谭小江

习题一

6. (1) $|1-\bar z_1 z_2|^2-|z_1-z_2|^2 = (1-\bar z_1 z_2)(1-z_1\bar z_2)-(z_1-z_2)(\bar z_1-\bar z_2)$

$= 1-\bar z_1 z_2 - z_1\bar z_2 + |z_1|^2|z_2|^2 - |z_1|^2 + z_2\bar z_1 + z_1\bar z_2 - |z_2|^2$

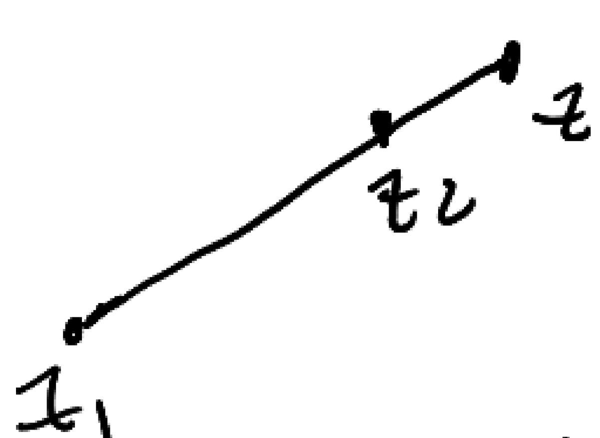
$= (1-|z_1|^2)(1-|z_2|^2)$

(2) 设 $z=\dfrac{z_1-z_2}{1-\bar z_1 z_2}$

$|z|^2 = \dfrac{|z_1-z_2|^2}{|1-\bar z_1 z_2|^2} = \dfrac{|z_1-z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)+|z_1-z_2|^2} < 1$

(3) 当 $z_1\ne z_2$ 且 $|z_1|=1$ or $|z_2|=1$ 时, $|z|^2=\dfrac{|z_1-z_2|^2}{|z_1-z_2|^2}=1$

7. $z_1=1+3i$ $z_2=-1+4i$ $\bar z_1=1-3i$ $\bar z_2=-1-4i$



$\dfrac{z-z_2}{z_1-z_2}\in\mathbb{R}$ 即 $\dfrac{z-z_2}{z_1-z_2}=\dfrac{\bar z-\bar z_2}{\bar z_1-\bar z_2}$

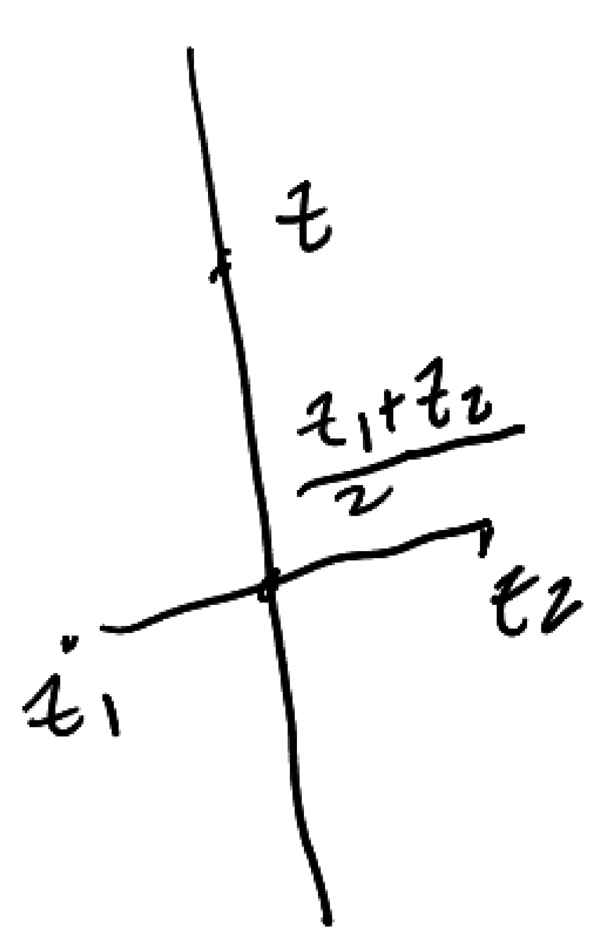
$(z-z_2)(\bar z_1-\bar z_2)=(\bar z-\bar z_2)(z_1-z_2)$

$z(\bar z_1-\bar z_2)-\bar z(z_1-z_2)-z_2(\bar z_1-\bar z_2)+\bar z_2(z_1-z_2)=0$

