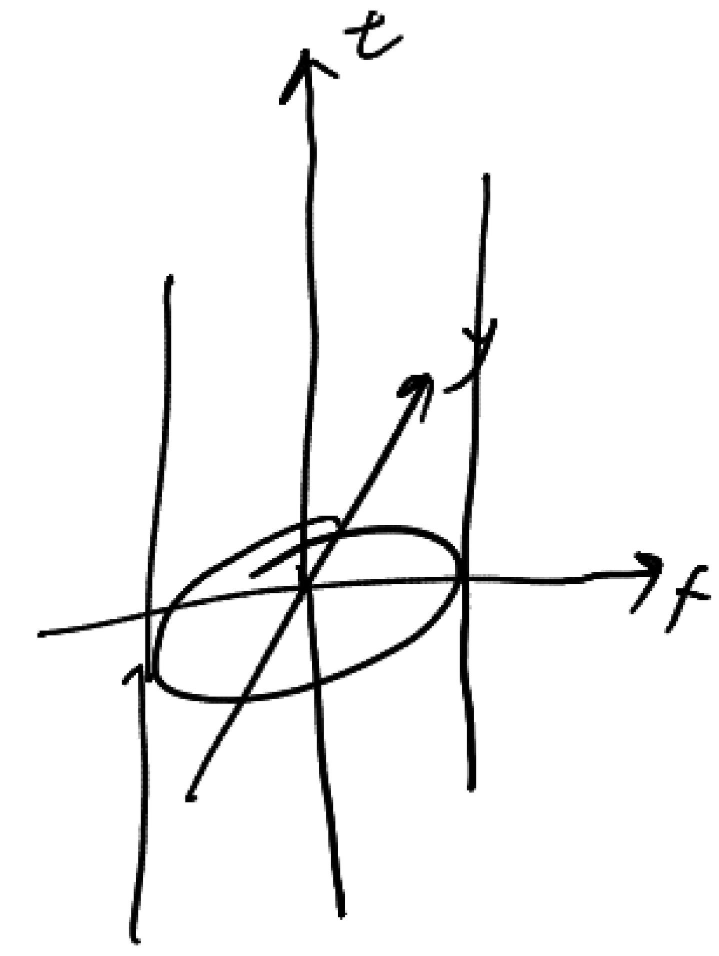
1. $\begin{cases} \partial_t u + x\partial_x u + y\partial_y u = 0 & (x,y)\in\Omega = \{z\in\mathbb{C} \mid |z|<1\} \\ u(x,y,0) = u_0(x,y) \end{cases}$

Need boundary condition?



Characteristic line is

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = y \end{cases}$$

When $x^2+y^2<1$

we have $\left(\frac{x}{e^t}\right)^2 + \left(\frac{y}{e^t}\right)^2 < 1$

$\Rightarrow$ No need boundard condition

$$\frac{d}{dt} u(x_0 e^t, y_0 e^t, t) = 0$$

$$\begin{cases} x = x_0 e^t \\ y = y_0 e^t \end{cases}$$

$$\Rightarrow \quad u(x,y,t) = u_0\left(\frac{x}{e^t}, \frac{y}{e^t}\right)$$

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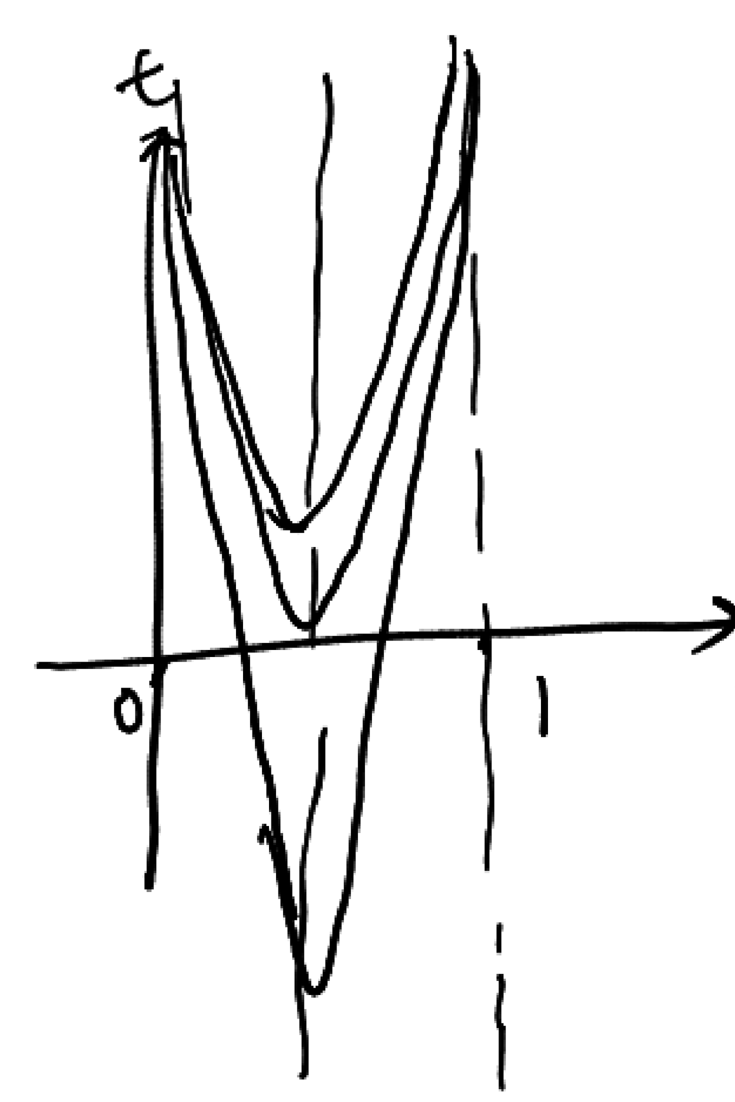
2. $\begin{cases} \partial_t u + x(x-1)\partial_x u = 0 & x\in(0,1) \quad t>0 \\ u(x,0) = u_0(x) \end{cases}$

