8.1 (3) $h(x)=\frac{1}{2}x^TAx+b^Tx+c$, $\operatorname{prox}_{th}(x)=(I+tA)^{-1}(x-tb)$

证明: 令邻近算子 $u=\operatorname{prox}_{th}(x)$ 最优性条件是

$$x-u\in t\partial(\frac{1}{2}u^TAu+b^Tu+c)$$

$$=t(Au+b)$$

$\Leftrightarrow$ $u(I+tA)=x-tb$

又 $A\succeq 0$ $t>0$ 有 $I+tA\succ 0$

$u=(I+tA)^{-1}(x-tb)$ #

(4) $h(x)=-\sum_{i=1}^{n}\ln x_i$, $\operatorname{prox}_{th}(x)_i=\frac{x_i+\sqrt{x_i^2+4t}}{2}$, $i=1,\cdots n$

证明: $u=\operatorname{prox}_{th}(x)$ 最优性条件是

$$x-u\in t\partial(-\sum_{i=1}^{n}\ln u_i)$$

$$=t(-\frac{1}{u_1},\cdots-\frac{1}{u_n})$$

$\Leftrightarrow$ $x_i-u_i=-\frac{t}{u_i}$, $i=1,\cdots n$ 且 $u_i>0$

$\Leftrightarrow$ $u_i^2-x_iu_i-t=0$, $i=1,\cdots n$ 且 $u_i>0$

$\Leftrightarrow$ $u_i=\frac{x_i+\sqrt{x_i^2+4t}}{2}$, $i=1,\cdots n$ (因为 $t>0$) #

8.3

(a) $f(x)=I_C(x)$, $\quad C=\{(x,t)\in\mathbb{R}^{n+1} \mid \|x\|_2\le t\}$

由定义，$\text{Prox}_f(x) = \arg\min_u \left(I_C(u)+\frac{1}{2}\|u-x\|^2\right)$

$= \arg\min_{u\in C}\|u-x\|^2$

$:= P_C(x) = P_{\|x\|_2\le t}(x)$ #

(b) $f(x)=\inf_{y\in C}\|x-y\|$，$C$闭凸集

由定义，$\text{Prox}_f(x) = \arg\min_u \left(d(u,C)+\frac{1}{2}\|u-x\|^2\right) := u_0$

记 $y_0=P_C(u)$，$d(u,C)+\frac{1}{2}\|u-x\|^2 = \|u-y_0\|+\frac{1}{2}\|u-x\|^2$

记 $u=y_0+tv$，$v=\frac{u-y_0}{\|u-y_0\|}$，$t\geq 0$

$$d(u,C)+\frac{1}{2}\|u-x\|^2 = t+\frac{1}{2}\|y_0-x+tv\|^2$$

$$=t+\frac{1}{2}\left(\|y_0-x\|^2+t^2+2t\,v^T(y_0-x)\right)$$

上式关于t为二次函数，对称轴

若 $\|x-y_0\|\le 1$，$\quad -1+v^T(x-y_0) \le -1+\|v\|\|x-y_0\| \le 0$

为 $t=-(1+v^T(y_0-x))$

故 $d(u,C)\geq\frac{1}{2}\|x-P_C(x)\|^2$ 取等当且仅当 $u=P_C(x)$. 此时 $\|x-P_C(x)\|\le 1$

若 $\|x-y_0\|>1$，$\quad d(u,C)\geq t+\frac{1}{2}\|y_0-x\|^2+\frac{1}{2}t^2-t\|y_0-x\|$

$$\geq \frac{\|y_0-x\|^2-(1-\|y_0-x\|)^2}{2} = -\frac{1}{2}+\|y_0-x\| \geq \|x-P_C(x)\|-\frac{1}{2}$$

当且仅当 $t=\|y_0-x\|-1$，$P_C(x)=P_C(u)$ 成立，即

$$u=x+\frac{P_C(x)-x}{\|P_C(x)-x\|} \quad \text{此时 } \|x-P_C(x)\|>1$$

综上，$\text{Prox}_f(x)=x+\min\left\{\frac{1}{d_C(x)},1\right\}(P_C(x)-x)$ #

(c) $f(x) = \frac{1}{2}\left(\inf_{y\in C} \|x-y\|\right)^2$. C 闭凸集

由定义

$u = \mathrm{prox}_f(x) = \arg\min_u \frac{1}{2} d^2(u, C) + \frac{1}{2} d^2(u, x)$

设 $y_0 = P_C(u)$

$\frac{1}{2}\left(d^2(u, C) + d^2(u, x)\right) = \frac{1}{2}\left(\|u-x\|^2 + \|u-y_0\|^2\right)$

$= \frac{1}{2}\left(2\|u - \frac{x+y_0}{2}\|^2 + \frac{1}{2}\|x-y_0\|^2\right)$

$\geq \frac{1}{4}\|x-y_0\|^2 \geq \frac{1}{4}\|x - P_C(x)\|^2$

且当 $u = \frac{1}{2}(x + P_C(x))$ 时

$P_C(u) = P_C(x)$，两个不等号均成立

故 $u = \frac{1}{2}(x + P_C(x)) = \mathrm{Prox}_f(x)$. #