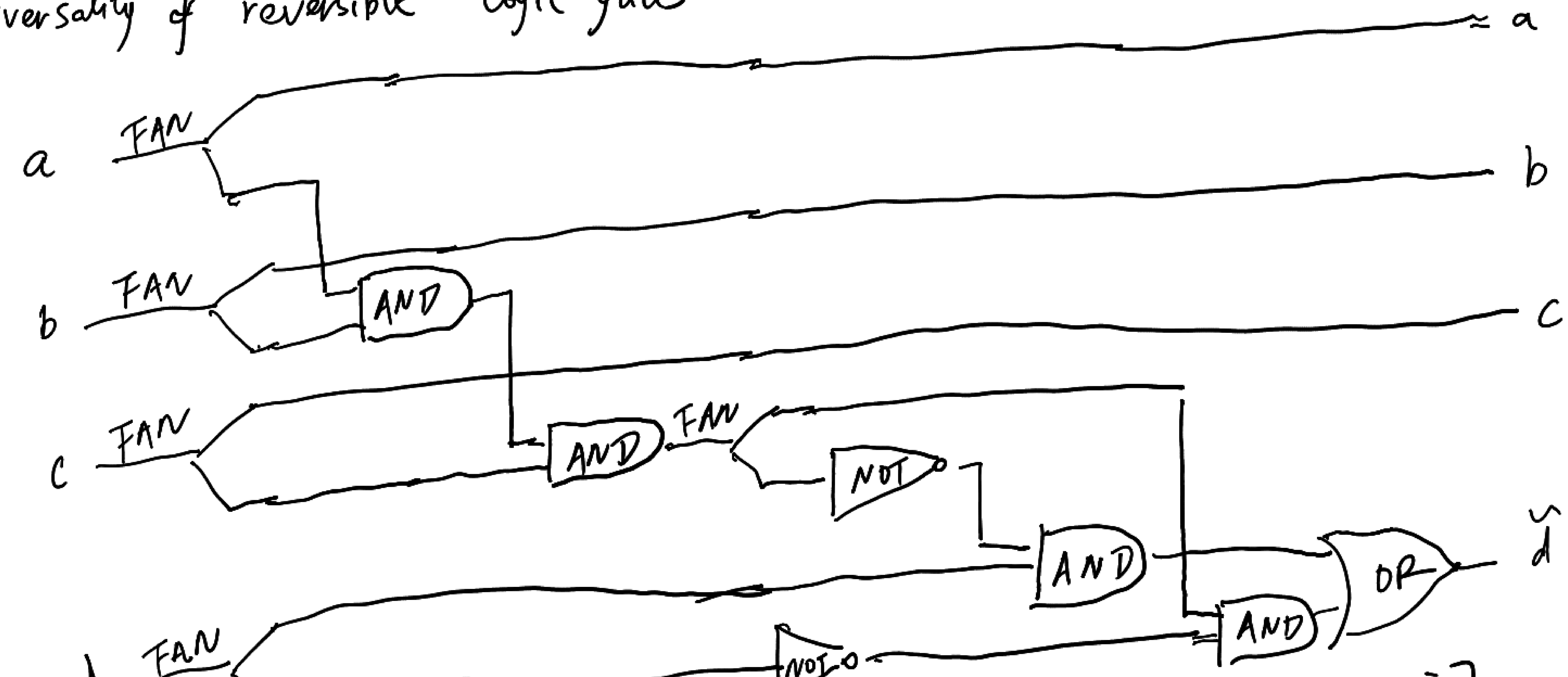
1. Universality of reversible Logic gates

(a)

