02/07/23

Απαλλακτική Εργασία Δυναμικά Συστήματα

* Εφόσον οι εκφωνήσεις αλλά και οι διαφάνειες του μαθήματος ήταν στα αγγλικά επέλυσα όλη την εργασία στα αγγλικά και μεγάλο μέρος των περιγραφών έχουν παρθεί από τις διαφάνειες

exercise 1

$$\dot{x} = f(x) = \lambda + x - x^3, \; \lambda \in \mathbb{R} \quad (1)$$

i) in order to determine the fixed points of the non-linear dynamical system, we should equal (1) with 0

$$\lambda + x - x^3 = 0$$

because finding close-form analytical solution for this equation is generally challenging and might not be possible for arbitrary values of λ, we'll use a computational approach as mentioned in the exercise presentation

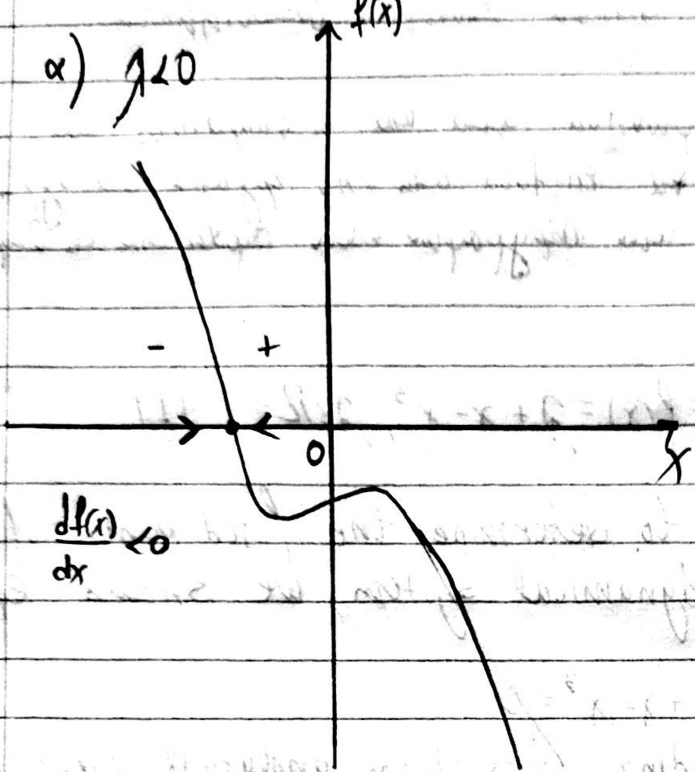
→ we chose a range of values for x such as $x \in [-2, 2]$ and a set of values for λ such as $\lambda > 0$, $\lambda < 0$, $\lambda = 0$, $\lambda = 1$ (the last just for evaluation purposes)

