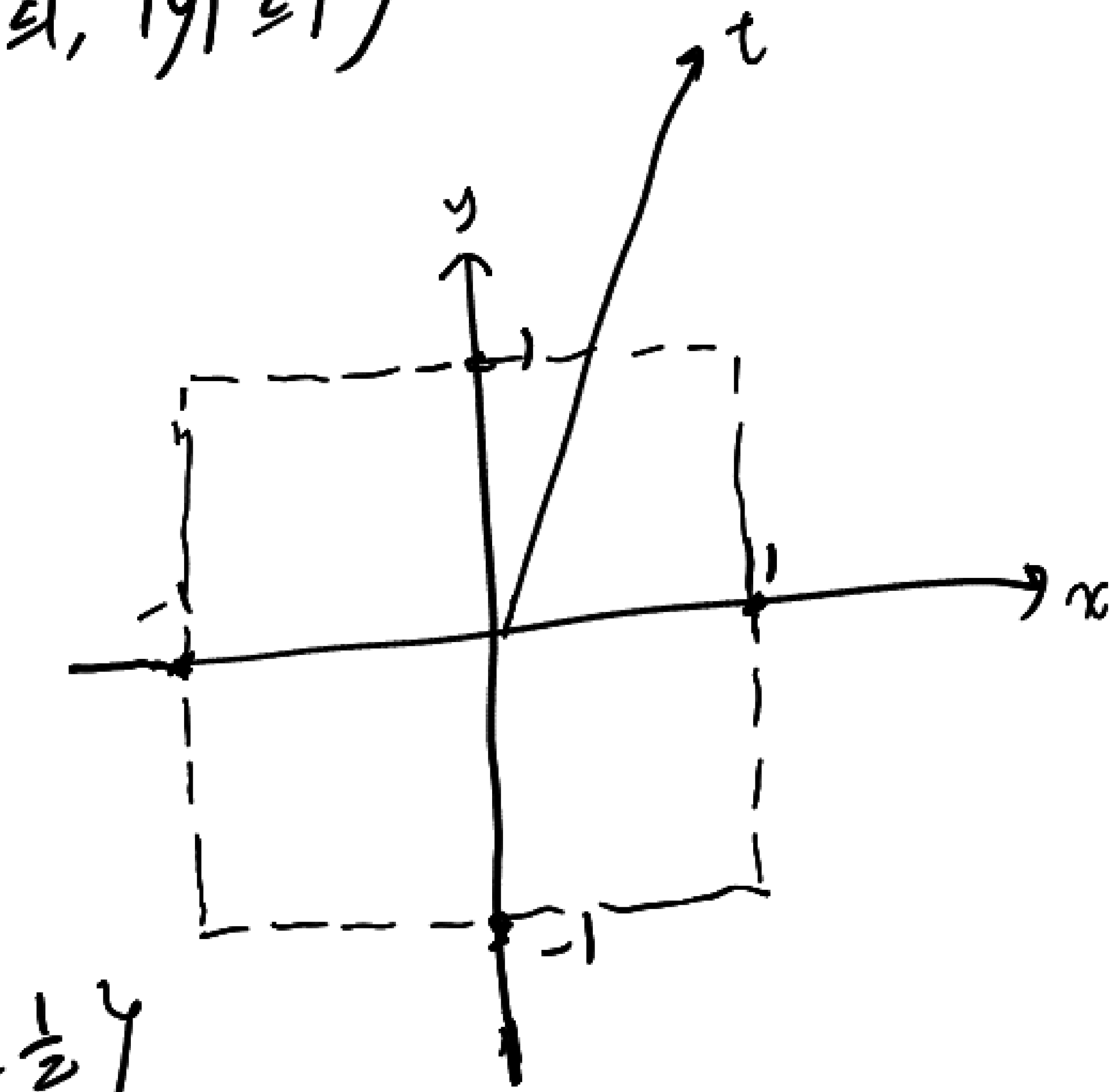
22 $u=u(x,y,t)$

$$\begin{cases} u_{tt}-4(u_{xx}+u_{yy})=0 & \text{in } \mathbb{R}^3_+ \\ u(x,y,0)=0 & \text{on } \mathbb{R}^2 \\ u_t(x,y,0)=\psi(x,y) & \text{on } \mathbb{R}^2 \end{cases}$$

