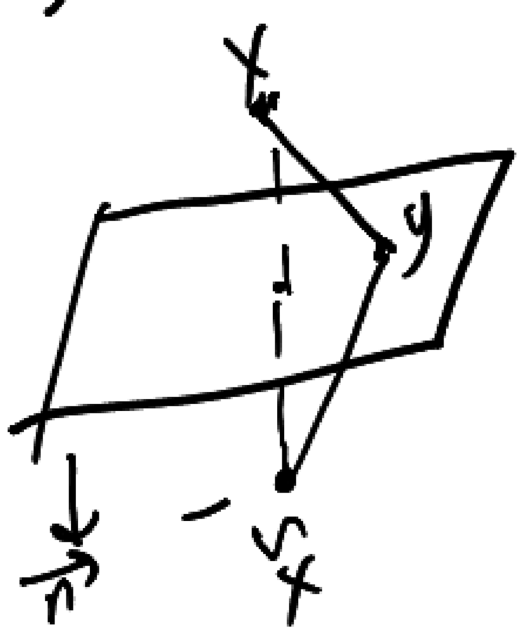
1. Green's function for $\mathbb{R}^3_+$

Solve $-\Delta u = 0$, $(x,y,z) \in \mathbb{R}^3_+$, $u(x,y,0) = h(x,y)$

By method of image,



$$\Phi(y-x) = \frac{1}{4\pi|x-y|}$$

$$\Phi(y-\tilde{x}) = \frac{1}{4\pi|y-\tilde{x}|} \qquad \tilde{x} = (x_1, x_2, -x_3)$$

$$G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x}) = \frac{1}{4\pi}\left(\frac{1}{|x-y|} - \frac{1}{|x-\tilde{y}|}\right)$$

Using Green's representation formula,

$$\frac{\partial G(x,y)}{\partial n} = -\frac{\partial}{\partial y_3}\left(\frac{1}{4\pi}\frac{1}{|x-y|} - \frac{1}{4\pi}\frac{1}{|x-\tilde{y}|}\right)$$

$$= \frac{1}{4\pi}\left(\frac{y_3-x_3}{|y-x|^3} - \frac{x_3+y_3}{|y-\tilde{x}|^3}\right) = \frac{-x_3}{2\pi|y-x|^3}$$

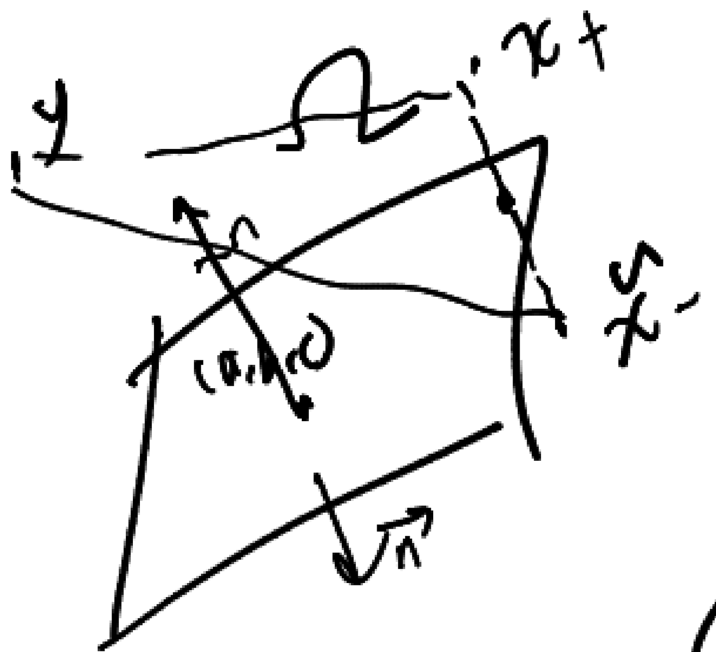
$$u(x_1, x_2, x_3) = -\int_{\partial\mathbb{R}^3_+} h(y_1, y_2)\,\frac{\partial G(x,y)}{\partial n}\,dy_1 dy_2$$

$$= \int_{\mathbb{R}^2} h(y_1, y_2)\,\frac{x_3}{2\pi\left(\sqrt{(y_1-x_1)^2 + (y_2-x_2)^2 + x_3^2}\right)^3}\,dy_1 dy_2 \qquad \#$$

