Lec 18

1. $\begin{cases} \overline{y_{n+1}} = y_n + h f(t_n, y_n) \\ y_{n+1} = y_n + \frac{h}{2}(f(t_n, y_n) + f(t_{n+1}, \overline{y_{n+1}})) \end{cases}$

$T(h) = O(h^p)$ 局部截断误差. $p = ?$

$$y_{n+1} = y_n + \frac{h}{2} f(t_n, y_n) + \frac{h}{2} f(t_n + h, y_n + h f(t_n, y_n))$$

$$= y_n + \frac{h}{2} f(t_n, y_n) + \frac{h}{2}(f(t_n, y_n) + h \frac{\partial f}{\partial x}(t_n, y_n) + h f(t_n, y_n) \frac{\partial f}{\partial y}(t_n, y_n)$$

$$+ \frac{1}{2}(f_{xx} h^2 + 2 f_{xy} h^2 f + f_{yy} h^2 f^2) + O(h^3))$$

$$= y_n + hf + \frac{1}{2} h^2 (f_x + f f_y) + \frac{h^3}{4}(f_{xx} + 2 f_{xy} f + f_{yy} f^2) + O(h^4)$$

$$y_{n+1} = y_n + hf + \frac{1}{2} h^2 (f_x + f f_y) + \frac{h^3}{3!}(f_{xx} + 2 f_{xy} f + f_x f_y + f_{yy} f^2 + f f_y^2) + O(h^4)$$

可见误差为 $O(h^3)$ $p = 3$

2. $y_{n+1} = y_n + \frac{h}{2}(f(t_n, y_n) + f(t_{n+1}, y_{n+1}))$

$T(h) = O(h^3)$

$$y_{n+1} = y_n + \frac{h}{2} f(t_n, y_n) + \frac{h}{2} f(t_n + h, y_n + hf + \frac{h^2}{2}(f_x + f f_y) + O(h^3))$$

$$= y_n + \frac{h}{2} f(t_n, y_n) + \frac{h}{2}(f(t_n, y_n) + h f_x + (hf + \frac{h^2}{2}(f_x + f f_y)) f_y$$

$$+ O(h^2))$$

$$= y_n + hf + \frac{h^2}{2}(f_x + f f_y) + O(h^3)$$

可见局部截断误差为 $O(h^3)$

3. 求阶: $y_{n+1} = y_n + \frac{h}{6}\left(4f(t_n,y_n) + 2f(t_{n+1}, y_{n+1}) + h\frac{d}{dt}\left(f(t,y(t))\right)\Big|_{(t,y)=(t_n,y_n)}\right)$

$$y_{n+1} = y_n + \frac{2h}{3} f(t_n,y_n) + \frac{h}{3} f\left(t_n+h, y_n + hf + \frac{h^2}{2}(f_x+ff_y) + O(h^3)\right)$$

$$+ \frac{h^2}{6}(f_x + ff_y)$$

$$= y_n + \frac{h^2}{6}(f_x+ff_y) + \frac{2h}{3}f + \frac{h}{3}\Big(f + hf_x + \left(hf + \frac{h^2}{2}(f_x+ff_y)+O(h^3)\right)f_y$$

$$+ \frac{1}{2} f_{xx} h^2 + \frac{1}{2} f_{yy}\left(hf+\frac{h^2}{2}(f_x+ff_y)+O(h^3)\right)^2$$

$$+ f_{xy}h\left(hf+\frac{h^2}{2}(f_x+ff_y)+O(h^3)\right)\Big)$$

$$= y_n + hf + h^2\left(\frac{f_x}{6}+\frac{ff_y}{6}+\frac{f_x}{3}+\frac{ff_y}{3}\right) + \frac{h^3}{6}\left(f_xf_y + ff_y^2 + 2f_{xy}f + f_{yy}f^2 + f_{xx}\right)$$

$$+ O(h^4)$$

方法是3阶的 ($O(h^4)$ 系数为 $\frac{1}{4!}=\frac{1}{24}$, 而左边 $O(h^3)$ 系数为 $\frac{1}{3!}$, 不会得到 $\frac{1}{24}$).

