4.2

3. 考虑 $\begin{cases} \frac{dy}{dx} = f(x,y) + \varepsilon_n \\ y(x_0) = y_0 \end{cases}$, $\varepsilon_n \to 0$

$$y_n = x_0 + \int_{x_0}^{x} (f(s,y) + \varepsilon_n)\, ds$$ 为等度连续且一致有界

由 Ascoli - Arzel 定理 存在子列使 $y_n \to \psi(x)$，$\psi(x)$ 即右行最大解

不妨设 ε_n 为这一子列

给定 $\varepsilon > 0$ 取 N 使 $|y_N - \psi(x)| < \frac{\varepsilon}{2}$，$\forall x \in (x_0 - h, x_0 + h)$.

考虑方程 $\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 + \sigma \end{cases}$ 解为 $\phi(x)$，记 $\Delta(x) = \phi(x) - \psi(x)$

则 $\Delta(x_0) = \sigma$　$\Delta'(x_0) = -\varepsilon_N = \lim_{t \to 0+} \frac{\Delta(x_0+t) - \Delta(x_0)}{t}$

设 $t < \delta$ 时 $\frac{\Delta(x_0+t) - \Delta(x_0)}{t} < -\frac{\varepsilon_N}{2}$，$\Delta(x_0+t) < -\frac{\varepsilon_N}{2} t + \Delta(x_0) < 0$，

当 $t > \frac{2\sigma}{\varepsilon_N}$ 取 σ 足够小，则 $t > \frac{2\sigma}{2\varepsilon_N}$ 时 $\phi(x)$ 与 $\psi(x)$ 有交点 (x_1, y_1)

由比较定理，$x > x_1$ 时，$\psi(x) < \phi(x) < y_N$，$|\phi(x) - \psi(x)| < \frac{\varepsilon}{2}$. 得证

4.3

1. 记 $v = \frac{\partial y}{\partial \eta} = v(x, \eta)$

$$y(x, \eta) = \eta + \int_0^x \sin(s\, y(s,\eta))\, ds$$

$$v = 1 + \int_0^x \cos(s\, y(s,\eta))\, s\, v(s)\, ds$$

$$\frac{\partial v}{\partial x} = x \cos(x\, y(x,\eta))\, v, \quad v(0) = 1.$$

$$v = e^{\int_0^x s \cos(s\, y(s,\eta))\, ds} > 0. \quad \#$$

2.

记 $\vec{u} = \dfrac{\partial \vec{y}(x; x_0, y_0)}{\partial x_0}$

$$V = \frac{\partial \vec{y}(x; x_0, y_0)}{\partial \vec{y}_0}.$$

我们有 $\dfrac{\partial \vec{u}}{\partial x} = \dfrac{\partial \vec{f}(x,y)}{\partial \vec{y}}\vec{u}$, $\vec{u}(x_0) = -\vec{f}(x_0, y_0)$

$$\frac{\partial V}{\partial x} = \frac{\partial \vec{f}(x,y)}{\partial \vec{y}} V, \quad \vec{V}(x_0) = I_n$$

记 $\theta(x) = \theta(x; x_0, y_0) = \vec{u} + V\vec{f}(x_0, y_0)$. $\theta(x_0) = -\vec{f}(x_0, y_0) + \vec{f}(x_0, y_0) = 0$

