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1. $w>0$ find periodicity T of

$$\ddot{x}(t)+w^2x(t)+ax^2(t)+bx^3(t)=0$$

$$x(0)=A,\quad \dot{x}(0)=0,\quad 0<A\ll 1$$

We use regular expansion for the problem.

$$x(t)=x_0+x_1A+x_2A^2+x_3A^3+\cdots$$

$$\ddot{x}(t)=\ddot{x}_0+\ddot{x}_1A+\ddot{x}_2A^2+\ddot{x}_3A^3+\cdots$$

$$w^2x(t)=w^2x_0+w^2x_1A+w^2x_2A^2+w^2x_3A^3+\cdots$$

$$a\,x^2(t)=a(x_0+x_1A+x_2A^2+x_3A^3+\cdots)^2$$

$$b\,x^3(t)=b(x_0+x_1A+x_2A^2+x_3A^3+\cdots)^3$$

$O(1)$: $\quad \ddot{x}_0+w^2x_0+ax_0^2+bx_0^3=0 \quad (1)$

$\quad x_0(0)=0 \quad \dot{x}_0(0)=0$

$O(A)$: $\quad \ddot{x}_1+w^2x_1+2ax_0x_1+3b\,x_0^2x_1=0 \quad (2)$

$\quad x_1(0)=1 \quad \dot{x}_1(0)=0$

$O(A^2)$: $\quad \ddot{x}_2+w^2x_2+a(2x_0x_2+x_1^2)+b(3x_0^2x_2+3x_1^2x_0)=0 \quad (3)$

$\quad x_2(0)=0 \quad \dot{x}_2(0)=0$

We find (1) is a non-linear 2nd-order ODE with initial condition. So $x_0\equiv 0$

(2): $\quad \ddot{x}_1+w^2x_1=0 \quad x_1(0)=1 \quad \dot{x}_1(0)=0$

$\quad \Rightarrow x_1=\cos wt$

(3): $\quad \ddot{x}_2+w^2x_2+a(\cos wt)^2=0 \quad x_2(0)=0 \quad \dot{x}_2(0)=0$

$$\Rightarrow x_2=\frac{a\cos(2tw)}{6w^2}-\frac{a}{2w^2}+\frac{a}{3w^2}\cos(wt)$$

$$x(t)=A\cos wt+A^2\left(\frac{a\cos(2wt)}{6w^2}-\frac{a}{2w^2}+\frac{a}{3w^2}\cos(wt)\right)+O(A^3)$$

$$\dot{x}(T)=-Aw\sin wT-\frac{A^2a\sin(2wT)}{3w}-\frac{A^2a}{3w}\sin(wT)+O(A^3)=0$$

Clearly if $T=\frac{2\pi}{w}+\alpha A+O(A^2)$, then $\alpha=0$

$$T=\frac{2\pi}{w}+sA^2+O(A^3) \qquad wT=2\pi+wsA^2+O(A^3)$$

So we need $O(A^3)$

$$\ddot{x}_3 + w^2 x_3 + a(2x_1x_2) + b(x_1^3) = 0 \qquad x_3(0)=0 \quad \dot{x}_3(0)=0$$

$$\ddot{x}_3 + w^2 x_3 + b(\cos wt)^3 + 2a(\cos wt)\left(\frac{a\cos 2wt}{6w^2} - \frac{a}{2w^2} + \frac{a}{3w^2}\cos(wt)\right) = 0$$

$$x_3 = \frac{a^2\cos 3wt}{48w^4} + \frac{5a^2 t\sin wt}{12w^3} - \frac{2a^2}{3w^4} + \frac{b\cos 3wt}{32w^2} - \frac{3bt\sin wt}{8w}$$
$$+ \left(\frac{31}{48}\frac{a^2}{w^4} - \frac{1}{32}\frac{b}{w^2}\right)\cos wt$$

$$\dot{x}_3(T) = -\frac{a^2\sin 3wt}{16w^3} + \frac{5a^2(\sin wt + tw\cos wt)}{12w^3} + \frac{-3wb\sin 3wt}{32w^2} - \frac{3b\sin wt + tw3bt\cos wt}{8w}$$
$$-w\left(\frac{31}{48}\frac{a^2}{w^4} - \frac{1}{32}\frac{b}{w^2}\right)\sin wt$$

$$-A^3w^2 s + \frac{5a^2\cdot 2\pi A^3}{12w^3} - \frac{6b\pi A^3}{8w} = 0 \qquad s = \frac{5\pi a^2}{6w^5} - \frac{3b\pi}{4w^3}$$

$$\underline{T = \frac{2\pi}{w} + \left(\frac{5\pi a^2}{6w^5} - \frac{3b\pi}{4w^3}\right)A^2 + O(A^3)} \quad \#$$

