14 $u=u(x,y,z,t)$ 是

$$\begin{cases} u_{tt}-a^2\Delta u=0 & \text{in } \mathbb{R}^3\times\mathbb{R}_+ \\ u|_{t=0}=f(x)+g(y) & \text{on } \mathbb{R}^3 \\ u_t|_{t=0}=\varphi(y)+\psi(z) & \text{on } \mathbb{R}^3 \end{cases} \quad a>0$$

求 $u(x,y,z,t)$

解：利用线性叠加原理，子问题退化为一维问题

$$\begin{cases} u_{tt}-a^2\Delta u=0 & \text{in } \mathbb{R}^3\times\mathbb{R}_+ \\ u|_{t=0}=f(x) & \text{on } \mathbb{R}^3 \\ u_t|_{t=0}=0 & \text{on } \mathbb{R}^3 \end{cases} \Rightarrow u_1=\frac{1}{2}[f(x+at)+f(x-at)]$$

$$\begin{cases} u_{tt}-a^2\Delta u=0 & \text{in } \mathbb{R}^3\times\mathbb{R}_+ \\ u|_{t=0}=g(y) & \text{on } \mathbb{R}^3 \\ u_t|_{t=0}=\varphi(y) & \text{on } \mathbb{R}^3 \end{cases} \Rightarrow u_2=\frac{1}{2}[g(y+at)+g(y-at)]+\frac{1}{2a}\int_{y-at}^{y+at}\varphi(\xi)\,d\xi$$

$$\begin{cases} u_{tt}-a^2\Delta u=0 & \text{in } \mathbb{R}^3\times\mathbb{R}_+ \\ u|_{t=0}=0 & \text{on } \mathbb{R}^3 \\ u_t|_{t=0}=\psi(z) & \text{on } \mathbb{R}^3 \end{cases} \Rightarrow u_3=\frac{1}{2a}\int_{z-at}^{z+at}\psi(\xi)\,d\xi$$

$$u=u_1+u_2+u_3=\frac{1}{2}[f(x+at)+f(x-at)+g(y+at)+g(y-at)]$$

$$+\frac{1}{2a}\int_{y-at}^{y+at}\varphi(\xi)\,d\xi+\frac{1}{2a}\int_{z-at}^{z+at}\psi(\xi)\,d\xi$$

由能量不等式，解唯一。#

15 $\Phi \in C^2(\mathbb{R})$ $\alpha \in \mathbb{R}^n$, $|\alpha| = 1$ 则 $\Phi(\alpha\cdot x + at)$ 满足

$$u_{tt} - a^2 \Delta u = 0 \quad (a > 0)$$

证明: $u = \Phi(\alpha \cdot x + at)$

$$u_t = a\,\Phi'(\alpha\cdot x + at)$$

$$u_{tt} = a^2\,\Phi''(\alpha\cdot x + at)$$

$$\nabla u = \alpha\,\Phi'(\alpha\cdot x + at)$$

$$\Delta u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\alpha_i \Phi'(\alpha\cdot x + at)\right)$$

$$= \sum_{i=1}^{n} \alpha_i^2\, \Phi''(\alpha\cdot x + at) = \Phi''(\alpha\cdot x + at) = \frac{1}{a^2} u_{tt}$$

#

17 $a>0$

(1) $\begin{cases} u_{tt} - a^2 \Delta u = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}_+ \\ u|_{t=0} = x^2(x+y), \quad u_t|_{t=0} = 0 & \text{on } \mathbb{R}^2 \end{cases}$

由 Poisson 公式

$$u(x,y,t) = \frac{1}{2\pi a t} \int_{(s_1-x)^2+(s_2-y)^2 \le a^2t^2} \frac{s_1^2(s_1+s_2) + 3s_1^2(s_1-x) + s_1^2(s_2-y)}{\sqrt{a^2t^2-(s_1-x)^2-(s_2-y)^2}} \, ds_1 \, ds_2$$

$$= \frac{1}{2\pi a t} \int_{\substack{r\le 1 \\ 0\le\theta\le 2\pi}} \frac{4(x+atr\cos\theta)^3 + 2(x+atr\cos\theta)^2(y+atr\sin\theta) - 3(x+atr\cos\theta)^2 x - (x+atr\cos\theta)^2 y}{\sqrt{1-r^2}} \cdot r\,dr\,d\theta$$

$$= \frac{1}{2\pi a t} \int_{\substack{r\le 1 \\ 0\le\theta\le 2\pi}} r \frac{(x+atr\cos\theta)^2(x+4atr\cos\theta) + (x+atr\cos\theta)^2(y+2atr\sin\theta)}{\sqrt{1-r^2}} \, dr\,d\theta$$

