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Lec 4

1. $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$

$n P_n(x) = (2n-1) x P_{n-1}(x) - (n-1) P_{n-2}(x)$

GF: $\sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1-2xt+t^2}}$

求导 $\frac{x-t}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2) \sum_{n=1}^{\infty} n P_n(x) t^{n-1} = (x-t) \sum_{n=0}^{\infty} P_n(x) t^n$

$\Rightarrow n P_n(x) - 2x(n+1) P_{n+1}(x) + (n+2) P_{n+2}(x) = -P_n(x) + x P_{n+1}(x)$

GF推导: $\frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n = \frac{1}{2^{n+1}\pi i} \oint_\gamma \frac{(z^2-1)^n}{(z-x)^{n+1}} dz$

$\sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{2\pi i} \oint_\gamma \sum_{n=0}^{\infty} \left(\frac{(z^2-1)t}{2(z-x)}\right)^n \frac{1}{z-x} dz = \frac{1}{\pi i} \oint_\gamma \frac{1}{2(z-x)-(z^2-1)t} dz$

$= \frac{1}{\sqrt{1-2xt+t^2}}$ (留数定理) #

2. (1) Chebyshev: $P_n(x) = \cos(n \cos^{-1} x)$, $w(x) = (1-x^2)^{-\frac{1}{2}}$, $[a,b]=[-1,1]$

Proof Obviously $P_0(x)=1$ $P_1(x)=x$

$$\boxed{P_{k+1}(x) = 2x P_k(x) - P_{k-1}(x) \quad (*)}$$

we know for $z \in \mathbb{C}$, $P_{k+1}(\cos z) = \cos(k+1)z$

$2z P_k(\cos z) - P_{k-1}(\cos z) = 2\cos z \cos kz - \cos(k-1)z = P_{k+1}(\cos z)$

$\cos z (\mathbb{C} \to \mathbb{C})$ is onto, $(*)$ is proved

$\int_{-1}^{1} P_n(x) P_m(x) \frac{1}{\sqrt{1-x^2}} dx = \int_{\pi}^{0} \cos nz \cos mz \frac{1}{\sin z} \left(-\frac{1}{\sin z}\right) dz$

$= \int_0^{\pi} \cos nz \cos mz \, dz = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0 \end{cases}$

#

(2) Laguerre $w(x)=e^{-x}$ $[a,b]=[0,+\infty)$

$$P_n(x)=\frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x})$$ (使用标准定义)

$$L_{n+1}(x)=\frac{(2n+1-x)L_n(x)-nL_{n-1}(x)}{n+1}$$

设 $L_n(x)=\sum_{j=0}^{n}a_j x^j$ 易知 $a_0=1$, $a_1=-n$, $a_n=\frac{(-1)^n}{n!}$

有 $(L_n, L_m)=0$, $m\neq n$ (见下)

$$xL_n=\sum_{j=1}^{n+1}b_jL_j \qquad (xL_n, L_m)=(L_n, xL_m)=b_m=0 \quad 若 m<n-1$$

$$\Rightarrow xL_n=aL_{n+1}+bL_n+cL_{n-1}$$

$$\begin{cases}0=a+b+c\\ 1=-a(n+1)-bn-(n-1)c\\ \frac{(-1)^n}{n!}=\frac{(-1)^{n+1}a}{(n+1)!}\end{cases}\Rightarrow\begin{cases}a=-(n+1)\\ b=2n+1\\ c=-n\end{cases}$$ #

设 $m<n$

$$\int_0^{\infty}e^{-x}x^mL_n(x)dx=\int_0^{\infty}x^m\frac{d^n}{dx^n}(x^ne^{-x})dx=(-1)^m m!\int_0^{\infty}\frac{d^{n-m}}{dx^{n-m}}(x^ne^{-x})dx$$

$$=(-1)^m m!\left.\frac{d^{n-m-1}}{dx^{n-m-1}}(x^ne^{-x})\right|_0^{+\infty}=0$$

而 $L_m(x)$ 为 m次多项式, $\int_0^{\infty}e^{-x}L_n(x)L_m(x)dx=0$

$$I_n=\int_0^{\infty}e^{-x}L_n(x)L_n(x)dx=\int_0^{\infty}e^{-x}L_n(x)\frac{(2n-1-x)L_{n-1}(x)-(n-1)L_{n-2}(x)}{n}dx$$

$$=-\frac{1}{n}\int_0^{\infty}xe^{-x}L_n(x)L_{n-1}(x)dx$$

$$=-\frac{1}{n}\int_0^{\infty}e^{-x}\left(-(n+1)L_{n+1}(x)+(2n+1)L_n(x)-nL_{n-1}(x)\right)L_{n-1}(x)dx$$

$$=\int_0^{\infty}e^{-x}(L_{n-1}(x))^2dx=I_{n-1}=\cdots=I_0=1$$

#

(3) Hermite $w(x) = e^{-x^2}$ $[a,b] = (-\infty, +\infty)$ $P_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$