$\psi|_\Omega=0 \quad \psi|_{\mathbb{R}^2-\Omega}>0 \quad \Omega=\{|x|\le 1,\ |y|\le 1\}$

指出 $t>0$ 时 $u(x,y,t)\equiv 0$ 的区域.



解

记是 Ω 的决定区域

$$F_\Omega=\{(x_0,y_0,t_0)\in\mathbb{R}^3_+ : D_{(x_0,y_0,t_0)}\subset\Omega\}$$

$$=\{(x_0,y_0,t_0)\mid (x-x_0)^2+(y-y_0)^2\le 4t_0^2\subseteq\Omega\}$$

$$=\{(x_0,y_0,t_0)\mid |x_0|\le 1-2t_0,\ |y_0|\le 1-2t_0,\ 0\le t_0\le \tfrac{1}{2}\}$$ #

26 Darboux 问题

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < t \\ u|_{x=0} = \varphi(t) & t \geq 0 \\ u|_{x=t} = \psi(t) & t \geq 0 \end{cases}$$

$\varphi(0) = \psi(0)$ φ, ψ 在 $[0, a]$ 给定 $a > 0$. 指出解存在的决定区域

解.

$$u(x,t) = F(x-t) + G(x+t)$$

$$\begin{cases} \varphi(t) = F(-t) + G(t) \\ \psi(t) = F(0) + G(2t) \end{cases} \Rightarrow \begin{cases} F(t) = \varphi(-t) - \psi(-\frac{t}{2}) + F(0) \\ G(t) = \psi(\frac{t}{2}) - F(0) \end{cases}$$

$$u(x,t) = \varphi(t-x) - \psi\left(\frac{t-x}{2}\right) + \psi\left(\frac{x+t}{2}\right)$$

$$\Rightarrow \begin{cases} 0 \leq t-x \leq a \\ 0 \leq t-x \leq 2a \\ 0 \leq t+x \leq 2a \end{cases}$$

决定区域 $R = \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ \mid 0 \leq t-x \leq a,\ 0 \leq t+x \leq 2a\}$

33

$$\begin{cases} u_{tt} - a^2 u_{xx} + b(x,t) u_x + c(x,t) u_t = f(x,t) & \text{in } \mathbb{R}^2_+ \\ u(x,0) = \varphi(x) & \text{on } \mathbb{R} \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R} \end{cases}$$

解唯一性. b, c 连续有界 $a > 0$

证. 只需考虑 $u_{tt} - a^2 u_{xx} + b(x,t) u_x + c(x,t) u_t = 0 \quad \text{in } \mathbb{R}^2_+$

$$\begin{cases} u_{tt} - a^2 u_{xx} + b(x,t) u_x + c(x,t) u_t = 0 & \text{in } \mathbb{R}^2_+ \\ u(x,0) = 0 & \text{on } \mathbb{R} \\ u_t(x,0) = 0 & \text{on } \mathbb{R} \end{cases}$$

的能量不等式.

在 $C(t)$ 上积分, $C(t) = \{t \leq T\} \cap C(x_0, t_0)$

$$\iint_{C(t)} u_{tt} u_t \, dx\,dt + \iint_{C(t)} \left[(-a^2) u_{xx} u_t + b(x,t) u_x u_t + c(x,t) u_t^2\right] dx\,dt = 0$$

我们记 $G(x,t) = \left(-a^2 u_x u_t, \frac{1}{2}(u_t^2 + a^2 u_x^2)\right) \quad G: \mathbb{R}^2 \to \mathbb{R}^2$

