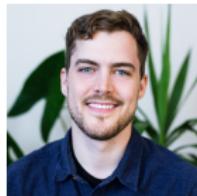


The latent variable proximal point algorithm for variational problems with inequality constraints

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Thomas Surowiec



29 January 2026

Numerical Analysis Group Internal Seminar

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Many phenomena are modelled by pointwise constrained infinite-dimensional optimization problems.

Unconstrained infinite-dimensional optimization

$$\min_u J(u) \Leftrightarrow \text{solve } J'(u) = 0 \text{ (a PDE).}$$



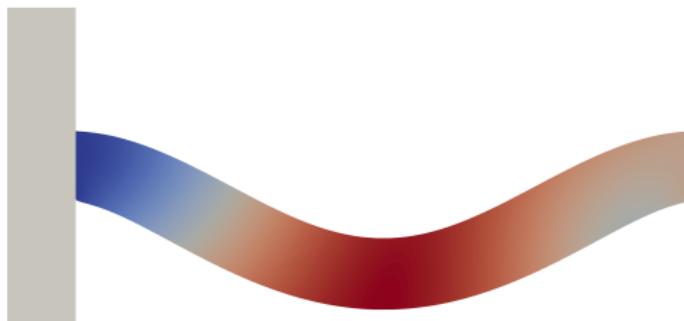
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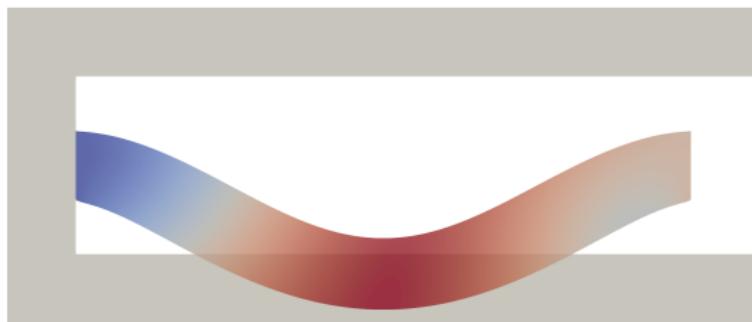
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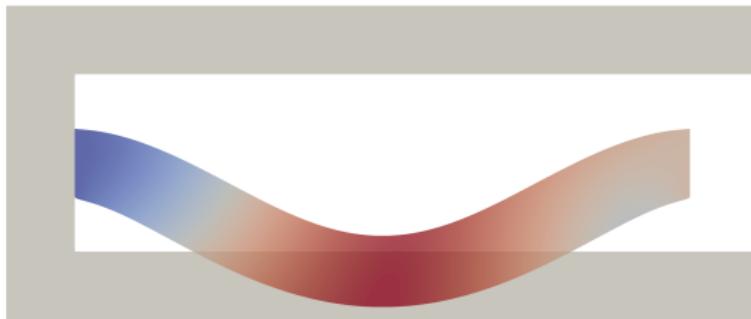
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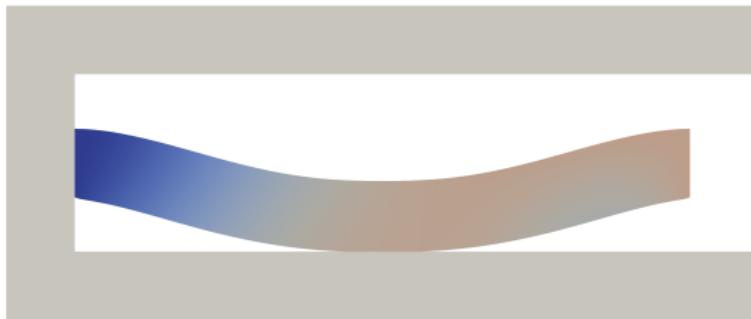
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$$\min_{u \in U} J(u) \text{ subject to } (Bu)(x) \in C(x) \text{ for almost every } x \in \Omega.$$

