ODE Homework #1

1.1

1(2) $y' + \frac{y}{x} = \frac{\sin x}{x^2} + \frac{\cos x \cdot x - \sin x}{x^2} = \frac{\cos x}{x}$

3. $y = C_1 e^x + C_2 x e^x$

$y' = C_1 e^x + C_2 e^x + C_2 x e^x = (C_1 + C_2) e^x + C_2 x e^x$

$y'' = C_1 e^x + C_2 e^x + C_2 e^x + C_2 x e^x = (C_1 + 2C_2) e^x + C_2 x e^x$

$\Rightarrow \boxed{y + y'' = 2y'}$

4. $C_1 x + (y - C_2)^2 = 0 \Rightarrow C_1 x + y^2 - 2C_2 y + C_2^2 = 0$

$C_1 + 2yy' - 2C_2 y' = 0$

$(y')^2 + (y) \cdot y'' - C_2 y'' = 0$

$\Rightarrow y^2 - 2C_2 y + C_2^2 = 2xyy' - 2C_2 xy'$

$C_2^2 + 2C_2(xy' - y) + y^2 - 2xyy' + x^2 y'^2 = x^2 y'^2$

$(C_2 + xy' - y)^2 = x^2 y'^2$

扎好可约去，若 $C_1 \neq 0$ 则 $y' \neq 0$

总有 $C_2 = y - 2xy' \Rightarrow yy'' - 2xy'y'' = (y')^2 + yy''$

故总有 $\boxed{2xy'' + y' = 0}$

5. $x^2 + y^2 + Dx + Ey + F = 0$

$2x + 2yy' + D + Ey' = 0$

$2 + 2(y')^2 + 2y \cdot y'' + Ey'' = 0$

$2 \cdot 2 \cdot y' \cdot y'' + 2y'y'' + 2y \cdot y''' + Ey''' = 0$

$y^{(3)}(2 + 2(y^{(1)})^2 + 2y\, y^{(2)}) = y^{(2)}(6 y^{(1)} y^{(2)} + 2y\, y^{(3)})$

