NA HW2 吕洋 2000010793

1. $\sum_{j=0}^{n} l_j(x) \equiv 1$, $\sum_{j=0}^{n} x_j^k\, l_j(x) = x^k$, $0 \le k \le n$

Proof. 考虑 1 和 $x^k, 0\le k\le n$ 的 Lagrange 插值. 由插值的存在唯一性定理

$$1 = a_0 l_0(x) + \cdots + a_n l_n(x)$$

令 $x = x_i$, $0\le i\le n \Rightarrow a_i = 1$, $0\le i\le n \Rightarrow 1 = \sum_{j=0}^{n} l_j(x)$

$$x^k = b_0 l_0(x) + \cdots + b_n l_n(x), \quad 0\le k\le n$$

令 $x = x_i$, $0\le i\le n \Rightarrow b_i = x_i^k$

故 $x^k = \sum_{j=0}^{n} x_j^k l_j(x)$ #

2. 见附图.

3. $f[x_0, \cdots x_k]$ 为 $f_0 \cdots f_k$ 线性组合 特别地

$$f[x_0, \cdots x_k] = \sum_{j=0}^{k} f(x_j) / w_{k+1}'(x_j)$$

Proof. $w_{k+1}(t) := (t-x_0)\cdots(t-x_k)$ $\quad w_{k+1}'(x_j) = \prod_{\substack{i=0\\ i\ne j}} (x_j - x_i)$

故 $w_{k+1}'(t) = \sum_{j=0}^{k} \frac{w_{k+1}(t)}{t - x_j}$

对 k 归纳. $k=0$, LHS $= f(x_0)$ RHS $= f(x_0)$

$k=1$ LHS $= \frac{f(x_1)-f(x_0)}{x_1-x_0}$ RHS $= \frac{f(x_0)}{x_0-x_1} + \frac{f(x_1)}{x_1-x_0}$

设 $k=t$ 成立 $(t\ge 1)$

$k=t+1$ 时

$$f[x_0 \cdots x_{t+1}] := \frac{f[x_1 \cdots x_{t+1}] - f[x_0 \cdots x_t]}{x_{t+1} - x_0} = A$$

当 $1\le i\le t$ 时, A 中 x_i 系数为 $\frac{1}{x_{t+1}-x_0}\left(\frac{1}{\prod_{\substack{j=1\\ j\ne i}}^{t+1}(x_i-x_j)} - \frac{1}{\prod_{\substack{j=0\\ j\ne i}}^{t}(x_i-x_j)}\right)$

$= \frac{1}{\prod_{\substack{j=0\\ j\ne i}}^{t+1}(x_i-x_j)}$ $\quad x_0$ 系数为 $\frac{1}{x_{t+1}-x_0}\left(-\frac{1}{(x_0-x_1)\cdots(x_0-x_t)}\right) = \frac{1}{(x_0-x_1)\cdots(x_0-x_{t+1})}$

x_{t+1} 系数为 $\frac{1}{x_{t+1}-x_0}\left(\frac{1}{(x_{t+1}-x_1)\cdots(x_{t+1}-x_t)}\right) = \frac{1}{(x_{t+1}-x_0)\cdots(x_{t+1}-x_t)}$

故 $k=t+1$ 成立 综上 证毕 #

4 (1) $M_n = \max_{[a,b]} |f^{(n)}(x)|$

$$\left|f^{(k)}(x) - \Pi_n^{(k)}(x)\right| \le \frac{1}{(n+1-k)!} M_{n+1} (b-a)^{n+1-k}$$

Proof. $f(x) - \Pi_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$

只需证 $\left|\left((x-x_0)\cdots(x-x_n)\right)^{(k)}\right| \le \frac{(n+1)!}{(n+1-k)!}(b-a)^{n+1-k}$

