Chap 2.

23. $X_1 \cdots X_n \sim Uniform(0, \theta)$

θ置信水平$1-\alpha$置信区间

0.08 0.28 0.53 0.91 0.89

θ置信水平0.95置信区间

记 $Q = \frac{X_{(n)}}{\theta}$ $X_{(n)} = \max_i X_i$

$P(Q \le t) = \prod_{i=1}^{n} P(X_i \le t\theta) = t^n$ 记 $C_n = \alpha^{\frac{1}{n}}$ $P(Q \le C_n) = \alpha$

$1-\alpha = P(C_n \le Q \le 1) = P(C_n \le \frac{X_{(n)}}{\theta} \le 1) = P(X_{(n)} \le \theta \le \frac{X_{(n)}}{C_n})$

区间 $[X_{(n)}, \frac{X_{(n)}}{\sqrt[n]{\alpha}}]$

置信区间为 $[0.91, \frac{0.91}{\sqrt[5]{0.05}}] = [0.91, 1.66]$

28. 12个样本 40 45 ⋯ 26 39

服从 $N(\mu, \sigma^2)$ 求μ置信水平0.95置信区间

$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim T(n-1)$ $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ $S = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$

记 $Pr(-c \le T \le c) = 0.95$

有 $Pr(\overline{X} - \frac{cS}{\sqrt{n}} \le \mu \le \overline{X} + \frac{cS}{\sqrt{n}}) = 0.95$

本问题中, $c = 1.796$

计算出 $\overline{X} = 36.33$ $S = 7.7146$

置信区间 $[32.33, 40.33]$

26 $X_1 \cdots X_n$ iid $F(x)$ $X_{(1)} \le \cdots \le X_{(n)}$

(1) $1 \le i \le n$

$$P(X_{(i)} \le x) = \frac{n!}{(i-1)!(n-i)!}\int_0^{F(x)} u^{i-1}(1-u)^{n-i}\,du$$

考虑 $X_{j_1} \le X_{j_2} \le \cdots \le X_{j_n}$ $(j_1 \cdots j_n) \in \sigma(n)$

有 $n!$ 种排列, $X_{(1)} \cdots X_{(i-1)}$ 及 $X_{(i+1)} \cdots X_{(n)}$ 内部排列不重要

即 $\frac{n!}{(i-1)!(n-i)!}$ 种, 其中 $X_{(i)} = v$ 时, $X_{(j)} \le v$, $j<i$

$X_{(j)} \ge v$ $j>i$

概率 $F(v)^{i-1}(1-F(v))^{n-i}$

$$P(X_{(i)} \le x) = \frac{n!}{(i-1)!(n-i)!}\int_{-\infty}^{x} F(v)^{i-1}(1-F(v))^{n-i}\,dF(v)$$

$$= \frac{n!}{(i-1)!(n-i)!}\int_0^{F(x)} u^{i-1}(1-u)^{n-i}\,du. \quad \#$$

(2) $F(x)$ 连续

$$\mathbb{E}[F(X_{(i)})] = \frac{i}{n+1}$$

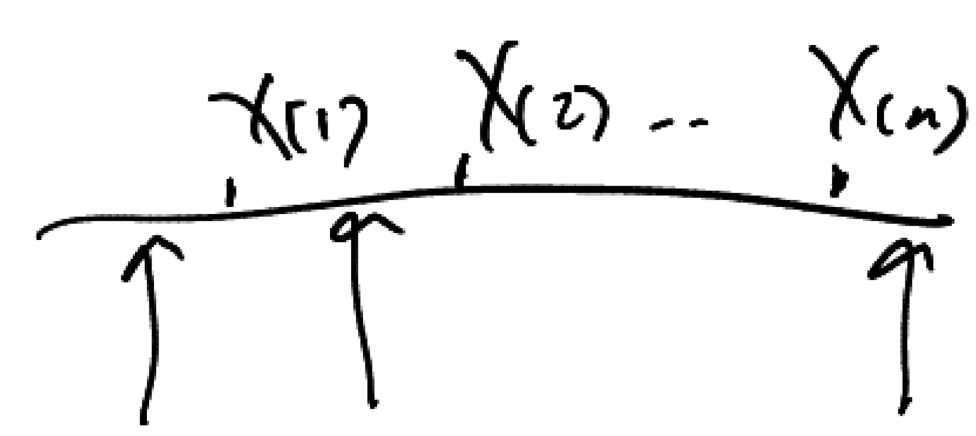
本质上 $F(X_{(i)}) = \mathbb{E}_{x \sim F(x)}\ x \le X_{(i)}$