4. $h_n := t_{n+1} - t_n > 0 \quad t_n \in [0,T]$

$$|e_{n+1}| \le (1+Lh_n)|e_n| + Ch_n^{p+1} \quad n=0,1,\dots,N-1$$

$|e_0| \sim O(h^p)$ 时仍有 $|e_n| \sim O(h^p)$ $\quad h=\max_k h_k$

$$|e_1| \le (1+Lh_0)|e_0| + Ch_0^{p+1}$$

$$|e_2| \le (1+Lh_1)|e_1| + Ch_1^{p+1} \le (1+Lh_1)(1+Lh_0)|e_0| + C\left(h_1^{p+1} + (1+Lh_1)h_0^{p+1}\right)$$

$$|e_3| \le (1+Lh_2)|e_2| + Ch_2^{p+1} \le \prod_{i=0}^{2}(1+Lh_i)|e_0| + C\left((1+Lh_2)(1+Lh_1)h_0^{p+1} + (1+Lh_2)h_1^{p+1} + h_2^{p+1}\right)$$

$$\vdots$$

$$|e_n| \le \prod_{i=0}^{n-1}(1+Lh_i)|e_0| + C\sum_{i=0}^{n-1}\prod_{j=i+1}^{n-1}(1+Lh_j)h_i^{p+1}$$

$$\le e^{LT}|e_0| + C\sum_{i=0}^{n-1}(1+Lh)^{n-i-1}h^{p+1}$$

$$\le e^{LT}|e_0| + C\frac{(1+Lh)^n-1}{L}h^p$$

$$\le e^{LT}|e_0| + \frac{C}{L}h^p(e^{LT}-1) = O(h^p)$$

Lec 19

1. $p(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_0 = 0$

x_i 为 m 重根

$y_n = n^{j-1} x_i^n \quad j=1,\cdots m$

$$y_{n+k} + a_{k-1} y_{n+k-1} + \cdots + a_0 y_n = 0$$

证: $x_i^{n+k} + a_{k-1} x_i^{n+k-1} + \cdots + a_0 x_i^n = 0 \quad (j=1)$

求导乘x $(n+k) x_i^{n+k} + (n+k-1) a_{k-1} x_i^{n+k-1} + \cdots + n a_0 x_i^n = 0 \quad , (j=2)$

求导乘x $(n+k)^2 x_i^{n+k} + (n+k-1)^2 a_{k-1} x_i^{n+k-1} + \cdots + n^2 a_0 x_i^n = 0 \quad . (j=3)$

$\vdots$

即证. #

$$\left(x(\cdots x(x(xp'(x))')' \cdots)' \Big|_{x=x_i} = 0, \ \text{求导次数} \le m-1 \right)$$

(见附件)

2. 梯形公式绝对稳定域 $\quad y_{n+1} = y_n + \frac{h}{2}(f(t_n,y_n) + f(t_{n+1}, y_{n+1}))$

$y_{n+1} = y_n + \frac{h\lambda}{2} y_n + \frac{h\lambda}{2} y_{n+1}$

$(1-\frac{z}{2}) y_{n+1} = (1+\frac{z}{2}) y_n$

$y_{n+1} = \frac{2+z}{2-z} y_n \qquad \left|\frac{2+z}{2-z}\right| < 1 \iff Re(z) < 0$

3. $R(z) = \left|1+z+\frac{z^2}{2}+\frac{z^3}{6}\right|, \left|1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}\right|$

绝对稳定域

4. $y_{n+2} - (1+\alpha) y_{n+1} + \alpha y_n = \frac{h}{12} \left((5+\alpha) f_{n+2} + 8(1-\alpha) f_{n+1} - (1+5\alpha) f_n \right)$

$-1 \le \alpha < 1$ 求绝对稳定域

$y' = \lambda y$

$$y_{n+2} - (1+\alpha) y_{n+1} + \alpha y_n = \frac{h}{12} \left((5+\alpha) \lambda y_{n+2} + 8(1-\alpha) \lambda y_{n+1} - (1+5\alpha) \lambda y_n \right)$$

$\lambda h = z$

$$y_{n+2} \left(1 - \frac{5+\alpha}{12} z\right) + y_{n+1} \left(-1 - \alpha - \frac{2}{3}(1-\alpha) z\right) + y_n \left(\alpha + \frac{1+5\alpha}{12} z\right) = 0$$

