

Scheduling and Amortization for MPC

Benjamin Levy
levyb3@rpi.edu
Rensselaer Polytechnic Institute
Troy, New York

Benjamin Sherman
shermb@rpi.edu
Rensselaer Polytechnic Institute
Troy, New York

Lindsey Kennard
kennal@rpi.edu
Rensselaer Polytechnic Institute
Troy, New York

Ana L. Milanova
milanova@cs.rpi.edu
Rensselaer Polytechnic Institute
Troy, New York

Muhammad Ishaq*
m.ishaq@ed.ac.uk
University of Edinburgh
Edinburgh, Scotland

Vassilis Zikas†
vzikas@inf.ed.ac.uk
University of Edinburgh
Edinburgh, Scotland

ABSTRACT

CCS CONCEPTS

• Theory of computation → Program analysis; Cryptographic protocols; • Security and privacy → Cryptography.

KEYWORDS

protocol mixing; linear programming; multiparty computation; program analysis; cryptography

ACM Reference Format:

Benjamin Levy, Benjamin Sherman, Lindsey Kennard, Ana L. Milanova, Muhammad Ishaq, and Vassilis Zikas. 2019. Scheduling and Amortization for MPC. In *2019 ACM SIGSAC Conference on Computer and Communications Security (CCS '19)*, November 11–15, 2019, London, United Kingdom. ACM, New York, NY, USA, ?? pages. <https://doi.org/10.1145/3319535.3339818>

1 INTRODUCTION

2 BACKGROUND AND PROBLEM STATEMENT

2.1 Scheduling in MPC

ANA: Here we need more on MPC before jumping to scheduling. MPC-source, basic assumptions about static loop bounds, etc.

For this treatment we make the following simplifying assumptions:

*This work was done in part while the author was at RPI.

†This work was done in part while the author was visiting UCLA and supported in part by DARPA and SPAWAR under contract N66001-15-C-4065 and by a SICS Cyber Nexus Research Exchanges grant.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. Copyrights for components of this work owned by others than ACM must be honored. Abstracting with credit is permitted. To copy otherwise, or republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee. Request permissions from permissions@acm.org.

CCS '19, November 11–15, 2019, London, United Kingdom

© 2019 Association for Computing Machinery.

ACM ISBN 978-1-4503-6747-9/19/11...\$15.00

<https://doi.org/10.1145/3319535.3339818>

- (1) All statements in the program execute using the same protocol (sharing). That is, there is no share conversion. ISHAQ: *This is not an assumption, this is our setting. We work in the single protocol world.*
- (2) All MPC instructions have the same unit cost, 1 unit. ISHAQ: *Don't we need to distinguish between local (cheap) vs. interactive (expensive) instructions? I don't see how we could assume cost to be a convex function.* ANA: *After discussion 1/12: replace single costs 1 with two levels of costs: α for non-local operations, e.g., MUL, and β for local ops, e.g., ADD.*
- (3) There is unlimited bandwidth—i.e., a single MPC-instruction costs as much as N amortized instructions, namely 1 unit. ISHAQ: *Comment from Vassilis: We assume there are infinite parallel capacities, this is the standard assumption in PRAM (Cryptographic Parallel RAM).* ANA: *PRAM assumption stays. We have to replace the unit cost of 1 with a suitable amortization function $f(n)$, which I don't think changes much in the cost modeling and analysis.*
- (4) MPC instructions scheduled in parallel benefit from amortization *only if* they are the same instruction. Given our previous assumption, 2 MUL instructions scheduled in parallel benefit from amortization and cost 1, however a MUL and a MUX instructions scheduled in parallel still cost 2. ISHAQ: *We need to reword this assumption to something like "K parallel MUL costs much less than K (because they get amortized), but any mix of K MUL and MUX still cost roughly K. Specifically, we should take away the low constants (2 and 1) because we know for low constants this is not true.* ANA: *Again, core assumption that MUL and MUX don't benefit from amortization stays. We have to replace constant costs with functions, as in (3).*

2.2 Scheduling in HPC

3 PRELIMINARIES

We assume arbitrarily nested loops in the MPC-source IR and read-only arrays. We assume that loops range from 0 to some constant N . Arrays are linearized (row-major order as in MOTION) and accesses are via functions of the induction variables of the enclosing loops. We write i_1, i_2, \dots, i_k to denote

the loop nest: i_1 is the outermost loop, i_2 , is immediately nested in i_1 , and so on until i_k . As an example, a statement nested in i_1, i_2, \dots, i_k can access array $A[f(i_1, i_2, \dots, i_k)]$. We write $A[i_1, i_2, \dots, i_k]$ interchangeably. **ANA: Need to state this more precisely.**

3.1 Pseudo ϕ -nodes

A pseudo ϕ -node $X_1 = \phi(X_0, X_2)$ in a loop header is evaluated during circuit generation. If it is the 0-th iteration, then the ϕ -node evaluates to X_0 , otherwise, it evaluates to X_2 .