$$\mathbb{E}[F(X_{(i)})] = \mathbb{E}_{X_1 \cdots X_n \sim F(x),\, x \sim F(x)}\ x \le X_{(i)}$$

$x, X_{(1)} \cdots X_{(n)}$ 为 $F(x)$ 中 iid $n+1$ 个样本

自然有 $\mathbb{E}[F(X_{(i)})] = \frac{i}{n+1}$ #

(因 $x, X_{(1)} \cdots X_{(n)}$ 地位均等)



Chap.3.

1. $X_1\ X_2\ X_3 \sim B(1,p)$

$$H_0: p=\frac{1}{2} \longleftrightarrow H_a: p=\frac{3}{4}$$

一个否定域 $W=\{(x_1,x_2,x_3): \sum x_i \geq 2\}$

求该否定域一、二类错误概率, $p=\frac{3}{4}$ 功效

Type I error: $$P_r\left(\sum X_i \geq 2 \mid p=\tfrac{1}{2}\right)=1-(1-p)^3-3p(1-p)^2=\frac{1}{2}$$

Type II error: $$P_r\left(\sum X_i < 2 \mid p=\tfrac{3}{4}\right)=(1-p)^3+3p(1-p)^2=\frac{5}{32}$$

$p=\frac{3}{4}$ power: $$\beta\left(\tfrac{3}{4}\right)=\mathbb{P}(T(X)\in R)=P\left(\sum X_i\geq 2 \mid p=\tfrac{3}{4}\right)=1-(1-p)^3-3p(1-p)^2$$
$$=\frac{27}{32}$$

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$$W=\{x: \varphi(x)=1\}=\varphi^{-1}(1)$$

2. $X_1 \cdots X_n \sim f(x)$

$$H_0: f(x)=f_0(x) \longleftrightarrow H_a: f(x)=f_1(x)$$

W 为 H_0 否定域 通过 $f_0(x)$, $f_1(x)$ 表示两类错误概率

Type I error: $P_r(x\in W \mid f(x)=f_0(x))$
$$=\int_{(x_1\cdots x_n)\in W} f_0(x_1)\cdots f_0(x_n)\,dx_1\cdots dx_n$$

Type II error: $P_r(x\notin W \mid f(x)=f_1(x))$
$$=\int_{(x_1\cdots x_n)\in W^c} f_1(x_1)\cdots f_1(x_n)\,dx_1\cdots dx_n$$
$$=1-\int_{(x_1\cdots x_n)\in W} f_1(x_1)\cdots f_1(x_n)\,dx_1\cdots dx_n$$

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3. $X \sim \text{Uniform}\,(\theta-\frac{1}{2},\ \theta+\frac{1}{2})$

$H_0: \theta \leq 3 \longleftrightarrow H_a: \theta \geq 4$

构造检验法，使功效函数 $p(\theta)$ 满足

$p(\theta)=0,\ \theta\leq 3,\quad p(\theta)=1,\ \theta\geq 4$

取 若 $X_1<3.5$ accept 若 $X_1>3.5$ reject

易知 $p(\theta)=\begin{cases}0, & \theta\leq 3\\ 1 & \theta\geq 4\end{cases}$

#

5 $X\sim f(x;\theta)$ $X_1 \cdots X_n \overset{iid}{\sim} X$

$H_0: \theta=\theta_0 \longleftrightarrow H_a: \theta=\theta_1 \quad \theta_0\neq\theta_1 \quad a, b>0$

