Iterative methods to solve linear systems

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$$Ax^* = b, x^* = ?$$

$$x^{(k+1)} = c + T \cdot x^{(k)} \to x^*, k \to \infty, T = ?\ c = ?\ x^{(0)} = ?$$

$$x^* = c + T \cdot x^*$$

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{i1}x_1 + \cdots + a_{in}x_n = b_i \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n \end{cases} \Leftrightarrow \begin{cases} x_1 = \dfrac{1}{a_{11}}\left(b_1 - \displaystyle\sum_{j=2}^{n} a_{1j}x_j\right) \\ \vdots \\ x_i = \dfrac{1}{a_{ii}}\left(b_i - \displaystyle\sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}x_j\right) \\ \vdots \\ x_n = \dfrac{1}{a_{nn}}\left(b_n - \displaystyle\sum_{j=1}^{n-1} a_{nj}x_j\right) \end{cases}$$

1) Jacobi: $$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

2) Gauss-Seidel: $$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

3) SOR : $$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \omega x_{i_{GS}}^{(k+1)}, \omega \in (0,2)$$

$$\omega = 1 \Rightarrow \text{Gauss-Seidel}$$

Example of a set of conditions for the real matrix A such that the above methods converge:

- $a_{ii}>0$, $i = 1, \dots, n$;
- A is symmetric ($A = A'$);
- A is stricty diagonally dominant: $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, i = 1, \dots, n.$

$x^{(k+1)} = c + T \cdot x^{(k)} \to x^*, k \to \infty, T =? \; c =? \; x^{(0)} =?$

$x^* = c + T \cdot x^*$

$A = M - N$, M - invertible, $N = M - A$

$Ax^* = b \Leftrightarrow (M - N)x^* = b \Leftrightarrow Mx^* = b + Nx^*$

$\Leftrightarrow x^* = c + Tx^*, c = M^{-1}b, T = M^{-1}N,$ $\quad M =? \; N =?$

1) Jacobi: $M = diag(diag(A))$
2) Gauss-Seidel: $M = tril(A)$
3) SOR: $M = \frac{1}{\omega} \cdot diag(diag(A)) + tril(A, -1)$

>>$norm(T, Inf)$ returns the number $\|T\|_\infty$.
If $\|T\|_\infty$<1, then the iterations converge (the reciprocal is not necessarily true).
The stoppoing criterion is given by the inequality: $\left\|x^{(k+1)} - x^*\right\|_\infty \leq \frac{\|T\|_\infty}{1-\|T\|_\infty} \cdot \left\|x^{(k+1)} - x^{(k)}\right\|_\infty$.
x^* is unknown $\Rightarrow$ the stopping criterion is $\frac{\|T\|_\infty}{1-\|T\|_\infty} \cdot \left\|x^{(k+1)} - x^{(k)}\right\|_\infty \leq \varepsilon$ and we return $x^{(k+1)}$.

Example of optimal ω, for A satisfying the conditions in the above example:
>>$TJ = diag(diag(A)) \backslash (diag(diag(A)) - A);\; rho = max(abs(eig(TJ)));\; omega = 2/(1 + sqrt(1 - rho^\wedge 2))$