Déduction naturelle

OPTION INFORMATIQUE - TP n° 4.5 - Olivier Reynet

À la fin de ce chapitre, je sais :

- ☞ lire un séquent
- ☞ décrire les règles d'introduction et d'élimination
- ☞ justifier les principaux raisonnements de la logique classique
- ☞ construire un arbre de preuve démontrant une formule simple

A Utilisation des règles d'inférence

Prouver les séquents suivants :

A1. $\vdash p \to p$

Solution :

$$\dfrac{\dfrac{}{p \vdash p}\,\text{ax}}{\vdash p \to p}\,{\to_i}$$

A2. $p, \neg p \vdash \bot$

Solution :

$$\dfrac{\dfrac{}{p, \neg p \vdash p}\,\text{ax} \qquad \dfrac{}{p, \neg p \vdash \neg p}\,\text{ax}}{p, \neg p \vdash \bot}\,{\neg_e}$$

A3. $p, q \vdash p \wedge q$

Solution :

$$\dfrac{\dfrac{}{p, q \vdash p}\,\text{ax} \qquad \dfrac{}{p, q \vdash q}\,\text{ax}}{p, q \vdash p \wedge q}\,{\wedge_i}$$

A4. $p \wedge q \vdash q \wedge p$

Solution :

$$\dfrac{\dfrac{\dfrac{}{p \wedge q \vdash p \wedge q}\,\text{ax}}{p \wedge q \vdash q}\,\wedge_e \quad \dfrac{\dfrac{}{p \wedge q \vdash p \wedge q}\,\text{ax}}{p \wedge q \vdash p}\,\wedge_e}{p \wedge q \vdash q \wedge p}\,\wedge_i$$

A5. $p \vee q \vdash q \vee p$

Solution :

$$\dfrac{\dfrac{}{p \vee q \vdash p \vee q}\,\text{ax} \quad \dfrac{\dfrac{}{p \vee q, p \vdash p}\,\text{ax}}{p \vee q, p \vdash q \vee p}\,\vee_i \quad \dfrac{\dfrac{}{p \vee q, q \vdash q}\,\text{ax}}{p \vee q, q \vdash q \vee p}\,\vee_i}{p \vee q \vdash q \vee p}\,\vee_e$$

A6. $q \vdash p \to q$

Solution :

$$\dfrac{\dfrac{}{q, p \vdash q}\,\text{ax}}{q \vdash p \to q}\,\to_i$$

A7. $p \wedge q \vdash p \to q$

Solution :

$$\dfrac{\dfrac{\dfrac{}{p \wedge q, p \vdash p \wedge q}\,\text{ax}}{p \wedge q, p \vdash q}\,\wedge_e}{p \wedge q \vdash p \to q}\,\to_i$$

A8. $p, q \wedge r \vdash p \wedge q$

Solution :

$$\dfrac{\dfrac{}{p, q \wedge r \vdash p}\,\text{ax} \quad \dfrac{\dfrac{}{p, q \wedge r \vdash q \wedge r}\,\text{ax}}{p, q \wedge r \vdash q}\,\wedge_e}{p, q \wedge r \vdash p \wedge q}\,\wedge_i$$

A9. $p \wedge q, r \wedge s \vdash p \wedge s$

Solution :

$$\dfrac{\dfrac{\dfrac{}{p \wedge q, r \wedge s \vdash p \wedge q}\,\text{ax}}{p \wedge q, r \wedge s \vdash p}\,\wedge_e \quad \dfrac{\dfrac{}{p \wedge q, r \wedge s \vdash r \wedge s}\,\text{ax}}{p \wedge q, r \wedge s \vdash s}\,\wedge_e}{p \wedge q, r \wedge s \vdash p \wedge s}\,\wedge_i$$

A10. $a \to \neg a \vdash \neg a$

Solution :

$$\dfrac{\dfrac{\dfrac{}{a \to \neg a, a \vdash a \to \neg a}\,\text{ax} \quad \dfrac{}{a \to \neg a, a \vdash a}\,\text{ax}}{a \to \neg a, a \vdash \neg a}\,{\to_e} \quad \dfrac{}{a \to \neg a, a \vdash a}\,\text{ax}}{a \to \neg a, a \vdash \bot}\,{\neg_e}}{a \to \neg a \vdash \neg a}\,{\neg_i}$$