3.2 Def-use Edges

The dependence graph has the following def-use edges:

- same-level forward $X \rightarrow Y$ where X and Y are in the same loop nest i_1, i_2, \dots, i_k . E.g., $d = \text{SUM}(S[i, j], C[j])$ to $p = \text{MUL}(d, d)$ in Biometric is a same-level edge. A ϕ node can be a source of a same-level forward edge but not a target.
- outer-to-inner forward $X \rightarrow Y$ where X is in an outer loop nest, i_1, i_2, \dots, i_j , and Y is in an inner one, $i_1, i_2, \dots, i_j, \dots, i_k$. A ϕ -node can be a source or a target of an outer-to-inner forward edge.
- inner-to-outer forward $X \rightarrow Y$ where X is a ϕ -node in an inner loop nest, $i_1, i_2, \dots, i_k, i_{k+1}$, and Y is in the enclosing loop nest i_1, i_2, \dots, i_k . E.g. $\text{sum}_0 = \phi(\text{sum}_1, 0)$ to $c = \text{CMP}(\text{sum}_0, \text{min}_0)$ is an inner-to-outer forward edge. Note that the source is *always* a ϕ -node in the immediately enclosing loop. The interpretation of this edge is that the use node Y uses the definition made in the last iteration of the inner loop. **BEN: The current representation of pseudo ϕ -nodes shows them attached to the loop header (i.e. in the loop). We may want to clarify that they are also evaluated at the loop's termination.**
- same-level back-edge $X \rightarrow Y$. Y is a ϕ -node in the header of the loop and X is a definition of the variable in the loop body. E.g., $\text{min}_1 = \text{MUX}(c, \text{sum}_1, \text{min}_1)$ to $\text{min}_0 = \phi(\text{min}_1, 10000)$ in Biometric is a same-level back-edge.
- inner-to-outer back-edge $X \rightarrow Y$: X and Y are both ϕ -nodes for some variable. The source X is in a loop nested into Y 's loop (not necessarily immediately).
- mixed forward edge $X \rightarrow Y$. X is a ϕ -node in some loop $i_1, i_2, \dots, i_k, i_{k+1}$ and Y is a node in a loop nested into i_1, i_2, \dots, i_k . We transform mixed forward edges as follows. Let x_j be the variable defined at the ϕ -node X . We add a variable and assignment $x'_j = x_j$ immediately after the i_1, i_2, \dots, i_k loop. Then we replace the use of x_j at Y with x'_j . This transforms a mixed forward edge into an "inner-to-outer" forward edge followed by an outer-to-inner forward edge. Thus Basic Vectorization handles one of "same-level", "inner-to-outer", or "outer-to-inner" def-use edges.

3.3 Helper Functions

We define $\text{closure}(n)$ where n is a ϕ -node. Intuitively, it computes the set of nodes (i.e., statements) that form a

dependence cycle with n . **ANA: Cycle(n) is probably a better name.** The closure of n is defined as follows:

- n is in $\text{closure}(n)$
- X is in $\text{closure}(n)$ if there is a same-level path from n to X , and $X \rightarrow n$ is a same-level back-edge.
- Y is in $\text{closure}(n)$ if there is a same-level path from n to Y and there is a same-level path from Y to some X in $\text{closure}(n)$.

We define the raise_dim (raise dimensions) and drop_dim (drop dimension) functions. Raise dimension "lifts" a lower-dimensional array (**ANA: The right term here is tensor, I am pretty sure!**) into a higher dimension one. This is necessary when a lower-dimensional array is used in a higher dimensional loop and, essentially, is just copying of values. For example, Biometric contains the statement $d = \text{SUB}(S[i, j], C[j])$, in the j -loop which is nested into the i -loop. Here C is a one-dimensional array, however, to vectorize across both loops it is necessary to turn it into a two-dimensional array: $C[i, j]$ becomes $[C[0], C[1], \dots, C[J], C[0], C[1], \dots, C[J], \dots, C[0], C[1], \dots, C[J]]$, which turns the row into a matrix of I identical rows. (We use capital letters to denote the upper bounds of loops, e.g., J is the upper bound of the j -loop and I is the upper bound of the i -loop.)

$\text{raise_dim}(A[i_1, \dots, i_k], i_j)$ is defined as follows. It results in a new $k+1$ -dimensional array A' where for every $0 \leq i_k < I_{k+1}$, $A'[i_1, \dots, i_j, \dots, i_k] = A[i_1, \dots, i_k]$. Adding n dimensions is trivially extended as a composition of n raise_dim that each adds a single dimension.

As expected, drop dimension turns a higher-dimensional array into a lower-dimensional one. The key use case is an inner-to-outer def-use edge. The code may define a variable, e.g., x in an inner loop, say j , then use this variable in an enclosing loop, say i . Our algorithm may vectorize the computation of x in the j -loop thus producing a vector $x[j]$ where $x[1]$ is the value of x after the 1st iteration and so on. Drop dimension states that the outer loop will use the value of the variable at the last iteration.

