习题2.5

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1.(1) $8y'^3=27y$

设$y'=p$ $8p^3=27y$

$24p^2\frac{dp}{dx}=27p$

$p\equiv0$ or $8p\frac{dp}{dx}=9$

$\downarrow$

$\boxed{y\equiv0}$

$9x=4p^2+C$

$y'=p=\frac{\sqrt{9x-C}}{2}$

$\boxed{y=\frac{1}{27}(9x-C)^{\frac{3}{2}}}$

(3) $y'^2+xy=y^2+xy'$

$(y'-y)(y'+y)=x(y'-y)$

$y'=y$ or $y'+y=x$

$y=Ce^x$ or $y=x-1+\frac{C}{e^x}$

$4e^{\frac{1}{2}\sqrt{4-xy}}(\sqrt{4-xy}-2)=Cy$

$\log p=\sqrt{p}x-C$

$p=Ce^{\sqrt{p}x}$

(5) $y+xy'=4\sqrt{y'}$

记$p=y'$ $y=x(-p)+4\sqrt{p}$

对x求导有 $\frac{dx}{dp}+\frac{-x+\frac{2}{\sqrt{p}}}{-2p}=0$

$\frac{dx}{dp}+\frac{x}{2p}=\frac{1}{p^{\frac{3}{2}}}$

$\frac{d}{dp}(\sqrt{p}x)=\frac{x}{2\sqrt{p}}+\sqrt{p}\frac{dx}{dp}=\sqrt{p}\left(\frac{x}{2p}+\frac{dx}{dp}\right)=\frac{1}{p}$

$x=\frac{\log p+C}{\sqrt{p}}$ 而 $p=\frac{-xy+8\pm4\sqrt{4-xy}}{x^2}=\frac{(\sqrt{4-xy}\pm2)^2}{x^2}$

$\Rightarrow \sqrt{4-xy}\pm2=2\log\frac{\sqrt{4-xy}\pm2}{x}+C$ #.

2.

(1) $2xy' - y = y' \ln(yy')$

$2xyy' - y^2 = yy' \ln(yy')$

$2z = y^2 \quad p = z' = yy'$

$2px - 2z = p \ln p$

对x求导， $2xp' = p' \ln p + p'$

$p' \equiv 0$ (舍) or $2x = \ln p + 1$

$p = e^{2x-1} = yy'$

$y\,dy = e^{2x-1} dx$

$\frac{1}{2} y^2 + C = \frac{1}{2} e^{2x-1}$

$y = \pm\sqrt{e^{2x-1} + C}$

(3) $y'^2 - 2xy' = x^2 - 4y$

$(y' - x)^2 = 2x^2 - 4y$

$p = y' - x \quad p^2 = 2x^2 - 4y$

$2p \frac{dp}{dx} = 4x - 4(p+x) = -4p$

$\frac{dp}{dx} = -2$ or $p = 0$

$y = \frac{1}{2} x^2 + C$ or $y = -\frac{1}{2} x^2 + C$

代回，得 $y = \pm \frac{1}{2} x^2$

3. $xy = a^2$ 即可

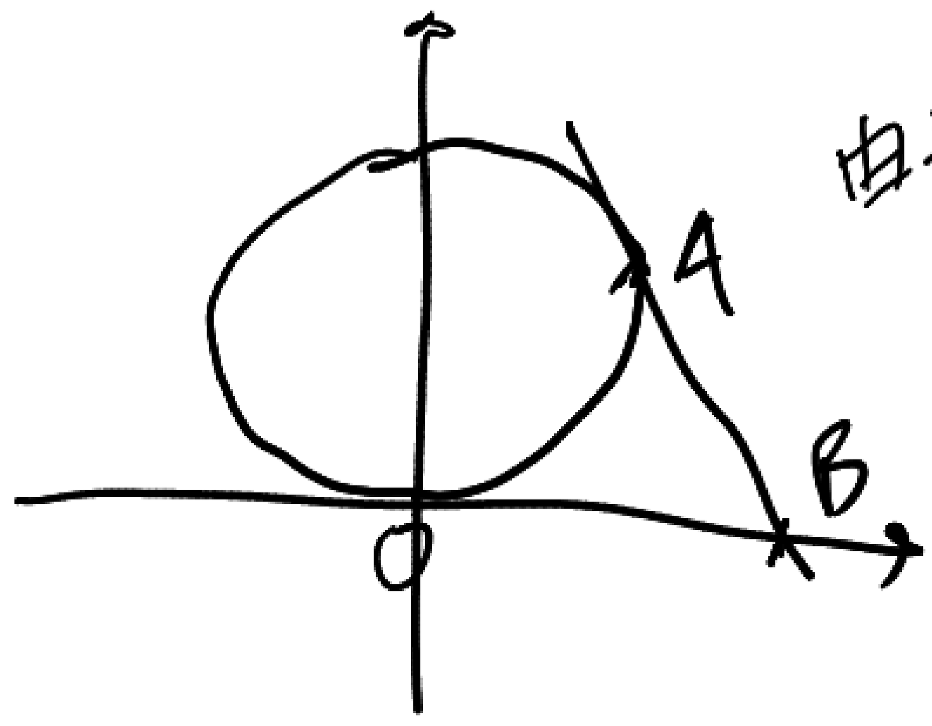
$(x_0, \frac{a^2}{x_0})$ 处切线 $y = -\frac{a^2}{x_0^2}(x - x_0) + \frac{a^2}{x_0}$