$L(\underline{x}, \theta)=\prod_{i=1}^{n} f(x_i, \theta)$

δ^*: $aL(\underline{x}, \theta_0) > bL(\underline{x}, \theta_1)$ 接受 H_0

$aL(\underline{x}, \theta_0) < bL(\underline{x}, \theta_1)$ 拒绝 H_0

α^*, β^*：一、二类错误概率

任意检验法 δ，α、β 为一、二类错误

$$a\alpha^*+b\beta^* \leq a\alpha+b\beta.$$

不妨设 $\gamma = \frac{b}{a}$

即找 $Pr(\text{Type I error}) + \gamma\, Pr(\text{Type I error})$ 的下界

$$\alpha + \gamma\beta := Pr(\text{Type I error}) + \gamma\, Pr(\text{Type II error})$$

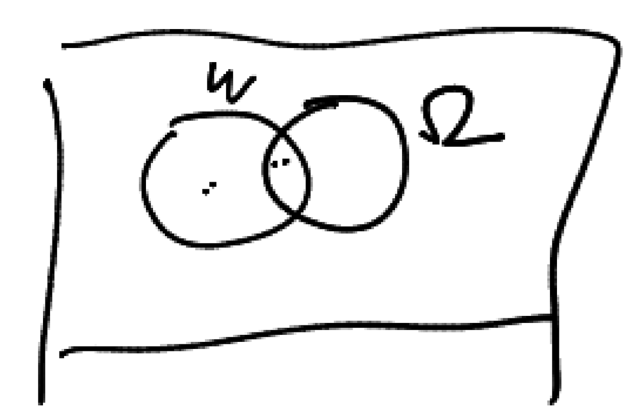
$$= Pr\left((x_1 \cdots x_n) \in W \mid \theta = \theta_0\right) + \gamma\, Pr\left((x_1 \cdots x_n) \in W^c \mid \theta = \theta_1\right)$$

$$= \int_W f(x_1, \theta_0) \cdots f(x_n, \theta_0)\, dx_1 \cdots dx_n + \gamma \int_{W^c} f(x_1, \theta_p) \cdots f(x_n, \theta_n)\, dx_1 \cdots dx_n$$

记 $\Omega = \{(x_1 \cdots x_n) \mid f(x_1, \theta_0) \cdots f(x_n, \theta_0) < \gamma f(x_1, \theta_1) \cdots f(x_n, \theta_n)\}$

$$\alpha + \gamma\beta = \int_{W \cap \Omega} L(\underline{x}, \theta_0)\, d\underline{x} + \int_{W \cap \Omega^c} L(\underline{x}, \theta_0)\, d\underline{x}$$

$$+ \gamma \int_{W^c \cap \Omega} L(\underline{x}, \theta_1)\, d\underline{x} + \gamma \int_{W^c \cap \Omega^c} L(\underline{x}, \theta_1)\, d\underline{x}$$



$$\geq \int_{W \cap \Omega} L(\underline{x}, \theta_0)\, d\underline{x} + \gamma \int_{W^c \cap \Omega^c} L(\underline{x}, \theta_1)\, d\underline{x}$$

$$+ \gamma \int_{W \cap \Omega^c} L(\underline{x}, \theta_1)\, d\underline{x} + \int_{W^c \cap \Omega} L(\underline{x}, \theta_0)\, d\underline{x}$$

$$= \int_{\Omega} L(\underline{x}, \theta_0)\, d\underline{x} + \gamma \int_{\Omega^c} L(\underline{x}, \theta_1)\, d\underline{x}$$

$$= \alpha^* + \gamma\beta^* \quad \#$$