3.36

$u \in C^{2,1}(\overline{Q_T})$

$$\begin{cases} u_t - u_{xx} = f(x,t) & (x,t) \in Q_T \\ u(x,0) = \varphi(x) & 0 \le x \le l \\ u(0,t) = u_x(l,t) = 0 & 0 \le t \le T \end{cases}$$

$$\sup_{0 \le t \le T} \int_0^l u_x^2(x,t)\,dx + \int_0^T \int_0^l u_t^2(x,t)\,dx\,dt$$

$$\le 2 \left\{ \int_0^l [\varphi'(x)]^2\,dx + \int_0^T \int_0^l f^2(x,t)\,dx\,dt \right\}$$

证明: $u_t^2 - u_{xx} u_t = f u_t = u_t^2 + \frac{1}{2}(u_x^2)_t - (u_t u_x)_x$

$$\frac{1}{2}\int_0^l u_x^2\,dx - \frac{1}{2}\int_0^l (\varphi'(x))^2\,dx + \int_0^t \int_0^l u_t^2\,dx\,dt = \int_0^t \int_0^l f u_t\,dx\,dt$$

$$\le \frac{1}{2}\int_0^t \int_0^l f^2\,dx\,dt + \frac{1}{2}\int_0^t \int_0^l u_t^2\,dx\,dt$$

$$\int_0^l u_x^2\,dx + \int_0^t \int_0^l u_t^2\,dx\,dt \le \int_0^l [\varphi'(x)]^2\,dx + \int_0^t \int_0^l f^2\,dx\,dt$$

对 $t \in [0,T]$ 取上确界即证. #

$$\left(\int_0^t \int_0^l (u_t u_x)_x\,dx\,dt = \int_0^t u_t(l,t)\,u_x(l,t)\,dt - \int_0^t u_t(0,t)\,u_x(0,t)\,dt = 0 \right).$$

#

3) $u\in C^{1,0}(\overline{Q_T})\cap C^{2,1}(Q_T)$

$$\begin{cases} u_t-a^2u_{xx}=f(x,t) & (x,t)\in Q_T \\ u(x,0)=\varphi(x) & 0\le x\le l \\ [-u_x+\alpha u]|_{x=0}=[u_x+\beta u]|_{x=l}=0 & 0\le t\le T \end{cases}$$

$a>0$, $\alpha\geq 0$, $\beta\geq 0$ $\quad M=M(T,a)$

$$\sup_{0\le t\le T}\int_0^l u^2(x,t)\,dx+\int_0^T\int_0^l u_x^2(x,t)\,dx\,dt\le M\left[\int_0^l\varphi^2(x)\,dx+\int_0^T\int_0^l f^2(x,t)\,dx\,dt\right]$$

证明:

$$u(u_t-a^2u_{xx})=\frac{1}{2}\left[(u^2)_t+a^2u_x^2\right]-a^2(uu_x)_x$$

$$\frac{1}{2}\int_0^l u^2dx-\frac{1}{2}\int_0^l\varphi^2dx+\frac{a^2}{2}\int_0^t\int_0^l u_x^2dxdt-a^2\int_0^t\int_0^l(uu_x)_xdxdt$$

$$=\int_0^t\int_0^l fu\,dxdt$$

$$\le\frac{1}{2}\int_0^t\int_0^l f^2dxdt+\frac{1}{2}\int_0^t\int_0^l u^2dxdt$$

我们有 $\int_0^l(uu_x)_xdx=uu_x\Big|_0^l=-\beta u^2(l,t)-\alpha u^2(0,t)\le 0$

因此有 $\int_0^l u^2dx-\int_0^l\varphi^2dx+a^2\int_0^t\int_0^l u_x^2dxdt\le\int_0^t\int_0^l f^2dxdt+\int_0^t\int_0^l u^2dxdt$

