1. 求函数 Fourier 变换

$$\frac{e^{-i\lambda x}}{-i\lambda} + (x-1)\, e^{-i\lambda x}$$

(1) $f(x) = \begin{cases} 0 & |x| > a \\ |x| & |x| \le a \end{cases} \quad (a > 0)$

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\left(\int_{-a}^{0} -x e^{-i\lambda x}\, dx + \int_{0}^{a} x e^{-i\lambda x}\, dx\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{-1+(1+ia\lambda)e^{-ia\lambda}}{\lambda^2} + \frac{-1+(1-ia\lambda)e^{ia\lambda}}{\lambda^2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{a\sin(a\lambda)}{\lambda} + \frac{\cos(a\lambda)-1}{\lambda^2}\right)$$

(3) $f(x) = \begin{cases} 0 & |x| > a \\ \sin\lambda_0 x & |x| \le a \end{cases} \quad a > 0,\ \lambda_0 > 0$

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} \sin\lambda_0 x\, e^{-i\lambda x}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} \sin\lambda_0 x\left[\cos\lambda x - i\sin\lambda x\right] dx$$

$$= \frac{-i}{\sqrt{2\pi}}\left[\frac{\sin(\lambda_0-\lambda)a}{\lambda_0-\lambda} - \frac{\sin(\lambda_0+\lambda)a}{\lambda_0+\lambda}\right]$$

(5) $f(x) = \cos x\, e^{-a|x|}, \quad a > 0$

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \cos x\, e^{-a|x|} e^{-i\lambda x}\, dx = \frac{1}{\sqrt{2\pi}}\left(\int_{0}^{+\infty} \cos x\, e^{-ax-i\lambda x}\, dx + \int_{-\infty}^{0} \cos x\, e^{ax-i\lambda x}\, dx\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_{0}^{+\infty} \frac{1}{2}\left[e^{-ax+i(1-\lambda)x} + e^{-ax-i(1+\lambda)x}\right] dx + \int_{0}^{+\infty} \frac{1}{2}\left[e^{-ax+i(\lambda+1)x} + e^{-ax+i(\lambda-1)x}\right] dx\right)$$

$$= \frac{1}{2\sqrt{2\pi}}\left(\frac{-1}{-a+i(1-\lambda)} + \frac{-1}{-a-i(1+\lambda)} + \frac{-1}{-a+i(\lambda+1)} + \frac{-1}{-a+i(\lambda-1)}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{a}{a^2+(1-\lambda)^2} + \frac{a}{a^2+(1+\lambda)^2}\right]$$

2. 用性质求 Fourier 变换

(2) $f(x) = x e^{-a|x|}$, $a>0$

$\hat{f}(\lambda) = i \frac{d}{d\lambda}(e^{-a|x|})^{\wedge} = i \frac{d}{d\lambda} \frac{2a}{a^2+\lambda^2} \frac{1}{\sqrt{2\pi}}$

$= -\frac{a i \lambda}{(a^2+\lambda^2)^2} \frac{2\sqrt{2}}{\sqrt{\pi}}$

$\mathcal{F}f' = i\lambda \mathcal{F}f$

$\mathcal{F}\, x f(x) = i\frac{d}{d\lambda}\mathcal{F}f$

$\mathcal{F}f(x-k) = e^{-i\lambda k}\mathcal{F}f$

$\mathcal{F}(f(kx)) = \frac{1}{|k|}\mathcal{F}f(\frac{\lambda}{k})$

(4) $f(x) = \sin(\lambda_0 x) e^{-a|x|}$, $a>0$, $\lambda_0>0$

利用定义计算,

$\hat{f}(\lambda) = \sqrt{\frac{2}{\pi}} \frac{a\lambda_0 i\lambda}{(a^2+(\lambda-\lambda_0)^2)(a^2+(\lambda+\lambda_0)^2)}$