Examples

- (Signorini.) Find $u : \Omega \rightarrow \mathbb{R}^d$,

$$\min_{u \in H_{\tilde{\sigma}}^1(\Omega)^d} \int_{\Omega} \frac{1}{2} (\mathbf{C}\epsilon(u)) : \epsilon(u) - f \cdot u \, dx \text{ subject to } u \cdot \tilde{n} \geq 0 \text{ on } \Gamma_T.$$

- (Obstacle problem.) Find $u : \Omega \rightarrow \mathbb{R}$,

$$\min_{u \in H_0^1(\Omega)} \int_{\Omega} \frac{1}{2} |\nabla u|^2 - fu \, dx \text{ subject to } u(x) \leq \varphi(x).$$

- (Elastic-plastic torsion.) Find $u : \Omega \rightarrow \mathbb{R}$,

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Solver wish-list

- Versatile for a number of different problems, e.g. $B = \text{Id}, \nabla, \nabla^2, J$ nonconvex etc.
- Flexible discretization requirements.
- Global and robust convergence including discretization-independent iteration counts.
- Minimal numerical ill-conditioning.
- Easy implementation.

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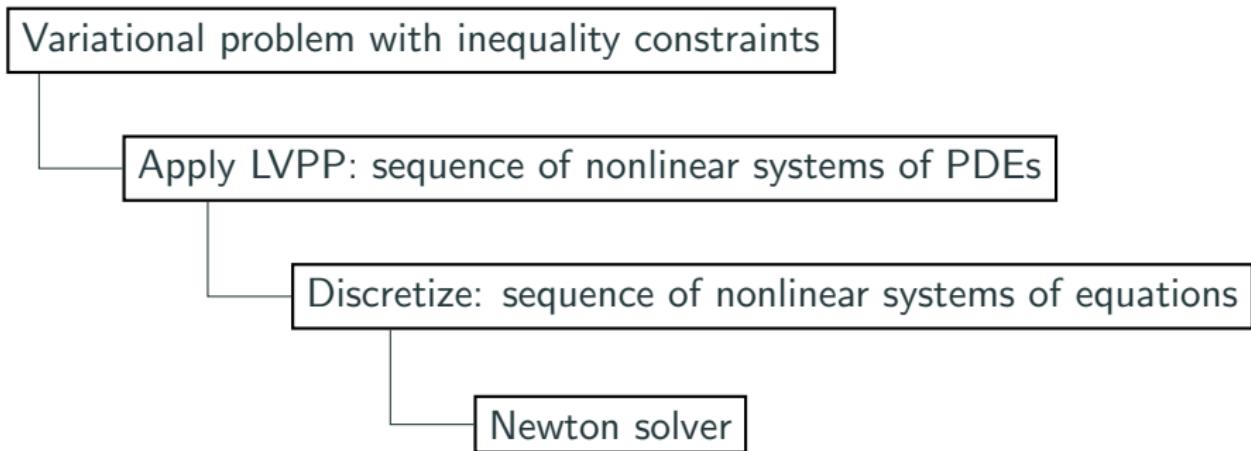
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Latent variable proximal point (LVPP) blueprint



Deriving the LVPP algorithm

$$\min_{u \in U} J(u) \text{ subject to } u \in K := \{v : (Bv)(x) \in C(x) \text{ for a.e. } x \in \Omega\}.$$

Bregman proximal point

$$\min_{u \in H_0^1(\Omega)} \int_{\Omega} \frac{1}{2} |\nabla u|^2 - fu \, dx \text{ subject to } u \leq \varphi \text{ a.e. in } \Omega.$$

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Bregman proximal point

First regularize the optimization problem via a *Bregman divergence*:

$$\min_{u \in U} J(u) + \frac{1}{\alpha} \int_{\Omega_d} R(Bu) - R(Bu^{k-1}) - \nabla R(Bu^{k-1})(Bu - Bu^{k-1}) d\mathcal{H}_d$$

$$R(u) = (\varphi - u) \log(\varphi - u) - (\varphi - u).$$

$$\min_{u \in H_0^1(\Omega)} \int_{\Omega} \frac{1}{2} |\nabla u|^2 - fu dx + \frac{1}{\alpha} \int_{\Omega} R(u) - R(u^{k-1}) - \nabla R(u^{k-1})(u - u^{k-1}) dx$$

