Lec03

1. Familiarize mean, median, min, max, cov, hist in matlab (omitted)

2. Sample from $U(S^2)$ in different ways. (See code attached)

3. Algo 2-6 overall rejection probability (Acc-Rej with general comparison function)

$X \sim p(x)$ S1: $X_i = F^{-1}(A Z_i)$. $Z_i \sim U[0,1]$

S2: $Y_i \sim U[0, f(X_i)]$ $X_i \sim$

S3: Accept if $0 \le Y_i < p(X_i)$

$p(\text{Accept} \mid X_i) = \frac{p(X_i)}{f(X_i)}$ We know PDF of X_i is $\frac{f(x)}{A}$.

$p(\text{-cept}) = \int p(\text{Accept} \mid X_i) \frac{f(X_i)}{A} dX_i = \frac{1}{A}$. $p(\text{reject}) = 1 - \frac{1}{A}$.

4. Envelope acc-rej

Gen $X \sim p(x)$ $g_\ell(x) \le p(x) \le M g_m(x)$ $g_m(x)$ is pdf

$M > 0$. $g_1(x) \ge 0$ simple.

S1. Generate $X \sim g_m(x)$, $U \sim U[0,1]$

S2 Accept X IF $U \le \frac{g_1(x)}{M g_m(x)}$

S3. ELSE acept X if $U \le \frac{p(x)}{M g_m(x)}$

generates X correctly. Advantage?

Advantage: Since $g_1(x)$ is faster to compute than $p(x)$, aception of smaller X is faster, overall accelerating sampling

Correctness: S2+S3 is basically S4: Accept X if $U \le \frac{p(x)}{M g_m(x)}$

S1 + S4 is just normal acc-rej algorithm.

Lec 04.

1. $X \sim U[0,1]$ $f(x)$ monotone

$$\mathrm{Cov}(f(X), f(1-X)) \le 0$$

$$\mathrm{Cov}(f(x), 1-f(x)) = \int_0^1 f(x) f(1-x)\,dx - \left(\int_0^1 f(x)dx\right)^2 = \int_0^1 \int_0^1 f(x) f(1-y)\,dx\,dy$$

$$- \int_0^1 \int_0^1 f(x) f(y)\,dx\,dy = \int_0^1 \int_0^1 f(x) f(1-x)\,dx\,dy - \int_0^1 \int_0^1 f(x) f(1-y)\,dx\,dy$$

$$= \int_0^1 dx \int_0^x f(x)\,[f(1-x) - f(1-y)]\,dy + \int_0^1 dy \int_0^y f(x)\,[f(1-x) - f(1-y)]\,dy$$

$$= \int_0^1 dx \int_0^x (f(x) - f(y))\,(f(1-x) - f(1-y))\,dy \le 0.$$

$$X = (X^{(1)}, X^{(2)})$$

2. Prove $\mathrm{Var} f(X) = \mathrm{Var}(\mathbb{E} f(X) | X^{(2)}) + \mathbb{E}\,\mathrm{Var}(f(X) | X^{(2)})$

We prove law of total variance:

$$\mathrm{Var} Y = \mathbb{E}\,\mathrm{Var}(Y|X) + \mathrm{Var}(\mathbb{E}(Y|X))$$

$$\mathrm{Var} Y = \mathbb{E} Y^2 - (\mathbb{E} Y)^2$$

$$\mathbb{E} Y^2 = \mathbb{E}(\mathbb{E}(Y^2|X)) = \mathbb{E}(\mathrm{Var}(Y|X) + \mathbb{E}(Y|X)^2)$$

$$\mathbb{E} Y^2 - (\mathbb{E} Y)^2 = \mathbb{E}(\mathrm{Var}(Y|X) + \mathbb{E}(Y|X)^2) - \mathbb{E}(\mathbb{E}(Y|X))^2$$

$$= \mathbb{E}\,\mathrm{Var}(Y|X) + \mathbb{E}(\mathbb{E}[Y|X]^2) - \mathbb{E}(\mathbb{E}(Y|X))^2$$

$$= \mathbb{E}\,\mathrm{Var}(Y|X) + \mathrm{Var}(\mathbb{E}(Y|X))$$

3. $D(f\|g) \geq 0$

$D(f\|g) = 0$ iff $f = g$ for pdf f, g.

Here $g > 0$, $0 \ln 0 = 0$

$$-D(f\|g) = \int_{\mathbb{R}} f(x) \log \frac{g(x)}{f(x)} \, dx$$

$$\leq \int_{\mathbb{R}} f(x) \left(\frac{g(x)}{f(x)} - 1 \right) dx$$

$$= 1 - 1 = 0$$

$$\Rightarrow D(f\|g) \geq 0$$

$$D(f\|g) = 0 \iff \log \frac{g(x)}{f(x)} = \frac{g(x)}{f(x)} - 1 \quad \text{a.e.}$$

$$\iff \frac{g(x)}{f(x)} \quad \text{a.e.}$$

For continuos pdf f, g $D(f\|g) = 0 \iff f = g$