1. Fundamental Solution to $\Delta^2 u = 0$

That is $\Delta^2 u = \delta(r)$

So $\Delta^2 u = 0$, $u \in \mathbb{R}^n \setminus \{0\}$

Let $V = \Delta u$, $\Delta V = \delta(r)$ Let $V = V(r)$

We have $\Delta V = r^{1-n}(r^{n-1} V'(r))'$

Case 1. $n > 4$ We have $V(r) = C_1 r^{2-n} + C_2$

$$\Delta u = r^{1-n}(r^{n-1} u'(r))' = C_1 r^{2-n} + C_2$$

$$\Rightarrow u(r) = A r^{4-n} + B r^{2-n} + C r^2 + D, \quad A, B, C, D \in \mathbb{R}$$

$V = \Delta u = 2(4-n) A r^{2-n} + 2Cn$ is fundamental solution to Laplace equation in $\mathbb{R}^n$ $\Rightarrow C = 0$, $2(4-n)A = \frac{1}{(n-2)\,W_{n-1}}$, W_{n-1} is area of S^{n-1}

$$A = \frac{1}{-2(n-4)(n-2)W_{n-1}} . \quad \text{WLOG } B = D = 0$$

$$u(r) = \Phi(x) = \frac{1}{-2(n-4)(n-2)W_{n-1}} |x|^{4-n}$$

Case 2. $n = 4$. $\Delta u = r^{-3}(r^3 u'(r))' = C_1 r^{-2} + C_2$

$$u(r) = D + C \log r + B r^2 + A r^{-2}$$

$V = \Delta u = \frac{2C}{r^2} + 8B$ $\quad \Delta V = \delta(r) \Rightarrow B = A = D = 0$, $C = \frac{1}{4W_3} = \frac{1}{8\pi^2}$

$$u(r) = \Phi(x) = \frac{1}{8\pi^2} \log|x|$$

Case 3. $n = 3$. $\Delta u = r^{-2}(r^2 u'(r))' = C_1 \frac{1}{r} + C_2$

$$u(r) = D + Cr + Br^2 + \frac{A}{r}$$

$r^{-2}(r^2 u')' = \frac{2C}{r} + 6B$ $\quad C = \frac{1}{2W_2} = \frac{1}{8\pi}$

$$u(r) = \Phi(x) = \frac{1}{8\pi} |x|$$

Case 4. $n = 2$. $\Delta u = r^{-1}(r u'(r))' = b \log r + C$

$$u(r) = D + C \log r + B r^2 + A\, r^2 \log r$$

$A = -\frac{1}{8\pi}$ $\quad u(r) = \Phi(x) = -\frac{1}{8\pi} |x|^2 \log|x|$

2. Fundamental solution for wave equation, $n=1,2$

2d: $\begin{cases} u_{tt} - u_{xx} = 0 & \mathbb{R}^2 \times (0,\infty) \\ u(x,0) = 0, & u_t(x,0) = \delta(x), \quad x \in \mathbb{R}^2 \end{cases}$

By method of descent,

$$u(x_1, x_2, t) = \frac{1}{2\pi} \int_{|y| \le 1} h(x - ty) (1 - |y|^2)^{-\frac{1}{2}} dy$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} h(x - ty) (1 - |y|^2)^{-\frac{1}{2}} \mathbb{1}_{|y| \le 1} dy$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} h(x - s) \left(1 - \frac{|s|^2}{t^2}\right)^{-\frac{1}{2}} \frac{1}{t} \mathbb{1}_{|s| \le t} ds$$

$$= h * g, \quad g(x,t) = \left(1 - \frac{|s|^2}{t^2}\right)^{-\frac{1}{2}} \frac{1}{t} \mathbb{1}_{|s| \le t}$$

$$= \frac{1}{\sqrt{t^2 - |s|^2}} \mathbb{1}_{|s| \le t}$$

is fundamental solution

1d: $\begin{cases} u_{tt} - u_{xx} = 0, & \mathbb{R} \times (0,\infty) \\ u(x,0) = 0, & u_t(x,0) = \delta(x), \quad x \in \mathbb{R} \end{cases}$