in the next page we will plot the graph of $f(x)$ for each λ

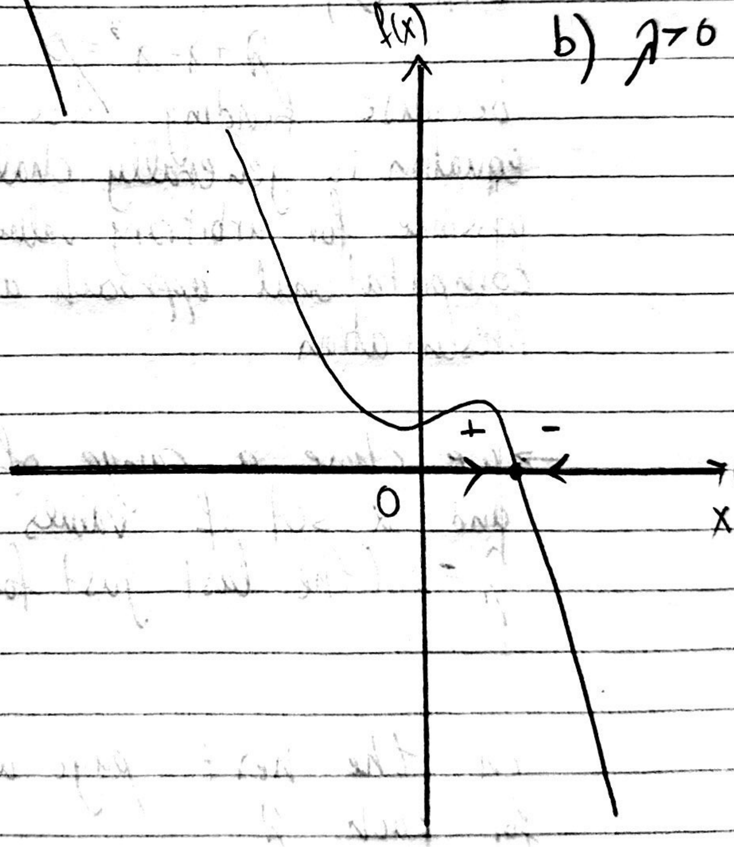
the diagrams where made with help of desmos calculator (online graph calculator)

