

Déduction naturelle

OPTION INFORMATIQUE - TP n° 4.5 - Olivier Reynet

À la fin de ce chapitre, je sais :

- lire un séquent
- décrire les règles d'introduction et d'élimination
- justifier les principaux raisonnements de la logique classique
- construire un arbre de preuve démontrant une formule simple

A Utilisation des règles d'inférence

Prouver les séquents suivants :

A1. $\vdash p \rightarrow p$

Solution :

$$\frac{\overline{p \vdash p}^{\text{ax}}}{\vdash p \rightarrow p} \rightarrow_i$$

A2. $p, \neg p \vdash \perp$

Solution :

$$\frac{\overline{p, \neg p \vdash p}^{\text{ax}} \quad \overline{p, \neg p \vdash \neg p}^{\text{ax}}}{p, \neg p \vdash \perp} \neg_e$$

A3. $p, q \vdash p \wedge q$

Solution :

$$\frac{\overline{p, q \vdash p}^{\text{ax}} \quad \overline{p, q \vdash q}^{\text{ax}}}{p, q \vdash p \wedge q} \wedge_i$$

A4. $p \wedge q \vdash q \wedge p$

Solution :

$$\frac{\frac{\overline{p \wedge q \vdash p \wedge q} \text{ ax}}{p \wedge q \vdash q} \wedge_e \quad \frac{\overline{p \wedge q \vdash p \wedge q} \text{ ax}}{p \wedge q \vdash p} \wedge_e}{p \wedge q \vdash q \wedge p} \wedge_i$$

A5. $p \vee q \vdash q \vee p$ **Solution :**

$$\frac{\overline{p \vee q \vdash p \vee q} \text{ ax} \quad \frac{\overline{p \vee q, p \vdash p} \text{ ax}}{p \vee q, p \vdash q \vee p} \vee_i \quad \frac{\overline{p \vee q, q \vdash q} \text{ ax}}{p \vee q, q \vdash q \vee p} \vee_i}{p \vee q \vdash q \vee p} \vee_e$$

A6. $q \vdash p \rightarrow q$ **Solution :**

$$\frac{\overline{q, p \vdash q} \text{ ax}}{q \vdash p \rightarrow q} \rightarrow_i$$

A7. $p \wedge q \vdash p \rightarrow q$ **Solution :**

$$\frac{\frac{\overline{p \wedge q, p \vdash p \wedge q} \text{ ax}}{p \wedge q, p \vdash q} \wedge_e}{p \wedge q \vdash p \rightarrow q} \rightarrow_i$$

A8. $p, q \wedge r \vdash p \wedge q$ **Solution :**

$$\frac{\overline{p, q \wedge r \vdash p} \text{ ax} \quad \frac{\overline{p, q \wedge r \vdash q \wedge r} \text{ ax}}{p, q \wedge r \vdash q} \wedge_e}{p, q \wedge r \vdash p \wedge q} \wedge_i$$

A9. $p \wedge q, r \wedge s \vdash p \wedge s$ **Solution :**

$$\frac{\frac{\overline{p \wedge q, r \wedge s \vdash p \wedge q} \text{ ax}}{p \wedge q, r \wedge s \vdash p} \wedge_e \quad \frac{\overline{p \wedge q, r \wedge s \vdash r \wedge s} \text{ ax}}{p \wedge q, r \wedge s \vdash s} \wedge_e}{p \wedge q, r \wedge s \vdash p \wedge s} \wedge_i$$

A10. $a \rightarrow \neg a \vdash \neg a$ **Solution :**

$$\frac{\frac{\frac{a \rightarrow \neg a, a \vdash a \rightarrow \neg a}{a \rightarrow \neg a, a \vdash \neg a} \text{ax}}{a \rightarrow \neg a, a \vdash \neg a} \rightarrow_e \quad \frac{\frac{a \rightarrow \neg a, a \vdash a}{a \rightarrow \neg a, a \vdash a} \text{ax}}{a \rightarrow \neg a, a \vdash a} \rightarrow_e}{\frac{a \rightarrow \neg a, a \vdash \perp}{a \rightarrow \neg a \vdash \neg a} \neg_i} \neg_e$$

