1. $m\ddot{x}(t) + \gamma\dot{x}(t) + kx(t) = q\cos(wt)$

$m, \gamma, k, q > 0$

(a) $\gamma = 0$

Homogeneous solution: $x(t) = C_1 \cos\sqrt{\frac{k}{m}}\,t + C_2 \sin\sqrt{\frac{k}{m}}\,t$

Special solution: let $x(t) = A\cos(wt)$

$m\ddot{x} + kx = -mAw^2\cos wt + kA\cos wt = q\cos wt \Rightarrow A = \frac{q}{k - mw^2}$ if $k \neq mw^2$

So if $k \neq mw^2$ $x(t) = C_1\cos\sqrt{\frac{k}{m}}\,t + C_2\sin\sqrt{\frac{k}{m}}\,t + \frac{q}{k-mw^2}\cos(wt)$

if $k = mw^2$ $x(t) = tA\sin(wt)$

$m\ddot{x} + kx = \quad mwA\cos wt - tAmw^2\sin wt + tkA\sin wt$

$A = \frac{q}{mw}$

$x(t) = C_1\cos\sqrt{\frac{k}{m}}\,t + C_2\sin\sqrt{\frac{k}{m}}\,t + \frac{q}{mw}(\sin wt)t$

(b) $\gamma \neq 0,\ q = 0$

$m\ddot{x}(t) + \gamma\dot{x}(t) + kx(t) = 0$

Case 1. $\gamma^2 > 4mk$ $\quad x(t) = C_1 e^{\frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m}t} + C_2 e^{\frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m}t}$

Case 2. $\gamma^2 = 4mk$ $\quad x(t) = (C_1 + C_2 t)\, e^{\frac{-\gamma}{2m}t}$

Case 3 $\gamma^2 < 4mk$ $\quad x(t) = C_1 e^{-\frac{\gamma}{2m}t}\cos\frac{\sqrt{4mk-\gamma^2}}{2m}t + C_2 e^{-\frac{\gamma}{2m}t}\sin\frac{\sqrt{4mk-\gamma^2}}{2m}t$

(c) $\gamma, q \neq 0$

Let $x(t) = A\cos wt + B\sin wt$

The result is $x(t)$ in (b) plus

$x^*(t) = \cos(wt)\frac{q(k-mw^2)}{(k-mw^2)^2 + \gamma^2 w^2} + \sin(wt)\frac{q\gamma w}{(k-mw^2)^2 + \gamma^2 w^2}$

#

2. (1) $x^{(4)}+4x=t^2e^t\cos t$

homogeneous solution: $\lambda^4+4=0 \quad \lambda=\pm1\pm i$

$x_0(t)=C_1e^t\cos t+C_2e^{-t}\cos t+C_3e^t\sin t+C_4e^{-t}\sin t$

Special Solution:

Let $x(t)=t[(a_1+a_2t+a_3t^2)\cos t+(a_4+a_5t+a_6t^2)\sin t]e^t$ ($1\pm i$ is root of characteristic polynomial)

After computation, we get

$x^{(4)}+4x=e^t\sin t(-8a_1-24a_2-24a_3-8a_4+24a_6)$
$+e^t\cos t(-8a_1+24a_3+8a_4+24a_5+24a_6)$
$+e^t t\sin t(-16a_2-72a_3-16a_5)+e^t t\cos t(-16a_2+16a_5+72a_6)$
$+e^t t^2\sin t(-24a_3-24a_6)+e^t t^2\cos t(24a_6+24a_3)$

$a_1=-\frac{5}{64} \quad a_2=0 \quad a_3=\frac{1}{48} \quad a_4=-\frac{5}{64} \quad a_5=\frac{3}{32} \quad a_6=-\frac{1}{48}$

$x(t)=(C_1\sin t+C_2\cos t)e^{-t}+(C_3+t^3/48-5t/64)\sin t\,e^t+(C_4-t^3/48+3t^2/32-5t/64)\cos t\,e^t$

(2)(a) $\ddot{x}+x=t^3\sin t$

homogeneous solution

$x_0(t)=C_1\cos t+C_2\sin t$

Special Solution:

Let $x(t)=t[(a_1+a_2t+a_3t^2+a_4t^3)\sin t+(a_5+a_6t+a_7t^2+a_8t^3)\cos t]$

$x''(t)+x(t)=\cos t(2a_1+2a_6)+\sin t(2a_2-2a_5)+t\cos t(4a_2+6a_7)$
$+t\sin t(6a_3+4a_6)+t^2\cos t(6a_3+12a_8)+t^2\sin t(12a_4-6a_7)$
$+t^3\cos t(8a_4)+t^3\sin t(-8a_3)$