#

(2) $\begin{cases} u_{tt} - a^2 \Delta u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}_+ \\ u|_{t=0} = x^2+y^2 z & \text{on } \mathbb{R}^3 \\ u_t|_{t=0} = 1+y & \text{on } \mathbb{R}^3 \end{cases}$

由 Kirchhoff 公式

$$u(x,y,z,t) = \frac{1}{4\pi a^2 t^2} \int_{\substack{(s_1-x)^2+(s_2-y)^2 \\ +(s_3-z)^2 \le a^2t^2}} \left[s_1^2 + s_2^2 s_3 + 2s_1(s_1-x) + 2s_2 s_3(s_2-y) + s_2^2(s_3-z) + t(1+s_2) \right] ds_1\, ds_2\, ds_3$$

$$= \frac{1}{4\pi a^2 t^2} \int_{\substack{r\le 1 \\ 0\le\varphi\le\pi \\ 0\le\theta\le 2\pi}} \Big[3(x+r\sin\varphi)^2 + 4(x+r\cos\varphi\cos\theta)(x+r\cos\varphi\sin\theta) + t(1+x+r\cos\varphi\cos\theta) - z(x+r\cos\varphi\cos\theta)^2 - 2(x+r\sin\varphi)x - 2(x+r\cos\varphi\cos\theta)(x+r\cos\varphi\sin\theta) - (x+r\cos\varphi\cos\theta)^2 z \Big] r^2 \sin\varphi \, dr\,d\theta\,d\varphi$$

#

18. $a>0$ $\varphi(x), \psi(x) \in C^2[0,+\infty)$, $g(t) \in C^2[0,+\infty)$

$g(0)=\varphi(0)$, $g'(0)=\psi(0)$, $g''(0)=a^2\varphi''(0)$

求
$$\begin{cases} u_{tt}-a^2u_{xx}=0 & \text{in } \mathbb{R}_+\times\mathbb{R}_+ \\ u(x,0)=\varphi(x), \quad u_t(x,0)=\psi(x) & x\geq 0 \\ u(0,t)=g(t) & , t\geq 0 \end{cases}$$

解. 作替换

$$u(x,t)=v(x,t)+g(t)$$

$$\begin{cases} v_{tt}-a^2v_{xx}=-g''(t) & \text{in } \mathbb{R}_+\times\mathbb{R}_+ \\ v(x,0)=\varphi(x)-g(0) & \text{on } \overline{\mathbb{R}_+} \\ v_t(x,0)=\psi(x)-g'(0) & \text{on } \overline{\mathbb{R}_+} \\ v(0,t)=0 & \text{on } \overline{\mathbb{R}_+} \end{cases}$$

当 $x\geq at$ 时

$$u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]-g(0)+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,d\xi-tg'(0)$$

$$-\int_0^t (t-\tau)g''(\tau)\,d\tau+g(t)$$

$$=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,d\xi$$

当 $x<at$ 时

$$u(x,t)=\frac{1}{2}[\varphi(x+at)-\varphi(at-x)]+\frac{1}{2a}\int_{at-x}^{x+at}\psi(\xi)\,d\xi$$

$$+\frac{1}{2a}\left[\int_{t-\frac{x}{a}}^{t}-2a(t-\tau)g''(\tau)\,d\tau+\int_0^{t-\frac{x}{a}}-2x\,g''(\tau)\,d\tau\right]$$

$$=\frac{1}{2}\varphi(x+at)-\frac{1}{2}\varphi(at-x)+\frac{1}{2a}\int_{at-x}^{at+x}\psi(\xi)\,d\xi+g(t)-g\left(t-\frac{x}{a}\right)$$

$$-t\left(g'(t)-g'\left(t-\frac{x}{a}\right)\right)+tg'(t)-\left(t-\frac{x}{a}\right)g'\left(t-\frac{x}{a}\right)$$

$$-\frac{x}{a}\left(g'\left(t-\frac{x}{a}\right)-g'(0)\right)$$

$$=\frac{1}{2}\varphi(x+at)-\frac{1}{2}\varphi(at-x)+\frac{1}{2a}\int_{at-x}^{at+x}\psi(\xi)\,d\xi+g(t)-g\left(t-\frac{x}{a}\right)$$

$$+\frac{x}{a}g'(0).$$