6.1

1. $\begin{cases} y''+p(x)y'+q(x)y=0 \\ y(x_0)=y_0, \quad y'(x_0)=y_0' \end{cases}$ (*)

p, q 在 (x_0-r, x_0+r) 解析 则 $y=y(x)$ 在 (x_0-r, x_0+r) 解析

记 $\vec{t}(x)=(y(x), y'(x))^T := (t_1(x), t_2(x))^T$

$\vec{t}(x_0)=(y_0, y_0')^T$

$\frac{d\vec{t}(x)}{dx} = (t_2, -p(x)t_2 - q(x)t_1)^T = \vec{f}(x, \vec{t})$

则 由 p, q 在 (x_0-r, x_0+r) 解析

故由 Cauchy 定理,

$\vec{t}(x)$ 在 x_0 某邻域有解析解 即 $t_1(x)=y(x)$ 有解析解

半径 $\rho = r(1-e^{-\frac{b}{3rM}})$. 以为常数 令 $b\to\infty$ ($\vec{t}(x)$ 在此解析)

有 $\rho\to r$. 故 $y(x)$ 在 (x_0-r, x_0+r) 解析.

6.2 2. $x=x_0$ 处两个线性无关解:

(1) $y''-xy'-y=0$, $x_0=0$

(2) $y''-xy'-y=0$, $x_0=1$

(1) 记 $y=\sum_{k=0}^{\infty} a_k x^k$ $\quad y'=\sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k$

$y''=\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k$

$k=0$, $a_0=2a_2$

$k\geq 1$, $(k+2)(k+1)a_{k+2}-ka_k-a_k=0$

$(k+2)a_{k+2}=a_k$

$a_3=\frac{1}{3}a_1$, $a_5=\frac{1}{5\cdot 3}a_1$, $\cdots$ $a_2=\frac{1}{2}a_0$, $a_4=\frac{1}{4\cdot 2}a_0$.

$y=a_1\sum_{k=0}^{\infty}\frac{1}{(2k+1)!!}x^{2k+1}+a_0\sum_{k=0}^{\infty}\frac{1}{(2k)!!}x^{2k}$

$y_1=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!!}x^{2k+1}$, $\quad y_2=\sum_{k=0}^{\infty}\frac{1}{(2k)!!}x^{2k}$

(2) $y=\sum_{k=0}^{\infty} b_k (x-1)^k$ $y'=\sum_{k=1}^{\infty} k b_k (x-1)^{k-1}$, $y''=\sum_{k=2}^{\infty} k(k-1) b_k (x-1)^{k-2}$

$\boxed{y''-(x-1)y'-y'-y=0}$

$k\geq 1$ 时, $(k+2)(k+1)b_{k+2}-kb_k-(k+1)b_{k+1}-b_k=0$

$k=0$, $2b_2-b_1-b_0=0$ 总结一下即

$k\geq 0$, $(k+2)b_{k+2}=b_{k+1}+b_k$ 可取

$b_0=0, b_1=1$ 及 $b_0=1, b_1=0$, 得到两个线性无关解.

$y=\sum_{k=0}^{\infty} b_k (x-1)^k$. (无法写成显式表达)

6.3

1. $x=-1,0,1$ 是常点, 正则奇点还是非正则奇点?

(1) $xy''+(1-x)y'+xy=0$ $\Rightarrow y''+\frac{1-x}{x}y'+y=0$

(2) $2x^4(1-x^2)y''+2xy'+3x^2y=0$ $\Rightarrow 2y''+\frac{y'}{x^3(1-x^2)}+\frac{3y}{x^2(1-x^2)}=0$

(3) $x^2(1-x^2)y''+2x^{-1}y'+4y=0$ $\Rightarrow y''+\frac{2}{x^3(1-x^2)}y'+\frac{4y}{x^2(1-x^2)}=0$

只需判断 p, q 极点阶数

结论:

	eq.1	eq.2	eq.3
-1	常	正则	正则
0	正则	非正则	非正则
1	常	正则	正则

2. 广义幂级数

(1) $2xy'' + y' + xy = 0$

0 是正则奇点

$$y = \sum_{k=0}^{\infty} C_k x^{k+\rho}, \quad C_0 \neq 0 \qquad y' = \sum_{k=0}^{\infty} (k+\rho) C_k x^{k+\rho-1}$$

$$y'' = \sum_{k=0}^{\infty} (k+\rho)(k+\rho-1) C_k x^{k+\rho-2}$$

$$\sum_{k=0}^{\infty} 2(k+\rho)(k+\rho-1) C_k x^{k+\rho-1} + \sum_{k=0}^{\infty} (k+\rho) C_k x^{k+\rho-1} + \sum_{k=2}^{\infty} C_{k-2} x^{k+\rho-1} = 0$$

$$2\rho(\rho-1) C_0 + \rho C_0 = 0 \Rightarrow \rho = 0 \text{ or } \frac{1}{2}$$

