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A NOTE ON MATHEMATICAL MODELS FOR LEARNING

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It is shown that a rational learning curve developed by Estes for paired associate learning is a special case of a more general function. The latter is the product of two functions and assumes that the discovery and fixation aspects of learning are independent. The indications are that the form of one of the functions has not been tested sharply by Estes in the paired associate learning setting.

In a recent paper Estes [2] shows that certain data on paired associate learning can be fitted very well by the rational learning curve

$$E_t = (1 - 1/N)(1 - c)^{t-1}, \tag{1}$$

where E_t is the number of errors made on the tth trial, N is the total number of alternative responses available, and c is a parameter, so that $(1 - c)^{t-1}$ is the number of *unlearned* items at the beginning of the tth trial.

Estes' data show that the error curves for different values of N (in particular, $N = 2$ and $N = 8$) are almost precisely similar; that is, they can be factored into products of a function of N and a function of t:

$$E_t = \phi(N)\psi(t), \tag{2}$$

where the particular form of (2) that Estes uses in fitting the data has already been shown in (1) above. That is, Estes makes the special assumptions that

$$\phi(N) = (1 - 1/N) \quad \text{and} \quad \psi(t) = (1 - c)^{t-1}, \tag{3}$$

and shows that, with these special assumptions, the functions fit the data well.

The fact that the learning curves for different N are similar, as hypothesized in (2), follows from the assumption that the number of trials required to learn a set of associations is independent of the size of the class of possible responses. The derivation does not depend on the forms of the functions ϕ and ψ; all it requires is that E_t be expressible as a product of two such functions. In psychological terms, the assumption underlying the derivation means simply that discovery and fixation are independent aspects of learning; it implies nothing about the nature of the fixation process. An equation equivalent to (2) was derived by Simon [3, p. 264] from this assumption, and was shown to be consistent with data from maze learning experiments, as well as rote learning experiments.

What Estes has shown, then, are three things: (i) the discovery and fixation aspects of learning are independent in his data, as in data analyzed previously in Simon [3]; (ii) the number of errors produced in discovery can be explained by a simple model of random search, from which the form of $\phi(N)$ follows; (iii) the number of trials required for fixation can be explained by a simple stimulus conditioning model, from which Estes obtains his form of $\psi(t)$.

What implications does this analysis have for the comparison Estes [2, pp. 136–140] makes between his learning model and the earlier models of Robertson and Thurstone? First, if we consider these models to be concerned only with the fixation process, we should compare them (suitably transformed to make E_t the dependent variable) with the function ψ, rather than with the product $\phi\psi$. By premultiplying the alternative function ψ with $(1 - 1/N)$, we could obtain fits to the data as good as those obtained with Estes' form of ψ. Moreover, the parameters estimated for $N = 8$ would still fit the data for $N = 2$—for the same reason that they do under Estes' assumptions.

We may still prefer Estes' form of ψ, since his function has only a single parameter, while those of Robertson and Thurstone have three and two, respectively. However, its claim for superiority as an explanation of the fixation process must rest entirely on its parsimony in parameters, since it is neither more nor less powerful than the alternatives in fitting data for different N without reestimating these parameters.

Of course there are other tests, beyond those applied by Estes, to determine whether the one-element stimulus sampling model defined by (1) fit the data better than other models. Bower, for example, has recently [1] tested the same model by examining certain conditional probabilities, i.e., sequential dependencies, of correct and wrong responses, and found that his model fitted the data better than did some alternative models. The point of the present note is not to argue that the form of $\psi(t)$ is untestable—which is patently false—but that it is not tested sharply in the analysis presented by Estes.

REFERENCES

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