www.desmos.com/calculator

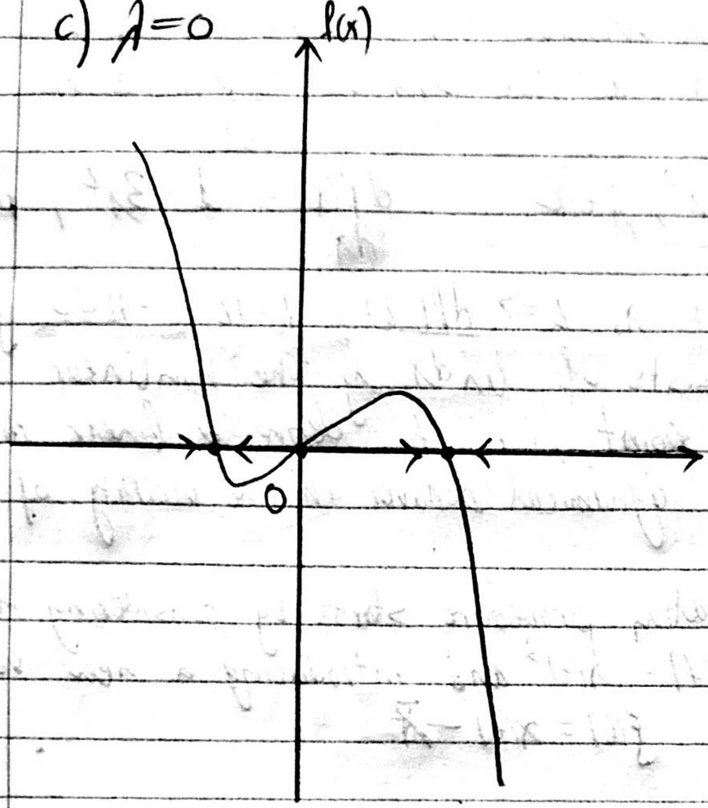
a) $\lambda < 0$



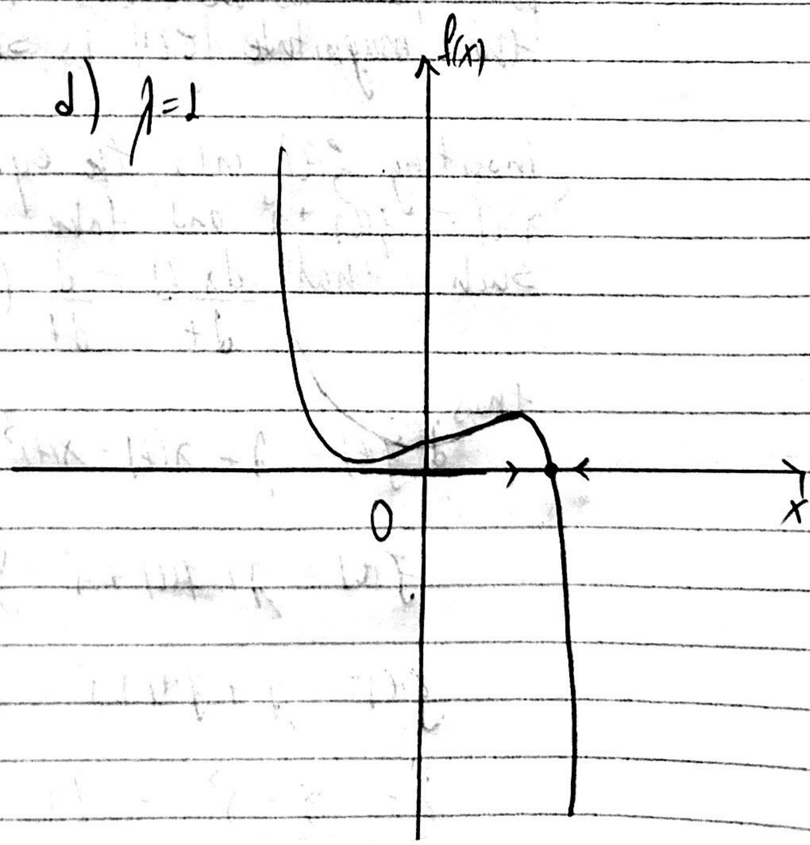
b) $\lambda > 0$



c) $\lambda = 0$



d) $\lambda = 1$



in order to analyze the stability of the fixed points we should examine the sign of $\frac{d f(x)}{dx}$ around them

$f(x) = \lambda + x - x^3$, $\lambda \in \mathbb{R}$ $\qquad \frac{d f(x)}{dx} = 1 - 3x^2$, but first

a standard procedure to determine the stability of a fixed point $\bar{x}$ consists of linearizing the nonlinear diff. eq. around this point, i.e to derive a linear diff. eq. that describes the dynamical behavior in the vicinity of $\bar{x}$ (fixed point)

the linearization procedure starts by considering the eq. $\dot{x}(t) = \lambda + x(t) - x(t)^3$ and introducing a new variable

$$\xi(t) = x(t) - \bar{x}$$

$\xi(t)$ represents the distance between $x(t)$ and the fixed point, and as we assume that we are in the vicinity of $\bar{x}$ the magnitude $|\xi(t)|$ is small

inserting $\xi(t)$ into the eq. may be done by solving for $x(t) = \xi(t) + \bar{x}$ and take its derivative w.r.t. time such that $\frac{dx(t)}{dt} = \frac{d}{dt}(\xi(t) + \bar{x}) = \frac{d\xi(t)}{dt} + \frac{d\bar{x}}{dt} \Rightarrow \dot{x}(t) = \dot{\xi}(t)$

thus,

$$\dot{\xi}(t) = \lambda + x(t) - x(t)^3 = \lambda + \{\xi(t) + \bar{x}\} - \{\xi(t) + \bar{x}\}^3 \Rightarrow$$

$$\dot{\xi}(t) = \lambda + \xi(t) + \bar{x} - \{\xi(t)^3 + 3\xi(t)\bar{x}(\xi(t) + \bar{x}) + \bar{x}^3\} \Rightarrow$$

$$\dot{\xi}(t) = \lambda + \dot{\xi}(t) + \bar{x} - \xi(t)^3 - 3\xi(t)^2\bar{x} - 3\xi(t)\bar{x}^2 - \bar{x}^3 \Rightarrow$$

$$\underbrace{\lambda + \bar{x} - x^3}_{:=0} + \xi(t) - \underbrace{\xi(t)^3 - 3\xi(t)^2 \bar{x}}_{\text{tiny}} - 3\xi(t)\bar{x}^2$$

($\bar{x}$ fixed point so it vanishes)