交坐标轴于 $(\frac{2a^2}{x_0}, 2x_0)$

$S_\triangle = \frac{1}{2} \frac{2a^2}{x_0} \cdot 2x_0 = 2a^2$

习题2.6

2.



观察得 $\Gamma: (x-1)^2+y^2=1$

由几何, 显然合题. #

4. $y=\int_0^x y(t)dt + x+1$

$\frac{dy}{dx}=y+1$

$\frac{dy}{y+1}=dx$

$x=\ln|y+1|+C$

$|y+1|=Ae^x$

$y=Ae^x-1$ or $-Ae^x-1$ 代入得 $y=2e^x-1$ or $-2e^x-1$

5. $y''+\sin y=0$

$y\equiv\pi$ 是 $x\to+\infty$ 时 $y\to\pi$ 的合题解. #

6. f 在 $(0,+\infty)$ 可导

$$\int_1^{xy} f(t)\,dt = x\int_1^y f(t)\,dt + y\int_1^x f(t)\,dt,\quad x,y>0$$

$f(1)=3$, $f(x)=?$

对 x 求导,有 $y f(xy) = \int_1^y f(t)\,dt + y f(x)$

再对 y 求导,有 $f(xy) + xy f'(xy) = f(y) + y f(x)$

$$f(x) + x f'(x) = 3 + f(x)$$

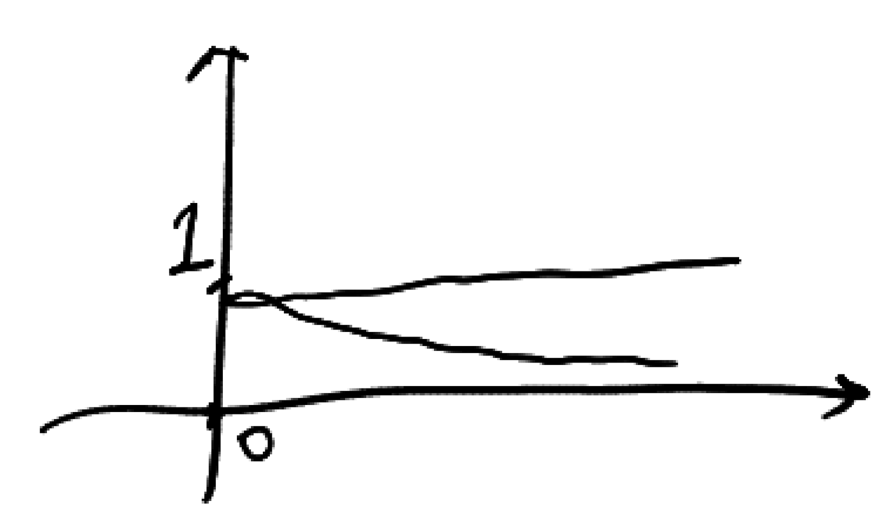
$$f'(x) = \frac{3}{x}$$

$$f(x) = 3\ln x + C,\quad f(1)=3$$

$$f(x) = 3\ln x + 3$$

7. $x>-1$ 时, $f'(x) + f(x) - \frac{1}{x+1}\int_0^x f(t)\,dt = 0$, $f(0)=1$

$x\geq 0$ 时, $e^{-x} \leq f(x) \leq 1$



易知 f' 可导, $\int_0^x f(t)\,dt = (x+1)(f(x)+f'(x))$

$$f(x) = f(x) + f'(x) + (x+1)(f'(x)+f''(x))$$

$$f'(x)(x+2) = -(x+1)f''(x)$$

$$\ln|f'(x)| = -(x+\ln|x+1|)+C$$

$$|f'(x)| = A\frac{e^{-x}}{x+1}$$ $f'(x)$ 连续, 设 $f'(x) = A\frac{e^{-x}}{x+1}$

代入 $x=0$, $f'(0)=-1$

$$f(x) = 1-\int_0^x \frac{e^{-t}}{t+1}\,dt \in [e^{-x}, 1].$$

习题 3.3

1.

