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APDE Fourier Transform (I)

1. f 2π periodic $f \in R[-\pi, \pi]$

(a) $\hat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\frac{\pi}{n})\, e^{-inx}\, dx$

$$\hat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x) - f(x+\tfrac{\pi}{n})]\, e^{-inx}\, dx$$

By definition, $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\; e^{-in(x-\frac{\pi}{n})}\, dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\tfrac{\pi}{n})\, e^{-inx}\, dx \quad \text{(used periodicity of } f\text{)}$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x) - f(x+\tfrac{\pi}{n})]\, e^{-inx}\, dx \quad \#$$

(b) $|f(x+h) - f(x)| \le C|h|^{\alpha}$, $0 < \alpha \le 1$, $C > 0$, $x, h \in \mathbb{R}$

Show $\hat{f}(n) = O(1/|n|^{\alpha})$

Using (a), $|\hat{f}(n)| \le \frac{1}{4\pi}\int_{-\pi}^{\pi} |f(x) - f(x+\frac{\pi}{n})|\, dx$

$$\le \frac{1}{4\pi}\int_{-\pi}^{\pi} C\left(\frac{\pi}{|n|}\right)^{\alpha} dx = \frac{C\pi^{\alpha}}{2}\frac{1}{|n|^{\alpha}} \quad \#$$

(c) $f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x}$, $0 < \alpha < 1$,

Prove (i) $|f(x+h) - f(x)| \le C|h|^{\alpha}$

(ii) $\hat{f}(N) = \frac{1}{N^{\alpha}}$, $N = 2^k$

(i) $$|f(x+h)-f(x)| = \left|\sum_{k=0}^{\infty} \frac{e^{i2^k(x+h)} - e^{i2^k x}}{2^{k\alpha}}\right| = \left|\sum_{k=0}^{\infty} \frac{e^{i2^k x}(e^{i2^k h}-1)}{2^{k\alpha}}\right|$$

$$= 2\left|\sum_{k=0}^{\infty} \frac{e^{i2^k x}\, e^{i2^{k-1}h} \sin 2^{k-1}h}{2^{k\alpha}}\right| \le 2\sum_{k=0}^{\infty} \frac{|\sin 2^{k-1}h|}{2^{k\alpha}} \le \sum_{k=0}^{\infty} \frac{2^k\,|h|}{2^{k\alpha}}$$

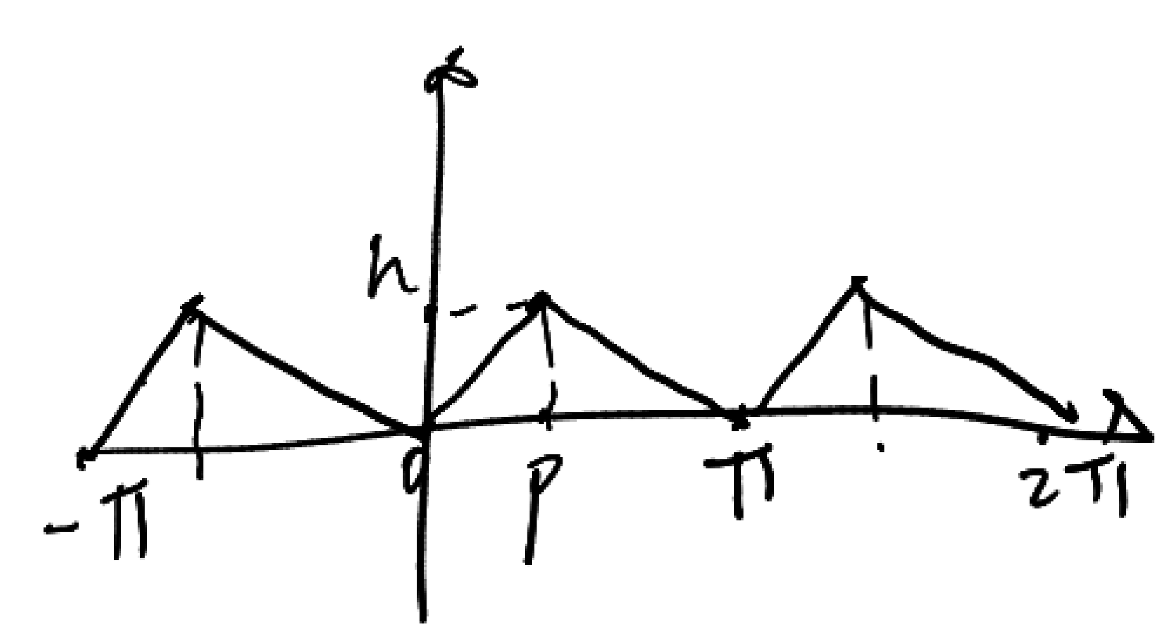
$$= \frac{|h|}{1-2^{1-\alpha}} \le C|h|^{\alpha}$$