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At each iteration, the (classical) Bregman proximal point algorithm seeks $u^k \in K$ satisfying the *smooth* PDE:

$$\alpha_k \langle J'(u^k), v \rangle + \langle \nabla R(Bu^k) - \nabla R(Bu^{k-1}), Bv \rangle = 0 \quad \forall v \in U. \quad (\text{BPP})$$

$$-\alpha_k \Delta u^k - \log(\varphi - u^k) = \alpha_k f - \log(\varphi - u^{k-1}).$$

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Numerically solving Eq. (BPP) via Newton's method performs poorly.

$$-\alpha_k \Delta u^k - \log(\varphi - u^k) = \alpha_k f - \log(\varphi - u^{k-1}).$$

The LVPP algorithm

Introduce a latent variable $\psi = \nabla R(Bu)$ and reformulate the primal equation Eq. (BPP) as a saddle point system.

The LVPP subproblem

Given ψ^{k-1} , for $k = 1, 2, \dots$, we seek (u^k, ψ^k) satisfying

$$\begin{aligned}\alpha_k \langle J'(u^k), v \rangle + \langle \psi^k, Bv \rangle &= \langle \psi^{k-1}, Bv \rangle \quad \forall v \in U, \\ Bu^k - (\nabla R)^{-1}(\psi^k) &= 0 \text{ a.e.,}\end{aligned}$$

- Pick proximal parameters α_k such that $\sum_{j=1}^k \alpha_j \rightarrow \infty$.
- Pick pointwise operator $(\nabla R)^{-1}$ such that $\nabla R(Bu)(x) \rightarrow \infty$ as $Bu(x) \rightarrow \partial C(x)$.

Generates two distinct approximations for Bu : Bu^k and $(\nabla R)^{-1}(\psi^k)$ (always feasible even after discretization).

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BPP: given u^{k-1} find $u^k \in H_0^1(\Omega)$ satisfying $\forall v \in H_0^1(\Omega)$,

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LVPP: given $\psi^{k-1} \in L^\infty(\Omega)$ find (u^k, ψ^k) satisfying,

$\forall (v, q) \in H_0^1(\Omega) \times L^\infty(\Omega)$,

$$\alpha_k(\nabla u^k, \nabla v) + (\psi^k, v) = \alpha_k(f, v) + (\psi^{k-1}, v),$$

$$(u^k, q) + (e^{-\psi^k}, q) = (\varphi, q).$$

Theorem (B. Keith, T. Surowiec, FoCM, 2024)

Suppose that Ω is an open, bounded and Lipschitz domain, $f \in L^\infty(\Omega)$ and $\varphi \in \{\phi \in H^1(\Omega) \cap C(\bar{\Omega}) : \Delta\phi \in L^\infty(\Omega)\}$, then

$$\|u^* - u^k\|_{H^1(\Omega)} \lesssim \left(\sum_{j=1}^k \alpha_j \right)^{-1/2}.$$

Note that $u^k \rightarrow u^*$ in $H^1(\Omega)$ even if $\alpha_k = 1$ for all $k \in \mathbb{N}$.

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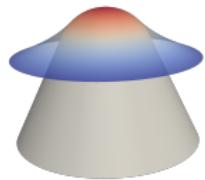
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Choices for $(\nabla R)^{-1}$

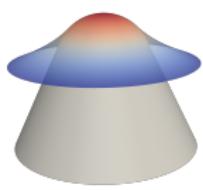
Feasible set K	B	$(\nabla R)^{-1}(\psi)$
$\{u \geq \phi\}$	id	$\phi + \exp \psi$
$\{\phi_1 \leq u \leq \phi_2\}$	id	$\frac{\phi_1 + \phi_2 \exp \psi}{1 + \exp \psi}$
$\{\text{tr } u \geq \phi\}$	tr	$\phi + \exp \psi$
$\{(\text{tr } u) \cdot n \leq \phi\}$	$\text{tr}(\cdot) \cdot n$	$\phi - \exp(-\psi)$
$\{ \nabla u \leq \phi\}$	∇	$\frac{\phi \psi}{\sqrt{1 + \psi ^2}}$
$\{u \geq 0, \sum_i u_i = 1\}$	id	$\frac{\exp \psi}{\sum_i \exp \psi_i}$
$\{\det(\nabla^2 u) \geq 0\}$	∇^2	$\exp \psi$

Examples

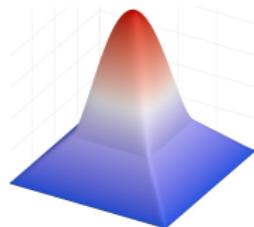


Obstacle, $u \leq \varphi$.

