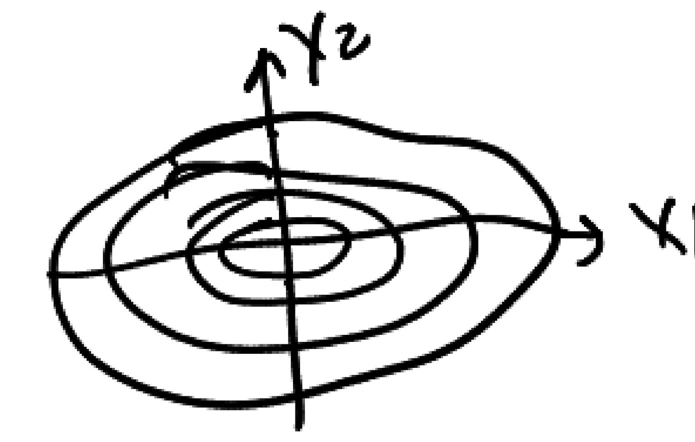
1. $\begin{cases} \dfrac{dx_1}{dt} = x_2(t) \\ \dfrac{dx_2}{dt} = -w^2(t)\, x_1(t) \end{cases}$ null solution stable or not



$$w(t) = \begin{cases} 0.4 & 2k\pi \le t < (2k+1)\pi \\ 0.6 & (2k-1)\pi \le t < 2k\pi \end{cases}$$

The system is 2π- periodic.

2. Find equilibria of $\begin{cases} \dot{x} = xy + 12 \\ \dot{y} = x^2 + y^2 - 25 \end{cases}$. find stability / type , draw phase curve

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}$$

Equilibria:

$(-4, 3)$ $\lambda_{1,2} = \dfrac{9 \pm \sqrt{137}}{2}$ $\lambda_1 > 0, \lambda_2 < 0$, unstable, sodde point

$(4, -3)$ $\lambda_{1,2} = \dfrac{-9 \pm \sqrt{137}}{2}$, $\lambda_1 < 0, \lambda_2 > 0$, unstable, sodde point

$(3, -4)$ $\lambda_{1,2} = -6 \pm \sqrt{22}$, $\lambda_1 < 0, \lambda_2 < 0$, asympotic stable, stable node

$(-3, 4)$ $\lambda_{1,2} = 6 \pm \sqrt{22}$. $\lambda_1 > 0, \lambda_2 > 0$, unstable, unstable node

3. Find equilibria of

$$\begin{cases} \dot{x} = -\sin y \\ \dot{y} = \sin x + \sin y \end{cases}$$ on torus $[0, 2\pi) \times [0, 2\pi)$

Stability / type of equilibrium, phase curve. Consider periodic extension.

$E_1(0, 0)$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

$\lambda_{1,2} = \frac{1 \pm \sqrt{3}i}{2}$ unstable focus

$E_2(0, \pi)$, $A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

$\lambda_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$, unstable, saddle point

$E_3(\pi, 0)$, $A = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$

$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$ unstable, saddle point

$E_4(\pi, \pi)$ $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

$\lambda_{1,2} = \frac{-1 \pm \sqrt{3}i}{2}$, stable focus.

4. Phase plane analysis of $\ddot{x}(t) + \dot{x}(t) = -\frac{\partial V}{\partial x}$, $V(x) = 1 - \cos x$

$x = x(t)$, $y = \dot{x}(t)$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -y - \sin x \end{cases}$$

Also, $H(x, \dot{x}) = V(x) + \frac{1}{2}\dot{x}^2$

$\frac{d}{dt} H(x, \dot{x}) \leq 0$

H acquires minima at E_{2k}

$E_k(k\pi, 0)$

$$A_k = \begin{pmatrix} 0 & 1 \\ -\cos k\pi & -1 \end{pmatrix}$$

$A_{2k} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ $\lambda_{1,2} = \frac{-1 \pm \sqrt{3}i}{2}$ stable focus

$A_{2k+1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ $\lambda_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$ (unstable) saddle point

#

5. $\begin{cases} \dot{x} = x^2 + y \\ \dot{y} = x - y + a \end{cases}$

(a) Find all equilibria & linearized system

$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 1 \\ 1 & -1 \end{pmatrix}$ linearized system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$

Case 1. $a < \frac{1}{4}$

$E_1 : \left(\frac{-1+\sqrt{1-4a}}{2}, \frac{2a-1+\sqrt{1-4a}}{2} \right)$, $A = \begin{pmatrix} -1+\sqrt{1-4a} & 1 \\ 1 & -1 \end{pmatrix}$

$E_2 : \left(\frac{-1-\sqrt{1-4a}}{2}, \frac{2a-1-\sqrt{1-4a}}{2} \right)$, $A = \begin{pmatrix} -1-\sqrt{1-4a} & 1 \\ 1 & -1 \end{pmatrix}$

Case 2. $a = \frac{1}{4}$

$E : \left(-\frac{1}{2}, -\frac{1}{4} \right)$ $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

Case 3. $a > \frac{1}{4}$

No equilibria

(b) Behavior of linearized system at equilibrium point

Case 1. $a < \frac{1}{4}$

For E_1, $\operatorname{Tr} A = -2+\sqrt{1-4a}$ $\det A = -\sqrt{1-4a}$

$\lambda_1 > 0$, $\lambda_2 < 0$, sadde point

For E_2, $\operatorname{Tr} A = -2-\sqrt{1-4a}$ $\det A = \sqrt{1-4a}$

$\operatorname{Tr} A^2 - 4 \det A > 0$, $\det A > 0$, $\operatorname{Tr} A < 0$

$\lambda_{1,2} = \frac{\operatorname{Tr} A \pm \sqrt{\operatorname{Tr}^2 A - 4 \det A}}{2} < 0$, stable node (asympotic stable)

Case 2. $a = \frac{1}{4}$ $\operatorname{Tr} A = -2$ $\det A = 1$, $\lambda_{1,2} = -1$, star node (asympotic stable)

(c) Describe bifurcation that occurs

$a = \frac{1}{4}$, chages equilibrium number