(ii) $$\hat f(2^k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-i2^k x}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m=0}^{\infty} \frac{e^{i2^m x}}{2^{m\alpha}}\, e^{-i2^k x}\,dx$$

$$= \frac{1}{2\pi}\sum_{m=0}^{\infty} \frac{1}{2^{m\alpha}}\int_{-\pi}^{\pi} e^{i(2^m-2^k)x}\,dx = \frac{1}{2\pi}\,\frac{1}{2^{k\alpha}}\int_{-\pi}^{\pi} 1\,dx = \frac{1}{2^{k\alpha}} = \frac{1}{N^{\alpha}}$$

2. $f: (0,\pi) \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{h}{p}x & x\in[0,p] \\ \frac{h(\pi-x)}{\pi-p} & x\in[p,\pi] \end{cases}$$



$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{h}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(2nx)\,dx$$

$$= \frac{2}{\pi}\left(\int_0^{p} \frac{h}{p}x\cos 2nx\,dx + \int_p^{\pi} \frac{h(\pi-x)}{\pi-p}\cos 2nx\,dx\right)$$

$$= \frac{2h}{\pi p}\int_0^{p} x\cos 2nx\,dx + \int_0^{\pi-p} \frac{2h\theta}{\pi(\pi-p)}\cos 2n\theta\,d\theta$$

$$= \frac{2h}{\pi p}\,\frac{1}{2n}\left(x\sin 2nx\Big|_0^p - \int_0^p \sin 2nx\,dx\right)$$

$$+ \frac{2h}{\pi(\pi-p)}\,\frac{1}{2n}\left(x\sin 2nx\Big|_0^{\pi-p} - \int_0^{\pi-p} \sin 2nx\,dx\right)$$

$$= \frac{h}{n\pi p}\left(p\sin 2np - \frac{1}{n}\sin^2 np\right) + \frac{h}{n\pi(\pi-p)}\left((p-\pi)\sin 2np - \frac{1}{n}\sin^2 np\right)$$

$$= \frac{h}{n^2 p(\pi-p)}\,\frac{\cos 2np - 1}{2}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin 2nx\,dx$$

$$= \frac{2}{\pi}\left(\int_0^{p} \frac{h}{p}x \sin 2nx\,dx + \int_p^{\pi} \frac{h(\pi-x)}{\pi-p}\sin 2nx\,dx\right)$$

$$= \frac{2h}{\pi p}\int_0^{p} x\sin 2nx\,dx - \frac{2h}{\pi(\pi-p)}\int_0^{\pi-p}\theta \sin 2n\theta\,d\theta$$

$$= \frac{2h}{\pi p}\left(-\frac{1}{2n}\right)\left(x\cos 2nx\Big|_0^{p} - \frac{\sin 2nx}{2n}\Big|_0^{p}\right)$$

$$- \frac{2h}{\pi(\pi-p)}\left(-\frac{1}{2n}\right)\left[x\cos 2nx\Big|_0^{\pi-p} - \frac{\sin 2nx}{2n}\Big|_0^{\pi-p}\right)$$

