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初始温度 $\varphi(x)$

(1) 侧表面绝热

(2) 介质常温 u_0

求 $u(x,t)$, $t\to+\infty$ 极限分布

(1) 方程列为

$$\begin{cases} u_t - u_{xx} = 0 & , 0<x\le l,\ t>0 \\ u(x,0) = \varphi(x) & , 0\le x\le l \\ u_x(0,t) = 0 \quad u_x(l,t)=0 & , t>0 \end{cases}$$

$$u(x,t) = \frac{1}{l}\int_0^l \varphi(x)\,dx + \sum_{n=1}^{\infty} \varphi_n e^{-\left(\frac{n\pi}{l}\right)^2 t} \cos\frac{n\pi}{l}x$$

$$\varphi_n = \frac{2}{l}\int_0^l \varphi(x)\cos\frac{n\pi}{l}x\,dx$$

$$|\varphi_n| \le \frac{2}{l}\int_0^l |\varphi(x)|\,dx$$ 由控制收敛定理

$$\lim_{t\to+\infty} u(x,t) = \frac{1}{l}\int_0^l \varphi(x)\,dx$$

(2) 方程列为

$$\begin{cases} u_t - u_{xx} = k(u_0 - u) & 0<x\le l \quad t>0 \\ u(x,0) = \varphi(x) & 0\le x\le l \\ u_x(0,t) = 0 \quad u_x(l,t) = 0 & t>0 \end{cases} \qquad k>0$$

记 $v = u - u_0$

$$\begin{cases} v_t - v_{xx} = -kv \\ v(x,0) = \varphi(x) - u_0 \\ v_x(0,t) = 0 \quad v_x(l,t) = 0 \end{cases}$$

$$\Rightarrow u(x,t) = u_0 + e^{-kt}\left[\frac{1}{l}\int_0^l \hat{\varphi}(x)\,dx + \sum_{n=1}^{\infty} \hat{\varphi}_n e^{-\left(\frac{n\pi}{l}\right)^2 t}\cos\frac{n\pi}{l}x\right]$$

$$\hat{\varphi}_n = \frac{2}{l}\int_0^l \hat{\varphi}(x)\cos\frac{n\pi}{l}x\,dx \qquad \hat{\varphi} = \varphi - u_0$$

$$|\hat{\varphi}_n| \le \frac{2}{l}\left(\int_0^l |\varphi|\,dx + \int_0^l |u_0|\,dx\right)$$ 由控制收敛定理

$$\lim_{t\to+\infty} u(x,t) = u_0$$

25 $v \in C^{2,1}(\Omega_T)$

$v_t - a^2 \Delta v \le 0 \quad (x,t) \in \Omega_t = \Omega \times (0,T] \quad a>0$

(1) $\max\limits_{\bar{\Omega}_T} v(x,t) = \max\limits_{\partial_p \Omega_T} v(x,t)$

证明: $L(v) = v_t - a^2 \Delta v \le 0$

由极值原理, $\max\limits_{\bar{\Omega}_T} v(x,t) = \max\limits_{\partial_p \Omega_T} v(x,t)$. #

(2) $\phi: \mathbb{R} \to \mathbb{R}$ 光滑凸, u 在 Ω_T 满足热方程 $v = \phi(u)$ 是下解

证: 已知 $u_t - a^2 \Delta u = 0$

则 $\phi(u)_t - a^2 \Delta \phi(u) = \phi'(u) u_t - a^2 \left(\phi'(u) \Delta u + |Du|^2 \phi''(u)\right)$

$= -a^2 |Du|^2 \phi''(u) \le 0$ #

(3) u 在 Ω_T 满足热方程 $v = a^2 |Du|^2 + u_t^2$ 在 Ω_T 是下解

证: $v_t - a^2 \Delta v$

$= 2u_t u_{tt} + 2a^2 \sum_{i=1}^n u_{x_i} u_{x_i t} - a^2 \Big(2u_t (\Delta u)_t + 2 \sum_{i=1}^n (u_{t x_i})^2$

$+ a^2 \big(\sum_{i=1}^n 2 u_{x_i x_j}^2 + 2 \sum_{i=1}^n u_{x_i} (\Delta u)_{x_i} \big) \Big)$

$\le 2u_t u_{tt} + 2a^2 \sum_{i=1}^n u_{x_i} u_{x_i t} - 2a^2 u_t (\Delta u)_t - 2a^4 \sum_{i=1}^n u_{x_i} (\Delta u)_{x_i}$

