Lec 9

1. $B_n(x)$ Bernoulli Polynomial

$$B_n(x) = (-1)^n B_n(1-x) \quad x \in \mathbb{R}, \quad n = 0, 1, \cdots$$

2. $\widetilde{B}_n(x)$ 为 $B_n(x)$ $[0,1]$ 至 $\mathbb{R}$ 周期延拓

$n \geq 2$, $$\widetilde{B}_{2m}(x) = 2(-1)^{m-1} \sum_{k=1}^{\infty} \frac{\cos 2\pi kx}{(2\pi k)^{2m}}$$

$$\widetilde{B}_{2m+1}(x) = 2(-1)^{m-1} \sum_{k=1}^{\infty} \frac{\sin 2\pi kx}{(2\pi k)^{2m+1}}$$

Lec 9

1. $\begin{cases} B_0(x)=1 \\ B_n'(x) = B_{n-1}(x), \quad \int_0^1 B_n(x)\,dx = 0 \end{cases}$

(1) $B_n(x) = (-1)^n B_n(1-x)$

用归纳法. $n=0$ 成立

设 $n<k$ 成立

$n=k$ 时　由于 $B_{k-1}(x) = (-1)^{k-1} B_{k-1}(1-x)$

$$\int_0^x B_{k-1}(t)\,dt = (-1)^{k-1}\int_0^x B_{k-1}(1-t)\,dt$$

$$B_k(x) - B_k(0) = (-1)^k \int_1^{1-x} B_{k-1}(s)\,ds$$

$$= (-1)^k \left[B_k(1-x) - B_k(1)\right]$$

$$B_k(x) = (-1)^k B_k(1-x) + C$$

$$0 = \int_0^1 B_k(x)\,dx = (-1)^k \int_0^1 B_k(1-x)\,dx + C \Rightarrow C=0$$

$$B_k(x) = (-1)^k B_k(1-x)$$

(2) 由于 $B_n \in C^\infty[0,1]$ 显然 B_n 的 Fourier 级数收敛

$B_n(0) = B_n(1)$ 且

设 $B_n(x) = \sum_{k\in\mathbb{Z}} a_{n,k} e^{2\pi i k x}$

$$a_{n,k} = \int_0^1 B_n(x) e^{-2\pi i k x}\,dx = -\frac{1}{2\pi i k}\left[B_n(x)e^{-2\pi i k x}\right]\Big|_0^1$$

$$+\frac{1}{2\pi i k}\int_0^1 B_{n-1}(x) e^{-2\pi i k x}\,dx$$

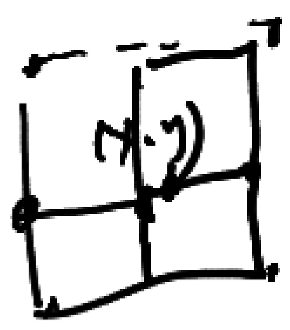
$\Rightarrow a_{n,k} = \frac{1}{2\pi i k} a_{n-1,k}$. 结合 $a_{0,k} = \begin{cases} 0 & k\neq 0 \\ 1 & k=0 \end{cases}$

$a_{n,0} = 0$ 立得所求. #

Lec10

1. $\frac{\partial^2 u}{\partial x \partial y} \approx \frac{\partial}{\partial y}\left(\frac{1}{2h}\left[u(x+h,y)-u(x-h,y)\right]\right)$

$$\approx \frac{1}{4h^2}\left[u(x+h,y+h)-u(x-h,y+h)-u(x+h,y-h)+u(x-h,y-h)\right]$$



由Taylor展开 $u(x+\varepsilon_1,y+\varepsilon_2)=u(x,y)+\frac{\partial u}{\partial x}\varepsilon_1+\frac{\partial u}{\partial y}\varepsilon_2+\frac{1}{2}[\varepsilon_1\ \varepsilon_2]\begin{bmatrix}u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{bmatrix}[\varepsilon_1\ \varepsilon_2]$