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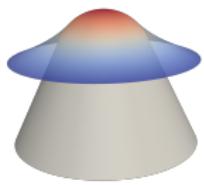


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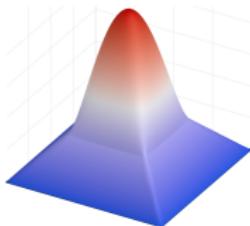


Gradient-type,
 $|\nabla u| \leq \varphi$.

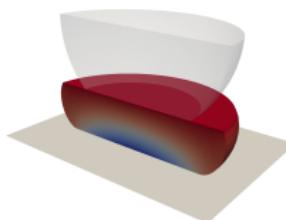
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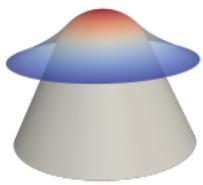


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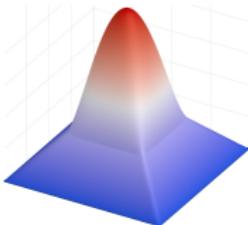


Signorini,
 $(u \cdot n)|_{\Gamma} \geq 0$.

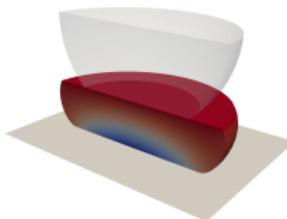
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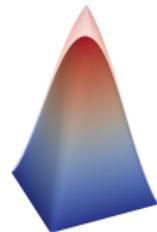
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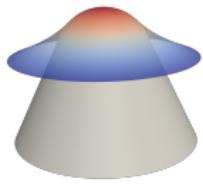


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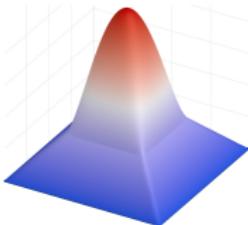


Thermoforming,
 $u \leq \varphi(u)$.

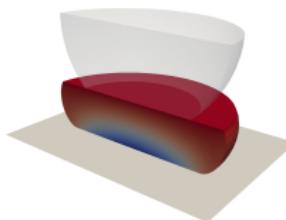
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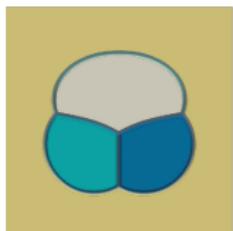
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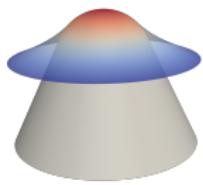


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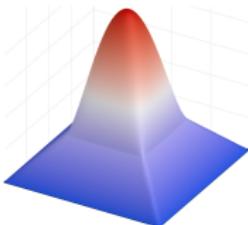


Cahn–Hilliard,
 $u_i \geq 0$, $\sum_i u_i = 1$.

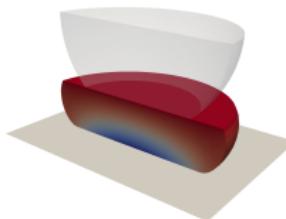
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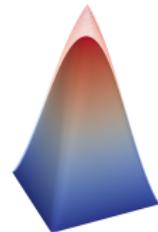
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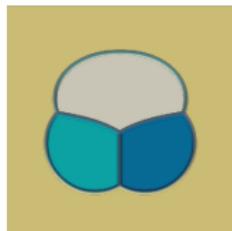
Gradient-type,
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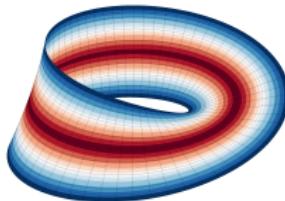
Signorini,
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Thermoforming,
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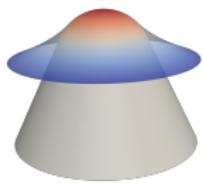


Cahn–Hilliard,
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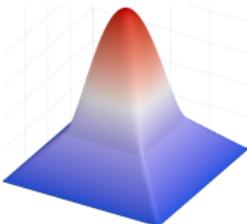


Eikonal, $|\nabla u| = 1$.