$2y^{(3)} + 2(y^{(1)})^2 y^{(3)} = 6 y^{(1)} (y^{(2)})^2$

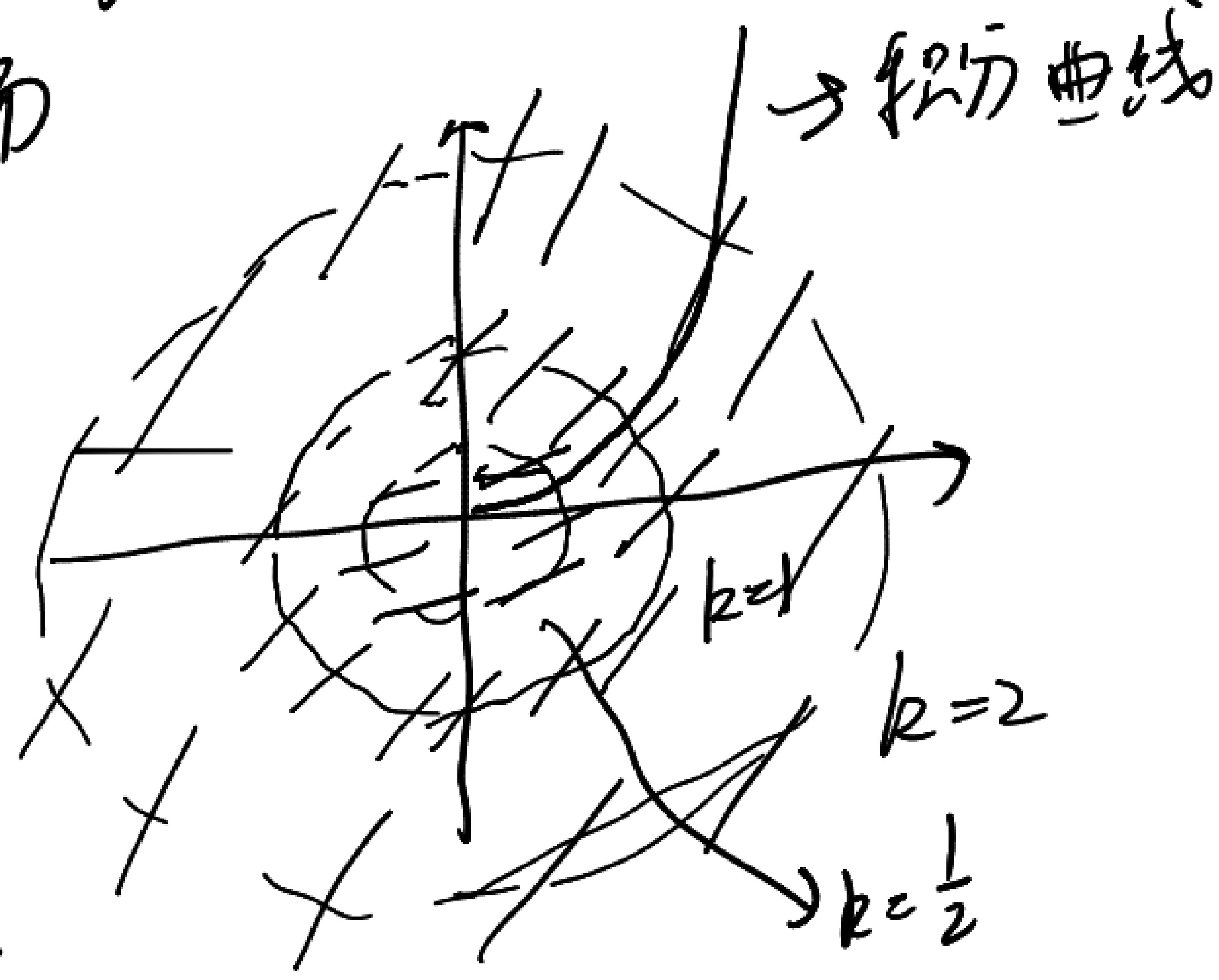
$\Rightarrow \boxed{y^{(3)} + (y^{(1)})^2 y^{(3)} = 3y^{(1)} (y^{(2)})^2}$

1.2

2 (2), $y' = x^2+y^2$

L_k 为 $x^2+y^2=k$

画出线素场



4, $y'=y-x^2$

$y''=y'-2x = y-x^2-2x=0$

轨迹 $\boxed{y=x^2+2x}$

2.1

1. $(4x^2y-y)dx+(3x+y)dy=0$

$\frac{\partial P}{\partial y}=4x^2-1 \neq \frac{\partial Q}{\partial x}=3$ 不恰当

3. $(ax-by)dx+(bx-cy)dy=0$

$\frac{\partial P}{\partial y}=-b \quad \frac{\partial Q}{\partial x}=b \quad b\neq 0$ 时不恰当

$b=0$ 时， $ax\,dx=cy\,dy \qquad cy^2=ax^2+t$ ，在 $c\neq 0$ 时有

$y=\pm\sqrt{\frac{1}{c}(ax^2+t)}$

$c=0$ 时无解

5. $3x^2(1+\ln y)dx-(2y-\frac{x^3}{y})dy=0$

$\frac{\partial P}{\partial y}=\frac{3x^2}{y}=\frac{\partial Q}{\partial x}=+\frac{3x^2}{y}$ 恰当

设 $\frac{\partial \phi}{\partial x}=3x^2(1+\ln y)$， $\frac{\partial \phi}{\partial y}=-2y+\frac{x^3}{y}$，

则 $\phi=x^3(1+\ln y)+\theta(y)$

$\frac{\partial \phi}{\partial y}=\theta'(y)+\frac{x^3}{y}=-2y+\frac{x^3}{y}$， $\theta(y)=-y^2+C$

$\phi(x,y)=x^3(1+\ln y)-y^2+C$

即

$x^3(1+\ln y)-y^2+C=0$

7. $2x(1+\sqrt{x^2-y})dx-\sqrt{x^2-y}\,dy=0$

$\frac{\partial P}{\partial y}=\frac{-x}{\sqrt{x^2-y}}=\frac{\partial Q}{\partial x}=\frac{-x}{\sqrt{x^2-y}}$

