17. $x=(x_1 \cdots x_n)\in\mathbb{R}^n_+$, Poisson kernel $K(x,y)$

$$\frac{z^{n-1}}{(z+i)^{\frac{n}{2}}(z-i)^{\frac{n}{2}}}$$

$$\int_{\mathbb{R}^{n-1}} K(x,y)\,dy=1$$

$y=(y_1 \cdots y_{n-1},0)\in\partial\mathbb{R}^n_+=\mathbb{R}^{n-1}$

$dy=dy_1\cdots dy_{n-1}$ $\quad K(x,y)=\frac{2x_n}{n\alpha(n)|y-x|^n}$

~~Proof~~. 只需证 $\int_{\mathbb{R}^{n-1}}\frac{1}{[(y_1-x_1)^2+\cdots+(y_{n-1}-x_{n-1})^2+x_n^2]^{\frac{n}{2}}}dy_1\cdots dy_{n-1}=\frac{n\alpha(n)}{2x_n}$

左式 $=\int_{\mathbb{R}^{n-1}}\frac{1}{[z_1^2+\cdots+z_{n-1}^2+x_n^2]^{\frac{n}{2}}}dz_1\cdots dz_{n-1}=\int_{\mathbb{R}^{n-1}}\frac{1}{(y_1^2+\cdots+y_{n-1}^2+1)^{\frac{n}{2}}}dy_1\cdots dy_{n-1}\cdot\frac{1}{x_n}$

$=\frac{1}{x_n}\int_{\mathbb{R}^{n-1}}\frac{1}{(1+|y|^2)^{\frac{n}{2}}}dy=\frac{1}{x_n}\int_0^\infty\frac{r^{n-2}}{(1+r^2)^{\frac{n}{2}}}(n-1)\alpha(n-1)\,dr$

整理，只需证 $\int_0^\infty\frac{r^{n-2}}{(1+r^2)^{\frac{n}{2}}}dr=\frac{\Gamma(\frac{n-1}{2})\sqrt{\pi}}{2\Gamma(\frac{n}{2})}$ 立得（由分部积分） #

这由 $\int_0^\infty\frac{r^{n-2}}{(1+r^2)^{\frac{n}{2}}}dr=\frac{n-3}{n-2}\int_0^\infty\frac{r^{n-4}}{(1+r^2)^{\frac{n-2}{2}}}dr$

18. $\begin{cases}-\Delta u=f(x,y), & (x,y)\in\Omega\\ u|_{\partial\Omega}=g(x,y)\end{cases}$ Green 函数

(1) Ω 上半平面

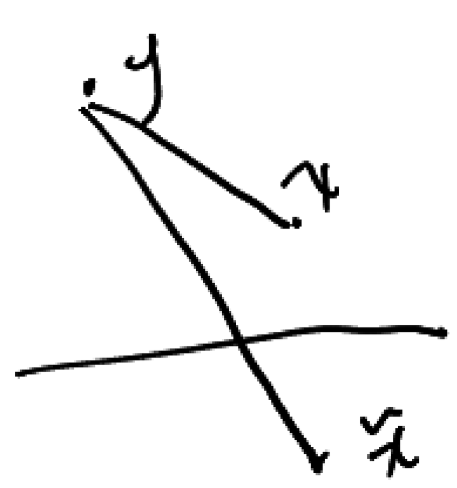
(2) Ω 第一象限

(1) 由定义 $G(x,y)=\Gamma(y-x)-\phi^x(y)$

考虑 $\phi^x(y):=\Gamma(y-\tilde{x})$，$\tilde{x}$ 为 x 关于 x 轴对称

则在上半平面 $\Delta\phi^x(y)=0$

且 x 轴上 $\phi^x(y)=\Gamma(y-\tilde{x})=\Gamma(y-x)$



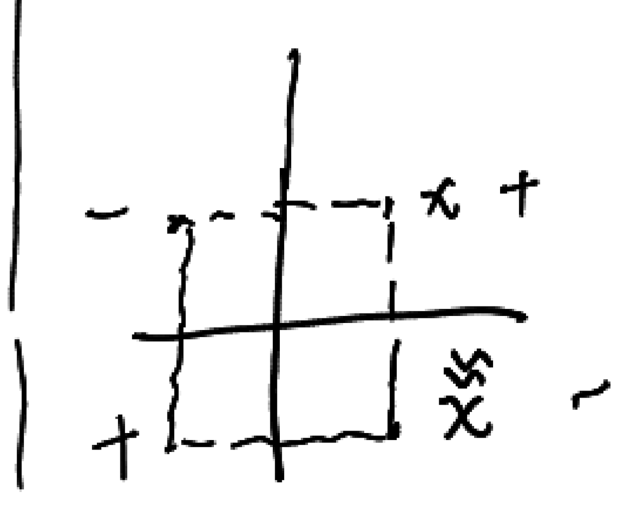
故 $G(x,y)=\Gamma(y-x)-\Gamma(y-\tilde{x})$