但 $\left|\left((x-x_0)\cdots(x-x_n)\right)^{(k)}\right| = \left|\sum_{i_1\cdots i_k} \prod_{j\neq i_1\cdots i_k}(x-x_j)\right|$

$$\le \sum_{i_1\cdots i_k}(b-a)^{n+1-k}$$

$$= \frac{(n+1)!}{(n+1-k)!}(b-a)^{n+1-k} \quad \#$$

其中 $i_1 \cdots i_k$ 不等

(2) $\nabla^+ f(x) = f(x+h) - f(x)$ 证明 $(\nabla^+)^k f = k!\, h^k f[x_0, \cdots x_k]$

Proof.

$$RHS = k!\, h^k f[x_0 \cdots x_k] = k!\, h^k \sum_{i=0}^{k} \frac{f(x_i)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_k)}$$

$$= k!\, h^k \sum_{i=0}^{k} \frac{f(x_0+ih)}{h^k\, i!\,(k-i)!\,(-1)^{k-i}}$$

$$= \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x_0+ih)$$

而 $\nabla^+ f(x) = f(x+h) - f(x)$

$(\nabla^+)^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$

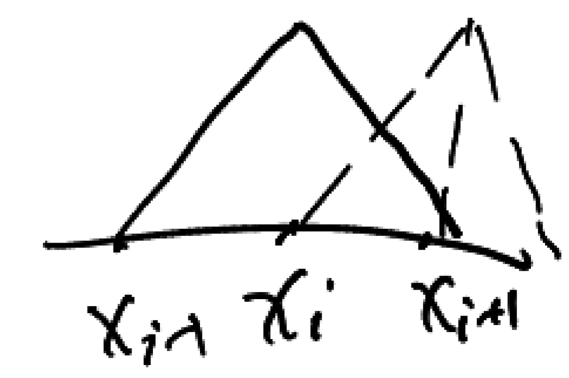
由归纳法易知

$$(\nabla^+)^k f(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x_0+ih)$$

故 LHS = RHS #

5. $(f,g)=\int_a^b fg\,dx$ 分片线性基 $\{l_j(x)\}$,

则刚度 $S_{ij}=(l_i',l_j')$ 质量 $M_{ij}=(l_i,l_j)$



先考虑 M. 当 $0\le i\le n-1$ 时,

$$M_{i,i+1}=\int_{x_i}^{x_{i+1}}\frac{x-x_{i+1}}{x_i-x_{i+1}}\,\frac{x-x_i}{x_{i+1}-x_i}\,dx$$

$$=-\frac{1}{(x_{i+1}-x_i)^2}\int_{x_i}^{x_{i+1}}(x-x_i)(x-x_{i+1})\,dx=\frac{x_{i+1}-x_i}{6}=M_{i+1,i}$$

$1\le i\le n-1$ 时, $M_{i,i}=\int_{x_i}^{x_{i+1}}\frac{(x-x_{i+1})^2}{(x_i-x_{i+1})^2}\,dx+\int_{x_{i-1}}^{x_i}\frac{(x-x_{i-1})^2}{(x_i-x_{i-1})^2}\,dx$

$$=\frac{x_{i+1}-x_i}{3}+\frac{x_i-x_{i-1}}{3}$$

$$=\frac{x_{i+1}-x_{i-1}}{3}$$

$M_{0,0}=\frac{x_1-x_0}{3}$ $\quad M_{n,n}=\frac{x_n-x_{n-1}}{3}$ $\quad$ 其余 $M_{i,j}=0$

再求 S. 当 $0\le i\le n-1$ 时,

$$S_{i,i+1}=\int_{x_i}^{x_{i+1}}\frac{1}{x_i-x_{i+1}}\,\frac{1}{x_{i+1}-x_i}\,dx=\frac{1}{x_i-x_{i+1}}=S_{i+1,i}$$

$1\le i\le n-1$ 时, $S_{i,i}=\int_{x_i}^{x_{i+1}}\frac{1}{(x_i-x_{i+1})^2}\,dx+\int_{x_{i-1}}^{x_i}\frac{1}{(x_i-x_{i-1})^2}\,dx=\frac{1}{x_{i+1}-x_i}+\frac{1}{x_i-x_{i-1}}$

$S_{0,0}=\frac{1}{x_1-x_0}$ $\quad S_{n,n}=\frac{1}{x_n-x_{n-1}}$ $\quad$ 其余 $S_{i,j}=0$.