$$= -\frac{h}{n\pi p}\left(p\cos 2np - \frac{\sin 2np}{2n}\right) + \frac{h}{n\pi(\pi-p)}\left((\pi-p)\cos 2np + \frac{\sin 2np}{2n}\right)$$

$$= \frac{h\sin 2np}{2n^2 p(\pi-p)}$$

$$f(x) = \frac{h}{4} + \sum_{n=1}^{\infty}\left(\frac{h}{2n^2 p(\pi-p)}(\cos 2np - 1)\right)\cos 2nx + \sum_{n=1}^{\infty}\frac{h\sin 2np}{2n^2 p(\pi-p)}\sin 2nx$$

$$= \frac{h}{4} + \frac{h}{2p(\pi-p)}\sum_{n=1}^{\infty}\left(\frac{\cos 2(n-p)x}{n^2} - \frac{\cos 2nx}{n^2}\right)$$

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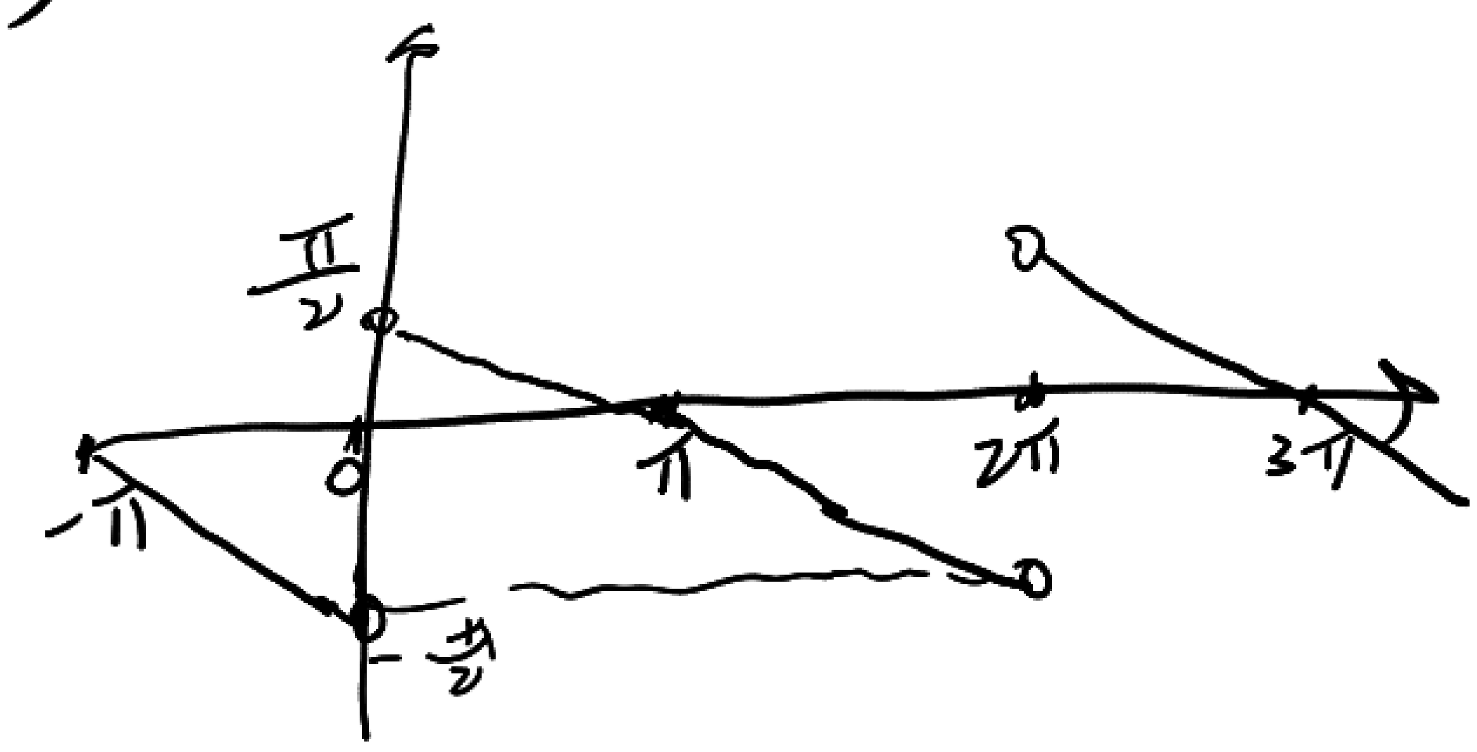
3. $f(x) = \frac{\pi - x}{2}$, $x \in (0, \pi)$, $f(0) = 0$, $f(x+2\pi) = f(x)$