$=-\frac{1}{2\pi}\ln|y-x|+\frac{1}{2\pi}\ln|y-\tilde{x}|$

$=\frac{1}{4\pi}\ln\frac{|y-\tilde{x}|^2}{|y-x|^2}$

$=\frac{1}{4\pi}\ln\frac{(x_1-y_1)^2+(x_2+y_2)^2}{(x_1-y_1)^2+(x_2-y_2)^2}$ #

(2)



同理考虑 $\phi^x(y)=\Gamma(y-\tilde{x}^1)+\Gamma(y-\tilde{x}^2)-\Gamma(y-\tilde{x}^3)$

$x=(x_1,x_2)$ 则

$\tilde{x}^1=(-x_1,x_2)$ $\quad\tilde{x}^3=(-x_1,-x_2)$

$\tilde{x}^2=(x_1,-x_2)$

易知第一象限 $\Delta\phi^x(y)=0$

而 y 在 x 轴时，记 $y-x=r$

$\Gamma(y-x)-\phi^x(y)=\Gamma(r)-\Gamma(r)-\Gamma(r)+\Gamma(r)=0$

y 在 y 轴同理.

故 $G(x,y)=-\frac{1}{4\pi}\ln[(x_1-y_1)^2+(x_2-y_2)^2]$

$+\frac{1}{4\pi}\ln[(x_1+y_1)^2+(x_2-y_2)^2]$

$+\frac{1}{4\pi}\ln[(x_1-y_1)^2+(x_2+y_2)^2]$

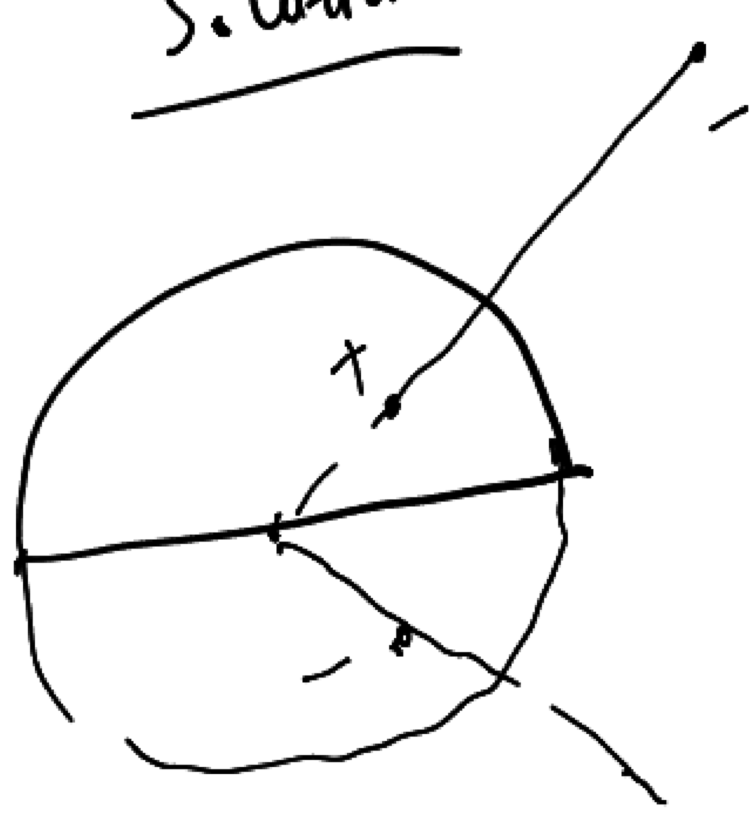
$-\frac{1}{4\pi}\ln[(x_1+y_1)^2+(x_2+y_2)^2]$

#

19 $B^+(R) = \{x=(x_1 \cdots x_n) \in \mathbb{R}^n \mid x_n > 0, |x| < R\}$ $n \geq 2$

$$\begin{cases} -\Delta u = f(x) & \text{on } B^+(R) \\ u|_{\partial B^+(R)} = g(x) & \end{cases}$$ Green 函数