记 $F(t)=\int_0^l\varphi^2(x)dx+\int_0^t\int_0^l f^2(x,\tau)dxd\tau$

$G(t)=\int_0^t\int_0^l u^2(x,t)\,dxdt \qquad G'(t)=\int_0^l u^2(x,t)\,dx$

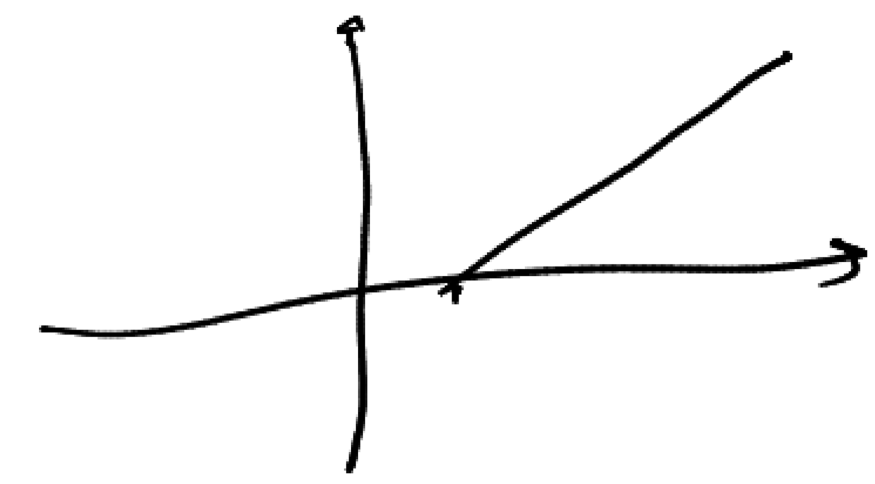
$G'(t)\le F(t)+G(t) \qquad G(0)=0\Rightarrow G(t)\le(e^t-1)F(t)$

有 $\int_0^t u^2dx+\int_0^t\int_0^l u_x^2dxdt\le\left[\int_0^l\varphi^2(x)\,dx+\int_0^t\int_0^l f^2(x,t)\,dxdt\right]\left(e^t+\frac{1}{a^2}+\frac{e^t-1}{a^2}\right)$

对 $t\in[0,T]$ 取上确界即可. #

4.3 特征线解 Cauchy 问题

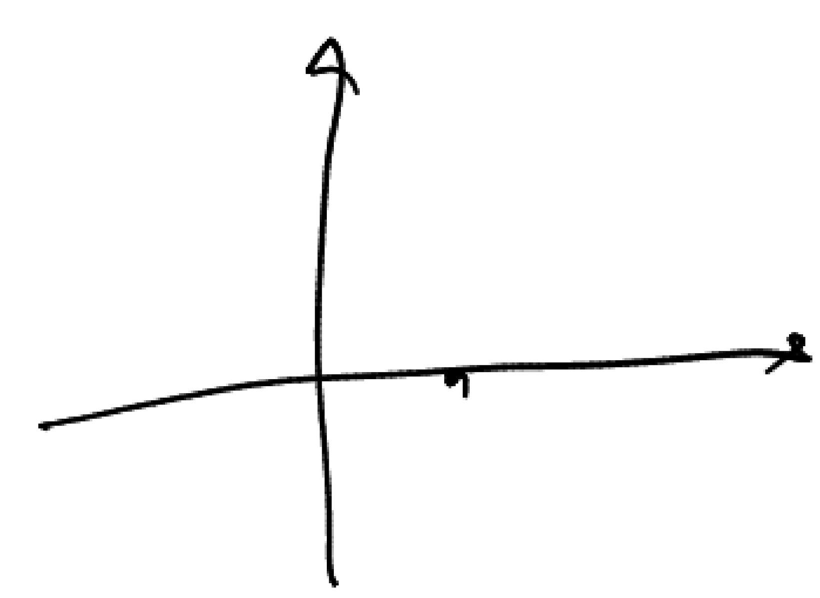
1.(1) $\begin{cases} u_t + 2u_x = 0, & (x,t) \in \mathbb{R}^2_+ \\ u(x,0) = x^2, & x \in \mathbb{R} \end{cases}$



记 $x = x_0 + 2s,\quad t = s$

$u_s = 2u_x + u_t = 0$

$\Rightarrow u(x,t) = u(x-2t, 0) = (x-2t)^2$



(3) $\begin{cases} 2u_t - u_x + xu = 0, & (x,t) \in \mathbb{R}^2_+ \\ u(x,0) = 2x e^{x^2/2}, & x \in \mathbb{R} \end{cases}$

记 $x = x_0 - \frac{s}{2} \quad t = s$

$u_s = -\frac{1}{2} u_x + u_t = -\frac{1}{2}(x_0 - \frac{s}{2}) u$

$u = e^{\frac{1}{8}s^2 - \frac{1}{2}x_0 s} (2x_0 e^{\frac{x_0^2}{2}})$

$u(x,t) = e^{\frac{1}{8}t^2 - \frac{1}{2}(x+\frac{t}{2})t} \cdot (2x+t) e^{\frac{1}{2}(x+\frac{t}{2})^2}$