$\text{drop_dim}(A[i_1, \dots, i_j, \dots, i_k], i_j)$ produces a k -dimensional array A' where $A'[i_1, \dots, i_k] = A'[i_1, \dots, i_j - 1, \dots, i_k]$. In our analysis, dimensions are always dropped at the end, and again, one can define dropping n dimensions as a composition of n drop_dim .

BEN: I'm not sure if I wrote up the above paragraphs correctly – the idea is that we're just copying values to extend over dimension j or only retain its last elements. The current writeup implies that $1 < j < k$ though, which isn't always true. Python implementations of $\text{raise_dim}()$ and $\text{drop_dim}()$ can be found here: https://github.com/milana2/ParallelizationForMPC/blob/master/compiler/compiler/motion_backend/reference_implementations.py

ANA: This section is still informal. Needs work to make more precise.

4 VECTORIZATION

ANA: Add back-edges into Phase 1. A back-edge from a non-phi-node in loop i to a phi-node in loop i 's header is a same-level edge. The only difference with the handling of normal same-level forward def-use is that the operand index will become $i - 1$. A back edge from a phi-node in loop j to a phi-node in loop i , where j is nested in i is an inner-to-outer edge and will require dropping dimension. We can show these are the only kinds of back-edges that may occur.

4.1 Basic Vectorization

{Phase 1: Raise dimension of scalar variables to corresponding loop nest. We may assume linear traversal of the MPC-source.}

```
for each MPC stmt :  $X = Op(Y_1, Y_2)$  in loop  $i_1, \dots, i_k$  do
  for each  $Y_i$  do
    case def-use edge  $stmt'(def\ of\ Y_i) \rightarrow stmt(def\ of\ X)$ 
    of
      same-level  $\rightarrow add\ Y'_i = Y_i$ 
      outer-to-inner  $\rightarrow add\ Y'_i[i_1, \dots, i_k] = raise\_dim(Y_i)$ 
      inner-to-outer  $\rightarrow add\ Y'_i[i_1, \dots, i_k] = drop\_dim(Y_i)$ 
  end for
  {Optimistically vectorize all.  $\vec{i}$  means vectorized dimension.}
```

change to $X[\vec{i}_1, \dots, \vec{i}_k] = Op(Y'_1[\vec{i}_1, \dots, \vec{i}_k], Y'_2[\vec{i}_1, \dots, \vec{i}_k])$

end for

{Phase 2: Recreating FOR loops for cycles; vectorizable statements hoisted up.}

```
for each dimension  $d$  from highest to 0 do
  for each  $\phi$ -node  $n$  in loop  $i_1, \dots, i_d$  do
    compute  $closure(n)$ 
  end for
  { $cl_1$  and  $cl_2$  intersect if they have common statement, or update same array; "intersect" definition can be expanded}
  while there are closure  $cl_1$  and  $cl_2$  that intersect do
    merge  $cl_1$  and  $cl_2$ 
  end while
  for each closure  $cl$  (after merge) do
    create FOR  $i_d = 0; \dots$  loop
    add  $\phi$ -nodes in  $cl$  to header block
    add target-less  $\phi$ -node for  $A$  if  $cl$  updates array  $A$ 
    add statements in  $cl$  to loop body in some order of dependences
    {Dimension is not vectorizable: }
    change  $\vec{i}_d$  to  $i_d$  in all statements in loop
    treat FOR loop as monolith node: change def-use edges accordingly. Some edges become same-level.
  end for
  for each target-less  $\phi$ -node  $A_1 = \phi(A_0, A_k)$  do
    in vectorizable stmts, replace use of  $A_1$  with  $A_0$ 
    discard  $\phi$ -node if not used in any  $cl$ 
  end for
end for
{Phase 3:}
add SIMD for simdfied dimensions
```

ISHAQ: We need to think of a way to handle 4th case for def-use, when a nested block B def is followed by another, different, nested block B' use. Note: The handwritten notes say we can break it into 2 edges. One idea to this end is to insert a def' node which is a copy of the original def. The def' should be placed in a block that contains both B and B' . ISHAQ: THE ABOVE COMMENT IS RESOLVED. After transformation to SSA, there should be an intermediate variable (phi node) that is assigned a value depending on whether control branched or not. Should we put this into the writeup?

4.2 Example: biometric

4.2.1 Benjamin's code with linear loops (MPC Source)

```
min_sum!1 = 10000
min_index!1 = 0
for i in range(0, N!0):
  min_sum!2 = PHI(min_sum!1, min_sum!4)
  min_index!2 = PHI(min_index!1, min_index!4)
  sum!2 = 0
  for j in range(0, D!0):
    sum!3 = PHI(sum!2, sum!4)
    d!3 = (S!0[(i * D!0) + j]) - C!0[j]
    p!3 = (d!3 * d!3)
    sum!4 = (sum!3 + p!3)
  !1!2 = (sum!3 < min_sum!2)
  min_sum!3 = sum!3
  min_index!3 = i
  min_sum!4 = MUX(!1!2, min_sum!3, min_sum!2)
  min_index!4 = MUX(!1!2, min_index!3, min_index!2)
```

4.2.2 Phase 1 of Basic Vectorization

The transformation preserves the dependence edges. It raises the dimensions of scalars and optimistically vectorizes all operations. The next phase discovers loop-carried dependences and removes affected vectorization.

