Lec 13

1. $G(x) = \ln(1+e^x)$ 在 $[a,b]$ 压缩无不动点

$G(x) = x \iff 1+e^x = e^x$

设 $a \le x < y \le b$ $|G(x) - G(y)| = |G'(\xi)||y-x|$

$= \frac{e^\xi}{1+e^\xi}|y-x| \le \frac{e^b}{1+e^b}|y-x|$

故为压缩映射.

2. $f(x)$ 充分光滑 $x^*$ 为 $m$ 重根 $m \ge 2$

(a) Newton 法一阶局部收敛. 设 $f(x) = (x-x^*)^m g(x)$, $g(x^*) \ne 0$

Newton 法可写作 $x_{k+1} = \varphi(x_k) = x_k - \frac{f(x_k)}{f'(x_k)}$ 为不动点迭代

$\varphi(x) = x - \frac{f(x)}{f'(x)}$ $\varphi'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}$

$$= \frac{(x-x^*)^m g(x)\left[m(m-1)(x-x^*)^{m-2}g(x) + m(x-x^*)^{m-1}g'(x) + m(x-x^*)^{m-1}g'(x) + (x-x^*)^m g''(x)\right]}{\left[m(x-x^*)^{m-1}g(x) + (x-x^*)^m g'(x)\right]^2}$$

$$= \frac{g(x)\left[m(m-1)g(x) + 2m(x-x^*)g'(x) + (x-x^*)^2 g''(x)\right]}{\left[mg(x) + (x-x^*)g'(x)\right]^2}$$ $\varphi'(x^*) = \frac{m-1}{m} \ne 0$

故为线性

(b) $S(x) = \frac{f(x)}{f'(x)}$, $S$ 的 Newton 法 至少二阶局部收敛

$$S(x) = \frac{(x-x^*)^m g(x)}{m(x-x^*)^{m-1}g(x) + (x-x^*)^m g'(x)} = \frac{(x-x^*)g(x)}{mg(x) + (x-x^*)g'(x)}$$

$$S'(x) = \frac{[g(x) + (x-x^*)g'(x)][mg(x) + (x-x^*)g'(x)] - (x-x^*)g(x)[(m+1)g'(x) + (x-x^*)g''(x)]}{[mg(x) + (x-x^*)g'(x)]^2}$$

$S(x^*) = 0$ $S'(x^*) = \frac{mg(x^*)^2}{m^2 g^2(x^*)} = \frac{1}{m} \ne 0$

故 $x^*$ 是单根. Newton 法至少 2 阶收敛.

(c) $x_{k+1} = x_k - m\dfrac{f(x_k)}{f'(x_k)}$ 至少2阶局部收敛

同理 $\varphi(x) = x - m\dfrac{f(x)}{f'(x)}$

$$\varphi'(x) = 1 - m\frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = (1-m) + m\frac{f(x)f''(x)}{[f'(x)]^2}$$

$\varphi'(x^*) = 1-m + m\cdot\dfrac{m-1}{m} = 0$ 故收敛速度最少2阶

3. $x_{k+1} = x_k - \dfrac{f(x_k)}{d}$ $k=0,1,2,\cdots$ 设 $f(x^*)=0$

(a) 什么d使局部收敛

看成不动点迭代

$\varphi(x) = x - \dfrac{f(x)}{d}$ $\varphi \in C^1$

若 $|\varphi'(x^*)|<1$ 即 $\left|1-\dfrac{f'(x^*)}{d}\right|<1$

若 $f'(x^*)>0$ 时 $d>\dfrac{f'(x^*)}{2}$

$f'(x^*)<0$ 时 $d<\dfrac{f'(x^*)}{2}$

可保证局部收敛

(b) 收敛速度

$\varphi'(x) = 1-\dfrac{f'(x)}{d}$

$f^{(k)}(x) = -\dfrac{f^{(k)}(x)}{d}$ . $k\geq 2$

由不动点迭代结论,

若 $d\neq f'(x^*)$, 则线性收敛

若 $d=f'(x^*)$, $f^{(k)}(x^*)\neq 0$, $f^{(k-1)}(x^*)=\cdots=f^{(2)}(x^*)=0$

则至少k阶收敛

(c) 是否存在 $\alpha$ 使二阶局部收敛

由(b)可以. 只要 $f'(x^*)\neq 0$，取 $\alpha = f'(x^*)$

则至少 2 阶局部收敛

Lec14 1. $v, w \in \mathbb{R}^n$，$A$ 可逆，$w^T A^{-1} v \neq -1$

证 $(A+vw^T)^{-1} = A^{-1} - \dfrac{A^{-1}v \cdot w^T A^{-1}}{1+w^T A^{-1} v}$

由于 $(A+vw^T)\left(A^{-1} - \dfrac{A^{-1} v w^T A^{-1}}{1+w^T A^{-1} v}\right)$

$= I_n + vw^T A^{-1} - \dfrac{vw^T A^{-1}}{1+w^T A^{-1} v} - \dfrac{vw^T A^{-1} vw^T A^{-1}}{1+w^T A^{-1} v}$

