13 $X_1 \dots X_n \overset{iid}{\sim} Poisson(\lambda)$

$\bar{X}$, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$ 为λ无偏估计 还有别的无偏估计吗!

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HW3

$E\bar{X} = \frac{1}{n}\sum_{i=1}^{n} EX_i = \frac{1}{n}\cdot n\lambda = \lambda$

$ES^2 = \frac{1}{n-1}\sum_{i=1}^{n} E(X_i-\bar{X})^2 = Var(X_i) = \lambda$

有其它无偏估计. 如 $\frac{1}{2}\bar{X} + \frac{1}{2(n-1)}\sum_{i=1}^{n}(X_i-\bar{X})^2$. #

15 $X_1 \dots X_n \sim Poisson(\lambda)$ 求 λ^2 的无偏估计

可取 $S = \frac{\sum_{i=1}^{n}(X_i^2 - X_i)}{n}$ $ES = \frac{1}{n}\sum_{i=1}^{n} EX_i^2 - \lambda$

$= \frac{1}{n}\sum_{i=1}^{n}(Var X_i + (EX_i)^2) - \lambda$

$= \frac{1}{n}\cdot n\cdot(\lambda+\lambda^2) - \lambda = \lambda^2$.

#

17. $X_1 \dots X_n \overset{iid}{\sim} f(x;\theta) = \begin{cases} \frac{\theta}{(1+x)^{\theta+1}}, & 0 < x < +\infty \\ 0 & x \le 0 \end{cases}$

$\theta \in (1,+\infty)$ 求 θ^{-1} 无偏估计

$L(X_1,\dots X_n;\theta) = \frac{\theta^n}{(1+X_1)^{\theta+1}\cdots(1+X_n)^{\theta+1}}$

$\log L(X_1,\dots X_n;\theta) = n\log\theta - (\theta+1)\log[(1+X_1)\cdots(1+X_n)]$

$0 = \frac{\partial \log L(X;\theta)}{\partial\theta} = \frac{n}{\theta} - \log[(1+X_1)\cdots(1+X_n)]$

$\hat{\theta} = \frac{n}{\sum_{i=1}^{n}\log(1+X_i)}$

取 $T = \frac{\sum_{i=1}^{n}\log(1+X_i)}{n}$

$ET = E\log(1+X) = \int_0^{+\infty}\frac{\log(1+x)\,\theta}{(1+x)^{\theta+1}}dx = -\int_0^{+\infty}\log(1+x)\,d\frac{1}{(1+x)^{\theta}}$

$= -\log(1+x)\frac{1}{(1+x)^{\theta}}\Big|_0^{+\infty} + \int_0^{+\infty}\frac{1}{(1+x)^{\theta+1}}dx = \frac{1}{\theta}$

故 T 即为所求 #

18. $X_1 \cdots X_n$ iid $f(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{1}{\theta}x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$\theta > 0$. 找出θ无偏估计方差尽可能大的下界

根据 Crámer-Rao bound

$$I(\theta) = E_\theta\left[\left(\frac{\partial l(X;\theta)}{\partial \theta}\right)^2\right] = \int_0^{+\infty} \left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2 \frac{1}{\theta} e^{-\frac{1}{\theta}x} dx = \frac{1}{\theta^2}$$

且满足条件

故 $Var(\hat{\theta}) \geq \frac{1}{n I(\theta)} = \frac{\theta^2}{n}$

考虑 $\bar{X} = \frac{\sum X_i}{n}$ $E\bar{X} = \int_0^{+\infty} \frac{x}{\theta} e^{-\frac{x}{\theta}} dx = \theta$

$$Var(\bar{X}) = \frac{1}{n^2} \sum Var(X_i) = \frac{1}{n} Var(X) = \frac{\theta^2}{n}$$