$z(\bar z_1-\bar z_2)-\bar z(z_1-z_2)$ $-z_2\bar z_1+\bar z_2 z_1=0$

$z(2+i)-\bar z(2-i)-14i=0$

9. 显然 z 在 z_1, z_2 中垂线



$\dfrac{z-\frac{z_1+z_2}{2}}{z_1-z_2}\in i\mathbb{R}$

$\Longleftrightarrow \dfrac{z-\frac{z_1+z_2}{2}}{z_1-z_2} = -\dfrac{\bar z-\frac{\bar z_1+\bar z_2}{2}}{\bar z_1-\bar z_2}$

$\Longleftrightarrow (z-\frac{z_1+z_2}{2})(\bar z_1-\bar z_2)+(\bar z-\frac{\bar z_1+\bar z_2}{2})(z_1-z_2)=0$

$\Longleftrightarrow z(\bar z_1-\bar z_2)+\bar z(z_1-z_2)+(|z_2|^2-|z_1|^2)=0$

$\Longleftrightarrow z(\bar z_1-\bar z_2)+\bar z(z_1-z_2)+|z_2|^2-|z_1|^2=0$ 故必要性得证

由于 $(z_1-z_2)\bar z_1+(\bar z_1-\bar z_2)z_2+|z_2|^2-|z_1|^2=0$

充分性: 若 $B\bar z_1+\bar B z_2+C=0$

又 $B\bar z+\bar B z+C=0$

有 $B(\bar z-\bar z_1)=\bar B(z_2-z)$

$\bar B(z-z_1)=B(\bar z_2-\bar z)$

$\Rightarrow B(\bar z_2-\bar z_1)=\bar B(z_2-z_1)$

设 $B=k(z_2-z_1)$, $\bar B=k(\bar z_2-\bar z_1)$

$k\in\mathbb{C}$, 则 $C=k(|z_2|^2-|z_1|^2)$

证毕.

12. 首先 $S \subset \bar{S}$ 必有 $diam(S) \le diam(\bar{S})$

下证 $diam(\bar{S}) \le diam(S)$

由于 $diam(\bar{S}) = \sup\limits_{z_1, z_2 \in \bar{S}} |z_1 - z_2| := A$

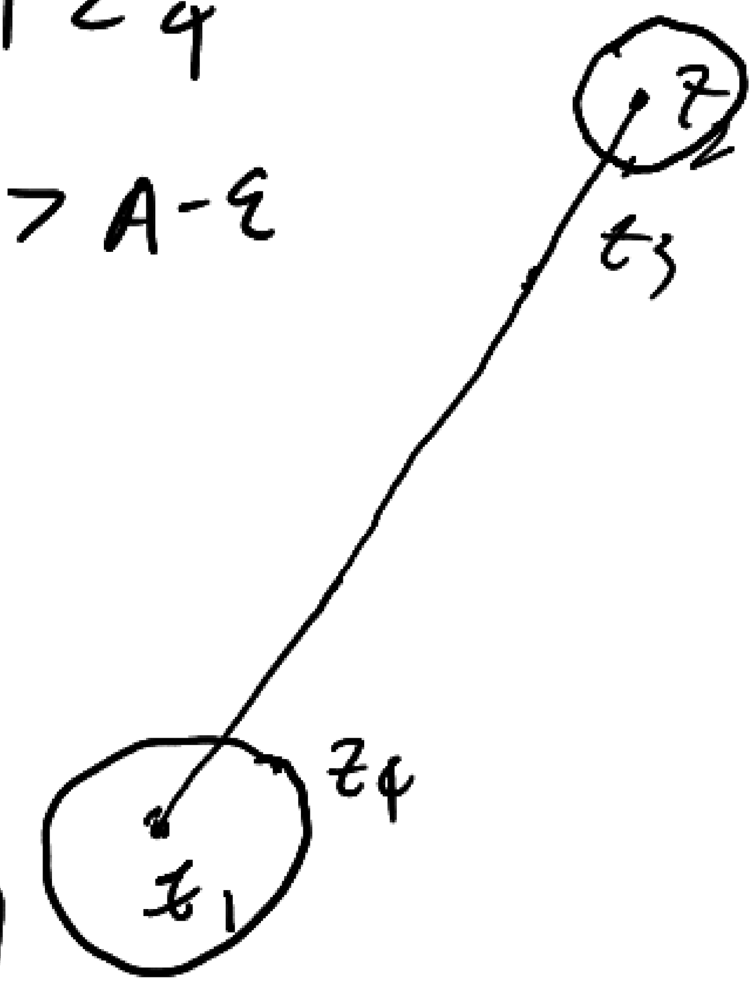
故 $\forall \varepsilon > 0$, $\exists z_1, z_2 \in \bar{S}$, $A - \frac{\varepsilon}{4} \le |z_1 - z_2| \le A$