$= I_n + \dfrac{(w^T A^{-1} v)\, vw^T A^{-1}}{1+w^T A^{-1} v} - \dfrac{(w^T A^{-1} v)\, vw^T A^{-1}}{1+w^T A^{-1} v} = I_n$ 得证

($C, D \in \mathbb{R}^{n\times n}$，$CD = I_n \Rightarrow C, D$ 可逆，$D = C^{-1}$)

Lec15. $f, g$ 周期 $2\pi$，分片光滑

1 $\quad \hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx \in \mathbb{C}$

$$(f*g)(x) = \int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy$$

(1) $\widehat{(f')}_k = ik\hat{f}_k$，$f \in C^1_{2\pi}$

证：$ik\hat{f}_k = \frac{ik}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$

$$\widehat{(f')}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x) e^{-ikx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}\, df(x) = \frac{-1}{2\pi}\int_{-\pi}^{\pi} f(x)\, de^{-ikx}\, dx$$

$$= \frac{ik}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$$

(2) $\widehat{f(x-a)}_k = e^{-ika}\hat{f}_k$

证：$\widehat{f(x-a)}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-a)\, e^{-ikx}\, dx = \frac{e^{-ika}}{2\pi}\int_{-\pi-a}^{\pi-a} f(y)\, e^{-iky}\, dy = e^{-ika}\hat{f}_k$

(3) $\widehat{f*g}_k = 2\pi \hat{f}_k \hat{g}_k$

证：$\widehat{(f*g)}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)\, g(y)\, e^{-ikx}\, dy\, dx = \frac{1}{2\pi}\int_{-\pi-t}^{\pi-t}\int_{-\pi}^{\pi} f(t)\, g(y) e^{-ik(y+t)}\, dy\, dt$

$$= \frac{1}{2\pi}\int_{-\pi-t}^{\pi-t} f(t) e^{-ikt}\int_{-\pi}^{\pi} g(y)\, e^{-iky}\, dy = 2\pi \hat{f}_k \hat{g}_k$$

2. $\quad a = (a_0, \cdots a_{N-1})^T \quad \hat{a} = c = (c_0, \cdots c_{N-1})^T$

$n = 0, 1, \cdots N-1, \quad b^{(n)}_k := a_k \cos\frac{2\pi k n}{N}$

$$d^{(n)}_k := c_{k+n} + c_{k-n}$$

证明 $\hat{b}^{(n)} = \frac{1}{2} d^{(n)}$

证：$b^{(n)} = \Lambda^{(n)} a$，$\Lambda^{(n)}$ 为对角阵，$\Lambda^{(n)}_{jj} = \cos\frac{2\pi n j}{N} = \frac{1}{2}\left(e^{i\frac{2\pi n j}{N}} + e^{-i\frac{2\pi n j}{N}}\right)$

$$= \frac{1}{2}\left(w^{nj} + w^{-nj}\right)$$

$(b^{(n)})^\wedge = F\Lambda^{(n)} a$

$$\Lambda_1^{(n)} = \begin{pmatrix} 1 & & & \\ & w^n & & \\ & & \ddots & \\ & & & w^{n(N-1)} \end{pmatrix} \quad \Lambda_2^{(n)} = \begin{pmatrix} 1 & & & \\ & w^{-n} & & \\ & & \ddots & \\ & & & w^{-n(N-1)} \end{pmatrix}$$

$$F\Lambda^n = \frac{1}{2}\left(F\Lambda_1^{(n)} + F\Lambda_2^{(n)}\right) = \frac{1}{2}\begin{pmatrix} 1 & \cdots & 1 \\ 1 & w \cdots & w^{N-1} \\ \vdots & & \\ 1 & w^{N-1} \cdots & w^{(N-1)^2} \end{pmatrix}\left(\begin{pmatrix} 1 & & \\ & w^n & \\ & & \ddots \\ & & & w^{n(N-1)} \end{pmatrix} + \begin{pmatrix} 1 & & \\ & w^{-n} & \\ & & \ddots \\ & & & w^{-n(N-1)} \end{pmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{pmatrix} 1 & w^n & \cdots & w^{n(N-1)} \\ 1 & w^{n+1} & & w^{(n+1)(N-1)} \\ \vdots & \vdots & & \vdots \\ 1 & w^{n+N-1} & \cdots & w^{(n+N-1)(N-1)} \end{pmatrix} + \begin{pmatrix} 1 & w^{-n} & \cdots & w^{-n(N-1)} \\ 1 & w^{-n+1} & & \vdots \\ \vdots & \vdots & & \\ 1 & w^{-n+(N-1)} & \cdots & w^{(-n+N-1)(N-1)} \end{pmatrix}\right)$$