Solution



由球的Green函数和上半平面Green函数易知，(如图)

$$G(x,y) = \Gamma(y-x) - \Gamma\left(\left(y - \frac{R^2 x}{\|x\|^2}\right)\frac{\|x\|}{R}\right)$$

$$-\Gamma(y-\tilde{x}) + \Gamma\left(\left(y - \frac{R^2 \tilde{x}}{\|x\|^2}\right)\frac{\|x\|}{R}\right),$$ 其中 $\tilde{x} = (x_1 \cdots x_{n-1}, -x_n)$

其中 $\Gamma(t) = \begin{cases} -\frac{1}{2\pi}\ln|t| & n=2 \\ \frac{1}{n(n-2)\alpha(n)|t|^{n-2}} & n \geq 3 \end{cases}$ #

21. $u \in C(\bar{\Omega}) \cap C^2(\Omega)$

$$\begin{cases} -\Delta u + c(x)u = f(x) & x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

(1) $c(x) \geq c_0 > 0$，有 $\max\limits_{\bar{\Omega}} |u| \leq \frac{1}{c_0} \sup\limits_{\Omega} |f|$

Proof 设 $|u|$ 在 $x_0 \in \Omega$ 取最大值 $M > 0$（否则不用做任何事）

$-\Delta u(x_0) + c(x_0)u(x_0) = f(x_0)$ 若 $u(x_0) > 0$

我们有 $tr(D^2u(x_0)) \leq 0$ $\quad f(x_0) \geq c(x_0)u(x_0) \geq c_0 M > 0$

$\frac{1}{c_0} \sup\limits_{\Omega} |f| \geq \frac{1}{c_0} f(x_0) \geq \frac{1}{c_0} c(x_0) u(x_0) \geq u(x_0) = M = \max\limits_{\bar{\Omega}} |u|$ #

若 $u(x_0) < 0$ $\quad f(x_0) \leq c(x_0)u(x_0)$ $\quad \sup\limits_{\Omega} |f| \geq c_0 M$ #

(2) $c(x) \geq 0$ 则 $\max\limits_{\bar{\Omega}} |u| \leq M \sup\limits_{\Omega} |f|$，$M = M(d)$.

Proof. 不妨设 $0 \in \Omega$ 记 $u(x) = (d^2 - |x|^2 + 1) v(x)$

$$-\Delta v + \left[c(x) + \frac{2}{d^2 - |x|^2 + 1}\right] v(x) + 4 \frac{Dv \cdot x}{d^2 - |x|^2 + 1} = \frac{f(x)}{d^2 - |x|^2 + 1}$$

设 $|v|$ 在 x_0 处取最大值 M 若 $v(x_0) > 0$

$\Delta v(x_0) \leq 0$, $Dv(x_0) = 0$ $\quad \frac{f(x)}{d^2 - |x|^2 + 1} \geq \left[c(x) + \frac{2}{d^2 - |x|^2 + 1}\right] v(x) \geq \frac{2}{d^2+1} M$

$M \leq \frac{d^2+1}{2} \sup\limits_{\Omega} |f|$ 若 $v(x_0) < 0$ 同理

总之 $\sup\limits_{\Omega} |u| \leq (d^2+1) \sup\limits_{\Omega} |v| \leq \frac{(d^2+1)^2}{2} \sup\limits_{\Omega} |f|$ #

(3) $c(x) < 0$ 最大模估计一般不成立

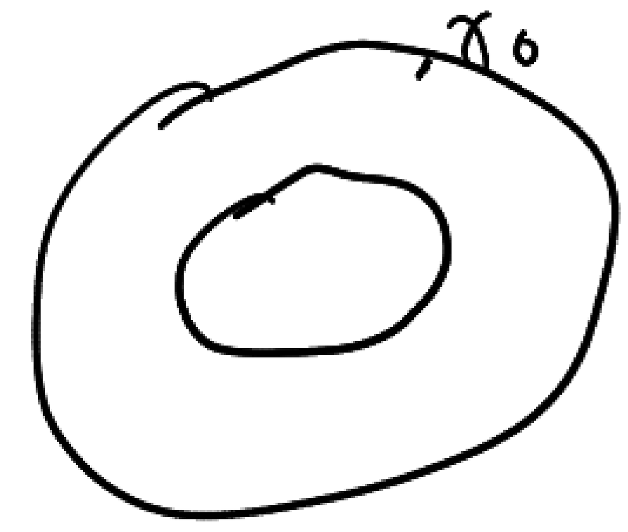
取 $c = -3$ $f \equiv 0$ 考虑 $\begin{cases} 3u + \Delta u = 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

例如 $u(x_1, x_2, x_3) = \sin x_1 \sin x_2 \sin x_3$ $\quad \Omega = (0, \pi)^3$

$\max\limits_{\bar{\Omega}} |u| = 1$ 矛盾.