($|\xi(t)|$ is small, so $|\xi(t)|^2 = \xi^2(t)$ even smaller) the eq that represents the linearization of the (1) eq around its fixed points takes the following familiar linear form for the variable $\xi(t)$ given by

$$\dot{\xi}(t) = \xi(t) - 3\xi(t)\tilde{x}^2 = \xi(t)(1-3\tilde{x}^2) = f'(\tilde{x})\xi(t)$$

we know that the solution for the above equation is given by $\xi(t) = ce^{f(\tilde{x})t}$ stable for $f(\tilde{x})<0$ and unstable for $f(\tilde{x})>0$

a) let $x_0 = -2 \Rightarrow \frac{df(-2)}{dx} = 1-12 = -11 < 0$

$f(-2) = -2-8+\lambda < 0$

So for $\lambda < 0$ the point is stable

b) let $x_0 = +2 \Rightarrow \frac{df(2)}{dx} = 1-12 = -11 < 0$

$f(2) = 2-8+\lambda = \lambda - 6$ $\begin{cases} >0 & \text{for } \lambda > -6 \\ <0 & \text{for } \lambda < -6 \end{cases}$

So for $\lambda > 0$ the point is stable

c) $\lambda = 0$

for $x = \{-1, 0, +1\} \Rightarrow \frac{df(-1)}{dx} = -2 < 0$ (stable)

$\frac{df(-1)}{dx} = -2 < 0$ (stable)

$\frac{df(0)}{dx} = 1 > 0$ (unstable)

ii) first we should consider the potential function $V(x)$ of $f(x)$ (given that $\dot{x}$ is a univariable dynamical system)

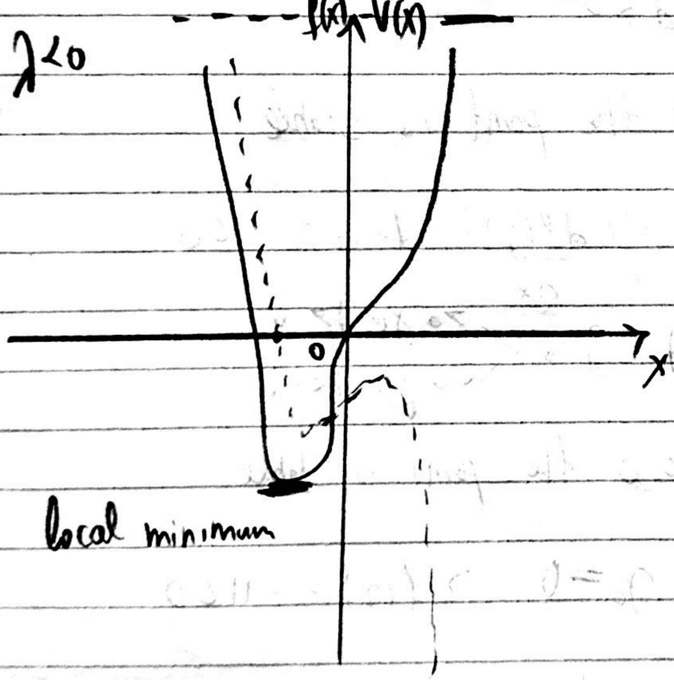
$$f(x) = \lambda + x - x^3 \qquad V'(x) = f(x)$$

