Lecture 2 (2011-10-13)**:**

Definition 2.1 (connected)**:**
A graph is called *connected* (zusammenhängend) if there exists a [s,t]-Path between all pairs of vertices $s, t \in V$.

Definition 2.2 (forrest, tree, spanning, forest problem, minimum spanning tree)**:**
A *forest* (Wald) is a graph that does not contain a cycle (Kreis). A connected forest is called a *tree* (Baum). A tree in a graph (as subgraph) is called *spanning* (aufspannend), if it contains all vertices.

Given a graph $G = (V, E)$ with edge weights $c_e \in \mathbb{R}$ for all $e \in E$, the task to find a forest $W \subset E$ such that $c(W) := \sum\limits_{e \in W}$ is maximal, is called the *Maximum Forest Problem* (Problem des maximalen Waldes). The task to find a tree $T \subset E$ which spans G and which weight $c(T)$ is minimal, is called the *Minimum Spanning Tree* (MST) problem (minimaler Spannbaum).

Lemma 2.3:
A tree $G = (V, E)$ with at leat 2 vertices has at least 2 vertices of degree 1.

Proof. Let v be arbitrary. Since G is connected, $deg(v) \geq 1$. Assume $deg(v) = 1$. So $\delta(v) = \{vw\}$. If $deg(w) = 1$, we found two vertices with $degree$ 1. If $deg(w) > 1$, there exist a neighbour of w, different from $v : u$. Now, again u has $degree$ 1 or higher. If we repeat this procedure we either find a vertix of degree 1 or find again *new* vertices. Hence, after at most $n - 1$ vertices we end up at a vertex of degree 1. Now, if $deg(v) \geq 2$, we do the same and find a vertex of degree 1, say w. Then repeat the above, staring from w to find a second vertex of degree 1. □

Corollary 2.4:
A tree $G = (V, E)$ with maximum degree Δ has at least Δ vertices of degree 1.

Lemma 2.5: (a) For every graph $G = (V, E)$ it holds that $2|E| = \sum\limits_{u \in V} deg(u)$

(b) for every tree $G = (v, E)$ it holds that $|E| = |V| - 1$.

Proof. (a) trivial

(b) Proof by induction. Clearly, if $|V| = 1$ or $|V| = 2$ it holds. Assumption: true for $n \geq 2$. Let G be a tree with $n + 1$ vertices. By Lemma 2.3, there exists a vertex $v \in G$ with $deg(v) = 1 . G - v = G[V \setminus \{v\}$ is a tree again with n vertices and thus $|E(G - v)| = V(G - v)| - 1$. Since G differs by one vertex and one edge from $G - v$, the claim holds got G as well.

□

Lemma 2.6:
If $G = (V, E)$ whith $|V| \geq 2$ has $|E| < |V| - 1, G$ is not connected.

Algorithm MST

$min_{x\in X} = -max_{x\in X} - c(x)$ ~~maximal forest~~
X spanning trees
$min_{x\in X} + (n-1)D = -max_{x\in X} - c(x)(n-1)D = max_{x\in X} \sum \underbrace{D - C_{ij}x_{ij}}_{\geq 0 if D \geq max_{ij\in E} c_{ij}}$

Theorem 2.7:
Kruskal's Algorithm returns the optimal solution.

Proof. Let T be Kruskal's tree and assume there exists a tree T' with $c(T') < c(T)$. Then there exist an edge $e' \in T' \setminus T$. Then $T \cup \{e'\}$ contains a cycle $\{e_1, e_2, \dots, e_k, e'\}$. Let $c_f = max_{i=1,\dots k} c_{l_i}$. At the moment Kruskal chooses edge f, edge e' cannot be added yet and therefore $c(e') \geq c(f)$. Now exchange e' by f in T'. Hence the number of differences beetween T' and T is reduced by one, $C(T'_{new}) \leq c(T') < c(T)$. Repeating the procedure results in $c(T) \leq \dots < c(T)$, a contradiction. $\square$

Lecture 3 (2011-10-17)**:**

Definition 2.7(+1):
The *running time of algorithms* (Laufzeit) of an algorithm is measured by the number of operations needed in worst case of a function of the input size. We use the $O(\cdot)$ notation (Big-O-notation) ot focus on the most important factor of the running time, ignoring constants and smaller factors.