设 $\frac{\partial \phi}{\partial x}=2x(1+\sqrt{x^2-y}) \quad \frac{\partial \phi}{\partial y}=-\sqrt{x^2-y}$

$\phi=\frac{2}{3}(x^2-y)^{\frac{3}{2}}+\gamma(x) \qquad \gamma'(x)+(x^2-y)^{\frac{1}{2}}\cdot 2x=2x(1+\sqrt{x^2-y}) \quad \gamma(x)=x^2+C$

$\Rightarrow \frac{2}{3}(x^2-y)^{\frac{3}{2}}+x^2=C$

$y=x^2-\left[\frac{3}{2}(C-x^2)\right]^{\frac{2}{3}}$

9. $\frac{y}{x}dx+(y^3+\ln x)dy=0$

$\frac{\partial P}{\partial y}=\frac{1}{x}=\frac{\partial Q}{\partial x}=\frac{1}{x}$.

设 $\frac{\partial \phi}{\partial x}=\frac{y}{x}$， $\frac{\partial \phi}{\partial y}=y^3+\ln x$

$\phi=\frac{y^4}{4}+y\cdot\ln x+\gamma(x)$

$\frac{\partial \phi}{\partial x}=\frac{y}{x}+\gamma'(x)=\frac{y}{x}$

$\Rightarrow \phi=\frac{y^4}{4}+y\ln x+C$

$\frac{y^4}{4}+y\ln x+C=0$

2.2

1.(4) $y'=\sqrt{4x+2y-1}$, $y(0)=1$

$\frac{dy}{dx}=\sqrt{4x+2y-1}$ 设 $u=4x+2y-1$ $\frac{du}{dx}=4+2\frac{dy}{dx}$

$\sqrt{u}=\frac{dy}{dx}=\frac{1}{2}(\frac{du}{dx}-4)$, $2\sqrt{u}+4=\frac{du}{dx}$, $dx=\frac{du}{2\sqrt{u}+4}$

$x=\sqrt{u}-2\ln(\sqrt{u}+2)+C$

$x=\sqrt{4x+2y-1}-2\ln(\sqrt{4x+2y-1}+2)+2\ln(3)-1$

(7) $3y^2y'+16x=2xy^3$, $x\to+\infty$时, $y(x)$有界.

$3y^2\frac{dy}{dx}=2xy^3-16x=2x(y^3-8)$

$\frac{3y^2}{y^3-8}dy=2xdx$ 及 $y\equiv2$

$x^2=\ln(y^3-8)+C$, $x\to\infty$时, $|y|\le M$, 不成立

故 $y\equiv2$

3. 首先我们指出 $y\equiv a$ 一定是原方程一个解

现假设 $\left|\int_{a+\varepsilon}^{a}\frac{1}{f(y)}dy\right|=+\infty$,

$y'=f(y)=\frac{dy}{dx}$ $y\ne a$时, $\frac{dy}{f(y)}=dx$

$x_0-x_1=\int_{x_1}^{x_0}dx=\int_{y(x_1)}^{a}\frac{1}{f(y)}dy$ 若 $\exists x_1\in\mathbb{R}$, $y(x_1)\ne a$

则 $|x_0-x_1|=\infty$, 矛盾! 故 $\forall x_1\in\mathbb{R}$, $y(x_1)=a$ 即解唯一.

反之, 若 $\exists\varepsilon\ne0$, $\left|\int_{a}^{a+\varepsilon}\frac{1}{f(y)}dy\right|$ 有限, 不妨设 $\varepsilon>0$,

且 $\int_{a}^{a+\varepsilon}\frac{1}{f(y)}dy=M=x_1-x_0$

$f(y)$ 在 $(a,a+\varepsilon)$ 不变号, 当 $x\in(x_0,x_1]$ 时, 存在唯一 $y(x)$,

$\int_{a}^{y(x)}\frac{1}{f(y)}dy=x-x_0$

两边对 x 求导, $1=\frac{1}{f(y)}y'$, 而此 $y(x)$ 不恒等于 a,

故解不唯一.