B Preuves intermédiaires

Prouver les séquents suivants :

B1. $p \rightarrow q \vdash \neg q \rightarrow \neg p$ **Solution :** On pose $\Gamma = p \rightarrow q, \neg q, p$.

$$\frac{\frac{\frac{\Gamma \vdash p}{\Gamma \vdash p} \text{ax} \quad \frac{\Gamma \vdash p \rightarrow q}{\Gamma \vdash q} \rightarrow_e}{\Gamma \vdash \neg q} \neg_e \quad \frac{\Gamma \vdash \perp}{p \rightarrow q, \neg q \vdash \neg p} \neg_i}{\frac{p \rightarrow q, \neg q \vdash \neg p}{p \rightarrow q \vdash \neg q \rightarrow \neg p} \rightarrow_i} \rightarrow_i$$

B2. $\neg a \vee b \vdash a \rightarrow b$ **Solution :**

$$\frac{\frac{\frac{\neg a \vee b, a \vdash \neg a \vee b}{\neg a \vee b, a \vdash \neg a \vee b} \text{ax} \quad \frac{\frac{\neg a \vee b, a \vdash \neg a}{\neg a \vee b, a \vdash \neg a} \text{ax}}{\frac{\neg a \vee b, a \vdash \neg a \vee b}{\neg a \vee b, a \vdash \neg a \vee b} \neg_e} \neg_e \quad \frac{\frac{\neg a \vee b, a \vdash \neg a \vee b}{\neg a \vee b, a \vdash \neg a \vee b} \neg_e \quad \frac{\neg a \vee b, a \vdash \perp}{\neg a \vee b, a \vdash \neg a \vee b} \perp_e}{\frac{\neg a \vee b, a \vdash b}{\neg a \vee b \vdash a \rightarrow b} \rightarrow_i} \vee_e$$

B3. $a \rightarrow b \vdash \neg a \vee b$ **Solution :**

$$\frac{\frac{a \rightarrow b \vdash \neg a \vee a}{a \rightarrow b \vdash \neg a \vee a} \text{te} \quad \frac{\frac{a \rightarrow b, \neg a \vdash \neg a}{a \rightarrow b, \neg a \vdash \neg a \vee b} \vee_i \quad \frac{\frac{a \rightarrow b, a \vdash a \rightarrow b}{a \rightarrow b, a \vdash a} \text{ax} \quad \frac{a \rightarrow b, a \vdash a}{a \rightarrow b, a \vdash a} \rightarrow_e}{\frac{a \rightarrow b, a \vdash b}{a \rightarrow b, a \vdash \neg a \vee b} \vee_i} \vee_e$$

B4. $a \rightarrow (b \rightarrow c) \vdash (a \wedge b) \rightarrow c$ **Solution :**

$$\begin{array}{c}
\frac{\frac{a \rightarrow (b \rightarrow c), a \wedge b \vdash a \wedge b}{a \rightarrow (b \rightarrow c), a \wedge b \vdash b} \text{ax} \quad \frac{\frac{a \rightarrow (b \rightarrow c), a \wedge b \vdash a \wedge b}{a \rightarrow (b \rightarrow c), a \wedge b \vdash a} \text{ax} \quad \frac{a \rightarrow (b \rightarrow c), a \wedge b \vdash a \rightarrow (b \rightarrow c)}{a \rightarrow (b \rightarrow c), a \wedge b \vdash b \rightarrow c} \text{ax}}{a \rightarrow (b \rightarrow c), a \wedge b \vdash b \rightarrow c} \wedge_e \quad \rightarrow_e \\
\frac{a \rightarrow (b \rightarrow c), a \wedge b \vdash b \rightarrow c}{a \rightarrow (b \rightarrow c), a \wedge b \vdash c} \rightarrow_e \\
\frac{a \rightarrow (b \rightarrow c), a \wedge b \vdash c}{a \rightarrow (b \rightarrow c) \vdash (a \wedge b) \rightarrow c} \rightarrow_i
\end{array}$$