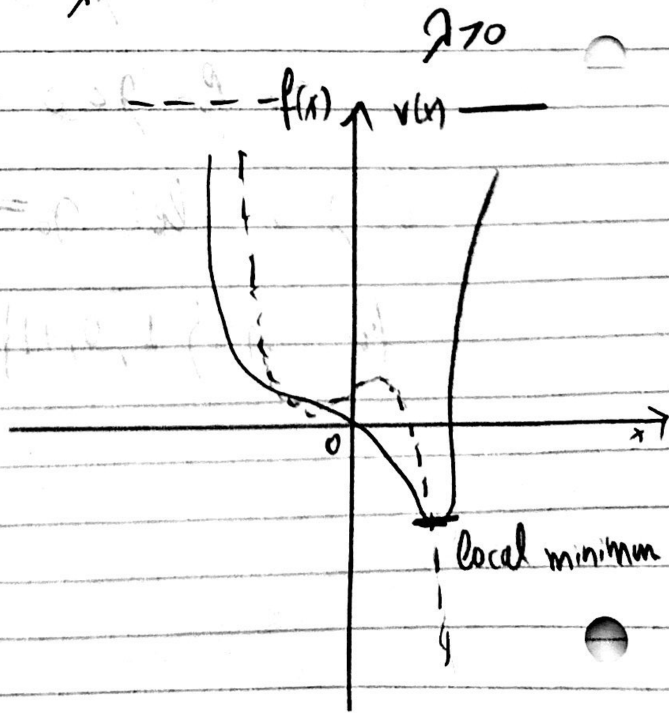
$$\text{so } \dot{x} = f(x) = -\frac{dV(x)}{dx} = -V'(x)$$

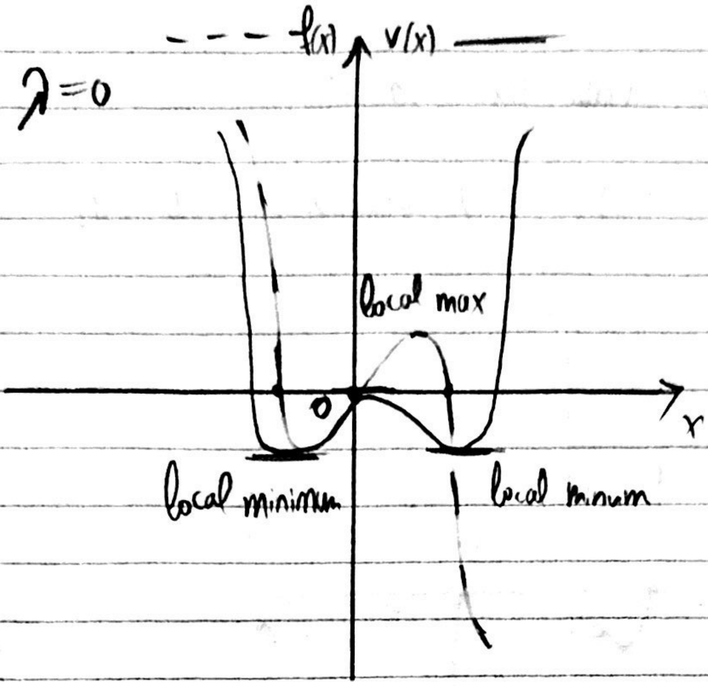
$$V(x) = -\int f(x)dx = -\int(\lambda + x - x^3)\,dx = -\left(\int \lambda\, dx + \int x\, dx - \int x^3 dx\right)$$

$$= -\lambda x - \frac{x^2}{2} + \frac{x^4}{4} + c$$

After we'll choose a range of values for x and $\dot{x}$ $[-2,2]$







we know local minima of $V(x)$ correspond to stable fixed points, while local maxima correspond to unstable fixed points

$V(-1) = \lambda - \frac{1}{4}$
$V(-2) = 2(\lambda+1) < 0$
$V(-0.5) = -0.5\lambda - 0.0625 > 0$