由于 $z_1, z_2 \in \bar{S}$, 存在 $z_3, z_4 \in S$, $|z_3 - z_1| < \frac{\varepsilon}{4}$, $|z_4 - z_2| < \frac{\varepsilon}{4}$

$|z_3 - z_4| \ge |z_1 - z_2| - |z_1 - z_3| - |z_1 - z_4| \ge A - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} > A - \varepsilon$

故 $diam(S) = \sup\limits_{z_5, z_6 \in S} |z_5 - z_6| \ge A - \varepsilon$

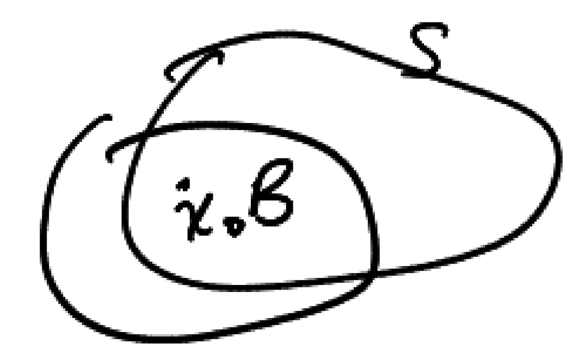
由 ε 任意性, $diam(S) \ge A = diam(\bar{S})$. #



15. 我们知道, 在子空间拓扑意义下, S 连通 $\Leftrightarrow$ S 没有非平凡既开又闭子集. 只需验证子空间拓扑开、闭集与本题所述一致

一方面, 设 $A \subseteq S$ 在 (S, d) 中相对闭. 考虑 $S \cap A^c$. $\forall x \in S \cap A^c$, x 不是 A 极限点, $\exists B(x, \varepsilon_x)$, $B(x, \varepsilon_x) \cap S \subset A^c$

取 $G = \bigcup\limits_x B(x, \varepsilon_x)$, $A^c = G \cap S \Rightarrow A$ 为 (S, d) 中闭集.



另一方面, 设 B 为 X 中开集, $S - B \cap S = S \cap B^c$

若 $x_0 \in S$, 且 x_0 为 $S \cap B^c$ 中极限点, 假设 $x_0 \in B$, 则 $\exists \varepsilon > 0$, $B(x_0, \varepsilon) \subseteq B$, x_0 不可能是 $S \cap B^c$ 极限点, 矛盾! $\Rightarrow x_0 \in B \cap S^c$. #

21.

$z \to z_0$ 时, $f(z)$ 收敛的 Cauchy 准则为

若 $\forall \varepsilon > 0$ 当 $|z - z_1| < \delta$, $|z - z_2| < \delta$, 有 $|f(z_1) - f(z_2)| < \varepsilon$

则 $z \to z_0$ 时 $f(z)$ 收敛

Proof. 首先易知 $\lim\limits_{z \to z_0} f(z) = b \Leftrightarrow \forall z_n \to z_0$, $z_n \ne z_0$, $\lim\limits_{n \to \infty} f(z_n) = b$.

只需考虑一个序列 z_n, $z_n \to z_0$, 但 $z_n \ne z_0$.

