*21 $X_1 \cdots X_n \sim f(x;\theta) = \frac{1}{2} e^{-|x-\theta|}, -\infty < \theta < +\infty$

找出 $\hat{\theta}_n$ 并证明 $\hat{\theta}_n$ 强相合

$$L(x_1, \cdots x_n; \theta) = \frac{1}{2^n} e^{-\sum_{i=1}^{n}|x_i-\theta|}$$

$$\Rightarrow \hat{\theta}_n = \text{Median}(X_1, \cdots X_n)$$

24. $X_1 \cdots X_n \sim B(1,p)$ 由 CLT 构造 n 值较大 p 的近似 $1-\alpha$ 置信水平的置信区间

解. $\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$

对 $B(1,p)$, $\mu = p$, $\sigma^2 = p(1-p)$

$$\sqrt{n}(\overline{X}_n - p) \xrightarrow{d} \sqrt{p(1-p)}\, Z, \quad Z \sim N(0,1)$$

近似: $\sqrt{n}(\overline{X}_n - p) \xrightarrow{d} \sqrt{\overline{X}_n(1-\overline{X}_n)}\, Z$

$$\frac{\overline{X}_n - p}{\sqrt{\frac{\overline{X}_n(1-\overline{X}_n)}{n}}} \approx Z \sim N(0,1)$$

设 $\mathbb{P}(-z_{1-\frac{\alpha}{2}} < Z < z_{1-\frac{\alpha}{2}}) = 1-\alpha$

$$\Rightarrow \mathbb{P}\left(\overline{X}_n - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\overline{X}_n(1-\overline{X}_n)}{n}} < p < \overline{X}_n + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\overline{X}_n(1-\overline{X}_n)}{n}}\right) \approx 1-\alpha$$

$$CI \left[\overline{X}_n - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\overline{X}_n(1-\overline{X}_n)}{n}},\ \overline{X}_n + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\overline{X}_n(1-\overline{X}_n)}{n}}\right]$$

其中 $z_{1-\frac{\alpha}{2}} =$

25 $X_1 \cdots X_n \sim N(\mu, \sigma^2)$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$Var(S^2) = \frac{2\sigma^4}{n-1} \quad (n \geq 2)$$

证：我们有 $\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \sim \chi^2(n-1)$ （ppt所讲）

$$\Rightarrow \quad S^2 \sim \frac{\sigma^2}{n-1} \chi^2(n-1)$$

$$Var(S^2) = \frac{\sigma^4}{(n-1)^2} Var(\chi^2(n-1))$$

$$= \frac{\sigma^4}{(n-1)^2} \cdot 2(n-1) = \frac{2\sigma^4}{n-1}$$

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求方差 σ^2 置信水平0.95置信区间，σ 置信水平0.95置信区间

由PPT，σ^2 置信区间 $\left[\frac{1}{\lambda_2} \sum_{i=1}^{n} (X_i - \bar{X})^2, \frac{1}{\lambda_1} \sum_{i=1}^{n} (X_i - \bar{X})^2\right]$

$Y \sim \chi^2(18)$，$P(Y < \lambda_1) = P(Y > \lambda_2) = 0.025$

$$\lambda_1 = 8.231 \qquad \lambda_2 = 31.526$$

置信区间为 $[418.8, 1603.9]$

σ 置信区间为 $\left[\sqrt{\frac{\sum (X_i - \bar{X})^2}{\lambda_2}}, \sqrt{\frac{\sum (X_i - \bar{X})^2}{\lambda_1}}\right]$

为 $[20.46, 40.05]$