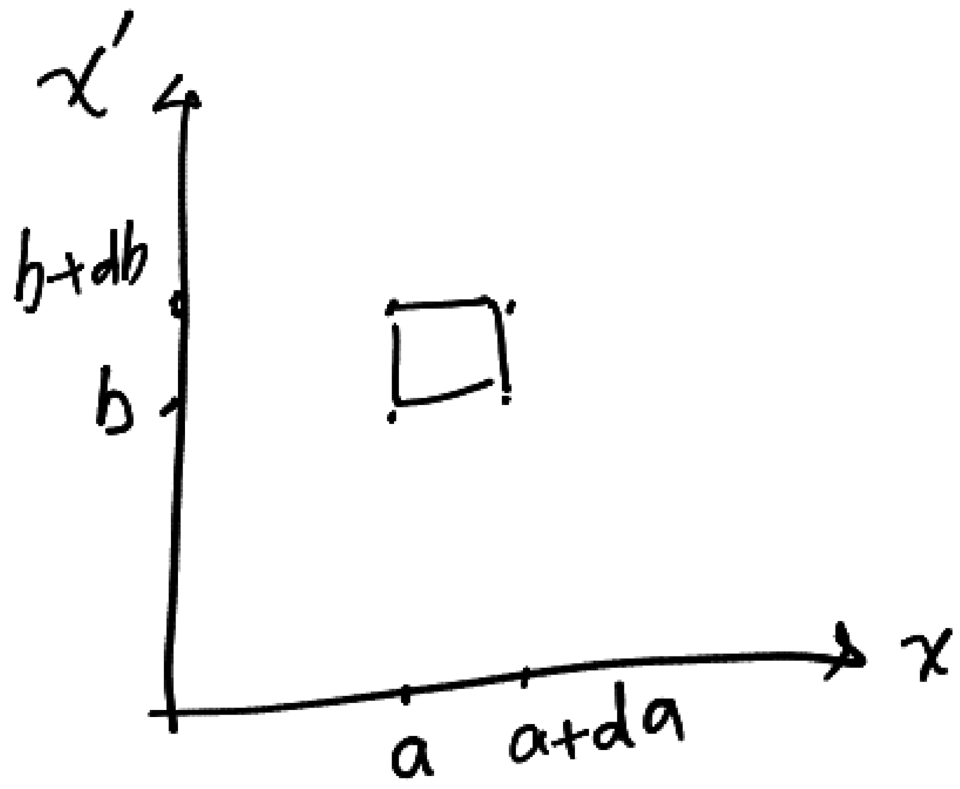
$$\frac{d\theta}{dx} = \frac{\partial \vec{f}(x,y)}{\partial \vec{y}}\vec{u} + \frac{\partial \vec{f}(x,y)}{\partial \vec{y}} V \vec{f}(x_0, y_0) = \frac{\partial \vec{f}(x,y)}{\partial \vec{y}}\theta(x) \quad (*)$$

给定 $x, \vec{y}$ 时, $\dfrac{\partial \vec{f}(x,y)}{\partial \vec{y}} = A(x)$ 是 $n\times n$ 由 x 连续函数构成矩阵. (由 $\vec{f} \in C^1(\mathbb{R}\times\mathbb{R}^n)$.)

任取开区间 (a,b), $(*)$ 在 (a,b) 上解存在唯一. 又 $\theta(x) \equiv 0$ 合题, 故就是 $\theta(x) \equiv 0$.

#

3. $x''(t) + cx'(t) + g(x) = p(t)$. $p(t+2\pi) = p(t)$. $g \in C^1(\mathbb{R})$. $c \in \mathbb{R}$.



$dS = da\,db$

设 $x = x(t)$ 满足初值 $x(0) = a$, $x'(0) = b$

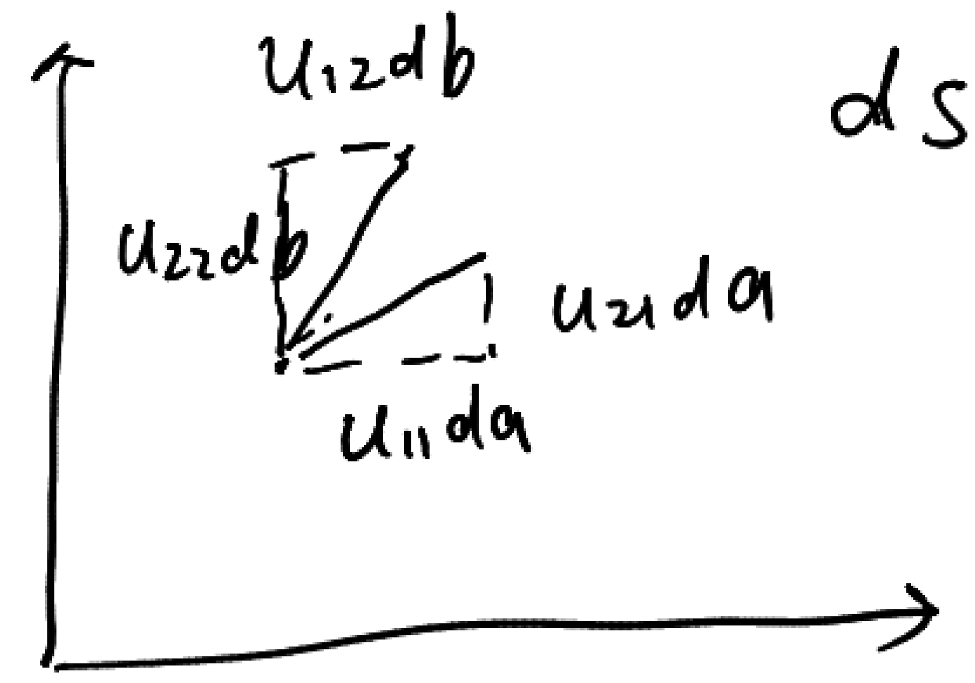
$y_1 = x(t)$ $y_2 = x'(t)$ $\vec{y} = (y_1, y_2)^T$