2. Green's function for $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid ax+by+cz>0\}$

Solve $-\Delta u = 0,\ (x,y,z)\in\Omega,\ u(x,y,z) = h(x,y,z)$

$\vec{n} = (-a,-b,-c)$



$\hat{x} = (x_1+ka,\ x_2+kb,\ x_3+kc)\ ,\ y\in\Omega$

where $k = \dfrac{-2(ax_1+bx_2+cx_3)}{a^2+b^2+c^2}$

$$G(x,y) = \Gamma(y-x) - \Gamma(y-\hat{x})$$
$$= \frac{1}{4\pi|y-x|} - \frac{1}{4\pi|y-\hat{x}|}$$

On plane $ax+by+cz=0$,

$$-\frac{\partial G(x,y)}{\partial \vec{n}} = \left(a\frac{\partial G(x,y)}{\partial y_1} + b\frac{\partial G(x,y)}{\partial y_2} + c\frac{\partial G(x,y)}{\partial y_3}\right)\frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{1}{\sqrt{a^2+b^2+c^2}}\Bigg(-a\frac{y_1-x_1}{4\pi|y-x|^3} - b\frac{y_2-x_2}{4\pi|y-x|^3} - c\frac{y_3-x_3}{4\pi|y-x|^3}$$
$$+a\frac{y_1-x_1-ka}{4\pi|y-x|^3} + b\frac{y_2-x_2-kb}{4\pi|y-x|^3} + c\frac{y_3-x_3-kc}{4\pi|y-x|^3}\Bigg)$$

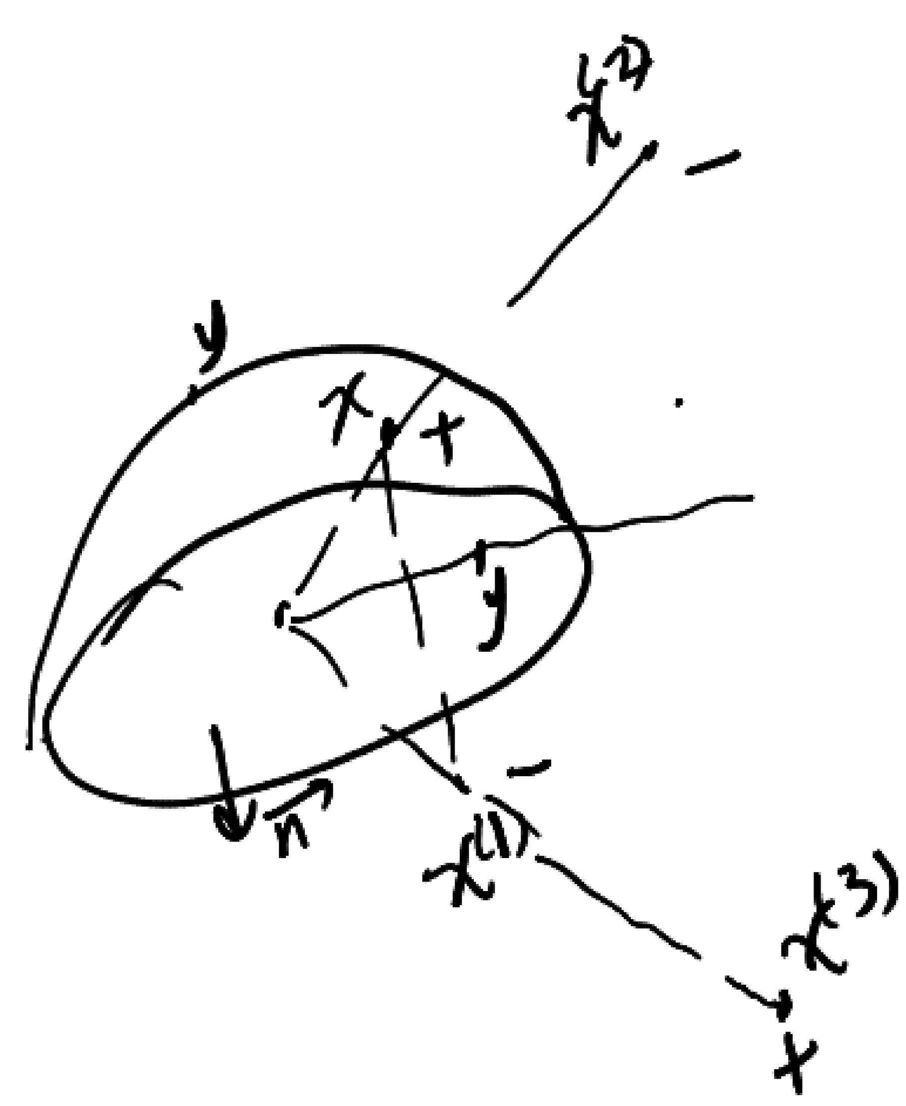
$$= \frac{1}{\sqrt{a^2+b^2+c^2}}\ \frac{(ax_1+bx_2+cx_3)}{2\pi|y-x|^3}$$

