$\Box u = u_{tt} - \Delta u \qquad \Box_a u = u_{tt} - a^2 \Delta u$

$\hat{g}(\lambda) = \int_{\mathbb{R}} g(x) e^{-2\pi i \lambda x} dx$

$\hat{h}(\lambda) = \int_{\mathbb{R}} h(x) e^{-2\pi i \lambda x} dx$

1. $$\begin{cases} u_{tt} + 2d\, u_t - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, \ u_t = h & \text{in } \mathbb{R} \times \{t = 0\} \end{cases}$$

$\hat{u}(\lambda, 0) = \hat{g}(\lambda) \qquad \hat{u}_t(\lambda, 0) = \hat{h}(\lambda)$

$$\frac{d^2}{dt^2} \hat{u}(\lambda, t) + (2d) \frac{d}{dt} \hat{u}(\lambda, t) + 4\pi\lambda^2 \hat{u}(\lambda, t) = 0$$

$\lambda_{1,2} = -d \pm \sqrt{d^2 - 4\pi\lambda^2} \rightarrow$ (Can be real or imaginary)

$$\hat{u}(\lambda, t) = C_1(\lambda) e^{(-d + \sqrt{d^2 - 4\pi\lambda^2})t} + C_2(\lambda) e^{(-d - \sqrt{d^2 - 4\pi\lambda^2})t}$$

$$\hat{g}(\lambda) = C_1(\lambda) + C_2(\lambda)$$

$$\hat{h}(\lambda) = (-d + \sqrt{d^2 - 4\pi\lambda^2}) C_1(\lambda) + (-d - \sqrt{d^2 - 4\pi\lambda^2}) C_2(\lambda)$$

$$C_1(\lambda) = \frac{\hat{h}(\lambda) + (d + \sqrt{d^2 - 4\pi\lambda^2}) \hat{g}(\lambda)}{2\sqrt{d^2 - 4\pi\lambda^2}}, \quad C_2(\lambda) = \frac{\hat{h}(\lambda) + (d - \sqrt{d^2 - 4\pi\lambda^2}) \hat{g}(\lambda)}{-2\sqrt{d^2 - 4\pi\lambda^2}}$$

$$u(x,t) = \int_{\mathbb{R}} \hat{h}(\lambda) \frac{e^{i(-d + \sqrt{d^2 - 4\pi\lambda^2})t} - e^{i(-d - \sqrt{d^2 - 4\pi\lambda^2})t}}{2\sqrt{d^2 - 4\pi\lambda^2}} e^{2\pi i \lambda x} d\lambda$$
$$+ \frac{1}{2} \int_{\mathbb{R}} \hat{g}(\lambda) \left[\left(1 + \frac{d}{\sqrt{d^2 - 4\pi\lambda^2}}\right) e^{i(-d + \sqrt{d^2 - 4\pi\lambda^2})t} + \left(1 - \frac{d}{\sqrt{d^2 - 4\pi\lambda^2}}\right) e^{i(-d - \sqrt{d^2 - 4\pi\lambda^2})t} \right] \cdot \left[e^{2\pi i \lambda x} \right] d\lambda$$

Unfortunately, no explicit formula can be given.

2. $u(x, y, t)$ be solution to

$$\begin{cases} \Box_2 u(x,y,t) = 0, & (x,y) \in \mathbb{R}^2, \ t > 0 \\ u|_{t=0} = \varphi(x,y) \\ u_t|_{t=0} = \phi(x,y) \end{cases}$$