24. $w(x) = e^{-a|x|^2} - e^{-aR^2}$ 证明 Hopf 引理, $a>0$

Proof. 只需证明 B_R^* 上 $\mathcal{L}w \le 0$ (1) 与 $\frac{\partial w}{\partial \vec{v}}\Big|_{x=x_0} < 0$ (2) 即可 记 $M=\sup\limits_{B_R}|c|$



(1):

$$\mathcal{L}w = -\Delta w + c(x) w$$
$$= c(x)[e^{-a|x|^2} - e^{-aR^2}] + 2a e^{-a|x|^2}[n - 2a|x|^2]$$
$$\le M[e^{-a\frac{R^2}{4}} - e^{-aR^2}] + 2a e^{-a\frac{R^2}{4}}(n - a\cdot\frac{R^2}{2})$$
$$e^{a\frac{R^2}{4}}\mathcal{L}w \le M(1-e^{-a\frac{3}{4}R^2}) + 2a(n - a\cdot\frac{R^2}{2})$$
$$\le M + 2a(n-a\cdot\frac{R^2}{2})$$ 取 a 充分大则 $\mathcal{L}w \le 0$

(2): $\frac{\partial w}{\partial x_i} = -2ax_i e^{-a|x|^2} = -2ax_i e^{-aR^2}$

记 $\vec{v} = (v_1 \cdots v_n)$ $\vec{v}\cdot\vec{n}>0$ 即 $\sum v_i x_i > 0$

而 $\frac{\partial w}{\partial \vec v} = \sum v_i w_i' = \sum -2a v_i x_i e^{-aR^2} = -2a e^{-aR^2}\sum v_i x_i < 0$

#

27. $\Omega \subseteq \mathbb{R}^n$ 有界开 $u\in C^2(\Omega)\cap C(\bar\Omega)$

$$\begin{cases} -\Delta u + |u|u = f \\ u|_{\partial\Omega} = g \end{cases}$$

$$\max_{\bar\Omega}|u| \le \max\{\max_{\partial\Omega}|g|, \sup_{\Omega} |f|^{\frac12}\}$$

设 $u(x_0) = \max\limits_{\bar\Omega}|u| = M$. 若 $x_0 \in \partial\Omega$, 没有需要证的.

否则 $x_0\in\Omega$ 若 $u(x_0)>0$, x_0处 $-\Delta u(x_0) + u^2(x_0) = f(x_0)$

且有 $\Delta u(x_0) = tr\, D^2 u(x_0) \le 0$ $f(x_0) \ge u^2(x_0) > 0$ $M \le \sqrt{f(x_0)} \le \sup\limits_{\Omega} |f|^{\frac12}$

若 $u(x_0)<0$ $-\Delta u(x_0) - u^2(x_0) = f(x_0)$

有 $\Delta u(x_0) \ge 0$ $f(x_0) \le -u^2(x_0) < 0$ $M^2 \le \sup\limits_{\Omega}|f|$