In the code below, all initializations (e.g., $min_sum!3 = i$), operations, and PHI nodes are *implicitly vectorized*. $raise_dim$ and $drop_dim$ statements, as well as propagation statements (e.g., $min_sum!3 = sum!3^$) may have slightly different interpretation. ANA: *This is vague, may need elaboration.*

Note the two different versions of $raise_dim$. ANA: *More here! One just adds a least-significant dimension. The other one may reshape an input array.*

```
min_sum!1 = 10000
min_sum!1^ = raise_dim(min_sum!1, (i:N!0))
min_index!1 = 0
min_index!1^ = raise_dim(min_index!1, (i:N!0))
S!0^ = raise_dim(S!0, ((i * D!0) + j), (i:N!0, j:D!0))
C!0^ = raise_dim(C!0, j, (i:N!0, j:D!0))
for i in range(0, N!0):
  min_sum!2 = PHI(min_sum!1^, min_sum!4)
  min_index!2 = PHI(min_index!1^, min_index!4)
  sum!2 = 0 // Will lift, when hoisted
  sum!2^ = raise_dim(sum!2, (j:D!0)) // Special form?
  for j in range(0, D!0):
    sum!3 = PHI(sum!2^, sum!4)
    d!3 = S!0^ - C!0^
```

```

p!3 = (d!3 * d!3)
sum!4 = (sum!3 + p!3)
sum!3^ = drop_dim(sum!3)
!1!2 = (sum!3^ < min_sum!2)
min_sum!3 = sum!3^
min_index!3 = i // Same-level, will lift when hoisted
min_sum!4 = MUX(!1!2, min_sum!3, min_sum!2)
min_index!4 = MUX(!1!2, min_index!3, min_index!2)

```

4.2.3 Phase 2 of Basic Vectorization

This phase analyzes statements from the innermost loop to the outermost. The key point is to discover loop-carried dependencies and re-introduce loops whenever dependencies make this necessary.

Starting at the inner phi-node `sum!3 = PHI(sum!2, sum!4)`, the algorithm first computes its closure. The closure amounts to the phi-node itself and the addition node `sum!4 = (sum!3 + p!3)`, accounting for the loop-carried dependency of the computation of `sum`. The algorithm replaces this closure with a FOR loop on `j` removing vectorization on `j`. Note that the SUB and MUL computations remain outside of the loop as they do not depend on phi-nodes that are part of cycles. The dependencies are from `p!3[I,J] = (d!3[I,J] * d!3[I,J])` to the monolithic FOR loop and from the FOR loop to `sum!3 = drop_dim(sum!3)`. (Lower case index, e.g., `i`, indicates non-vectorized dimension, while uppercase index, e.g., `I` indicates vectorized dimension.)

After processing inner loop code becomes:

```

min_sum!1 = 10000
min_sum!1^ = raise_dim(min_sum!1, (i:N!0))
min_index!1 = 0
min_index!1^ = raise_dim(min_index!1, (i:N!0))
S!0^ = raise_dim(S!0, ((i * D!0) + j), (i:N!0,j:D!0))
C!0^ = raise_dim(C!0, j, (i:N!0,j:D!0))
for i in range(0, N!0):
    min_sum!2[I] = PHI(min_sum!1^[I], min_sum!4[I])
    min_index!2[I] = PHI(min_index!1^[I], min_index!4[I])
    sum!2 = [0,...,0]
    sum!2^ = raise_dim(sum!2, (j:D!0))
    d!3[I,J] = S!0^[I,J] - C!0^[I,J]
    p!3[I,J] = (d!3[I,J] * d!3[I,J])
    for j in range(0, D!0):
        sum!3[I,j] = PHI(sum!2^[I,j], sum!4[I,j-1])
        sum!4[I,j] = (sum!3[I,j] + p!3[I,j])
    sum!3^ = drop_dim(sum!3)
    !1!2[I] = (sum!3^[I] < min_sum!2[I])
    min_sum!3 = sum!3^
    min_index!3 = [0,1,2,...,N!0-1]
    min_sum!4[I] = MUX(!1!2[I], min_sum!3[I], min_sum!2[I])
    min_index!4[I] = MUX(!1!2[I], min_index!3[I], min_index!2[I])

```

When processing the outer loop two closures arise, one for `min_sum!2[I] = PHI(...)` and one for `min_index!2[I] = PHI(...)`. Since the two closures *do not* intersect, we have two distinct FOR-loops on `i`:

```

min_sum!1 = 10000
min_sum!1^ = raise_dim(min_sum!1, (i:N!0))
min_index!1 = 0
min_index!1^ = raise_dim(min_index!1, (i:N!0))
S!0^ = raise_dim(S!0, ((i * D!0) + j), (i:N!0,j:D!0))
C!0^ = raise_dim(C!0, j, (i:N!0,j:D!0))