B Preuves intermédiaires

Prouver les séquents suivants :

B1. $p \to q \vdash \neg q \to \neg p$

Solution : On pose $\Gamma = p \to q, \neg q, p$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p}\,\text{ax} \quad \dfrac{}{\Gamma \vdash p \to q}\,\text{ax}}{\Gamma \vdash q}\,{\to_e} \quad \dfrac{}{\Gamma \vdash \neg q}\,\text{ax}}{\Gamma \vdash \bot}\,{\neg_e}}{p \to q, \neg q \vdash \neg p}\,{\neg_i}}{p \to q \vdash \neg q \to \neg p}\,{\to_i}$$

B2. $\neg a \vee b \vdash a \to b$

Solution :

$$\dfrac{\dfrac{}{\neg a \vee b, a \vdash \neg a \vee b}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{}{\neg a \vee b, a, \neg a \vdash a}\,\text{ax} \quad \dfrac{}{\neg a \vee b, a, \neg a \vdash \neg a}\,\text{ax}}{\neg a \vee b, a, \neg a \vdash \bot}\,{\neg_e}}{\neg a \vee b, a, \neg a \vdash b}\,{\bot_e} \quad \dfrac{}{\neg a \vee b, a, b \vdash b}\,\text{ax}}{\neg a \vee b, a \vdash b}\,{\vee_e}}{\neg a \vee b \vdash a \to b}\,{\to_i}$$

B3. $a \to b \vdash \neg a \vee b$

Solution :

$$\dfrac{\dfrac{}{a \to b \vdash \neg a \vee a}\,\text{te} \quad \dfrac{\dfrac{}{a \to b, \neg a \vdash \neg a}\,\text{ax}}{a \to b, \neg a \vdash \neg a \vee b}\,{\vee_i} \quad \dfrac{\dfrac{\dfrac{}{a \to b, a \vdash a \to b}\,\text{ax} \quad \dfrac{}{a \to b, a \vdash a}\,\text{ax}}{a \to b, a \vdash b}\,{\to_e}}{a \to b, a \vdash \neg a \vee b}\,{\vee_i}}{a \to b \vdash \neg a \vee b}\,{\vee_e}$$

B4. $a \to (b \to c) \vdash (a \wedge b) \to c$

Solution :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \to (b \to c), a \wedge b \vdash a \wedge b}\,\text{ax}}{a \to (b \to c), a \wedge b \vdash b}\,\wedge_e \quad \dfrac{\dfrac{}{a \to (b \to c), a \wedge b \vdash a \wedge b}\,\text{ax}}{a \to (b \to c), a \wedge b \vdash a}\,\wedge_e \quad \dfrac{}{a \to (b \to c), a \wedge b \vdash a \to (b \to c)}\,\text{ax}}{a \to (b \to c), a \wedge b \vdash b \to c}\,\to_e}{a \to (b \to c), a \wedge b \vdash c}\,\to_e}{a \to (b \to c) \vdash (a \wedge b) \to c}\,\to_i$$

B5. $(a \wedge b) \to c \vdash a \to (b \to c)$

Solution : On pose $\Gamma = (a \wedge b) \to c, a, b$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash a}\,\text{ax} \quad \dfrac{}{\Gamma \vdash b}\,\text{ax}}{\Gamma \vdash a \wedge b}\,\wedge_i \quad \dfrac{}{\Gamma \vdash (a \wedge b) \to c}\,\text{ax}}{(a \wedge b) \to c, a, b \vdash c}\,\to_e}{(a \wedge b) \to c, a \vdash b \to c}\,\to_i}{(a \wedge b) \to c \vdash a \to (b \to c)}\,\to_i$$

B6. $a \to (b \to c), b \to a \vdash b \to c$

Solution : On pose $\Gamma = a \to (b \to c), b \to a, b$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash b}\,\text{ax} \quad \dfrac{}{\Gamma \vdash b \to a}\,\text{ax}}{\Gamma \vdash a}\,\to_e \quad \dfrac{}{\Gamma \vdash a \to (b \to c)}\,\text{ax}}{\Gamma \vdash b \to c}\,\to_e \quad \dfrac{}{\Gamma \vdash b}\,\text{ax}}{\Gamma \vdash c}\,\to_e}{a \to (b \to c), b \to a \vdash b \to c}\,\to_i$$

