_3 = 2, c_4 = 0.5$, converges to a periodic orbit. We plot only after the first 1000 iterations so that we can see the dynamics of the system after it has reached the attractor. However this isn't always the case if we adjust the parameter c_4 from 0.5 to 0.7 we now see that the attractor is in fact a fixed point at $(0, 0)$. Reducing c_4 to 0.3 results in an attractor that is a limiting cycle including only 4 different (x, y) points while reducing c_4 further to 0.1 results in another continuous limiting cycle similar to that of $c_4 = 0.5$ but of slightly different shape.

This leads to the question of how changing all the initial parameters might effect the attractor. If we now set $c_1 = 0.9, c_2 = -0.6013, c_3 = 2, c_4 = 0.5$, The phase portrait in Figure 5 shows a very different type of limiting cycle, after 1000 iterations. This is called a strange attractor, it is a chaotic orbit with a fractal structure.

Given a system described by the Tinkerbell map, take a small region A in the (x, y) plane does the map expand or contract this region? A small area at time $t + 1$ is related to an area at time t by the relation

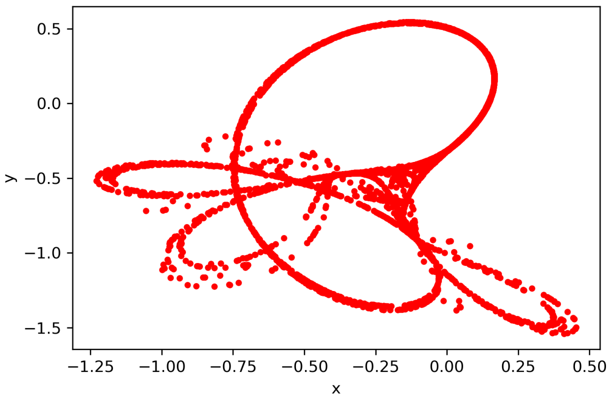
$$\delta A_{t+1} = \det \left[\frac{\partial(x, y)_{t+1}}{\partial(x, y)_t} \right] \delta A_t$$

So we have that $|\delta A_t|$ is contracted or expanded by a factor $|\det [J]\,|$ given as a function of x and y where J is the Jacobian of the map evaluated at $(x, y)_t$. For the Tinkerbell map

$$J = \begin{bmatrix} 2x + c_1 & -2y + c_2 \\ 2y + c_3 & 2x + c4 \end{bmatrix}$$

and $|\det [J]\,| = 4x^2 + 2x(c_1 + c_4) + 4y^2 + 2y(c_2 + c_3)$. If $|\det [J]\,| < 0$ then the are contracts and if $|\det [J]\,| > 0$ the area expands.

Figure 5: Phase portrait after 1000 iterations for Tinkerbell map with $c_1 = 0.9, c_2 = -0.6013, c_3 = 2, c_4 = 0.5,$



3 A model of sheared fluid flow

We will now consider the following 2-d model.

$$\dot{\boldsymbol{u}} = A\boldsymbol{u} + ||\boldsymbol{u}||B\boldsymbol{u}, \ A = \begin{bmatrix} -1/R & 1 \\ 0 & -2/R \end{bmatrix}, \ B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{3}$$

where $\boldsymbol{u} = (u_1, u_2)$ and $||\boldsymbol{u}|| = \sqrt{u_1^2 + u_2^2}$, $R > 0$. This model will represent a flow. Here the linear term involving A amplifies energy transiently while the non-linear term involving B rotates energy in the $\boldsymbol{u_1 u_2}$ plane, this sort of model holds many similarities to the equations of fluid mechanics. We suppose that u_1 represents disturbances to the main flow and u_2 motions going perpendicular to the boundary wall.

Generally as flow rates increase flows transition from smooth laminar flow to turbulent(chaotic) flow, one fixed point of the system is $(u_1, u_2) = (0, 0)$, this suggests no disturbances or flow perpendicular to the wall so it represents laminar flow, smooth and parallel to the boundary. Integrating the above 2-d model in time gives the phase plot as shown in Figure 6, showing an attracting non-linear fixed point not equal to zero, so this can be considered the turbulence. After linearisation about the the solution $\boldsymbol{u} = 0$ giving $\dot{\boldsymbol{u}} = A\boldsymbol{u}$, taking eigenvalues of A suggests that the fixed point $(0, 0)$ is stable, but where exactly is the boundary between the laminar and turbulent flow?

