31. $u, u_x \in C(\overline{Q_T}) \cap C^{2,1}(Q_T)$

$$\begin{cases} u_t - u_{xx} = f(x,t) & \text{in } Q_T \\ u(x,0) = \varphi(x) & 0 \le x \le l \\ [-u_x + \alpha u]\big|_{x=0} = g_1(t) & 0 \le t \le T \\ [u_x + \beta u]\big|_{x=l} = g_2(t) & 0 \le t \le T \end{cases}$$

$\alpha, \beta \ge 0$ 求 $\max\limits_{Q_T} |u_x|$ 估计

解:

我们有 $\max\limits_{Q_T} |u| \le C(F+B)$, $F = \sup\limits_{Q_T} |f|$, $B = \max\{\max\limits_{[0,l]} |\varphi|, \max\limits_{[0,T]} |g_1|, \max\limits_{[0,T]} |g_2|\}$

记 $v = u_x$

$$\begin{cases} v_t - v_{xx} = f_x(x,t) & \text{in } Q_T \\ v(x,0) = \varphi'(x) & 0 \le x \le l \\ v(0,t) = \alpha u(0,t) - g_1(t) & 0 \le t \le T \\ v(l,t) = g_2(t) - \beta u(l,t), & 0 \le t \le T \end{cases}$$

有 $\max\limits_{Q_T} |u_x| \le \max\limits_{Q_T} |f_x| T + \max\{\max\limits_{[0,l]} |\varphi_x|, \max\limits_{[0,T]} |g_1| + \alpha C(F+B),$

$\max\limits_{[0,T]} |g_2| + \beta C(F+B)\}$,

$F = \sup\limits_{Q_T} |f|$, $B = \max\{\max\limits_{[0,T]} |g_1|, \max\limits_{[0,T]} |g_2|, \max\limits_{[0,l]} |\varphi|\}$.

$C = \max\{T, 1 + \frac{1+2T}{l} + \frac{l}{4}\}$

($u \in C^{2,1}(Q_T)$, $u_x t \in C^{1,0}(Q_T)$ 故 $u_x(x,t) \rightrightarrows \varphi(x)$ 当 $t \to 0$

故 $\varphi(x)$ 可导)

#.

33. $u \in C^{1,0}(\overline{Q}_T) \cap C^{2,1}(Q_T)$

$$\begin{cases} u_t - u_{xx} = 0 & (x,t) \in Q_T \\ u(x,0) = 0 & 0 \le x \le l \\ u_x + h(u_0 - u)\big|_{x=0} = 0 & 0 \le t \le T \\ u\big|_{x=l} = 0 & 0 \le t \le T \end{cases}$$

$h, u_0 > 0$

(1) $0 \le u(x,t) \le u_0$ in Q_T

(2) $u = u_h(x,t)$ 关于 h 单调递增

证明：(1) 由于 $Lu = u_t - u_{xx} = 0$

由极值原理，u 在 $\overline{Q}_T$ 最大值在 $\partial_p Q_T$ 取得

若最大值在 $\{0\} \times [0,T]$ 取得，在 $(0, t_0)$ 取得，则 $u_x\big|_{t=t_0} \le 0$

$hu = u_x + hu_0 \le hu_0 \quad u \le u_0 \quad$ 故 $\max\limits_{Q_T} u \le u_0$

若最小值在 $\{0\} \times [0,T]$ 取得，在 $(0, t_0)$ 取得，则 $u_x\big|_{t=t_0} \ge 0$

$hu = u_x + hu_0 \ge 0 \quad u \ge 0 \quad$ 故 $\min\limits_{Q_T} u \ge 0$ #

(2) 设 $h_1 > h_2 > 0$，u_{h_1}, u_{h_2} 为对应解

$w = u_{h_1} - u_{h_2}$

$$\begin{cases} w_t - w_{xx} = 0, & (x,t) \in Q_T \\ w(x,0) = 0, & 0 \le x \le l \\ w_x + (h_1 - h_2)u_0 = h_1 w + (h_1 - h_2) u_{h_2}, & x = 0 \\ w\big|_{x=l} = 0 \end{cases}$$

考虑 w 在 $x=0$ 最小值，在 $(0, t_0)$ 取得 $\quad w_x\big|_{t=t_0} \ge 0$

$h_1 w + (h_1 - h_2) u_{h_2} \ge (h_1 - h_2) u_0$

$h_1 w \ge (h_1 - h_2)(u_0 - u_{h_2}) \ge 0$

$w \ge 0$

$\Rightarrow w\big|_{\overline{Q}_T} \ge 0$ 由极值原理得证 #

34. $u \in C(\overline{Q_T}) \cap C^{2,1}(Q_T)$

$$\begin{cases} u_t - u_{xx} = -u^2 + bu & (x,t) \in Q_T \\ u(x,0) = \varphi(x) & 0 \le x \le l \\ u(0,t) = u(l,t) = 0 & 0 \le t \le T \end{cases}$$

$b = b(x,t) \in C(\overline{Q_T})$ $\varphi \in C[0,l]$, $\varphi \geq 0$

证明 $0 \le u(x,t) \le M \max\limits_{[0,l]} \varphi(x)$, $M = M(T, \max\limits_{Q_T} |b|)$