由此立得

$$(b^{(n)})^\wedge_k = \frac{1}{2}\left(C_{k+n} + C_{k-n}\right)$$

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3. $a = (a_0, \cdots a_{N-1})^T \quad b_k = a_{-k} = a_{N-k}$ 证明 $\hat{\hat{a}} = Nb$

$$F = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & w \cdots & w^{N-1} \\ \vdots & & \vdots \\ 1 & w^{N-1} \cdots & w^{(N-1)^2} \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ 1 & w \cdots & w^{N-1} \\ \vdots & & \vdots \\ 1 & w^{N-1} \cdots & w^{(N-1)^2} \end{pmatrix}$$

$$\hat{\hat{a}} = F^2 a \qquad (F^2)_{ij} = \sum_{k=0}^{N-1} w^{(i+j)k} = \begin{cases} 0 & i+j \neq 0, N \\ N & i+j = 0, N \end{cases}$$

$$F^2 = \begin{pmatrix} N & 0 & \cdots & 0 \\ 0 & 0 & \cdots & N \\ 0 & 0 & \cdots N & 0 \\ \vdots & & & \vdots \\ N & 0 & \cdots & 0 \end{pmatrix}$$ 故 $\hat{\hat{a}} = Nb$.

4. $a = (a_0 \cdots a_{N-1})^T \quad b_k = \hat{a}_{k+1} - \hat{a}_k \quad k = 0, \cdots N-1$

$C_k = a_k(W_N^k - 1)$ 则有 $b = \hat{c}$

证：只需证 $b^\vee = c$ 记 $S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & \cdots & 0 & 1 \\ 1 & 0 & \cdots & & 0 \end{pmatrix}$

则 $b = S\hat{a} - \hat{a}$

$b^\vee = (S\hat{a})^\vee - a = F^{-1}SFa - a$

$$F^{-1}SF = \frac{1}{N}\begin{pmatrix} 1 & \cdots & 1 \\ 1 & w^{-1} \cdots & w^{-(N-1)} \\ \vdots & & \vdots \\ 1 & w^{-(N-1)} \cdots & w^{-(N-1)^2} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & 1 \\ & & & & \\ 1 & 0 & \cdots & & 0 \end{pmatrix}\begin{pmatrix} 1 & \cdots & 1 \\ 1 & w \cdots & w^{N-1} \\ \vdots & & \vdots \\ 1 & w^{N-1} \cdots & w^{(N-1)^2} \end{pmatrix}$$

$$F^{-1}SF = \frac{1}{N}\begin{pmatrix} 1 & - & - & 1 \\ 1 & w^{-1} & \cdots & w^{-(N-1)} \\ \vdots \\ 1 & w^{-(N-1)} & \cdots & w^{-(N-1)^2} \end{pmatrix}\begin{pmatrix} 1 & w & \cdots & w^{N-1} \\ 1 & w^2 & \cdots & w^{2(N-1)} \\ \vdots \\ 1 & w^{N-1} & \cdots & w^{(N-1)^2} \\ 1 & \cdots & \cdots & 1 \end{pmatrix}$$

$$(F^{-1}SF)_{ij} = \frac{1}{N}\sum_{k=0}^{N-1} w^{-ik}\, w^{j(k+1)} = \frac{w^j}{N}\sum_{k=0}^{N-1} w^{(j-i)k} = \begin{cases} w^i & i=j \\ 0 & i\neq j \end{cases}$$

$$\Rightarrow b^{\vee} = F^{-1}SFa - a = (W_N^0 \cdots W_N^{N-1})\,a - a. \quad \Rightarrow b = \check{c}. \quad \#$$

Lec 16

1. DST

$$F_k = \sum_{j=1}^{N-1} f_j \sin\frac{jk\pi}{N} \qquad k=1,\cdots N-1$$

逆变换 $f_j = \frac{2}{N}\sum_{k=1}^{N-1} F_k \sin\frac{jk\pi}{N}$ , $j=1,\cdots N-1$