Expand solution until $t = O(1/\varepsilon)$

2. $\begin{cases} \ddot{y} + y = \varepsilon\left(\dot{y} - \frac{1}{3}(\dot{y})^3\right), \quad t>0 \\ y(0)=0, \ \dot{y}(0)=a \end{cases}$

We use Poincare-Lindsteadt $\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau}$

$$y(t,\tau) = y_0(t,\tau) + \varepsilon y_1(t,\tau) + \varepsilon^2 y_2(t,\tau) + O(\varepsilon^3)$$

differential operator $L_\varepsilon = \left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau}\right)^2 + I - \varepsilon\left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau}\right)$

function $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \qquad L_\varepsilon y = \frac{-\varepsilon}{3}\left[\left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau}\right)(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots)\right]^3$

$O(1)$: $\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0$

$O(\varepsilon)$: $2\frac{\partial^2 y_0}{\partial t\partial \tau} - \frac{\partial y_0}{\partial t} + \frac{1}{3}\left(\frac{\partial y_0}{\partial t}\right)^3 + \frac{\partial^2 y_1}{\partial t^2} + y_1 = 0$

$$y_0 = A(\tau)e^{it} + A^*(\tau)e^{-it}$$

$$\frac{\partial^2 y_1}{\partial t^2} + y_1 = \frac{\partial y_0}{\partial t} - \frac{2\partial^2 y_0}{\partial t\partial \tau} + \frac{1}{3}\left(\frac{\partial y_0}{\partial t}\right)^3$$

$$= iA(\tau)e^{it} - iA^*(\tau)e^{-it} - 2iA'(\tau)e^{it} - 2i[A^*(\tau)]'e^{-it}$$
$$+ \frac{1}{3}\left(iA(\tau)e^{it} - iA^*(\tau)e^{-it}\right)^3$$

$$= iA(\tau)e^{it} - iA^*(\tau)e^{-it} - 2iA'(\tau)e^{it} - 2i\,A^*(\tau)'e^{-it}$$
$$+ \frac{1}{3}(-i)A^3(\tau)e^{3it} - iA^2(\tau)A^*(\tau)e^{it} - iA(\tau)(A^*(\tau))^2e^{-it}$$
$$+ \frac{i}{3}(A^*(\tau))^3 e^{-3it}$$

$$\Rightarrow \begin{cases} iA(\tau) - 2iA'(\tau) - iA^2(\tau)A^*(\tau) = 0 \\ iA^2(\tau) - 2i(A^*(\tau))' - iA(\tau)(A^*(\tau))^2 = 0 \\ A'(\tau) = \frac{1}{2}A(\tau) - \frac{1}{2}A^2(\tau)A^*(\tau) \end{cases}$$

Let $A(\tau) = R(\tau)e^{i\theta(\tau)}$ $\quad R, \theta \in \mathbb{R}$

$$R'(\tau)e^{i\theta(\tau)} + iR(\tau)e^{i\theta(\tau)}\theta'(\tau) = \frac{1}{2}R(\tau)e^{i\theta(\tau)} - \frac{1}{2}R^3(\tau)e^{i2\theta(\tau)}e^{-i\theta(\tau)}$$
$$R'(\tau) + iR(\tau)\theta'(\tau) = \frac{1}{2}R(\tau) - \frac{1}{2}R^3(\tau)$$

$$\begin{cases} R'(\tau) - \frac{1}{2}R(\tau) + \frac{1}{2}R^3(\tau) = 0 \\ R(\tau)\theta'(\tau) = 0 \end{cases}$$

$$\theta(\tau) = \theta(0) \qquad R(\tau) = \frac{e^{\frac{\tau}{2}}}{\sqrt{R(0) + e^{\tau}}}$$

$$A(\tau) = \frac{e^{\frac{\tau}{2}}}{\sqrt{R(0)+e^{\tau}}}e^{i\theta(0)} \qquad y_0 = \frac{e^{\frac{\tau}{2}}}{\sqrt{R(0)+e^{\tau}}}e^{i\theta(0)+it} + \frac{e^{\frac{\tau}{2}}}{\sqrt{R(0)+e^{\tau}}}e^{-i\theta(0)-it}$$

