

Lecture 2 (2011-10-13):

Definition 2.1 (connected):

A graph is called *connected* (zusammenhängend) if there exists a $[s,t]$ -Path between all pairs of vertices $s, t \in V$.

Definition 2.2 (forest, tree, spanning, forest problem, minimum spanning tree):

A *forest* (Wald) is a graph that does not contain a cycle (Kreis). A connected forest is called a *tree* (Baum). A tree in a graph (as subgraph) is called *spanning* (aufspannend), if it contains all vertices.

Given a graph $G = (V, E)$ with edge weights $c_e \in \mathbb{R}$ for all $e \in E$, the task to find a forest $W \subset E$ such that $c(W) := \sum_{e \in W} c_e$ is maximal, is called the *Maximum Forest Problem* (Problem des maximalen Waldes).

The task to find a tree $T \subset E$ which spans G and which weight $c(T)$ is minimal, is called the *Minimum Spanning Tree* (MST) problem (minimaler Spannbaum).

Lemma 2.3:

A tree $G = (V, E)$ with at least 2 vertices has at least 2 vertices of degree 1.

Proof. Let v be arbitrary. Since G is connected, $\deg(v) \geq 1$. Assume $\deg(v) = 1$. So $\delta(v) = \{vw\}$. If $\deg(w) = 1$, we found two vertices with degree 1. If $\deg(w) > 1$, there exist a neighbour of w , different from $v : u$. Now, again u has degree 1 or higher. If we repeat this procedure we either find a vertex of degree 1 or find again *new* vertices. Hence, after at most $n - 1$ vertices we end up at a vertex of degree 1. Now, if $\deg(v) \geq 2$, we do the same and find a vertex of degree 1, say w . Then repeat the above, starting from w to find a second vertex of degree 1. \square

Corollary 2.4:

A tree $G = (V, E)$ with maximum degree Δ has at least Δ vertices of degree 1.

Lemma 2.5: (a) For every graph $G = (V, E)$ it holds that $2|E| = \sum_{u \in V} \deg(u)$

(b) for every tree $G = (V, E)$ it holds that $|E| = |V| - 1$.

Proof. (a) trivial

(b) Proof by induction. Clearly, if $|V| = 1$ or $|V| = 2$ it holds. Assumption: true for $n \geq 2$. Let G be a tree with $n + 1$ vertices. By Lemma 2.3, there exists a vertex $v \in G$ with $\deg(v) = 1$. $G - v = G[V \setminus \{v\}]$ is a tree again with n vertices and thus $|E(G - v)| = |V(G - v)| - 1$. Since G differs by one vertex and one edge from $G - v$, the claim holds for G as well. \square

Lemma 2.6:

If $G = (V, E)$ with $|V| \geq 2$ has $|E| < |V| - 1$, G is not connected.

Algorithm MST

$\min_{x \in X} = -\max_{x \in X} - c(x)$ maximal forest

X spanning trees

$$\min_{x \in X} + (n-1)D = -\max_{x \in X} - c(x)(n-1)D = \max_{x \in X} \sum \underbrace{D - C_{ij}x_{ij}}_{\geq 0 \text{ if } D \geq \max_{ij \in E} c_{ij}}$$

Theorem 2.7:

Kruskal's Algorithm returns the optimal solution.

Proof. Let T be Kruskal's tree and assume there exists a tree T' with $c(T') < c(T)$. Then there exist an edge $e' \in T' \setminus T$. Then $T \cup \{e'\}$ contains a cycle $\{e_1, e_2, \dots, e_k, e'\}$. Let $c_f = \max_{i=1, \dots, k} c_{l_i}$. At the moment Kruskal chooses edge f , edge e' cannot be added yet and therefore $c(e') \geq c(f)$. Now exchange e' by f in T' . Hence the number of differences between T' and T is reduced by one, $c(T'_{new}) \leq c(T') < c(T)$. Repeating the procedure results in $c(T) \leq \dots < c(T)$, a contradiction. \square

Lecture 3 (2011-10-17):

Definition 2.7(+1):

The *running time of algorithms* (Laufzeit) of an algorithm is measured by the number of operations needed in worst case of a function of the input size. We use the $O(\cdot)$ notation (Big-O-notation) to focus on the most important factor of the running time, ignoring constants and smaller factors.