证:

$$A = \begin{pmatrix} \sin\frac{\pi}{N} & \cdots & \sin\frac{N-1}{N}\pi \\ \vdots & & \vdots \\ \sin\frac{N-1}{N}\pi & \cdots & \sin\frac{(N-1)^2}{N}\pi \end{pmatrix} \qquad B = \frac{2}{N}\begin{pmatrix} \sin\frac{\pi}{N} & \cdots & \sin\frac{N-1}{N}\pi \\ \vdots \\ \sin\frac{N-1}{N}\pi & \cdots & \sin\frac{(N-1)^2}{N}\pi \end{pmatrix}$$

只需证 $AB = I_n$

$$(AB)_{sj} = \frac{2}{N}\sum_{k=1}^{N-1} \sin\frac{sk\pi}{N}\sin\frac{jk\pi}{N} = \frac{1}{2}\sum_{k=1}^{N-1}\left[\cos\frac{(s-j)k\pi}{N} - \cos\frac{(s+j)k\pi}{N}\right]$$

$$= \left(\frac{1}{2}\mathrm{Re}\sum_{k=1}^{N-1} e^{i\frac{(s-j)}{N}k\pi} - \frac{1}{2}\mathrm{Re}\sum_{k=1}^{N-1} e^{i\frac{s+j}{N}k\pi}\right)\frac{2}{N} = \left(\frac{1}{2}\mathrm{Re}\sum_{k=1}^{N-1} e^{i\frac{(s-j)k\pi}{N}} + \frac{1}{2}\right)\frac{2}{N}$$

$$= \begin{cases} 0 & s\neq j \\ \frac{N}{2} & s=j \end{cases} \qquad \#$$

2. DCT

$$F_k = \frac{1}{2}(f_0 + (-1)^k f_N) + \sum_{j=1}^{N-1} f_j \cos\frac{jk\pi}{N} \quad k=0,\cdots N$$

证 $f_j = \frac{2}{N}\left(\frac{1}{2}(F_0 + (-1)^j F_N) + \sum_{k=1}^{N-1} F_k \cos\left(\frac{jk\pi}{N}\right)\right) \quad j=0,1,\cdots N$

记 DCT 矩阵为 A

$$A_{ij} = \begin{cases} \cos\frac{ij\pi}{N} & j\neq 0, N \\ \frac{1}{2}\cos\frac{ij\pi}{N} & j=0 \text{ or } N \end{cases}$$

记 $B = \frac{2}{N}A \quad 1\le j \le N-1$ 时

$$(AB)_{ij} = \frac{2}{N}\sum_{k=0}^{N} A_{ik}A_{kj} = \frac{2}{N}\left(\sum_{k=1}^{N-1}\cos\frac{ik\pi}{N}\cos\frac{jk\pi}{N} + \frac{1}{2}(-1)^{i+j} + \frac{1}{2}\right)$$

$$= \frac{2}{N}\left(\frac{1}{2}\sum_{k=1}^{N-1}\left(\cos\frac{(i+j)k\pi}{N} + \cos\frac{(i-j)k\pi}{N}\right) + \frac{1}{2}(-1)^{i+j} + \frac{1}{2}\right)$$

$$= \frac{2}{N}\left(\begin{cases} -1 & 2|i-j,\ i\neq j \\ 0 & 2\nmid i-j \\ \frac{N}{2}-1 & i=j \end{cases} + \frac{(-1)^{i+j}+1}{2}\right) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$(AB)_{i0} = \frac{1}{N}\left(\sum_{k=1}^{N-1}\cos\frac{ik\pi}{N} + \frac{(-1)^i+1}{2}\right) = \begin{cases} 1 & i=0 \\ 0 & i\neq 0 \end{cases}$$

$$(AB)_{iN} = \frac{1}{N}\left(\frac{1}{2}\sum_{k=1}^{N-1}\left(\cos\frac{(i+N)k\pi}{N} + \cos\frac{(i-N)k\pi}{N}\right) + \frac{1}{2}(-1)^{i+N} + \frac{1}{2}\right) = \begin{cases} 1 & i=N \\ 0 & i\neq N \end{cases}$$

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$$\boxed{\cos a + \cdots + \cos(a+(n-1)d) = \frac{\sin\left(\frac{nd}{2}\right)}{\sin\frac{d}{2}}\cos\left(a + \frac{n-1}{2}d\right)}$$

3. DQWCT

$$F_k = \sum_{j=0}^{N-1} f_j \cos\frac{\pi k(j+\frac{1}{2})}{N} \quad k=0,1,\cdots N-1$$

$$f_j = \frac{2}{N}\left(\sum_{k=0}^{N-1}\right)' F_k \cos\frac{\pi k(j+\frac{1}{2})}{N} \quad j=0,1,\cdots N-1$$