for $\lambda < 0$
so local minima at $\bar{x} = -1$

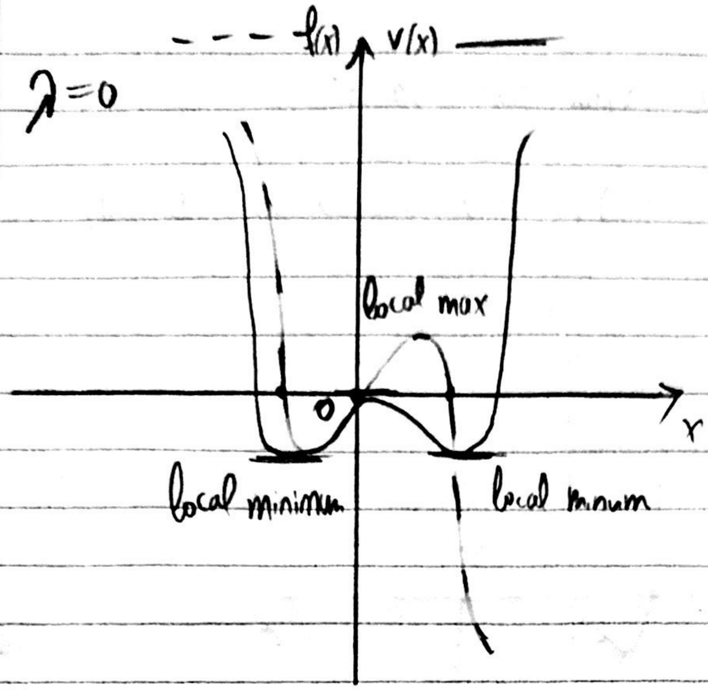
$V(0) = 0$
$V(2) = -2\lambda - 2 + 8$
$V(1) = -1/4 - \lambda < 0$

for $\lambda > 0$
so local minimum at $\bar{x} = +1$

$V(-1) = -\frac{1}{4} < 0$
$V(-2) = -2 + 8 = 6 > 0$
$V(-0.5) = -1/8 + 1/32 < 0$
$V(0) = 0$
$V(2) = 6 > 0$
$V(1) = -1/4 < 0$

for $\lambda = 0$

So local minimum at $\bar{x} = -1, +1$
and local maximum at $\bar{x} = 0$



we know local minima of $V(x)$ correspond to stable fixed points, while local maxima correspond to unstable fixed points

$V(-1) = \lambda - \frac{1}{4}$
$V(-2) = 2(\lambda+1) < 0$
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for $\lambda < 0$
so local minima at $\bar{x} = -1$

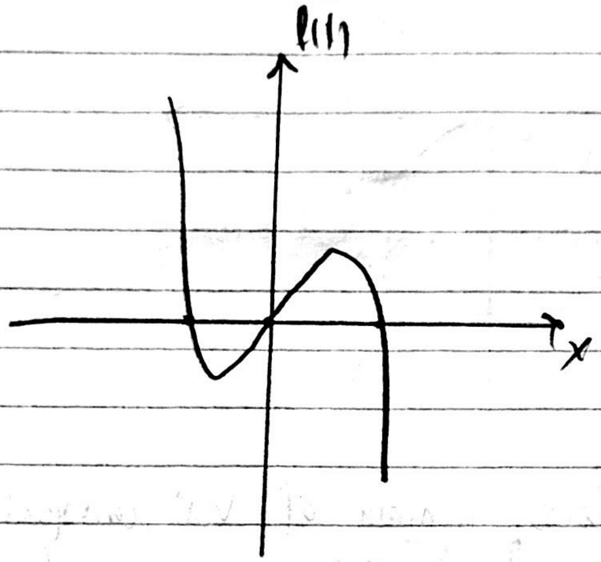
$V(0) = 0$
$V(2) = -2\lambda - 2 + 8$
$V(1) = -1/4 - \lambda < 0$
for $\lambda > 0$
so local minimum at $\bar{x} = +1$

$V(-1) = -\frac{1}{4} < 0$
$V(-2) = -2 + 8 = 6 > 0$
$V(-0.5) = -1/8 + 1/32 < 0$
$V(0) = 0$
$V(2) = 6 > 0$
$V(1) = -1/4 < 0$
for $\lambda = 0$

So local minimum at $\bar{x} = -1, +1$
and local maximum at $\bar{x} = 0$

iii) if $\lambda=0$ then $f(x) = x - x^3$

$f(x) = 0 \Rightarrow x(1-x^2) = 0 \Rightarrow x_1 = 0,\ x_2 = +1,\ x_3 = -1$



- You can find the implementation of iv and v in the same file named ex1.m (matlab) in which I followed the exact same approach we followed in the lectures (I used code provided on the gunet-eclass) exercise 2