(a) Fourier Series of f

(b) $\max_{[0,\frac{\pi}{N}]} S_N(f)(x) - \frac{\pi}{2} = \int_0^{\pi} \frac{\sin t}{t}\,dt - \frac{\pi}{2}$, $N \to \infty$

(c) Draw conclusion from (a) and (b)



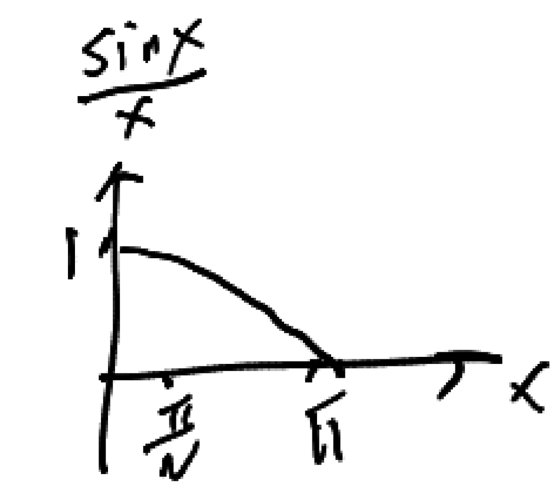
(a) $a_n = 0$ ($f(x)\cos 2nx$ is odd)

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi}{2}\sin nx - \frac{x}{2}\sin nx\right) dx$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} x \sin nx\,dx$$