$u(x,t) = (2x+t) e^{\frac{1}{2}x^2}$

(5) $\begin{cases} u_t + A\cdot Du + cu = 0, & (x,t)\in\mathbb{R}^{n+1}_+ \\ u(x,0) = \varphi(x), & x\in\mathbb{R}^n \end{cases}$

设 $x = (x_{10} + A_1 s, \cdots x_{n0} + A_n s)$, $t = s$

$$u_s = u_t + A\cdot Du = -cu$$

$$u = e^{-cs}\varphi(x_0)$$

$$u(x_1, \cdots x_n) = e^{-ct}\varphi(x_1 - A_1 t, \cdots x_n - A_n t)$$

其中 $A = (A_1 \cdots A_n)^T$

\#

3. (1) $u = u(x,y)$ 在凸连通 $\Omega \subset \mathbb{R}^2$

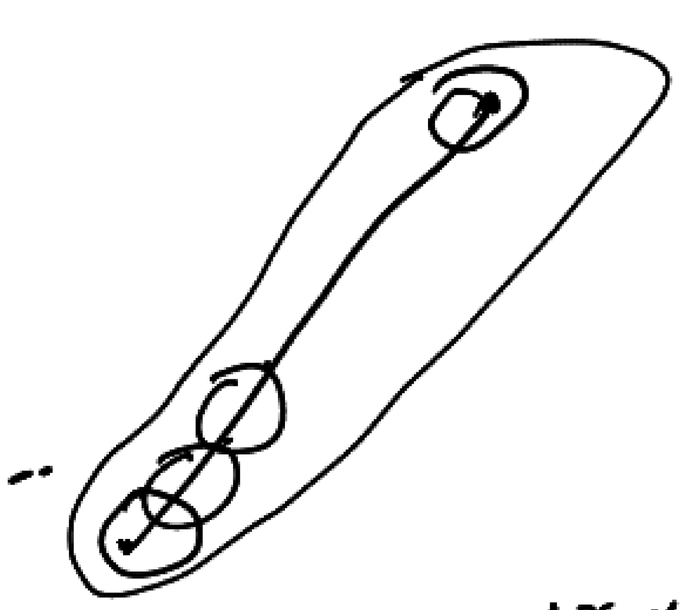
$u_{xy} = 0$ 求 u

任取 $(x_0, y_0) \in \Omega$ 设 $(x,y)\in\Omega$

由 Ω 的凸性, $A := \{(x_0 + t(x - x_0),\ y_0 + t(y - y_0)) \mid t\in[0,1]\} \in \Omega$

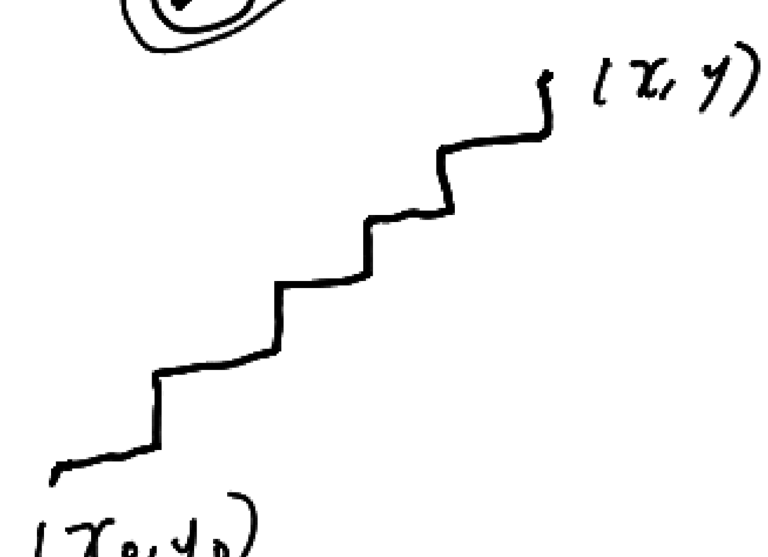
由于 Ω 是区域, $\exists\ (x_1, r_1) \cdots (x_m, r_m)$, $x_i \in \mathbb{R}^2$, $r_i > 0$

$\Omega \supseteq \bigcup_i B(x_i, r_i) \supseteq A$



存在折线 $\overline{A_1A_2 \cdots A_s}$, A_1A_2 水平, A_2A_3 垂直, A_3A_4 水平, …

使 $\overline{A_1 \cdots A_s} \subseteq \bigcup_i B(x_i, r_i)$



$$u(x,y) = u(x_0,y_0) + \sum_i \int_{y_i}^{y_{i+1}} u_y(x_i, t)\,dt + \sum_i \int_{x_i}^{x_{i+1}} u_x(t, y_i)\,dt$$

$$= u(x_0, y_0) + \int_{y_0}^{y} A(t)\,dt + \int_{x_0}^{x} B(t)\,dt$$

$$:= \phi_1(x) + \phi_2(y)$$ \#

(其中 $A(t) = u_y(x,t)$,
$B(t) = u_x(t,y)$).