By method of descent

$$u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} h(s) ds = \frac{1}{2} \int_{-t}^{t} h(x-u) du$$

$$= \frac{1}{2} \int_{\mathbb{R}} h(x-u) \mathbb{1}_{[-t,t]} du = h * g$$

$g(x) = \frac{1}{2} \mathbb{1}_{[-t,t]}(x)$ is fundamental solution

$$= \frac{1}{2} H(t - |x|)$$

3. $u(x) = v(r)$ nonlinear elliptic equation

$$-\Delta u = u^p, \quad x \in \mathbb{R}^n, \quad p>1$$

$$v(r)^p = -\sum_{i=1}^n v(r)_{x_i x_i} = -\sum_{i=1}^n (v_r r_{x_i})_{x_i} = -\sum_{i=1}^n \left(v_r \frac{x_i}{r}\right)_{x_i}$$

$$= -v_{rr} - v_r \frac{n-1}{r}$$

$$v'' + v' \frac{n-1}{r} + v^p = 0$$

Use Emden-Fowler transform:

$$t = \log r \qquad x(t) = e^{\frac{2t}{p-1}} v(e^t)$$

$$v(r) = e^{-\frac{2\log r}{p-1}} x(\log r) = r^{-\frac{2}{p-1}} x(\log r)$$

$$v'(r) = x'(\log r)\, r^{-\frac{p+1}{p-1}} - \frac{2}{p-1} r^{-\frac{p+1}{p-1}} x(\log r)$$

$$v''(r) = x''(\log r)\, r^{-\frac{2p}{p-1}} + x'(\log r)\left(-\frac{p+1}{p-1}\right) r^{-\frac{2p}{p-1}}$$
$$- \left[x'(\log r) \frac{2}{p-1} r^{-\frac{2p}{p-1}} - \frac{2(p+1)}{(p-1)^2} r^{-\frac{2p}{p-1}} x(\log r)\right]$$
$$= -\frac{n-1}{r}\left[x'(\log r)\, r^{-\frac{p+1}{p-1}} - \frac{2}{p-1} r^{-\frac{p+1}{p-1}} x(\log r)\right] - r^{-\frac{2p}{p-1}} x(\log r)$$

We have

$$x'' r^{-\frac{2p}{p-1}} - \frac{p+1}{p-1} x' r^{-\frac{2p}{p-1}} - \frac{2}{p-1} x' r^{-\frac{2p}{p-1}} + \frac{2(p+1)}{(p-1)^2} r^{-\frac{2p}{p-1}} x$$
$$+ (n-1) x' r^{-\frac{2p}{p-1}} - \frac{2(n-1)}{(p-1)} r^{-\frac{2p}{p-1}} x + r^{-\frac{2p}{p-1}} x = 0$$

$$(p-1)^2 x'' + (p-1)[(n-1)(p-1) - (p+3)]\, x' + [p^2+3 - 2(n-1)(p-1)]\, x = 0$$

$$\lambda_{1,2} = \frac{-[(n-1)(p-1) - (p+3)] \pm (p-1)\sqrt{n^2-4n}}{2(p-1)}$$

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$\Rightarrow v(r) = \begin{cases} r^{-\frac{2}{p-1}} \left(c_1 r^{\lambda_1} + c_2 r^{\lambda_2}\right), & c_1, c_2 \in \mathbb{R},\ n \geq 4 \\ r^{-\frac{2}{p-1}} r^{-\frac{(n-1)(p-1)-(p+3)}{2(p-1)}} C_1 \cos\left((p-1)\sqrt{4n-n^2}\, r + C_2\right) & C_1, C_2 \in \mathbb{R},\ n < 4 \end{cases}$$