B5. $(a \wedge b) \rightarrow c \vdash a \rightarrow (b \rightarrow c)$ **Solution :** On pose $\Gamma = (a \wedge b) \rightarrow c, a, b$.

$$\begin{array}{c}
\frac{\frac{\Gamma \vdash a}{\Gamma \vdash a} \text{ax} \quad \frac{\Gamma \vdash b}{\Gamma \vdash b} \text{ax}}{\Gamma \vdash a \wedge b} \wedge_i \quad \frac{\Gamma \vdash (a \wedge b) \rightarrow c}{\Gamma \vdash (a \wedge b) \rightarrow c} \text{ax} \\
\frac{\Gamma \vdash a \wedge b \quad \Gamma \vdash (a \wedge b) \rightarrow c}{(a \wedge b) \rightarrow c, a, b \vdash c} \rightarrow_e \\
\frac{(a \wedge b) \rightarrow c, a, b \vdash c}{(a \wedge b) \rightarrow c, a \vdash b \rightarrow c} \rightarrow_i \\
\frac{(a \wedge b) \rightarrow c, a \vdash b \rightarrow c}{(a \wedge b) \rightarrow c \vdash a \rightarrow (b \rightarrow c)} \rightarrow_i
\end{array}$$

B6. $a \rightarrow (b \rightarrow c), b \rightarrow a \vdash b \rightarrow c$ **Solution :** On pose $\Gamma = a \rightarrow (b \rightarrow c), b \rightarrow a, b$.

$$\begin{array}{c}
\frac{\frac{\Gamma \vdash b}{\Gamma \vdash b} \text{ax} \quad \frac{\Gamma \vdash b \rightarrow a}{\Gamma \vdash b \rightarrow a} \text{ax}}{\Gamma \vdash a} \rightarrow_e \quad \frac{\Gamma \vdash a \rightarrow (b \rightarrow c)}{\Gamma \vdash a \rightarrow (b \rightarrow c)} \text{ax} \\
\frac{\Gamma \vdash a \quad \Gamma \vdash a \rightarrow (b \rightarrow c)}{\Gamma \vdash b \rightarrow c} \rightarrow_e \quad \frac{\Gamma \vdash b}{\Gamma \vdash b} \text{ax} \\
\frac{\Gamma \vdash b \rightarrow c \quad \Gamma \vdash b}{\Gamma \vdash c} \rightarrow_e \\
\frac{\Gamma \vdash c}{a \rightarrow (b \rightarrow c), b \rightarrow a \vdash b \rightarrow c} \rightarrow_i
\end{array}$$

B7. $p \rightarrow (q \vee r), \neg q, \neg r \vdash \neg p$ **Solution :** On pose $\Gamma = p \rightarrow (q \vee r), \neg q, \neg r$.

$$\begin{array}{c}
\frac{\frac{\Gamma, p \vdash p \rightarrow (q \vee r)}{\Gamma, p \vdash q \vee r} \text{ax} \quad \frac{\Gamma, p \vdash p}{\Gamma, p \vdash p} \text{ax}}{\Gamma, p \vdash q \vee r} \rightarrow_e \quad \frac{\Gamma, p, q \vdash q}{\Gamma, p, q \vdash \perp} \text{ax} \quad \frac{\Gamma, p, q \vdash \neg q}{\Gamma, p, q \vdash \perp} \text{ax} \quad \frac{\Gamma, p, r \vdash r}{\Gamma, p, r \vdash \perp} \text{ax} \quad \frac{\Gamma, p, r \vdash \neg r}{\Gamma, p, r \vdash \perp} \text{ax}}{\Gamma, p \vdash \perp} \vee_e \quad \neg_e \\
\frac{\Gamma, p \vdash \perp}{p \rightarrow (q \vee r), \neg q, \neg r \vdash \neg p} \neg_i
\end{array}$$

B8. $p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$

Solution: On pose $\Gamma = p \rightarrow (q \rightarrow r), p, \neg r$.