(1)

$$\left|y_n(x)-y_0\right| = \left|\int_{x_0}^{x-d_n} f(s,y_n(s))\,ds\right|$$
$$\le \int_{x_0}^{x-d_n} \left|f(s,y_n(s))\right| ds$$
$$\le Mh$$

$\{y_n(x)\}$一致有界

$$\left|y_n(x_1)-y_n(x_2)\right| = \left|\int_{x_0}^{x_1-d_n} f(s,y_n(s))\,ds - \int_{x_0}^{x_2-d_n} f(s,y_n(s))\,ds\right|$$
$$= \left|\int_{x_2-d_n}^{x_1-d_n} f(s,y_n(s))\,ds\right| \le M|x_1-x_2|$$

$\{y_n(x)\}$等度连续，

由 A-A 定理，$y_n(x)$ 有一致收敛子列 $y_{n_j}(x)$

$$y_{n_j}(x) = y_0 + \int_{x_0}^{x-d_{n_j}} f(s,y_{n_j}(s))\,ds$$

$j\to\infty$ 时，$y_{n_j}(x) \rightrightarrows \phi(x) = y_0 + \int_{x_0}^{x} f(s,\phi(s))\,ds$ 故原方程有解

(2) 只需证明 $\{y_n(x)\}$ 一致收敛. $m>n$ 时

$$|y_m(x)-y_n(x)| = \begin{cases} 0 & x\in[x_0,\ x_0+\frac{1}{m}] \\ \left|\int_{x_0}^{x-\frac{1}{m}} f(s,y_m(s))\,ds\right|, & x\in[x_0+\frac{1}{m},\ x_0+\frac{1}{n}). \\ \left|\int_{x_0}^{x-\frac{1}{m}} f(s,y_m(s))\,ds - \int_{x_0}^{x-\frac{1}{n}} f(s,y_n(s))\,ds\right|, & x\in[x_0+\frac{1}{n},\ x_0+n] \end{cases}$$

$\forall \varepsilon>0,\ n>\frac{M}{\varepsilon} \Rightarrow$

$$\left|\int_{x_0}^{x-\frac{1}{m}} f(s,y_m(s))\,ds\right| \le \left(\frac{1}{n}-\frac{1}{m}\right)M, \quad x\in[x_0+\frac{1}{m},\ x_0+\frac{1}{n}].$$

当 $x\in[x_0+\frac{1}{n},\ x_0+n)$，

$$|y_m(x)-y_n(x)| = \left|\int_{x_0}^{x-\frac{1}{n}} \left(f(s,y_m(s)) - f(s,y_n(s))\right)ds + \int_{x-\frac{1}{n}}^{x-\frac{1}{m}} f(s,y_n(s))\,ds\right|$$

$$|f(s,y_m(s))-f(s,y_n(s))| \le L|y_m(s)-y_n(s)|$$

$$|y_m(x)-y_n(x)| \le \int_{x_0}^{x-\frac{1}{n}} L|y_m(s)-y_n(s)|\,ds + \left|\int_{x-\frac{1}{n}}^{x-\frac{1}{m}} f(s,y_n(s))\,ds\right|$$

记 $a_k = \int_{x_0}^{x_0+\frac{k}{n}} L|y_m(s)-y_n(s)|\,ds + \left|\int_{x-\frac{1}{n}}^{x-\frac{1}{m}} f(s,y_n(s))\,ds\right|$

$\exists N,\ a_1 < \varepsilon$

$x_0+\frac{1}{n} \le x \le x_0+\frac{2}{n}$ 时，$|y_m(x)-y_n(x)| \le a_1$

$x_0+\frac{2}{n} \le x \le x_0+\frac{3}{n}$ 时，$|y_m(x)-y_n(x)| \le a_2 + L\int_{x_0+\frac{1}{n}}^{x_0+\frac{2}{n}} |y_m(s)-y_n(s)|\,ds$

$$\le a_1 + \frac{L}{n}a_1 = (1+\frac{L}{n})a_1$$

$x_0+\frac{k-1}{n} \le x \le x_0+\frac{k}{n}$ 时，$|y_m(x)-y_n(x)| \le a_{k-1} + L\int_{x_0+\frac{k-1}{n}}^{x_0+\frac{k}{n}} |y_m(s)-y_n(s)|\,ds$

$$\le (1+\frac{L}{n})a_{k-1}$$

$\Rightarrow\ |y_m(x)-y_n(x)| \le (1+\frac{L}{n})^n a_1 \le e^L a_1,\ \forall x.$

取 N 使得 $n>N$ 时，$a_1<\varepsilon$ 故 $y_n(x) \rightrightarrows y(x)$. #.