```

```

sum!2 = [0,...,0]
sum!2^ = raise_dim(sum!2, (j:D!0))
d!3[I,J] = S!0^[I,J] - C!0^[I,J]
p!3[I,J] = (d!3[I,J] * d!3[I,J])

```

```

for j in range(0, D!0):
    sum!3[I,j] = PHI(sum!2^[I,j], sum!4[I,j-1])
    sum!4[I,j] = (sum!3[I,j] + p!3[I,j])

```

```

sum!3^ = drop_dim(sum!3)
min_index!3 = [0,1,2,...,N!0-1]
min_sum!3 = sum!3^

```

```

for i in range(0, N!0):
    min_sum!2[i] = PHI(min_sum!1^[i], min_sum!4[i-1])
    !1!2[i] = (sum!3^[i] < min_sum!2[i])
    min_sum!4[i] = MUX(!1!2[i], min_sum!3[i], min_sum!2[i])

```

```

for i in range(0, N!0):
    min_index!2[i] = PHI(min_index!1^[i], min_index!4[i-1])
    min_index!4[i] = MUX(!1!2[i], min_index!3[i], min_index!2[i])

```

4.2.4 Phase 3 of Basic Vectorization

And this phase adds SIMD operations:

```

min_sum!1 = 10000
min_sum!1^ = raise_dim(min_sum!1, (i:N!0))
min_index!1 = 0
min_index!1^ = raise_dim(min_index!1, (i:N!0))
S!0^ = raise_dim(S!0, ((i * D!0) + j), (i:N!0,j:D!0))
C!0^ = raise_dim(C!0, j, (i:N!0,j:D!0))

```

```

sum!2 = [0,...,0]
sum!2^ = raise_dim(sum!2, (j:D!0))
d!3[I,J] = SUB_SIMD(S!0^[I,J], C!0^[I,J])
p!3[I,J] = MUL_SIMD(d!3[I,J] * d!3[I,J])

```

```

for j in range(0, D!0):
    sum!3[I,j] = PHI(sum!2^[I,j], sum!4[I,j-1])
    sum!4[I,j] = ADD_SIMD(sum!3[I,j], p!3[I,j])

```

```

sum!3^ = drop_dim(sum!3)
min_index!3 = [0,1,...,N!0-1]
min_sum!3 = sum!3^

```

```

for i in range(0, N!0):
    min_sum!2[i] = PHI(min_sum!1^[i], min_sum!4[i-1])
    !1!2[i] = CMP(sum!3^[i], min_sum!2[i])
    min_sum!4[i] = MUX(!1!2[i], min_sum!3[i], min_sum!2[i])

```

```

for i in range(0, N!0):
    min_index!2[i] = PHI(min_index!1^[i], min_index!4[i-1])
    min_index!4[i] = MUX(!1!2[i], min_index!3[i], min_index!2[i])

```

ANA: Begin Work-in-progress notes on implementation. Just some thoughts...

For now, I suggest we go with a "naive" implementation and try to do only minimal optimizations. All arrays are stored in the program as one-dimensional arrays. Arrays lifted from scalars have dimensionality of the loop enclosure in the original code, e.g., `sum!3` is two dimensional `[N!0,D!0]` (but represented linearly in row-major order; all these "copies"

are arrays of references to shares). Naturally, all writes into these arrays are canonical writes, i.e., $A[i, j] = \dots$ and no things like $A[i + 1, j - 1] = \dots$.

The loop "for j in range..." is perhaps the most interesting. First, consider "projection", e.g., $p!3[I, j]$. We fix j (the current iteration) and iterate over I using the row-major formula; it returns the j -th column, a one-dimensional column vector. I think this can be generalized across any mix of vectorized vs. non-vectorized dimensions but I'm not 100% sure.

```
// project the (j-1)-st column vector:
A = sum!4[I, j-1]
// project j-th column from p!3:
B = p!3[I, j]
// we get a N!0-size vector of references to NEW shares:
C = ADD_SIMD(A, B)
// Writes j-th column of sum!4 with the new references:
sum!4[I, j] = C
// may eventually optimize since C is A
```

There are three kinds of arrays (for now all kept internally as one-dimensional arrays, but that's under discussion).

- **Scalars:** These are scalar variables we lift into arrays for the purposes of vectorization. For those, all writes are canonical writes and all reads are canonical reads. We may apply both raise dimension and drop dimension on these.
- **Read-only input arrays:** Read-only inputs. There are NO writes, while we may have non-canonical reads, $f(i, j, k)$. Phase 1 of Basic vectorization will add raise dimension operation at the beginning of the function and raise dimension may reshape arrays. If there are multiple "views" of the input array, there could be multiple raise dimension statements to create each one of these views. The invariant is that at reads in loops, the reads of "views" of the original input array will be canonical. Only raise dimension applies.
- **Read-write output arrays:** Writes are canonical (by restriction) but reads can be non-canonical. Conjecture: do nothing. Dependence analysis takes care of limiting vectorization so non-canonical access will work. Still work in progress. We may apply both raise and drop dimension.