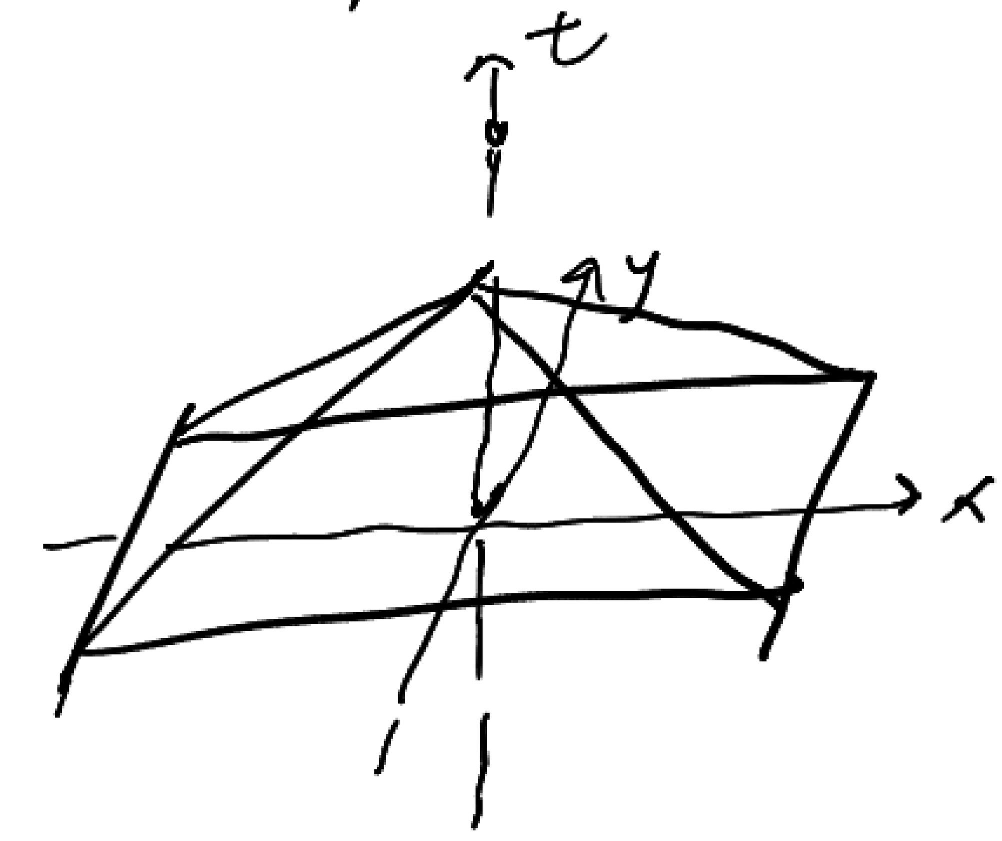
$\Omega = \{(x,y) \mid |x| \le 1, |y| \le 1\}$

$$\varphi(x,y) = \begin{cases} 0 & (x,y) \in \Omega \\ 10 & \text{else} \end{cases}$$

Find domain over which $u(x,y,t) \equiv 0$

We know $u(x,y,t) = t M_t \phi(x,y) + \frac{\partial}{\partial t}[t M_t \varphi(x,y)]$. $\quad \phi \ge 0$, Wave equation in 2d follows weak Huygens principle. wave speed=2

So answer is $\{(x,y,t) \mid |x| \le 1-s, |y| \le 1-s, t = \frac{s}{2}\}$



Cone in (x,y,t) space

3. $\begin{cases} u_t = x^2 u_{xx} + a x u_x & x \in (0,\infty),\ t>0 \\ u(x,0) = u_0(x) \end{cases}$

Hint: $x = e^{-y}$, $-\infty < y < \infty$ $\quad y = -\log x$

$u(x,t) := v(y,t) \qquad u_t = v_t$

$u_x = v_y\, y_x = -\frac{1}{x} v_y$

$u_{xx} = \left(-\frac{1}{x} v_y\right)_x = \frac{1}{x^2} v_y - \frac{1}{x} v_{yy}\, y_x = \frac{1}{x^2} v_y + \frac{1}{x^2} v_{yy}$

$v_t = v_y + v_{yy} - a v_y = (1-a) v_y + v_{yy}$

$v(y,0) = u_0(x) = u_0(e^{-y}) := u_1(y)$

$\frac{d}{dt} \hat{v}(\xi,t) = \hat{v}(\xi,t)(1-a) 2\pi i \xi - \hat{v}(\xi,t) 4\pi^2 \xi^2$

$= \hat{v}(\xi,t)\left[(1-a) 2\pi i \xi - 4\pi^2 \xi^2\right]$

$\hat{v}(\xi,0) = \hat{u}_1(\xi)$

$\hat{v}(\xi,t) = \hat{u}_1(\xi)\, e^{[(1-a)2\pi i \xi - 4\pi^2 \xi^2] t}$

記 $\hat{f}(y,t) = e^{[(1-a)2\pi i \xi - 4\pi^2 \xi^2] t}$ 則 $v = f * u_1$

$f(y,t) = \int_{\mathbb{R}} e^{[2\pi i (1-a)\xi - 4\pi^2 \xi^2] t} e^{2\pi i \xi y}\, d\xi$