(6) $f(x) = e^{-ax^2+ibx+c}$, $a>0$, $b, c \in \mathbb{R}$

$= e^{-(\sqrt{a}(x-\frac{ib}{2a}))^2} e^{c-\frac{b^2}{4a}}$

$(\hat{e^{-x^2}} = \frac{1}{\sqrt{2}} e^{-\frac{\lambda^2}{4}})$

$\hat{f}(\lambda) = e^{c-\frac{b^2}{4a}} \frac{1}{\sqrt{2a}} e^{-\frac{\lambda^2}{4a}} e^{\frac{b\lambda}{2a}}$

(8) $f(x) = \frac{x}{a^2+x^2}$ $a>0$

$\left(\frac{1}{a^2+x^2}\right)^{\wedge} = \sqrt{\frac{\pi}{2}}\frac{1}{a} e^{-a|\lambda|}$

$\Rightarrow \hat{f}(\lambda) = \begin{cases} -i\sqrt{\frac{\pi}{2}} e^{-a\lambda} & \lambda>0 \\ i\sqrt{\frac{\pi}{2}} e^{a\lambda} & \lambda<0 \end{cases}$

3. 求Fourier逆变换

(1) $F(\lambda) = e^{-a^2\lambda^2 t}$ $t>0$ 参数，$a>0$ 常数

$$\hat{F}(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-a^2\lambda^2 t} e^{i\lambda x} d\lambda = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-(a\sqrt{t}\lambda - \frac{ix}{2a\sqrt{t}})^2} d\lambda \cdot e^{\frac{-x^2}{4a^2 t}}$$

$$= \frac{1}{a\sqrt{2t}} e^{-\frac{x^2}{4a^2 t}}$$

(2) $F(\lambda) = e^{-(a^2\lambda^2 + ib\lambda + c)t}$ $t>0$ 参，$a \in \mathbb{R}_+$，$b, c \in \mathbb{R}$ 常

$$\hat{F}(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-a^2\lambda^2 t - ib\lambda t + i\lambda x - ct} d\lambda$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-(a\sqrt{t}\lambda - \frac{i(x-bt)}{2a\sqrt{t}})^2} d\lambda \, e^{\frac{-(x-bt)^2}{4a^2 t} - ct}$$

$$= \frac{1}{\sqrt{2t}\,a} e^{-\frac{(x-bt)^2}{4a^2 t} - ct}$$

(3) $F(\lambda) = e^{-|\lambda| y}$，$y>0$ 参

$$\hat{F}(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-|\lambda| y + i\lambda x} d\lambda$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_0^{+\infty} e^{\lambda(ix-y)} d\lambda + \int_{-\infty}^0 e^{\lambda(ix+y)} d\lambda\right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{y}{x^2+y^2}$$

4. (1) $\begin{cases} u_t - a^2 u_{xx} + b u_x + c u = f(x,t) & (x,t) \in \mathbb{R}_+^2 \\ u(x,0) = \varphi(x) & x \in \mathbb{R} \end{cases}$ $\quad a \in \mathbb{R}_+,\ b, c \in \mathbb{R}$

有 $\begin{cases} \frac{d}{dt}\hat{u} + a^2\lambda^2 \hat{u} + ib\lambda \hat{u} + c\hat{u} = \hat{f}(\lambda,t) \\ \hat{u}(\lambda,0) = \hat{\varphi}(\lambda) \end{cases}$

有 $\hat{u}(\lambda,t) = e^{-(a^2\lambda^2+ib\lambda+c)t}\left(\int_0^t e^{(a^2\lambda^2+ib\lambda+c)s}\hat{f}(\lambda,s)\,ds + \hat{\varphi}(\lambda)\right)$

注意到 $\left(e^{-(a^2\lambda^2+ib\lambda+c)t}\right)^\vee = \frac{1}{\sqrt{2t}\,a} e^{-\frac{(x-bt)^2}{4a^2t}-ct} := \sqrt{2\pi}\,K(x,t)$