4. $f(t^{\alpha s}x, t^{\beta s}y) = t^{d s} f(x,y)$

$t>0,\ \alpha,\beta>0,\ \alpha+\beta=1,\ s\in\mathbb{R}$ $\qquad d_0 = d_1+\beta-\alpha$

设 $y = u x^{\frac{\beta}{\alpha}}$

$P(x, u x^{\frac{\beta}{\alpha}})\,dx + Q(x, u x^{\frac{\beta}{\alpha}})\,dy = 0$

$x^{\frac{d_0}{\alpha}} P(1,u)\,dx + x^{\frac{d_1}{\alpha}} Q(1,u) \quad (du\ x^{\frac{\beta}{\alpha}} + u \frac{\beta}{\alpha} x^{\frac{\beta}{\alpha}-1} dx) = 0$

$dx\,(x^{\frac{d_0}{\alpha}} P(1,u) + x^{\frac{d_1}{\alpha}} Q(1,u)\, u\ \frac{\beta}{\alpha}) + du\,(x^{\frac{\beta}{\alpha}} x^{\frac{d_1}{\alpha}} Q(1,u)) = 0$

$dx\,(P(1,u) + u\,Q(1,u)\frac{\beta}{\alpha}) + du\ (x\,Q(1,u)) = 0$

$$\frac{dx}{x} = \frac{du\ Q(1,u)}{P(1,u)+u\,Q(1,u)\frac{\beta}{\alpha}}$$

2.3

1(3) $(xy+e^x)\,dx - x\,dy = 0$

$xy' = xy + e^x$

$y' = y + \frac{e^x}{x}$

$e^{-x}y' = e^{-x}y + \frac{1}{x}$

$(e^{-x}y)' = \frac{1}{x}$

$e^{-x}y = \ln|x| + C$

$y = e^x(\ln|x| + C)$

(5) $(1-2xy)\,y' = y(y-1)$

$(1-2xy)\,dy = y(y-1)\,dx$

$1-2xy = (y^2-y)\frac{dx}{dy}$

$\frac{1}{2y} = x + \frac{y-1}{2}\frac{dx}{dy}$

$\frac{dx}{dy} + \frac{2}{y-1}x = \frac{1}{y(y-1)}$

$(y-1)^2\frac{dx}{dy} + 2(y-1)x = \frac{y-1}{y}$

$((y-1)^2 x)'_y = \frac{y-1}{y}$

$(y-1)^2 x = y - \ln|y| + C$

即所有积分曲线

2. $y'\sin 2x = 2(y+\cos x)$, $x\to\frac{\pi}{2}$ 时 y 有界

$$y' = \frac{2y}{\sin 2x} + \frac{2\cos x}{\sin 2x} = \frac{y}{\sin x\cos x} + \frac{1}{\sin x} \qquad x\in(0,\frac{\pi}{2})$$

$$y'\frac{\cos x}{\sin x} = \frac{y}{\sin^2 x} + \frac{\cos x}{\sin^2 x}$$

$$\left(y\frac{\cos x}{\sin x}\right)' = \frac{\cos x}{\sin^2 x}$$

$$y\frac{\cos x}{\sin x} = \frac{-1}{\sin x} + C \qquad y = \frac{C\sin x - 1}{\cos x}$$

$\Rightarrow C=1$, $y = \dfrac{\sin x - 1}{\cos x}$ 在 $\frac{\pi}{2}$ 附近有界

3. $xy' + ay = f(x)$, $a>0$, $\lim\limits_{x\to 0} f(x) = b$.

$$y' + \frac{a}{x}y = \frac{f(x)}{x}$$

$$e^{a\ln|x|}y' + \frac{a}{x}e^{a\ln|x|}y = \frac{f(x)}{x}e^{a\ln|x|}$$

$$\left(e^{a\ln|x|}y\right)' = \frac{f(x)}{x}e^{a\ln|x|}$$

$x>0$ 时 $\quad x^a y = \displaystyle\int_0^x f(t)t^{a-1}dt + C$

$$y = \frac{\int_0^x f(t)t^{a-1}dt + C}{x^a} \quad (*)$$

$x\to 0^+$ 有界，必有 $C=0$

此时 $\lim\limits_{x\to 0^+} y \overset{L'Hopital}{=} \lim\limits_{x\to 0^+} \dfrac{f(x)x^{a-1}}{ax^{a-1}} = \dfrac{b}{a}$.