B7. $p \to (q \vee r), \neg q, \neg r \vdash \neg p$

Solution : On pose $\Gamma = p \to (q \vee r), \neg q, \neg r$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, p \vdash p \to (q \vee r)}\,\text{ax} \quad \dfrac{}{\Gamma, p \vdash p}\,\text{ax}}{\Gamma, p \vdash q \vee r}\,\to_e \quad \dfrac{\dfrac{}{\Gamma, p, q \vdash q}\,\text{ax} \quad \dfrac{}{\Gamma, p, q \vdash \neg q}\,\text{ax}}{\Gamma, p, q \vdash \bot}\,\neg_e \quad \dfrac{\dfrac{}{\Gamma, p, r \vdash r}\,\text{ax} \quad \dfrac{}{\Gamma, p, r \vdash \neg r}\,\text{ax}}{\Gamma, p, r \vdash \bot}\,\neg_e}{\Gamma, p \vdash \bot}\,\vee_e}{p \to (q \vee r), \neg q, \neg r \vdash \neg p}\,\neg_i$$

B8. $p \to (q \to r), p, \neg r \vdash \neg q$

Solution : On pose $\Gamma = p \to (q \to r), p, \neg r$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma, q \vdash \neg r}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, q \vdash p}\,\text{ax} \qquad \dfrac{}{\Gamma, q \vdash p \to (q \to r)}\,\text{ax}}{\Gamma, q \vdash q \to r}\,\to_e \qquad \dfrac{}{\Gamma, q \vdash q}\,\text{ax}}{\Gamma, q \vdash r}\,\to_e}{\Gamma, q \vdash \bot}\,\neg_e}{p \to (q \to r), p, \neg r \vdash \neg q}\,\neg_i$$

C Preuves plus complexes

Prouver les séquents suivants :

C1. $q \to r, \neg q \to \neg p \vdash p \to r$

Solution : On pose $\Gamma = q \to r, \neg q \to \neg p, p$.

$$\dfrac{\dfrac{}{\Gamma \vdash q \neg q}\,\text{te} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, q \vdash q \to r}\,\text{ax} \qquad \dfrac{}{\Gamma, q \vdash q}\,\text{ax}}{\Gamma, q \vdash r}\,\to_e \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg q \vdash \neg q \to \neg p}\,\text{ax} \qquad \dfrac{}{\Gamma, \neg q \vdash \neg q}\,\text{ax}}{\Gamma, \neg q \vdash \neg p}\,\to_e \qquad \dfrac{}{\Gamma, \neg q \vdash p}\,\text{ax}}{\Gamma, \neg q \vdash \bot}\,\neg_e}{\Gamma, \neg q \vdash r}\,\bot_e}{\Gamma \vdash r}\,\vee_e}{q \to r, \neg q \to \neg p \vdash p \to r}\,\to_i$$

C2. $(p \wedge q) \to r \vdash (p \to r) \vee (q \to r)$

Solution : On pose $\Gamma = (p \wedge q) \to r$ et $\psi = (p \to r) \vee (q \to r)$

$$\dfrac{\dfrac{}{\Gamma \vdash p \neg p}\,\text{te} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, p, q \vdash \Gamma}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma, p, q \vdash p}\,\text{ax} \qquad \dfrac{}{\Gamma, p, q \vdash q}\,\text{ax}}{\Gamma, p, q \vdash p \wedge q}\,\wedge_i}{\Gamma, p, q \vdash r}\,\to_i}{\Gamma, p \vdash q \to r}\,\vee_i}{\Gamma, p \vdash \psi} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg p, p \vdash p}\,\text{ax} \qquad \dfrac{}{\Gamma, \neg p, p \vdash \neg p}\,\text{ax}}{\Gamma, \neg p, p \vdash \bot}\,\neg_e}{\Gamma, \neg p, p \vdash r}\,\bot_e}{\Gamma, \neg p \vdash p \to r}\,\to_i}{\Gamma, \neg p \vdash \psi}\,\vee_i}{\Gamma \vdash \psi}\,\vee_e$$