1° $\rho = \frac{1}{2}$ $\quad 2 \cdot \frac{3}{2} \cdot \frac{1}{2} C_1 + \frac{3}{2} C_1 = 0, \ C_1 = 0$

$k \geq 2$, $\quad C_k (k+\frac{1}{2})(2k) + C_{k-2} = 0 \qquad C_3 = C_5 = \cdots = 0$

$$C_{2k} = \frac{-C_{2k-2}}{2k(4k+1)} \qquad C_{2k} = \frac{(-1)^k}{(2k)!!(4k+1)(4k-3)\cdots 1} C_0$$

2° $\rho = 0$ 退化为幂级数

$$y = \sum_{k=0}^{\infty} C_k x^k \quad y' = \sum_{k=1}^{\infty} k C_k x^{k-1} \quad y'' = \sum_{k=2}^{\infty} k(k-1) C_k x^{k-2}$$

$k=0$, $C_1 = 0$

$k \geq 2$, $\quad 2k(k-1)C_k + kC_k + C_{k-2} = 0$

$$C_{2k} = \frac{-C_{2k-2}}{2k(4k-1)} \qquad C_{2k} = \frac{(-1)^k C_0}{(2k)!!(4k-1)\cdots 3}$$

(2) $xy'' + y' - y = 0$

同上有 $\displaystyle\sum_{k=0}^{\infty} (k+\rho)(k+\rho-1) C_k x^{k+\rho-1} + \sum_{k=0}^{\infty} (k+\rho) C_k x^{k+\rho-1} - \sum_{k=0}^{\infty} C_k x^{k+\rho} = 0$

$x^{\rho-1}$ 系数: $\rho \cdot (\rho-1) C_0 + \rho C_0 = 0 \Rightarrow \rho = 0$ 退化为幂级数

$$\sum_{k=2}^{\infty} k(k-1) C_k x^{k-1} + \sum_{k=1}^{\infty} k C_k x^{k-1} - \sum_{k=0}^{\infty} C_k x^k = 0$$

x^0: $C_1 - C_0 = 0$

$x^k (k \geq 1)$: $\quad (k+1) k C_{k+1} + (k+1) C_{k+1} - C_k = 0$

$$C_k = \frac{C_{k-1}}{k^2} \quad k \geq 1$$

$$C_n = \frac{C_0}{(n!)^2}$$

7.1

1. 定理 7.1 中，若 $R(x) > Q(x)$ 则有 $\exists \bar{x} \in (x_1, x_2)$, $\psi(\bar{x}) = 0$

设 x_1, x_2 为 ϕ 相邻零点，不妨设 $\phi(x) > 0$, $x \in (x_1, x_2)$. 有 $\phi'(x_1) > 0$, $\phi'(x_2) < 0$

若不然，则不妨设 $\psi(x) > 0$, $x \in (x_1, x_2)$ 且 $\psi(x_1) = 0$ or $\psi(x_2) = 0$. 记 $V(x) = \psi(x)\phi'(x) - \phi(x)\psi'(x)$

$V'(x) + p(x)V(x) = (R(x) - Q(x))\phi(x)\psi(x) > 0$, $x \in (x_1, x_2)$

$$\frac{d}{dx}\left(e^{\int_{x_1}^{x} p(t)dt} V(x)\right) > 0, \quad x \in (x_1, x_2)$$

$$e^{\int_{x_1}^{x_2} p(t)dt} V(x_2) > e^{\int_{x_1}^{x_1} p(t)dt} V(x_1) = V(x_1) = \psi(x_1)\phi'(x_1)$$

$$\| $$

$e^{\int_{x_1}^{x_2} p(t)dt} \psi(x_2)\phi'(x_2)$ 若 $\psi(x_1) = 0$ 则 $\psi(x_2) < 0$ 若 $\psi(x_2) = 0$ 则 $\psi(x_1) < 0$

均矛盾. 证毕!

2. $y'' + Q(x)y = 0$ $Q \in C(\mathbb{R})$ $\exists m, M > 0$,

$m < Q(x) < M$.

由推论 7.2, $y = \phi(x)$ 无限振动. 若 $x_2 - x_1 \geq \frac{\pi}{\sqrt{m}}$ 设 $\bar{x}_2 - \bar{x}_1 = \frac{\pi}{\sqrt{m}}$

$x_1 \leq \bar{x}_1 < \bar{x}_2 \leq x_2$

将原方程与 $y'' + my = 0$ 对比

解 $y = \sin(\sqrt{m}(x - \bar{x}_1))$ 相邻解 $\bar{x}_1, \bar{x}_2$ 之间 $(\bar{x}_1, \bar{x}_2)$ 无 $\phi(x)$ 根

矛盾!