Example 2.7(+2):

If the running time is $3n \cdot \log n + 26n$, the algorithm runs in $O(n \cdot \log n)$. If the running time is $3n \cdot \log n + 25n^2$, the algorithm runs in $O(n^2)$.

For graph Problems, the running is expressed in the number of vertices $n = |V|$ and the number of edges $m = |E|$. Sometimes m is approximated by n^2 .

Example 2.7(+3) (Kruskal's Algorithm):

First, the edges are sorted according to nondecreasing weights. This can be done in $O(m \cdot \log m)$. Next, we repeatedly select an edge or reject its selection until $n - 1$ edges are selected. Since the last selected edge might be after m steps, this routine is performed at most $O(m)$ times.

Checking whether the end nodes of $\{u, v\}$ are already in the same tree can be done in constant time, if we label the vertices of the trees selected so far: $r(u) = \# \text{trees containing } u$. If $r(u) \neq r(v)$, the trees are connected by $\{u, v\}$ to a new tree.

Without going into details, the resetting of labels in one of the old trees, can be done $O(\log n)$ on average. Since this update has to be done at most $n - 1$ times, it takes $O(n \cdot \log n)$.

Overall, Kruskal runs in

$$O(n \log m + m + n \cdot \log n) = O(m \cdot \log m) = O(m \cdot \log n^2) = O(m \cdot \log n)$$

Definition 2.7(+4) (Shortest paths in acyclic digraphs):

A directed graph (digraph) $D = (V, A)$ is called *acyclic* (azyklisch) if it does not contain any *directed cycles*, i.e. a *chain* (Kette) $(v_0, a_1, v_1, a_2, v_2, \dots, a_k, v_k)$, $k \geq 0$, with $a_i(v_{i-1}, v_i) \in A$ and $v_k = v_0$. In particular, D does not contain *antiparallel* arcs: if $(u, v) \in A$, $(v, u) \notin A$. With $\delta_D^+(v)$ we denote the arcs leaving vertex v :

$$\delta_D^+(v) = \{(u, w) \in A : u = v\}$$

similarly:

$$\delta_D^-(v) = \{(u, w) \in A : w = v\}$$

are the arcs entering v .

The *outdegree* of v is $\deg_D^+(v) = |\delta_D^+(v)|$ (assuming simple digraph)

The *indegree* of v is $\deg_D^-(v) = |\delta_D^-(v)|$

Definition 3.1:

The *shortest path* problem in a acyclic digraph is, given an acyclic digraph $D = (V, A)$, a length function $C : A \rightarrow \mathbb{R}$ and two vertices $s, t \in V$, find a $[s, t]$ -path of minimal length.

Question 1:

Does there exist a $[s, t]$ -path at all?

Theorem 3.2:

A digraph $D = (V, A)$ is acyclic, if and only if there exists a permutation $\sigma : V \rightarrow \{1, \dots, n\}$ of the vertices such that $\deg_{D[v_1, \dots, v_n]}^-(v_i) = 0$ for all $i = 1, \dots, n$ with $v_i = \sigma^{-1}(i)$.

Proof. By induction:

For digraph with $|V| = 1$, the statement is true. Assume the statement is true for all digraphs with $|V| \leq n$ and consider $D = (V, A)$ acyclic with $n + 1$ vertices. If there does not exist a vertex with $\deg_D^-(v) = 0$, a directed cycle can be detected by following incoming arcs backwards until a vertex is repeated, a contradiction regarding the acyclic property of D .

Hence, let v be a vertex with $\deg_D^-(v) = 0$. Set $v_1 = v$. The digraph $D - v_1$ has n vertices and is acyclic, and thus has a permutation (v_2, \dots, v_{n+1}) with

$$\deg_{D[v_1, \dots, v_{n+1}]}^-(v_i) = 0 \quad \forall i = 2, \dots, n + 1$$

Now, (v_1, \dots, v_{n+1}) is a permutation fulfilling the condition.

