18. 分离变量 $a>0$ $A_1, A_2 \in \mathbb{R}$ constant

(1) $\begin{cases} u_t - u_{xx} = u & 0<x\le\pi,\ t>0 \\ u(x,0) = \sin x & 0\le x\le\pi \\ u(0,t)=0 \quad u(\pi,t)=0 & t\ge 0 \end{cases}$

满足 $u(x,0) = \sin x$

$\Rightarrow u(x,t) = \sin x$ #

记 $u(x,t) = X(x)\,T(t)$

$X(x)T'(t) - X''(x)T(t) = X(x)T(t)$

$\frac{T'(t)}{T(t)} = \frac{X''(x)+X(x)}{X(x)} = -\lambda$

$T'(t) + \lambda T(t) = 0$ $\quad X(0) = X(\pi) = 0$

$X''(x) + (\lambda+1)X(x) = 0$

$$u(x,t) = \sum_{n=1}^{\infty} T_n(0)\, e^{-(n^2-1)t} \sin nx$$

(2) $\begin{cases} u_t - a^2 u_{xx} = 0 & 0<x<\pi,\ t>0 \\ u(x,0) = \cos x & 0\le x\le\pi \\ u_x(0,t)=0,\ u_x(\pi,t)=0 & t\ge 0 \end{cases}$

记 $u(x,t) = X(x)\,T(t)$

$X(x)T'(t) - a^2 X''(x)T(t) = 0$ $\quad \frac{T'(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda$

$T' + a^2\lambda T = 0$ $\quad X'(0) = X'(\pi) = 0$ 解此 S-T 问题

$X'' + \lambda X = 0$

$$u(x,t) = \sum_{n=1}^{\infty} T_n(0)\, e^{-n^2a^2t} \cos(nx)$$

满足初值 $u(x,0) = \cos x$

故 $u(x,t) = e^{-a^2t}\cos x$ #

19. $a>0$ 求 Green 函数

(2) $$\begin{cases} u_t - a^2 u_{xx} = f(x,t) & 0\le x\le l,\ t>0 \\ u(x,0) = \varphi(x) & 0\le x\le l \\ u_x(0,t) = g_1(t) \quad u_x(l,t) = g_2(t) & ,\ t\ge 0 \end{cases}$$

使用分离变量法 $u(x,t) = X(x)T(t)$

$$\begin{cases} T' + a^2\lambda T = 0 \\ X'' + \lambda X = 0 \quad X'(0) = X'(l) = 0 \end{cases}$$

$\delta(x-\xi, t-\tau) = \delta(x-\xi)\,\delta(t-\tau)$

$X_n(x) = \sqrt{\frac{2}{l}}\cos\frac{n\pi}{l}x$

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

$$\frac{dT_n(t)}{dt} + \lambda_n a^2 T_n(t) = \delta(t-\tau) X_n(\xi)$$

$$T_n(t) = X_n(\xi)\, e^{-a^2\left(\frac{n\pi}{l}\right)^2(t-\tau)} H(t-\tau)$$

$$\Rightarrow G(x,t,\xi,\tau) = \frac{2}{l}\sum_{n=0}^{\infty}\cos\frac{n\pi\xi}{l}\cos\frac{n\pi x}{l}\, e^{-a^2\left(\frac{n\pi}{l}\right)^2(t-\tau)} H(t-\tau)$$

19(3) 求 Green 函数 $a>0$

$$\begin{cases} u_t - a^2 u_{xx} = f(x,t) & 0<x\le l,\ t>0 \\ u(x,0) = \varphi(x) & 0\le x\le l \\ -u_x(0,t) + u(0,t) = g_1(t),\ u(l,t) = g_2(t) & t\ge 0 \end{cases}$$

$u(x,t) = X(x)T(t)$

$$\begin{cases} T' + a^2\lambda T = 0 \\ X'' + \lambda X = 0 \quad -X'(0) + X(0) = 0,\ X(l) = 0 \end{cases}$$

$\tan\mu_n l = -\mu_n \quad \lambda_n = \mu_n^2 \quad X_n(x) = \sin\mu_n x + \mu_n\cos\mu_n x$

$$\Rightarrow G(x,t,\xi,\tau) = \frac{2}{l}\sum_{n=1}^{\infty}(\sin\mu_n x \quad \mu_n\cos\mu_n x)(\sin\mu_n\xi + \mu_n\cos\mu_n\xi) + \mu_n\cos\mu_n x$$

$$e^{-\mu_n^2 a^2(t-\tau)} H(t-\tau)$$