ANA: End Work-in-progress notes.

4.3 Correctness Argument

4.4 Towards Extension of Basic Vectorization

4.4.1 Removal of Infeasible Edges

Array writes limit vectorization as they sometimes introduce infeasible loop-carried dependencies. Consider the following example: *ANA: Have to add citation to Aiken's paper*

```
for i in range(N):
  A[i] = B[i] + 10;
  B[i] = A[i] * D[i-1];
  C[i] = A[i] * D[i-1];
  D[i] = B[i] * C[i];
```

In Cytron's SSA this code (roughly) translates into for i in range(N):

1. $A_0 = \phi(A, A_1)$
2. $B_0 = \phi(B, B_1)$
3. $C_0 = \phi(C, C_1)$
4. $D_0 = \phi(D, D_1)$
5. $A_1 = A_0$; $A_1[i] = B_0[i] + 10$; {equiv. to Update}
6. $B_1 = B_0$; $B_1[i] = A_1[i] * D_0[i-1]$;
7. $C_1 = C_0$; $C_1[i] = A_1[i] * D_0[i-1]$;
8. $D_1 = D_0$; $D_1[i] = B_1[i] * C_1[i]$;

There is a cycle around $B_0 = \phi(B, B_1)$ that includes statement $A_1[i] = B_0[i] + 10$; and that statement won't be vectorized even though in fact there is no loop-carried dependency from the write of $B_1[i]$ at 8 to the read of $\dots = B_0[i]$ at 6.

The following algorithm removes certain infeasible loop-carried dependencies that are due to array writes. Consider a loop with index $0 \leq j < J$ nested at i, j, k . Here i represents the enclosing loops of j and k represents the enclosed loops in j .

```
for each array A written in loop j do
  { including enclosed loops in j }
  dep = False
  for each pair def:  $A_m[f(i, j, k)] = \dots$ , and use:  $\dots = A_n[f'(i, j, k)]$  in loop j do
    if  $\exists i, j, j', k, k'$ , s.t.  $0 \leq i < I$ ,  $0 \leq j, j' < J$ ,  $0 \leq k, k' < K$ ,  $j < j'$ , and  $f(i, j, k) = f'(i, j', k')$  then
      dep = True
    end if
  end for
  if dep == False then
    remove back edge into A's  $\phi$ -node in loop j.
  end if
end for
```

ISHAQ: Note to self: This algorithm is an instantiation for j loop, the one for k loop will be exactly the same, modulo variable name..

Consider a loop j enclosed in some fixed i . Only if an update (definition) $A_m[f(i, j, k)] = \dots$ at some iteration j references the *same* array element as a use $\dots = A_n[f'(i, j, k)]$ at some later iteration j' , we may have a loop-carried dependence for A due to this def-use pair. (In contrast, Cytron's algorithm inserts a loop-carried dependency every time there is an array update.) The algorithm above examines all def-use pairs in loop j , including defs and uses in nested loops, searching for values i, j, j', k, k' that satisfy $f(i, j, k) = f'(i, j', k')$. If such values exist for some def-use pair, then there is a potential loop-carried dependence on A ; otherwise there is not and we can remove the spurious backward edge thus "freeing up" statements for vectorization.

Consider the earlier example. There is a single loop, i . Clearly, there is no pair i and i' , where $i < i'$ that make $i = i'$ (due to the def-use pairs of A 6-8 and 6-10). Therefore, we remove the back edge from 6 to 1. Analogously, we remove the back edges from 8 to 2 and 10 to 3. However, there are many values $i < i'$ that make $i = i' - 1$ and the back edge from 12 to 4 remains (def-use pairs for D). As a result of

removing these spurious edges, Basic Vectorization will find that statement 6 is vectorizable. Statements 8, 10 and 12 will correctly appear in the FOR loop.

4.4.2 Array MUX refinement

ANA: TODO: I think we should implement this.

Next, the algorithm refines array MUX statements. MPC-source after Cytron's SSA may result in statements $A_j = \text{MUX}(\dots, A_k, A_l)$, which imply that any index of A can be written at this point and therefore there is a loop-carried dependency. In some cases the MUX can be refined to just a single index or a pair of indices, e.g., $A_j[i] = \text{MUX}(c, A_k[i], A_l[i])$.