In reverse, if there exists a permutation (v_1, \dots, v_{n+1}) , $\deg_D^-(v_1) = 0$ and there cannot exist a directed cycle containing v_1 . By induction, neither cycles containing $v_i, i = 2, \dots, n + 1$ exist. \square

Theorem 3.3:

A $[s, t]$ -path exists in a acyclic Digraph $D = (V, A)$ if and only if in all permutations $\sigma : V \rightarrow \{1, \dots, n\}$ with $\deg_{D[v_1, \dots, v_n]}^-(v_i) = 0$ for all $i = 1, \dots, n$, it holds that $\sigma(s) < \sigma(t)$.

Proof. Assume there exists a permutation σ with $\sigma(s) > \sigma(t)$. Since outgoing arcs only go to higher ordered vertices, there does not exist a path from s to t in D .

In reverse, if there does not exist a path from s to t , we order all vertices with paths to t first, followed by t and s afterwards. \square

Question 2:

How do we find the shortest $[s, t]$ -path if it exists?

To simplify notation, let $V = \{1, \dots, n\}$, $s = 1$, $t = n$ and $(i, j) \in A \Rightarrow i < j$. Let $D(i)$ be the distance from i to n and $NEXT(i)$ be the next vertex on the shortest path from i to n .

Bellmann's Algorithm

- (a) $D(i) = \{\infty : i < n \text{ and } NEXT(i) = NIL, 0 : i = n\}$
- (b) FOR $i = n - 1$ DOWNT0 1 DO
- (c) $D(i) = \min_{j=i+1, \dots, n} \{D(j) + c(i, j)\}$ with $c(i, j) = \infty$ if $(i, j) \notin A$

$$(d) \quad \text{NEXT}(i) = \operatorname{argmin}_{j=i+1, \dots, n} \{D(j) + c(i, j)\}$$

Theorem 3.4:

Bellmann's Algorithm is correct and runs in $O(m + n)$ time.

Proof. Every path from 1 to n passes through vertices of increasing ID. Assume there exists a path (a_1, \dots, a_k) with $\sum_{i=1}^k c(a_i) < D(1)$. Let $a_1 = (1, j_1)$. Since $D(1) \leq c(a_1) + D(j_1)$, it should hold that

$$\sum_{i=2}^k c(a_i) < D(j_1)$$

But $D(j_1) \leq c(a_2) + D(j_2)$ with $a_2 = (j_1, j_2)$, etc.

In the end, $c(a_k) < D(j_{k-1})$ but $D(j_{k-1}) \leq c(a_k) + D(n) = c(a_k)$, contradiction. \square

Lecture 4 (2011-10-20):

Theorem 3.5:

Bellman's Algorithm is correct and runs in $O(m + n) = O(n)$.

Proof. of runtime:

$$D(i) = \min_{(i,j) \in A} D(j) + D(i,j)$$

\Rightarrow Every arc is considered once, and thus overall $O(m)$ computations are needed. Initialization costs $O(n)$. \square

Bemerkung 3.5(+1):

The running time does not contain the time to find the permutation.

Observation 1: We not only found the shortest path from 1 to n , but also from i to n , $i = 2, \dots, n$.

Observation 2: We can use a similar procedure for the shortest path from 1 to i , $i = 2, \dots, n$. (with $PREV(i)$ for previous instead of $NEXT(i)$).

Question 3:

Can we find a shortest path from 1 to i in a digraph that is not acyclic, i.e. it contains cycles?

Theorem 4.1:

The Moore-Bellman-Algorithm returns the shortest paths from 1 to $i = 1, \dots, n$ provided D does not contain negative-weighted directed cycles.

Proof. We call an arc $(i,j) \in A$ an *upgoing* arc (Aufwärtsbogen) if $i < j$ and a *downgoing* arc (Abwärtsbogen) if $i > j$.

A shortest path from 1 to i contains at most $n - 1$ arcs. If an upgoing arc is followed by a downgoing arc (or vice versa), we have a *change of direction* (Richtungswechsel). With at most $n - 1$ arcs, at most $n - 2$ changes of direction are possible.

Let $D(i, m)$ be the value of $D(i)$ at the end of the m -th iteration. We will show (and this is enough):

$$D(i, m) = \min\{c(W) : W \text{ is the directed } [1, i]\text{-path with at most } m \text{ changes of directions}\}$$

We prove it by induction on m .