特征方程两根 $|\lambda_1| < 1$ $|\lambda_2| < 1$

绝对稳定域

$$\left\{ z \in \mathbb{C} \,\middle|\, \left| \frac{1+\alpha + \frac{2}{3}(1-\alpha) z \pm \sqrt{\left(1+\alpha+\frac{2}{3}(1-\alpha) z\right)^2 - 4\left(1 - \frac{5+\alpha}{12} z\right)\left(\alpha + \frac{1+5\alpha}{12} z\right)}}{2\left(1 - \frac{5+\alpha}{12} z\right)} \right| < 1 \right\}$$

lec 20

1. $$\begin{cases} y_{n+1} = y_n + \frac{h}{2}(k_2 + k_3) \\ k_1 = f(t_n, y_n) \\ k_2 = f(t_n + \alpha h, y_n + \alpha h k_1) \\ k_3 = f(t_n + (1-\alpha) h, y_n + (1-\alpha) h k_2) \end{cases}$$

为二阶, $\forall \alpha$

$$y_{n+1} = y_n + \frac{h}{2} f(t_n + \alpha h, y_n + \alpha h f_n) + \frac{h}{2} f(t_n + (1-\alpha) h, y_n + (1-\alpha) h [f_n + \alpha h f_x + \alpha f_n f_y h + O(h^2)])$$

$$= y_n + \frac{h}{2}(f_n + \alpha h f_x + \alpha f_n f_y h + O(h^2)) + \frac{h}{2}(f_n + (1-\alpha) h f_x + (1-\alpha) f_y h [f_n + \alpha h f_x + \alpha f_n f_y h + O(h^2)] + O(h^2))$$

$$= y_n + h\left(\frac{f_n}{2} + \frac{f_n}{2}\right) + h^2 \left(\frac{\alpha}{2} f_x + \frac{\alpha}{2} f_n f_y + \frac{1-\alpha}{2} f_x + \frac{1-\alpha}{2} f_y f_n\right) + O(h^3) = y_n + h f_n + \frac{h^2}{2}(f_x + f_n f_y) + O(h^3)$$

故方法二阶

2. Butcher 表 (A, b, c) s 级 Runge-Kutta

$$R(z) = 1 + z b^T (I - zA)^{-1} \cdot 1 = \frac{\det(I - zA + z\mathbf{1} \cdot b^T)}{\det(I - zA)}$$

写出 Runge-Kutta 算法

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

$$y_{n+1} = y_n + h(b_1 K_1 + \cdots + b_s K_s)$$

$$K_i = f\left(x_n + c_i h,\ y_n + h \sum_{j=1}^{s} a_{ij} K_j\right)$$

$$K_i = \lambda\left(y_n + h \sum_{j=1}^{s} a_{ij} K_j\right), \quad i = 1, \cdots s$$

$$(I - zA) K = \lambda y_n \mathbf{1} \qquad K = (I - zA)^{-1} \lambda y_n \mathbf{1}$$

$$y_{n+1} = y_n + h b^T K = y_n + h b^T (I - zA)^{-1} \lambda y_n \mathbf{1} = y_n (1 + z b^T (I - zA)^{-1} \mathbf{1})$$

$$R(z) = \det(1 + z b^T (I - zA)^{-1} \cdot 1) = \frac{\det(I - zA + z\mathbf{1} \cdot b^T)}{\det(I - zA)}$$

Lec 21

$R(z)$ rational function

$|R(z)| < 1$, $Re(z) < 0 \iff R(z)$ 在 $Re(z) < 0$ 解析, $|R(z)| \le 1$, $Re(z) = 0$

$\Rightarrow$: 显然 $R(z)$ 在 $Re(z) \le 0$ 解析

让 $Re(z) \to 0^-$ 有 $|z| \le 1$, $Re(z) = 0$.

$\Leftarrow$: 作一共形映射 γ 将 $D = \{z : Re(z) < 0\}$

映到 $T = \{z : |z| < 1\}$

$R^* = R \circ \gamma$ 在 $|z| < 1$ 解析 记 $C = \{|z| = 1\}$

因此 $R^*(C) \subseteq \overline{T}$ 为一条简单闭曲线

R^* 不可能是解析的 (否则在 ∞ 前 让 $|Im(z)| \to \infty$ 不在)

在 $|z| > 1$ 有奇点 故 $|z| > 1$ 的象在 $R^*(C)$ 外面

$|z| < 1$ 象在 $R^*(C)$ 里面

故 $|z| < 1 \Rightarrow |R^*(z)| < 1$

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