当 $n \ge N_1$ 时, $|z_{n_1} - z_0| < \delta$

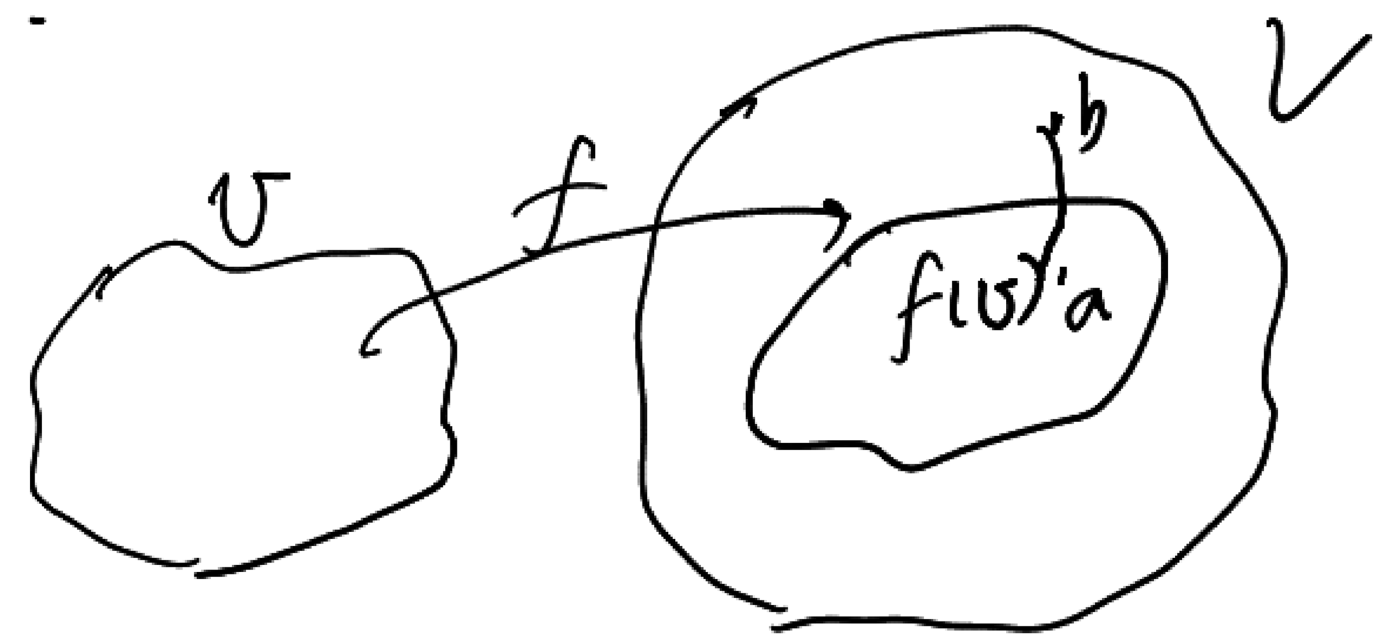
$n \ge N_2$ 时 $|z_{n_2} - z_0| < \delta$

故 $|f(z_{n_1}) - f(z_{n_2})| < \varepsilon$,

这表明 $\{f(z_n)\}$ 是 Cauchy 序列

故 $f(z_n)$ 收敛. 显然, 此极限对不同 z_n 是唯一的. #

17.



假如 $f(U) \neq V$

设 $a \in f(U)$, $b \in V - f(U)$

V 是道路连通的.

存在 $\gamma: [0,1] \to [a,b]$, $\gamma(0)=a$, $\gamma(1)=b$

设 $K = \gamma[0,1]$ 为紧集, $A = K \cap f(U) = f(f^{-1}(K))$ 在 V 中紧

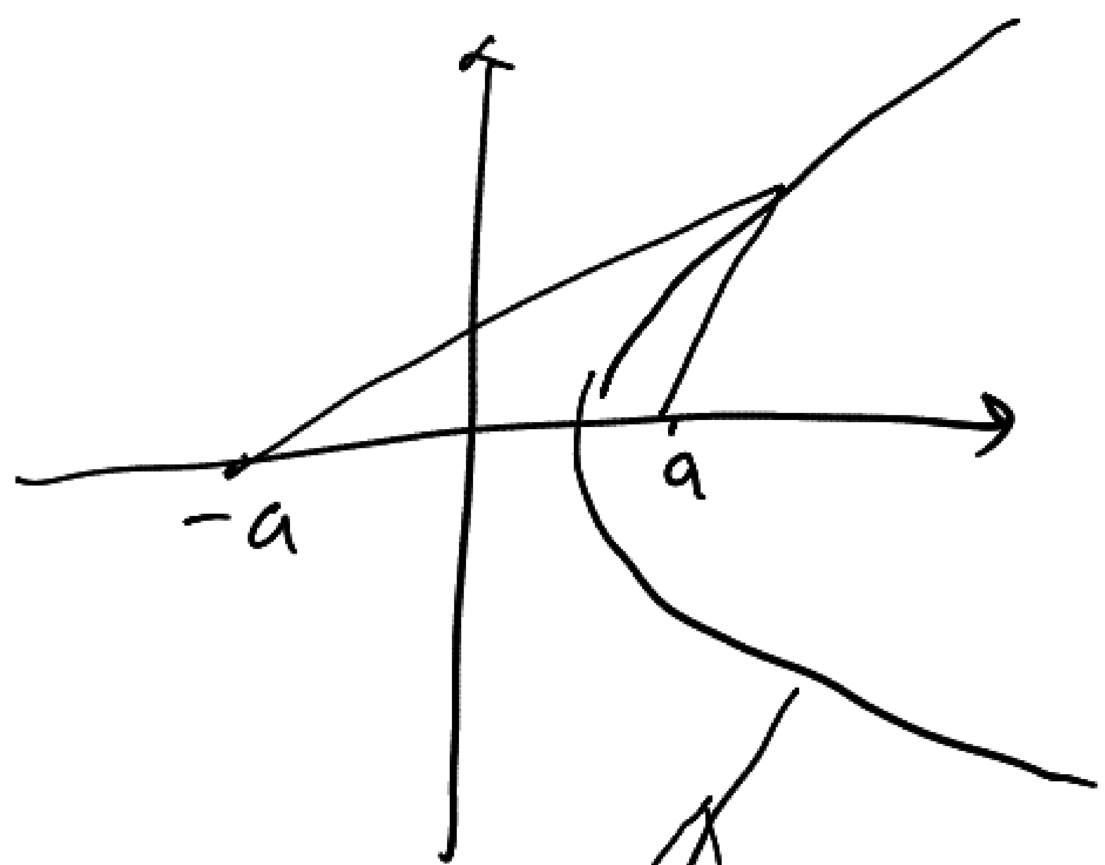
故 A 在 K 闭, 但 $A = f(U) \cap K$ 在 K 开, 故 $A = K$ or ϕ (均不成立)