由复分析,(换一种解法)

$$P_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = (-1)^n e^{x^2} \frac{n!}{2\pi i} \oint_\gamma \frac{e^{-z^2}}{(z-x)^{n+1}} dz$$

$$\sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n = \frac{e^{x^2}}{2\pi i} \oint_\gamma \sum_{n=0}^{\infty} \frac{(-t)^n e^{-z^2}}{(z-x)^{n+1}} dz = \frac{e^{x^2}}{2\pi i} \oint_\gamma \frac{e^{-z^2}}{z-(x-t)} dz$$

$$= e^{2xt-t^2}$$

对 t 求导 $$\sum_{n=1}^{\infty} \frac{P_n(x)\, t^{n-1}}{(n-1)!} = e^{2xt-t^2}(2x-2t)$$

$$= 2(x-t) \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n$$

$$\Rightarrow \frac{P_{n+1}(x)}{n!} = \frac{2x P_n(x)}{n!} - 2\frac{P_{n-1}(x)}{(n-1)!}$$

$$P_{n+1}(x) = 2x P_n(x) - 2n P_{n-1}(x)$$

$m < n$

$$\int_{-\infty}^{+\infty} e^{-x^2} x^m (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) dx = (-1)^n \int_{-\infty}^{+\infty} x^m \frac{d^n}{dx^n}(e^{-x^2}) dx$$

(分步积分)
$$= (-1)^n (-1)^m m! \int_{-\infty}^{+\infty} \frac{d^{n-m}}{dx^{n-m}}(e^{-x^2}) dx = 0$$

$$I_n = \int_{-\infty}^{+\infty} e^{-x^2} P_n^2(x) dx = \int_{-\infty}^{+\infty} e^{-x^2} P_n(x) \left(2x P_{n-1}(x) - (2n-2) P_{n-2}(x)\right) dx$$

$$= \int_{-\infty}^{+\infty} e^{-x^2} \left(P_{n+1}(x) + 2n P_{n-1}(x)\right) P_{n-1}(x) dx$$

$$= 2n I_{n-1} = 2^n n! I_0 = 2^n n! \sqrt{\pi}$$ #

Lec 5

$$T_n(x) = \begin{cases} \cos(n \arccos x), & |x| \le 1 \quad (c) \\ \cosh(n \operatorname{arccosh} x), & x \ge 1 \quad (d) \\ (-1)^n \cosh(n \operatorname{arccosh}(-x)), & x \le -1 \quad (e) \end{cases}$$

1. $T_n(x)$ is defined on $\mathbb{R}$

$$T_n(x) = \begin{cases} \cos(n \cos^{-1} x), & |x| \le 1 \quad (a) \\ \frac{1}{2}\left[(x-\sqrt{x^2-1})^n + (x+\sqrt{x^2-1})^n\right], & |x| \ge 1 \quad (b) \end{cases}$$

Proof. One can easy verify $T_n^* : \mathbb{C} \to \mathbb{C}$, $\cos z \mapsto \cos nz$ is well defined, satisfies $T_{n+1}^*(z) = 2z T_n^*(z) - T_{n-1}^*(z)$, $T_0^*(z)=1$, $T_1^*(z)=z$

So $T^* = T$

(a)(c) Let $x = \cos y$, $y \in [0, \pi)$ $\quad T_n(x) = \cos(ny) = \cos(n \arccos x)$

(b): Let $\cos z = \frac{e^{iz}+e^{-iz}}{2} = x \quad e^{iz} = x \pm \sqrt{x^2-1}$

$T_n(x) = \cos(nz) = \frac{(e^{iz})^n + (e^{-iz})^n}{2} = \frac{1}{2}\left[(x-\sqrt{x^2-1})^n + (x+\sqrt{x^2-1})^n\right]$

(d) 设 $y = \operatorname{arccosh} x \quad \cosh y = x = \cos(-iy)$

$T_n(x) = \cos(-iny) = \cosh(ny)$

(e) 设 $y = \operatorname{arccosh}(-x) \quad \cosh y = -x = \cos(-iy) \quad x = -\cos(iy) = \cos(iy+\pi)$

$T_n(x) = \cos(iny + n\pi) = (-1)^n \cos(iny) = (-1)^n \cosh(ny)$

#

2. $f \in [a,b]$ $\qquad U_k(f) = \underset{P \in P_k}{\arg\min} \max_{x \in [a,b]} |P(x) - f(x)|$

(1) Find $U_0(f)$

(2) $f \in C^2[a,b]$ $f'' > 0$ find $U_1(f)$

(3) $U_k(f+g) = U_k(f) + U_k(g)$?

