Problem 1

(a) $\left| \mathbb{E}_S [e_{n,h}(x)] \right| = \left| \mathbb{E}_S \left(\hat{p}_{n,h}(x) - \frac{p_j^*}{h} + \frac{p_j^*}{h} - p^*(x) \right) \right|$

$\leq \left| \mathbb{E}_S \left(\hat{p}_{n,h}(x) - \frac{p_j^*}{h} \right) \right| + \left| \frac{p_j^*}{h} - p^*(x) \right|$

$= \left| \frac{1}{nh} \mathbb{E}_S \sum_{i=1}^{n} 1(X_i \in B_j) - \frac{p_j^*}{h} \right| + \left| \frac{1}{h} \int_{B_j} p^*(y)\, dy - p^*(x) \right|$

$= \left| \frac{1}{nh} n\, p_j^* - \frac{p_j^*}{h} \right| + \left| \frac{1}{h} \int_{B_j} [p^*(y) - p^*(x)]\, dy \right|$

$\leq \frac{1}{h} \int_{B_j} L^* |y-x|\, dy \leq \frac{L^*}{h} \int_{B_j} h\, dy = h L^*$

$\mathrm{Var}_S [e_{n,h}(x)] = \mathrm{Var}_S \left(\frac{1}{h} \frac{\sum_{i=1}^{n} 1(X_i \in B_j)}{n} \right)$

$= \frac{1}{n^2 h^2} \mathrm{Var}_S \sum_{i=1}^{n} 1(X_i \in B_j)$

$= \frac{n\, p_j^* (1 - p_j^*)}{n^2 h^2}$ (Because $1(X_i \in B_j) \sim B(1, p_j^*)$)

$= \frac{p_j^* (1 - p_j^*)}{n h^2}$

(b) $\mathbb{E}_S \int_0^1 e_{n,h}^2(x)\, dx = \int_0^1 \mathbb{E}_S\, e_{n,h}^2(x)\, dx = \int_0^1 [\mathbb{E}_S\, e_{n,h}(x)]^2 + \mathrm{Var}_S\, e_{n,h}(x)\, dx$

$\leq h^2 L^{*2} + \int_0^1 \frac{p_j^*}{n h^2}\, dx = h^2 L^{*2} + \frac{1}{n h^2} \sum_j p_j^* h = h^2 L^{*2} + \frac{1}{nh}$ #