$$\frac{dy_1}{dt} = y_2 \quad \frac{dy_2}{dt} = p(t) - cy_2 - g(y_1)$$

$\vec{f}(t, y_1, y_2) = (y_2, p(t) - cy_2 - g(y_1))$. $\vec{u} = \dfrac{\partial \vec{y}}{\partial \vec{y}_0}$

$$\begin{pmatrix} u_{11}' & u_{12}' \\ u_{21}' & u_{22}' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g'(y_1) & -c \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} (*) \quad \left(u_{11} = \frac{dx}{da} \quad u_{12} = \frac{dx}{db}, u_{21} = \frac{dx'}{da} \quad u_{22} = \frac{dx'}{db}\right)$$

$$\begin{pmatrix} u_{11}(0) & u_{12}(0) \\ u_{21}(0) & u_{22}(0) \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$



$dS' = |\det u|\,da\,db$

记 $u(t) = \det u = u_{11}u_{22} - u_{21}u_{12}$

由 $(*)$ $\dfrac{du}{dx} = -cu$

$u(t) = e^{-ct}$

$u(2\pi) = |\det u| = e^{-2\pi c}$ 证毕!

4. (1) $y = \mu - 1 + \int_0^x (2s + \mu y^2)\, ds$

$\frac{\partial y}{\partial \mu} = 1 + \int_0^x (\quad y^2 + 2\mu \cdot y \frac{\partial y}{\partial \mu})\, ds$

$\mu = 0$ 时 $y(0) = -1$, $y' = 2x$, $y = x^2 - 1$

记 $y_\mu = \frac{\partial y}{\partial \mu}$, $y_\mu = 1 + \int_0^x (s^2-1)^2\, ds$

$\Rightarrow y_\mu \Big|_{\mu=0} = \frac{x^5}{5} - \frac{2x^3}{3} + x + 1$

(2) 记 $\vec{u} = (y_1, y_2) = (y, y')$

记 $V_1 = \frac{\partial y_1}{\partial \mu}$ $V_2 = \frac{\partial y_2}{\partial \mu}$

$y_1 = 1 + \int_1^x y_2\, ds$, $y_2 = \int_1^x (\frac{2}{s} - \frac{2}{y_1})\, ds$

$\mu = 1$ 时, $y = x$ 是解, 由高维 Picard 定理, 解存在唯一.

且 $V_1(1) = 0$, $V_1'(1) = 1$

$V_1 = \int_1^x V_2\, ds$

$V_2 = \int_1^x \frac{2}{y_1^2} V_1\, ds$

$\frac{dV_1}{dx} = V_2$ $\frac{dV_2}{dx} = \frac{2}{x^2} V_1$

$V_1 = ax + \frac{b}{x^2}$ $V_1(1) = 0$, $V_1'(1) = 1$

$\Rightarrow V_1 = \frac{1}{3}(x - \frac{1}{x^2}) = \frac{\partial y}{\partial \mu}\Big|_{\mu=1}$.

