1. $\begin{cases} \partial_t^2 u = 9\partial_x^2 u, & x \in \mathbb{R} \\ u|_{t=0} = 0 \\ \partial_t u|_{t=0} = (1+x^2)^\alpha \end{cases}$

Find $\alpha \in \mathbb{R}$, $\lim\limits_{t\to\infty} u(0,t)$ exists and is finite

According to D'Alembert formula for wave equation,

$$u(x,t) = \frac{1}{6}\int_{x-3t}^{x+3t} (1+\xi^2)^\alpha \, d\xi$$

$$u(0,t) = \frac{1}{6}\int_{-3t}^{3t} (1+\xi^2)^\alpha \, d\xi$$

$\lim\limits_{t\to\infty} u(0,t)$ exist, finite $\Leftrightarrow \int_0^{+\infty} (1+x^2)^\alpha dx$ converges Obviously $\alpha < 0$

If $\alpha < -\frac{1}{2}$

$$\int_0^{+\infty} (1+x^2)^\alpha dx = \int_0^{+\infty} \frac{1}{(1+x^2)^{-\alpha}} dx \leq \int_0^{+\infty} \frac{1}{x^{-2\alpha}} dx < +\infty$$

If $\alpha \geq -\frac{1}{2}$

$$\int_1^{+\infty} (1+x^2)^\alpha dx = \int_1^{+\infty} \frac{1}{(1+x^2)^{-\alpha}} dx \geq \int_1^{+\infty} \frac{1}{(2x^2)^{-\alpha}} dx = +\infty$$

So the wanted α range is $(-\infty, -\frac{1}{2})$

2. $\begin{cases} \partial_t u - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

$f * g = (\hat{f}\,\hat{g})^\vee$

$\widehat{f*g} = \hat{f}\,\hat{g}$

Find solution (c is constant)

$u = (\hat{u})^\vee =$

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} u(x) e^{-2\pi i \xi \cdot x}\, dx$$

$$\frac{d}{dt}\hat{u}(\xi,t) - \left[(2\pi i \xi_1)^2 + \cdots + (2\pi i \xi_n)^2\right]\hat{u}(\xi,t) + c\,\hat{u}(\xi,t) = \hat{f}(\xi,t)$$

$$\frac{d}{dt}\hat{u}(\xi,t) + \left(c + 4\pi^2(\xi_1^2 + \cdots + \xi_n^2)\right)\hat{u}(\xi,t) = \hat{f}(\xi,t)$$

(Let $S = c + 4\pi^2(\xi_1^2 + \cdots + \xi_n^2)$)

$$\frac{d\hat{u}}{dt} + S\hat{u} = \hat{f}$$

$$\frac{d}{dt}\left(e^{St}\hat{u}\right) = Se^{St}\hat{u} + e^{St}(\hat{f} - S\hat{u}) = e^{St}\hat{f}$$

$$e^{St}\hat{u} = \int_0^t e^{S\tau}\hat{f}\, d\tau + \hat{g}$$

$$\hat{u}(\xi,t) = \int_0^t e^{-S(t-\tau)}\hat{f}(\xi,\tau)\, d\tau + e^{-St}\hat{g}(\xi)$$

$$\Rightarrow u(x,t) = h(t) * g + \int_0^t h(t-\tau) * f\, d\tau$$

$$= \frac{1}{\sqrt{(4\pi t)^n}} e^{-ct} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4t}} g(x-y)\, dy + \int_0^t e^{-c(t-\tau)} \frac{1}{\sqrt{4^n \pi^n (t-\tau)^n}} e^{-\frac{|y|^2}{4(t-\tau)}} f(x-y)\, dy\, d\tau$$

#

$$\left(\begin{aligned} \hat{h}(t) &= e^{-St} = e^{-ct} e^{-4\pi^2|\xi|^2 t} \\ h(t) &= e^{-ct} \frac{1}{2^n\sqrt{\pi^n t^n}} e^{-\frac{|x|^2}{4t}} \end{aligned}\right)$$

3. Q is $n\times n$ SPD matrix.

Calculate Fourier Transform of $e^{-\langle Qx, x\rangle}$

$$f(x) = e^{-x^TQx}$$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-x^TQx} e^{-2\pi i \xi\cdot x}\, dx$$

Let $Q = P\Lambda P^T$ where P is $n\times n$ orthogonal matrix, Λ diagonal

Let $y = P^Tx$ i.e. $x = Py$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-y^T\Lambda y} e^{-2\pi i \xi^T P y}\, dy \quad (|\det P| = 1)$$

Let $\Lambda = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn}\end{pmatrix}$ $P = (P_1 \cdots P_n)$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n (a_{ii}y_i^2 + 2\pi i\, \xi\cdot P_i y_i)}\, dy$$

$$= \prod_{i=1}^n \int_{\mathbb{R}} e^{-(a_{ii}y_i^2 + 2\pi i\, \xi\cdot P_i y_i)}\, dy_i$$

$$= \prod_{i=1}^n \left(\sqrt{\frac{\pi}{a_{ii}}}\, e^{-\frac{\pi^2(\xi^T P_i)^2}{a_{ii}}} \right)$$

$$= \frac{\sqrt{\pi^n}}{\sqrt{|\det Q|}} e^{-\pi^2 \sum_{i=1}^n \frac{[(P^T\xi)_i]^2}{a_{ii}}}$$

$$= \frac{\sqrt{\pi^n}}{\sqrt{|\det Q|}} e^{-\pi^2 \xi^T P \Lambda^{-1} P^T \xi}$$

$$= \frac{\sqrt{\pi^n}}{\sqrt{|\det Q|}} e^{-\pi^2 \xi^T Q^{-1} \xi}$$