故 $f(U) = V$.

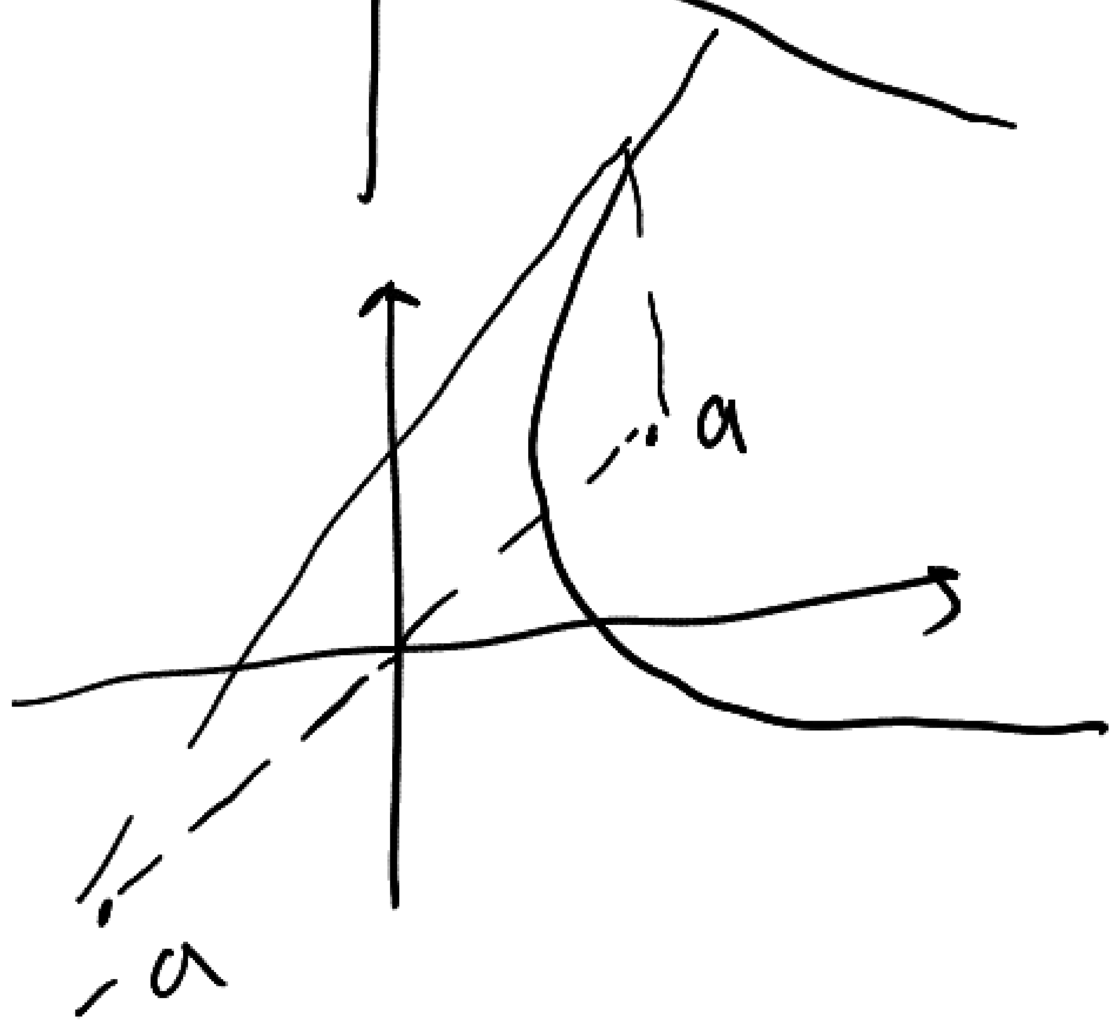
Conway

P4 Ex3.

(I)



(II)



$a\in\mathbb{R}$时，z轨迹是半长轴为c，半焦距为a的标准双曲线右半支

当$a\in\mathbb{C}$时，设a模长$|a|$，辐角θ

则轨迹为(I)中以$|a|$为参数的轨迹逆时针绕原点旋转θ所得.

P20 Ex.6.

Example 1: $\mathbb{Q}$, $d(x,y)=|x-y|$

其中Cauchy列 1.4, 1.41, 1.414, 1.4142, … 不收敛

Example 2: $(0,1]$, $d(x,y)=|x-y|$,

Cauchy列 $a_n=\frac{1}{10^n}$ 不收敛

Example 3: $\mathcal{R}[-1,1]$, $d(f,g)=\int_{-1}^{1}|f-g|$

考虑 $f_n(x)=\begin{cases} n & x\in[-\frac{1}{n^2},\frac{1}{n^2}] \\ 0 & x\in[-1,1]\setminus[-\frac{1}{n^2},\frac{1}{n^2}] \end{cases}$

设 $n>m$, $d(f_n,f_m)\le\int_{-1}^{1}|f_n|+\int_{-1}^{1}|f_m|\le\frac{4}{m}$ 故$\{f_n\}$ Cauchy. 但$f(x)$不在.

Ex8. $\{x_{n_k}\}$收敛，$\{x_n\}$为Cauchy列

我们设 $x_{n_k}\to a$ 则 $\forall\varepsilon>0$ $\exists K$, $\forall k>K$, $|x_{n_k}-a|<\varepsilon$.

而$\{x_n\}$为Cauchy 故$\exists N$, $\forall n,m>N$, $|x_n-x_m|<\varepsilon$

现在当$n>N$时，取$k>K$且k充分大 使$n_k\ge k>N$

则 $|x_n-a|\le|x_n-x_{n_k}|+|x_{n_k}-a|<2\varepsilon$

