1. telegraph equation

$$\begin{cases} u_{tt} - a^2 u_{xx} + 2bu_t + cu = 0 & (x,t) \in (0,L) \times (0,\infty) \\ u = \phi(x),\ u_t = \psi(x) & x \in (0,L),\ t=0 \\ u(0,t) = u(L,t) = 0 \end{cases}$$

Let $u(x,t) = V(x)\,T(t)$

$$V(x)T''(t) - a^2 V''(x)T(t) + 2b\,V(x)T'(t) + c\,V(x)T(t) = 0$$

$$V(x)\left(T''(t) + 2bT'(t)\right) = T(t)\left(a^2V''(x) - cV(x)\right)$$

$$\frac{T''(t) + 2bT'(t)}{T(t)} = \frac{a^2V''(x) - cV(x)}{V(x)} = \lambda$$

$$V''(x) = \frac{\lambda + c}{a^2}V(x),\quad V(0) = V(L) = 0$$

$$V(x) = \sin\frac{k\pi x}{L},\quad \lambda_k = -\frac{k^2\pi^2a^2}{L^2} - c,\quad k = 1,2,\cdots$$

$$T_k''(t) + 2bT_k'(t) + \left(\frac{k^2\pi^2a^2}{L^2} + c\right)T_k(t) = 0$$

$$\lambda_{1,2} = -b \pm \sqrt{b^2 - \frac{k^2\pi^2a^2}{L^2} - c}$$ In context of telegraph eq., WLOG let λ be imaginary.

$$T_k(t) = e^{-bt}\left(a_k\cos\sqrt{c + \frac{k^2\pi^2a^2}{L^2} - b^2}\,t + b_k\sin\sqrt{c + \frac{k^2\pi^2a^2}{L^2} - b^2}\,t\right)$$

$$u(x,t) = \sum_{k=1}^{\infty} e^{-bt}\left(a_k\cos\sqrt{c + \frac{k^2\pi^2a^2}{L^2} - b^2}\,t + b_k\sin\sqrt{c + \frac{k^2\pi^2a^2}{L^2} - b^2}\,t\right)\sin\frac{k\pi x}{L} \quad (1)$$

$$\phi(x) = \sum_{k=1}^{\infty} a_k\sin\frac{k\pi x}{L}\qquad a_k = \frac{2}{L}\int_0^L \phi(x)\sin\frac{k\pi x}{L}\,dx \quad (2)$$

$$\psi(x) = \sum_{k=1}^{\infty}\left(-b\,a_k + b_k\sqrt{c + \frac{k^2\pi^2a^2}{L^2} - b^2}\right)\sin\frac{k\pi x}{L}$$

$$b_k = \frac{1}{\sqrt{c + \frac{k^2\pi^2a^2}{L^2} - b^2}}\left(\frac{2}{L}\int_0^L \left(\psi(x) + b\phi(x)\right)\sin\frac{k\pi x}{L}\,dx\right) \quad (3)$$

(2), (3) 代入 (1) 即所求 #

2. Solve

$$\begin{cases} u_{tt} - a^2 u_{xx} = A\cos\frac{\pi x}{L}\sin wt, \quad (x,t) \in (0,L)\times(0,+\infty) \\ u_x(0,t) = u(L,t) = 0 \\ u(x,0) = u_t(x,0) = 0 \end{cases}$$

We first solve homogeneous problem:

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 \\ u_x(0,t) = u(L,t) = 0 \\ u(x,0) = u_t(x,0) = 0 \end{cases}$$

$$u(x,t) = U(x)T(t)$$

$$U(x)T''(t) - a^2 U''(x)T(t) = 0$$

$$\frac{U(x)}{U''(x)} = a^2\frac{T(t)}{T''(t)} = \frac{1}{\lambda}$$

$$U''(x) - \lambda U(x) = 0, \quad U'(0) = U(L) = 0$$

$$U_k(x) = \cos\frac{(k+\frac{1}{2})\pi}{L}x \qquad \lambda_k = -\frac{(k+\frac{1}{2})^2\pi^2}{L^2}$$

$$T_k(t) = a_k\cos\frac{(k+\frac{1}{2})\pi a}{L}t + b_k\sin\frac{(k+\frac{1}{2})\pi a}{L}t$$

$$u(x,t) = \sum_{k=0}^{\infty}\left(a_k\cos\frac{(k+\frac{1}{2})\pi a t}{L} + b_k\sin\frac{(k+\frac{1}{2})\pi a t}{L}\right)\cdot\cos\frac{(k+\frac{1}{2})\pi}{L}x$$

$$0 = u(x,0) = \sum_{k=0}^{\infty} a_k\cos\frac{(k+\frac{1}{2})\pi}{L}x$$

$$0 = u_t(x,0) = \sum_{k=0}^{\infty} b_k\frac{(k+\frac{1}{2})\pi a}{L}\cos\frac{(k+\frac{1}{2})\pi x}{L}$$

$$\Rightarrow a_k = b_k = 0$$

For inhomogeneous problem, expand terms:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)\cos\frac{(k+\frac{1}{2})\pi}{L}x$$

$$u(x,0) = 0 \Rightarrow u_k(0) = 0$$

$$u_t(x,0) = 0 \Rightarrow u_k'(0) = 0$$

$$\sum_{k=0}^{\infty} u_k''(t)\cos\frac{(k+\frac{1}{2})\pi}{L}x + a^2\frac{(k+\frac{1}{2})^2\pi^2}{L^2}u_k(t)\cos\frac{(k+\frac{1}{2})\pi x}{L} = \sum_{k=0}^{\infty} f_k(t)\cos\frac{(k+\frac{1}{2})\pi}{L}x$$

$$u_k''(t) + \frac{a^2(k+\frac{1}{2})^2\pi^2}{L^2}u_k(t) = f_k(t)$$

Using (*),

$$u_k(t) = \frac{(-1)^{k+1}4(2k+1)}{\pi(2k+3)(2k-1)}A\left(\frac{\sin wt}{\frac{(k+\frac{1}{2})^2\pi^2 a^2}{L^2} - w^2} - \frac{w\sin\frac{(k+\frac{1}{2})\pi a t}{L}}{\frac{(k+\frac{1}{2})\pi a}{L}\left(\frac{(k+\frac{1}{2})^2\pi^2a^2}{L^2} - w^2\right)}\right)$$

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)\cos\frac{(k+\frac{1}{2})\pi x}{L}$$

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$$A\cos\frac{\pi x}{L}\sin wt = \sum_{k=0}^{\infty} f_k(t)\cos\frac{(k+\frac{1}{2})\pi}{L}x$$

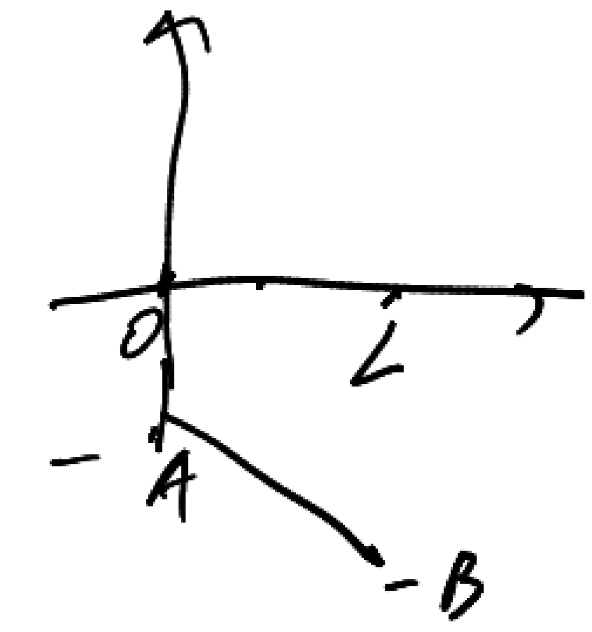
$$f_k(t) = \frac{2}{L}A\sin wt\int_0^L \cos\frac{\pi x}{L}\cos\frac{(k+\frac{1}{2})\pi x}{L}\,dx$$