CCCNOT

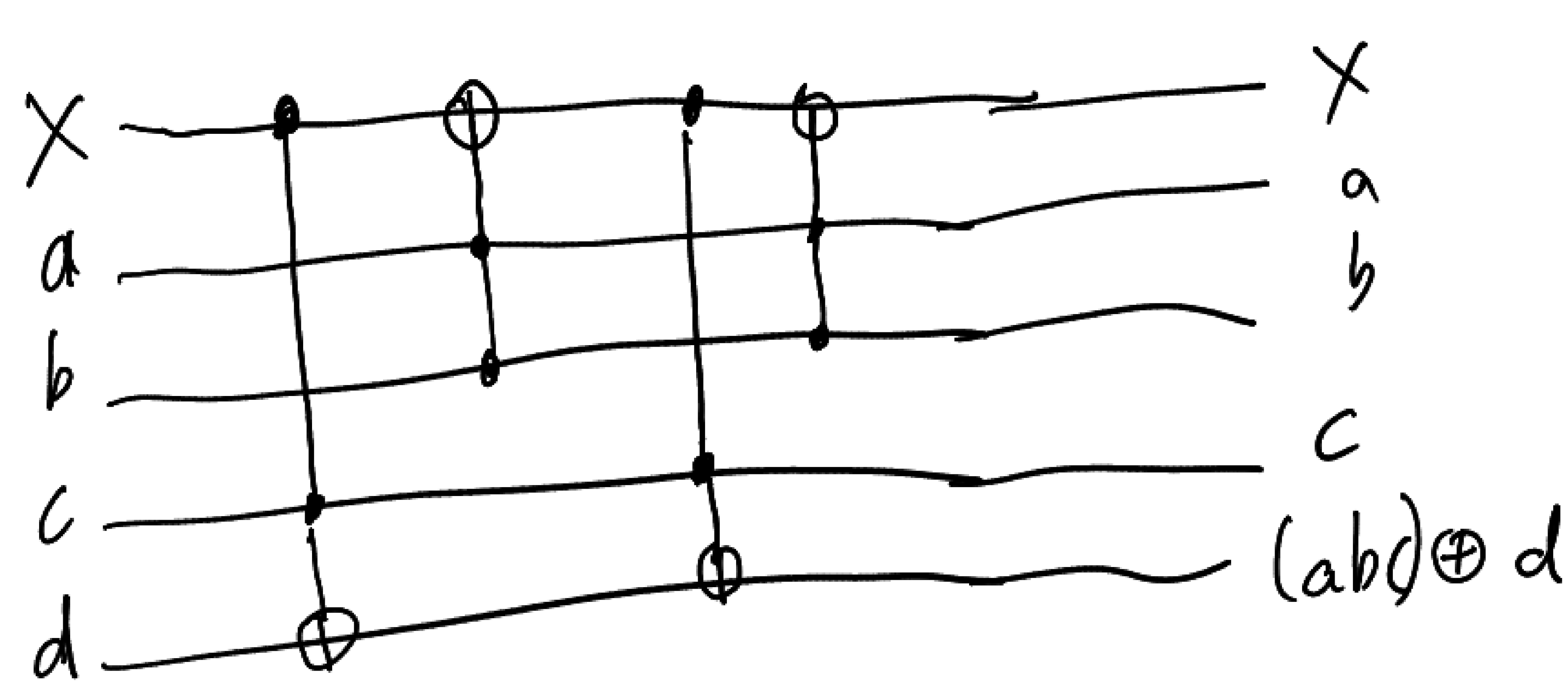


直接验证知这是 CCCNOT门 #

($\hat{d} = abc \oplus d$.

(b) Implement CCCNOT using Toffoli:

任取 $X \in \{0,1\}$,



直接验证即可 #

(c) Toffoli cannot be implemented by CNOT: 以下在 $\mathbb{F}_2$ 中考虑: 固定 $x_1, \cdots x_n \in \mathbb{F}_2$

若不然, 设初始有 $(a, b, c, x_1, \cdots, x_n)$ 经过某变换 T 变为 $(a, b, c+ab, x_1, \cdots x_n)$ 而 $T = T_1 \cdots T_m$ 任一步 T_i 将中间状态 S_{i-1} 中某两个元素 u, v 变为 $u, u+v$

因此 $T(a) = S_{00} a + S_{01} b + S_{02} c + S_{03} = a$, $S_{0i} \in \mathbb{F}_2$ (1)

$T(b) = S_{10} a + S_{11} b + S_{12} c + S_{13} = b$, $S_{1i} \in \mathbb{F}_2$ (2)

$T(c) = S_{20} a + S_{21} b + S_{22} c + S_{23} = c + ab$, $S_{2i} \in \mathbb{F}_2$ (3)

这意味着 $0 + 0 = 0 + S_{21} \cdot 1 = S_{20} \cdot 1 + 0$ ((3)中令 $ab = 0$)

$S_{22} c + S_{23} = c + ab$, $\forall a, b, c \in \mathbb{F}_2$ 这不可能 矛盾!

证毕 #

2 . Product and entangled states

(a) 设 $(a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle) = \frac{2}{3}|00\rangle - \frac{1}{3}|01\rangle + \frac{2}{3}|11\rangle$

$a, b, c, d \in \mathbb{C}$

有 $\begin{cases} ac = \frac{2}{3} \\ bc = 0 \\ ad = -\frac{1}{3} \\ bd = \frac{2}{3} \end{cases}$

但 $\frac{2}{3} = ac \Rightarrow c \neq 0$

$\frac{2}{3} = bd \Rightarrow b \neq 0$

而 $bc = 0$ 矛盾!