$$= \frac{1}{2\pi n}\int_0^{2\pi} x\, d\cos nx$$

$$= \frac{1}{n}$$



f has finite discontinuoties $f(0) = \frac{f(0+)+f(0-)}{2}$

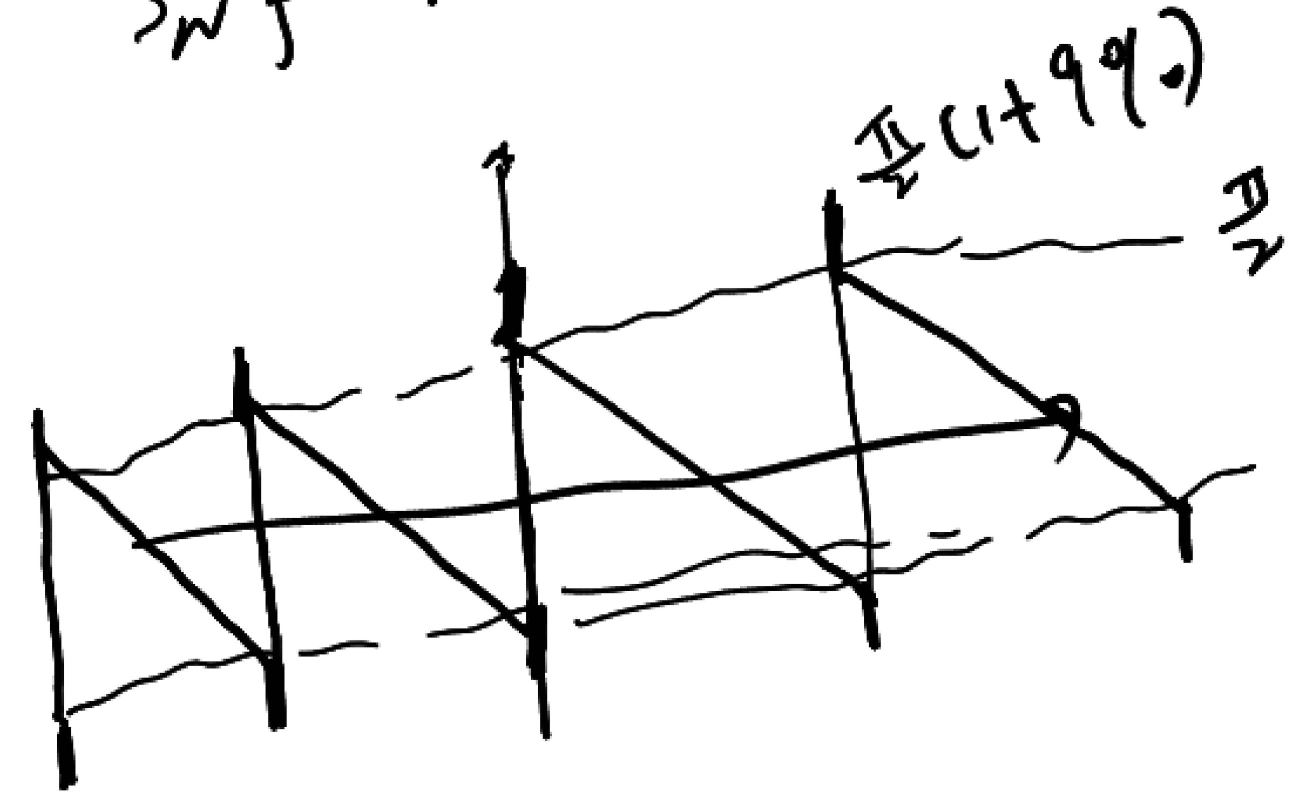
$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

(b) $\lim_{N\to\infty} \max_{[0,\frac{\pi}{N}]} S_N(f)(x) = \lim_{N\to\infty} \max_{[0,\frac{\pi}{N}]} \sum_{k=1}^{N} \frac{\sin kx}{k} = \lim_{N\to\infty} \sum_{k=1}^{N} \frac{N}{k\pi} \sin\frac{k\pi}{N} \cdot \frac{\pi}{N}$

$= \int_0^{\pi} \frac{\sin t}{t}\,dt$ (Riemann sum)

(c) When N is large, $\max_{[0,\frac{\pi}{N}]} S_N(f)(x) - \frac{\pi}{2}$ is $9\%\,\pi$

This means $S_N f$ has a 9% jump around discontinuous point of f

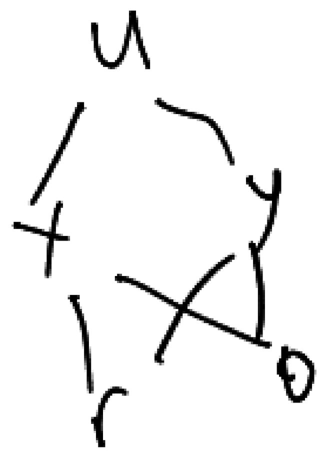


4. (a) $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$

(b) $\left|\frac{\partial u}{\partial x}\right|^2+\left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2+\frac{1}{r^2}\left|\frac{\partial u}{\partial\theta}\right|^2$

(a) $x = r\cos\theta$ $\quad \frac{\partial}{\partial r} = \frac{\partial}{\partial x}\cos\theta + \frac{\partial}{\partial y}\sin\theta$