$= 2u_t (u_t - a^2 \Delta u)_t + 2a^2 \sum_{i=1}^n u_{x_i} (u_t - a^2 \Delta u)_{x_i} = 0$ #

27. $\mathcal{L}u = u_t - u_{xx} + |u_x|$

$u, v \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$

$\mathcal{L}u \leq \mathcal{L}v,\ u|_{\Gamma} \leq v|_{\Gamma} \Rightarrow u \leq v$ on $\overline{Q_T}$

证明：记 $w = u - v$

考虑 w 在 $\overline{Q_T}$ 最大值. 若是在 $\partial_p Q_T$ 取得 则 $u \leq v$ on $\overline{Q_T}$

若是在 Q_T 上取得,

记 $r(x,t) = w(x,t) - \varepsilon t$, $\varepsilon > 0$

则 考虑 r 在 $(x_0, t_0) \in Q_T$ 取最大值

$r_x(x_0, t_0) = 0 \quad r_{xx}(x_0, t_0) \leq 0$

$r_t(x_0, t_0) \geq 0 \Rightarrow w_t(x,t) > 0$

与 $u_t - u_{xx} + |u_x| \leq v_t - v_{xx} + |v_x|$ 矛盾

故 $\max\limits_{\overline{Q_T}} r(x,t) = \max\limits_{\partial_p Q_T} r(x,t) \leq 0$

令 $\varepsilon \to 0$ $\max\limits_{\overline{Q_T}} w(x,t) \leq 0$ 不等式也成立 #

30. $u \in C^{1,0}(\overline{Q_T}) \cap C^{2,1}(Q_T)$

$$\begin{cases} u_t - u_{xx} = 0 & (x,t) \in Q_T \\ u(x,0) = \varphi(x) & 0 \le x \le l \\ u(0,t) = u(l,t) = 0 & 0 \le t \le T \end{cases}$$

(1) $\max\limits_{(0,T)} |u_x(0,t)| \le C$, $\max\limits_{(0,T)} |u_x(l,t)| \le C$ 　C 仅依赖 $\|\varphi\|_{C^1[0,l]}$

证：有 $\varphi(0) = \varphi(l) = 0$ 　$|u(x,0)| \le (l-x) \max\limits_{[0,l]} |\varphi'(y)|$

$|u(x,0)| \le x \max\limits_{[0,l]} |\varphi'(y)|$

由比较原理，（分别用 $\pm x \max\limits_{[0,l]} |\varphi'(y)|$ 及 $\pm(l-x)\max\limits_{[0,l]} |\varphi'(y)|$ 代替 u）

有 $|u(x,t)| \le x \max\limits_{[0,l]} |\varphi'(y)|$, $|u(x,t)| \le (l-x) \max\limits_{[0,l]} |\varphi'(y)|$

$$|u_x(0,t)| = \lim_{\Delta x \to 0} \frac{|u(\Delta x, t)|}{\Delta x} \le \max_{[0,l]} |\varphi'(y)|$$

$$|u_x(l,t)| = \lim_{\Delta x \to 0} \frac{|u(l-\Delta x, t)|}{\Delta x} \le \max_{[0,l]} |\varphi'(y)|$$. 取 $C = \max\limits_{[0,l]} |\varphi'(y)|$ 即可

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(2) $u_x \in C^{2,1}(Q_T)$ 　$\max\limits_{\overline{Q_T}} |u_x(x,t)| \le \overline{C}$, $\overline{C}$ 仅依赖 $\|\varphi\|_{C^1[0,l]}$

证：记 $v(x,t) = u_x(x,t)$ 满足定解问题

$$\begin{cases} v_t - v_{xx} = 0 & 0 \le x \le l,\ t > 0 \\ v(x,0) = \varphi'(x) & 0 \le x \le l \\ v(0,t) = u_x(0,t),\ v(l,t) = u_x(l,t), & 0 \le t \le T \end{cases}$$

则由定理 3.1 有估计

$$\max_{\overline{Q_T}} |u| \le B = \max\left\{ \max_{[0,l]} |\varphi'(x)|,\ \max_{[0,T]} |u_x(0,t)|,\ \max_{[0,T]} |u_x(l,t)| \right\}$$

$$\le \max\left\{ C,\ \max_{[0,l]} |\varphi'(x)| \right\} := \overline{C} \quad \#$$

其中 C 为 (1) 问中常数