DQWCT 矩阵 A $\quad A_{kj} = \cos\frac{\pi k(j+\frac{1}{2})}{N}$

矩阵 B $\quad B_{jk} = \begin{cases} \frac{2}{N}\cos\frac{\pi k(j+\frac{1}{2})}{N} & k>0 \\ \frac{1}{N} & k=0 \end{cases}$

$j>0$ 时

$$(AB)_{ij} = \sum_{k=0}^{N-1} A_{ik}B_{kj} = \frac{2}{N}\sum_{k=0}^{N-1}\cos\frac{\pi i(k+\frac{1}{2})}{N}\cos\frac{\pi j(k+\frac{1}{2})}{N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\cos\frac{\pi(i+j)(k+\frac{1}{2})}{N} + \cos\frac{\pi(i-j)(k+\frac{1}{2})}{N}\right)$$

$= \frac{1}{N}\left( \quad 0 \quad + \begin{cases} 0 & i \neq j \\ N & i = j \end{cases} \right) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$(AB)_{i0} = \frac{1}{N}\sum_{k=0}^{N-1} A_{ik} = \frac{1}{N}\sum_{k=0}^{N-1} \cos\frac{i\pi(k+\frac{1}{2})}{N} = \frac{1}{N}\begin{cases} N & i=0 \\ 0 & i \neq 0 \end{cases} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

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4. (Reference TA's answer)

$A[u_0 \cdots u_N]^T = [f_0 \cdots f_N]^T$

$u_1 - u_0 = f_0$

$u_{j-1} - 2u_j + u_{j+1} = f_j$

$u_{N-1} - u_N = f_N$

作DCT

$u_j = \frac{2}{N}\left(\frac{1}{2}U_0 + \sum_{k=1}^{N} U_k \cos\left(\frac{\pi}{N+1}(j+\frac{1}{2})k\right)\right)$

$f_j = \frac{2}{N}\left(\frac{1}{2}F_0 + \sum_{k=1}^{N} F_k \cos\left(\frac{\pi}{N+1}(j+\frac{1}{2})k\right)\right)$

$u_1 - u_0 = \frac{2}{N}\sum_{k=1}^{N} U_k\left(\cos\left(\frac{\pi}{N+1}\frac{3}{2}k\right) - \cos\left(\frac{\pi}{N+1}\frac{1}{2}k\right)\right)$

$= f_0 = \frac{2}{N}\left(\frac{1}{2}F_0 + \sum_{k=1}^{N} F_k \cos\left(\frac{\pi}{N+1}\frac{1}{2}k\right)\right)$

$u_{j-1} - 2u_j + u_{j+1} = \frac{2}{N}\sum_{k=1}^{N} U_k\left(\cos\left(\frac{\pi}{N+1}(j-\frac{1}{2})k\right) + \cos\frac{\pi}{N+1}(j+\frac{3}{2})k\right.$

$\left. - 2\cos\frac{\pi}{N+1}(j+\frac{1}{2})k\right) = f_j = \frac{2}{N}\left(\frac{1}{2}F_0 + \sum_{k=1}^{N} F_k \cos\left(\frac{\pi}{N+1}(j+\frac{1}{2})k\right)\right)$

$u_{N-1} - u_N = \frac{2}{N}\sum_{k=1}^{N} U_k\left(\cos\frac{\pi}{N+1}(N-\frac{1}{2})k - \cos\frac{\pi}{N+1}(N+\frac{1}{2})k\right)$

$= f_N = \frac{2}{N}\left(\frac{1}{2}F_0 + \sum_{k=1}^{N} F_k \cos\frac{\pi}{N+1}(N+\frac{1}{2})k\right)$

有 $\frac{2}{N}\sum_{k=1}^{N} -4U_k \sin^2\frac{\pi k}{2(N+1)} \cos\frac{\pi}{N+1}(j+\frac{1}{2})k = \frac{2}{N}\left(\frac{1}{2}F_0 + \sum_{k=1}^{N} F_k \cos\frac{\pi}{N+1}(j+\frac{1}{2})k\right)$

$-4U_j \sin^2\frac{\pi j}{2(N+1)} = F_j$

因此特征值 $-4\sin^2\frac{\pi j}{2(N+1)}$

特征向量 $[\frac{1}{2} \cdots \frac{1}{2}]^T \quad j=0$

$[\cos\frac{\pi}{N+1}\frac{1}{2}j, \cdots \cos\frac{\pi}{N+1}(N+\frac{1}{2})j]^T, \ j \neq 0$