$y = r\sin\theta$ $\quad \frac{\partial}{\partial \theta} = -\frac{\partial}{\partial x} r\sin\theta + \frac{\partial}{\partial y} r\cos\theta$



$$\frac{\partial^2}{\partial r^2} = \cos\theta\frac{\partial}{\partial r}\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial r}\frac{\partial}{\partial y}$$
$$= \cos\theta\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\cos\theta + \frac{\partial}{\partial y}\sin\theta\right)$$
$$+\sin\theta\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\cos\theta + \frac{\partial}{\partial y}\sin\theta\right)$$
$$= \frac{\partial^2}{\partial x^2}\cos^2\theta + \frac{\partial^2}{\partial y^2}\sin^2\theta + 2\frac{\partial^2}{\partial x\partial y}\sin\theta\cos\theta$$

$$\frac{\partial^2}{\partial \theta^2} = -r\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial x}\right) + r\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial y}\cos\theta\right)$$
$$= -r\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial x}\left(-\frac{\partial}{\partial x}r\sin\theta + \frac{\partial}{\partial y}r\sin\theta\right)\right)$$
$$+r\left(-\sin\theta\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial y}\left(-\frac{\partial}{\partial x}r\sin\theta + \frac{\partial}{\partial y}r\cos\theta\right)\right)$$
$$= -r\cos\theta\frac{\partial}{\partial x} - r\sin\theta\frac{\partial}{\partial y} + r^2\sin^2\theta\frac{\partial^2}{\partial x^2} + r^2\cos^2\theta\frac{\partial}{\partial y}$$
$$-2r^2\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y}$$

$$\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} = \frac{\partial^2}{\partial x^2}\cos^2\theta + \frac{\partial^2}{\partial y^2}\sin^2\theta + \frac{\partial^2}{\partial x\partial y}\sin2\theta$$
$$+\frac{\partial}{\partial x}\frac{\cos\theta}{r} + \frac{\partial}{\partial y}\frac{\sin\theta}{r} + \sin^2\theta\frac{\partial^2}{\partial x^2} + \cos^2\theta\frac{\partial}{\partial y}$$
$$-\sin2\theta\frac{\partial^2}{\partial x\partial y} - \frac{\cos\theta}{r}\frac{\partial}{\partial x} - \frac{\sin\theta}{r}\frac{\partial}{\partial y} = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

(b) $\left|\frac{\partial u}{\partial r}\right|^2+\frac{1}{r^2}\left|\frac{\partial u}{\partial\theta}\right|^2 = \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\left(-\frac{\partial u}{\partial x}r\sin\theta + \frac{\partial u}{\partial y}r\cos\theta\right)^2$

$$= \left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin2\theta + \left(\frac{\partial u}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\cos^2\theta$$
$$-\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin2\theta$$
$$= \left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2$$

Solve BVP

5. $$\begin{cases} \partial_t u + t\,\partial_x u = 0, & x \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases}$$

We have

$$\begin{cases} \frac{d}{dt}\hat{u}(\xi,t) + t \cdot 2\pi i \xi\, \hat{u}(\xi,t) = 0 \\ \hat{u}(\xi,0) = \hat{u}_0(\xi) \end{cases}$$

$$\ln \hat{u} - \ln \hat{u}(0) = -\pi i \xi t^2$$

$$\hat{u} = \hat{u}(0)\, e^{-\pi i \xi t^2}$$

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)\, e^{-\pi i \xi t^2}$$

$$u(x,t) = (\hat{u})^\vee = (\hat{u}_0(\xi)\, e^{-\pi i \xi t^2})^\vee$$

$$= u_0 * g(x)$$

where $\hat{g} = e^{-\pi i \xi t^2}$

$$g(x,t) = \int_{\mathbb{R}} e^{-\pi i \xi t^2} e^{2\pi i x \xi}\, d\xi$$

$$= \delta\left(x - \frac{t^2}{2}\right)$$

$$u(x,t) = \int_{\mathbb{R}} u_0(x-s)\, \delta\left(s - \frac{t^2}{2}\right) ds = u_0\left(x - \frac{t^2}{2}\right).$$