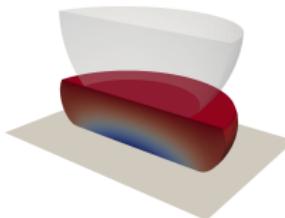
Examples



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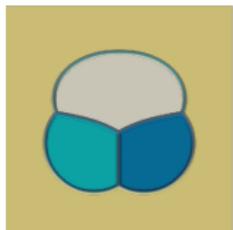
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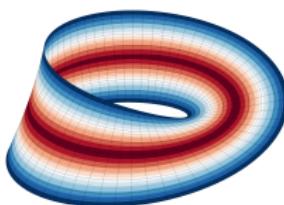
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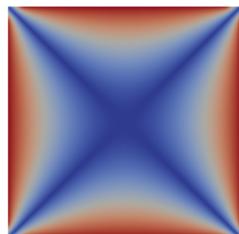
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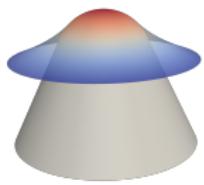


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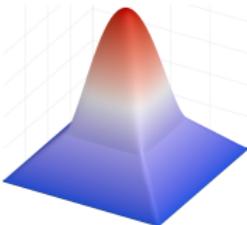


Nematic liquid crystals,
 $|\lambda_i(Q)| \leq 1/2$.

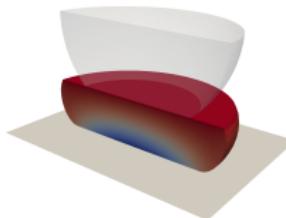
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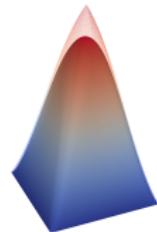
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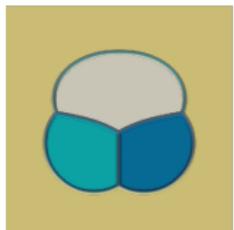
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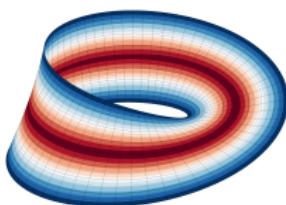
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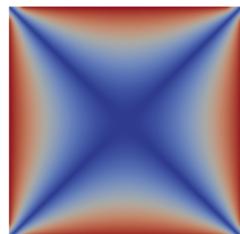
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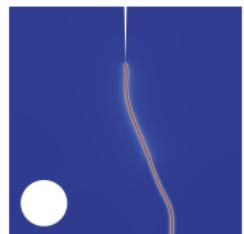
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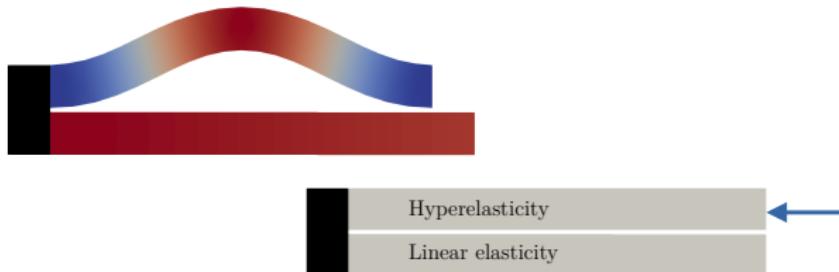


Contact problems



Contact problems

Solution #1



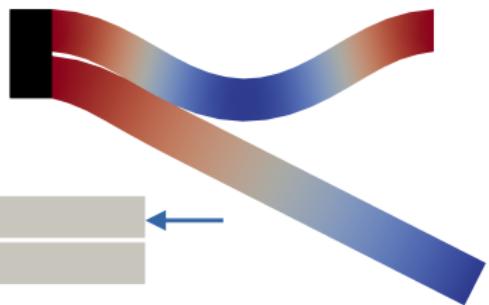
Multiple solutions to a generalized Signorini problem.