$$\Rightarrow u(x) = \int_{ay_1+by_2+cy_3=0} \frac{ax_1+bx_2+cx_3}{\sqrt{a^2+b^2+c^2}}\cdot\frac{1}{2\pi|y-x|^3}\,h(y)\,dy$$

3. Green's function for hemisphere in $\mathbb{R}^3$

$$\begin{cases} -\Delta u = 0 & |x| < R,\ z > 0 \\ u(x, y, 0) = h(x, y) & \\ u(x, y, z) = 0, & x^2 + y^2 + z^2 = R^2 \end{cases}$$

Using method of image, we have



$$G(x, y) = \Phi(y - x) - \Phi\left(\frac{|x|}{R}(y - x^{(2)})\right) - \Phi(y - x^{(1)}) + \Phi\left(\frac{|x|}{R}(y - x^{(3)})\right)$$

$$x^{(1)} = (x_1, x_2, -x_3) \qquad x^{(2)} = \frac{R^2 x}{|x|^2} \qquad x = (x_1, x_2, x_3)$$

$$x^{(3)} = \frac{R^2 (x_1, x_2, -x_3)}{|x|^2}$$

$$G(x, y) = \frac{1}{4\pi |y - x|} - \frac{R}{4\pi |x|\, |y - x^{(2)}|} - \frac{1}{4\pi} \frac{1}{|y - x^{(1)}|} + \frac{1}{4\pi} \frac{R}{|x|\, |y - x^{(3)}|}$$

On plane $z = 0$,

$$-\frac{\partial G(x, y)}{\partial n} = \frac{\partial G(x, y)}{\partial y_3} = -\frac{y_3 - x_3}{4\pi |y - x|^3} + \frac{R}{4\pi |x|} \frac{y_3 - x_3 \frac{R^2}{|x|^2}}{|y - x^{(2)}|^3} + \frac{y_3 + x_3}{4\pi |y - x^{(1)}|^3} - \frac{R}{4\pi |x|} \frac{y_3 + x_3 \frac{R^2}{|x|^2}}{|y - x^{(3)}|^3}$$

$$= \frac{x_3}{2\pi |y - x|^3} + \frac{-x_3 R^3}{2\pi |x|^3 \left| y - \frac{R^2 x}{|x|^2} \right|^3}$$

$$= \frac{x_3}{2\pi |y - x|^3} - \frac{x_3}{2\pi \left| \frac{|x|}{R} y - \frac{x}{|x|} R \right|^3}$$

$$\Rightarrow u(x) = -\int_{\substack{|y| \le R \\ y \in \mathbb{R}^2}} \frac{\partial G(x, y)}{\partial n} h(y)\, dy$$

$$= -\int_{\substack{|y| \le R \\ y \in \mathbb{R}^2}} \frac{x_3}{2\pi} \left(\frac{1}{|y - x|^3} - \frac{1}{\left| \frac{|x|}{R} y - \frac{x}{|x|} R \right|^3} \right) h(y)\, dy$$