Example 2.7(+2):
If the running time is $3n \cdot \log n + 26n$, the algorithm runs in $O(n \cdot \log n)$. If the running time is $3n \cdot \log n + 25n^2$, the algorirthm runs in $O(n^2)$.

For graph Problems, the running is expressed in the number of vertices $n = |V|$ and the number of edges $m = |E|$. Sometimes m is approximated by n^2.

Example 2.7(+3) (Kruskal's Algorithm)**:**
First, the edged are sorted according to nondecreasing weights. This can be done in $O(m \cdot \log m)$. Next, we repeatedly select an edge or reject its selection until $n-1$ edges are selected. Since the last selected edge might be after m steps, this routine is performed at most $O(m)$ times.

Checking whether the end nodes of $\{u, v\}$ are already in the same tree can be done in constant time, if we label the vertices of the trees selected so far: $r(u) = \#trees\, containing\, u$. If $r(u) \neq r(v)$, the trees are connected by $\{u, v\}$ to a new tree.

Without going into details, the resetting of labels in one of the old trees, can be done $O(\log n)$ on average. Since this update has to be done at most $n-1$ times, it takes $O(n \cdot \log n)$.

Overall, Kruskal runs in

$$O(n \log m + m + n \cdot \log n) = O(m \cdot \log m) = O(m \cdot \log n^2) = O(m \cdot \log n)$$

Definition 2.7(+4) (Shortest paths in acyclic digraphs)**:**
A directed graph (digraph) $D = (V, A)$ is called *acyclic* (azyklisch) if it does not contain any *directed cycles*, i.e. a *chain* (Kette) $(v_0, a_1, v_1, a_2, v_2, \ldots a_k, v_k)$, $k \geq 0$, with $a_i(v_{i-1}, v_i) \in A$ and $v_k = v_0$. In particular, D does not contain *antiparallel* arcs: if $(u, v) \in A$, $(v, u) \notin A$. With $\delta_D^+(v)$ we denote the arcs leaving vertex v:

$$\delta_D^+(v) = \{(u, w) \in A : u = v\}$$

similarly:

$$\delta_D^-(v) = \{(u, w) \in A : w = v\}$$

are the arcs entering v.

The *outdegree* of v is $\deg_D^+(v) = |\delta^+(v)|$ (assuming simple digraph)

The *indegree* of v is $\deg_D^-(v) = |\delta^-(v)|$

Definition 3.1:
The *shortest path* problem in a acyclic digraph is, given an acyclic digraph $D = (V, A)$, a length function $C : A \to \mathbb{R}$ and two vertices $s, t \in V$, find a $[s, t]$-path of minimal length.

Question 1:
Does there exist a $[s, t]$-path at all?

Theorem 3.2:
A digraph $D = (V, A)$ is acyclic, if and only if there exists a permutaion $\sigma : V \to \{1, .., n\}$ of the vertices such that $\deg^-_{D[v_1,...,v_n]}(v_i) = 0$ for all $i = 1, \dots, n$ with $v_i = \sigma^{-1}(i)$.

Proof. By induction:

For digraph with $|V| = 1$, the statement is true. Assume the statement is true for all digraphs with $|V| \leq n$ and consider $D = (V, A)$ acyclic with $n + 1$ vertices. If there does not exist a vertex with $\deg^-_D(v) = 0$, a directed cycle can be detected by following incoming arcs backwards until a vertex is repeated, a contradiction regarding the acyclic property of D.

