1. $X_i \overset{iid}{\sim} U[0,1]$

$$\lim_{n\to\infty} \frac{n}{X_1^{-1}+\cdots+X_n^{-1}}, \quad \lim_{n\to\infty} \sqrt[n]{X_1\cdots X_n}, \quad \lim_{n\to\infty} \sqrt{\frac{X_1^2+\cdots+X_n^2}{n}}$$

$\mathbb{E} X_i^2 = \frac{1}{3}$ SLLN $\Rightarrow$ $\overline{X_i^2} \xrightarrow{a.s.} \frac{1}{3}$. $\Rightarrow \lim_{n\to\infty} \sqrt{\frac{X_1^2+\cdots+X_n^2}{n}} = \frac{1}{\sqrt{3}}$ a.s.

$\mathbb{E} \log X = \int_0^1 \log x \, dx = -1$ SLLN $\Rightarrow$ $\overline{\log X_n} \xrightarrow{a.s.} -1$

$$\lim_{n\to\infty} \sqrt[n]{X_1\cdots X_n} = \frac{1}{e} \text{ a.s.}$$

$Y_i = \frac{1}{X_i}$ is supported on $(1,+\infty)$ $\mathbb{E} Y_i^+ = +\infty$ $\mathbb{E} Y_i^- = 0$

$(Y_i^+ = \max(Y_i, 0),\ Y_i^- = \min(Y_i, 0))$.

$$\Rightarrow \lim_{n\to\infty} \frac{Y_1+\cdots+Y_n}{n} = \infty \text{ a.s.}$$

$$\lim_{n\to\infty} \frac{n}{\frac{1}{X_1}+\cdots+\frac{1}{X_n}} = 0 \text{ a.s.}$$

2. $X_n \overset{iid}{\sim} \mu$ $\mathbb{E} X_1 = 0$

$$Z_n = \frac{X_1+\cdots+X_n}{\sqrt{n}} \xrightarrow{d} X. \quad Z_{2n} = \frac{X_1+\cdots+X_{2n}}{\sqrt{2n}} \xrightarrow{d} X$$

$f_X(\xi)$ is characteristic func. of X

(a) $f(\xi) = f^2(\xi/\sqrt{2})$

Let g be charcteristic func. of X_i

$$Z_n \xrightarrow{d} X \Rightarrow \mathbb{E} e^{i\xi \frac{X_1+\cdots+X_n}{\sqrt{n}}} = g^n(\xi/\sqrt{n}) \to f(\xi)$$

$$Z_{2n} \xrightarrow{d} X \Rightarrow \mathbb{E} e^{i\xi \frac{X_1+\cdots+X_{2n}}{\sqrt{2n}}} = g^{2n}(\xi/\sqrt{2n}) \to f(\xi)$$

$$f^2(\xi/\sqrt{2}) = \left(\lim_{n\to\infty} g^n(\xi/\sqrt{2n})\right)^2 = \lim_{n\to\infty} g^{2n}(\xi/\sqrt{2n}) = f(\xi)$$

#

(b) if $f \in C^2(\mathbb{R})$, $f(\xi)$ is charactristic func of a gaussian r.v.

For $m = 2, 3, \cdots$ we also have

$f(\xi) = f^m(\xi/\sqrt{m})$ Let $f(x) = e^{g(x)}$, $x \in \mathbb{R}$

So $g(\xi) = m\, g(\xi/\sqrt{m})$, $g(\sqrt{m}\, t) = m g(t)$, $g(mt) = m^2 g(t)$, $m = 2, 3, \cdots$ $t \in \mathbb{R}$

So $g(x) = g(1) x^2$ holds for x is integers $\to$ rationals $\xrightarrow{f \text{ continus}}$ all reals

So $f(\xi)$ takes form of $e^{-k\xi^2}$, the characteristic function of gaussian r.v.

(c) Replace $1/\sqrt{n}$ with $1/n$, X Corresponds to Cauchy-Lorentz distribution if $f(\xi) = f(-\xi)$ or $f(\xi) \equiv 1$

$f(\xi) \equiv 1 \Leftrightarrow \delta$ distribution; $f(\xi) = f(-\xi) \Rightarrow f \in \mathbb{R}$

We also have $f(\xi) = f^m(\xi/m) > 0$, $m = 2, 3, \cdots$

Let $f(\xi) = e^{g(\xi)}$ $\Rightarrow g(m\xi) = m\, g(\xi)$

Similarly we have g is linear

$f(\xi) = e^{k\xi}$ which is charactaristic func of Cauchy r.v.

(d) Replace $1/\sqrt{n}$ with $1/n^\alpha$ Given $f(\xi) = f(-\xi)$, what can be known about $f(\xi)$. range of α?