Contact problems

Solution #1



Solution #2



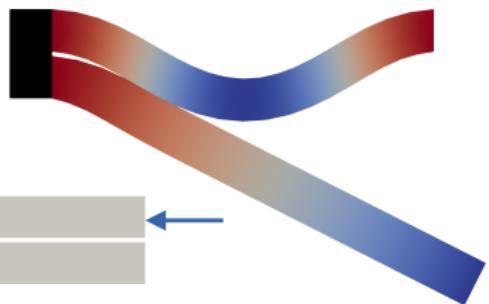
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Solution #3



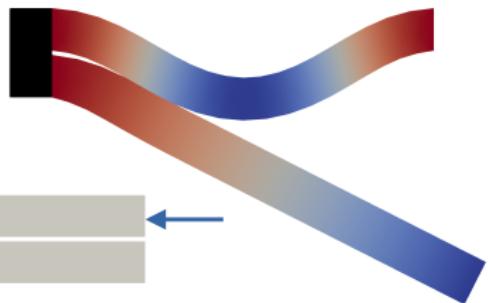
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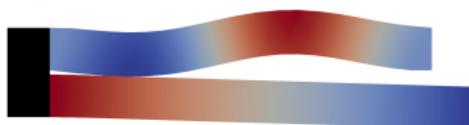
Solution #2



Solution #3



Solution #4

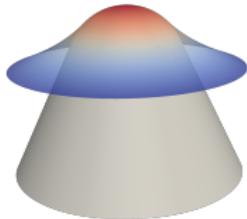


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Obstacle problem solver comparisons

Method	Degree $p = 1$			Degree $p = 2$		
	h	$h/2$	$h/4$	h	$h/2$	$h/4$
LVPP	15	13	12	15	16	12
Active Set (PETSc)	11	16	25			
Trust-Region (Galahad)	6	12	19			Not bound preserving
Interior Point (IPOPT)	9	9	8			
IPOPT without Hessian	90	260	500			

Number of linear system solves for popular solvers using various mesh sizes h .

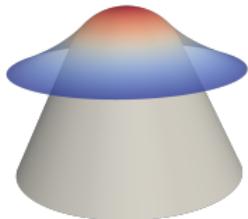


Obstacle ϕ (grey) and
membrane u
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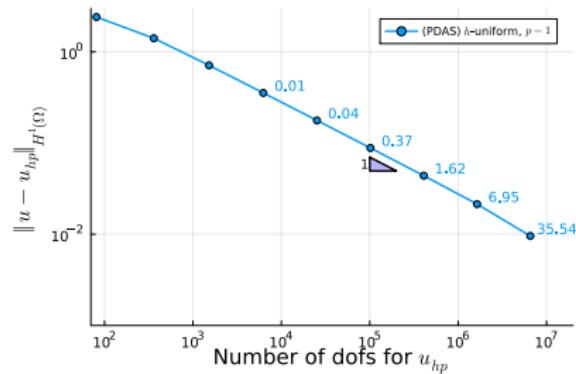
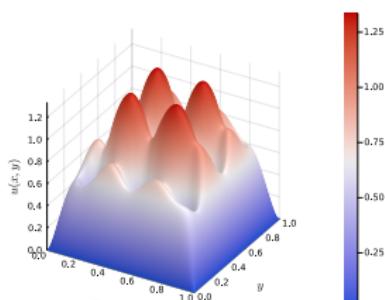
Mesh size h	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
Finite Difference	10	15	13	15	16	16
Degree p	8	16	24	32	40	48
Spectral Method	16	17	16	16	16	15

Number of linear system solves for the proximal finite difference and spectral methods.