$R(t)$ = Romeo's love or hate for Juliet at time t
$J(t)$ = Juliet's love or hate for Romeo at time t

$R(t) > 0, \; J(t) > 0 \Rightarrow$ love
$R(t) < 0, \; J(t) < 0 \Rightarrow$ hate

$$\dot{R}(t) = \alpha \cdot J(t), \; \alpha > 0$$
$$\dot{J}(t) = -b\, R(t), \; b > 0 \quad (1)$$

i) $\begin{cases} \dot{R}(t) = 0 \\ \dot{J}(t) = 0 \end{cases} \Rightarrow \begin{cases} a\, J(t) = 0 \\ -b\, R(t) = 0 \end{cases} \Rightarrow \begin{cases} J(t) = 0 \\ R(t) = 0 \end{cases}$

$R = 0, \; J = 0$ is the fixed point of the system

in order to define its stability we should examine the eigenvalues of the system at this point

we can represent the system of linear first order diff equations in matrix form as

$$\begin{bmatrix} \dot{R}(t) \\ \dot{J}(t) \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & -b \end{bmatrix} \begin{bmatrix} J(t) \\ R(t) \end{bmatrix} = \dot{x} = Lx \quad (2)$$

it is a fundamental theorem that states that the general solution of a system of linear equations of the form defined in eq (2) is completely determined by the eigenvalues λ and eigenvectors v of the coefficient matrix and given by

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

thus, we are now in the position of calculating the general Solution in eq (1)

to begin with we will determine the expression for the characteristic polynomial of the coefficient matrix

$$p(\lambda) = |L - \lambda I_1| \Rightarrow p(\lambda) = \begin{vmatrix} \alpha-\lambda & 0 \\ 0 & -b-\lambda \end{vmatrix} \Rightarrow p(\lambda) = (a-\lambda)(b-\lambda)$$

$$\Rightarrow p(\lambda) = -\alpha b - \alpha\lambda + b\lambda + \lambda^2$$

$$\Rightarrow p(\lambda) = \lambda^2 + \lambda(b-\alpha) - \alpha b$$

after, the roots of the characteristic polynomial will be computed

the discriminant $\Delta p(\lambda) = \beta^2 - 4\alpha\gamma$

$$\Delta p(\lambda) = (b-\alpha)^2 - 4 \cdot 1(-\alpha b)$$
$$\Delta p(\lambda) = (b-\alpha)^2 + 4\alpha b$$
$$\Delta p(\lambda) = (b+\alpha)^2 > 0$$

So the roots are $\lambda_{1,2} = -\frac{\beta \pm \sqrt{\Delta}}{2\alpha}$

$$\lambda_{1,2} = -(b-\alpha) \pm \sqrt{(\alpha+b)^2}$$

$$\lambda_{1,2} = \frac{\alpha - b + \alpha + b}{2} = \alpha$$

$$\frac{\alpha - b - \alpha - b}{2} = -b$$

now we are able to find the eigen vectors of the coefficient matrix

$$Lv = \lambda v \Rightarrow \begin{bmatrix} \alpha & 0 \\ 0 & -b \end{bmatrix} \begin{bmatrix} u_e \\ u_j \end{bmatrix} = \lambda \begin{bmatrix} u_e \\ u_j \end{bmatrix} \quad (3)$$

eq yields

$$\alpha u_e = \lambda u_j \text{ and } -b u_j = \lambda u_e \; (4)$$

by construction eq (4) are linearly dependent and we can find the eigenvectors from either one

thus, by letting

$$v_k = \begin{bmatrix} u_e^k \\ u_j^k \end{bmatrix} = \begin{bmatrix} u_e^k \\ \frac{\lambda u_e^k}{-b} \end{bmatrix} = u_r^k \begin{bmatrix} 1 \\ \frac{\lambda}{-b} \end{bmatrix}, k \in \{1,2\}$$

