1. 设 $u = \underset{v \in \mathcal{A}_g}{\arg\min} \iint_\Omega \sqrt{1+|Dv|^2}\, dxdy$

设 test function $\varphi \in C_0^1(\Omega)$

$$f(\varepsilon) = \iint_\Omega \sqrt{1+|D(u+\varepsilon\varphi)|^2}\, dxdy$$

有 $\left.\frac{df(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0$ 而 $\frac{df(\varepsilon)}{d\varepsilon} = \iint_\Omega \frac{\sum_{i=1}^{2} (u+\varepsilon\varphi)_{x_i} \varphi_{x_i}}{\sqrt{1+|D(u+\varepsilon\varphi)|^2}}\, dxdy$

故 $\iint_\Omega \frac{Du \cdot D\varphi}{\sqrt{1+|Du|^2}}\, dxdy = 0$ 由 Gauss - Green,

$$\iint_\Omega \frac{u_x\varphi_x + u_y\varphi_y}{\sqrt{1+u_x^2+u_y^2}}\, dxdy = 0$$

$$\iint_\Omega \left[\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_x^2+u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+u_x^2+u_y^2}}\right)\right]\varphi \, dxdy = 0$$

由 φ 任意性, $\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|Du|^2}}\right) = 0$. #

2. 设二次函数 $f(x) = x^TAx + b^Tx + c$, A 为 $n\times n$ 对称阵, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$.

$A = (a_{ij})_{n\times n}$

易知 $\frac{\partial^2 f}{\partial x_i^2} = 2a_{ii}$ 故 $\Delta f(x) = 0 \iff \sum_{i=1}^{n} a_{ii} = 0 \iff Tr(A) = 0$

故 $\{x^TAx + b^Tx + c \mid tr(A) = 0, A \in S^n, b \in \mathbb{R}^n, c \in \mathbb{R}\}$ 为所求线性空间.

#

4 考虑 $\phi(r) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r} u(x)\, dS(x) + \frac{1}{n(n-2)\alpha(n)} \int_{B_r} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f(x)\, dx$

只需证明 $\phi(r)$ 是常数

$$\frac{d\phi(r)}{dr} = \frac{-1}{n\alpha(n) r^{n-1}} \int_{B_r} f(x)\, dx + \frac{1}{n(n-2)\alpha(n)} \int_{\partial B_r} \frac{f(x)}{|x|^{n-2}}\, dS(x) - \frac{1}{n(n-2)\alpha(n)} \cdot$$

$$\left[\frac{-(n-2)}{r^{n-1}} \int_{B_r} f(x)\, dx + \frac{1}{r^{n-2}} \int_{\partial B_r} f(x)\, dS(x) \right]$$

$$= \frac{1}{n(n-2)\alpha(n) r^{n-2}} \int_{\partial B_r} f(x)\, dS(x) - \frac{1}{n(n-2)\alpha(n) r^{n-2}} \int_{\partial B_r} f(x)\, dS(x) = 0$$

故 $u(0) = \lim_{r \to 0^+} \phi(r) = \phi(R) = \frac{1}{n\alpha(n) R^{n-1}} \int_{\partial B_R} g(x)\, dS(x) + \frac{1}{n(n-2)\alpha(n)}$

$\int_{B_R} \left(\frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}} \right) f(x)\, dx$ #

6. (1) 考虑 $\phi(r) = ⨍_{\partial B(x,r)} v(y)\, dS(y)$

$$\phi'(r) = \frac{r}{n} ⨍_{B(x,r)} \Delta v(y)\, dy \geq 0$$

$$\phi(r) = ⨍_{B(x,r)} v(y) dy \geq \lim_{r \to 0^+} \phi(r) = v(x) \quad (\text{因 } v \in C^2(\Omega)).$$

(2) 与定理 2.5 (1) 证法完全相同 #

(3) 我们有 $\Delta \phi(u) = \phi'(u) \Delta u + \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 \phi''(u) \geq 0$ 故 $\phi(u)$ subharmonic

(4) $\Delta u = 0 \quad v = |Du|^2$

$$\Delta v = \sum_{i=1}^{n} v_{x_i x_i} = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} (u_{x_i x_j})^2 + \sum_{j=1}^{n} \sum_{i=1}^{n} u_{x_j} u_{x_j x_i x_i}$$

$$= 2 \sum_{i=1}^{n} \sum_{j=1}^{n} (u_{x_i x_j})^2 + \sum_{j=1}^{n} u_{x_j} (\Delta u)_{x_j} = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} (u_{x_i x_j})^2 \geq 0$$

#

7. 由于 $\{u_n\}$ 在 $\partial\Omega$ 上一致收敛，

$\forall \varepsilon>0\ \exists N, \forall n,m>N, \quad |u_n(x)-u_m(x)|<\varepsilon, \quad \forall x\in\partial\Omega$. 即 $\sup\limits_{\partial\Omega}|u_n-u_m|\leq\varepsilon$

考虑函数 $u_n(x)-u_m(x)$ 为调和函数

故 $\sup\limits_{\bar\Omega}|u_n(x)-u_m(x)| = \sup\limits_{\partial\Omega}|u_n(x)-u_m(x)|\leq\varepsilon$

即 $\forall x\in\bar\Omega, |u_n(x)-u_m(x)|\leq\varepsilon$ 　故 $\{u_n(x)\}$ 在 $\bar\Omega$ 一致收敛到 $u(x)$

下证 $u(x)$ 在 Ω 调和 　显然 $u(x)\in C(\Omega)$.

