Lec 7

1. 复合 Cotes C_n 复合 Simpson S_n

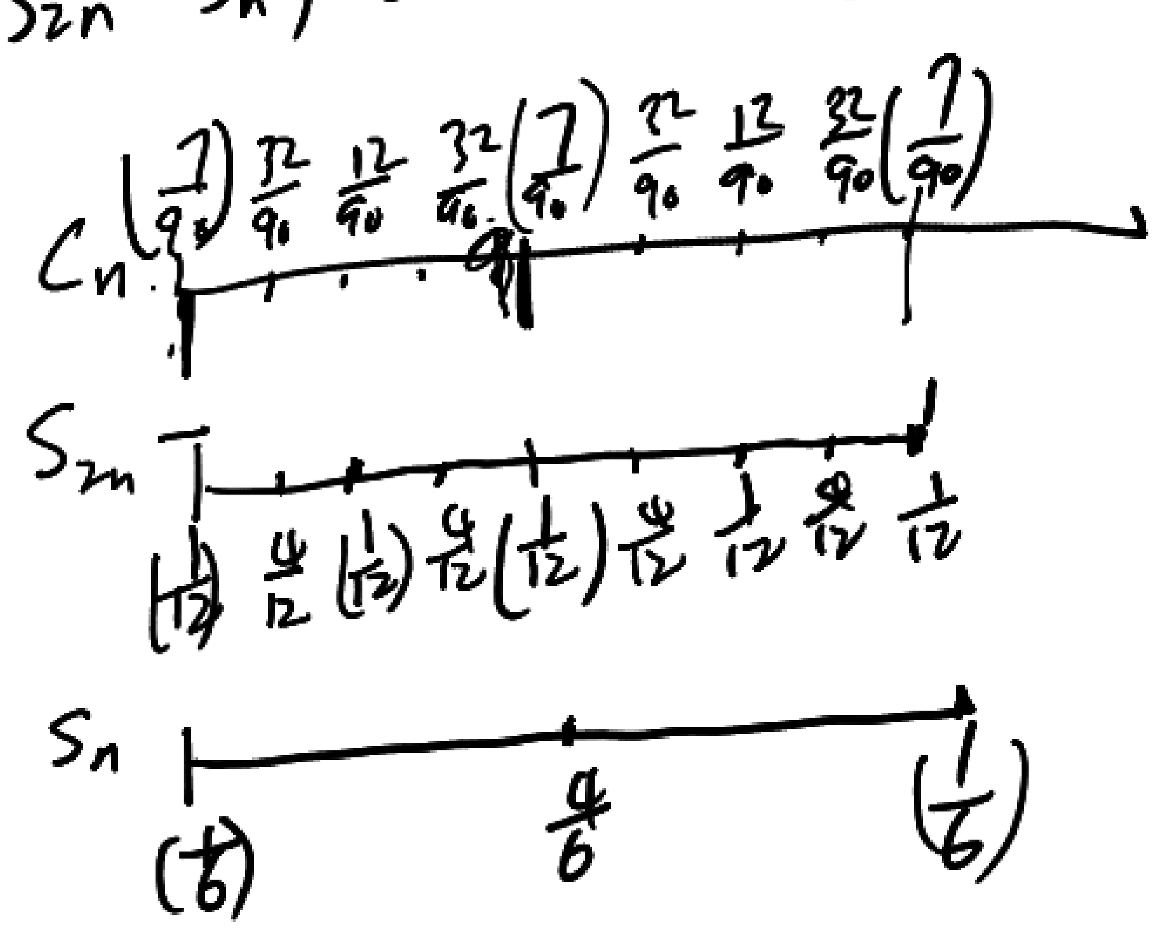
但这个 C_n 好像叫 Boole's rule

$$C_n = \frac{4^2 S_{2n} - S_n}{4^2 - 1}$$

$$C_n = \sum_{i=0}^{n-1}\left[\frac{7}{90}f(x_i) + \frac{32}{90}f(x_{i+\frac14}) + \frac{12}{90}f(x_{i+\frac12}) + \frac{32}{90}f(x_{i+\frac34}) + \frac{7}{90}f(x_{i+1})\right]h$$

$$S_n = \sum_{i=0}^{n-1}\left[\frac16 f(x_i) + \frac46 f(x_{i+\frac12}) + \frac16 f(x_{i+1})\right]h$$

$\frac{1}{15}(16S_{2n} - S_n)$ 是一个 $f\left(a + \frac{k}{2(n-1)}(b-a)\right)$, $k = 0, 1, \cdots 2(n-1)$ 的线性组合，6



根据此图验证即可，其中加(·)为×2.