Hence, let v be a vertex with $\deg^-_D(v) = 0$. Set $v_1 = v$. The digraph $D - v_1$ has n vertices and is acyclic, and thus has a permutation $(v_2, \dots, v_{n+1})$ with

$$\deg^-_{D[v_i,...,v_{n+1}]}(v_i) = 0 \qquad \forall i = 2, \dots, n + 1$$

Now, $(v_1, \dots, v_{n+1})$ is a permutation fulfilling the condition.

In reverse, if there exists a permutation $(v1, \dots, v_{n+1})$, $\deg^-_D(v_1) = 0$ and there cannot exist a directed cycle containing v_1. By induction, neither cycles containing $v_i, i = 2, \dots, n + 1$ exist. □

Theorem 3.3:
A $[s, t]$-path exists in a acyclic Digraph $D = (V, A)$ if and only if in all permutations $\sigma : V \to \{1, \dots, n\}$ with $\deg^-_{D[v_i,...,v_n]}(v_i) = 0$ for all $i = 1, \dots, n$, it holds that $\sigma(s) < \sigma(t)$.

Proof. Assume there exists a permutation σ with $\sigma(s) > \sigma(t)$. Since outgoing arcs only go to higher ordered vertices, there does not exist a path from s to t in D.

In reverse, if there does not exist a path from s to t, we order all vertices with paths to t first, followed by t and s afterwards. □

Question 2:
How do we find the shortest $[s, t]$-path if it exists?

To simplify notation, let $V = \{1, \dots, n\}, s = 1, t = n$ and $(i, j) \in A \Rightarrow i < j$. Let $D(i)$ be the distance from i to n and $NEXT(i)$ be the next vertex on the shortest path from i to n.

Bellmann's Algorithm

(a) $D(i) = \{\infty : i < n and NEXT(i) = NIL, 0 : i = n\}$

(b) FOR $i = n - 1$ DOWNTO 1 DO

(c) $\quad D(i) = \min\limits_{j=i+1,...,n} \{D(j) + c(i, j)\}$ with $c(i, j) = \infty$ if $(i, j) \notin A$

(d) $NEXT(i) = argmin_{j=i+1,...,n}\{D(j) + c(i,j)\}$

Theorem 3.4:
Bellmann's Algorithm is correct and runs in $O(m+n)$ time.

Proof. Every path from 1 to n passes through vertices of increasing ID. Assume there exists a path $(a_1, \ldots, a_k)$ with $\sum_{i=1}^{k} c(a_i) < D(1)$. Let $a_1 = (1, j_1)$. Since $D(1) \leq c(a_1) + D(j_1)$, it should hold that

$$\sum_{i=2}^{2} c(a_i) < D(j_1)$$

But $D(j_1) \leq c(a_2) + D(j_2)$ with $a_2 = (j_1, j_2)$, etc.
In the end, $c(a_k) < D(j_{k-1})$ but $D(j_{k-1}) \leq c(a_k) + D(n) = c(a_k)$, contradiction. □

Lecture 4 (2011-10-20)**:**

Theorem 3.5:
Bellman's Algorithm is correct and runs in $O(m+n) = O(n)$.

Proof. of runtime:

$$D(i) = \min_{(i,j)\in A} D(j) + D(i,j)$$

$\Rightarrow$ Every arc is considered once, and thus overall $O(m)$ computations are needed. Initialization costs $O(n)$. $\square$

Bemerkung 3.5(+1):
The running time does not contain the time to find the permutation.

Observation 1: We not only found the shortest path from 1 to n, but also from i to n, $i = 2, \dots, n$.

Observation 2: We can use a similar procedure for the shortest path from 1 to i, $i = 2, \dots, n$. (with $PREV(i)$ for previous instead of $NEXT(i)$).

Question 3:
Can we find a shortest path from 1 to i in a digraph that is not acyclic, i.e. it contains cycles?

Theorem 4.1:
The Moore-Bellman-Algorithm returns the shortest paths from 1 to $i = 1, \dots, n$ provided D does not contain negative-weighted directed cycles.