故 $x_n\to a$.

P24 Ex4. 设 $E_1, \cdots E_k$ 均为 Compact sets.

设 $\bigcup_\alpha O_\alpha$ 为 $E_1 \cup \cdots \cup E_k$ 的一个开覆盖，即 $E_1 \cup \cdots \cup E_k \subset \bigcup_\alpha O_\alpha$,

O_α 为开集．那么 $\forall 1 \le i \le k$, $E_i \subset E_1 \cap \cdots \cap E_k \subset \bigcup_\alpha O_\alpha$，由于 E_i 紧性

存在 $O_{i_1} \cdots O_{is_i}$ 使得 $E_i \subset \bigcup_{j=1}^{s_i} O_{is_i}$

于是 $E_1 \cup \cdots \cup E_k \subset \bigcup_{i=1}^{k} \bigcup_{j=1}^{s_i} O_{is_i}$ 故 $E_1 \cup \cdots \cup E_k$ 紧.

Ex6. 设 A 是 totally bounded set, $A \subset X$

由于 $\forall \varepsilon > 0$, $\exists \bigcup_{i=1}^{N} B(x_i, \frac{\varepsilon}{2}) \supseteq X$

设 $x \in \overline{A}$. 则 $\exists y_2 \in A$, $d(x, y_2) < \frac{\varepsilon}{2}$ 设 $d(y_2, x_i) < \frac{\varepsilon}{2}$

$d(x, x_i) \le d(x, y_2) + d(y_2, x_i) < \varepsilon$

$x \in B(x_i, \varepsilon)$

$\overline{A} \subseteq \bigcup_{i=1}^{N} B(x_i, \varepsilon)$. #.