(2) $a>0$ $\xi = x+at$, $\eta = x-at$

$u_{tt} - a^2 u_{xx} = 0 \Longleftrightarrow u_{\xi\eta} = 0$

证: $x = \frac{1}{2}(\xi+\eta)$

$t = \frac{1}{2a}(\xi-\eta)$

有 $u_{\xi\eta} = (u_\xi)_\eta = \left(\frac{1}{2}u_x + \frac{1}{2a}u_t\right)_\eta = \frac{1}{2}(u_\eta)_x + \frac{1}{2a}(u_\eta)_t$

$= \frac{1}{2}\left(\frac{1}{2}u_x - \frac{1}{2a}u_t\right)_x + \frac{1}{2a}\left(\frac{1}{2}u_x - \frac{1}{2a}u_t\right)_t$

$= \frac{1}{4a^2}(a^2 u_{xx} - u_{tt})$ #

(3) D'Alembert 公式证明, 利用 (1), (2).

D'Alembert: $u(x,t) = \frac{1}{2}\left[\varphi(x+at) + \varphi(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,d\xi$

证明: 由 (1), (2), 可假设

$u(x,t) = f(x+at) + g(x-at)$

$\varphi(x) = u(x,0) = f(x) + g(x)$

$\psi(x) = u_t(x,0) = af'(x) - ag'(x)$

$\psi(x) = a(\varphi'(x) - g'(x)) - ag'(x)$

$g'(x) = \frac{1}{2}\varphi'(x) - \frac{1}{2a}\psi(x)$

$g(x) = \frac{1}{2}[\varphi(x) - \varphi(0)] - \frac{1}{2a}\int_0^x \psi(t)\,dt$

$f(x) = \frac{1}{2}[\varphi(x) + \varphi(0)] + \frac{1}{2a}\int_0^x \psi(t)\,dt$

$u(x,t) = \frac{1}{2}\varphi(x+at) + \frac{1}{2}\varphi(0) + \frac{1}{2a}\int_0^{x+at}\psi(t)\,dt$

$+ \frac{1}{2}\varphi(x-at) - \frac{1}{2}\varphi(0) - \frac{1}{2a}\int_0^{x-at}\psi(t)\,dt$

证毕 #

5. $\vec{E}=(E_1, E_2, E_3)$ $\vec{B}=(B_1, B_2, B_3)$

$$\begin{cases} \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \operatorname{curl}\vec{B} & (a) \\ \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = -\operatorname{curl}\vec{E} & (b) \\ \operatorname{div}\vec{E} = \operatorname{div}\vec{B} = 0 & (c) \end{cases}$$

2) $u_{tt} - c^2 \Delta u = 0$ $u = \{E_1, E_2, E_3, B_1, B_2, B_3\}$.

证明：考虑 E_1

$$\frac{\partial^2 E_1}{\partial t^2} = c\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)_t = c\left(\frac{\partial^2 B_3}{\partial y \partial t} - \frac{\partial^2 B_2}{\partial z \partial t}\right) \quad (\text{由 } (a))$$

$$= -c^2\left[\left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right)_y - \left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x}\right)_z\right] \quad (\text{由 } (b))$$

$$= -c^2\left[\frac{\partial^2 E_2}{\partial x \partial y} + \frac{\partial^2 E_3}{\partial x \partial z} - \frac{\partial^2 E_1}{\partial y^2} - \frac{\partial^2 E_1}{\partial z^2}\right]$$

$$= -c^2\left[-\frac{\partial^2 E_1}{\partial x^2} - \frac{\partial^2 E_1}{\partial y^2} - \frac{\partial^2 E_1}{\partial z^2}\right] \quad (\text{由 } (c))$$

$$= c^2 \Delta E_1$$

$u = \{E_2, E_3, B_1, B_2, B_3\}$ 时同理可证. #

7. 直接求解

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t) \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