Proof. We call an arc $(i,j) \in A$ an *upgoing* arc (Aufwärtsbogen) if $i < j$ and a *downgoing* arc (Abwärtsbogen) if $i > j$.

A shortest path from 1 to i contains at most $n-1$ arcs. If an upgoing arc is followed by a downgoing arc (or vice versa), we have a *change of direction* (Richtungswechsel). With at most $n-1$ arcs, at most $n-2$ changes of direction are possible.

Let $D(i,m)$ be the value of $D(i)$ at the end of the m-th iteration. We will show (and this is enough):

$$D(i,m) = min\{c(W) : W \text{ is the directed } [1,i]\text{-path with at most } m \text{ changes of directions}\}$$

We prove it by induction on m.

- For $m = 0$, the algorithm is equivalent to Bellman's algorithm for acyclic grpahs. Thus, $D(i,0)$ is the length of the shortest path without any changed of direction.
- Now, let us assume, that the statement is true for $m \geq 0$ and the subroutine is executed for the $m+1$-st time. The set of $[1,i]$-paths with at most $m+1$ changes of direction consists of

(a) $[1, i]$-paths with $\leq m$ changes of direction

(b) $[1, i]$-paths with exactly $m + 1$ changes of direction

$\Rightarrow D(i, m)$

- Since every path starts with an upgoing arc $(1, k)$, the last arc after $m + 1$ changes is either a downgoing arc if $m + 1$ is odd or an upgoing arc if $m + 1$ is even. We restrict ourselves to $m + 1$ odd ($m + 1$ even is similar).

  To compute the minimum length path in (b) we use an additional induction on $i = n, n - 1, ..., j + 1$. Since every path ending at n ends with an upgoing arc, there do not exist such $[1, n]$-paths. Hence, $D(n, m + 1) = D(n, m)$.

  Now assume that $D(k, m + 1)$ is correctly computed for $i \leq k \leq n$. The shortest path from 1 to $i - 1$ with exactly $m + 1$ changes ends with a downgoing arc $(j, i - 1)$, $j > i - 1$.

  $D(j, m + 1)$ is already computed correctly. If $PREV(j) > j$, no change of direction is required in j and $D(i - 1, m + 1) = D(j, m + 1) + c(j, i - 1)$ If $PREV(j) < j$, the last avec of the $[1, j]$-Path is upgoing, and thus $D(i - 1, m + 1) = D(j, m) + c(j, i)$. The last change of direction at j is thus, in worst case, the $(m + 1)$-st change. Hence, $D(i - 1, m + 1)$ fulfills the statement.

□

Remark 1:
In fact, the algorithm finds the minimum length of a chain (kette) with at most $n - 2$ changes of direction. In case of negative weighted cycles these might be in a chain several times.

In case no negative weighted cycles exist, the min. length chains are indeed paths. Hence, the algoithm only works correctly if *all* cycles are non-negative weighted.

Remark 2:
If a further executing of the subroutine ($m = n - 1$) results in at least one change of a value $D(i)$, then the digraph contains negative weighted cycles.

Remark 3:
A more efficient implementation is given by E'sopo-Pape-Variant.

Dijkstra's Algorithm for non-negative weights

Theorem 4.2:
Dijkstra returns the shortest paths from 1 to $i, i = 1 \ldots n$, provided all weights ≥ 0.

Proof. Each step, one vetex is moved from T to S. At the end of a step, $D(j)$ is the shortest path from 1 to j via vertices in S.

If $S = V (T = \emptyset)$, $D(i)$ is thus the shortiest $[1, i]$-path □

Lecture 5 (2011-10-24)**:**

Shortest pahts between all pairs of vertices

Solution 1: Apply Moore-Bellman or Dijkstra to all vertices i as starting vertex

Solution 2: Apply Floyd's Algorithm

Notation:

$$\begin{aligned} w_{ij} &= \text{length of the shortest } [i,j]\text{-path, } i \neq j \\ w_{ii} &= \text{length of the shortest directed cycle containing } i \\ p_{ij} &= \text{predecessor of } j \text{ on the shortest } [i,j]\text{-path (cycle)} \\ W &= (w_{ij}) \text{ is the } \textit{shortest path length matrix} \end{aligned}$$

Theorem 5.1:

The Floyd Algorithm works correctly if and only if $D = (V, A)$ does not contain any negative weighted cycles.