有 $\iint_{C(t)} u_{tt} u_t \, dx\,dt - a^2 \iint_{C(t)} u_{xx} u_t \, dx\,dt$

$$= \iint_{C(t)} \operatorname{div} G \, dx\,dt = \iint_{\partial C(t)} G \cdot \vec{n} \, dS(y)$$

$$= \frac{1}{2} \int_{C(t)} (u_t^2 + a^2 u_x^2)\Big|_{t=T} dx - \frac{1}{2} \int_{C(t)} (u_t^2 + a^2 u_x^2)\Big|_{t=0} dx$$

$$+ \int_{C(t)} G\left(\frac{x - x_0}{|x - x_0|} + a\right) dy$$

存在 $M > 0$, 记 $E_0 = \int_{C(t)} (v_t^2 + a^2 v_x^2)\Big|_{t=0} dx$, $S(\tau) = \int_{C(t)} (v_t^2 + a^2 v_x^2)\Big|_{t=\tau} dx$

使得 $S'(t) - E_0 \leq M S(t)$

$S'(t) \leq M S(t)$ 且 $S(0) = 0 \Rightarrow S(t) \leq 0$

$\Rightarrow v_t^2 = v_x^2 = 0$

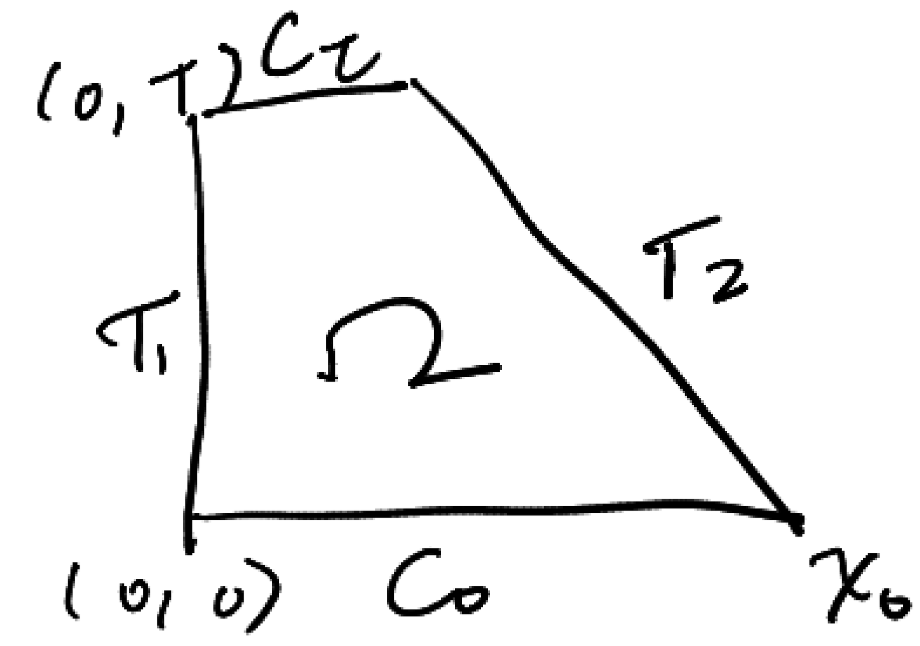
由初值条件, $v \equiv 0$ 唯一性得证.

3) 半无界问题能量不等式.

考虑在 $\{0\le x\le x_0-at,\ 0\le t\le T\}=\Omega$ 上积分

$$\int_{C_\tau}\frac{1}{2}(u_t^2+a^2u_x^2)\Big|_{t=\tau}dx-\int_{C_0}\frac{1}{2}(u_t^2+a^2u_x^2)\Big|_{t=0}dx$$

$$+\int_{T_1}a^2u_xu_t\Big|_{x=0}dt+\int_{T_2}\tilde{F}\cdot(t,x_0-x)dy$$



$$\le\frac{1}{2}\iint_\Omega f^2(t,x)\,dx\,dt+\frac{1}{2}\iint_\Omega u_t^2\,dx\,dt$$