解： $\left(\frac{d}{dt} - a\frac{d}{dx}\right)\left(\frac{d}{dt} + a\frac{d}{dx}\right)u = f(x,t)$

$$\begin{cases} \frac{du}{dt} + a\frac{du}{dx} = v \\ \frac{dv}{dt} - a\frac{dv}{dx} = f(x,t) \end{cases}$$

$u(x,0) = \varphi(x)$

$v(x,0) = \psi(x) + a\varphi'(x)$

设 $x = x_0 - as \quad t = s$

$\frac{d}{ds}v = v_t - av_x = f(x_0 - as, s)$

$$v(x,t) = \int_0^t f(x+at-as, s)\,ds + \psi(x+at) + a\varphi'(x+at)$$

设 $x = x_0 + ar, \ t = r$

$\frac{d}{dr}u = u_t + au_x = v(x_0+ar, r)$

$$u = \int_0^t v(x-at+ar, r)\,dr + u(x-at, 0)$$

$$u(x,t) = \int_0^t \int_0^r f(x-at+2ar-as, s)\,ds\,dr + \int_0^t \psi(x-at+2ar)\,dr$$

$$+ a\int_0^t \varphi'(x-at+2ar)\,dr$$

$$= \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,d\xi + \frac{1}{2}\left[\varphi(x+at) + \varphi(x-at)\right]$$

$$+ \frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\,d\xi$$

8.

(a) $u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$

$+\frac{1}{2a}\int_0^t d\tau\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)d\xi$

当 $\varphi\in C^2(\mathbb{R})$, $\psi\in C^1(\mathbb{R})$, $f\in C^1(\mathbb{R}\times\overline{\mathbb{R}_+})$

有 $u\in C^2(\mathbb{R}\times\overline{\mathbb{R}_+})$ 且 $u_{tt}-a^2u_{xx}=f(x,t)$, $u(x,0)=\varphi(x)$, $u_t(x,0)=\psi(x)$

证: 易见 $u(x,0)=\varphi(x)$

$u_t=\frac{1}{2}[a\varphi'(x+at)-a\varphi'(x-at)]+\frac{1}{2}\psi(x+at)+\frac{1}{2}\psi(x-at)$

$+\frac{1}{2a}\left(\int_0^t af(x+a(t-\tau),\tau)+af(x-a(t-\tau),\tau)\,d\tau\right)$

有 $u_t(x,0)=\psi(x)$

$u_{tt}=\frac{1}{2}[a^2\varphi''(x+at)+a^2\varphi''(x-at)]+\frac{a}{2}\psi'(x+at)-\frac{a}{2}\psi'(x-at)$

$+\frac{1}{2}\left(2f(x,t)+a\int_0^t f_1'(x+a(t-\tau),\tau)d\tau-a\int_0^t f_1'(x+a(t-\tau),\tau)\,d\tau\right)$

$u_x=\frac{1}{2}[\varphi'(x+at)+\varphi'(x-at)]+\frac{1}{2a}(\psi(x+at)-\psi(x-at))$

$+\frac{1}{2a}\int_0^t[f(x+a(t-\tau),\tau)-f(x-a(t-\tau),\tau)]d\tau$

$u_{xx}=\frac{1}{2}[\varphi''(x+at)+\varphi''(x-at)]+\frac{1}{2a}(\psi'(x+at)-\psi'(x-at))$

$+\frac{1}{2a}\int_0^t[f_1'(x+a(t-\tau),\tau)-f_1'(x-a(t-\tau),\tau)]\,d\tau$

$\Rightarrow u_{tt}=a^2u_{xx}+f(x,t)$

$u_{tx}=\frac{1}{2}[a\varphi''(x+at)-a\varphi''(x-at)]+\frac{1}{2}(\psi'(x+at)+\psi'(x-at)$

$+\frac{1}{2}\int_0^t f_1'(x+a(t-\tau),\tau)+f_1'(x-a(t-\tau),\tau)\,d\tau$

由 u_{xx}, u_{tx}, u_{tt} 表达式, $u\in C^2(\mathbb{R}\times\overline{\mathbb{R}_+})$. #

(b) φ, ψ, f 是关于 x 奇，偶，周期 l 函数，

u 也是相应函数.

证：奇，偶是平凡的

考虑 $f(x+l,t)=f(x)$ $\quad \varphi(x+l)=\varphi(x)$, $\psi(x+l)=\psi(x)$

$$u(x+l,t)=\frac{1}{2}\left[\varphi(x+l+at)+\varphi(x+l-at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi+l)\,d\xi$$

$$+\frac{1}{2a}\int_0^t d\tau\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi+l,\tau)\,d\xi \quad = u(x,t).$$

#