D contains a negative weighted cycle if and only if one of the diagonal elements $w_{ii} < 0$.

Proof. Let W^k be the matrix W after iteration k, with W^0 being the initial matrix. By induction on $k = 0, \dots, n$ we show that W^k is the matrix of shortest path lengths with vertices $1, \dots, k$ as *possible* internal vertices, provided D does not contain a negative cycle on these vertices.

If D has a negative cycle, then $w_{ii}^k < 0$ for an $i \in \{1, \dots, n\}$

For $k = 0$, the statement clearly true.

Assume, it is correct for $k \geq 0$, and we have executed the $(k+1)$st iteration.

It holds that $w_{ij}^{k+1} = \min\{w_{ij}^k, w_{i,k+1}^k + w_{k+1,j}^k\}$. Note that, provided no negative cycle exists, $w_{i,k+1}^{k+1}$ does not have any vertex $k+1$ as internal vertex, and thus $w_{i,k+1}^{k+1} = w_{i,k+1}^k$ (similarly, $w_{k+1,j}^{k+1} = w_{k+1,j}^k$).

$w_{i,k+1}^k$ is the minimal length of a $[i, k+1]$-path with $\{1, \dots, k\}$ as allowed internal vertices. Similarly, $w_{k+1,j}^k$.

Thus, $w_{i,k+1}^k + w_{k+1,j}^k$ is the minimal length of an $[i,j]$-path (not necessarily simple) containing $k+1$ (mandatory) and $\{1, \dots, k\}$ (voluntary). If the shortest path from i to j using $\{1, \dots, k+1\}$ does not contain $k+1$, it only contains $\{1, \dots, k\}$ (voluntary) and, hence, w_{ij}^k is the right value.

What remains to show is that the connection of the $[i, k+1]$-path with the $[l+1, j]$-path is indeed a simple path.

Let K be this chain. After removal of cycles, the chain K contains (of course) a simple $[i,j]$-path $\bar{K}$. Since such cycles may only contain vertices from $\{1, \dots, k+1\}$, one cycle must contain $k+1$. If this cycle is not negatively weighted, then path $\bar{K}$ is shorter and $w_{ij} < w_{i,k+1}^k + w_{k+1,j}^k$.

If this cycle is negatively weighted, $w_{k+1,k+1}^k < 0$ (the cycle only contains internal vertices from $\{1, \dots, k\}$) and algorithm would have stopped earlier. $\square$

Min-Max-Theorems for combinatorial Optimization Problems

From "Optimierung A": Duality of linear programs

$$\max_{\text{s. t.}, Ax \leq b, x \geq 0} c^T x = \min_{\text{s.t.}, A^T y \geq c, y \geq 0} b^T y$$

For several combinatorial problems $\min\{c(x) : x \in X\}$

We can define a second set Y and a function $b(y)$ with $\max\{b(y) : y \in Y\} = \min\{c(x) : x \in X\}$ where Y and b(y) have a graph theoretical interpretation.

Existence of such a "Dual" Problem indicates often that the problem can be solved "efficiently". For the shortest path problem several max-min-theorems exist.

Definition 5.2:
An *(s,t)-cut* (Schnitt) in a digraph $D = (V, A)$ with $s, t \in V$ is a subset $B \subset A$ of the arcs with the property that every (s,t)-path contains at least one arc of B.