若 $u(x_0)=0$ 没有要证的

#

30. $\Omega \subset \mathbb{R}^n$ 有界开 $u \in C^2(\Omega) \cap C(\bar{\Omega})$

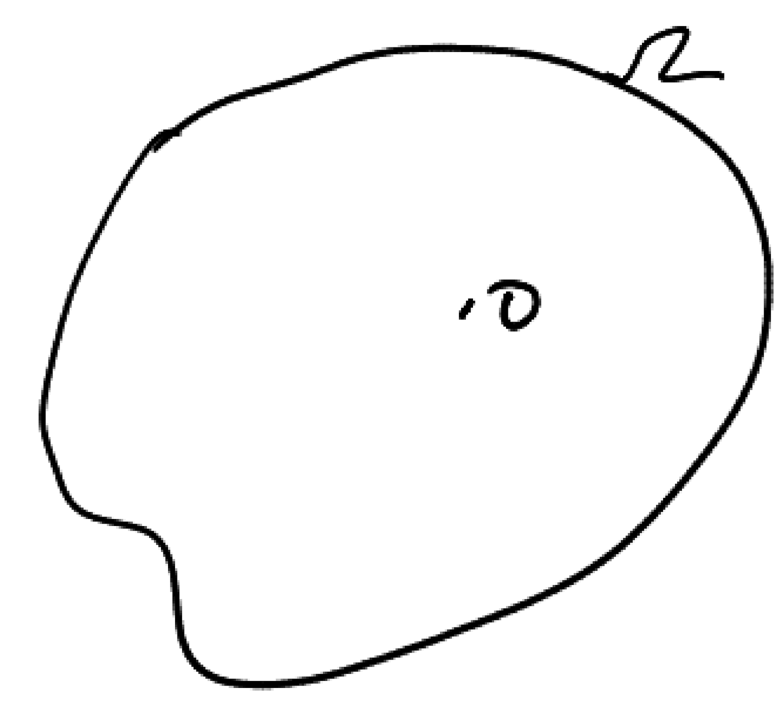
$$\begin{cases} -\Delta u + A \cdot Du = f(x) & \text{on } \Omega \\ u = g(x) & \text{on } \partial\Omega \end{cases}$$

$A : \Omega \to \mathbb{R}^n$ 连续有界向量函数 $f(x) \geq 0$, $g(x) \geq 0$

则 $u(x) \geq 0$ on $\bar{\Omega}$.

<u>Proof</u> 记 $\mu = \sup\limits_{\Omega} |A(x)| + 1$ $\quad w(x) = u(x) + \varepsilon(e^{\mu d} - e^{\mu x_1})$, $d = \sup\limits_{x,y \in \Omega} |x-y|$

不妨设 $0 \in \Omega$.



$$\begin{aligned} \Delta w &= \Delta u - \varepsilon \mu^2 e^{\mu x_1} \\ &= A \cdot Du - f(x) - \varepsilon \mu^2 e^{\mu x_1} \\ &= A_2 w_2' + \cdots + A_n w_n' - f(x) + A_1(w_1' + \mu \varepsilon e^{\mu x_1}) \\ &\qquad - \varepsilon \mu^2 e^{\mu x_1} \\ &= A \cdot Dw - f(x) + \varepsilon \mu e^{\mu x_1}(A_1 - \mu) \end{aligned}$$

$Lw = -\Delta w + A \cdot Dw = f(x) + \varepsilon \mu e^{\mu x_1}(\mu - A_1) > 0$

设 x_0 为 w 在 $\bar{\Omega}$ 上最小值

若 $x_0 \in \partial\Omega$, 则 $w|_{\partial\Omega}(x_0) = g(x_0) + \varepsilon(e^{\mu d} - e^{\mu x_1}) \geq 0 \Rightarrow w|_{\Omega} \geq 0$

若 $x_0 \in \Omega$ 则 $Dw(x_0) = 0$ $\quad \Delta w(x_0) \geq 0$

$Lw = -\Delta w + A \cdot Dw \leq 0$ 矛盾!

故 Ω上 $w \geq 0$, $u(x) \geq \varepsilon(e^{\mu x_1} - e^{\mu d})$

令 $\varepsilon \to 0^+$ $\quad u(x) \geq 0$ on Ω. #

36. $R_+^2 = \{(x,y)\in R^2 \mid y>0\}$

$$\begin{cases} -\Delta u = f(x,y) & \text{on } R_+^2 \\ u|_{y=0} = g(x) & x\in R \end{cases}$$ 在 $C^2(R_+^2)\cap C(\overline{R_+^2})$ 有界解唯一

Proof

只需证明方程 $\begin{cases} -\Delta u = 0 \\ u|_{y=0} = 0 \end{cases}$ 的有界解 u 必为0 先考虑方程 $\begin{cases} -\Delta u = 0 & \text{on } B_R^+ \\ u|_{y=0} = 0 & \text{on } \partial B_R^+ \end{cases}$ 有界解