setting $u_x^k = 1$, we finally get that

$$v_k = \begin{bmatrix} 1 \\ -\frac{\lambda}{b} \end{bmatrix}$$

thus the first eigen vector will be given as

$$v_1 = \begin{bmatrix} 1 \\ -\frac{\lambda_1}{b} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{\alpha}{b} \end{bmatrix}$$

likewise the second eigenvector $v_2 = \begin{bmatrix} 1 \\ -\frac{\lambda_2}{b} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

the stability of the fixed point will depend on the sign of the eigen values
given that $\lambda_1 = \alpha > 0$ and $\lambda_2 = -b < 0$ $(b > 0)$
then the stability is indeterminate (saddle point)
(given that eigenvalues are both real numbers)

ii) $t=0$ $\begin{cases} R(0) = R_0 \\ J(0) = J_0 \end{cases}$

$\dot{R}(t) = \alpha J(t) \Rightarrow \int \dot{R}(t)\, dt = \int \alpha J(t) dt$

$\int \dot{R}(t)\, dt = \alpha \int J(t) dt \Rightarrow R(t) = \alpha \int J(t)\, dt + c_1$ (4)

Similarly $\int \dot{J}(t)\, dt = \int -b R(t)\, dt$

$\Rightarrow J(t) dt = -b \int R(t) dt + c_2$ (5)

we know that $R(0) = R_0$ and $J(0) = J_0$

So (4) $\Rightarrow R(0) = \alpha \int J(0) dt + c_1 \Rightarrow R_0 = \alpha \int J_0\, dt + c_1$

$\Rightarrow c_1 = R_0 - \alpha J_0$

(5) $\Rightarrow J(0) = -b \int R(0) dt + c_2 \Rightarrow J_0 = -b \int R_0 dt + c_2$

$\Rightarrow c_2 = J_0 + b R_0$

So (4) $\Rightarrow R(t) = \alpha \int J(t)\, dt + R_0 - \alpha J_0$

(5) $\Rightarrow J(t) = -b \int R(t) dt + J_0 + b R_0$

ii) in order to achieve simultaneously love, it must be $R(t) > 0$ and $J(t) > 0$

So we need to find the conditions under which both variables are positive

the behavior of the system will depend on the initial conditions $R(0)$ and $J(0)$, we can analyze the stability of the system and the behaviour of the solutions over time

→ we know that the system has one saddle point from (i) and we also know that for the initial conditions $t=0$, $R(0)=R_0$ $J(0)=J_0$ the solutions are

$$R(t) = a\int J(t)\,dt + R_0 - aJ_0$$
$$J(t) = -b\int R(t)\,dt + J_0 + bR_0$$

we need to determine the time intervals and calculate the total time where both $R(t)$ and $J(t)$ are positive

the we can compute the percentage

- for this purpose I created a matlab program which does the above procedure for a certain set of initial conditions, using ode45 (a matlab solver for diff eq which I found on an online video)

in the next page I'll summarize what is the program doing in love_system.m a class is created which contains the static function love-or-hate(y,a,b) used to calculate the derivatives of R, J and return a column vector with the values dRt and dJt

in ex2_iii.m, after we define the time interval, parameters and set initial conditions we solve the diff eq system by using ode45, a matlab function for solving ordinary diff eq., which takes as an argument the love_or_hate function (as an anonymous one) as well as the time interval and the initial conditions

its output are two variable t and y, where t is a column vector containing the time points and y is a matrix where each column represent the values of the variables R and J at a specific time

the y argument inside love_or_hate is returned from ode45

then we iterate through the values R and J during the whole time interval to find the positive_interval (love) and calculate the percentage

specifically the y(1) and y(2) are the initial values and are extracted from the state vector y which is returned from the ode 45 solver (internally)