5.1

1. "$\Leftarrow$": $\vec{y}(x) = \Phi(x)\Phi^{-1}(x_0)\vec{y_0} + \int_{x_0}^x \Phi(x)\Phi^{-1}(s)\vec{f}(s, \vec{y}(s))\, ds$

则 $\vec{y}(x_0) = \vec{y_0} + \int_{x_0}^{x_0} \cdots ds = \vec{y_0}$

$\frac{d\vec{y}}{dx} = A(x)\Phi(x)\Phi^{-1}(x_0)\vec{y_0} + \vec{f}(x, \vec{y}(x)) + A(x)\Phi(x)\int_{x_0}^x \Phi^{-1}(s)\vec{f}(s, \vec{y}(s))\, ds$

$= \vec{f}(x, \vec{y}) + A(x)\Phi(x)\Phi^{-1}(x_0)\vec{y_0} + A(x)(\vec{y} - \Phi(x)\Phi^{-1}(x_0)\vec{y_0})$

$= \vec{f}(x, \vec{y}) + A(x)\vec{y}$

"$\Rightarrow$": 已知 $\begin{cases} \frac{d\vec{y}}{dx} = A(x)\vec{y} + f(x, \vec{y}) \\ \vec{y}(x_0) = \vec{y_0} \end{cases}$

不妨记 $\vec{y}(x) = \Phi(x)\vec{c}(x)$, 其中 $c(x) = \Phi^{-1}(x)\vec{y}(x)$.

则 $c(x_0) = \Phi^{-1}(x_0)\vec{y_0}$

$A(x)\Phi(x)\vec{c}(x) + \vec{f}(x, \vec{y}) = A(x)\Phi(x)\vec{c}(x) + \Phi(x)\frac{dc}{dx}$

$\frac{dc}{dx} = \Phi^{-1}(x)\vec{f}(x, \vec{y})$

$c(x) = \int_{x_0}^x \Phi^{-1}(s)\vec{f}(s, \vec{y}) + \Phi^{-1}(x_0)\vec{y_0}$ 得证.

3. 否则，该方程组通解为

$$\begin{cases} y_1 = C_1 + C_2 x + C_3 x^2 \\ y_2 = 0 \\ y_3 = 0 \end{cases}$$

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{pmatrix} = \begin{pmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{11}(x) y_1 \\ 0 \\ 0 \end{pmatrix}$$

但 $\frac{dy_1}{dt} = a_{11}(x) y_1$ 代入 $y_1 = 1$ 得 $a_{11}(x) \equiv 0$ 故解只能为 $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. 矛盾! #

4. $$\begin{cases} \frac{dx}{dt} = \frac{2}{t} x + 1 \\ \frac{dy}{dt} = \frac{1}{t} x + y \end{cases}$$

记 $x = tu$

$$2u + 1 = \frac{dx}{dt} = u + t\frac{du}{dx}$$

$$u + 1 = t\frac{du}{dx}$$

$$\Rightarrow u = At - 1$$

$$x = At^2 - t$$

2) $\frac{dy}{dt} = At - 1 + y$

$$\Rightarrow (e^{-t} y)' = Ate^{-t} - e^{-t}$$

$$e^{-t} y = -Ae^{-t}(t+1) + e^{-t} + C$$

$$y = -A(t+1) + 1 + Ce^t$$

综上

$$\begin{cases} x = At^2 - t \\ y = -At - A + 1 + Ce^t \end{cases}$$

$A, C \in \mathbb{R}$.

5. 记 $v=\dfrac{\partial\vec{\varphi}(x;x_0,y_0)}{\partial\vec{y_0}}$ 我们有 $\dfrac{dv}{dx}=\dfrac{\partial\vec{f}(x,y)}{\partial\vec{y}}v \quad v(x_0)=I_n$

记 $D=\det v=\det\dfrac{\partial\vec{\varphi}(x;x_0,y_0)}{\partial\vec{y_0}}$ 由行列式展开. 记 $A=(a_{ij})=\dfrac{\partial\vec{f}(x,y)}{\partial\vec{y}}$

$$\frac{dD}{dx}=\sum_{i=1}^{n}\begin{vmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \vdots \\ v_{i1}' & \cdots & v_{in}' \\ \vdots & & \vdots \\ v_{n1} & \cdots & v_{nn}\end{vmatrix}=\operatorname{tr}A\cdot D$$ (与 Wronski 推导一样, 只不过 $A=A(x,y)$).