u. 考虑 $w(x,y) = -\varepsilon \ln[x^2+(y+1)^2] - u(x,y)$ $\varepsilon>0$

$\Delta w = -\Delta u - \varepsilon\Delta \ln[x^2+(y+1)^2] = 0$

$w|_{y=0} = -\varepsilon \ln(x^2+1) \le 0$ 故 w 不可能在 B_R^+ 取得正最大值

B_R^+ 上有 $u(x,y) \ge -\varepsilon \ln[x^2+(y+1)^2] \ge -2\varepsilon \ln(R+1)$

考虑 $s(x,y) = u(x,y) - \varepsilon \ln[x^2+(y+1)^2]$

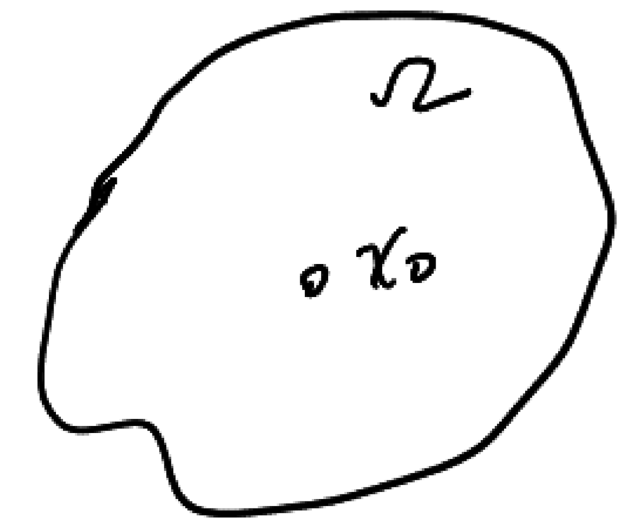
$\Delta s = 0$ $s|_{y=0} = -\varepsilon \ln(x^2+1) \le 0$ 故 s 不可能在 B_R^+ 取正最大值

B_R^+ 上有 $u(x,y) \le \varepsilon \ln[x^2+(y+1)^2] \le 2\varepsilon \ln(R+1)$

B_R^+ 上 $|u| \le 2\varepsilon \ln(R+1)$ 令 $\varepsilon\to 0^+$, B_R^+ 上 $u\equiv 0$

令 $R\to+\infty \Rightarrow R_+^2$ 上 $u\equiv 0$ #

39. $\Omega \subset \mathbb{R}^n$ $n \geq 3$ 有界开集 $x_0 \in \Omega$ $u \in C^2(\Omega) \cap C(\bar{\Omega})$



$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ u|_{\partial\Omega} = g \end{cases}$$

$v \in C^2(\Omega \setminus \{x_0\})$

$$\begin{cases} -\Delta v = f & \text{on } \Omega \setminus \{x_0\} \\ v|_{\partial\Omega} = g \end{cases}$$ v 有界

则 $u(x) \equiv v(x)$ on $\bar{\Omega} \setminus \{x_0\}$.

Proof 记 $s(x) = u(x) - v(x)$ $\begin{cases} -\Delta s = 0 & \text{on } \Omega \setminus \{x_0\} \\ s = 0 & \text{on } \partial\Omega \end{cases}$ s 在 $\Omega \setminus \{x_0\}$ 上有界

只需证 $\bar{\Omega} \setminus \{x_0\}$ 上 $s \equiv 0$

记 $w(x) = -\dfrac{\varepsilon}{|x - x_0|^{n-2}} + s(x)$

$\Delta w = \Delta s = 0$

$w|_{\partial\Omega} = -\dfrac{\varepsilon}{|x - x_0|^{n-2}} \leq 0$

若 w 在 $\Omega \setminus \{x_0\}$ 有最大值 $M > 0$ 记 $O = \{x \in \Omega \setminus \{x_0\} \mid u(x) = M\}$ 用同于强极值原理证明知 O 相对于 $\Omega \setminus \{x_0\}$ 开

$O \neq \varphi$ O 相对于 $\Omega \setminus \{x_0\}$ 闭.

故 $O = \Omega \setminus \{x_0\}$ 与 $w|_{\partial\Omega} \leq 0$ 矛盾!

故 $\Omega \setminus \{x_0\}$ 上 $s(x) \leq \dfrac{\varepsilon}{|x - x_0|^{n-2}}$

同理考虑 $t(x) = \dfrac{-\varepsilon}{|x - x_0|^{n-2}} - s(x)$ 知 $\Omega \setminus \{x_0\}$ 上 $s(x) \geq \dfrac{-\varepsilon}{|x - x_0|^{n-2}}$

$\Omega \setminus \{x_0\}$ 上 $|s(x)| \leq \dfrac{\varepsilon}{|x - x_0|^{n-2}}$ 令 $\varepsilon \to 0^+$, $\Omega \setminus \{x_0\}$ 上 $s \equiv 0$

故 $\bar{\Omega} \setminus \{x_0\}$ 上 $s \equiv 0$ #