(1). Let $M=\max\limits_{[a,b]} f$ $m=\min\limits_{[a,b]} f$ 易知 $V_0(f)=\frac{M+m}{2}$

(2) Let $g(x)=f(x)-cx-d$

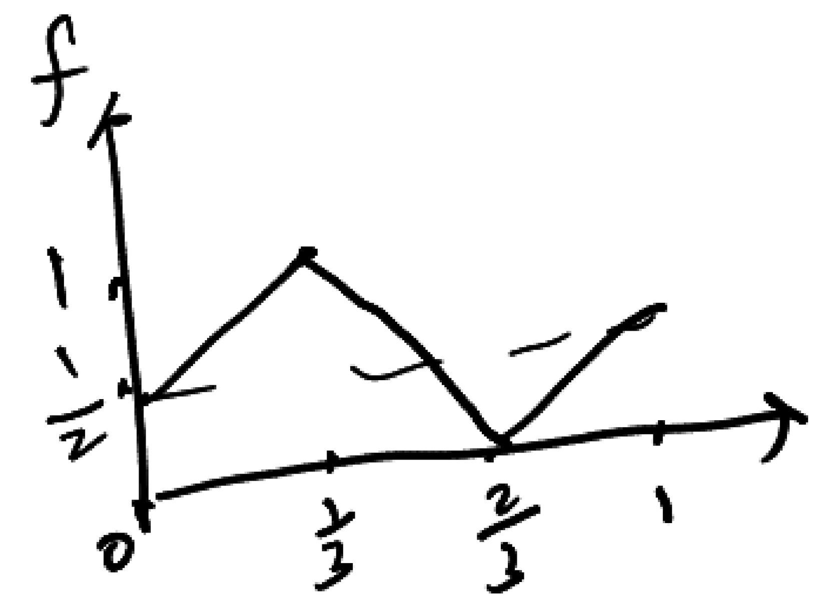
显然 $d=\frac{\max\limits_{[a,b]}[f(x)-cx]+\min\limits_{[a,b]}[f(x)-cx]}{2}$

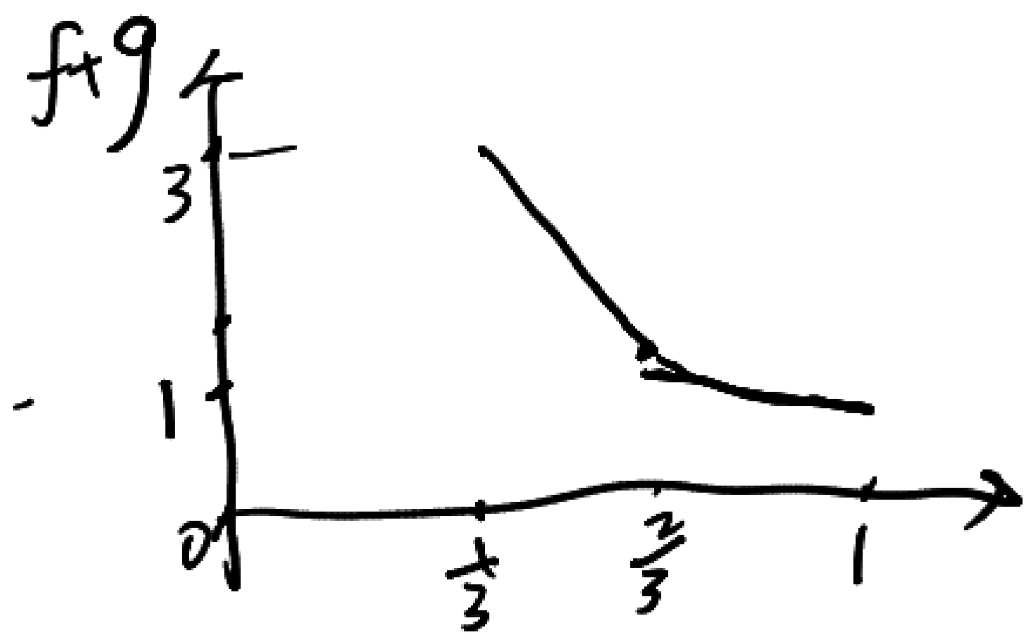
$g''>0$ 但 根据 一致逼近的结论, g 必有 三个 极值点, 易知 只能为 $g(a)=g(b)=M$, $g(\xi)=-M$, $a<\xi<b$