Stated otherwise, for every cut B, there exists a vertex set $W \subset V$ such that

- $s \in W,\ t \in V \setminus W$
- $\delta^+(w) = \{(i,j) \in A : i \in W, j \in V \setminus W\} \subseteq B$

Theorem 5.3:
Let $D = (V, A)$ be a digraph, $c(a) = 1\, \forall a \in A, s, t \in V, s \neq t$. Then the minimum length of a $[s,t]$-path equals the maximum number of arc-disjoint (s,t)-cuts.

Proof. Folows from **??** ☐

Theorem 5.4:
Let $D = (V, A)$ be a digraph, $c(a) \in \mathbb{Z}_+ \forall a \in A \wedge s, t \in V \wedge s \neq t$. Then the min length of an $[s,t]$-path equals the maximum number d of (not necessarily different) (s,t)-cuts $C_1, \ldots, C_d$ such that every arc $a \in A$ is contained in at most $c(a)$ cuts.

Proof. We define (s,t)-cuts $C_i = \delta^+(v_i)$ with $V_i = \{v \in V : \exists (s,v)\text{-path with } c(P) \leq i-1\}$

$$\begin{aligned} v_1 &= \{s\} \\ v_2 &= \{5, 3, 4\} \\ v_3 &= \{5, 2, 3, 4\} \\ v_4 &= v_3 \cup \{6\} \end{aligned}$$

(for the example graph on the board)

The shortest $[s,t]$-path P consists of arcs $a_1, \ldots a_k$ with arc a_j contained in (s,t)-cuts C_i, $i \in \{\sum_{l=1}^{j-1} c(a_l) + 1, \ldots, \sum_{l=1}^{j} a(a_l)\}$: exactly $c(a)$ cuts. ☐

Lecture 6 (2011-10-26)**:**

Knapsack problem

Definition 6.1:
The *Knapsack problem* is defined by a set of items $N = \{1, ..., n\}$ weights $a_i \in \mathbb{N}$, value $c_i \in \mathbb{N}$, and a bound $b \in \mathbb{N}$. We search for a subset $S \subset \mathbb{N}$ such that

$$a(S) = \sum_{i \in S} a_i \leq b \text{ and } c(S) = \sum_{i \in S} c_i \text{ maximum}$$

Appreach 1: Greedy algorithm

Idea: Items with small weight but high value are the most atrractive ones.

Procedure:

(a) Sort the items such that $\frac{c_1}{a_1} \leq \frac{c_2}{a_2} \leq ... \leq \frac{c_n}{a_n}$.
Set $S = \emptyset$.

(b) For $i = 1$ to n do if $(a(s) + a_i \leq b)$ then
$S = S \cup \{i\}$
endif
endfor

(c) return S and $c(S)$

Theorem 6.2:
The greedy algorithm does *not* guarantee an optimal solution.

Proof. Let $b = 10$, $n = 6$

i	2	3	4	5	6
a_i	9	2	2	2	2
c_i	19	4	4	4	4

Greedy: $S = \{1\}$, $c(s) = 20$
Optimal: $S = \{2, 3, 4, 5, 6, \}$, $c(S) = 20$ □

Approach 2: Integer Linear Programming

The set of solutions X of a combinatorial optimization problem can (almost always) be written as the intersection of integer points in $\mathbb{N}_0^n$ and a polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$

Let $x \in \{0, 1\}^n$ be a vector representing all solutions of the knapsack problem:

$$x_i = \begin{cases} 1 & \text{if} i \in S \\ 0 & \text{otherwise} \end{cases}$$

$X = \{0,1\} \cap \{x \in \mathbb{R}^n : \sum\limits_{i=0}^{n} a_i x_i \leq b\}$

Knapsack: $\max \sum i = 0^n c_i x_i$

The *linear relaxation* (Lineare Relaxierung) of an ILP is the linear program optained by relaxing the integrality of the variables:

$\max \sum_{i=1}^{n} c_i x_i$

s. t. $\sum_{i=1}^{n} a_i x_i \leq b, 0 \leq x_i \leq 1 \qquad \forall i \in \{1, ..., n\}$