This is to reduce the dimensionality of SIMD-ified computation. Technically, $A_j = \text{MUX}(\dots, A_k, A_l)$ is a SIMDified operation that can be carried out in parallel "in one round". However, particularly when A is a multi-dimensional array, there is substantial increase in the size of the arrays (vectors) we send to SIMD operations. Refining to an update to a specific index would reduce the size of those vectors. Note that this is a heuristic that handles a common case, but not all cases of array updates. *ANA: TODO: Simplify this algorithm, taking into account the restriction to canonical updates. It should be handling all cases.*

```

for each stmt:  $A_j = \text{MUX}(c, A_k, A_l)$  in the MPC-source seq.
do
   $i_1 = \text{find\_update}(A_k)$  { Is null when  $A_k = \phi(\dots)$  }
   $i_2 = \text{find\_update}(A_l)$  { Is null when  $A_l = \phi(\dots)$  }
  if  $i_1 == i_2$  or  $i_1$  is null or  $i_2$  is null then
    { With our restrictions on writes we must have  $i_1 = i_2$ . }
    replace stmt with
       $A_j = A_{j-1}; A_j[i_1] = \text{MUX}(c, A_k[i_1], A_l[i_1])$ 
  else
    stmt stays as is
  end if
end for

```

4.5 Extension of Basic Vectorization with Array Writes

ANA: TODO: This section is work in progress!

4.5.1 Restricting Array Writes

For now, we restrict array updates to *canonical updates*. Assume (for simplicity) a two-dimensional array $A[I, J]$. A canonical update is the following:

```

for i in range(I):
  for j in range(J):
    ...
     $A[i, j] = \dots$ 
    ...

```

The update $A[i, j]$ can be nested into an inner loop and there may be multiple updates, i.e., writes to $A[i, j]$. However, update such as $A[i - 1, j] = \dots$ or $A[i - 1, j - 1] = \dots$, etc., is not allowed. Additionally, while there could be several different loops that perform canonical updates, they must be of the same dimensionality, i.e., an update of higher or lower dimension, e.g., $A[i, j, k] = \dots$ is not allowed. We assign the

canonical dimensionality to each array in the obvious way. This restriction simplifies reasoning in this early stage of the compiler; we will look to relax the restriction in future work.

Reads through an arbitrary formula, such as $A[i - 1]$ for example, are allowed, however, we assume the programmer ensures that the formula is within the bounds of the array.

4.5.2 Changes to Basic Vectorization

One change to Basic vectorization is the expansion of dimension if the array write or read occurs in a nested loop. That is, if there is an update $A[i, j] = \dots$ that occurs in loop nest i, j, k , $A[i, j]$ will be rewritten into $A[i, j, k]$. Similarly, a read $A[f(i, j)]$ will be rewritten into $A[f(i, j), k]$. *ANA: TODO: This notion of $f(i, j)$ needs a precise definition. For now I'm hoping the hand waving works.* This can be a preprocessing step or it can be incorporated into the Basic Vectorization algorithm.

The other change concerns def-use edges $X \rightarrow Y$ where X defines and Y uses an array variable (e.g., the definition can be an update $A_2 = \text{update}(A_1, \dots)$ or a pseudo ϕ -node $A_2 = \phi(A_0, A_1)$). These edges are not handled in the same way as in Basic Vectorization, specifically, we do not expand dimension as we do for scalars in Basic Vectorization according to the loop nest. A key invariant is that the dimension of an array A cannot drop below A 's canonical dimensionality. Below we enumerate the cases of def-use edges.

- (1) same-level $X \rightarrow Y$. We do nothing, just propagate the array, which happens to be of the right dimension. *ANA: There might be some opportunities to do copy propagation optimization and save some cycles, but let's leave this for later.*
- (2) inner-to-outer $X \rightarrow Y$. Let A be the array defined at X . If the dimensionality of A is greater than its canonical dimensionality, then add $\text{drop_dim}(\dots)$ at Y , as in Basic Vectorization. Otherwise, do nothing.
- (3) outer-to-inner $X \rightarrow Y$. Add $\text{raise_dim}(\dots)$ (at X) as in Basic Vectorization.
- (4) "mixed" $X \rightarrow Y$. We assume that the mixed edge is transformed into an inner-to-outer followed by outer-to-inner edge before we perform vectorization, just as with Basic vectorization.

4.5.3 Examples with Array Writes

Example 1. After Phase 1 of Basic Vectorization the Aiken's array write example will be (roughly) as follows:

```

for i in range(N):
  1.  $A_1 = \phi(A_0, A_2)$ 
  2.  $B_1 = \phi(B_0, B_2)$ 
  3.  $C_1 = \phi(C_0, C_2)$ 
  4.  $D_1 = \phi(D_0, D_2)$ 
  5.  $A_2 = \text{update}(A_1, \vec{i}, B_1[\vec{i}] + 10);$ 
  6.  $B_2 = \text{update}(B_1, \vec{i}, A_2[\vec{i}] * D_1[\vec{i} - 1]);$ 
  7.  $C_2 = \text{update}(C_1, \vec{i}, A_2[\vec{i}] * D_1[\vec{i} - 1]);$ 
  8.  $D_2 = \text{update}(D_1, \vec{i}, B_2[\vec{i}] * C_2[\vec{i}]);$ 

```

There are no nested loops, thus, array accesses remain as is. Phase 2 computes the closure of 4; $cl = \{4, 6, 7, 8\}$

while 5 is vectorizable. Recall that 1,2, and 3 are targetless phi-nodes. Since the closure cl includes updates to B and C , the corresponding phi-nodes are added to the closure, i.e., the FOR loop. The uses of A_1 and B_1 in the vectorized statement turn into uses of A_0 and B_0 respectively. (But note that A_0 is irrelevant; the updates writes into array A_2 in parallel.)