记 $\tilde{f}(t-x_0-x)=F\left(\sqrt{1+a^2}\left(\frac{x_0-x}{|x_0-x|},a\right)\right)$

有 $G'(t)-E_0\le G(t)+F(t)+U(t)=G(t)+H(t)$

其中 $G(t)=\iint(u_t^2+a^2u_x^2)\,dx\,dt$

$G'(t)\le e^{t}(H(t)+E_0)$

若解不唯一，设 u_1,u_2 为两解，$w=u_1-u_2$.

$$\begin{cases}U^{w}(t)=0\\ H^{w}(t)=F^{w}(t)=0\quad E_0=0\quad G'(\tau)=0\end{cases}$$

有 $w_t^2+a^2w_x^2=0$ $\quad w_t=w_x=0\Rightarrow w\equiv0$ 解唯一成立.

37 $\begin{cases} u_{tt}-a^2u_{xx}=0 & \text{in } \mathbb{R}_T^2 \\ u(x,0)=\varphi(x) & \text{on } \mathbb{R} \\ u_t(x,0)=\psi(x) & \text{on } \mathbb{R} \end{cases}$

$a>0 \quad \varphi,\psi\in C^\infty(\mathbb{R})$

(1) $x_1<x_2 \quad t\geq 0$

$$\int_{x_1}^{x_2}[u_t^2(x,t)+a^2u_x^2(x,t)]\,dx \leq \int_{x_1-at}^{x_2+at}[\psi^2(x)+a^2|\varphi'(x)|^2]\,dx$$

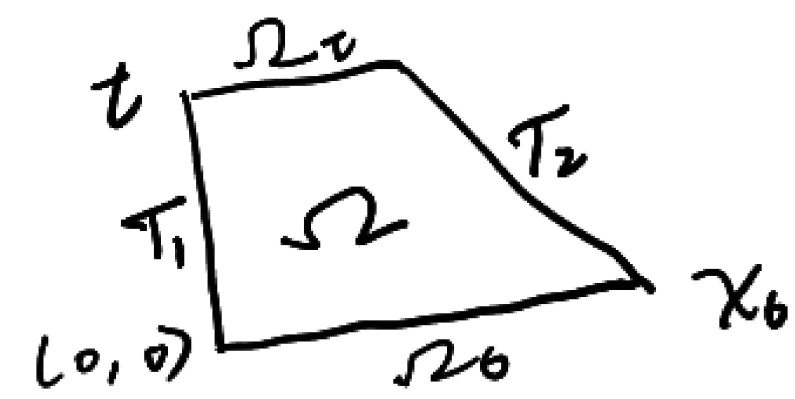
考虑 $\Omega_t=\{(x,\tau)\in\mathbb{R}_+^2 \mid 0\leq\tau\leq t,\ x_1-at+a\tau\leq x\leq x_2+at-a\tau\}$

$$\iint_{\Omega_t}\operatorname{div}F\,dxdt=0$$

$$\iint_{\Omega_t}(u_t^2+a^2u_x^2)\,dx\leq\int_{\Omega_0}(u_t^2+a^2u_x^2)\,dx$$

即 $\displaystyle\int_{x_1}^{x_2}u_t^2(x,t)+a^2u_x^2(x,t)\,dx\leq\int_{x_1-at}^{x_2+at}[\psi^2(x)+a^2|\varphi'(x)|^2]\,dx$

#.



