1. $C_{st} = \sum_{i=1}^{n} a_{si} b_{it}$

Let $D = \sum_{m=1}^{K} L^{(m)} R^{(m)} = \sum_{m=1}^{K} \frac{1}{K p_{i_m}} A_{:,i_m} B_{i_m,:}$.

$D_{st} = \sum_{m=1}^{k} \frac{1}{K p_{i_m}} a_{s,i_m} b_{i_m,t}$

$\mathbb{E} D_{st} = \frac{1}{K} \sum_{m=1}^{K} \mathbb{E} \frac{1}{p_{i_m}} a_{s,i_m} b_{i_m,t} = \frac{1}{K} \sum_{m=1}^{K} \left( p_1 \cdot \frac{1}{p_1} a_{s1} b_{1t} + \dots + p_n \frac{1}{p_n} a_{s,n} b_{s,n} \right)$

$= C_{st}$ Thus using $D$ to approximate $C$ is valid

Quantifying accuracy:

$\mathrm{Var}\, D_{st} = \frac{1}{K^2} \sum_{m=1}^{K} \mathrm{Var}\left( \frac{1}{p_{i_m}} a_{s,i_m} b_{i_m,t} \right) = \frac{1}{K^2} \sum_{m=1}^{K} \left[ \mathbb{E} \left( \frac{1}{p_{i_m}} a_{s,i_m} b_{i_m,t} \right)^2 - C_{st}^2 \right]$

$= \frac{1}{K^2} \sum_{m=1}^{K} \left[ \sum_{i=1}^{n} \frac{1}{p_i} a_{si}^2 b_{it}^2 - \left( \sum_{i=1}^{n} a_{si} b_{it} \right)^2 \right] = \frac{1}{K} \left[ \sum_{i=1}^{n} \frac{1}{p_i} a_{si}^2 b_{it}^2 - \left( \sum_{i=1}^{n} a_{si} b_{it} \right)^2 \right]$

$\mathbb{E} \left| C_{st} - D_{st} \right| \leq \sqrt{\mathbb{E} (C_{st} - D_{st})^2} = \sqrt{\mathrm{Var}\, D_{st}} \leq \frac{1}{\sqrt{K}} M$

where $M = \max_{s,t} \sum_{i=1}^{n} \frac{1}{p_i} a_{si}^2 b_{it}^2 - \left( \sum_{i=1}^{n} a_{si} b_{it} \right)^2 > 0$

Thus we have error bound

2. $I(f) = \int_0^1 f(x) dx$ Midpoint rule can be written as


$I_N(f) = h \sum_{i=0}^{N-1} f((i+\frac{1}{2})h), \quad h = \frac{1}{N}$

$I(f) - I_N(f) = \sum_{i=0}^{N-1} \int_{ih}^{(i+1)h} f(x) dx - f((i+\frac{1}{2})h) h$

$= \sum_{i=0}^{N-1} \int_{ih}^{(i+1)h} \left[ f((i+\frac{1}{2})h) + f'((i+\frac{1}{2})h)(x - (i+\frac{1}{2})h) + O(h^2) \right]$

$- f((i+\frac{1}{2})h)h$

$= \sum_{i=0}^{N-1} O(h^3)$

$= O(h^2).$