$\operatorname{tr}A=\operatorname{tr}\dfrac{\partial\vec{f}(x,\vec{\varphi}(x;x_0,y_0))}{\partial\vec{y}}$, $D(0)=1$.

故 $D=e^{\int_{x_0}^{x}\operatorname{tr}\left(\frac{\partial\vec{f}}{\partial\vec{y}}(s,\vec{\varphi}(s;x_0,y_0))\right)ds}$. #

5.2

1 (1) $A=\begin{pmatrix}2&1\\3&4\end{pmatrix}$ $\lambda I-A=\begin{pmatrix}\lambda-2&-1\\-3&\lambda-4\end{pmatrix}$ $|\lambda I-A|=\lambda^2-6\lambda+5=0$

$\lambda_1=1 \quad \lambda_2=5$

$(A-I)x=0 \quad \begin{pmatrix}1&1\\3&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=0 \quad \vec{v_1}=(1,-1)^T$

$(A-5I)x=0 \quad \begin{pmatrix}-3&1\\3&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=0 \quad \vec{v_2}=(1,3)^T$

通解 $c_1\begin{pmatrix}e^x\\-e^x\end{pmatrix}+c_2\begin{pmatrix}e^{5x}\\3e^{5x}\end{pmatrix}$

(3) $A=\begin{pmatrix}2&-1&1\\1&2&-1\\1&-1&2\end{pmatrix}$ $\lambda I-A=\begin{pmatrix}\lambda-2&1&-1\\-1&\lambda-2&1\\-1&1&\lambda-2\end{pmatrix}$

$|\lambda I-A|=(\lambda-2)[(\lambda-2)^2-1]+[(\lambda-2)+1]-(1+(\lambda-2))$
$=(\lambda-2)^3-(\lambda-2)=0 \quad \lambda=1,2,3$

$(A-I)x=0 \quad \begin{pmatrix}1&-1&1\\1&1&-1\\1&-1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0 \quad \vec{v_1}=(0,1,1)^T$

$(A-2I)x=0 \quad \begin{pmatrix}0&-1&1\\1&0&-1\\1&-1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0 \quad \vec{v_2}=(1,1,1)^T$

$(A-3I)x=0 \quad \begin{pmatrix}-1&-1&1\\1&-1&-1\\1&-1&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0 \quad \vec{v_3}=(1,0,1)^T$

通解

$c_1\begin{pmatrix}0\\e^x\\e^x\end{pmatrix}+c_2\begin{pmatrix}e^{2x}\\e^{2x}\\e^{2x}\end{pmatrix}+c_3\begin{pmatrix}e^{3x}\\0\\e^{3x}\end{pmatrix}$

(5)

$$A=\begin{pmatrix}4&-1&-1\\1&2&-1\\1&-1&2\end{pmatrix}\quad |\lambda I-A|=(\lambda-3)^2(\lambda-2)=0\quad \lambda=2\Rightarrow v_1=(1,1,1)^T$$

$\lambda=3$ 解 $(A-\lambda I)^2 v=0$

即 $\begin{pmatrix}1&-1&-1\\1&-1&-1\\1&-1&-1\end{pmatrix}^2 v=0$, $\begin{pmatrix}1&-1&-1\\1&-1&1\\1&-1&1\end{pmatrix}v=0$

让 $v_1^0=(1,1,0)^T\quad v_1'=0$

$v_2^0=(1,0,1)^T\quad v_2'=0$

通解 $C_1\begin{pmatrix}e^{2x}\\e^{2x}\\e^{2x}\end{pmatrix}+C_2\begin{pmatrix}e^{3x}\\e^{3x}\\0\end{pmatrix}+C_3\begin{pmatrix}e^{3x}\\0\\e^{3x}\end{pmatrix}$

(7)

$$A=\begin{pmatrix}3&-2&-1\\3&-4&-3\\2&-4&0\end{pmatrix}$$

$|\lambda I-A|=0\Rightarrow(\lambda+5)(\lambda-2)^2=0\quad \lambda=-5\Rightarrow v_1=(1,3,2)^T$

$\lambda=2$, $A-2I=\begin{pmatrix}1&-2&-1\\3&-6&-3\\2&-4&-2\end{pmatrix}\quad (A-2I)^2v=0$, $7\begin{pmatrix}1&-2&-1\\3&-6&-3\\2&-4&-2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$