证明: 记 $c(x,t) = u(x,t) - b(x,t)$ $c(x,t) \in C(\overline{Q_T})$ 设 $c(x,t) \geq -c_0$

$$\begin{cases} u_t - u_{xx} + c(x,t)u = 0 & 0 < x < l, t > 0 \\ u|_{\partial_p Q_T} \geq 0 \end{cases}$$

引理: $u \geq 0$

从而 $\begin{cases} u_t - u_{xx} \le Bu & ,(x,t) \in Q_T \\ u(x,0) = \varphi(x) & ,0 \le x \le l \\ u(0,t) = u(l,t) = 0 & ,0 \le t \le T \end{cases}$

其中 $B = \max\limits_{Q_T} |b(x,t)|$

记 $v(x,t) = e^{-Bt} u(x,t)$

$$v_t - v_{xx} = e^{-Bt} u_t - Be^{-Bt} u - e^{-Bt} u_{xx}$$
$$= e^{-Bt}(u_t - Bu - u_{xx}) \le 0$$

$v(x,0) = \varphi(x)$, $0 \le x \le l$

$v(0,t) = v(l,t) = 0$

由极值原理, $\max\limits_{Q_T} v(x,t) \le \max\limits_{[0,l]} v(x)$

$0 \le u(x,t) \le e^{BT} \max\limits_{[0,l]} v(x)$ 得证.

引理的证明:

记 $r(x,t) = e^{-c_0 t} u(x,t)$ $r|_{\partial_p Q_T} \geq 0$

$$r_t - r_{xx} + (c(x,t) + c_0) r = e^{-c_0 t} u_t - c_0 e^{-c_0 t} u - e^{-c_0 t} u_{xx} + e^{-c_0 t} u (c(x,t) + c_0) = 0$$

由极值原理, Q_T 上 $r(x,t) \geq 0$ $\Rightarrow$ Q_T 上 $u \geq 0$

得证.

35.

$$\begin{cases} u_t - a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u = f(x,t) & (x,t)\in\mathbb{R}^2_T \\ u(x,0) = \varphi(x), & x\in\mathbb{R} \end{cases}$$

有界解唯一 $a(x,t)\geq a_0>0$，$c(x,t)\geq 0$ a,b,c 有界

证：考虑 $Q_T^L=(-L,L)\times(0,T]$

设 $u_1\geq u_2$ u_1,u_2 满足 Q_T^L 上的原方程，且有界，$|w|\leq M$

记 $w=u_1-u_2$ w 有界

$$\begin{cases} w_t - a(x,t)w_{xx} + b(x,t)w_x + c(x,t)w = 0 & \text{on } \mathbb{R}\times\mathbb{R}_+ \quad (*) \\ w(x,0) = 0 & x\in\mathbb{R}. \end{cases}$$

先证明 Q_T^L 上 $(*)$ 的比较原理：

设 Q_T^L 上

$$\begin{cases} w_t - a(x,t)w_{xx} + b(x,t)w_x + c(x,t)w \leq 0 \\ w(x,0)\leq 0, \ x\in\mathbb{R} \\ w(L,t)\leq 0,\ w(-L,t)\leq 0, \quad 0\leq t\leq T \end{cases}$$

若 $w-\varepsilon t$ 最大值在 $-L<x_0<L$，$0<t_0\leq T$ 取得，

有 $c(x_0,t_0)w\leq -w_t\leq -\varepsilon$

则 $w_t-\varepsilon\geq 0$，$w_x=0$，$w_{xx}\leq 0$

$\Rightarrow w<0$ 否则 $w-\varepsilon t$ 最大值在 $t=0$ or $x=\pm L$ 取得

总之有 $\sup\limits_{Q_T^L} w\leq \varepsilon T$ 令 $\varepsilon\to 0^+$ 即有 $\sup\limits_{Q_T^L} w\leq 0$ 得证.

下考虑 $\theta(x,t)=\dfrac{Me^t(x^2+2At+B^2)}{L^2}$，$|a|\leq A$，$|b|\leq B$

$$\theta_t - a\theta_{xx} + b\theta_x + c\theta = \frac{M}{L^2}\big(e^t(x^2+2At+B^2+2A) + be^t(2x) - 2ae^t + ce^t(x^2+2At+B^2)\big)$$

$$= \frac{Me^t}{L^2}\big(x^2+2At+B^2+2A+2bx-2a+cx^2+2Act+B^2c\big)$$

$$= \frac{Me^t}{L^2}\big[(x+b)^2+B^2-b^2+2(A-a)+2At+2Act+c(x^2+B^2c)\big]\geq 0$$

$$\theta(x,0) = \frac{Me^t(x^2+B^2)}{L^2}\geq 0$$

$$\theta(\pm L,t) = \frac{Me^t(L^2+2At+B^2)}{L^2}\geq M\geq |w(x,t)|$$

因此 $|w(x,t)| \le Me^{t}(x^2+2At+B^2)/L^2$ (比较定理)

固定 (x_0,t_0)，$t_0 \le T$ 和 L 充分大，

$$|w(x_0,t_0)| \le Me^{t_0}(x^2+2At+B^2)/L^2$$

令 $L\to\infty$ $\Rightarrow w(x_0,t_0)=0$

故 $w\equiv 0$

证毕