Initial value

$$\begin{cases} y_0(0,0) = 0 \Rightarrow \dfrac{e^{i\theta(0)}}{\sqrt{R(0)+1}} + \dfrac{e^{-i\theta(0)}}{\sqrt{R(0)+1}} = 0 \\ a = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\varepsilon\right)(y_0 + \varepsilon y_1)\Big|_{(0,0)} = \frac{\partial y_0}{\partial t}\Big|_{(0,0)} = i\left(\dfrac{e^{i\theta(0)}}{\sqrt{R(0)+1}} - \dfrac{e^{-i\theta(0)}}{\sqrt{R(0)+1}}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \cos\theta(0) = 0 \qquad \theta(0) = \frac{3\pi}{2} \\ R(0) = \dfrac{4\sin^2\theta(0)}{a^2} - 1 = \dfrac{4-a^2}{a^2} \end{cases}$$

$$y = \frac{-e^{\frac{\tau}{2}}}{\sqrt{\frac{4-a^2}{a^2} + e^{\tau}}}e^{i(t+\frac{\pi}{2})} + \frac{-e^{\frac{\tau}{2}}}{\sqrt{\frac{4-a^2}{a^2}+e^{\tau}}}e^{-i(t+\frac{\pi}{2})}$$
$$= \frac{2e^{\frac{\varepsilon t}{2}}\sin t}{\sqrt{\frac{4-a^2}{a^2} + e^{\varepsilon t}}}$$

3. $\begin{cases} \ddot{y} + \varepsilon(1+\gamma\cos y)\dot{y} + \sin y = \varepsilon\alpha, & t>0 \\ y(0) = \dot{y}(0) = 0 \end{cases}$

$\gamma > 0$, α constant. Expand until $t = O(1/\varepsilon)$

We use Poincare-Lindstedt method

$y(t,\tau) = y_0(t,\tau) + \varepsilon y_1(t,\tau) + \varepsilon^2 y_2(t,\tau) + \cdots$

$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}$

Linearize the ODE: (This approximation is enough)

$$\ddot{y} + \varepsilon(1+\gamma)\dot{y} + y - \varepsilon\alpha = 0$$

Differential Operator:

$\mathcal{L}_\varepsilon = \left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau}\right)^2 + (\varepsilon + \varepsilon\gamma + 1)I$

$y = y_0(t,\tau) + \varepsilon y_1(t,\tau) + \varepsilon^2 y_2(t,\tau) + \cdots$

$O(\varepsilon^2)$: $\frac{\partial^2 y_0}{\partial \tau^2} + 2\frac{\partial^2 y_1}{\partial t \partial \tau} + (1+\gamma)y_1 + \frac{\partial^2 y_2}{\partial t^2} + y_2 = 0$

$\mathcal{L}_\varepsilon y = \varepsilon\alpha$

$O(1)$: $\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0$ (1)

$O(\varepsilon)$: $2\frac{\partial^2 y_0}{\partial t \partial \tau} + (1+\gamma)y_0 + \frac{\partial^2 y_1}{\partial t^2} + y_1 = \alpha$ (2)

(1): $y_0 = A(\tau)e^{it} + A^*(\tau)e^{-it}$

(2): $\frac{\partial^2 y_1}{\partial t^2} + y_1 = -2\frac{\partial^2 y_0}{\partial t \partial \tau} - (1+\gamma)y_0 + \alpha$

Some Computation gives $y_0 \equiv 0$

$\frac{\partial^2 y_1}{\partial t^2} + y_1 = \alpha$ $\quad y_1 = A(\tau)e^{it} + A^*(\tau)e^{-it} + \alpha$

$\frac{\partial^2 y_2}{\partial t^2} + y_2 = -2\frac{\partial^2 y_1}{\partial t \partial \tau} - (1+\gamma)y_1$

$= -2iA'(\tau)e^{it} + 2i A^*(\tau)' e^{-it}$

$-(1+\gamma)A(\tau)e^{it} - (1+\gamma)A^*(\tau)e^{-it} - (1+\gamma)\alpha$

We have

$$\begin{cases} -2iA'(\tau) - (1+r)A(\tau) = 0 \\ 2i A^*(\tau)' - (1+r)A^*(\tau) = 0 \end{cases} \qquad A(\tau) = e^{\frac{i(1+r)}{2}\tau} A(0)$$

$$y_1 = C_1 e^{\frac{i(1+r)}{2}\tau} e^{it} + C_2 e^{-\frac{i(1+r)}{2}\tau} e^{-it} + \alpha$$

$$\begin{cases} 0 = C_1 + C_2 + \alpha \\ 0 = \frac{\partial y_1}{\partial t} = iC_1 e^{\frac{i(1+r)}{2}\tau} e^{it} - i e^{-\frac{i(1+r)}{2}\tau} e^{-it} \Big|_{(t,\tau)=0} \end{cases}$$

$$\Rightarrow C_1 = C_2 = -\frac{\alpha}{2}$$

$$\underline{y = \varepsilon y_1 = \varepsilon\left[-\alpha\cos\left(\frac{(1+r)}{2}\varepsilon t + t\right) + \alpha\right]}$$ #