故 此为纠缠态

(b) 有 $\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\right) = \frac{1}{2}|00\rangle - \frac{i}{2}|01\rangle + \frac{i}{2}|10\rangle + \frac{1}{2}|11\rangle$

不是纠缠态

(c) 设 $(a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle) = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$

$a, b, c, d \in \mathbb{C}$

有 $\begin{cases} ac = \frac{1}{2} \\ bc = \frac{1}{2} \\ ad = -\frac{1}{2} \\ bd = \frac{1}{2} \end{cases}$

但 $abcd = ac \cdot bd = \frac{1}{4} = bc \cdot ad = -\frac{1}{4}$ 矛盾

故为纠缠态

3. Unitary operations and measurements

(a) $H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ $\quad I \otimes H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$$|\phi\rangle = (I \otimes H)|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} \end{pmatrix}$$

(b) $|\phi\rangle = \frac{\sqrt{2}}{6}|00\rangle + \frac{\sqrt{2}}{2}|01\rangle + \frac{\sqrt{2}}{3}|10\rangle - \frac{\sqrt{2}}{3}|11\rangle$

$= \frac{\sqrt{5}}{3}|0\rangle\left(\frac{1}{\sqrt{10}}|0\rangle + \frac{3}{\sqrt{10}}|1\rangle\right) + \frac{2}{3}|1\rangle\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$ normalized expression

故有 $\frac{5}{9}$ 概率 $|0\rangle$，系统变为 $|0\rangle\left(\frac{1}{\sqrt{10}}|0\rangle + \frac{3}{\sqrt{10}}|1\rangle\right)$，第2个qubit为 $\frac{1}{\sqrt{10}}|0\rangle + \frac{3}{\sqrt{10}}|1\rangle$

(c) $|\phi\rangle = \frac{\sqrt{10}}{6}\left(\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right)|0\rangle + \frac{\sqrt{26}}{6}\left(\frac{3}{\sqrt{13}}|0\rangle - \frac{2}{\sqrt{13}}|1\rangle\right)|1\rangle$

故有 $\frac{5}{18}$ 概率 $|0\rangle$，第1个qubit为 $\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$

(d). $|00\rangle$ 概率 $\frac{1}{18}$ $|01\rangle$ 概率 $\frac{1}{2}$ $|10\rangle$ 概率 $\frac{2}{9}$ $|11\rangle$ 概率 $\frac{2}{9}$

$P_r(\text{qubit } 1 = 0) = \frac{1}{18} + \frac{1}{2} = \frac{5}{9}$

$P_r(\text{qubit } 2 = 0) = \frac{2}{9} + \frac{1}{18} = \frac{5}{18}$ 与(b),(c)符合.

4 Distinguishing quantum States

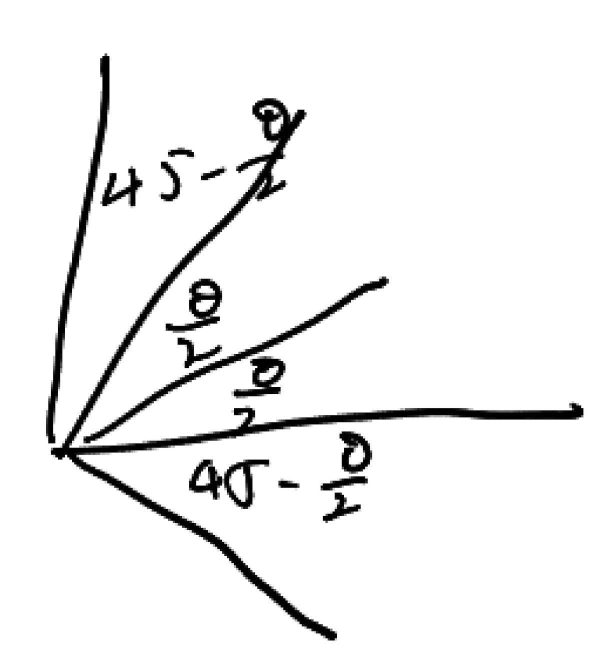
对 $|\psi_1\rangle = |0\rangle$ 及 $|\psi_2\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$,