Need boundary condition?



Characteristic line is

$$\frac{dx}{dt} = x(x-1)$$

$$x(1-x) = Ae^{-t}, \quad A>0$$

Need boundary condition, say $u(\frac{1}{2}, t) = g(t)$, $t\geq 0$, $g(0) = u_0(\frac{1}{2})$

$$\Rightarrow u(x_0,t_0) = \begin{cases} u_0\left(\frac{1}{2} - \sqrt{x_0(x_0-1)e^{t_0}+\frac{1}{4}}\right), & 4x_0(x_0-1)e^{t_0}+1 \geq 0, \ x_0 \leq \frac{1}{2} \\ u_0\left(\frac{1}{2} + \sqrt{x_0(x_0-1)e^{t_0}+\frac{1}{4}}\right), & 4x_0(x_0-1)e^{t_0}+1 \geq 0, \ x_0 > \frac{1}{2} \\ g(t_0 + \ln(4x_0(1-x_0))), & 4x_0(x_0-1)e^{t_0}+1 < 0 \end{cases}$$

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3 (1) $x\partial_x u + y\partial_y u = 2u \qquad u(x,1) = g(x)$

Charalteristic line is

$$\begin{cases} \dfrac{dx}{ds} = x \\ \dfrac{dy}{ds} = y \\ \dfrac{du}{ds} = 2u \end{cases} \Rightarrow \begin{cases} x = x_0 e^s \\ y = y_0 e^s \\ u = u_0 e^{2s} \end{cases}$$

$u(x_0 e^s, y_0 e^s) = u_0 e^{2s}$

$y_0 e^s = 1 \Rightarrow s = -\ln y_0$

$u(\frac{x_0}{y_0}, 1) = g(\frac{x_0}{y_0}) = u_0 \frac{1}{y_0^2}$

$u_0 = y_0^2 g(\frac{x_0}{y_0})$

$\Rightarrow u(x,y) = y^2 g(\frac{x}{y})$ (*)

$(y \neq 0)$

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(2) $x u_x + 2y u_y + u_z = 3u \qquad u(x,y,0) = g(x,y)$

Charalteristic line is

$$\begin{cases} \dfrac{dx}{ds} = x \\ \dfrac{dy}{ds} = 2y \\ \dfrac{dz}{ds} = 1 \\ \dfrac{du}{ds} = 3u \end{cases}$$

$u(x_0 e^s, y_0 e^{2s}, s + z_0) = u_0 e^{3s}$

$u(x_0 e^{-z_0}, y_0 e^{-2z_0}, 0) = g(x_0 e^{-z_0}, y_0 e^{-2z_0})$

$= u_0 e^{-3z_0}$

$\Rightarrow u(x,y,z) = e^{3z} g(\frac{x}{e^z}, \frac{y}{e^{2z}})$

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(3) $u u_x + u_y = 1 \qquad u(x,x) = \frac{x}{2}$

Characteristic line is

$$\begin{cases} \frac{dx}{ds} = u(s) \\ \frac{dy}{ds} = 1 \\ \frac{du}{ds} = 1 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}s^2 + u_0 s + x_0 \\ y = s + y_0 \\ u = s + u_0 \end{cases}$$

$$u\left(\frac{1}{2}s^2 + u_0 s + x_0,\ y_0 + s\right) = u_0 + s$$

Boundary condition $\begin{cases} \frac{1}{2}s^2 + u_0 s + x_0 = y_0 + s \\ y_0 + s = 2(u_0 + s) \end{cases}$

$$\Rightarrow s = y_0 - 2u_0$$

$$\frac{1}{2}(y_0 - 2u_0)^2 + u_0 y_0 - 2u_0^2 + x_0 = 2y_0 - 2u_0$$

$$u_0 = \frac{y_0^2 - 4y_0 + 2x_0}{2(y_0 - 2)}$$

$$\Rightarrow u(x,y) = \frac{y^2 - 4y + 2x}{2(y-2)}$$

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