$v_1^0=(0,1,-2)^T\quad v_2^0=(1,0,1)^T$

$v_1'=0\quad v_2'=0$

通解 $C_1\begin{pmatrix}e^{-5x}\\3e^{-5x}\\2e^{-5x}\end{pmatrix}+C_2\begin{pmatrix}0\\e^{2x}\\-2e^{2x}\end{pmatrix}+C_3\begin{pmatrix}e^{2x}\\0\\e^{2x}\end{pmatrix}$

2(1)

$$\frac{d}{dx}\begin{pmatrix}y\\z\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}y\\z\end{pmatrix}+e^x\begin{pmatrix}2\\1\end{pmatrix}$$

通解 $C_1\begin{pmatrix}e^x\\e^x\end{pmatrix}+C_2\begin{pmatrix}e^{-x}\\-e^{-x}\end{pmatrix}$

设 $\begin{pmatrix}y(x)\\z(x)\end{pmatrix}=\begin{pmatrix}C_1(x)e^x+C_2(x)e^{-x}\\C_1(x)e^x-C_2(x)e^{-x}\end{pmatrix}$

$y'=C_1'(x)e^x+C_1(x)e^x-C_2(x)e^{-x}+C_2'(x)e^{-x}=C_1(x)e^x-C_2(x)e^{-x}+2e^x$

$C_1'e^x+C_2'e^{-x}=2e^x$

$z'=C_1'e^x+C_1e^x+C_2e^{-x}-C_2'e^{-x}=C_1e^x+C_2e^{-x}+e^x$

$C_1'e^x-C_2'e^{-x}=e^x$

$\Rightarrow\begin{cases}C_1'=\frac{3}{2}\\C_2'=\frac{1}{2}e^{2x}\end{cases}$ 积之 $\begin{cases}C_1=\frac{3}{2}x+a\\C_2=\frac{1}{4}e^{2x}+b\end{cases}$

通解为

$C_1\begin{pmatrix}e^x\\e^x\end{pmatrix}+C_2\begin{pmatrix}e^{-x}\\-e^{-x}\end{pmatrix}$

$+\begin{pmatrix}\frac{3}{2}xe^x+\frac{1}{4}e^x\\\frac{3}{2}xe^x-\frac{1}{4}e^x\end{pmatrix}$

3(1)

$$\frac{d}{dx}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} \tan^2 x - 1 \\ \tan x \end{pmatrix}$$

$\lambda_1 = i \quad \lambda_2 = -i$

$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad v_1 = (1,\ i)^T$

$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad v_2 = (i\ \ 1)^T$

齐次解 $C_1\begin{pmatrix} e^{ix} \\ ie^{ix} \end{pmatrix} + C_2\begin{pmatrix} ie^{ix} \\ e^{ix} \end{pmatrix}$

实部取,有 $C_1\begin{pmatrix} \cos x \\ -\sin x \end{pmatrix} + C_2\begin{pmatrix} \sin x \\ \cos x \end{pmatrix}$

设 $\begin{pmatrix} y \\ z \end{pmatrix} = C_1(x)\begin{pmatrix} \cos x \\ -\sin x \end{pmatrix} + C_2(x)\begin{pmatrix} \sin x \\ \cos x \end{pmatrix}$

$$\frac{d}{dx}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} C_1'\cos x - C_1\sin x + C_2'\sin x + C_2\cos x \\ -C_1'\sin x - C_1\cos x + C_2'\cos x - C_2\sin x \end{pmatrix}$$

$$= \begin{pmatrix} -C_1\sin x + C_2\cos x + \tan^2 x - 1 \\ -C_1\cos x - C_2\sin x + \tan x \end{pmatrix}$$

$$\begin{cases} C_1'\cos x + C_2'\sin x = \tan^2 x - 1 \\ -C_1'\sin x + C_2'\cos x = \tan x \end{cases}$$

$$\Rightarrow \begin{cases} C_1' = -\cos x \\ C_2' = \sin x \tan^2 x \end{cases}$$

取 $\begin{cases} C_1(x) = -\sin x \\ C_2(x) = \cos x + \dfrac{1}{\cos x} \end{cases}$

$$\begin{pmatrix} y \\ z \end{pmatrix} = C_1\begin{pmatrix} \cos x \\ -\sin x \end{pmatrix} + C_2\begin{pmatrix} \sin x \\ \cos x \end{pmatrix} + \begin{pmatrix} \tan x \\ 2 \end{pmatrix}$$

#

4 要使 $\frac{dX}{dt} = AX + f(t)$ 的形式,至少 解 X 是以 w 为周期 (给定 f 以 w 为周期) $\Longleftrightarrow \exists X(0),\ X(0) = X(w)$.