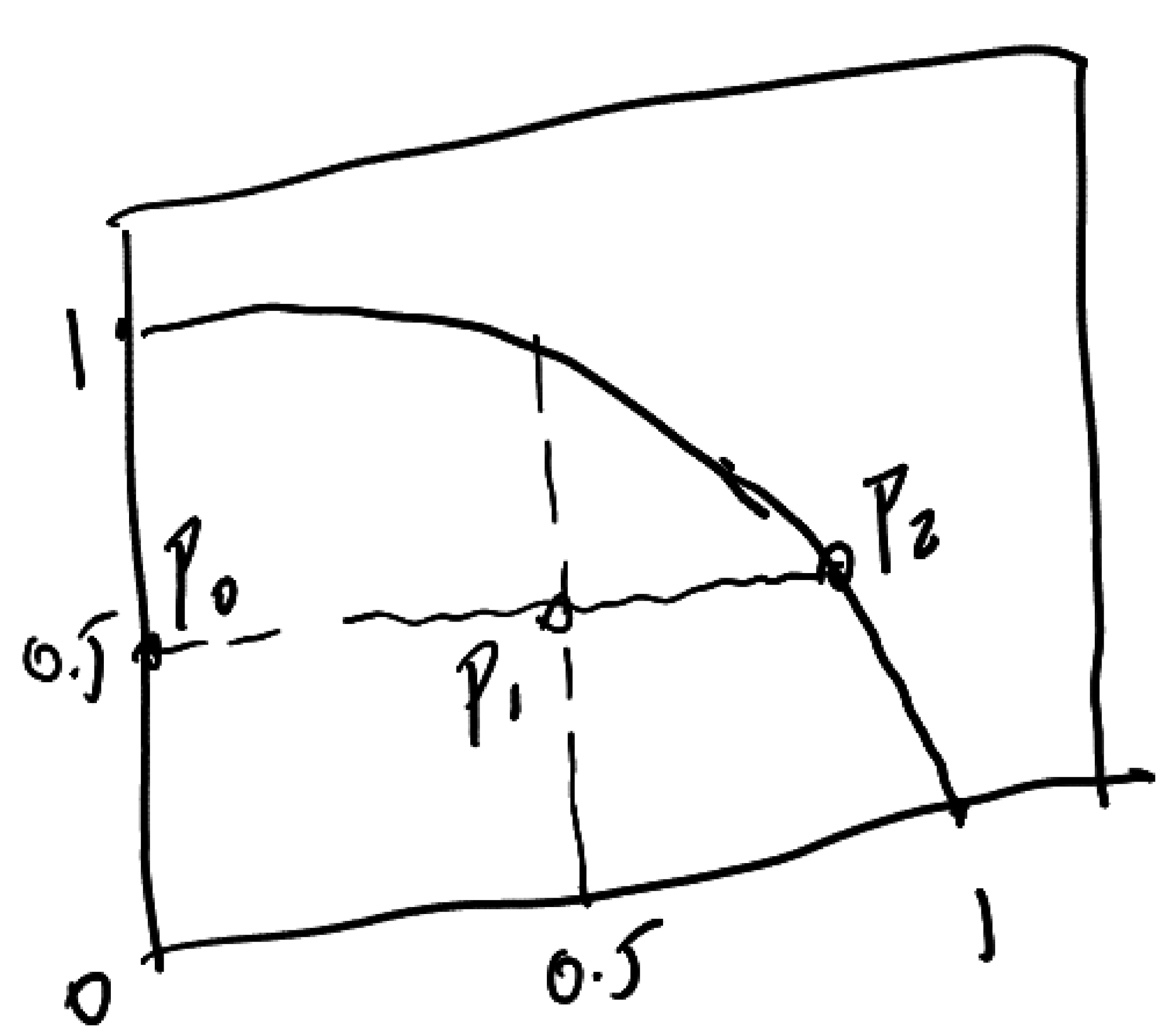
$$+\frac{1}{3!}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}u_{x_i x_j x_k}\varepsilon_i\varepsilon_j\varepsilon_k$$

$$+\frac{1}{4!}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}u_{x_i x_j x_k x_l}\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_l+O(h^5)$$