(2) $t\geq 0$

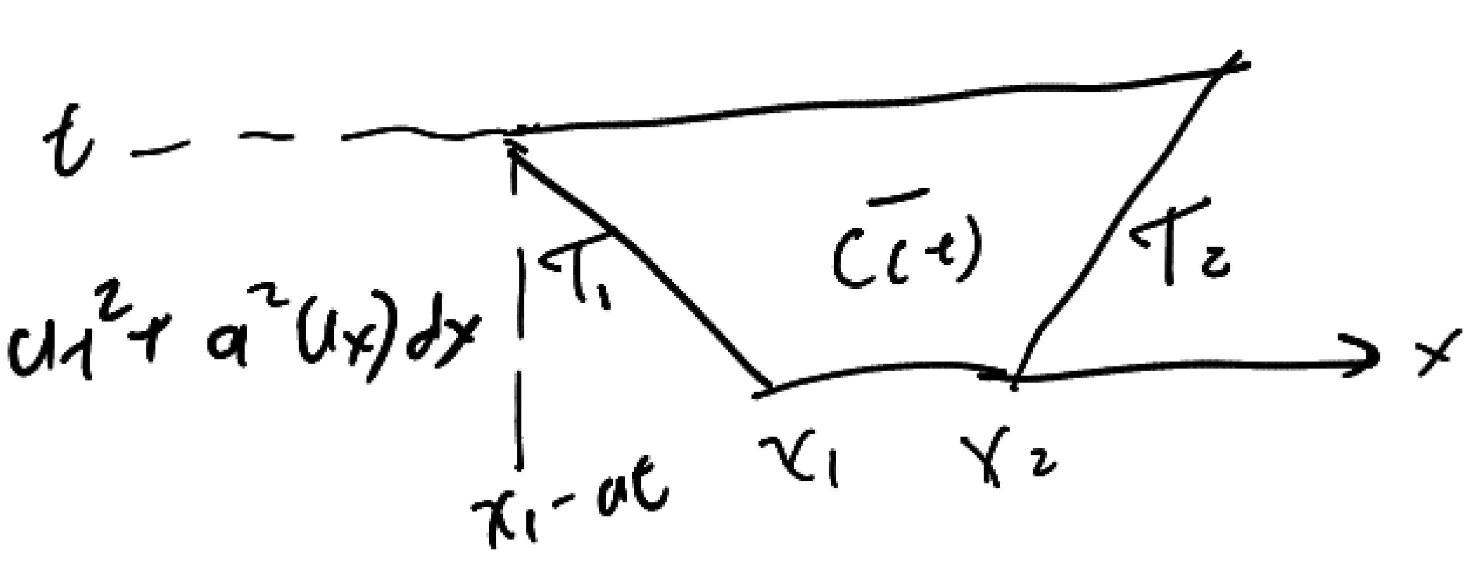
$$\int_{-\infty}^{+\infty}[u_t^2(x,t)+a^2u_x^2(x,t)]\,dx\leq\int_{-\infty}^{+\infty}[\psi^2(x)+a^2|\varphi'(x)|^2]\,dx$$

令 $t_1\to-\infty$, $t_2\to+\infty$ 即可.

(3) (2)中实际为恒等式.

令 $F(t)=\displaystyle\int_{-\infty}^{+\infty}(u_t^2+a^2u_x^2)\,dx$, $F(t)$ 单调递减

在右图所示区域有



$$\iint_{C(t)}\operatorname{div}F\,dxdt=0$$

$$\int_{\overline{C(t)}}\left(\tfrac{1}{2}u_t^2+a^2u_x\right)dx-\int_{C_0}\tfrac{1}{2}(u_t^2+a^2u_x)\,dx$$

$$+\int_{T_1\cup T_2}\vec{F}\cdot\vec{n}\,ds=0$$

有 $\displaystyle\int_{T_1\cup T_2}\vec{F}\cdot\vec{n}\,ds\leq 0$, $\displaystyle\int_{-\infty}^{+\infty}u_t^2(x,t)+a^2u_x^2(x,t)\,dx \geq\int_{-\infty}^{+\infty}(\psi^2(x)+a^2|\varphi'(x)|^2)\,dx$

即证.