$= \int_{\mathbb{R}} e^{\xi 2\pi i (t(1-a)+y)} e^{-4\pi^2 \xi^2 t}\, d\xi$

$= \int_{\mathbb{R}} e^{-4\pi^2 t \left(\xi - \frac{i(t(1-a)+y)}{4\pi t}\right)^2} e^{-\frac{(t(1-a)+y)^2}{4t}}\, d\xi$

$= \frac{1}{2\sqrt{\pi t}} e^{-\frac{(t(1-a)+y)^2}{4t}}$

$u(x,t) = v(y,t) = \int_{\mathbb{R}} u_1(y-s) f(s,t)\, ds = \int_{\mathbb{R}} u_0\left(\frac{1}{x} e^{-s}\right) e^{-\frac{(t(1-a)+s)^2}{4t}}\, ds$ #

4. $\mu > 0$, Explicit form of Yukawa potential in $d=3$

$$-\Delta Y_\mu(x) + \mu^2 Y_\mu(x) = \delta(x), \quad x \in \mathbb{R}^3$$

$$4\pi^2 |\lambda|^2 \hat{Y}_\mu(\lambda) + \mu^2 \hat{Y}_\mu(\lambda) = 1$$

$$\hat{Y}_\mu(\lambda) = \frac{1}{4\pi|\lambda|^2 + \mu^2} = \int_0^\infty e^{-(4\pi|\lambda|^2 + \mu^2)t} \, dt$$

$$Y_\mu(x) = \int_0^\infty e^{-\mu^2 t} \int_{\mathbb{R}^3} e^{-4\pi|\lambda|^2 t} \cdot e^{2\pi i \lambda \cdot x} \, dx$$

$\downarrow$ Heat kernel

$$= \int_0^\infty e^{-\mu^2 t - \frac{x^2}{4t}} (4\pi t)^{-\frac{3}{2}} \, dt$$

$$= \int_0^\infty e^{-\frac{x^2}{2}\left(\frac{1}{\sqrt{2t}} - \frac{\mu}{|x|}\sqrt{2t}\right)^2} e^{-\mu|x|} (4\pi t)^{-\frac{3}{2}} \, dt$$

$$= \frac{e^{-\mu|x|}}{(2\pi)^{\frac{3}{2}}} \int_0^\infty e^{-\frac{x^2}{2}\left(u - \frac{\mu}{|x|}\frac{1}{u}\right)^2} \, du$$

Meanwhile, $$Y_\mu(x) = \frac{e^{-\mu|x|}}{(2\pi)^{\frac{3}{2}}} \int_0^\infty e^{-\frac{x^2}{2}\left(s - \frac{\mu}{|x|}\frac{1}{s}\right)^2} \frac{1}{s^2} \frac{\mu}{|x|} \, ds$$

$$\Rightarrow 2Y_\mu(x) = \frac{e^{-\mu|x|}}{(2\pi)^{\frac{3}{2}}} \int_0^\infty e^{-\frac{x^2}{2}\left(s - \frac{\mu}{|x|}\frac{1}{s}\right)^2} \left(1 + \frac{1}{s^2} \frac{\mu}{|x|}\right) ds$$

$$= \frac{e^{-\mu|x|}}{(2\pi)^{\frac{3}{2}}} \int_0^\infty e^{-\frac{x^2}{2}\left(s - \frac{\mu}{|x|}\frac{1}{s}\right)^2} \, d\left(s - \frac{\mu}{|x|}\frac{1}{s}\right)$$

$$= \frac{e^{-\mu|x|}}{(2\pi)^{\frac{3}{2}}} \int_0^\infty e^{-\frac{x^2 w^2}{2}} \, dw = \frac{e^{-\mu|x|}}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{2\pi}{|x|}}$$

$$= \frac{e^{-\mu|x|}}{2\pi|x|}$$

$$\Rightarrow Y_\mu(x) = \frac{e^{-\mu|x|}}{4\pi|x|}$$