$$= 2A\sin wt\int_0^1 \cos\pi x\cos(k+\tfrac{1}{2})\pi x\,dx$$

$$= A\sin wt\int_0^1 \left[\cos(k+\tfrac{3}{2})\pi x + \cos(k-\tfrac{1}{2})\pi x\right]dx$$

$$= \frac{(-1)^{k+1}4(2k+1)}{\pi(2k+3)(2k-1)}A\sin wt \quad (*)$$

3. $\begin{cases} u_{tt} - a^2 u_{xx} = f(x) & (x,t) \in (0,L)\times(0,+\infty) \\ u(0,t) = A \\ u(L,t) = B \\ u(x,0) = u_t(x,0) = 0 \end{cases}$



Let $Q(x,t) = u(x,t) - \frac{L-x}{L}A - \frac{x}{L}B$ We have

$$\begin{cases} Q(0,t)=0 \quad Q(L,t)=0 \\ Q(x,0) = -\frac{L-x}{L}A - \frac{x}{L}B \quad Q_t(x,0) = 0 \\ Q_{tt} - a^2 Q_{xx} = u_{tt} - a^2 u_{xx} = f(x) \end{cases}$$

eigenfunctions $\sin\frac{k\pi x}{L}$, $k=1,2,\cdots$

$$f(x) = \sum_{k=1}^{\infty} f_k \sin\frac{k\pi x}{L}, \quad f_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx$$

$$Q(x,t) = \sum_{k=1}^{\infty} u_k(t)\sin\frac{k\pi x}{L}$$

$$u_{tt} - a^2 u_{xx} = \sum_{k=1}^{\infty}\left(u''_k(t) + \frac{a^2k^2\pi^2}{L^2}u_k(t)\right)\sin\frac{k\pi x}{L} = \sum_{k=1}^{\infty} f_k \sin\frac{k\pi x}{L}$$

$$u_k''(t) + \frac{a^2k^2\pi^2}{L^2}u_k(t) = f_k$$

$$0 = u_t(x,0) = \sum_{k=1}^{\infty} u_k'(0)\sin\frac{k\pi x}{L}, \quad u_k'(0) = 0$$

$$-\frac{L-x}{L}A - \frac{x}{L}B = \sum_{k=1}^{\infty} u_k(0)\sin\frac{k\pi x}{L}$$

$$u_k(0) = \frac{2}{L}\int_0^L\left(-\frac{L-x}{L}A - \frac{x}{L}B\right)\sin\frac{k\pi x}{L}\,dx = 2\int_0^1\left(-(1-x)A - Bx\right)\sin k\pi x\,dx$$

$$= -2A\int_0^1 \sin k\pi x\,dx + 2(A-B)\int_0^1 x\sin k\pi x\,dx = \frac{2(A-B)(-1)^{k+1}}{k\pi} + \frac{2A((-1)^k-1)}{k\pi}$$

$$= \frac{2B(-1)^k - 2A}{k\pi}$$

$$u_k(t) = a_k\cos\frac{ak\pi}{2}t + b_k\sin\frac{ak\pi}{L}t + \frac{L^2 f_k}{a^2k^2\pi^2}$$

$$a_k = \frac{2B(-1)^k - 2A}{k\pi} - \frac{2L}{a^2k^2\pi^2}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx, \quad b_k = 0$$

$$\Rightarrow u(x,t) = \sum_{k=1}^{\infty}\left(\frac{2B(-1)^k-2A}{k\pi} - \frac{2L}{a^2k^2\pi^2}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx\right)\cos\frac{ak\pi}{L}t\sin\frac{k\pi x}{2}$$

$$+\frac{L-x}{L}A + \frac{x}{L}B \quad \#$$