#

4. $\begin{cases} -u''+2u=f, & x\in(0,1) \\ u'(0)=a, \quad u'(1)=b & a,b\in\mathbb{R} \end{cases}$

(1) Directly Solve

Use variation of parameter method $u''-2u=-f$

$C_1(x)=\sin\sqrt{2}x \quad C_2(x)=\cos\sqrt{2}x$

Wronski $W(x)=C_1(x)C_2'(x)-C_1'(x)C_2(x)=-\sqrt{2}$

Special solution $y_p(x)=\sin\sqrt{2}x\int\frac{\cos\sqrt{2}x\,(-f(x))}{-\sqrt{2}}dx+\cos\sqrt{2}x\int\frac{\sin\sqrt{2}x\,f(x)}{-\sqrt{2}}dx$

$$u(x)=d_1\sin\sqrt{2}x+d_2\cos\sqrt{2}x+\frac{1}{\sqrt{2}}\sin\sqrt{2}x\int_0^x\cos\sqrt{2}t\,f(t)\,dt-\frac{1}{\sqrt{2}}\cos\sqrt{2}x\int_0^x\sin\sqrt{2}t\,f(t)\,dt$$

d_1, d_2 should satisfy

$$\begin{cases} u'(0)=a \\ u'(1)=b \end{cases} \Rightarrow \begin{cases} d_1=\frac{a}{\sqrt{2}} \\ d_2=\left(\sqrt{2}\cos a-b+\cos\sqrt{2}\int_0^1\cos\sqrt{2}t\,f(t)\,dt+\frac{\sin\sqrt{2}\cos\sqrt{2}f(1)}{\sqrt{2}}\right. \\ \left.+\sin\sqrt{2}\int_0^1\sin\sqrt{2}t\,f(t)\,dt-\frac{\cos\sqrt{2}}{\sqrt{2}}\sin\sqrt{2}\,f(1)\right)\frac{1}{\sqrt{2}\sin\sqrt{2}} \end{cases}$$ #

(2) Find Green function, use it to solve

Let $\mathcal{L}=-\frac{d^2}{dx^2}$ The inhomogenous Sturm-Liouville problem

$$(*)\begin{cases} \mathcal{L}u+2u=\delta(x) \\ u'(0)=u'(1)=0 \end{cases} \text{ has eigen problem } \begin{cases} \mathcal{L}u_n=\lambda_n u_n \\ u_n'(0)=u_n'(1)=0 \end{cases}$$

Its eigen pairs $u_n(x)=\sqrt{2}\cos\pi n x$, $\lambda_n=\pi^2n^2$

Expand under $\{u_n(x)\}$, $\underline{G(x,y)=\sum_{n=1}^{\infty}\frac{u_n(x)u_n(y)}{\lambda_n+2}=2\sum_{n=1}^{\infty}\frac{\cos\pi nx\cos\pi ny}{2+n^2\pi^2}}$ (for $(*)$)

We have $\mathcal{L}G(x,y)+2G(x,y)=\delta(x-y)$.

Noting Green's identity

$$\int_{\partial[0,1]}u\frac{\partial G(x,y)}{\partial n}-G(x,y)\frac{\partial u}{\partial n}\,dy=\int_{[0,1]}u\,\Delta G(x,y)-G(x,y)\,\Delta u\,dx$$

$$aG(x,0)-bG(x,1)=-u(x)+\int_{[0,1]}G(x,y)f(y)\,dy$$

$$\underline{u(x)=\int_0^1G(x,y)f(y)\,dy-aG(x,0)+bG(x,1)}$$

for orginal problem #

$\boxed{\text{observation: (2) is fourier expansion of (1).}}$