7.1. 考虑 $\min_x \ x^3$

$s.t. \ x=0$

该问题有局部极小值 $x=0$

二次罚函数 $x^3+\frac{1}{2}\sigma x^2$ 无界

7.2 罚函数 $P_E(x,\sigma_k) = -x_1x_2x_3 + \frac{1}{2}\sigma_k(x_1+2x_2+3x_3-60)^2$

给定 σ_k 求 x^{k+1}

该问题光滑. 没有下界. 局部极小必要条件

$$\begin{cases} \frac{\partial P_E(x,\sigma_k)}{\partial x_1} = -x_2x_3 + \sigma_k(x_1+2x_2+3x_3-60) = 0 \\ \frac{\partial P_E(x,\sigma_k)}{\partial x_2} = -x_1x_3 + 2\sigma_k(x_1+2x_2+3x_3-60) = 0 \\ \frac{\partial P_E(x,\sigma_k)}{\partial x_3} = -x_1x_2 + 3\sigma_k(x_1+2x_2+3x_3-60) = 0 \end{cases}$$

$\Rightarrow (x_1^*, x_2^*, x_3^*) = (0,30,0), (60,0,0), (0,0,20),$

$(9\sigma_k - \sqrt{81\sigma_k^2-3600\sigma_k}, \ \frac{9}{2}\sigma_k - \frac{1}{2}\sqrt{81\sigma_k^2-3600\sigma_k}, \ 3\sigma_k - \sqrt{9\sigma_k^2-400\sigma_k})$ --- (*)

$(9\sigma_k + \sqrt{81\sigma_k^2-3600\sigma_k}, \ \frac{9}{2}\sigma_k + \frac{1}{2}\sqrt{81\sigma_k^2-3600\sigma_k}, \ 3\sigma_k + \sqrt{9\sigma_k^2-400\sigma_k})$ --- (**)

其中 $(0,30,0)$ 不是局部极小. 因为 $(-\varepsilon, 30+2\varepsilon, -\varepsilon)$ 值严格小于 $(0,30,0)$ 值, $\forall 0<\varepsilon<1$.

$(60,0,0)$ $(0,0,20)$ 同理不是局部极小

而 (**) 对应解 $k\to\infty$ 时不收敛, 故取 (*) 对应解

故 $x^{k+1} = (9\sigma_k - \sqrt{81\sigma_k^2-3600\sigma_k}, \ \frac{9}{2}\sigma_k - \frac{1}{2}\sqrt{81\sigma_k^2-3600\sigma_k}, \ 3\sigma_k - \sqrt{9\sigma_k^2-400\sigma_k})$

$k\to\infty$ 时 $x^{k+1} \to (20, 10, \frac{20}{3}) := x^*$

由定理 7.2, 因为 $\nabla c_i(x^*) = (1,2,3) \neq 0$

故 $\lambda^* = \lim_{k\to\infty}(-\sigma_k c_i(x^{k+1})) = \lim_{k\to\infty} -\sigma_k(27\sigma_k - 3\sqrt{81\sigma_k^2-3600\sigma_k} - 60)$

$= -\frac{200}{3}$

对原问题考虑 KKT 条件 $\mathcal{L}(x,\lambda) = -x_1x_2x_3 + \lambda(x_1+2x_2+3x_3-60)$

$$\begin{cases} 0 = \frac{\partial \mathcal{L}}{\partial x_1} = -x_2x_3 + \lambda \\ 0 = \frac{\partial \mathcal{L}}{\partial x_2} = -x_1x_3 + 2\lambda \\ 0 = \frac{\partial \mathcal{L}}{\partial x_3} = -x_1x_2 + 3\lambda \\ x_1+2x_2+3x_3-60=0 \end{cases} \Rightarrow \begin{matrix} (x_1^*, x_2^*, x_3^*) = (20, 10, \frac{20}{3}) \\ \lambda^* = \frac{200}{3} \end{matrix}$$

故 原问题无下界，且最优解与 Lagrange 乘子与罚问题至极限得出相同.

最后考虑最优点处

Hessian

$$\nabla^2_{xx} P_E(x,\sigma) = \begin{pmatrix} \sigma & -x_3+2\sigma & -x_2+3\sigma \\ -x_3+2\sigma & 4\sigma & -x_1+6\sigma \\ -x_2+3\sigma & -x_1+6\sigma & 9\sigma \end{pmatrix} = \begin{pmatrix} \sigma & -\frac{20}{3}+2\sigma & 3\sigma-10 \\ -\frac{20}{3}+2\sigma & 4\sigma & 6\sigma-20 \\ 3\sigma-10 & 6\sigma-20 & 9\sigma \end{pmatrix}$$

特征多项式

$$f(\lambda) = \lambda^3 - \frac{4900\lambda}{9} - 14\lambda^2\sigma + \frac{9800\lambda}{3}\sigma - 1200\sigma + \frac{8000}{3}$$

故 $\nabla^2_{xx} P_E(x,\sigma) \succ 0 \iff \sigma > \frac{20}{9}$