Method of characteristics

(1)

$$\begin{cases} \partial_t u + 2t\,\partial_x u = 0 & x \in \mathbb{R},\ t>0 \\ u(x,0) = e^{-x^2} \end{cases}$$

$$V_t = U_x \dot{X}(t) + U_t = u_x(\dot{X}(t) - 2t) = 0$$

$$v(t) = u(X(t), t)$$

$$\Rightarrow X(t) = t^2 + C$$

$$u(t^2 + C, t) = u(C, 0) = e^{-C^2}$$

$$u(x,t) = e^{-(x-t^2)^2}, \quad x \in \mathbb{R},\ t>0$$

(2)

$$\begin{cases} \partial_t u + \partial_t v + 3\partial_x u + 2\partial_x v = 0 \\ -\partial_t u + \partial_t v + 5\partial_x u + 2\partial_x v = 0, \quad x \in \mathbb{R},\ t>0 \\ u(x,0) = \sin x, \quad v(x,0) = e^x, \quad x \in \mathbb{R} \end{cases}$$

记 $U = [u, v]$

$$\begin{pmatrix} 3 & 2 \\ 5 & 2 \end{pmatrix} \partial_x U + \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \partial_t U = 0$$

$$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 5 & 2 \end{pmatrix} \partial_x U + \partial_t U = 0$$

$$\begin{pmatrix} -1 & 0 \\ 4 & 2 \end{pmatrix} \partial_x U + \partial_t U = 0$$

$$\begin{pmatrix} -1 & 0 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix} = 0$$

$$-u_x + u_t = 0$$

$$4u_x + 2v_x + v_t = 0$$

$$\Rightarrow u(x,t) = \sin(x-t)$$

$$2v_x + v_t = -4\cos(x-t)$$

记 $x = x(s)$, $t = t(s)$ $\quad V = V(s)$

$$\frac{dx}{ds} = 2 \quad \frac{dt}{ds} = 1 \quad \frac{dV}{ds} = -4\cos(x(s) - t(s))$$

让 $t = s \quad x = 2s + x_0$

有 $\frac{dV}{ds} = -4\cos(s + x_0)$

$$V(0) = e^{x_0}$$

$$V(s) = -4\sin(s+x_0) + e^{x_0} + 4\sin x_0$$

$$V(2s+x_0, s) = -4\sin(s+x_0) + e^{x_0} + 4\sin x_0$$

$$V(x,t) = -4\sin(x-t) \Rightarrow + e^{x-2t} + 4\sin(x-2t)$$

(3) $\begin{cases} \partial_t u + x\,\partial_x u = u \\ u(x,0) = u_0 \end{cases}$

让 $\begin{cases} \dfrac{dx}{ds} = x & (1) \\ \dfrac{dt}{ds} = 1 & (2) \end{cases}$

有 $\dfrac{du}{ds} = u_x \cdot x + u_t = u$ (3)

设 $x(0) = x_0$ $t(0) = 0$

有 $\underline{x(s) = x_0 e^s}$ $\underline{t(s) = s}$

设 $u(0) = u_0(x_0)$

有 $\underline{u(s) = u_0(x_0)\, e^s}$

$u(x_0 e^s, s) = u_0(x_0)\, e^s$

$u(x,t) = u_0\left(\frac{x}{e^t}\right) e^t$

(4) $\begin{cases} \partial_t^2 u - a^2 \partial_x^2 u = f(x,t), & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x), & \partial_t u(x,0) = h(x) \end{cases}$

$a > 0$

我们证明 D'Alembert 公式

$$u(x,t) = \frac{1}{2}[g(x+at) + g(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} h(\xi)\,d\xi + \frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\,d\xi$$

$$\left(\frac{d}{dt} - a\frac{d}{dx}\right)\left(\frac{d}{dt} + a\frac{d}{dx}\right) u = f(x,t)$$

$$\begin{cases} \dfrac{du}{dt} + a\dfrac{du}{dx} = v & u(x,0) = g(x) \\ \dfrac{dv}{dt} - a\dfrac{dv}{dx} = f(x,t) & v(x,0) = h(x) + a g'(x) \end{cases}$$

$$v(x,t) = \int_0^t f(x+at-as,\, s)\,ds + h(x+at) + a\,g'(x+at)$$

记 $x = x_0 + ar \quad t = r$

$$\frac{d}{dr} u = u_t + a u_x = v(x_0+ar,\, r)$$

$$u = \int_0^t v(x-at+ar,\, r)\,dr + u(x-at,\, 0)$$

$$u(x,t) = \int_0^t \int_0^r f(x-at+2ar-as,\, s)\,ds\,dr + \int_0^t h(x-at+2ar)\,dr$$

$$+ a\int_0^t g'(x-at+2ar)\,dr$$

$$= \frac{1}{2a}\int_{x-at}^{x+at} h(\xi)\,d\xi + \frac{1}{2}\left[g(x+at) + g(x-at)\right]$$

$$+ \frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\,d\xi$$