$$\frac{\frac{\frac{\Gamma, q \vdash p}{\Gamma, q \vdash p} \text{ax} \quad \frac{\Gamma, q \vdash p \rightarrow (q \rightarrow r)}{\Gamma, q \vdash q \rightarrow r} \rightarrow_e}{\Gamma, q \vdash q} \text{ax} \quad \frac{\Gamma, q \vdash q}{\Gamma, q \vdash r} \rightarrow_e}{\Gamma, q \vdash \perp} \neg_e \quad \frac{\Gamma, q \vdash \perp}{p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q} \neg_i$$

C Preuves plus complexes

Prouver les séquents suivants :

$$\text{C1. } q \rightarrow r, \neg q \rightarrow \neg p \vdash p \rightarrow r$$

Solution : On pose $\Gamma = q \rightarrow r, \neg q \rightarrow \neg p, p$.

$$\frac{\frac{\frac{\Gamma \vdash q \neg q}{\Gamma \vdash q \neg q} \text{te} \quad \frac{\frac{\Gamma, q \vdash q \rightarrow r}{\Gamma, q \vdash r} \text{ax} \quad \frac{\Gamma, q \vdash q}{\Gamma, q \vdash q} \text{ax}}{\Gamma, q \vdash r} \rightarrow_e \quad \frac{\frac{\frac{\Gamma, \neg q \vdash \neg q \rightarrow \neg p}{\Gamma, \neg q \vdash \neg p} \text{ax} \quad \frac{\Gamma, \neg q \vdash \neg q}{\Gamma, \neg q \vdash \neg q} \text{ax}}{\Gamma, \neg q \vdash \neg p} \rightarrow_e \quad \frac{\Gamma, \neg q \vdash \perp}{\Gamma, \neg q \vdash r} \perp_e}{\Gamma \vdash r} \vee_e}{\frac{\Gamma \vdash r}{q \rightarrow r, \neg q \rightarrow \neg p \vdash p \rightarrow r} \rightarrow_i} \quad \frac{\Gamma, \neg q \vdash p}{\Gamma, \neg q \vdash p} \text{ax} \neg_e$$

C2. $(p \wedge q) \rightarrow r \vdash (p \rightarrow r) \vee (q \rightarrow r)$

Solution : On pose $\Gamma = (p \wedge q) \rightarrow r$ et $\psi = (p \rightarrow r) \vee (q \rightarrow r)$

$$\begin{array}{c}
\frac{\Gamma \vdash p \supset p}{\Gamma \vdash p \supset p} \text{te} \quad \frac{\frac{\frac{\Gamma, p, q \vdash \Gamma}{\Gamma, p, q \vdash \Gamma} \text{ax} \quad \frac{\frac{\Gamma, p, q \vdash p}{\Gamma, p, q \vdash p} \text{ax} \quad \frac{\Gamma, p, q \vdash q}{\Gamma, p, q \vdash q} \text{ax}}{\Gamma, p, q \vdash p \wedge q} \wedge_i \quad \frac{\frac{\frac{\Gamma, \neg p, p \vdash p}{\Gamma, \neg p, p \vdash p} \text{ax} \quad \frac{\Gamma, \neg p, p \vdash \neg p}{\Gamma, \neg p, p \vdash \neg p} \text{ax}}{\Gamma, \neg p, p \vdash \perp} \perp_e}{\Gamma, \neg p, p \vdash r} \rightarrow_i}{\Gamma, \neg p \vdash p \rightarrow r} \rightarrow_i}{\Gamma, \neg p \vdash \psi} \vee_e \\
\frac{\frac{\frac{\Gamma, p, q \vdash r}{\Gamma, p \vdash q \rightarrow r} \rightarrow_i \quad \frac{\Gamma, p \vdash q \rightarrow r}{\Gamma, p \vdash \psi} \vee_i}{\Gamma \vdash p \supset p} \text{te} \quad \frac{\Gamma, p \vdash \psi}{\Gamma \vdash \psi} \vee_e
\end{array}$$