6. $f\in C^4[a,b]$ 求三次 $p(x)$

$p(x_i)=f(x_i)$ $\quad i=0,1,2$

$p'(x_1)=f'(x_1)$

$a\le x_0<x_1<x_2\le b$ $\quad$ 并求 p 逼近 f 的截断误差

考虑基 $\alpha_1(x)$ $\alpha_2(x)$ $\alpha_3(x)$ $\alpha_4(x)$ （Hermite 插值）

$\alpha_1(x_0)=1$ $\alpha_1(x_1)=\alpha_1'(x_1)=0$, $\alpha_1(x_2)=0$

$$\alpha_1(x)=\frac{(x-x_2)(x-x_1)^2}{(x_0-x_2)(x_0-x_1)^2}$$

有 $f(x)=f(x_0)\alpha_1(x)+f(x_2)\alpha_2(x)+f(x_1)\alpha_3(x)+f'(x_1)\alpha_4(x)$

$\alpha_2(x_2)=1$ $\alpha_2(x_1)=\alpha_2'(x_1)=0$ $\alpha_2(x_0)=0$

$$\alpha_2(x)=\frac{(x-x_0)(x-x_1)^2}{(x_2-x_0)(x_2-x_1)^2}$$

$\alpha_3(x_1)=1$ $\alpha_3'(x_1)=0$ $\alpha_3(x_0)=0$ $\alpha_3(x_2)=0$

$$\alpha_3(x)=\frac{(x-x_0)(x-x_2)((x-x_1)+A)}{A(x_1-x_0)(x_1-x_2)},\quad A=\frac{(x_0-x_1)(x_1-x_2)}{2x_1-x_0-x_2}$$

$\alpha_4(x_1)=0$ $\alpha_4'(x_1)=1$ $\alpha_4(x_0)=0$ $\alpha_4(x_2)=0$

$$\alpha_4(x)=\frac{(x-x_0)(x-x_2)(x-x_1)}{(x_1-x_0)(x_1-x_2)}$$

我们有 $R(x)=f(x)-p(x)=K(x)(x-x_1)^2(x-x_0)(x-x_2)$ 设 $x\neq x_0,x_1,x_2$

记 $F(t)=R(t)-K(x)(t-x_1)^2(t-x_0)(t-x_2)$

$F(t)$ 在 $[a,b]$ 上有零点 x_0, x_1, x_1, x_2, x

由于 $f\in C^4[a,b]$，由 Rolle 定理，

$\exists\xi\in(a,b)$ $F^{(4)}(\xi)=0$ $\Rightarrow$ $K(x)=\frac{f^{(4)}(\xi)}{4!}$

$$|f(x)-p(x)|=\frac{|f^{(4)}(\xi)|}{24}\left|(x-x_1)^2(x-x_0)(x-x_2)\right|$$
$$\leq\frac{M}{24}\frac{(b-a)^2}{4}(b-a)^2=\frac{M}{96}(b-a)^4$$

NA HW3

1. $H(x) = y_0^{(0)} h_0^0(x) + y_0^{(1)} h_0^1(x) + y_1^{(0)} h_1^0(x) + y_1^{(1)} h_1^1(x)$

$h_i^0(x_j) = \delta_{ij}, \quad (h_i^0)'(x_j) = 0 \quad h_i^1(x_j) = 0, \quad (h_i^1)'(x_j) = \delta_{ij}$

