45. $u(x,t) \in C^1(\bar{Q}_T) \cap C^2(Q_T)$

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t) & (x,t) \in Q_T \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) & 0 \le x \le l \\ u_x(0,t) = 0 \quad u_x(l,t) = 0 & 0 \le t \le T \end{cases}$$

$a > 0$ 推导能量不等式.

解 $F(x,t) = \left(-a^2 u_t u_x, \frac{1}{2}(u_t^2 + a^2 u_x^2)\right)$

$$\iint_{Q_\tau} \operatorname{div} F \, dx dt = \iint_{Q_\tau} f(x,t) u_t \, dx dt = \int_{\partial Q_\tau} \vec{F}(x,t) \cdot \vec{n} \, dS_y$$

$$= \int_{T_1} (-a^2 u_t u_x)\Big|_{x=l} dt + \int_{T_2} u_t u_x \Big|_{x=0} dt + \int_{T_3} \frac{1}{2}(u_t^2 + a^2 u_x^2)\Big|_{t=\tau} dx$$

$$+ \int_{T_4} -\frac{1}{2}(u_t^2 + a^2 u_x^2)\Big|_{t=0} dx = \int_{T_3} \frac{1}{2} u_t^2(x,\tau) + a^2 u_x^2(x,\tau) \, dx$$

$$+ \int_{T_4} \left[\frac{1}{2} u_t^2(x,0) + a^2 u_x^2(x,0)\right] dx \le \frac{\varepsilon}{2} \iint_{Q(\tau)} f^2(x,t) \, dx dt + \frac{1}{2\varepsilon} \iint_{Q(\tau)} u_t^2 \, dx dt$$

$\forall \varepsilon > 0$

记 $\Psi(\tau_0) = \iint_{Q_{\tau}} f^2(x,t) \, dx dt \qquad F(\tau_0) = \iint_{Q_{\tau_0}} (u_t^2 + a^2 u_x^2) \, dx dt$

$$F'(\tau_0) + \int_{T_4} [\psi^2(x) + a^2 \varphi^2(x)] dx \le \varepsilon F(\tau_0) + \frac{1}{\varepsilon} \int_{Q_{\tau_0}} f^2(x,t) \, dx dt$$

$F(0) = 0 \qquad F'(\tau_0) \le e^{\varepsilon \tau_0} \left[\frac{1}{\varepsilon} \iint_{Q_{\tau_0}} f^2(x,t) \, dx dt + \int_{T_4} [\psi^2(x) + a^2 \varphi^2(x)] dx\right]$

记 $\varepsilon = \frac{1}{\tau_0} \qquad F'(\tau_0) \le e\left[\tau_0 \iint_{Q_{\tau_0}} f^2(x,t) \, dx d\tau + \int_0^l [\psi^2(x) + a^2 \varphi'^2(x)] dx\right.$

47. $u(x,t) \in C^1(\overline{Q_T}) \cap C^2(Q_T)$

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t) & (x,t) \in Q_T \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) & 0 \le x \le 1 \\ u_x|_{x=0} = 0,\ (u_x + u)|_{x=1} = 0 & 0 \le t \le T \end{cases}$$

$a > 0$ 推导能量不等式

类似上题

$$RHS = a^2 \int_{T_1} -u_t u_x|_{x=1}\, dt + a^2 \int_{T_2} u_t u_x|_{x=0}\, dt + \frac{1}{2}\int_{T_3} (u_t^2 + a^2 u_x^2)|_{t=\tau}\, dx$$

$$- \frac{1}{2}\int_{T_4} (u_t^2 + a^2 u_x^2)|_{t=0}\, dx = a^2 \int_{T_1} u_t u|_{x=1}\, dt + \frac{1}{2}\int_{T_3} (u_t^2 + a^2 u_x^2)|_{t=\tau}\, dx$$

$$- \frac{1}{2}\int_{T_4} [\psi^2(x) + a^2 \varphi'^2(x)]\, dx$$

$$LHS \le \frac{\varepsilon}{2} \iint_{Q} f^2(x,t)\, dx\, dt + \frac{1}{2\varepsilon} \iint_{Q_T} u_t^2\, dx\, dt$$

$$F'(t_0) \le a^2 (1,\tau) + \varepsilon F(t_0) + \frac{1}{\varepsilon} \tilde{F}(t_0) + \int_0^1 \psi^2(x) + a^2 \varphi'^2(x)\, dx + a^2 \varphi^2(1)$$

综上，我们有 $F'(t_0) \le c\left[t_0 \iint_{Q_{t_0}} f^2(x,t)\, dx\, dt + \int_0^1 \psi^2(x) + a^2 \varphi'^2(x)\, dx + \varphi^2(1) a^2\right]$

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50 $u(x,t) \in C(\overline{Q_T})$ 是

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 & (x,t) \in (0,l) \times (0,+\infty) \\ u(x,0) = \varphi(x) & x \in [0,l] \\ u_t(x,0) = \psi(x) & t \in [0,l] \\ u(0,t) = u(l,t) = 0 & t \in [0,+\infty) \end{cases}$$ 广义解

(1) $u(x,0) = \varphi(x)$

$$\iint_{Q_T} u(\zeta_{tt} - a^2 \zeta_{xx})\, dx dt + \int_0^l \varphi\, \zeta_t(x,0)\, dx - \int_0^l \psi\, \zeta(x,0)\, dx = 0 \quad (*)$$

我们曾经证所有广义解满足

$$u(x,t) = \frac{1}{2}\left[\varphi(x+at) + \varphi(x-at)\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\, ds \quad (\triangle)$$

考虑 $(\bar{x},\bar{t}) \in U$

$$\zeta_\varepsilon(x,t) = \alpha\left(\frac{x-t-\bar{x}+\bar{t}}{\varepsilon}\right) \alpha\left(\frac{\bar{x}+\bar{t}-x-t}{\varepsilon}\right)$$

其中 $\alpha: \mathbb{R} \to \mathbb{R}$ 是 C^2，单增函数 $\alpha(s) = 1,\ s \geq 1,\ \alpha(s) = 0,\ s \leq 0$

让 $\varepsilon \to 0^+$ 代入 $(*)$

即有 $(\triangle)$

令 $t \to 0$，$u(x,0) = \frac{1}{2}[\varphi(x) + \varphi(x)] = \varphi(x)$

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(2) 若 $u \in C^1(\overline{Q_T})$，$u_t(x,0) = \psi(x)$

由(1)求导即有

$$u_t(x,0) = \frac{1}{2}\left[\psi(x) + \psi(x)\right] = \psi(x)$$

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