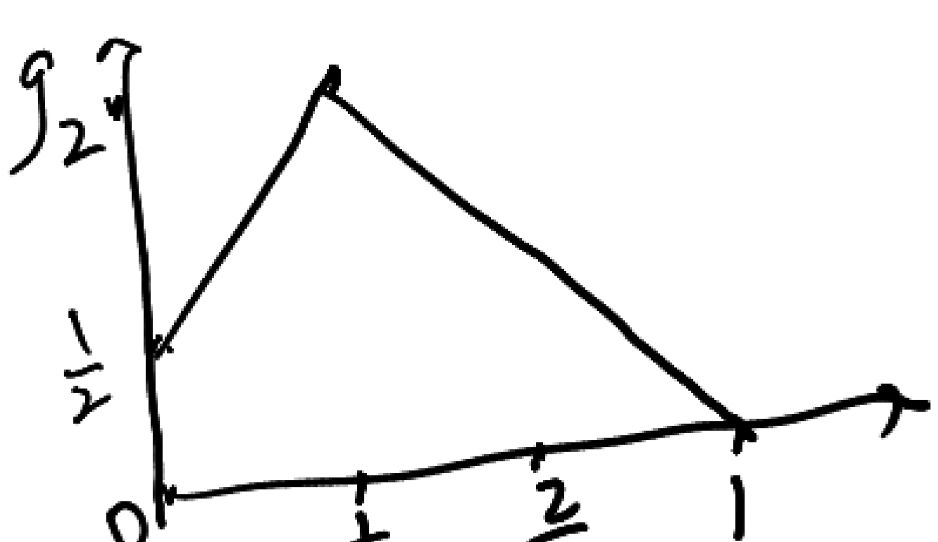
$\Rightarrow f(a)-ca=f(b)-cb$

$c=\frac{f(a)-f(b)}{a-b}$ $d=\frac{\max\limits_{[a,b]}\left[f(x)-\frac{f(a)-f(b)}{a-b}x\right]+\min\limits_{[a,b]}\left[f(x)-\frac{f(a)-f(b)}{a-b}x\right]}{2}$

(3) 不对. 取 $k=0$ $a=0, b=1$







$V_0(f)=\frac{1}{2}$

$V_0(g)=1$

$V_0(f+g)=\frac{7}{4}\ne V_0(f)+V_0(g)$

Lec 6

1. $x_k = x_0 + kh$, $k=0,1,\cdots n$, $h>0$

(a) $f \in C^n[x_0, x_n]$

$$f[x_0, x_1, \cdots x_n] \sim O(1) , h \ll 1$$

由 $f[x_0, \cdots x_n] = \frac{f^{(n)}(\xi)}{n!}$ 立得

(b) f right-continous piecewise-smooth function on $[x_0, x_n]$

exists unique first class jump point $x^* \in (x_0, x_n)$

Prove $f[x_0, x_1, \cdots x_n] \sim O(h^{-n})$, $h \ll 1$

$n=1$ $f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1 - x_0} \approx \frac{M}{h}$ 其中 $M = f(x^*_+) - f(x^*_-)$

易知有 $M_2 > 0$, $f[x_0, x_1] < \frac{M_2}{h}$ $\Rightarrow f[x_0, x_1] \sim O(h^{-1})$

一般情况 设 $x_i < x^* < x_{i+1}$ 设 $n = k+1$.

$$f[x_0, x_1, \cdots x_{k+1}] = \frac{f[x_1, \cdots x_{k+1}] - f[x_0, \cdots x_k]}{x_{k+1} - x_0}$$

$$= \frac{\frac{f[x_2, \cdots x_{k+1}] - f[x_1, \cdots x_k]}{x_{k+1} - x_1}}{O(h)}$$

$$= \frac{\frac{f[x_3, \cdots x_{k+1}] - f[x_2, \cdots x_k]}{x_{k+1} - x_2} - \frac{f[x_2, \cdots x_k] - f[x_1, \cdots x_{k-1}]}{x_{k+1} - x_1}}{O(h^2)}$$

$$= \cdots = \frac{f[x_i] - f[x_{i+1}]}{O(h^{k+1})} + O(h^{-k})$$

$$= O(h^{-(k+1)}) = O(h^{-n}) \#$$