谭小江

28. $z = \frac{1}{w}$

$dz = -\frac{1}{w^2}dw$, $dw = -w^2 dz$

$$\frac{4|dw|^2}{(1+|w|^2)^2} = \frac{4|w|^4|dz|^2}{\left(1+\frac{1}{|z|^2}\right)^2} = \frac{4|dz|^2}{(1+|z|^2)^2}$$

Conway

P6 ex 7. $z = re^{i\theta}$ $z^n = r^n e^{in\theta}$

若 $Re(z) \geq 0$，则 $0 \leq \arg(z) \leq \frac{\pi}{2}$ or $\frac{3\pi}{2} \leq \arg(z) \leq 2\pi$

排除取等号的情况后，若 $0 < \arg(z) < \frac{\pi}{4}$，则考虑 n 使

$\arg(nz) \leq \frac{\pi}{2}$ 但 $\arg((n+1)z) > \frac{\pi}{2}$ 则 $\arg((n+1)z) < \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4} < \pi$

$Re(z^{n+1}) < 0$

若 $\frac{\pi}{4} < \arg(z) < \frac{\pi}{2}$，则 $Re(z^2) < 0$

$\frac{3\pi}{2} < \arg(z) < 2\pi$ 时完全类似考虑。最终仅有 $\arg(z) = 0$，即 $z \in \mathbb{R}_{\geq 0}$ 成立.

P10 ex.1.

$$d(z,z')^2 = 2 - 2(x_1x_1' + x_2x_2' + x_3x_3')$$

$$= 2 - 2\,\frac{(z+\bar z)(z'+\bar z') - (z-\bar z)(z'-\bar z') + (|z|^2-1)(|z'|^2-1)}{(1+|z|^2)(1+|z'|^2)}$$

$$= 4\,\frac{z\bar z + z'\bar z' - z'\bar z - z\bar z'}{(1+|z|^2)(1+|z'|^2)} = \frac{4|z-z'|^2}{(1+|z|^2)(1+|z'|^2)}$$

$$d(z,\infty)^2 = x_1^2 + x_2^2 + (x_3-1)^2 = \frac{(z+\bar z)^2}{(1+|z|^2)^2} - \frac{(z-\bar z)^2}{(|z|^2+1)^2} + \frac{2^2}{(|z|^2+1)^2}$$

$$= \frac{4+4|z|^2}{(1+|z|^2)^2} = \frac{4}{(1+|z|^2)}$$

ex.2.

$$x_1=\frac{z+\bar z}{|z|^2+1}\qquad x_2=\frac{-i(z-\bar z)}{|z|^2+1}\qquad x_3=\frac{|z|^2-1}{|z|^2+1}$$

$z=0,\quad x=(0,0,-1)$

$z=1+i,\quad x=\left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$

$z=3+2i,\quad x=\left(\frac{3}{7},\frac{2}{7},\frac{13}{15}\right)$

ex.5.

$$x_w=\frac{z+z'+\bar z+\bar z'}{1+|z+z'|^2}=\frac{\frac{x_1+ix_2}{1-x_3}+\frac{x_1'+ix_2'}{1-x_3'}+\frac{x_1-ix_2}{1-x_3}+\frac{x_1'-ix_2'}{1-x_3'}}{1+\left|\frac{x_1+ix_2}{1-x_3}+\frac{x_1'+ix_2'}{1-x_3'}\right|^2}$$

$$=\frac{\frac{2x_1}{1-x_3}+\frac{2x_1'}{1-x_3'}}{1+\left(\frac{x_1}{1-x_3}+\frac{x_1'}{1-x_3'}\right)^2+\left(\frac{x_2}{1-x_3}+\frac{x_2'}{1-x_3'}\right)^2}$$

$$y_w=\frac{-i(z_1+z_2-\bar z_1-\bar z_2)}{1+|z_1+z_2+\bar z_1+\bar z_2|^2}=\frac{\frac{2x_2}{1-x_3}+\frac{2x_2'}{1-x_3'}}{1+\left(\frac{x_1}{1-x_3}+\frac{x_1'}{1-x_3'}\right)^2+\left(\frac{x_2}{1-x_3}+\frac{x_2'}{1-x_3'}\right)^2}$$

$$z_w=\frac{|z_1+z_2|^2-1}{|z_1+z_2|^2+1}=\frac{\left(\frac{x_1}{1-x_3}+\frac{x_1'}{1-x_3'}\right)^2+\left(\frac{x_2'}{1-x_3'}+\frac{x_2}{1-x_3}\right)^2-1}{1+\left(\frac{x_1}{1-x_3}+\frac{x_1'}{1-x_3'}\right)^2+\left(\frac{x_2}{1-x_3}+\frac{x_2'}{1-x_3'}\right)^2}.$$

P17 ex.5.