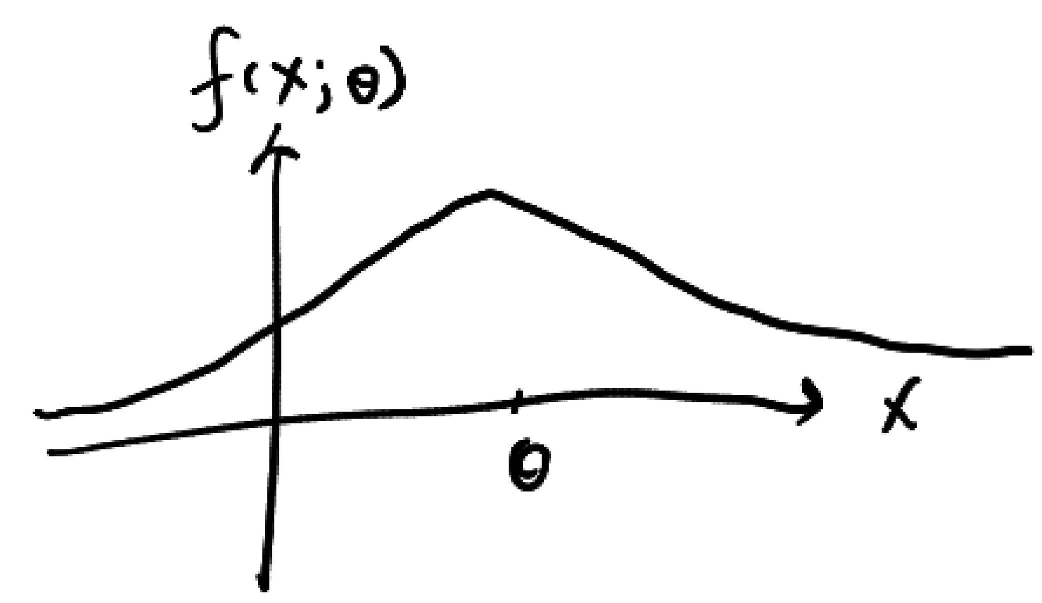
故最大下界为 $\frac{\theta^2}{n}$. #

19. $X_1 \cdots X_n \sim f(x;\theta) = \frac{1}{\pi(1+(x-\theta)^2)}$ $\theta \in \mathbb{R}$ 找出θ合适的估计

由于 Cauchy 分布没有 finite moments, 因此不适用 $\bar{X}$, S^2 等

分布关于θ对称, 在θ处最大

可取 $x = \text{Median}(X_1, \cdots X_n)$ 作为估计 #



22. $X_1 \cdots X_n \overset{iid}{\sim} Poisson(\lambda)$, $\lambda > 0$ $\bar{X}$为λ强相合估计

由SLLN,

$\bar{X}_n \xrightarrow{a.s.} \lambda$

即 $Pr(\lim_{n\to\infty} \bar{X}_n = \lambda) = 1$ $\bar{X}$为λ强相合估计

#

20. $$P_\theta(X=x) = \begin{cases} \theta^x(1-\theta)^{1-x} & \theta \text{ rational} \\ \theta^{1-x}(1-\theta)^x & \theta \text{ irrational} \end{cases}$$

$\theta \in (0,1)$ MLE $\hat{\theta}_n$ is not Consistant.

$$L(X_1, \cdots X_n; \theta) = \begin{cases} \theta^{\sum X_i}(1-\theta)^{n-\sum X_i} & \theta \text{ rational} \\ \theta^{n-\sum X_i}(1-\theta)^{\sum X_i} & \theta \text{ irrational} \end{cases}$$

$$\hat{\theta} = \frac{\sum X_i}{n} \;/\; 1 - \frac{\sum X_i}{n}$$

有 $\hat{\theta}_n = \frac{\sum X_i}{n}$ (MLE estimator)

但 $\theta \notin \mathbb{Q}$时 $P_\theta(X=1) = 1-\theta$, $P_\theta(X=0) = \theta$

故 $\hat{\theta}_n = \frac{\sum X_i}{n} \xrightarrow{P} 1-\theta \neq \theta$ 故 $\hat{\theta}_n$ 不相合 #

14. $X_1 \cdots X_n \sim B(m, \theta)$ 求θ^2无偏估计

$\mathbb{E}X = m\theta$ $Var(X) = m\theta - m\theta^2$

$\theta^2 = \frac{1}{m}(\mathbb{E}X - Var X)$

取 $T = \frac{1}{m}\left(\bar{X} - \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2\right)$

即可 #