1. $A_2 = \text{update}(A_0; \vec{i}, \text{ADD_SIMD}(B_0[\vec{i}], [10, \dots]))$ {Fully vectorized, size N.}
- FOR $i=0; i \leq N; i++$; { MOTION loop }
2. $B_1 = \phi(B_0, B_2)$
3. $C_1 = \phi(C_0, C_2)$
4. $D_1 = \phi(D_0, D_2)$
5. $B_2 = \text{update}(B_1, i, \text{MUL}(A_2[i], D_1[i - 1]))$
6. $C_2 = \text{update}(C_1, i, \text{MUL}(A_2[i], D_1[i - 1]))$
7. $D_2 = \text{update}(D_1, i, \text{MUL}(B_2[i], C_1[i]))$

Example 2. Consider MPC-source of Histogram after removal of infeasible edges and redundant phi-nodes, and MUX refinement:

```
res!1 = []
for i: plaintext in range(0, num_bins):
    res!2 =  $\phi$ (res!1, res!3)
    ...
res!4 = res!2 {Added due to the "mixed" def-use edge.}
for i: plaintext in range(0, num_bins):
    for j: plaintext in range(0, N):
        res!5 =  $\phi$ (res!4, res!7)
        !2!3 = (A[j] == i)
        !3!3 = (res!5[i] + B[j])
        res!6 = res!5; res!6[i] = !3!3
        res!7 = res!6; res!7[i] = MUX(!2!3, res!6[i], res!5[i])
```

After Phase 1 of Basic Vectorization:

```
res!1 = []
for i: plaintext in range(0, num_bins):
    res!2 =  $\phi$ (res!1, res!3)
    ...
res!4 = res!2 {Added due to the "mixed" def-use edge.}
Because the canonical dimensionality of res is 1, we DO NOT drop dimension of res!2.}
A[i, j] = raise_dim(A[j], i) {Copying row A num_bin times. Reusing name.}
B[i, j] = raise_dim(B[i], j) {Copying column B N times. Reusing name.}
res!4[i, j] = raise_dim(res!4[i], j) {Reusing name.}
for i: plaintext in range(0, num_bins):
    for j: plaintext in range(0, N):
        res!5 =  $\phi$ (res!4, res!7) {Do nothing. res!4 has been raised to proper dimension.}
        !2!3[ $\vec{i}, \vec{j}$ ] = EQ(A[ $\vec{i}, \vec{j}$ ], [[0,0...], [1,1...], ..., [num_bins-1, num_bins-1...]])
        !3!3[ $\vec{i}, \vec{j}$ ] = ADD(res!5[ $\vec{i}, \vec{j}$ ], B[ $\vec{i}, \vec{j}$ ])
        res!6 = res!5 {Do nothing, just copy.}
        res!6[ $\vec{i}, \vec{j}$ ] = !3!3[ $\vec{i}, \vec{j}$ ]
        res!7 = res!6
        res!7[ $\vec{i}, \vec{j}$ ] = MUX(!2!3[ $\vec{i}, \vec{j}$ ], res!6[ $\vec{i}, \vec{j}$ ], res!5[ $\vec{i}, \vec{j}$ ])
```

After Phase 2 and Phase 3. The EQ operation is vectorizable across both dimensions, and the rest of the computation is vectorizable across the i -dimension.

```
...
!2!3[ $\vec{i}, \vec{j}$ ] = EQ_SIMD(A[ $\vec{i}, \vec{j}$ ], [[0,0...], [1,1...], ..., [num_bins-1, num_bins-1...]])
FOR j=0; j ≤ N; j++; {MOTION loop}
    res!5 =  $\phi$ (res!4, res!7) {Do nothing, just copy.}
    !3!3[ $\vec{i}, j$ ] = ADD_SIMD(res!5[ $\vec{i}, j$ ], B[ $\vec{i}, j$ ])
    res!6 = res!5 {Do nothing, just copy.}
    res!6[ $\vec{i}, j$ ] = !3!3[ $\vec{i}, j$ ]
    res!7 = res!6
    res!7[ $\vec{i}, j$ ] = MUX_SIMD(!2!3[ $\vec{i}, j$ ], res!6[ $\vec{i}, j$ ], res!5[ $\vec{i}, j$ ])
```

5 DIVIDE-AND-CONQUER

ANA: TODO: Now that we have broken FOR loops into smaller chunks, we can add Divide-and-conquer reasoning with Z3 and implement this additional transform.

6 IMPLEMENTATION AND EVALUATION

7 FUTURE WORK

8 CONCLUSIONS