如第一项

$\frac{1}{15}(16S_{2n} - S_n)$ 系数为 $2\times\frac{16}{15}\times\frac{1}{12} - \frac{1}{15}\times\frac16\times 2$

$= \frac{7}{45}$

C_n 系数为 $2\times\frac{7}{90} = \frac{7}{45}$.

#

2. (1) 内接正n边形 p_n，外切正n边形 q_n

$$p_n = n\sin\frac{\pi}{n}$$

$$q_n = n\tan\frac{\pi}{n}$$

(2) $p_n = a_0 + a_1h^2 + a_2h^4 + \cdots$

$q_n = b_0 + b_1h^2 + b_2h^4 + \cdots$

$h = \frac1n$ $\quad a_0, b_0 = ?$

$$p_n = \frac1h\sin h\pi = \frac1h\left(h\pi - \frac{(h\pi)^3}{3!} + \frac{(h\pi)^5}{5!} - \cdots\right) = \pi - \frac{h^2\pi^3}{6} + \frac{h^4\pi^5}{120} + \cdots$$

$$q_n = \frac1h\tan h\pi = \frac1h\left(h\pi + \frac13(h\pi)^3 + \frac{2}{15}(h\pi)^5 + \cdots\right)$$

$$= \pi + \frac13h^2\pi^3 + \frac{2}{15}h^4\pi^5 + \cdots$$

$$(|h| < \tfrac12)$$

$$a_0 = b_0 = \pi$$

(3) $p_6=3$ $p_{12}=3.1058$ 用Richardson外推给出 π 更好近似

$$A_{i+1}(h)=\frac{2^{k_i}A_i(\frac{h}{2})-A_i(h)}{2^{k_i}-1} \qquad \pi=A_{i+1}(h)+O(h^{k_{i+1}})$$

$p_{\frac{1}{6}}=3$ $p_{\frac{1}{12}}=3.1058$

$$p^*=\frac{2^2\cdot 3.1058-3}{2^2-1}=3.141067$$

$q_6=3.4641$ $q_{12}=3.2154$

$q_{\frac{1}{6}}=3.4641$ $q_{\frac{1}{12}}=3.2154$

$$q^*=\frac{2^2\cdot 3.2154-3.4641}{2^2-1}=3.1325$$

(4) Richardson外推，密率精度，$n\geq$?

$\frac{355}{113}=3.1415929203J\ldots$ 假设用 P_n 计算

考虑 $\frac{4P_{2n}-P_n}{3}\sim O(\frac{1}{n^4})+\pi\approx 3.1415929203J\cdots$ error $\sim 10^{-7}$

$O(\frac{1}{n^4})\sim 10^{-7}$

我们取小数点后6位准确，第7位有误差，选择到用(32,64)边形

如果 q_n 的话，经试验用的边数更多

$$\frac{4P_{64}-P_{32}}{3}=3.14159204J7\cdots$$

故 $n\geq 64$

#

(代码：
```
def cal(n):
    return (4.0/3.0)*2*n*np.sin(np.pi/(2*n)) - (1.0/3.0)*n*np.sin(np.pi/n))
```

Lec 8

1. $x_k, k=0,1\cdots n \in [a,b]$ 不等. f 充分光滑

$$\frac{d}{dx} f[x_0,\cdots,x_n,x] = f[x_0,\cdots x_n, x, x]$$

$$\lim_{u\to x} \frac{f[u,x_0,\cdots,x_n]-f[x_0,\cdots,x_n,x]}{u-x} = \lim_{u\to x} \frac{f[x_0,\cdots x_n,u]-f[x_0,\cdots x_n,x]}{u-x} = \frac{d}{dx} f[x_0,\cdots x_n,x]$$

$$\parallel$$

$$\lim_{u\to x} f[x,x_0,\cdots x_n,u] = f[x_0,\cdots x_n,x,x]$$

#

2. Simpson 推导 $\int_a^b\int_c^d f(x,y)\,dy\,dx$，写出余项

由 Simpson,

$$\int_c^d f(x,y)\,dy = \frac{d-c}{6}\left(f(x,c)+4f(x,\frac{c+d}{2})+f(x,d)\right)+O(d-c)^5$$