设用正交基 $|u\rangle, |v\rangle$ 作测量，测得 $|u\rangle$ 认为是 $|\psi_1\rangle$，测得 $|v\rangle$ 认为是 $|\psi_2\rangle$

判断成功概率

由Bayes法则),

$P = \frac{1}{2}P(\text{success}|\psi=\psi_1) + \frac{1}{2}P(\text{success}|\psi=\psi_2) = \frac{1}{2}|\langle\psi_1|u\rangle|^2 + \frac{1}{2}|\langle\psi_2|v\rangle|^2$



设 $|u\rangle = \cos\frac{\alpha}{2}|0\rangle + e^{i\varphi_1}\sin\frac{\alpha}{2}|1\rangle$ $\varphi_1, \varphi_2 \in [0, 2\pi)$

$|v\rangle = \cos\frac{\beta}{2}|0\rangle + e^{i\varphi_2}\sin\frac{\beta}{2}|1\rangle$ $\alpha, \beta \in [0, \pi]$

$\langle u|v\rangle = 0 \Rightarrow \cos\frac{\alpha}{2}\cos\frac{\beta}{2} + e^{i(\varphi_1-\varphi_2)}\sin\frac{\alpha}{2}\sin\frac{\beta}{2} = 0$

$|u\rangle\langle u| + |v\rangle\langle v| = I_2 \Rightarrow \cos^2\frac{\alpha}{2} + \cos^2\frac{\beta}{2} = 1 \Rightarrow \alpha + \beta = \pi$

及 $e^{i\varphi_1} = -e^{i\varphi_2}$ 不妨设 $\varphi_2 = \varphi_1 + \pi$

$|v\rangle = \sin\frac{\alpha}{2}|0\rangle - e^{i\varphi_1}\cos\frac{\alpha}{2}|1\rangle$

$$2P = \cos^2\frac{\alpha}{2} + \|\cos\theta\sin\frac{\alpha}{2} - \sin\theta\cos\frac{\alpha}{2}e^{i\varphi_1}\|^2$$

$$= \cos^2\frac{\alpha}{2} + (\cos\theta\sin\frac{\alpha}{2} - \sin\theta\cos\frac{\alpha}{2}\cos\varphi_1)^2 + \sin^2\theta\cos^2\frac{\alpha}{2}\sin^2\varphi_1$$

$$= \cos^2\frac{\alpha}{2} + \cos^2\theta\sin^2\frac{\alpha}{2} + \sin^2\theta\cos^2\frac{\alpha}{2} - 2\cos\theta\sin\theta\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\cos\varphi_1$$

$$\overset{\varphi_1=\pi}{\leq} \cos^2\frac{\alpha}{2} + (\cos\theta\sin\frac{\alpha}{2} + \sin\theta\cos\frac{\alpha}{2})^2 = \cos^2\frac{\alpha}{2} + \sin^2(\theta+\frac{\alpha}{2})$$

$$= \frac{1+\cos\alpha}{2} + \frac{1-\cos(\alpha+2\theta)}{2} = 1 + \frac{\cos\alpha - \cos(\alpha+2\theta)}{2} = 1 + \sin(\alpha+\theta)\sin\theta$$

$$\overset{\alpha=\frac{\pi}{2}-\theta}{\leq} 1+\sin\theta$$

取等条件即 $|u\rangle = \cos(\frac{\pi}{4}-\frac{\theta}{2})|0\rangle - \sin(\frac{\pi}{4}-\frac{\theta}{2})|1\rangle$

$|v\rangle = \sin(\frac{\pi}{4}-\frac{\theta}{2})|0\rangle + \cos(\frac{\pi}{4}-\frac{\theta}{2})|1\rangle$

此时 $P = P_{max} = \frac{1+\sin\theta}{2}$

5 Teleporting through a Hadarmard gate

(a) 由于 $\begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 & 0 \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ 0 & 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt2} \\ 0 \\ 0 \\ \frac{1}{\sqrt2} \end{pmatrix} = \begin{pmatrix} \frac12 \\ \frac12 \\ \frac12 \\ -\frac12 \end{pmatrix}$

故 $(I\otimes H)|\beta_{00}\rangle = \frac12|00\rangle + \frac12|01\rangle + \frac12|10\rangle - \frac12|11\rangle$