38 $u \in C^2(\overline{\mathbb{R}^2_+})$

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 & \text{in } \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) = \varphi(x) & \text{on } \mathbb{R} \\ u_t(x,0) = \psi(x) & \text{on } \mathbb{R} \end{cases}$$

$$k(t) = \frac{1}{2}\int_{-\infty}^{+\infty} u_t^2(x,t)\,dx \qquad p(t) = \frac{a^2}{2}\int_{-\infty}^{+\infty} u_x^2(x,t)\,dx$$

(1) $k(t)+p(t)$ 与 t 无关

(2) t 足够大有 $k(t)=p(t)$

证:

(1) $k(t)+p(t) = \frac{1}{2}\int_{-\infty}^{+\infty}\left(u_t^2 + a^2 u_x^2\right)dx$

$= \int_{-\infty}^{+\infty}\left(\psi^2(x) + a^2|\varphi'(x)|^2\right)dx$ 与 t 无关

(2) $u(x,t) = F(x+at) + G(x-at)$

$u_t(x,t) = aF'(x+at) - aG'(x-at)$

$u_x(x,t) = F'(x+at) + G'(x-at)$

$u_t - au_x = a\varphi'(x-at) + \psi(x-at)$

$u_t + au_x = a\varphi'(x+at) + \psi(x+at)$

由于 φ, ψ 具有紧支集, t 充分大时,

有 $u_t - au_x \equiv 0$ 或 $u_t + au_x \equiv 0$

$$k(t) - p(t) = \frac{1}{2}\int_{-\infty}^{+\infty}(u_t - au_x)(u_t + au_x) = 0$$

#

40 $a, h > 0 \quad A_1, A_2 \in \mathbb{R}$

(3)
$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & 0 < x < l \quad t > 0 \\ u(x,0) = \cos\frac{\pi x}{l} \quad u_t(x,0) = 0, & 0 \le x \le l \\ u_x(0,t) = 0, \quad u_x(l,t) = 0, & t \geq 0 \end{cases}$$

$u(x,t) = X(x)\, T(t)$

$X(x) = a \sin\sqrt{-\lambda}\,\hat{x} + b\cos\sqrt{-\lambda}\,\hat{x} \quad a = 0 \quad \sqrt{-\lambda} = k \in N$

$u(x,t) = \sum_{k=0}^{\infty} \left(c_k \sin ak\hat{t} + d_k \cos ak\hat{t}\right)\cos k\hat{x}$

$\hat{t} = \frac{\pi t}{l} \quad \hat{x} = \frac{\pi x}{l}$

$u(x,0) = \sum_{k=0}^{\infty} d_k \cos k\hat{x} = \cos\frac{\pi x}{l}, \quad u_t(x,0) = \sum_{k=0}^{\infty} \frac{ak\pi}{l} c_k \cos\frac{k\pi x}{l} = 0$

$\Rightarrow u(x,t) = \cos\frac{\pi a t}{l}\cos\frac{\pi x}{l}$

(5)
$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi, \quad t > 0 \\ u(x,0) = 0, \quad u_t(x,0) = 0 & 0 \le x \le \pi \\ u_x(0,t) = A_1 t, \quad u_x(\pi,t) = A_2 t & , t \geq 0 \end{cases}$$

$v(x,t) = X(x)\, T(t) \quad X(x) = \cos kx$

$v(x,t) = \sum_{k=0}^{\infty} T_k(t)\cos kx$

$$\begin{cases} \sum_{k=0}^{\infty} T_k''(t)\cos kx - a^2 T_k(t)(-k^2)\cos kx = -\frac{A_1 - A_2}{\pi} t a^2 \\ \sum_{k=0}^{\infty} T_k(0)\cos kx = 0 \quad \sum_{k=0}^{\infty} T_k'(0)\cos kx = \frac{1}{2\pi}\left[A(x-\pi)^2 - A_2 x^2\right] \end{cases}$$

$\Rightarrow v(x,t) = \sum_{k=0}^{\infty} -\frac{2}{ak^2}\left(A_2(-1)^k - A_1\right)\sin akt \cos kx + \frac{\pi}{3}(A_1 - A_2)t$

$u(x,t) = v(x,t) - \frac{1}{2\pi}\left[A_2(x-\pi)^2 t - A_2 x^2 t\right]$

41(4) $\begin{cases} u_{tt} - a^2 u_{xx} = 0 & 0<x<l \quad t>0 \\ u(x,0) = 1, \ u_t(x,0) = 0 & , 0\le x \le l \\ u_x(0,t) = A\sin\omega t, \ u(l,t) = 1, & t\ge 0 \end{cases}$ 共振和不共振.

