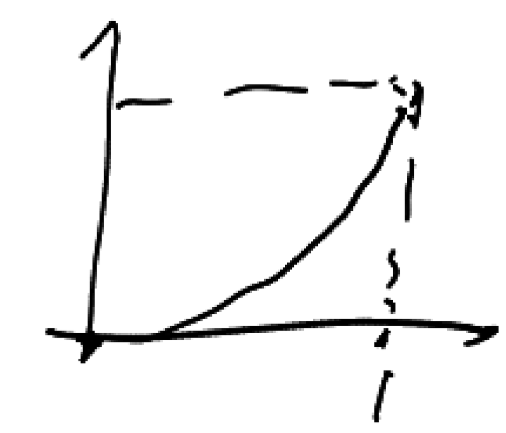
1. $T = \int_0^1 \frac{\sqrt{1+u'(x)^2}}{x}\,dx \qquad u(0)=0,\ u(1)=1$

(a) $\frac{\delta T}{\delta u} = 0 \Rightarrow$ Snell law

(b) Find optimal $u(x)$



(a) Use E-L equation, let $L(x,u,u') = \frac{\sqrt{1+u'^2}}{x}$

$$0 = \frac{\delta T}{\delta u} = \frac{\partial L}{\partial u} - \frac{d}{dx}\frac{\partial L}{\partial u'} = -\frac{d}{dx}\left(\frac{u'(x)}{x\sqrt{1+u'(x)^2}}\right)$$

Snell's law: $\frac{u'(x)}{x\sqrt{1+u'(x)^2}} = C$

(b): $u'(x)^2 = C^2 x^2 (1+u'(x)^2)$

$A = C^2$

$$v(x) = \sqrt{\frac{Ax^2}{1-Ax^2}}.$$

$$u(x) = \int_0^x \sqrt{\frac{At^2}{1-At^2}}\,dt$$

$$\int_0^1 \sqrt{\frac{At^2}{1-At^2}}\,dt = 1 \Rightarrow A = 1$$

$$u(x) = \int_0^x \frac{t}{\sqrt{1-t^2}}\,dt = 1-\sqrt{1-x^2}$$

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2. $u(0) = u(1) = 0 \quad \int_0^1 u(x)\,dx = A$

Show $\min \int_0^1 u'(x)^2 dx = 12A^2$

Lagrange multiplier m, Solve E-L equation for u, verify $A = -\frac{m}{24}$

$\int_0^1 u'(x)^2 dx \geq 0$ So infinum exists. Use Lagrange multiplier:

$$\mathcal{L}(u,m) = \int_0^1 u'(x)^2 dx + m\left(\int_0^1 u(x)\,dx - A\right) = \int_0^1 \left(u'(x)^2 + m\,u(x)\right)dx - mA$$

$$\nabla_u \mathcal{L}(u,m) = \frac{\partial L}{\partial u} - \frac{d}{dx}\frac{\partial L}{\partial u'}$$

$$= m - 2u''(x) = 0$$

$$u(x) = \frac{1}{4}mx^2 + Cx + D$$

Combine $u(0) = u(1) = 0$, $D = 0$, $C = -\frac{m}{4}$

$$A = \int_0^1 u(x)\,dx = \frac{m}{12} - \frac{m}{4}\frac{1}{2} = -\frac{m}{24}$$

In this case $\int_0^1 u'(x)^2 dx = \int_0^1 \left(\frac{1}{2}mx - \frac{m}{4}\right)^2 dx$

$$= \frac{1}{4}m^2 \cdot \frac{1}{3} - \frac{m^2}{8} + \frac{m^2}{16}$$

$$= \frac{m^2}{48} = 12A^2$$

We have proved $\min_u \int_0^1 u'(x)^2 dx = 12A^2$

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3. $\Omega = (0,\pi)\times(0,\pi)$

$$\begin{cases} -\Delta u(x) = \lambda u(x) & \text{in } \Omega \\ \dfrac{\partial u(x)}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

(a) eigenvalues real, nonnegative

(b) Compute eigenvalue, eigenfunctions

(a) Let $\mathcal{L}u = -\Delta u$, $\quad (u,v) = \int_\Omega u v \, dx$

$$(\mathcal{L}u, \bar{u}) = \int_\Omega -\Delta u \, \bar{u} \, dx = \int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} \bar{u} \, dA = \int_\Omega |\nabla u|^2 dx$$

$$(\mathcal{L}\bar{u}, u) = \int_\Omega -\Delta \bar{u} \, u dx = \int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} \frac{\partial \bar{u}}{\partial n} u \, dA = \int_\Omega |\nabla u|^2 dx$$

$$(\mathcal{L}u, \bar{u}) = \lambda (u, \bar{u}) = (\mathcal{L}\bar{u}, u) = \bar{\lambda}(u, \bar{u})$$

$$(\lambda - \bar{\lambda})(u, \bar{u}) = 0 \qquad u \not\equiv 0 \text{ in } \Omega \Rightarrow \lambda = \bar{\lambda}, \ \lambda \in \mathbb{R}$$

$$\lambda = \frac{(\mathcal{L}u, \bar{u})}{(u, \bar{u})} = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx} \geq 0$$

(b) $u(x) = u_1(x_1)\, u_2(x_2)$

$$-u_1''(x_1)\, u_2(x_2) - u_1(x_1)\, u_2''(x_2) = \lambda\, u_1(x_1)\, u_2(x_2)$$

$$-\frac{u_1''(x_1)}{u_1(x_1)} - \frac{u_2''(x_2)}{u_2(x_2)} = \lambda$$

$$\begin{cases} u_1''(x_1) = -\lambda_1 u_1(x_1) \\ u_1'(0) = u_1'(\pi) = 0 \end{cases} \Rightarrow \begin{matrix} u_1(x) = \cos k_1 x \\ \lambda_1 = k_1^2 \end{matrix}$$

$$\begin{cases} u_2''(x_2) = -\lambda_2 u_2(x_2) \\ u_2'(0) = u_2'(\pi) = 0 \end{cases} \Rightarrow \begin{matrix} u_2(x) = \cos k_2 x \\ \lambda_2 = k_2^2 \end{matrix}$$

$$\lambda = (k_1^2 + k_2^2) \qquad u(x) = \cos k_1 x_1 \cos k_2 x_2 \qquad k_1, k_2 \in \mathbb{Z}$$

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4. $-(xu')' = \frac{\lambda}{x} u, \quad 1 < x < b$, $u(1) = u(b) = 0$ Hint : $x = e^s$

Find eigenvalue and eigenfunction

$$\begin{cases} x^2 u'' + x u' + \lambda u = 0, & 1 < x < b \\ u(1) = u(b) = 0 \end{cases}$$

This is Cauchy - Euler ODE, $s = \ln x$

$$u(x) = V(s) = V(\ln x)$$

$$u_x = V_s s_x = \frac{1}{x} V_s$$

$$u_{xx} = -\frac{1}{x^2} V_s + \frac{1}{x^2} V_{ss}$$

$$x^2 u'' + x u' + \lambda u = -V_s + V_{ss} + V_s + \lambda V = 0$$

$$V_{ss} = -\lambda V$$

$$V(0) = V(\ln b) = 0$$

$$\Rightarrow V(s) = \sin\left(\frac{k\pi s}{\ln b}\right) \qquad \lambda_k = \frac{k^2 \pi^2}{(\ln b)^2}$$

$$u_k(x) = \sin\left(\frac{k\pi \ln x}{\ln b}\right)$$

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