40. $\Omega \subseteq \mathbb{R}^n, n \geq 3$ 有界开集

$$\begin{cases} -\Delta u(x) + A(x)\cdot Du(x) + c(x)u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$A: \Omega \to \mathbb{R}^n$ 及 $c(x)$ 在 Ω 上连续有界 $c(x) - \frac{1}{4}|A(x)|^2 > 0$

energy estimate $\Rightarrow$ 唯一性

<u>Proof</u>. 设 u_1, u_2 均为解

$u = u_1 - u_2$

则 $\begin{cases} -\Delta u(x) + A(x)\cdot Du(x) + c(x)u(x) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

$$-\int_\Omega \Delta u\, u\, dx + \int_\Omega u \cdot A(x)\cdot Du(x)\, dx + \int_\Omega c(x) u^2(x)\, dx = 0$$

$$\int_\Omega \left(|Du|^2 + c(x)u^2(x)\right) dx = \left|\int_\Omega u\, A(x)\cdot Du(x)\, dx\right|$$

$$\leq \int_\Omega \left(\frac{1}{4}|A(x)|^2 u^2 + |Du|^2\right) dx,$$

$$\int_\Omega u^2 \left(c(x) - \frac{1}{4}|A(x)|^2\right) \leq 0 \quad \because c(x) - \frac{1}{4}|A(x)|^2 > 0$$

而 $c(x)$, u, $A(x)$ 在 Ω 上连续,

故 $u^2\left(c(x) - \frac{1}{4}|A(x)|^2\right) \equiv 0$, $\Rightarrow u \equiv 0$ #

41. $\Omega \subset \mathbb{R}^n$, $n\geq 3$ 有界开 Ω上 $c(x)\geq 0$ 能量估计

$$\begin{cases} -\Delta u(x) + c(x)u(x) = f(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \vec{n}} = g(x) & \text{on } \partial\Omega \end{cases}$$

在 $C^2(\Omega)\cap C^1(\bar{\Omega})$ 解相差常数唯一

Proof. 设 u_1, u_2 为解 $u = u_1 - u_2$

$$\begin{cases} -\Delta u + c(x) u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \vec{n}} = 0 & \text{on } \partial\Omega \end{cases}$$

$$-\int_\Omega \Delta u \cdot u \, dx + \int_\Omega c(x) u^2 dx = 0$$

$$\int_\Omega \left(|Du|^2 + c(x) u^2\right) dx = 0$$

而 $c(x)$, u, Du 在 Ω 上连续. $c(x)\geq 0$

故 Ω 上恒有 $|Du|^2 + c(x)u^2 = 0 \geq |Du|^2$

即 Ω 上 $Du\equiv 0$, $u\equiv C_1$

(需要 Ω 连通, 否则只能证 Ω 连通分支 Ω_k 上 u 为常数).

#