10. $u(x)$ 是 $B(0,R)$ 调和函数，在 $\overline{B(0,R)}$ 连续

$$M = \int_{B(0,R)} u^2 \, dx$$

(a) $|u(0)| \le \left[\dfrac{M}{\alpha(n) R^n} \right]^{\frac{1}{2}}$

Proof.

$$|u(0)| = \left| \fint_{B(0,R)} u(x)\, dx \right| = \frac{1}{\alpha(n) R^n} \left| \int_{B(0,R)} u \, dx \right| \le \frac{1}{\alpha(n) R^n} \cdot$$

$$\left[\int_{B(0,R)} u^2 dx \int_{B(0,R)} 1 \, dx \right]^{\frac{1}{2}} = \frac{M^{\frac{1}{2}}}{[\alpha(n) R^n]^{\frac{1}{2}}}$$

，由 Cauchy-Schwarz 不等式.

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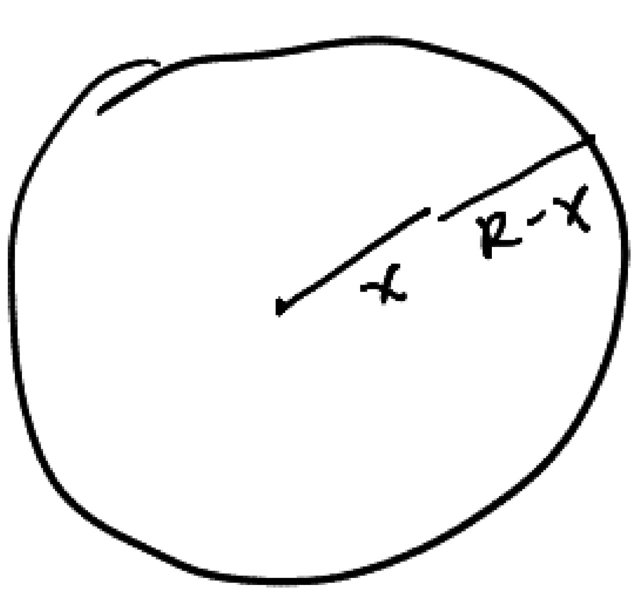
(b) $|u(x)| \le \left[\dfrac{M}{\alpha(n)(R-|x|)^n} \right]^{\frac{1}{2}}$

Proof.



$$u(x) = \left| \fint_{B(x,R-|x|)} u(y)\, dy \right| = \frac{\left| \int_{B(x,R-|x|)} u(y)\, dy \right|}{\alpha(n)(R-|x|)^n}$$

$$\le \frac{\left[\int_{B(x,R-|x|)} u^2(y)\, dy \cdot \int_{B(x,R-|x|)} 1 \, dy \right]^{\frac{1}{2}}}{\alpha(n)(R-|x|)^n}$$

$$\le \frac{M^{\frac{1}{2}}}{\alpha(n)^{\frac{1}{2}} (R-|x|)^{\frac{n}{2}}}$$

，由 Cauchy-Schwarz 不等式 #

14. $u \in C^2(\mathbb{R}^n)$ $r>0$,

$$u_r(x) = \frac{1}{N\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y)\, dS(y)$$

$$\Delta u_r = (\Delta u)_r$$

<u>Proof</u>. $u_r(x) = \dfrac{1}{N\omega_n} \displaystyle\int_{\partial B(0,1)} u(x+rz)\, dS(z)$

$$\frac{\partial u_r(x)}{\partial x_i} = \frac{1}{N\omega_n} \int_{\partial B(0,1)} u_i'(x+rz)\, dS(z)$$

$$\frac{\partial^2 u_r(x)}{\partial^2 x_i} = \frac{1}{N\omega_n} \int_{\partial B(0,1)} u_i''(x+rz)\, dS(z) = \frac{1}{N\omega_n r^{n-1}} \int_{\partial B(0,1)} u_{ii}(y)\, dS(y)$$

$$\Delta u_r(x) = \sum_{i=1}^{n} \frac{\partial^2 u_r(x)}{\partial^2 x_i} = \frac{1}{N\omega_n r^{n-1}} \int_{\partial B(0,1)} \sum_{i=1}^{n} u_{ii}(y)\, dS(y)$$

$$= \frac{1}{N\omega_n r^{n-1}} \int_{\partial B(0,1)} \Delta u(y)\, dS(y) = (\Delta u)_r \quad \#$$

Extra. u 在 B_1 调和 $A(r) = ⨍_{B_r} u^2 dx$ 关于 r 单增.

<u>Proof</u>. $A(r) = \dfrac{1}{\alpha(n) r^n} \displaystyle\int_{B_r} u^2(x)\, dx$

$$= \frac{1}{\alpha(n)} \int_{B_1} u^2(ry)\, dy$$

$$A'(r) = \frac{1}{\alpha(n)} \int_{B_1} 2u \sum_{i=1}^{n} u_i(ry)\, y_i\, dy$$

$$= \frac{2}{\alpha(n)} \int_{B_1} u\, \nabla u \cdot y\, dy = \frac{2}{\alpha(n)} \int_0^1 dr \int_{\partial B_r} u\, \nabla u \cdot y\, dS(y)$$

$$= \int_0^1 dr \int_{B_r} \operatorname{div}(u \nabla u)\, dy = \int_0^1 dr \int_{B_r} |\nabla u|^2\, dy \geq 0$$ (利用到 B_1 上 $\Delta u = 0$)

故 $A(r)$ 关于 r 单增. #