$u(x+h,y+h)-u(x-h,y+h)-u(x+h,y-h)+u(x-h,y-h)$

$=4h^2 u_{xy}+O(h^4)$ 误差 $O(h^4)$

2. $P_0(0,0.5)$ $P_1(0.5,0.5)$ $P_2\left(\frac{\sqrt{3}}{2},0.5\right)$



由Taylor展开

$u(x+\Delta,y)=u(x,y)+\Delta\frac{\partial u}{\partial x}+\frac{\Delta^2}{2!}\frac{\partial^2 u}{\partial x^2}+\frac{\Delta^3}{3!}\frac{\partial^3 u}{\partial y^3}+\cdots$

$$u(x+\lambda_1\Delta,y)=u(x,y)+\lambda_1\Delta u_x+\frac{\lambda_1^2\Delta^2}{2!}u_{xx}+O(\Delta^3)$$

$$u(x+\lambda_2\Delta,y)=u(x,y)+\lambda_2\Delta u_x+\frac{\lambda_2^2\Delta^2}{2!}u_{xx}+O(\Delta^3)$$

$$\Rightarrow \lambda_2^2 u(x+\lambda_1\Delta,y)-\lambda_1^2 u(x+\lambda_2\Delta,y)=(\lambda_2^2-\lambda_1^2)u(x,y)+\lambda_1\lambda_2(\lambda_2-\lambda_1)u_x+O(\Delta^3)$$

$$u_x \approx \frac{\lambda_2^2 u(x+\lambda_1\Delta,y)-\lambda_1^2 u(x+\lambda_2\Delta,y)-(\lambda_2^2-\lambda_1^2)u(x,y)}{\lambda_1\lambda_2(\lambda_2-\lambda_1)}$$

误差 $O(\Delta^3)$. 本题取 $\Delta=0.5$ $\lambda_1=-1$ $\lambda_2=\frac{\sqrt{3}-1}{2}$

$$\left.\frac{\partial u}{\partial x}\right|_{P_1} \approx -\sqrt{3}\,u(P_1)+2u(P_2)-(2-\sqrt{3})\,u(P_0)$$ #

Lec 11

1, 3 见程序代码

2. $\mathrm{Cov}(f(x), f(1-x))$

$$= \int_0^1 f(x) f(1-x)\, dx - \left[\int_0^1 f(x) dx\right]^2$$

$$= \int_0^1\int_0^1 f(x) f(1-x)\, dx dy - \int_0^1\int_0^1 f(x) f(y)\, dx dy$$

$$= \int_0^1\int_0^1 f(x) f(1-x)\, dx dy - \int_0^1\int_0^1 f(x) f(1-y)\, dx dy$$

$$= \int_0^1 dx \int_0^x f(x) [f(1-x) - f(1-y)]\, dy + \int_0^1 dy \int_0^y f(x) [f(1-x) - f(1-y)]\, dy$$

$$= \int_0^1 dx \int_0^x [f(x) - f(y)] [f(1-x) - f(1-y)]\, dy \le 0$$

$E I_N = I(f)$

可取 $I_N = \frac{1}{2N} \sum_{i=1}^{N} (f(x_i) + f(1-x_i))$

$\mathrm{Var}\, I_N = \frac{1}{2N} (\mathrm{Var}(f) + \mathrm{Cov}(f(x), f(1-x))) \le \frac{1}{2N} \mathrm{Var} f$

Lec 12

1. $A(\sigma \to \sigma') = \dfrac{1}{1 + e^{-\beta H(\sigma) + \beta H(\sigma')}}$ $\quad p(\sigma, \sigma') = G(\sigma, \sigma') A(\sigma, \sigma')$

detailed balance 条件

$$\pi(\sigma) P(\sigma, \sigma') = G(\sigma, \sigma') \pi(\sigma) A(\sigma, \sigma')$$

$$= G(\sigma, \sigma') \frac{e^{-\beta H(\sigma)}}{1 + e^{-\beta H(\sigma) + \beta H(\sigma')}}$$

$$= G(\sigma', \sigma) \frac{1}{e^{\beta H(\sigma)} + e^{\beta H(\sigma')}}$$

$$= G(\sigma', \sigma) \pi(\sigma') A(\sigma', \sigma)$$

$$= \pi(\sigma') P(\sigma', \sigma)$$ #