- For $m = 0$, the algorithm is equivalent to Bellman's algorithm for acyclic graphs. Thus, $D(i, 0)$ is the length of the shortest path without any change of direction.
- Now, let us assume, that the statement is true for $m \geq 0$ and the subroutine is executed for the $m + 1$ -st time. The set of $[1, i]$ -paths with at most $m + 1$ changes of direction consists of

- (a) $[1, i]$ -paths with $\leq m$ changes of direction
- (b) $[1, i]$ -paths with exactly $m + 1$ changes of direction

$\Rightarrow D(i, m)$

- Since every path starts with an upgoing arc $(1, k)$, the last arc after $m + 1$ changes is either a downgoing arc if $m + 1$ is odd or an upgoing arc if $m + 1$ is even. We restrict ourselves to $m + 1$ odd ($m + 1$ even is similar).

To compute the minimum length path in (b) we use an additional induction on $i = n, n - 1, \dots, j + 1$. Since every path ending at n ends with an upgoing arc, there do not exist such $[1, n]$ -paths. Hence, $D(n, m + 1) = D(n, m)$.

Now assume that $D(k, m + 1)$ is correctly computed for $i \leq k \leq n$. The shortest path from 1 to $i - 1$ with exactly $m + 1$ changes ends with a downgoing arc $(j, i - 1)$, $j > i - 1$.

$D(j, m + 1)$ is already computed correctly. If $PREV(j) > j$, no change of direction is required in j and $D(i - 1, m + 1) = D(j, m + 1) + c(j, i - 1)$. If $PREV(j) < j$, the last arc of the $[1, j]$ -Path is upgoing, and thus $D(i - 1, m + 1) = D(j, m) + c(j, i)$. The last change of direction at j is thus, in worst case, the $(m + 1)$ -st change. Hence, $D(i - 1, m + 1)$ fulfills the statement.

□

Remark 1:

In fact, the algorithm finds the minimum length of a chain (kette) with at most $n - 2$ changes of direction. In case of negative weighted cycles these might be in a chain several times.

In case no negative weighted cycles exist, the min. length chains are indeed paths. Hence, the algorithm only works correctly if *all* cycles are non-negative weighted.

Remark 2:

If a further executing of the subroutine ($m = n - 1$) results in at least one change of a value $D(i)$, then the digraph contains negative weighted cycles.

Remark 3:

A more efficient implementation is given by E'sopo-Pape-Variant.

Dijkstra's Algorithm for non-negative weights

Theorem 4.2:

Dijkstra returns the shortest paths from 1 to i , $i = 1 \dots n$, provided all weights ≥ 0 .

Proof. Each step, one vertex is moved from T to S . At the end of a step, $D(j)$ is the shortest path from 1 to j via vertices in S .

If $S = V(T = \emptyset)$, $D(i)$ is thus the shortest $[1, i]$ -path

□

Lecture 5 (2011-10-24):

Shortest paths between all pairs of vertices

Solution 1: Apply Moore-Bellman or Dijkstra to all vertices i as starting vertex

Solution 2: Apply Floyd's Algorithm

Notation:

w_{ij} = length of the shortest $[i, j]$ -path, $i \neq j$

w_{ii} = length of the shortest directed cycle containing i

p_{ij} = predecessor of j on the shortest $[i, j]$ -path (cycle)

$W = (w_{ij})$ is the *shortest path length matrix*

Theorem 5.1:

The Floyd Algorithm works correctly if and only if $D = (V, A)$ does not contain any negative weighted cycles.

D contains a negative weighted cycle if and only if one of the diagonal elements $w_{ii} < 0$.

Proof. Let W^k be the matrix W after iteration k , with W^0 being the initial matrix. By induction on $k = 0, \dots, n$ we show that W^k is the matrix of shortest path lengths with vertices $1, \dots, k$ as *possible* internal vertices, provided D does not contain a negative cycle on these vertices.

If D has a negative cycle, then $w_{ii}^k < 0$ for an $i \in \{1, \dots, n\}$

For $k = 0$, the statement clearly true.

Assume, it is correct for $k \geq 0$, and we have executed the $(k + 1)$ st iteration.

