Calculus of Variation

(1) Find Euler-Lagrange Eq.

$$I[u]=\int_0^1 [u'(x)^2+e^{u(x)}]dx$$

$$I[u+\varepsilon v]=\int_0^1 [(u'+\varepsilon v')^2+e^{u+\varepsilon v}]dx$$

$$\frac{dI[u+\varepsilon v]}{d\varepsilon}=\int_0^1 (v(x)e^{u+\varepsilon v}+2(u'+\varepsilon v')v')dx$$

$$\left.\frac{dI[u+\varepsilon v]}{d\varepsilon}\right|_{\varepsilon=0}=0 \iff \int_0^1 v(x)e^{u(x)}+2u'(x)v'(x)dx=0$$

分部积分 $\int_0^1 v(x)e^{u(x)}=-2\int_0^1 u'(x)dv(x)=-2\left(u'v\Big|_0^1-\int_0^1 v(x)u''(x)dx\right)$

$$\int_0^1 v(x)(e^{u(x)}-2u''(x))dx=0$$

E-L: $e^{u(x)}-2u''(x)=0$

$$I[u]=\int_\Omega (yu_x^2+u_y^2)dxdy$$

$$L(u,u_x,u_y)=yu_x^2+u_y^2$$

E-L equation is

$$\frac{\partial L}{\partial u}=\left(\frac{\partial L}{\partial u_x}\right)_x+\left(\frac{\partial L}{\partial u_y}\right)_y$$

$$\frac{\partial L}{\partial u}=0,\quad \frac{\partial L}{\partial u_x}=2yu_x \quad \frac{\partial L}{\partial u_y}=2u_y$$

$$0=2yu_{xx}+2u_{yy}$$

i.e. $u_{yy}+yu_{xx}=0$

$$I[u] = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 \, dx\, dy$$

$$L(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2$$

此时的 E-L 方程为

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial L}{\partial u_y} + \frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial u_{xx}} + \frac{\partial^2}{\partial x \partial y}\frac{\partial L}{\partial u_{xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial L}{\partial u_{yy}} = 0$$

$$\frac{\partial L}{\partial u_{xx}} = 2u_{xx} \qquad (2u_{xx})_{xx} = 2u_{xxxx}$$

$$\frac{\partial L}{\partial u_{xy}} = 4u_{xy} = 4u_{xxyy}$$

$$\frac{\partial L}{\partial u_{yy}} = 2u_{yy} = 2u_{yyyy}$$

E-L 方程为

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$$

(2) $$A[v] = \int_0^1 \int_0^T \frac{1}{2} v_x v_t - (v_x)^3 - \frac{1}{2}(v_{xx})^2 \, dx\, dt$$

$A[u] = \min A[v]$, find E-L equation

$\psi = u_x$, find Eq. for ψ.

$$L(v, v_t, v_{tt}, v_x, v_{xx}, v_{xt}) = \frac{1}{2} v_x v_t - (v_x)^3 - \frac{1}{2}(v_{xx})^2$$

此时 E-L 方程为

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\frac{\partial L}{\partial u_t} - \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial u_{xx}} + \frac{\partial^2}{\partial x \partial t}\frac{\partial L}{\partial u_{xt}} + \frac{\partial^2}{\partial t^2}\frac{\partial L}{\partial u_{tt}} = 0$$

$$\frac{\partial L}{\partial u_x} = \frac{1}{2}u_t - 3u_x^2$$

$$\frac{\partial L}{\partial v_t} = \frac{1}{2}u_x$$

$$\frac{\partial L}{\partial u_{xx}} = -u_{xx}$$

$$-\frac{1}{2}u_{xt} - \frac{1}{2}u_{xt} + 6u_x u_{xx} - u_{xxxx} = 0$$

$$u_{xt} + u_{xxxx} = 6u_x u_{xx}$$

$\psi = u_x$ 满足 $\psi_t + \psi_{xxx} = 6\psi\psi_x$

This is Korteweg-De Vries Equation.

(3) $\begin{cases} -\Delta u(x) = f(x) & x \in \Omega \\ \frac{\partial u}{\partial n} + a(x) u(x) = 0, & x \in \partial\Omega \end{cases}$

Find $I[v]$ minimized by above B.V.P. solution

$$-\int_{\Omega} \Delta u \, v = \int_{\Omega} f v$$

$$\int_{\Omega} \nabla u \cdot \nabla v - \int_{\partial\Omega} (\nabla u \cdot n) v = \int_{\Omega} f v$$

$$\int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial\Omega} \alpha u v = \int_{\Omega} f v$$

$$a(u,v) = \int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial\Omega} \alpha u v \quad f(v) = \int_{\Omega} f v$$

由 Riesz 表示定理,

u 是变分问题

$$F(v) = \frac{1}{2} \int_{\Omega} \|\nabla v\|^2 + \int_{\partial\Omega} \alpha(x) v^2(x) dx - \int_{\Omega} f(x) v(x) dx$$

的变分极小点.