$v(x,t) = u(x,t) + \lambda_1(x) g_1(t) + \lambda_2(x) g_2(t)$

$g_1(t) = \sin\omega t \quad \lambda_1(x) = -A(x-l) \quad \lambda_2(x) = -1 \quad g_2(t) = 1$

$\begin{cases} v_{tt} - a^2 v_{xx} = -A(x-l)(-\omega^2)\sin\omega t & 0<x<l, \ t>0 \\ v(x,0) = 0 \quad v_t(x,0) = A\omega(x-l) & 0\le x\le l \\ v_x(0,t) = 0 \quad v(l,t) = 0 & t\ge 0 \end{cases}$

$X(x) = a\sin\sqrt{-\lambda}\,\hat{x} + b\cos\sqrt{-\lambda}\,\hat{x} \quad \hat{x} = \frac{\pi x}{l}, \ a=0, \ \lambda = -k^2$

$v(x,t) = \sum_{k=0}^{\infty} T_k(t)\cos(k+\frac{1}{2})\frac{\pi x}{l}$

不共振条件，解得 $T_k(t) = \frac{\sin\omega t \ 2A\omega^2 l}{\left(\omega^2 - (k+\frac{1}{2})^2\frac{a^2\pi^2}{l^2}\right)[(k+\frac{1}{2})\pi]^2} + C_k \sin(k+\frac{1}{2})\frac{a\pi}{l}t$

$T_k'(0) = \frac{2A\omega l}{[(k+\frac{1}{2})\pi]^2}$

44 $\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t) & 0<x<l, \ t>0 \\ u(x,0) = \varphi(x) \quad u_t(x,0) = \psi(x) & 0\le x\le l \\ -u_x + \alpha u|_{x=0} = g_1(t) & t\ge 0 \\ u_x + \beta u|_{x=l} = g_2(t) & t\ge 0 \end{cases}$

$\alpha>0$ 将边值齐次化.

(1) $\alpha>0 \ \beta>0$ (2) $\alpha=\beta=0$ (3) $\alpha>0 \ \beta=0$

$v(x,t) = u(x,t) + \lambda_1(x) g_1(t) + \lambda_2(x) g_2(t)$

$\Leftrightarrow \begin{cases} g_1(t) - \lambda_1'(x) g_1(t) - \lambda_2'(x) g_2(t) + \alpha(\lambda_1(x) g_1(t) + \lambda_2(x) g_2(t)) = 0 \\ g_2(t) + \lambda_1'(x) g_1(t) + \lambda_2'(x) g_2(t) + \beta(\lambda_1(x) g_1(t) + \lambda_2(x) g_2(t)) = 0 \end{cases}$

$\Leftrightarrow \begin{cases} -\lambda_1'(0) + \alpha\lambda_1(0) = 0 \\ -\lambda_2'(0) + \alpha\lambda_2(0) = 0 \end{cases} \quad \begin{cases} 1 + \lambda_2'(l) + \beta\lambda_2(l) = 0 \\ \lambda_1'(l) + \beta\lambda_1(l) = 0 \end{cases}$

例如 (1) $\lambda_1(x) = -\frac{(x-l)^2}{\alpha l^2 + 2l} \quad \lambda_2(x) = -\frac{(x-l)^2}{\beta l^2 + 2l}$

(2) $\lambda_1(x) = -\frac{(x-l)^2}{2l} \quad \lambda_2(x) = -\frac{x^2}{2l}$ (3) $\lambda_1(x) = -\frac{1}{\alpha} \quad \lambda_2(x) = -\frac{1}{\alpha} - x$