(b) 设 $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$

$|\varphi\rangle = (I\otimes H)|\beta_{00}\rangle$

则 $|\psi\rangle|\varphi\rangle = (a_0|0\rangle + a_1|1\rangle)(\frac12|00\rangle + \frac12|01\rangle + \frac12|10\rangle - \frac12|11\rangle) = \frac{a_0}{2}|000\rangle + \frac{a_1}{2}|100\rangle + \frac{a_0}{2}|001\rangle + \frac{a_1}{2}|101\rangle + \frac{a_0}{2}|010\rangle + \frac{a_1}{2}|110\rangle - \frac{a_0}{2}|011\rangle - \frac{a_1}{2}|111\rangle$

$= \frac{a_0}{2\sqrt2}(|\beta_{00}\rangle + |\beta_{10}\rangle)|0\rangle + \frac{a_1}{2\sqrt2}(|\beta_{01}\rangle - |\beta_{11}\rangle)|0\rangle + \frac{a_0}{2\sqrt2}(|\beta_{00}\rangle + |\beta_{10}\rangle)|1\rangle + \frac{a_1}{2\sqrt2}(|\beta_{01}\rangle - |\beta_{11}\rangle)|1\rangle + \frac{a_0}{2\sqrt2}(|\beta_{01}\rangle + |\beta_{11}\rangle)|0\rangle$

$+ \frac{a_1}{2\sqrt2}(|\beta_{00}\rangle - |\beta_{10}\rangle)|0\rangle - \frac{a_0}{2\sqrt2}(|\beta_{01}\rangle + |\beta_{11}\rangle)|1\rangle - \frac{a_1}{2\sqrt2}(|\beta_{00}\rangle + |\beta_{10}\rangle)|1\rangle$

$= \frac12|\beta_{00}\rangle(\frac{a_0+a_1}{\sqrt2}|0\rangle + \frac{a_0-a_1}{\sqrt2}|1\rangle) + \frac12|\beta_{01}\rangle(\frac{a_0+a_1}{\sqrt2}|0\rangle + \frac{a_1-a_0}{\sqrt2}|1\rangle)$

$+ \frac12|\beta_{10}\rangle(\frac{a_0-a_1}{\sqrt2}|0\rangle + \frac{a_0+a_1}{\sqrt2}|1\rangle) + \frac12|\beta_{11}\rangle(\frac{-a_1+a_0}{\sqrt2}|0\rangle + \frac{-a_1-a_0}{\sqrt2}|1\rangle)$

由于 $|a_0+a_1|^2 + |a_0-a_1|^2 = 2(|a_0|^2+|a_1|^2) = 2$

故有 $\frac{1}{4}$ 概率，Bob 态 $\frac{(a_0+a_1)}{\sqrt{2}}|0\rangle + \frac{(a_0-a_1)}{\sqrt{2}}|1\rangle$，Alice 测量得 $|\beta_{00}\rangle$

$\frac{1}{4}$ 概率 Bob 态 $\frac{(a_0+a_1)}{\sqrt{2}}|0\rangle + \frac{(a_1-a_0)}{\sqrt{2}}|1\rangle$，Alice 测量得 $|\beta_{01}\rangle$

$\frac{1}{4}$ 概率 Bob 态 $\frac{(a_0-a_1)}{\sqrt{2}}|0\rangle + \frac{(a_0+a_1)}{\sqrt{2}}|1\rangle$，Alice 测量得 $|\beta_{10}\rangle$

$\frac{1}{4}$ 概率 Bob 态 $\frac{(a_1-a_0)}{\sqrt{2}}|0\rangle + \frac{(a_0+a_1)}{\sqrt{2}}|1\rangle$，Alice 测量得 $|\beta_{11}\rangle$

(c) $|\beta_{00}\rangle$：用 I 作用 即 $H|\psi\rangle$

$|\beta_{01}\rangle$：用 Z 作用 $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} \frac{a_0+a_1}{\sqrt{2}} \\ \frac{a_1-a_0}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{a_0+a_1}{\sqrt{2}} \\ \frac{a_0-a_1}{\sqrt{2}} \end{pmatrix} = H|\psi\rangle$

$|\beta_{10}\rangle$：用 X 作用 $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} \frac{a_0-a_1}{\sqrt{2}} \\ \frac{a_0+a_1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{a_0+a_1}{\sqrt{2}} \\ \frac{a_0-a_1}{\sqrt{2}} \end{pmatrix} = H|\psi\rangle$

$|\beta_{11}\rangle$：用 ZX 作用 $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} \frac{a_1-a_0}{\sqrt{2}} \\ \frac{a_0+a_1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{a_0+a_1}{\sqrt{2}} \\ \frac{a_0-a_1}{\sqrt{2}} \end{pmatrix} = H|\psi\rangle$

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