Considering now the values of $||\boldsymbol{u}||$ (amplitude) for solutions to equation (3) with $R = 25$ starting from six different initial vectors $\boldsymbol{u}(0) = (0, \epsilon)$, $\epsilon = 10^{-7}, 10^{-6}..., 10^{-2}$ (Figure 6) for $||\boldsymbol{u}(\boldsymbol{0})|| \leq 10^{-4}$ the curves appear to be approximate translations of one another, suggesting a linear relationship. However at $||\boldsymbol{u}(\boldsymbol{0})|| \leq 10^{-3}$ the non linear behaviour is evident, for this initial amplitude and higher the curves don't decay to zero but instead rapidly grow to a limiting amplitude of 1. Taking a look at other values of R (Figure 7) shows that increasing R reduces the initial value of $\boldsymbol{u}(0)$ that is necessary for the amplitude to rapidly grow to the critical value ≈ 1

These results show us something interesting, there is a threshold initial $||\boldsymbol{u}(0)||$ at which the amplitude stops decaying and grows towards the critical value, resulting in turbulent flow for $R = 25$ this is between 10^{-3} and 10^{-4} . This threshold value changes depending on the value of R as this increases the initial amplitude of the flow required to eventually result in turbulence gets smaller and smaller, only tiny perturbations to the laminar flow are required to result in chaos.

Figure 6: Turbulent fixed point of the flow map for $R = 25$ (**left**) and $||\boldsymbol{u}(t)||$ for solutions to equation (3) with $\boldsymbol{u}(0) = (0, \epsilon)$, $\epsilon = 10^{-7}, 10^{-6}..., 10^{-2}$ and $R = 25$ (**right**)

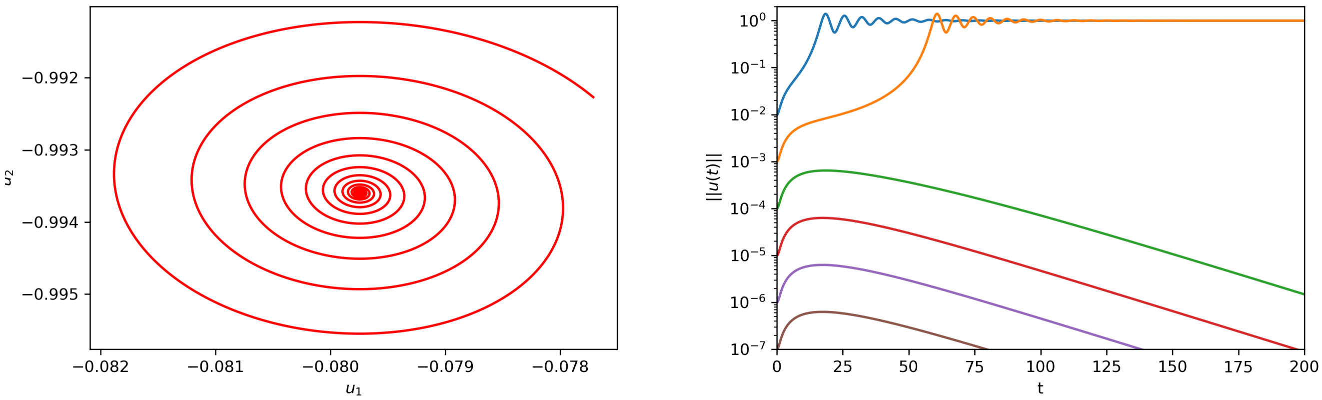


Figure 7: $||\boldsymbol{u}(t)||$ for solutions to equation (3) with $\boldsymbol{u}(0) = (0, \epsilon)$, $\epsilon = 10^{-7}, 10^{-6}..., 10^{-2}$ and $R = 12.5$ (left) and $R = 50$ (right)