It holds that $w_{ij}^{k+1} = \min\{w_{ij}^k, w_{i,k+1}^k + w_{k+1,j}^k\}$. Note that, provided no negative cycle exists, $w_{i,k+1}^{k+1}$ does not have any vertex $k + 1$ as internal vertex, and thus $w_{i,k+1}^{k+1} = w_{i,k+1}^k$ (similarly, $w_{k+1,j}^{k+1} = w_{k+1,j}^k$).

$w_{i,k+1}^k$ is the minimal length of a $[i, k + 1]$ -path with $\{1, \dots, k\}$ as allowed internal vertices. Similarly, $w_{k+1,j}^k$.

Thus, $w_{i,k+1}^k + w_{k+1,j}^k$ is the minimal length of an $[i, j]$ -path (not necessarily simple) containing $k + 1$ (mandatory) and $\{1, \dots, k\}$ (voluntary). If the shortest path from i to j using $\{1, \dots, k + 1\}$ does not contain $k + 1$, it only contains $\{1, \dots, k\}$ (voluntary) and, hence, w_{ij}^k is the right value.

What remains to show is that the connection of the $[i, k + 1]$ -path with the $[k + 1, j]$ -path is indeed a simple path.

Let K be this chain. After removal of cycles, the chain K contains (of course) a simple $[i, j]$ -path \bar{K} . Since such cycles may only contain vertices from $\{1, \dots, k + 1\}$, one cycle must contain $k + 1$. If this cycle is not negatively weighted, then path \bar{K} is shorter and $w_{ij} < w_{i,k+1}^k + w_{k+1,j}^k$.

If this cycle is negatively weighted, $w_{k+1,k+1}^k < 0$ (the cycle only contains internal vertices from $\{1, \dots, k\}$) and algorithm would have stopped earlier. \square

Min-Max-Theorems for combinatorial Optimization Problems

From "Optimierung A": Duality of linear programs

$$\max_{\text{s. t.}, Ax \leq b, x \geq 0} c^T x = \min_{\text{s. t.}, A^T y \geq c, y \geq 0} b^T y$$

For several combinatorial problems $\min\{c(x) : x \in X\}$

We can define a second set Y and a function $b(y)$ with $\max\{b(y) : y \in Y\} = \min\{c(x) : x \in X\}$ where Y and $b(y)$ have a graph theoretical interpretation.

Existence of such a "Dual" Problem indicates often that the problem can be solved "efficiently". For the shortest path problem several max-min-theorems exist.

Definition 5.2:

An (s, t) -cut (Schnitt) in a digraph $D = (V, A)$ with $s, t \in V$ is a subset $B \subset A$ of the arcs with the property that every (s, t) -path contains at least one arc of B .

Stated otherwise, for every cut B , there exists a vertex set $W \subset V$ such that

- $s \in W, t \in V \setminus W$
- $\delta^+(w) = \{(i, j) \in A : i \in W, j \in V \setminus W\} \subseteq B$

Theorem 5.3:

Let $D = (V, A)$ be a digraph, $c(a) = 1 \forall a \in A, s, t \in V, s \neq t$. Then the minimum length of a $[s, t]$ -path equals the maximum number of arc-disjoint (s, t) -cuts.

Proof. Follows from ?? □

Theorem 5.4:

Let $D = (V, A)$ be a digraph, $c(a) \in \mathbb{Z}_+ \forall a \in A \wedge s, t \in V \wedge s \neq t$. Then the min length of an $[s, t]$ -path equals the maximum number d of (not necessarily different) (s, t) -cuts C_1, \dots, C_d such that every arc $a \in A$ is contained in at most $c(a)$ cuts.

Proof. We define (s, t) -cuts $C_i = \delta^+(v_i)$ with $V_i = \{v \in V : \exists (s, v)\text{-path with } c(P) \leq i - 1\}$

$$\begin{aligned} v_1 &= \{s\} \\ v_2 &= \{5, 3, 4\} \\ v_3 &= \{5, 2, 3, 4\} \\ v_4 &= v_3 \cup \{6\} \end{aligned}$$

(for the example graph on the board)

The shortest $[s, t]$ -path P consists of arcs a_1, \dots, a_k with arc a_j contained in (s, t) -cuts $C_i, i \in \{\sum_{l=1}^{j-1} c(a_l) + 1, \dots, \sum_{l=1}^j c(a_l)\}$: exactly $c(a)$ cuts. □