High-order FEM discretizations

Observations

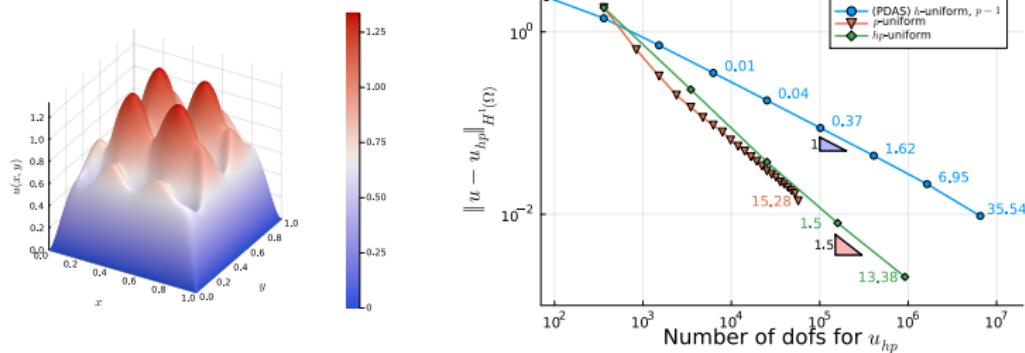
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Thermoforming solver comparison

The thermoforming quasi-variational inequality seeks $u : \Omega \rightarrow \mathbb{R}$ minimizing

$$\min_{u \in H_0^1(\Omega)} \int_{\Omega} \frac{1}{2} |\nabla u|^2 - fu \, dx \text{ subject to } u \leq \varphi(T) := \Phi_0 + \xi T, \quad (1a)$$

where Φ_0 and ξ are given and T satisfies

$$-\Delta T + \beta T = g(\Phi_0 + \xi T - u), \quad \partial_{\nu} T = 0 \text{ on } \partial\Omega. \quad (1b)$$

LVPP subproblem

Given ψ^{k-1} , we seek (u^k, T^k, ψ^k) satisfying for all
 $(v, q, w) \in H_0^1(\Omega) \times L^\infty(\Omega) \times H^1(\Omega)$

$$(\nabla T^k, \nabla q) + \beta(T^k, q) = (g(e^{-\psi^k}), q), \quad (2a)$$

$$\alpha_k(\nabla u^k, \nabla v) + (\psi^k, v) = \alpha_k(f, v) + (\psi^{k-1}, v), \quad (2b)$$

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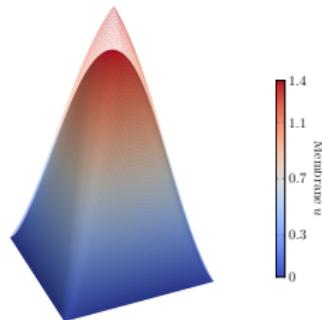
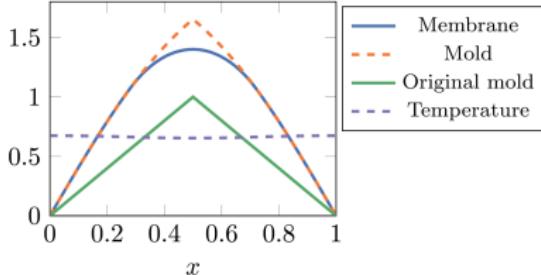
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Thermoforming solver comparison



Solver	Outer loop	Linear system solves	Run time (s)
LVPP	13	20	61.70
Moreau–Yosida Penalty	14	51	78.01
Semismooth Active Set	7	236	112.60
Fixed Point	164	8493	3633.72

The performance of 4 solvers, terminating when $\|u^k - u^{k-1}\|_{H^1(\Omega)} \leq 10^{-5}$.

Conclusions

- Many pointwise constraints can be effectively handled by LVPP resulting in a nonlinear system of smooth PDEs.
- Observed discretization-independent number of linear system solves.
- LVPP has a simple mechanism for enforcing pointwise constraints on the discrete level (without the need for a projection).
- Ease of implementation — the algorithm reduces to the repeated solve of a smooth nonlinear system of PDEs.
- Robust numerical performance since convergence occurs even if α_k is kept small.
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Thank you for listening!

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