写出 $h_i^j(x), \ i,j=0,1$

$$h_0^0(x) = \frac{(x-x_1)^2\left(\frac{2(x-x_0)}{x_1-x_0}+1\right)}{(x_0-x_1)^2} \qquad h_1^0(x) = \frac{(x-x_0)^2\left(\frac{2(x-x_1)}{x_0-x_1}+1\right)}{(x_1-x_0)^2}$$

$$h_0^1(x) = \frac{(x-x_0)(x-x_1)^2}{(x_0-x_1)^2} \qquad h_1^0(x) = \frac{(x-x_1)(x-x_0)^2}{(x_1-x_0)^2}$$ 均由性质直接唯一决定

2. (1) 二次样条的存在唯一性

(2) 高次样条的可能性

(1) 在 $x_0 < x_1 < \cdots < x_n$ 上样条 值为 $y_0, y_1, \cdots y_n$, 导数 $D_0, D_1, \cdots D_n$

$S_i(t) = a_i + b_i t + c_i t^2, \ i=0,\cdots n-1, \ t\in[0,1]$

$S_i(0) = a_i = y_i \quad , i=0,\cdots n-1$

$S_i(1) = a_i + b_i + c_i = y_{i+1} \quad , i=0,\cdots n-1$

$S_i'(0) = D_i = b_i \quad , i=0,\cdots n-1$

$S_i'(1) = D_{i+1} = b_i + 2c_i \quad , i=0,\cdots n-1$

$$\begin{cases} a_i = y_i \\ b_i = D_i \\ c_i = y_{i+1} - y_i - D_i \end{cases} i=0,\cdots n-1$$ 且 $D_i + D_{i+1} = 2y_{i+1} - 2y_i, \ i=0,\cdots n-1$

再补充 $D_0 = s_0$ (给定任一点导数都可)

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & \ddots & \\ & & 1 & 1 \end{pmatrix}\begin{pmatrix} D_0 \\ \vdots \\ D_{n-1} \end{pmatrix} = \begin{pmatrix} s_0 \\ 2y_1 - 2y_0 \\ \vdots \\ 2y_n - 2y_{n-1} \end{pmatrix}$$ 这方程组解存在唯一

故 (a_i, b_i, c_i) 存在唯一

(2) 是可能的

考虑 k 次的多项式, $n+1$ 个点 未知数有 $n(k+1)$ 个

$C^{k-1} \Rightarrow (n-1)k$ 个方程 节点 2 个方程, 有 $k-1$ 个自由度

n 次 B-spline $M_n(x_0, \cdots x_n; x)$ 即为实例.

3. 三次样条 $S(x)$

$$S'(x_i)=y_i', \quad i=1,\cdots n$$

如何给出边界条件让 $S(x)$ 唯一确定

设 $x_1<\cdots<x_{n-1}<x_n$ 导数 $D_1,\cdots D_n$ $(D_i=y_i')$

$$S_i(x)=a_i+b_ix+c_ix^2+d_ix^3, \quad x\in[0,1], \quad i=1,\cdots n-1$$

$$S_i'(0)=b_i=D_i \quad i=1,\cdots n-1$$

$$S_i'(1)=b_i+2c_i+3d_i=D_{i+1} \quad i=1,\cdots n-1$$

$$S_i''(1)=S_{i+1}''(0) \Rightarrow 2c_i+6d_i=2c_{i+1} \quad i=1,\cdots n-2$$

$$S_i(0)=a_i=y_i \quad i=1,\cdots n-1$$

$$S_i(1)=a_i+b_i+c_i+d_i=y_{i+1} \quad i=1,\cdots n-1$$

有
$$\begin{cases} a_i=y_i \\ b_i=D_i \\ c_i=3(y_{i+1}-y_i)-2D_i-D_{i+1} \\ d_i=2(y_i-y_{i+1})+D_i+D_{i+1} \end{cases}$$

$$3y_{i+2}-3y_i=D_i+D_{i+2}+4D_{i+1}, \quad i=1,\cdots n-2 \quad (*)$$

n 个插值点 $y_1\cdots y_n$ $n-2$ 个方程

可取边界条件为 $S(x_1)=y_1$ $S(x_2)=y_2$ 这样(*)有唯一解.

