4. $f_0(x) = 1_{[0,1]}$ $f_1(x) = 2x\, 1_{[0,1]}$

Single observation X

$H_0: f(x) = f_0(x) \longleftrightarrow H_a: f(x) = f_1(x)$

检验水平 0.1，求使二类错误概率 β 最小的检验法

否定域 $W_0 = \{X : X \in [0, 0.9]\}$ 此时 size $\alpha = 0.1$

考虑任一检验 H_1 否定域 W，size $= 0.1 \Rightarrow m(W) = 0.9$ $(W \subseteq [0,1])$

power $1-\beta = \int_W 2x\,dx \le \int_{[0,0.9]} 2x\,dx = 0.81$

故这是所求 UMP 检验（或者 N-P 引理，$(2x)^\alpha 1_{[0,1]}$，$\alpha = 0.1$）.

6. $X \sim N(\mu_0, \sigma^2)$ μ_0 已知 $X_1 \cdots X_n \sim X$

(1): $H_0: \sigma^2 \le \sigma_0^2 \longleftrightarrow H_a: \sigma^2 > \sigma_0^2$

(2) $H_0: \sigma^2 \le \sigma_1^2$ or $\sigma^2 \ge \sigma_2^2 \longleftrightarrow H_a: \sigma_1^2 < \sigma^2 < \sigma_2^2$

UMP 检验（检验水平 α）

$T(X) = \sum_{i=1}^n (X_i - \mu_0)^2$

(1) $W = \{(X_1, \cdots X_n) \mid \sum_{i=1}^n (X_i - \mu_0)^2 > C\}$

$\alpha = \mathbb{P}(\sum (X_i - \mu_0)^2 > C \mid \sigma_0^2)$

$\frac{1}{\sigma_0^2} \sum (X_i - \mu_0)^2 \sim \chi^2(n)$

设 $\chi^2(n)$ PDF 为 $f(x;n)$ 取 $\int_{-\infty}^{\frac{C}{\sigma_0^2}} f(x;n) = 1-\alpha$

此 W 即为所求 UMP

(2) $W = \{(x_1, \cdots x_n) \mid C_1 < \sum_{i=1}^{n}(x_i - \mu_0)^2 < C_2\}$

$\alpha = \mathbb{P}((x_1 \cdots x_n) \in W \mid \sigma_1) = \mathbb{P}((x_1, \cdots x_n) \in W \mid \sigma_2)$

$\int_{\frac{C_1}{\sigma_1^2}}^{\frac{C_2}{\sigma_1^2}} f(x;n)\,dx = \alpha$ $\quad\quad$ $\int_{\frac{C_1}{\sigma_2^2}}^{\frac{C_2}{\sigma_2^2}} f(x;n)\,dx = \alpha$

此即所求 UMP

7. $X \sim N(\mu, 1)$ $\quad x_1 \cdots x_n \sim X$

$H_0: \mu \leq \mu_1$ or $\mu \geq \mu_2 \longleftrightarrow H_a: \mu_1 < \mu < \mu_2$

UMP 检验

$T(x) = \sum_{i=1}^{n} x_i$

$W = \{(x_1, \cdots x_n) \mid C_1 < \sum_{i=1}^{n} x_i < C_2\}$

$\alpha = \mathbb{P}((x_1 \cdots x_n) \in W \mid \mu = \mu_1)$

$\alpha = \mathbb{P}((x_1 \cdots x_n) \in W \mid \mu = \mu_2)$

$\sum_{i=1}^{n} x_i \sim N(n\mu_i, \sqrt{n}) = n\mu_i + \sqrt{n}\,Z$

$C_1 < \sum_{i=1}^{n} x_i < C_2 \iff \frac{C_1 - n\mu_i}{\sqrt{n}} < Z < \frac{C_2 - n\mu_i}{\sqrt{n}}$

$\Phi\left(\frac{C_2 - n\mu_1}{\sqrt{n}}\right) - \Phi\left(\frac{C_1 - n\mu_1}{\sqrt{n}}\right) = \alpha$