故 $x_2 - x_1 < \frac{\pi}{\sqrt{m}}$

若 $x_2 - x_1 \leq \frac{\pi}{\sqrt{M}}$ 与 $y'' + My = 0$ 比较

$y = \sin(\sqrt{M}(x - x_1))$ 必有一零点 $x_3 \in (x_1, x_2)$

但 $y = \sin(\sqrt{M}(x - x_1))$ 零点 $x = x_1 + \frac{k\pi}{\sqrt{M}}$

$\frac{\pi}{\sqrt{M}} \geq x_2 - x_1 > x_3 - x_1 \geq \frac{\pi}{\sqrt{M}}$ 矛盾! 故 $x_2 - x_1 > \frac{\pi}{\sqrt{M}}$.

3. $y(x)$, $z(x)$ 为 $y'' + q(x)y = 0$, $z'' + Q(x)z = 0$ 解

满足 $y(x_0) = z(x_0)$, $y'(x_0) = z'(x_0)$

(x_0, x_1) 上 $Q(x) > q(x)$, $y(x) > 0$, $z(x) > 0$ 则 $\frac{z(x)}{y(x)}$ 在 (x_0, x_1) 上 $\searrow$.

$$\begin{cases} zy'' + q(x)yz = 0 \\ yz'' + Q(x)yz = 0 \end{cases}$$

$(Q(x) - q(x))y(x)z(x) = (y'z - yz')'$, $x \in (x_0, x_1)$.

记 $V(x) = y'(x)z(x) - y(x)z'(x)$. $V(x_0) = 0$ $V'(x) > 0$, $x \in (x_0, x_1)$.

$\Rightarrow V(x) > 0$. $x \in (x_0, x_1)$. $\left(\frac{z(x)}{y(x)}\right)' < 0$ #.

7. $q(x)\le 0$ 连续. $y''+q(x)y=0$ 满足 $y(0)=a$, $y(1)=b$ 解存在唯一

若 $a\neq 0, b=0$ 此解在 $[0,1]$ 严格单调.

唯一：易知 y 非振荡 若 y_1, y_2 均为解 记 $z=y_1-y_2$，则 z 也为解

$z(0)=0$, $z(1)=0$ 与 z 非振荡矛盾!

存在：考虑满足初值 $y_2(0)=0$, $y_2'(0)=\pm 1$ 的解 $\phi(x)$

其解在 $x>0$ 时恒正 ($y'(0)=1$) 或恒负 ($y'(0)=-1$)

考虑满足 $y_3(0)=0$ $y_3'(0)=\mu$ 的解. 由存在唯一性, 其为 $\mu\phi(x)$

故 $y_3(1)=\mu y_2(1)$

综上, 存在存在解 满足 $y(0)=0$, $y(1)=\mu$, $\forall \mu\in\mathbb{R}$.

任取一个满足 $y(0)=a$ 的解 y_4. 设 $y_4(1)=c$ 取 $y_5(0)=0$, $y_5(1)=b-c$

则 y_4+y_5 合题.

单调：此时 y 在 $[0,1)$ 上恒正或恒负 $y''=-q(x)y$ 恒正或恒负

$y>0 \Rightarrow y''>0 \Rightarrow y'\nearrow$ 且 $y'(1)<0 \Rightarrow y'<0 \Rightarrow y$ 严格减

$y<0 \Rightarrow y''<0 \Rightarrow y'\searrow$ 且 $y'(1)>0 \Rightarrow y'>0 \Rightarrow y$ 严格增

7.2

1. (1) $\begin{cases} y''+\lambda y=0 \\ y'(0)=0, y(1)=0 \end{cases}$ (2) $\begin{cases} y''+\lambda y=0 \\ y(0)=0, y'(1)=0 \end{cases}$

Case 1. $\lambda=-a^2$, $(a\neq 0)$ $y=a_1e^{ax}+a_2e^{-ax}$ $y'=a_1ae^{ax}-a_2ae^{-ax}$

(1): $\begin{cases} a_1a-a_2a=0 \\ a_1e^a+a_2e^{-a}=0 \end{cases} \Rightarrow y\equiv 0$ (2): $\begin{cases} a_1+a_2=0 \\ a_1ae^a-a_2ae^{-a}=0 \end{cases} \Rightarrow y\equiv 0$

Case 2. $\lambda=0$ y 为线性函数. (1): $y\equiv 0$ (2): $y\equiv 0$

Case 3. $\lambda=a^2$ $(a\neq 0)$ $y=a_1\cos ax+a_2\sin ax$ $y'=-aa_1\sin ax+aa_2\cos ax$