有 $u(x,t) = \int_{\mathbb{R}} K(x-y,t)\varphi(y)\,dy + \int_0^t \int_{\mathbb{R}} K(x-y,t-s) f(y,s)\,dy\,ds$

(2) $\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \mathbb{R}_+^2 \\ u(x,0) = \varphi(x), & x \in \mathbb{R} \end{cases}$ $\quad \varphi$ 连续有界，求有界解

有 $\begin{cases} -\lambda^2\hat{u} + \frac{d^2}{dy^2}\hat{u} = 0 \\ \hat{u}(\lambda,0) = \hat{\varphi}(\lambda) \end{cases}$ 解得 $\hat{u}(\lambda,y) = \hat{\varphi}(\lambda)e^{-|\lambda|y}$ (有界性)

$u(x,y) = \left(\hat{u}(\lambda,y)\right)^\vee = \left(\hat{\varphi}(\lambda)\left(\sqrt{\frac{2}{\pi}}\frac{y}{x^2+y^2}\right)^\wedge\right)^\vee = \frac{1}{\pi}\int_{\mathbb{R}} \varphi(t)\frac{y}{(x-t)^2+y^2}\,dt$ #

易知 $u(x,y)$ 有界.

6. $\begin{cases} u_{tt} - a^2 u_{xx} = 0 & (x,t) \in R_+^2 \\ u(x,0) = \varphi(x), & x \in R \\ u_t(x,0) = \psi(x), & x \in R \end{cases}$

解：对原式作 Fourier 变换：

$$\boxed{\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda(x-x_0)}\, d\lambda = \delta(x-x_0)}$$

$$\frac{d^2\hat{u}}{dt^2} + a^2\lambda^2\hat{u} = 0$$

$$\hat{u}(\lambda,0) = \hat{\varphi}(\lambda)$$

$$\frac{d\hat{u}(\lambda,0)}{dt} = \hat{\psi}(\lambda)$$

$$\hat{u}(\lambda,t) = A(\lambda)\cos(a\lambda t) + B(\lambda)\sin(a\lambda t)$$

$$\hat{\varphi}(\lambda) = A(\lambda)$$

$$\hat{\psi}(\lambda) = a\lambda B(\lambda)$$

$$\hat{u}(\lambda,t) = \hat{\varphi}(\lambda)\cos(a\lambda t) + \frac{1}{a\lambda}\hat{\psi}(\lambda)\sin(a\lambda t)$$

先求 $\cos(a\lambda t)^{\vee} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \cos(a\lambda t) e^{i\lambda x}\, d\lambda = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left[\frac{1}{2}e^{i\lambda(at+x)} + \frac{1}{2}e^{i\lambda(-at+x)}\right] d\lambda$

$= \frac{1}{\sqrt{2\pi}}\left(\pi\delta(x-at) + \pi\delta(x+at)\right) = \sqrt{\frac{\pi}{2}}\left[\delta(x-at) + \delta(x+at)\right]$

再求 $\left(\frac{1}{a\lambda}\sin(a\lambda t)\right)^{\vee}$：

设 $f(x) = \begin{cases} k & |x| \le A \\ 0 & |x| > A \end{cases}$ $\quad \hat{f}(\lambda) = k\sqrt{\frac{2}{\pi}}\frac{\sin\lambda A}{\lambda}$ （例题）

故 $\left(\frac{1}{a\lambda}\sin(a\lambda t)\right)^{\vee} = \frac{1}{a}\sqrt{\frac{\pi}{2}}\mathbb{1}_{|x|\le at}$

故 $u(x,t) = \frac{1}{2}\int_{\mathbb{R}}\left[\delta(y-at) + \delta(y+at)\right]\varphi(x-y)\, dy + \frac{1}{2a}\int_{\mathbb{R}}\mathbb{1}_{|x-y|\le at}\,\psi(y)\, dy$

$= \frac{1}{2}\left[\varphi(x-at) + \varphi(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(y)\, dy$