4. $B_{i,k}(x) = \dfrac{x_{i+k+1} - x_i}{k+1} M_{k+1}[x_i, \cdots x_{i+k+1}; x]$ $k \geq 0$

$$B_{i,0}(x) = \begin{cases} 1 & x \in [x_i, x_{i+1}) \\ 0 & \text{else} \end{cases}$$

$$B_{i,k}(x) = \frac{x - x_i}{x_{i+k} - x_i} B_{i,k-1}(x) + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} B_{i+1,k-1}(x), \quad k = 1, 2, \cdots \quad (*)$$

$$\sum_{i=J-k}^{J} B_{i,k}(x) = 1, \quad x_J \leq x < x_{J+1} \qquad \sum_{i=-k}^{n-1} B_{i,k}(x) = 1, \quad x_0 \leq x < x_n$$

Proof.

$$B_{i,0}(x) = (x_{i+1} - x_i) M_1[x_i, x_{i+1}; x] = (x_{i+1} - x_i)\left(\frac{1_{x_i > x}}{x_i - x_{i+1}} + \frac{1_{x_{i+1} > x}}{x_{i+1} - x_i}\right)$$

$$= \begin{cases} 1, & x \in [x_i, x_{i+1}) \\ 0, & \text{else} \end{cases}$$

记 $M_k^*[a_1, \cdots a_{k+1}; x]$ 为 $(k+1)(y-x)_+^k$ 的 k次差商.

(*) 留到后面证

由于 $B_{i,k}(x) = \dfrac{x_{i+k+1} - x_i}{k+1} \dfrac{M_k^*[x_{i+1}, \cdots x_{i+k+1}; x] - M_k^*[x_i, \cdots x_{i+k}; x]}{x_{i+k+1} - x_i}$

$$= \frac{1}{k+1}\left(M_k^*[x_{i+1}, \cdots x_{i+k+1}; x] - M_k^*[x_i, \cdots x_{i+k}; x]\right)$$

$$\sum_{i=J-k}^{J} B_{i,k}(x) = \frac{1}{k+1}\left(M_k^*[x_{J+1}, \cdots x_{J+k+1}; x] - M_k^*[x_{J-k}, \cdots x_J; x]\right) = 0 - 0 = 0$$

$$(x_J \leq x < x_{J+1})$$

$$\sum_{i=-k}^{n} B_{i,k}(x) = \frac{1}{k+1}\left(M_k^*[x_{n+1}, \cdots x_{n+k+1}; x] - M_k^*[x_{-k}, \cdots x_0; x]\right) = 0 - 0 = 0$$

$$(x_0 \leq x < x_n)$$

回到 (*) 的证明.

$$\frac{1}{k+1} M_{k+1}(y; x) = (y-x)_+^k = (y-x) \frac{1}{k} M_k(y; x)$$

由高阶差商的 Leibniz 公式.

$$\frac{1}{k+1} M_{k+1}[x_i, \cdots x_{i+k+1}; x] = \frac{1}{k}(x_i - x) \frac{M_k[x_i, \cdots x_{i+k}; x] - M_k[x_{i+1}, \cdots x_{i+k+1}; x]}{x_i - x_{i+k+1}}$$

$$+ \frac{1}{k} M_k[x_{i+1}, \cdots x_{i+k+1}; x]$$

即 $\dfrac{B_{i,k}(x)}{x_{i+k+1} - x_i} = \dfrac{B_{i+1,k-1}(x)}{x_{i+k+1} - x_{i+1}} + \dfrac{x_i - x}{x_{i+k+1} - x_i}\left(-\dfrac{B_{i,k-1}(x)}{x_{i+k} - x_i} + \dfrac{B_{i+1,k-1}(x)}{x_{i+k+1} - x_{i+1}}\right)$

整理即 (*). #