$x<0$ 时， $\quad |x|^a y = \displaystyle\int_0^x \frac{f(t)}{t}|t|^a dt + C_2$

同理 $C_2=0$ ，(*) 式同样成立. 总之有 $\lim\limits_{x\to 0} y = \dfrac{b}{a}$.

4. $y' - 2y\cos^2 x + \sin x = 0$

$$e^{-(x+\sin x\cos x)} y' - 2y\cos^2 x\, e^{-(x+\sin x\cos x)} + \sin x\, e^{-(x+\sin x\cos x)} = 0$$

$$\left(e^{-(x+\sin x\cos x)} y\right)' = -\sin x\, e^{-(x+\sin x\cos x)}$$

$$y(x) = e^{x+\sin x\cos x}\left(-\int_0^x \sin t\, e^{-(t+\sin t\cos t)}\,dt + C\right)$$

周期解必有界（因$y(x)$是C^1函数）

易知$x\to+\infty$时，$\int_0^x \sin t\, e^{-(t+\sin t\cos t)}\,dt$收敛，故 $C = \int_0^{+\infty} \sin t\, e^{-(t+\sin t\cos t)}$

$$y(x) = e^{x+\sin x\cos x}\int_x^{+\infty} \sin t\, e^{-(t+\sin t\cos t)}\,dt$$

$$y(x+2\pi) = e^{x+\sin x\cos x}\, e^{2\pi}\int_{x+2\pi}^{+\infty} \sin t\, e^{-t} e^{-\sin t\cos t}\,dt$$

$y(x) = y(x+2\pi)$，周期2π

5. $$x' e^t + x e^t = f(t) e^t$$

$$(x e^t)' = f(t) e^t$$

$$x e^t = \int_0^t f(t_1) e^{t_1}\,dt_1 + C$$

$$x = \left(\int_0^t f(t_1) e^{t_1}\,dt_1 + C\right) e^{-t}$$

由Cauchy收敛准则知 $\lim_{t\to-\infty}\int_0^t f(s)e^s\,ds$ 收敛 $, t\to-\infty)$

$$\left(\left|\int_0^{t_1} f(s)e^s\,ds - \int_0^{t_2} f(s)e^s\,ds\right| = \left|\int_{t_1}^{t_2} f(s)e^s\,ds\right| \le M\left(e^{t_2}-e^{t_1}\right) \le Me^{-N} < \varepsilon\right)$$

令$t\to-\infty$，只能有 $C = -\int_0^{-\infty} f(s)e^s\,ds$

即 $x = \int_{-\infty}^t f(s)e^{s-t}\,ds$. 设 $f(s+T) = f(s)$，$\forall s$

则 $x(t+T) - x(t) = \int_{-\infty}^{t+T} f(s)e^{s-t-T}\,ds - \int_{-\infty}^t f(s)e^{s-t}\,ds = 0$（换变量）

$\forall t.$

6. $x=0$ 时 $y=0$

$x\neq 0$ 时 $y'-(2x+\frac{1}{x})y=x$

$$e^{-(x^2+\ln|x|)}y'-(2x+\frac{1}{x})e^{-(x^2+\ln|x|)}=e^{-(x^2+\ln|x|)}x$$

$$\frac{1}{|x|e^{x^2}}y=\int_1^x\frac{x_1}{e^{x_1^2}|x_1|}dx_1+C$$

当 $x\to+\infty$ 时，$y=xe^{x^2}\left(\int_0^x e^{-x_1^2}dx_1+C\right)$ 有界

x 一项 $C=-\int_0^{+\infty}e^{-x_1^2}dx_1=-\frac{\sqrt{\pi}}{2}$

此时 $\lim\limits_{x\to+\infty}y=\lim\limits_{x\to+\infty}\frac{e^{-x^2}\quad x^2\,(e^{x^2})^2}{-(e^{x^2}+2x^2e^{x^2})}=\lim\limits_{x\to+\infty}\frac{x^2}{-(1+2x^2)}=-\frac{1}{2}$.

