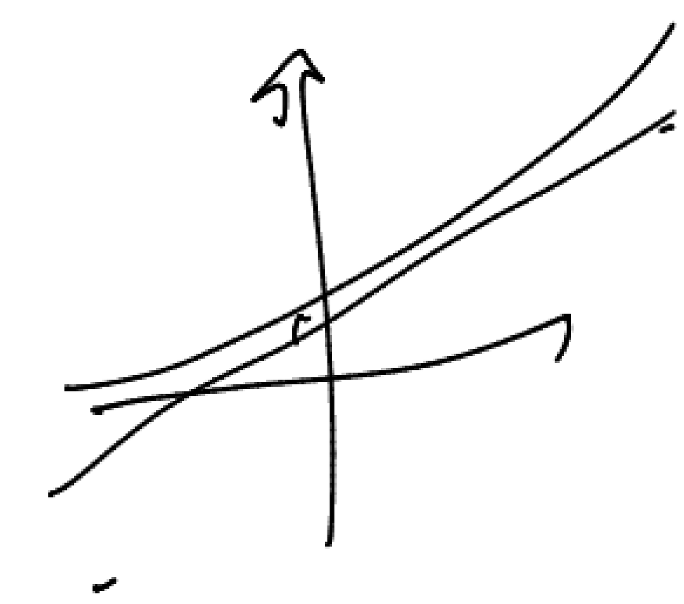
Lec 2

1. Prove Second B-C Lema

$\sum_{n=1}^{\infty} P(A_n) = +\infty$. A_n mutually independent, $\Rightarrow P(w \in \Omega,\ w \in A_n \text{ i.o.}) = 1$

Proof. $1 - P(w \in \Omega,\ w \in A_n \text{ i.o.})$

$$= P\left(\bigcup_n \bigcap_{k=n}^{\infty} A_k^c\right)$$



$$= \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} A_k^c\right)$$

$$= \lim_{n\to\infty} \prod_{k=n}^{\infty} (1 - P(A_k))$$

$$\le \lim_{n\to\infty} \prod_{k=n}^{\infty} e^{-P(A_k)} = \lim_{n\to\infty} e^{-\sum_{k=n}^{\infty} P(A_k)} = 0 \quad \#.$$

2. $X \sim P(\lambda)$ $Y \sim P(\mu)$ X, Y independent, $X + Y \sim P(\lambda+\mu)$

Proof. Consider characteristic function

$$f_X(\xi) = e^{\lambda(e^{i\xi}-1)}$$

$$f_Y(\xi) = e^{\mu(e^{i\xi}-1)}$$

X, Y independent $\Rightarrow$ $f_{X+Y}(\xi) = e^{(\lambda+\mu)(e^{i\xi}-1)}$

$\Rightarrow$ $X + Y \sim P(\lambda+\mu)$. #

3. $X \sim P(\lambda)$ $Y \sim P(\mu)$ independent

Find $P(X \mid X+Y)$, $P(Y \mid X+Y)$

$$P(X=k \mid X+Y=N) = \frac{P(X+Y=N,\ X=k)}{P(X+Y=N)} = \frac{P(Y=N-k)\,P(X=k)}{P(X+Y=N)}$$

$$= \frac{e^{-\lambda}\frac{\lambda^k}{k!}\, e^{-\mu}\frac{\mu^{N-k}}{(N-k)!}}{e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^N}{N!}} = \binom{N}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right)^{N-k} \sim B\left(N, \frac{\lambda}{\lambda+\mu}\right)$$

$P(Y \mid X+Y)$ is similar. #

$X \sim \mathcal{E}(\lambda)$

4. (1) $P(X > s+t \mid X > s) = P(X > t)$, $s, t > 0$

$$P(X > s+t \mid X > s) = \frac{P(X > s+t) P(X > s \mid X > s+t)}{P(X > s)} = \frac{P(X > s+t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} = P(X > t)$$

(2) $P(X > s+t) = P(X > s) P(X > t)$. $\forall s, t > 0$

$\exists \lambda > 0$, $X \sim \mathcal{E}(\lambda)$

Denote $g(x) = \ln F(x)$. $F(x) = P(X > x)$

$g(s+t) = g(s) + g(t)$, $\forall s.t > 0$

g is left-continuous By Cauchy's functional equation

$g(x) = g(1) x := -\lambda x$ $\quad P(X > x) = e^{-\lambda x}$

$F(x) = 1 - e^{-\lambda x}$ $\quad$ PDF is $\lambda e^{-\lambda x}$. $X \sim \mathcal{E}(\lambda)$.

5. Wick Thm. $(X_1 \cdots X_n)$ mean $= 0$

$$\mathbb{E} X_1 \cdots X_k = \begin{cases} \sum \prod \mathbb{E} X_i X_j & 2 \mid k \\ 0 & 2 \nmid k \end{cases}$$

$(X_1 \cdots X_n) \sim N(0, \Sigma)$ Let $v \in \mathbb{R}^n$

As $-x^T \Sigma^{-1} x/2 + v^T x - v^T \Sigma v/2 = -(x - \Sigma v)^T \Sigma^{-1} (x - \Sigma v)/2$

MGF $\frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \int e^{-x^T \Sigma^{-1} x/2 + v^T x} dx = e^{v^T \Sigma v/2}$

$$\mathbb{E}[X_1 \cdots X_n] = \partial_{v_1 \cdots v_n} e^{v^T \Sigma v/2} \Big|_{v_1 = \cdots = v_n = 0}$$

if $2 \nmid n$, $\mathbb{E}[X_1 \cdots X_n] = 0$

else $n = 2m$ expanding $\frac{1}{m!} (v^T \Sigma v/2)^m$ we are done.

$\mathbb{E}[X_1 \cdots X_n]$ is coefficient of $v_1 \cdots v_n$ in Taylor expansion of $e^{v^T \Sigma v/2}$

6. A_n m. independent $P(\bigcup_n A_n) = 1$ $P(A_n) < 1$,

Prove $P(A_n \text{ i.o.}) = 1$

$$P(A_n \text{ i.o.}) = P\left(\bigcup_n \bigcap_{k=n}^{\infty} A_k^c\right)$$

$$= \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} A_k^c\right)$$

$$= \lim_{n\to\infty} \prod_{k=n}^{\infty} P(A_k)^c$$

$$= \lim_{n\to\infty} e^{-\left(\sum_{k=n}^{\infty} -\ln P(A_k^c)\right)}$$

$$= \lim_{n\to\infty} e^{-\infty} = 0. \quad \#$$

$$1 = P(\bigcup_n A_n)$$

$$\Rightarrow P\left(\bigcap_n A_n^c\right) = 0$$

$$= \prod_n P(A_n^c)$$

$$\Rightarrow \sum_{n=1}^{\infty} -\ln P(A_n^c) = +\infty$$

$\forall n$

7. Binomial $\to$ Poisson $\to$ Normal distribution (See code)

Find suitable param. regime.

Poisson λ $B(n,p)$ $np = \lambda$ $p = \frac{\lambda}{n}$

Normal $N(\lambda, \lambda)$

① $B \to P$: Fix $\lambda = 1$. $n \to \infty$

② $P \to N$: $\lambda \to \infty$