而 $X(t) = e^{tA}\left(X_0 + \int_0^t e^{-sA}\vec{f}(s)\,ds\right)$

$$X(0) = X(w) = e^{wA}X(0) + e^{wA}\int_0^w e^{-sA}\vec{f}(s)\,ds$$

$$\Longleftrightarrow (I_n - e^{wA})X(0) = e^{wA}\int_0^w e^{-sA}\vec{f}(s)\,ds$$

$\Longleftrightarrow I_n - e^{wA}$ 可逆. ,这样 $X(0)$ 总是有解,否则调整 $\vec{f}(s)$ 即可.

$\Longleftrightarrow e^{wA}$ 不以 1 为特征值

设 $\mathrm{Spec}(A) = \{\lambda_1, \dots \lambda_t\}$

由线性代数, $\mathrm{Spec}(wA) = \{e^{w\lambda_1}, \dots e^{w\lambda_t}\}$ 设 $\lambda_j = \alpha_j + i\beta_j$

$$e^{w\lambda_j} = e^{w\alpha_j}e^{iw\beta_j} = 1 \Longleftrightarrow w\beta_j = 2k\pi,\ w\alpha_j = 0$$

故 充要条件 $\mathrm{Re}(\lambda_j) \neq 0,\ \forall j$.

5.3

1.(1) $y''+y'-2y=0 \quad \lambda^2+\lambda-2=0 \quad \lambda=1,-2$

$y(x)=Ae^{x}+Be^{-2x} \quad A,B\in\mathbb{R}$

(3) $y^{(4)}-5y^{(2)}+4y=0$

$\lambda^4-5\lambda^2+4=0 \quad (\lambda^2-1)(\lambda^2-4)=0$

$y(x)=Ae^{x}+Be^{-x}+Ce^{2x}+De^{-2x}, \quad A,B,C,D\in\mathbb{R}.$

(5) $y''-4y'+8y=e^{2x}+\sin 2x$

先解齐次方程，得到 $y(x)=Ae^{(2+2i)x}+Be^{(2-2i)x}$. $A,B\in\mathbb{C}$

实数化，得 $y(x)=e^{2x}(a\cos 2x+b\sin 2x)$，$a,b\in\mathbb{R}$.

求一个 $y''-4y'+8y=e^{2x}$ 的解，设为 $y_1=ae^{2x}$

$(4a-8a+8a)e^{2x}=e^{2x} \quad a=\frac{1}{4} \quad y_1=\frac{1}{4}e^{2x}$

求一个 $y''-4y'+8y=\sin 2x$ 的解.

设 $y=a\cos 2x+b\sin 2x$ $-8b\cos 2x+8a\cos 2x+8b\sin 2x$

$-4a\cos 2x-4b\sin 2x+8a\sin 2x$ $=\sin 2x$

$\begin{cases}-4a-8b+8a=0\\-4b+8a+8b=1\end{cases} \quad \begin{cases}a=\frac{1}{10}\\b=\frac{1}{20}\end{cases} \quad y_2=\frac{1}{10}\cos 2x+\frac{1}{20}\sin 2x$

$y^*=\frac{1}{4}e^{2x}+\frac{1}{10}\cos 2x+\frac{1}{20}\sin 2x \Rightarrow y=C_1e^{2x}\cos 2x+C_2e^{2x}\sin 2x+\frac{1}{4}e^{2x}+\frac{1}{10}\cos 2x+\frac{1}{20}\sin 2x.$