任取 $x\in\Omega$, $r>0$ s.t. $B(x,r)\subseteq\Omega$

$$u_n(x)=⨍_{\partial B(x,r)} u_n(y)\,dS(y)$$

令 $n\to\infty$ 注意到一致收敛性，有

$$u(x) = ⨍_{\partial B(x,r)} u(y)\,dS(y),$$ u 满足 mean value property

故 u 调和 #

11 当 $|x|=t$ 时，取 $r=1-t$，有

$$|u_{x_i}(x)| \leq \frac{n+1}{\alpha(n)r^{n+1}}\int_{B(x,r)}|u(y)|\,dy$$. 设 $|u|\leq M$ 在 $B(0,1)$ 上

$$\text{2)}\quad |u_{x_i}(x)| \leq \frac{n+1}{\alpha(n)r^{n+1}}\alpha(n)r^n M = \frac{n+1}{r}M$$

$$|Du| = \sqrt{\sum_{i=1}^n u_{x_i}^2(x)} \leq \frac{\sqrt{n}(n+1)M}{r}$$

$$(1-|x|)|Du| \leq (1-t)\cdot\frac{\sqrt{n}(n+1)M}{1-t} = \sqrt{n}(n+1)M$$

故 $\sup\limits_{x\in B}(1-|x|)|Du|<+\infty$. #

12 (1) 考虑 $\theta(x) = u(x) - \inf_{B(0,R)} u$ 在 $B(0,\frac{R}{2})$ 上调和函数，且 $\theta(x) \geq 0$

由 Harnack 不等式，$\exists C_1 > 0$

$$\sup_{B(0,\frac{R}{2})} \theta(x) \leq 3^n \inf_{B(0,\frac{R}{2})} \theta(x)$$

$$w(\tfrac{R}{2}) = \sup_{B(0,\frac{R}{2})} \theta(x) - \inf_{B(0,\frac{R}{2})} \theta(x) \leq \sup_{B(0,\frac{R}{2})} \theta(x) - \frac{1}{3^n} \sup_{B(0,\frac{R}{2})} \theta(x)$$

$$\leq (1-\frac{1}{3^n}) \sup_{B(0,R)} \theta(x) = (1-\frac{1}{3^n}) w(R)$$

(2) 设 $\frac{R_0}{2^i} \leq R < \frac{R_0}{2^{i-1}}$，有 $\frac{1}{2^i} \leq \frac{R}{R_0} < \frac{1}{2^{i-1}}$

$w(R) \leq \eta^{i-1} w(2^{i-1}R)$ 令 $\eta = \frac{1}{2^\alpha}$ 则 $\alpha > 0$

$$w(R) \leq \frac{1}{\eta} w(2^{i-1}R)\left(\frac{1}{2^i}\right)^\alpha \leq \frac{1}{\eta} w(2^{i-1}R)\left(\frac{R}{R_0}\right)^\alpha \leq \frac{2M_0}{\eta}\left(\frac{R}{R_0}\right)^\alpha$$

取 $C = \frac{2}{\eta}$ 则 $w(R) \leq CM_0\left(\frac{R}{R_0}\right)^\alpha < C(M_0+1)\left(\frac{R}{R_0}\right)^\alpha$ #

(只需证有 $\alpha \in (0,+\infty)$ 合题，因 $\frac{R}{R_0} \leq 1$).

13. $|u(x)| \leq C_1|x|^m + C_2$ 任取 $r > 0$，任取所有 $m+1$ 时 全指标 α

$$|D^\alpha u(x)| \leq \frac{C_{m+1}}{r^{n+m+1}} \int_{B(x,r)} |u(y)|\,dy$$

$$\leq \frac{C_{m+1}C_2}{r^{n+m+1}} V(B(x,r)) + \frac{C_{m+1}C_1}{r^{n+m+1}} \int_{B(x,r)} |y|^m\,dy$$

$$\leq \frac{C_{m+1}C_2\,\alpha(n)}{r^{m+1}} + \frac{C_{m+1}C_1}{r^{n+m+1}} V(B(x,r))\,(|x|+r)^m$$

$$= \frac{C_{m+1}C_2\,\alpha(n)}{r^{m+1}} + \frac{C_{m+1}C_1\,\alpha(n)\,(|x|+r)^m}{r^{m+1}}$$

令 $r \to +\infty$，有 $D^\alpha u(x) = 0$ 故 $D^\beta u(x) = 0$, $\forall |\beta| > |\alpha|$.

由于 u 调和 考虑 u 的 Taylor 展开

$$u(x) = \sum_{|\alpha|=0}^{\infty} \frac{D^\alpha u(0)}{\alpha!}(x-x_0)^\alpha = \sum_{|\alpha|=0}^{m} \frac{D^\alpha u(0)}{\alpha!}(x-x_0)^\alpha.$$

u 为至多 m 次调和多项式. #