习题1

4.

(1) $u, v \in C^1(\Omega) \cap C(\overline{\Omega})$,

$$\int_\Omega u_{x_i} v\,dx = -\int_\Omega u v_{x_i}\,dx + \int_{\partial\Omega} u v n_i\,dS(x)$$

Proof: 记 $\vec{F} = (0, \cdots 0, \underset{i位}{uv}, 0, \cdots 0)$ 则 $\vec{F} \in C^1(\Omega) \cap C(\overline{\Omega})$

由 Gauss-Green 公式,

$$\int_\Omega \operatorname{div}\vec{F}\,dx = \int_{\partial\Omega} \vec{F}\cdot\vec{n}\,dS(x).$$

此即 $\displaystyle\int_\Omega \frac{\partial(uv)}{\partial x_i}\,dx = \int_{\partial\Omega} u v n_i\,dS(x)$

即 $\displaystyle\int_\Omega u_{x_i} v\,dx = -\int_\Omega v_{x_i} u\,dx + \int_{\partial\Omega} u v n_i\,dS(x)$

(2) $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$,

$$\int_\Omega \Delta u\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}}\,dS(x)$$

Proof: $Du = \left(\frac{\partial u}{\partial x_1}, \cdots \frac{\partial u}{\partial x_n}\right) \in C^1(\Omega) \cap C(\overline{\Omega})$

则由 Gauss-Green 公式,

$$\int_\Omega \operatorname{div}\overrightarrow{Du}\,dx = \int_{\partial\Omega} \overrightarrow{Du}\cdot\vec{n}\,dS(x) = \int_{\partial\Omega}\left(\frac{\partial u}{\partial x_1}n_1 + \cdots + \frac{\partial u}{\partial x_n}n_n\right)dS(x)$$

此即 $\displaystyle\int_\Omega \Delta u\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}}\,dS(x)$ #

(3) $u \in C^1(\Omega) \cap C(\bar{\Omega})$, $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$,

$$\int_\Omega \vec{Du} \cdot \vec{Dv}\, dx = -\int_\Omega u \Delta v\, dx + \int_{\partial\Omega} u \frac{\partial v}{\partial \vec{n}}\, dS(x)$$

Proof: 记 $\vec{F} = (uv_{x_1}, \cdots uv_{x_n})$ 则 $\vec{F} \in C^1(\Omega) \cap C(\bar{\Omega})$

由 Gauss-Green公式,

$$\int_\Omega \operatorname{div} \vec{F} \cdot dx = \int_{\partial\Omega} \vec{F} \cdot \vec{n}\, dS(x)$$

$$\int_\Omega \sum_{i=1}^n uv_{x_i x_i} + u_{x_i} v_{x_i}\, dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \vec{n}}\, dS(x)$$

也即 $\int_\Omega \vec{Du} \cdot \vec{Dv}\, dx = -\int_\Omega u \Delta v\, dx + \int_{\partial\Omega} u \frac{\partial v}{\partial \vec{n}}\, dS(x)$

(4) $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$,

$$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}}\right) dS(x)$$

Proof: 记 $\vec{F} = (uv_{x_1} - vu_{x_1}, \cdots, uv_{x_n} - vu_{x_n})$

$\vec{F} \in C^1(\Omega) \cap C(\bar{\Omega})$ $\qquad u_{y_1y_1} + u_{y_2y_2} = 0$

由 Gauss-Green 公式,

$$\int_\Omega \sum_{i=1}^n \left(u_{x_i} v_{x_i} + u v_{x_i x_i} - v_{x_i} u_{x_i} - v u_{x_i x_i}\right) dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}}\right) dS(x)$$

整理立得. #.

6. 化为标准型

(1) $u_{xx} + 2u_{xy} + 2u_{yy} = 0$ Elliptic PDE

设 $\vec{y} = B\vec{x}$

我们有 $\nabla_y := \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix} = B^T \nabla_x := B^T \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$

原方程 $\nabla_x^T A \nabla_x \vec{x} = 0$ ，其中 $A = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$

化为 $\nabla_y^T B A B^T \nabla_y \vec{y} = 0$

只需 $BAB^T=-I$ 取 $B=\begin{pmatrix}1&0\\-1&1\end{pmatrix}$ 即可 此时 $y_1=x$ $y_2=-x+y$

(就是高代中二次型标准型)