$$\Rightarrow \int_a^b f(x,y)\,dy\,dx = \frac{d-c}{6}\cdot\frac{b-a}{6}\left(f(a,c)+4f(\frac{a+b}{2},c)+f(b,c)\right)$$
$$+\frac{d-c}{6}\cdot\frac{b-a}{6}\left(4f(a,\frac{c+d}{2})+16f(\frac{a+b}{2},\frac{c+d}{2})+4f(b,\frac{c+d}{2})\right)$$
$$+\frac{d-c}{6}\frac{b-a}{6}\left(f(a,d)+4f(\frac{a+b}{2},d)+f(b,d)\right).$$
$$+O(b-a)^5+O(d-c)^5$$

#

3. $p(x) \in P_n$ $\int_a^b p(x) x^k = 0$, $k = 0, 1, \cdots n-1$

(1) p 在 (a,b) 有 n 个实单根

考虑 p 在 (a,b) 内全部奇数的实根 $x_1, \cdots x_k$

$p_n(x) \prod_{i=1}^{k} (x - x_i)$ 在 (a,b) 不变号

$\Rightarrow \int_a^b p_n(x) \prod_{i=1}^{k} (x - x_i) dx \neq 0$

但 $p_n(x)$ 与所有次数 $\leq n-1$ 次的多项式正交

$\Rightarrow k \geq n$ 但 $p(x) \in \mathbb{P}_n$ $\Rightarrow k = n$ #

(2) $[a,b]$ n点积分公式，以 $p(x) = 0$ 根为节点，代数精度 $2n-1$ 阶

设 $h(x) \in \mathbb{P}_{2n-1}$ 设 $x_1 \cdots x_n$ 为 $p(x) = 0$ 的根

$h(x) = p_n(x) q(x) + r(x)$

有 $\deg q \leq n-1$, $\deg r \leq n-1$

$\int_a^b h(x) dx = \int_a^b p_n(x) q(x) + r(x) dx = \int_a^b r(x) dx$

$\deg r \leq n-1$ 作 Lgrange 插值 $r(x) = \sum_{i=1}^{n} l_i(x) r(x_i)$

$\int_a^b r(x) dx = \sum_{i=1}^{n} \int_a^b l_i(x) dx \, r(x_i) := \sum_{i=1}^{n} r(x_i) w_i$

$w_i = \int_a^b l_i(x) dx$ 与 h 无关

但 $h(x_i) = r(x_i)$

故 $\int_a^b h(x) dx = \int_a^b r(x) dx = \sum_{i=1}^{n} h(x_i) w_i$, $i = 1, \cdots n$

再考虑 $f(x) = \left(\prod_{i=1}^{n}(x-x_i)\right)^2$

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i) \iff 0 = \int_a^b \left(\prod_{i=1}^{n}(x-x_i)\right)^2 dx \quad 矛盾$$

$\Rightarrow$ 精度 $2n-1$

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4. $a \le x_0 < \cdots < x_n \le b$ $\exists!\, r_0, \cdots r_n$

$$\sum_{k=0}^{n} r_k\, p(x_k) = \int_a^b p(x)\,dx, \quad \forall p(x) \in P_n$$

存在性

取 $l_i(x) = \dfrac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)}$ $\quad r_i = \int_a^b l_i(x)\,dx$

$$\int_a^b p(x)\,dx = \int_a^b \sum_{k=0}^{n} l_k(x)\,p(x_k)\,dx = \sum_{k=0}^{n} r_k\, p(x_k)$$

唯一性 若 $s_0, \cdots s_n$ 也满足上述性质

则 $\sum_{k=0}^{n} r_k\, p(x_k) = \int_a^b p(x)\,dx = \sum_{k=0}^{n} s_k\, p(x_k)$ $, \forall p \in \mathbb{P}_n$

$$\sum_{k=0}^{n}(r_k - s_k)\, p(x_k) = 0. \quad \forall p \in \mathbb{P}_n$$

取 $p_i(x_j) = \delta_{ij} \in \mathbb{P}_n \Rightarrow r_i - s_i = 0$ $, i = 0, \cdots n$

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