$\Rightarrow x(t)=C_1\cos t+C_2\sin t+(-\frac{3t}{8}+\frac{t^3}{4})\sin t+(\frac{3t^2}{8}-\frac{t^4}{8})\cos t$

(b) $\ddot{x}-x=t^3\sin t$

homogeneous solution

$x(t)=C_1e^t+C_2e^{-t}$

Special solution

Let $x(t)=[(a_1+a_2t+a_3t^2+a_4t^3)\sin t+(a_5+a_6t+a_7t^2+a_8t^3)\cos t]$

$$x''(t) - x(t) = \sin t(-2a_1 + 2a_3 - 2a_6) + \cos t(2a_2 - 2a_5 + 2a_7)$$
$$+ t\sin t(-2a_2 + 6a_4 - 4a_7) + t\cos t(4a_3 - 2a_6 + 6a_8)$$
$$+ t^2\sin t(-2a_3 + 6a_8) + t^2\cos t(6a_4 - 2a_7) + t^3\sin t(-2a_4) + t^3\cos t(-2a_8) = t^3\sin t$$
$$\Rightarrow x(t) = C_1e^t + C_2e^{-t} + \sin t\left(\frac{3}{2}t - \frac{1}{2}t^3\right) + \cos t\left(-\frac{3}{2}t^2\right)$$

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3. Picard Iteration

$\ddot{x} + x = 0$, $x(0) = 1$, $\dot{x}(0) = 0$

Let $y = \dot{x}$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}, \quad \begin{pmatrix}x(0)\\ y(0)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$$\phi_0(t) = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$$\phi_1(t) = \begin{pmatrix}1\\ 0\end{pmatrix} + \int_0^t \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} ds = \begin{pmatrix}1\\ -t\end{pmatrix}$$

$$\phi_2(t) = \begin{pmatrix}1\\ 0\end{pmatrix} + \int_0^t \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1\\ -s\end{pmatrix} ds = \begin{pmatrix}1 - \frac{t^2}{2}\\ -t\end{pmatrix}$$

$$\phi_3(t) = \begin{pmatrix}1\\ 0\end{pmatrix} + \int_0^t \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 - \frac{s^2}{2}\\ -s\end{pmatrix} ds = \begin{pmatrix}1 - \frac{t^2}{2}\\ -t + \frac{t^3}{3!}\end{pmatrix}$$

$$\phi_{2k}(t) = \begin{pmatrix}1 - \frac{t^2}{2} + \cdots + (-1)^k\frac{t^{2k}}{(2k)!}\\ -t + \frac{t^3}{3!} + \cdots + (-1)^k\frac{t^{2k-1}}{(2k-1)!}\end{pmatrix},$$

$$\phi_{2k+1}(t) = \begin{pmatrix}1 - \frac{t^2}{2} + \cdots + (-1)^k\frac{t^{2k}}{(2k)!}\\ -t + \frac{t^3}{3!} + \cdots + (-1)^{k+1}\frac{t^{2k+1}}{(2k+1)!}\end{pmatrix}$$

(easy by induction)

finally, $\phi_k(t) \to \begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}$ as $k \to \infty$

4. $\begin{cases} \dot{x}(t) = f(x) \\ x|_{t=0} = x_0 \end{cases}$

f is locally Lipschitz $(0, T)$ maximal interval of existence

$T < \infty$ $\lim\limits_{t \to T^-} |x(t)| = \infty$

Proof.

We claim that as $t \to T^-$ either one of three conditions hold:

(i) $\lim\limits_{t \to T^-} x(t) = c$

(ii) $\lim\limits_{t \to T^-} x(t) = +\infty$

(iii) $\lim\limits_{t \to T^-} x(t) = -\infty$

If neither holds,

$\exists\ x_{n_k} \to T^-$, $f(x_{n_k}) \to a$

$x_{n_t} \to T^-$, $f(x_{n_t}) \to b$, $a < b$ (mathematical analysis)

if $a = -\infty$ or $b = +\infty$, noticing continuity of f. we get another finite limit point

WLOG, $|a|, |b| < \infty$

Let $|f| \le M$, when $x \in [0, T)$, $y \in [a-1, b+1]$

$0 < \varepsilon < \frac{b-a}{4}$ $|f(x_{n_k}) - a| < \varepsilon$, $|f(x_{n_t}) - b| < \varepsilon$ $|n_k - T| < \delta$, $|n_t - T| < \delta$

But $|f(x_{n_k}) - f(x_{n_t})| \le M |x_{n_k} - x_{n_t}| \le M\delta < \frac{\varepsilon}{4}$

Contradiction.

If (i) holds, $(0, T)$ is not the maximal existence interval

So either (ii) or (iii) holds

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