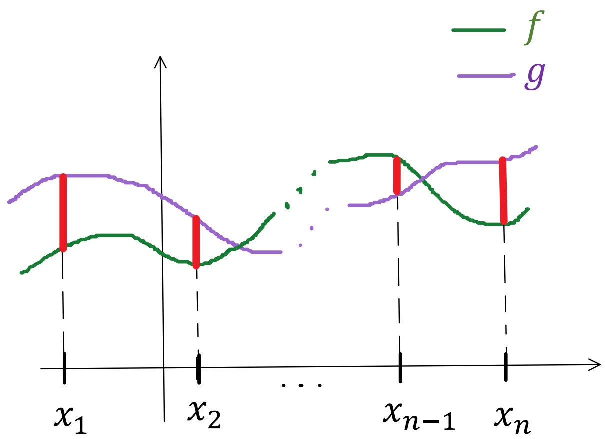
Least squares

Wednesday, April 8, 2020 12:18 PM

Nodes: $x_1, \dots, x_n$

$\langle f, g\rangle$ = inner product of functions f and g

$= \sum_{i=1}^{n} f(x_i) \cdot g(x_i)$



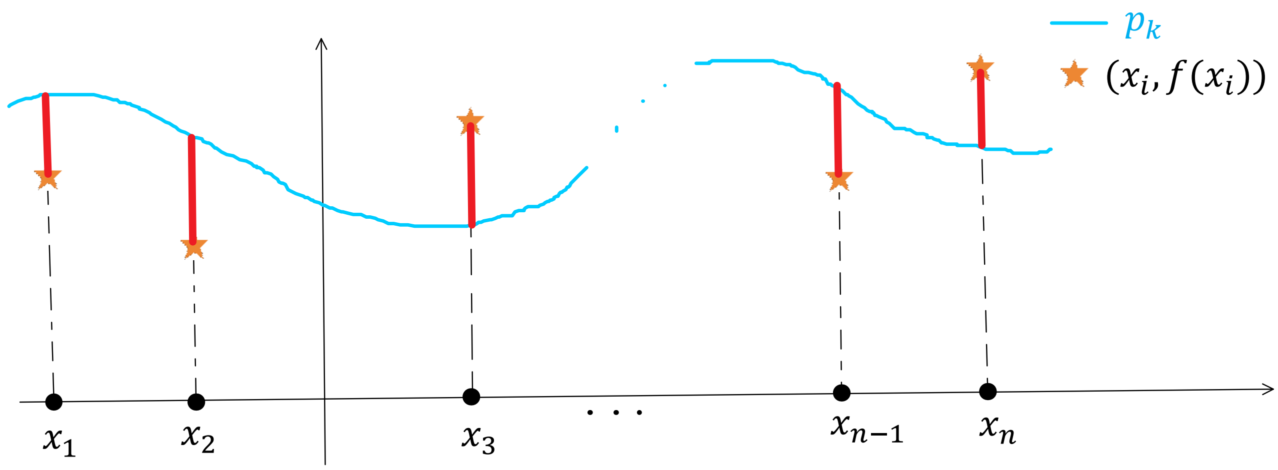
$\|f\| = \sqrt{\langle f, f\rangle}$=norm of f