$u_{y_1y_1}+u_{y_2y_2}=0$

(2) $u_{xx}+2u_{xy}+u_{yy}=0$ Parabolic PDE

同理

$A=\begin{pmatrix}-1&-1\\-1&-1\end{pmatrix}$

要 $BAB^T=-\begin{pmatrix}1&\\&0\end{pmatrix}$

取 $B=\begin{pmatrix}1&0\\-1&1\end{pmatrix}$ $\begin{cases}y_1=x\\y_2=-x+y\end{cases}$

$u_{y_1y_1}=0$

(3) $u_{xx}+4u_{xy}+u_{yy}=0$

同理 $A=\begin{pmatrix}-1&-2\\-2&-1\end{pmatrix}$

要 $BAB^T=\begin{pmatrix}-1&\\&1\end{pmatrix}$

取 $B=\begin{pmatrix}1&\\-\frac{2}{\sqrt{3}}&\frac{1}{\sqrt{3}}\end{pmatrix}$

故 $\begin{cases}y_1=x\\y_2=-\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}y\end{cases}$

$-u_{y_1y_1}+u_{y_2y_2}=0$

7. 化为标准型

(1) $\sum_{i=1}^{n} u_{x_i x_i} + \sum_{1\le i<j\le n} u_{x_i x_j} = 0$ elliptic

其中 $J = \mathbb{1}_{n\times n}$,

Spec(J): 0 ... 0 n

特征向量 (1, ..., -1) (0...1,-1) (1, ... 1)

Spec(A): $-\frac{1}{2}$ ... $-\frac{1}{2}$ $-\frac{n+1}{2}$

$$A = \begin{pmatrix} -1 & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & -1 & & & -\frac{1}{2} \\ \vdots & & \ddots & & -\frac{1}{2} \\ -\frac{1}{2} & & & -\frac{1}{2} & -1 \end{pmatrix} = -\frac{1}{2}I - \frac{1}{2}J$$

有特征分解

$A = C^T \Lambda C$, $\Lambda = \mathrm{diag}(-\frac{1}{2}, \cdots -\frac{1}{2}, -\frac{n+1}{2})$. $C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \cdots & 0 & \frac{1}{\sqrt{n}} \\ \vdots & & \vdots & \vdots \\ & & \frac{1}{\sqrt{2}} & \\ -\frac{1}{\sqrt{2}} & & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{n}} \end{pmatrix}$

而 Λ

$= D^T(-I)D$ $\quad D = \mathrm{diag}\left(\frac{1}{\sqrt{2}} \cdots \frac{1}{\sqrt{2}}, \sqrt{\frac{n+1}{2}}\right)$ $\Rightarrow DCA(DC)^T = -I$

$$\Rightarrow B = DC = \begin{pmatrix} \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{\sqrt{2n}} \\ 0 & \frac{1}{2} & & \vdots & \vdots \\ \vdots & & \ddots & \frac{1}{2} & \frac{1}{\sqrt{2n}} \\ 0 & & & & \\ -\sqrt{n+1} & -\sqrt{n+1} & \cdots & & \sqrt{\frac{n+1}{2n}} \end{pmatrix}$$

$\vec{y} = B\vec{x}$

方程为 $\frac{\partial^2 u}{\partial y_1^2} + \cdots + \frac{\partial^2 u}{\partial y_n^2} = 0$

(2) $\sum_{1\le j<i\le n} u_{x_i x_j} = 0$ hyperbolic

Spec(J): 0 ... 0 n

特征向量 (1 ... -1) ... (0...1 -1) (1 ... 1)

Spec(A): -1 ... -1 n-1

$$A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} = J - I$$

$A = C^T \Lambda C$, $\Lambda = \mathrm{diag}(-1, \cdots -1, n-1)$ $\quad C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \cdots & 0 & \frac{1}{\sqrt{n}} \\ \vdots & & \vdots & \vdots \\ & & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{n}} \\ -\frac{1}{\sqrt{2}} & & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{n}} \end{pmatrix}$

$CAC^T = \Lambda = D^T \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} D$

$$B = DC = \begin{pmatrix} \frac{1}{\sqrt{2}} & \cdots & & \frac{1}{\sqrt{n}} \\ \vdots & & & \\ 0 & \cdots & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{n}} \\ -\sqrt{\frac{n-1}{2}} & \cdots & -\sqrt{\frac{n-1}{2}} & \sqrt{\frac{n-1}{n}} \end{pmatrix}$$

其中 $D = \mathrm{diag}(1, \cdots 1, \sqrt{n-1})$

$\vec{y} = B\vec{x}$ 方程 $-u_{y_1y_1} \cdots - u_{y_{n-1}y_{n-1}} + u_{y_ny_n} = 0$.