2.(1) 设 $2n$. 折线为 $l_1, l_2, \dots l_n$

易见 $l_1: y=0,\quad 0\le x\le \frac{1}{n}$

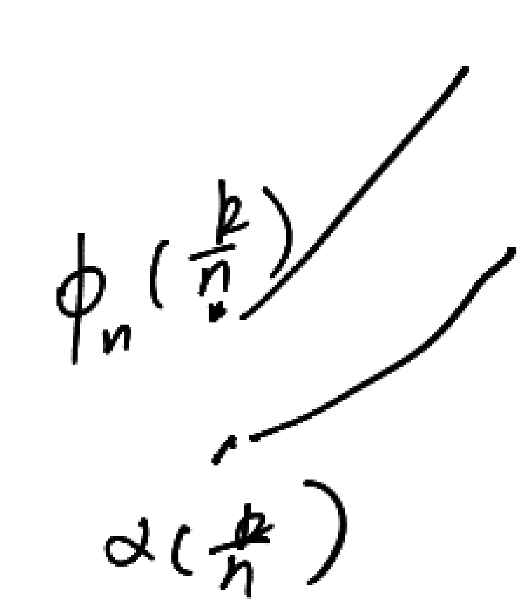
$l_2: y=\frac{1}{n}\cos n\pi(x-\frac{1}{n}),\ \frac{1}{n}\le x\le \frac{2}{n}$

$x=\frac{2}{n}$ 时，$y=\frac{1}{n^2}$

我们知道 $n\ge 6$ 时，$\alpha(\frac{2}{n}) = \int_0^{\frac{2}{n}} e^{-\frac{1}{s^2}}ds = \int_0^{\frac{1}{n}} e^{-\frac{1}{s^2}}ds + \int_{\frac{1}{n}}^{\frac{2}{n}} e^{-\frac{1}{s^2}}ds$

$\le \frac{1}{n}e^{-n^2} + \frac{1}{n}e^{-\frac{n^2}{4}} \le \frac{1}{n}\cdot\frac{1}{n^2+1} + \frac{1}{n}\cdot\frac{1}{1+\frac{n^2}{4}} \le \frac{1}{n^2}$

$n=4$，$\int_0^{\frac{1}{2}} e^{-\frac{1}{s^2}}ds < \frac{1}{4}$ 故 l_2 端点在 $y=\alpha(x)$ 上方



$l_3: y=\frac{1}{n^2}+\frac{2}{n}(x-\frac{2}{n}),\ \frac{2}{n}\le x\le\frac{3}{n}$

$k\ge 2$，设 $x\le\frac{k}{n}$ 时，$\phi_n(x)>\alpha(x)$

$\phi_n(\frac{k}{n})>\alpha(\frac{k}{n})$　　$\frac{k}{n}\le x\le\frac{k+1}{n}$ 时，$\alpha'(x)=e^{-\frac{1}{x^2}}$

只需证明 $e^{-(\frac{n}{k+1})^2} \le \frac{k}{n}$，即有 $\phi_n(x)>\alpha(x)$，$x\le\frac{k+1}{n}$.

设 $r=\left(\frac{n}{2x}\right)^2$, $\frac{n^2}{4}\le r\le 1$

只需证 $e^{-r}\le\frac{1}{\sqrt{r}}-\frac{1}{n}$

$$g(r):=e^{-r}-\frac{1}{\sqrt{r}}+\frac{1}{n}$$

$r=1,\ \frac{n^2}{4}$ 时易知 $g(r)\le 0$, $1<r<\frac{n^2}{4}$ 时,

$g'(r)=0\Rightarrow e^{-r}=\frac{1}{2}r^{-\frac{3}{2}}$, 此时 $g(r)=\frac{1}{2}r^{-\frac{3}{2}}-r^{-\frac{1}{2}}+\frac{1}{n}\le 0$

故 n 为偶数时, 当 $x\in[\frac{2}{n},1]$, 总有 $\phi_n(x)\geq\alpha(x)$.

(II) n 为奇数的情况完全对称 ($\alpha(x)$ 与 y 无关)

(III) 因此我们知道 $\phi_{2k}(x)\geq\alpha(x)$

$\phi_{2k+1}(x)\le-\alpha(x)$

若 $\{\phi_n(x)\}$ 收敛, $\lim\limits_{k\to\infty}\phi_{2k}(x)\geq\alpha(x)$

$$\begin{array}{c}\parallel\end{array}$$

$\lim\limits_{k\to\infty}\phi_{2k+1}(x)\le-\alpha(x)$ 矛盾!

(IV) 初值问题有解 $y=\pm\frac{1}{2}x^2$, 不是唯一的. #