Lecture 2:
2011-10-13

Definition 2.1 (connected)**:**
A graph is called *connected* (zusammenhängend) if there exists a [s,t]-Path between all pairs of vertices $s, t \in V$.

Definition 2.2 (forrest, tree, spanning, forest problem, minimum spanning tree)**:**
A *forest* (Wald) is a graph that does not contain a cycle (Kreis). A connected forest is called a *tree* (Baum). A tree in a graph (as subgraph) is called *spanning* (aufspannend), if it contains all vertices.

Given a graph $G = (V, E)$ with edge weights $c_e \in \mathbb{R}$ for all $e \in E$, the task to find a forest $W \subset E$ such that $c(W) := \sum\limits_{e \in W}$ is maximal, is called the *Maximum Forest Problem* (Problem des maximalen Waldes). The task to find a tree $T \subset E$ which spans G and which weight $c(T)$ is minimal, is called the *Minimum Spanning Tree* (MST) problem (minimaler Spannbaum).

Lemma 2.3:
A tree $G = (V, E)$ with at leat 2 vertices has at least 2 vertices of degree 1.

Proof. Let v be arbitrary. Since G is connected, $deg(v) \geq 1$. Assume $deg(v) = 1$. So $\delta(v) = \{vw\}$. If $deg(w) = 1$, we found two vertices with $degree$ 1. If $deg(w) > 1$, there exist a neighbour of w, different from $v : u$. Now, again u has $degree$ 1 or higher. If we repeat this procedure we either find a vertix of degree 1 or find again *new* vertices. Hence, after at most $n - 1$ vertices we end up at a vertex of degree 1. Now, if $deg(v) \geq 2$, we do the same and find a vertex of degree 1, say w. Then repeat the above, staring from w to find a second vertex of degree 1. $\square$

Corollary 2.4:
A tree $G = (V, E)$ with maximum degree Δ has at least Δ vertices of degree 1.

Lemma 2.5: (a) For every graph $G = (V, E)$ it holds that $2|E| = \sum\limits_{u \in V} deg(u)$

(b) for every tree $G = (v, E)$ it holds that $|E| = |V| - 1$.

Proof. (a) trivial

(b) Proof by induction. Clearly, if $|V| = 1$ or $|V| = 2$ it holds. Assumption: true for $n \geq 2$. Let G be a tree with $n + 1$ vertices. By Lemma 2.3, there exists a vertex $v \in G$ with $deg(v) = 1. G - v = G[V \setminus \{v\}$ is a tree again with n vertices and thus $|E(G - v)| = V(G - v)| - 1$. Since G differs by one vertex and one edge from $G - v$, the claim holds got G as well.

$\square$

Lemma 2.6:
If $G = (V, E)$ whith $|V| \geq 2$ has $|E| < |V| - 1, G$ is not connected.

Algorithm MST

$min_{x \in X} = -max_{x \in X} - c(x)$ ~~maximal forest~~
X spanning trees
$min_{x \in X} + (n-1)D = -max_{x \in X} - c(x)(n-1)D = max_{x \in X} \sum \underbrace{D - C_{ij}x_{ij}}_{\geq 0 if D \geq max_{ij \in E} c_{ij}}$

Theorem 2.7:
Kruskal's Algorithm returns the optimal solution.

Proof. Let T be Kruskal's tree and assume there exists a tree T' with $c(T') < c(T)$. Then there exist an edge $e' \in T' \setminus T$. Then $T \cup \{e'\}$ contains a cycle $\{e_1, e_2, \dots, e_k, e'\}$. Let $c_f = max_{i=1,\dots k} c_{l_i}$. At the moment Kruskal chooses edge f, edge e' cannot be added yet and therefore $c(e') \geq c(f)$. Now exchange e' by f in T'. Hence the number of differences beetween T' and T is reduced by one, $C(T'_{new}) \leq c(T') < c(T)$. Repeating the procedure results in $c(T) \leq \dots < c(T)$, a contradiction. $\square$