(1): $\begin{cases} aa_2=0 \\ a_1\cos a+a_2\sin a=0 \end{cases} \Rightarrow$ $y\equiv 0$ or $a=(n+\frac{1}{2})\pi$, $\lambda_n=(n+\frac{1}{2})^2\pi^2$

$y_n=k\cos(n+\frac{1}{2})\pi x$

(2): $\begin{cases} a_1=0 \\ aa_2\cos a=0 \end{cases} \Rightarrow y\equiv 0$ or $a=(n+\frac{1}{2})\pi$, $\lambda_n=(n+\frac{1}{2})^2\pi^2$

$y_n=k\sin(n+\frac{1}{2})\pi x$

2. BVP $\begin{cases} x^2y'' - \lambda x y' + \lambda y = 0 \\ y(1)=0,\ y(2)=0 \end{cases}$ 无非零解. $\lambda \in \mathbb{R}$

令 $x = e^t$ 记 $y = y(x) = z(t)$ 则 $z(t) = y(e^t)$

$z'(t) = e^t y'(e^t)$

$z(0) = 0 \quad z(\ln 2) = 0$

$z''(t) = e^t y'(e^t) + e^{2t} y''(e^t)$

$$e^{2t} y''(e^t) - \lambda e^t y'(e^t) + \lambda y(e^t) = 0$$

$$z''(t) - z'(t) - \lambda z'(1) + \lambda z(t) = 0$$

$$z''(t) - (\lambda+1) z'(t) + \lambda z(t) = 0$$

特征根 $x_1 = 1 \quad x_2 = \lambda$

Case1. $\lambda \neq 1$ 通解. $z(t) = a e^t + b e^{\lambda t}$ $\begin{cases} a+b=0 \\ 2a + b2^\lambda = 0 \end{cases}$ $a(2-2^\lambda)=0$ $\Rightarrow \lambda = 1$ or $a=0$ 均不合题.

Case2. $\lambda = 1$. 通解. $z(t) = (a+bt)e^t$

$\begin{cases} a = 0 \\ 2(a + b\ln 2) = 0 \end{cases} \Rightarrow a=b=0$ 综上证毕

4. 非齐次线性 ODE 的 S-L BVP

$$\begin{cases} y'' + (\lambda r(x) + q(x))y = f(x) \\ y(0)\cos\alpha - y'(0)\sin\alpha = 0,\ y(1)\cos\beta - y'(1)\sin\beta = 0 \end{cases}$$

r, q 连续, $r > 0$. λ 不是特征值时, 有且仅有一解.

$\lambda = \lambda_m$ 是特征值时, 有解

$$\Longleftrightarrow \int_0^1 f(s)\,\phi_m(s)\,ds = 0$$

考虑方程解 $\phi(x)=\sum_{n=0}^{\infty}C_n\phi_n(x)$，$C_n=\int_0^1\phi(x)\phi_n(x)r(x)dx$，$\phi$ 是非齐次

解 $\Leftrightarrow f(x)=\left(\frac{d^2}{dx^2}+q(x)+\mu r(x)\right)\sum_{n=0}^{\infty}C_n\phi_n(x)$

$$=\sum_{n=0}^{\infty}C_n\left(\frac{d^2}{dx^2}+q(x)\right)\phi_n(x)+\sum_{n=0}^{\infty}\mu C_n r(x)\phi_n(x)$$

$$=\sum_{n=0}^{\infty}C_n\left(-\lambda_n r(x)\phi_n(x)\right)+\sum_{n=0}^{\infty}\mu C_n r(x)\phi_n(x)$$

$$=\sum_{n=0}^{\infty}C_n(\mu-\lambda_n)r(x)\phi_n(x)$$

$$\int_0^1 f(x)\phi_m(x)dx=\int_0^1\sum_{n=0}^{\infty}C_n(\mu-\lambda_n)r(x)\phi_n(x)dx$$

$$=\sum_{n=0}^{\infty}C_n(\mu-\lambda_n)\int_0^1 r(x)\phi_n(x)\phi_m(x)dx$$

$$=\sum_{n=0}^{\infty}C_n(\mu-\lambda_n)\delta_{m,n}$$

$$=C_m(\mu-\lambda_m)$$

若 $\forall n$，$\mu\neq\lambda_n$，则 ϕ 是解 $\Leftrightarrow C_m=\frac{1}{\mu-\lambda_m}\int_0^1 f(x)\phi_m(x)dx$，$\forall m$

若 $\exists n$，$\mu=\lambda_n$ 则 ϕ 是解 $\Rightarrow \int_0^1 f(x)\phi_n(x)dx=0$

此时取 $C_m=\frac{1}{\mu-\lambda_m}\int_0^1 f(x)\phi_m(x)dx$，$m\neq n$

$C_n\in\mathbb{R}$ 即可.

证毕.