综上，$y=xe^{x^2}\left(\int_0^x e^{-t^2}dt-\int_0^{+\infty}e^{-t^2}dt\right)$，$\lim\limits_{x\to+\infty}y=-\frac{1}{2}$.

7. 设 $f(x)+f'(x)=g(x)$，$|g(x)|\leq 1$

$(e^x f(x))'=e^x g(x)$

$$f(x)=\frac{\int_0^x e^t g(t)dt+C}{e^x}$$

由与P5相同的理由知 $C=-\int_0^{-\infty}e^t g(t)dt$

$$f(x)=\int_{-\infty}^x e^{t-x}g(t)dt$$

$$|f(x)|\leq\left|\int_{-\infty}^x e^{t-x}dt\right|=1\ .\ \#$$

习题2-4

1. (1) $(x^2+y^2+x)\,dx+y\,dy=0$

$$\frac{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}}{-Q}=\frac{2y}{y}=2 \quad \text{只与}x\text{有关}$$

$$e^{2x}(x^2+y^2+x)\,dx+e^{2x}y\,dy=0$$

$$\frac{\partial\phi}{\partial y}=e^{2x}y$$

$$\phi=\frac{1}{2}e^{2x}y^2+\theta(x)$$

$$\frac{\partial\phi}{\partial x}=e^{2x}y^2+\theta'(x)=e^{2x}(x^2+y^2+x)$$

$$\theta'(x)=e^{2x}(x^2+x)$$

$$\theta(x)=\frac{1}{2}x^2e^{2x}$$

$$\phi(x,y)=\frac{1}{2}e^{2x}(y^2+x^2)=C.$$

(3) $y\,dy=(x\,dy+y\,dx)\sqrt{1+y^2}$

$$dx\,(y\sqrt{1+y^2})+dy\,(x\sqrt{1+y^2}-y)=0$$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-\frac{y^2}{\sqrt{1+y^2}}$$

$$\frac{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}}{P}=-\frac{y}{1+y^2}$$

$$\mu(y)=-\frac{1}{2}\ln(1+y^2) \qquad e^{\mu(y)}=\frac{1}{\sqrt{1+y^2}}$$

$$dx\cdot y+dy\left(x-\frac{y}{\sqrt{1+y^2}}\right)=0$$

$$\frac{\partial\phi}{\partial x}=y \qquad \frac{\partial\phi}{\partial y}=x-\frac{y}{\sqrt{1+y^2}}$$

$$\phi=xy+\theta(y)$$

$$\frac{\partial\phi}{\partial y}=x+\theta'(y)=x-\frac{y}{\sqrt{1+y^2}}$$

$$\theta(y)=-\sqrt{1+y^2}$$

$$xy-\sqrt{1+y^2}=C.$$

(5) $y\,dx-x\,dy=2x^3\tan\frac{y}{x}\,dx$

$$d\left(\frac{y}{x}\right)=-2x\tan\frac{y}{x}\,dx$$

$$\frac{d\left(\frac{y}{x}\right)}{\tan\frac{y}{x}}=-2x\,dx$$

$$\ln\left|\sin\frac{y}{x}\right|=-x^2+C$$

得 $\left|\sin\frac{y}{x}\right|=e^{-x^2+C}$

或 $y=k\pi x,\ k\in\mathbb{Z}.$

(7) $(x^2-y^2+y)\,dx+x(2y-1)\,dy=0$

$$\frac{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}}{-Q}=-\frac{2}{x} \quad \text{只与}x\text{有关}$$

$$\frac{x^2-y^2+y}{x^2}\,dx+\frac{(2y-1)}{x}\,dy=0$$

$$\frac{\partial\phi}{\partial x}=\frac{x^2-y^2+y}{x^2} \qquad \frac{\partial\phi}{\partial y}=\frac{2y-1}{x}$$

$$\phi=\frac{y^2-y}{x}+\theta(x)$$

$$\frac{\partial\phi}{\partial x}=-\frac{y^2-y}{x^2}+\theta'(x)=\frac{-y^2+y+x^2}{x^2}$$

$$\theta'(x)=1 \qquad \theta(x)=x+C$$

$$\frac{y^2-y}{x}+x=C$$