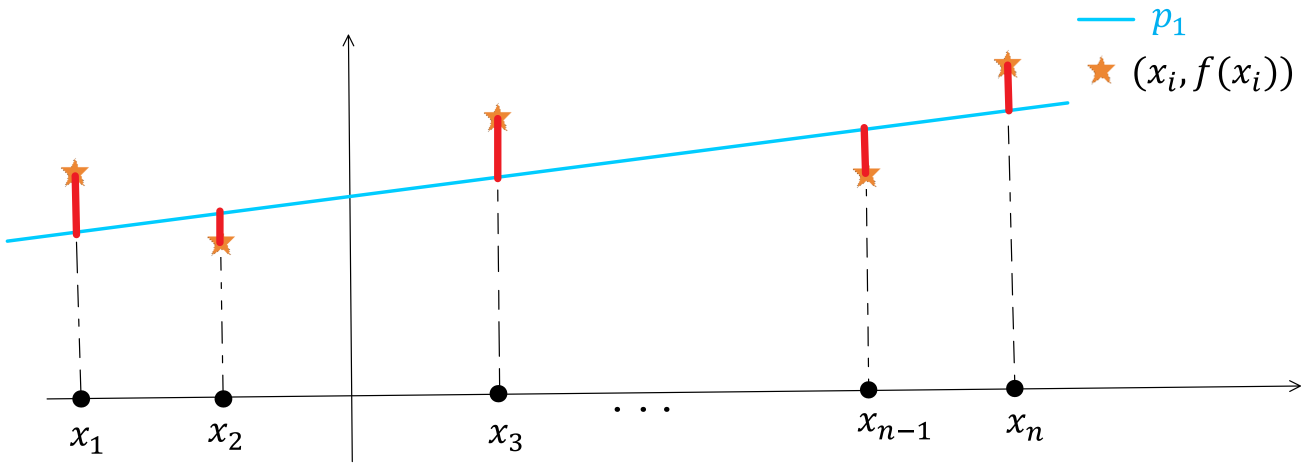
$= \sqrt{\sum_{i=1}^{n} f^2(x_i)}$

$\|f - g\|$ =distance between f and g

$= \sqrt{\sum_{i=1}^{n} \left(f(x_i) - g(x_i)\right)^2}$

? Find the polynomial p_k of degree $\le k$ such that $\|f - p_k\| = \min_{p \in \mathbb{P}_k} \|f - p\|$ is <u>minimal</u>, where $\mathbb{P}_k$ is the space of polynomials of degree $\le k$.





$p_k = a_k g_k + \dots + a_1 g_1 + a_0 g_0$, where $g_0, \dots, g_k$ is <u>base</u> of $\mathbb{P}_k$; ex.: $g_i(x) = x^i, i = 0, \dots, k$.

$E^2(a_k, \dots, a_0) = \|f - p_k\|^2$ is minimal $\Rightarrow \frac{\partial E^2}{\partial a_i}(a_k, \dots, a_0) = 0, i = 0, \dots, k$

$\Leftrightarrow$ <u>normal equations:</u> $\sum_{i=0}^{n} a_i \langle g_i, g_k\rangle = \langle f, g_k\rangle, i = 0, \dots, k$

$\rightarrow$ <u>linear system</u> with unique solution.

<u>Remark:</u> $k = n - 1 \Rightarrow p_k$ is the Lagrange interpolation polynomial.

Example 6.5. *In robotics and many other applications, one often encounters the* Procrustes problem *or one of its variants (see [45], Chapter 23). Consider a given body (e.g., a pyramid like in Figure 6.3) and a copy of the same body. Assume that we know the coordinates of m points $\boldsymbol{x}_i$ on*

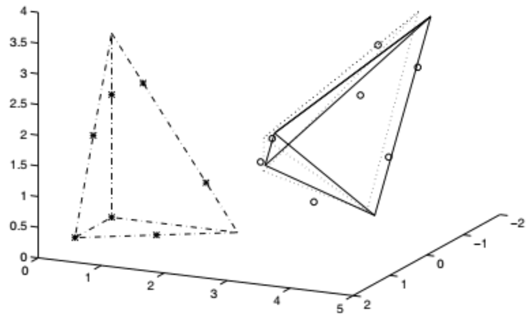


Figure 6.3. *Procrustes or registration problem*

the first body, and that the corresponding points $\boldsymbol{\xi}_i$ have been measured on the other body in another position in space. We would like to rotate and translate the second body so that it can be superimposed onto the first one as well as possible. In other words, we seek an orthogonal matrix Q (the product of three rotations) and a translation vector $\boldsymbol{t}$ such that $\boldsymbol{\xi}_i \approx Q\boldsymbol{x}_i + \boldsymbol{t}$ for $i = 1, \ldots, m$.

[45] W. Gander and J. Hřebíček. *Solving Problems in Scientific Computing using Maple AND Matlab*. Springer, 3rd edition, 1997.