2(1) $y''-2y'+y=x^{-1}e^{x}$

齐次解为 $y=(ax+b)e^{x}$ 设 $y=(a(x)x+b(x))e^{x}$

$y'=e^{x}(a(x)x+b(x)+a(x)+a'(x)x+b'(x))$ 记 $a'(x)x+b'(x)=0$

$y''=e^{x}(a'(x)x+a(x)+b'(x)+a'(x))$

$y''-2y'+y=e^{x}(a'x+a+b'+a'+ax+b-2ax-2b-2a-2a'x-2b')$

$=e^{x}(-a'x-a-b'-b))$.

$$a(x) = -\int \frac{f(t)y_2(t)}{y_1(t)y_2'(t)-y_1'(t)y_2(t)}dt = -x$$

$$b(x) = \int \frac{f(t)y_1(t)}{y_1(t)y_2'(t)-y_1'(t)y_2(t)}dt = \log x$$

$$y = x\log x e^x + C_1 e^x + C_2 e^x x$$

周期性:

简单变形,存式

$$y(x) = \int_0^{+\infty} f(x-s)\frac{e^{\lambda_2 s}-e^{\lambda_1 s}}{\lambda_2-\lambda_1}ds$$

$y(x+T) = y(x)$. ⇒ 若 f 以 T 为周期.

3. 齐次方程解为 $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$

$$W(x) = e^{\lambda_1 x}e^{\lambda_2 x}\lambda_2 - \lambda_1 e^{\lambda_1 x}e^{\lambda_2 x} = (\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)x}$$

$$a(x) = -\int \frac{f(t)e^{\lambda_2 t}}{(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)t}}dt = \frac{1}{\lambda_1-\lambda_2}\int_0^x f(t)e^{-\lambda_1 t}dt$$

$$b(x) = \int \frac{f(t)e^{\lambda_1 x}}{(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)}}dt = \frac{1}{\lambda_2-\lambda_1}\int_0^x f(t)e^{-\lambda_2 t}dt$$

通解 $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x} + \frac{1}{\lambda_1-\lambda_2}\left(\int_0^x f(t)e^{\lambda_1(x-t)}dt - \int_0^x f(t)e^{\lambda_2(x-t)}dt\right)$

$$= \frac{e^{\lambda_1 x}}{\lambda_1-\lambda_2}\int_{-\infty}^x f(t)e^{-\lambda_1 t}dt + \frac{e^{\lambda_2 x}}{\lambda_2-\lambda_1}\int_{-\infty}^x f(t)e^{-\lambda_2 t}dt$$

$$\left(\text{取 } A = \frac{1}{\lambda_1-\lambda_2}\int_{-\infty}^0 e^{-\lambda_1 t}f(t)dt \quad B = \frac{1}{\lambda_2-\lambda_1}\int_{-\infty}^0 e^{-\lambda_2 t}f(t)dt\right)$$

该解有界.

因 $\left|\int_{-\infty}^x f(t)e^{-\lambda_1 t}dt\right| \le M\cdot\frac{e^{-\lambda_1 x}}{-\lambda_1}$

$\left|\int_{-\infty}^x f(t)e^{-\lambda_2 t}dt\right| \le M\frac{e^{-\lambda_2 x}}{-\lambda_2}$

易知其余解无界,且 $x\to+\infty$ 趋于有界解.

4. $y'' = x^2 y$, $y(0)=1$, $y'(0)=0$

首先该方程存在级数解 $\phi(x) = \sum_{k=0}^{\infty} a_{4k}x^{4k}$, $a_0=1$, $a_{4k} = \frac{a_{4(k-1)}}{4k(4k-1)}$

易知 $a_{4k} < \frac{1}{(2k)!}$ $\sqrt[4k]{a_{4k}} < \left(\frac{1}{\sqrt[2k]{(2k)!}}\right)^{\frac{1}{2}} \to 0\ (k\to\infty)$ 故收敛半径 $R=+\infty$.

此函数 >0 且为偶函数.

通解为 $y = C_1\phi(x) + C_2\int_0^x \frac{1}{\phi^2(s)}ds$

$x=0 \Rightarrow 1 = C_1 + C_2$

求导 $\Rightarrow y'(x) = C_1\phi'(x) + C_2\frac{1}{\phi^2(x)}$, $0 = C_2$.

故 $y = \phi(x)$ IE毕.