$\Phi\left(\frac{C_2 - n\mu_2}{\sqrt{n}}\right) - \Phi\left(\frac{C_1 - n\mu_1}{\sqrt{n}}\right) = \alpha.$

此即所求 UMP 检验

14. $f_n(x) = \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}} \left(1+\dfrac{x^2}{n}\right)^{-\frac{n+1}{2}}$

$$\lim_{n\to\infty} f_n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

证:

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}} \sim \frac{\sqrt{\frac{4\pi}{n+1}}\left(\frac{n+1}{2e}\right)^{\frac{n+1}{2}}}{\sqrt{\frac{4\pi}{n}}\left(\frac{n}{2e}\right)^{\frac{n}{2}}\sqrt{n\pi}} e^{-\frac{x^2}{2}}$$

$$\sim \frac{\left(1+\frac{1}{n}\right)^{\frac{n}{2}}\left(\frac{n+1}{2e}\right)^{\frac{1}{2}}}{\sqrt{n\pi}} e^{-\frac{x^2}{2}} \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (n\to\infty).$$

8. $X_1 \cdots X_n \sim Poisson(\lambda)$ 检验水平 α

$H_0: \lambda = \lambda_0 \leftrightarrow H_a: \lambda > \lambda_0$ UMP 检验

$$P_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda} = \frac{e^{-\lambda}}{k!} e^{k\ln\lambda}$$

$$t(X_1,\cdots X_n) = \sum_{i=1}^n X_i$$

$$W(X_1,\cdots X_n) = \left\{(X_1,\cdots X_n) \,\middle|\, \sum_{i=1}^n X_i > C\right\}.$$

$$\alpha = \mathbb{P}\left(\sum_{i=1}^n X_i > C \,\middle|\, \lambda=\lambda_0\right) = 1 - \sum_{s=0}^{C} \frac{(n\lambda_0)^s}{s!} e^{-n\lambda_0}$$

每一对 $C \in \mathbb{Z}$, $\alpha \in (0,1)$ 对应一个检验水平的 UMP test

11. $X_1 \dots X_n \sim N(\mu, 1)$

$H_0: \mu \le 0 \leftrightarrow H_a: \mu > 0$

$\sum X_i \sim N(n\mu, \sqrt{n}) = n\mu + \sqrt{n} Z > C$

$0.025 = \mathbb{P}\left(Z > \frac{C - n\mu}{\sqrt{n}}\right)$

(1) $\alpha = 0.025$ UMP, power $\rho(\mu)$

$T(X) = \sum X_i$ $\quad W = \{(x_1 \dots x_n) \mid \sum X_i > C\}$ $\quad \frac{C}{\sqrt{n}} = 1.96$ $\quad Z > 1.96 - \sqrt{n}\mu$

UMP: $W = \{(X_1 \dots X_n) \mid \sum X_i > 1.96\sqrt{n}\}$ $\quad \rho(\mu) = 1 - \Phi(1.96 - \sqrt{n}\mu)$

(2) $\mu \ge 0.5$ 时 $\rho(\mu) \ge 0.9$, n 至少多大

即 $\Phi(1.96 - 0.5\sqrt{n}) \le 0.1$, $n \ge 43$

$(\Phi^{-1}(0.1) = -1.28155)$

(3) $\mu \le -0.1$ 时 $\rho(\mu) \le 0.001$, n 至少多大

即 $\Phi(1.96 + 0.1\sqrt{n}) \ge 0.999$

$n \ge 128$

$(\Phi^{-1}(0.999) = 3.09023)$

10. $X_1 \dots X_n \sim \text{Uniform}[0, \theta]$

$H_0: \theta = \theta_0 \leftrightarrow H_a: \theta > \theta_0$ UMP 检验

$L(X_1, \dots X_n \mid \theta) = 1_{X_1 \le \theta} \cdots 1_{X_n \le \theta} = 1_{\max(X_1, \dots X_n) \le \theta}$

$W = \{(X_1, \dots X_n) \mid \max(X_1, \dots X_n) > C\}$

$\alpha = 1 - \left(\frac{C}{\theta_0}\right)^n$

$C = \theta_0 (1-\alpha)^{\frac{1}{n}}$