设 $a\in F$ 考虑 $B=\{b\in F,\ \exists z_1,\cdots z_n\in F,\ d(z_i,z_{i+1})<\varepsilon\}$ 显然 $B\neq\phi$.

首先B是开集(相对)，理由如下：设 $b\in B$，则 $N(b,\frac{\varepsilon}{2})\subseteq B$

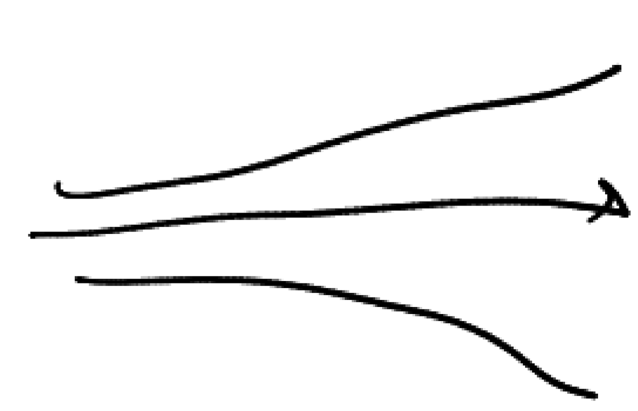
其次B是闭集(相对). 理由如下：设 z_0 为B的一个极限点，则 $z_0\in F$，则 $\exists z_1\in B$,

$d(z_1,z_0)<\varepsilon$ 故 $z_0\in B$

故 $B=F$

F为闭集的条件是 $\boxed{\text{不必要的}}$

若此性质成立，F未必连通 考虑 $\{(x,y)\in\mathbb{R}^2:\ y=\pm e^x\}$



P28

ex2 若 $\forall \varepsilon>0, \exists \delta>0, \forall |x_1-x_2|<\delta$, $|f(x_1)-f(x_2)|<\varepsilon$, $|g(x_1)-g(x_2)|<\varepsilon$

则此时 $|(f+g)(x_1)-(f+g)(x_2)| < |f(x_1)-f(x_2)|+|g(x_1)-g(x_2)| < 2\varepsilon$.

若 $|f(x_1)-f(x_2)| < K|x_1-x_2|$, $|g(x_1)-g(x_2)| < K|x_1-x_2|$

则 $|(f+g)(x_1)-(f+g)(x_2)| < 2K|x_1-x_2|$.

ex4. 答案是肯定的. 给定 $\varepsilon>0$

$\exists \delta_1$, $|x_1-x_2|<\delta_1$ 时, $|f(x_1)-f(x_2)|<\varepsilon$, $\exists \delta_2$, $|y_1-y_2|<\delta_2$ 时, $|g(y_1)-g(y_2)|<\delta$

此时 $|f(g(y_1))-f(g(y_2))|<\varepsilon$.

对于Lipschitz: $|f(g(x_1))-f(g(x_2))| < K_1|g(x_1)-g(x_2)| < K_1K_2|x_1-x_2|$.

ex8. 若 $f: X\to\Omega$ continuous. 给定 $\varepsilon>0$,

$\forall x\in X, \exists \varepsilon_x, \forall x_1\in B(x,\varepsilon_x)$, $d(f(x_1), f(x))<\varepsilon$.

考虑 $\bigcup\limits_{x} B(x,\varepsilon_x)\supseteq X$ 有 Lebesgue number δ 和 有限子覆盖

$\bigcup\limits_{i=1}^{N} B(x_i,\varepsilon_{x_i})\supseteq X$. 设 $|x_1-x_2|<\delta$, $B(x_1,\delta)\subset B(x_i,\varepsilon_{x_i})$

for certain i. 则 $d(f(x_1),f(x_2)) \le d(f(x_1),f(x_i)) + d(f(x_2),f(x_i)) < 2\varepsilon$.

故 f uniformly continous.

ex9. 若 (X,d) 不连通, 存在 X 既开又闭非平凡子集 A.

由紧性, A是紧的, $X-A$闭 $d(X, X-A)>0 := d_0$

取 $\varepsilon=\frac{d_0}{10}$, $a\in A$, $b\in X-A$ 即矛盾!