We have $f(\xi) = f^m(\xi/m^\alpha) > 0$, $m = 2, 3, \cdots$

$f(\xi) = e^{g(\xi)}$

$g(\xi) = m\, g(\xi/m^\alpha)$

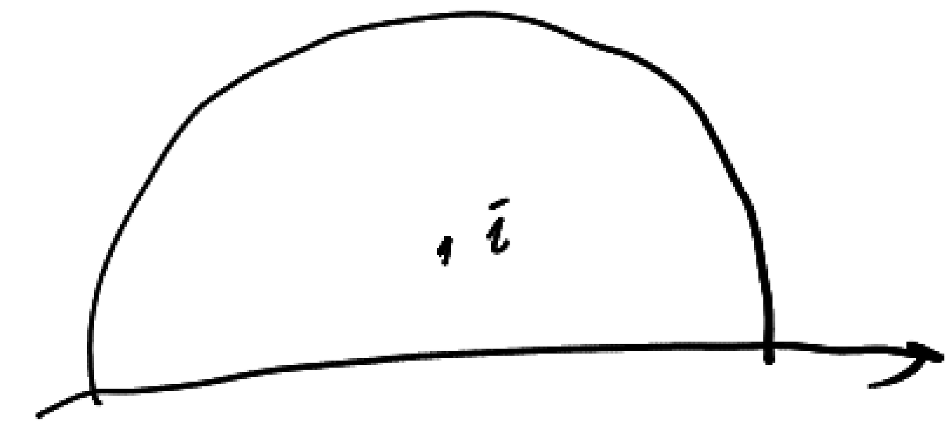
$g(m^\alpha t) = m\, g(t)$, $t \in \mathbb{R}$, $m = 2, 3, \cdots$

If $\alpha = \frac{1}{p}$ $p = 1, 2, \cdots$

$g(mt) = m^p g(t)$ $\Rightarrow g(x) = k x^p$

We can deduce $f(\xi) = e^{-k x^p}$

3. $X_n \overset{iid}{\sim} p(x) = \frac{1}{\pi(1+x^2)}, x \in \mathbb{R}$



$\mathbb{E}X_1 = 0 \quad \mathbb{E}|X_1| = \infty, \ \mathbb{E}(X_1)^2 = \infty$

$S_n/n \sim X_1$. WLLN/ SLLN fails.

$\mathbb{E}|X_1| = 2\int_0^\infty \frac{x\,dx}{\pi(1+x^2)} = +\infty \qquad \mathbb{E}X_1^2 = 2\int_0^\infty \frac{x^2 dx}{\pi(1+x^2)} = +\infty$

X_1 characteristic func. $f_{X_1}(\xi) = \mathbb{E}e^{i\xi X_1} = \int_{\mathbb{R}} e^{i\xi x}\frac{1}{\pi(1+x^2)}dx = e^{-\xi}$

$f_{S_n}(\xi) = f_{X_1}^n(\xi/n) = (e^{-\xi/n})^n = e^{-\xi}$

$\Rightarrow S_n/n \sim X_1$ #

4. $h(x)$ $x=0$ only maximum, $h' \in C'(0,+\infty)$, $h'(0) < 0$, $h(x) < h(0)$. $x>0$

$h(x) \to -\infty$, Prove $\int_0^\infty e^{th(x)}dx$ converges to the leading order:

$$\int_0^\infty e^{th(x)}dx \sim (-th'(0))^{-1} e^{th(0)} \quad \text{as } t\to\infty$$

<u>Proof</u>. WLOG Let $h(0)=0$

$\forall \varepsilon>0 \ \exists \delta>0, \ \forall x \in [0,\delta]$
$|h(x) - (h'(0)x)| \le \varepsilon$ and $h(x) \le -c, \ x \ge \delta$

$\int_0^\infty e^{th(x)}dx = \int_0^\delta e^{th(x)}dx + \int_\delta^\infty e^{th(x)}dx$

$\le \int_0^\infty e^{th'(0)x+\varepsilon}dt - \int_\delta^\infty e^{th'(0)x}dt \qquad + \int_\delta^\infty e^{h(x)}dx\, e^{-c(t-1)}$

$\le e^\varepsilon(-th'(0))^{-1} + O(e^{-ct}) + \frac{e^{th'(0)\delta}}{th'(0)}$

$\Rightarrow \limsup_{t\to\infty} \int_0^\infty e^{th(x)}dx \le (-th'(0))^{-1}e^\varepsilon$

$\varepsilon \to 0^+ \quad \lim_{t\to\infty}\int_0^\infty e^{th(x)}dx \le -(th'(0))^{-1}$

Another direction is similar. #

5. $I(x)$ for $N(\mu, \sigma^2)$ and $Exp(\lambda)$.

$N(\mu, \sigma^2)$: MGF $M_X(t) = e^{t(\mu + \frac{1}{2}\sigma^2 t)}$ $\Lambda(s) = t(\mu + \frac{1}{2}\sigma^2 t)$

$$I(x) = \sup_{s \in \mathbb{R}} (sx - \Lambda(s))$$

$$= \sup_{s \in \mathbb{R}} (sx - s\mu - \frac{1}{2}\sigma^2 s^2)$$

$$s^* = \frac{x-\mu}{\sigma^2}$$

$$I(x) = \frac{(x-\mu)^2}{2\sigma^2}$$

$Exp(\lambda)$: MGF $M_X(t) = \frac{1}{1 - \frac{t}{\lambda}}$, $t < \lambda$

$$I(x) = \sup_{s \in \mathbb{R}} (sx - M(s))$$

$$= \sup_{s \in \mathbb{R}} sx + \log(1 - \frac{s}{\lambda})$$

$$x + \frac{-\frac{1}{\lambda}}{1 - \frac{s^*}{\lambda}} = 0$$

$$s^* = \lambda - \frac{1}{x}$$

$$I(x) = \lambda x - 1 - \log(\lambda x)$$