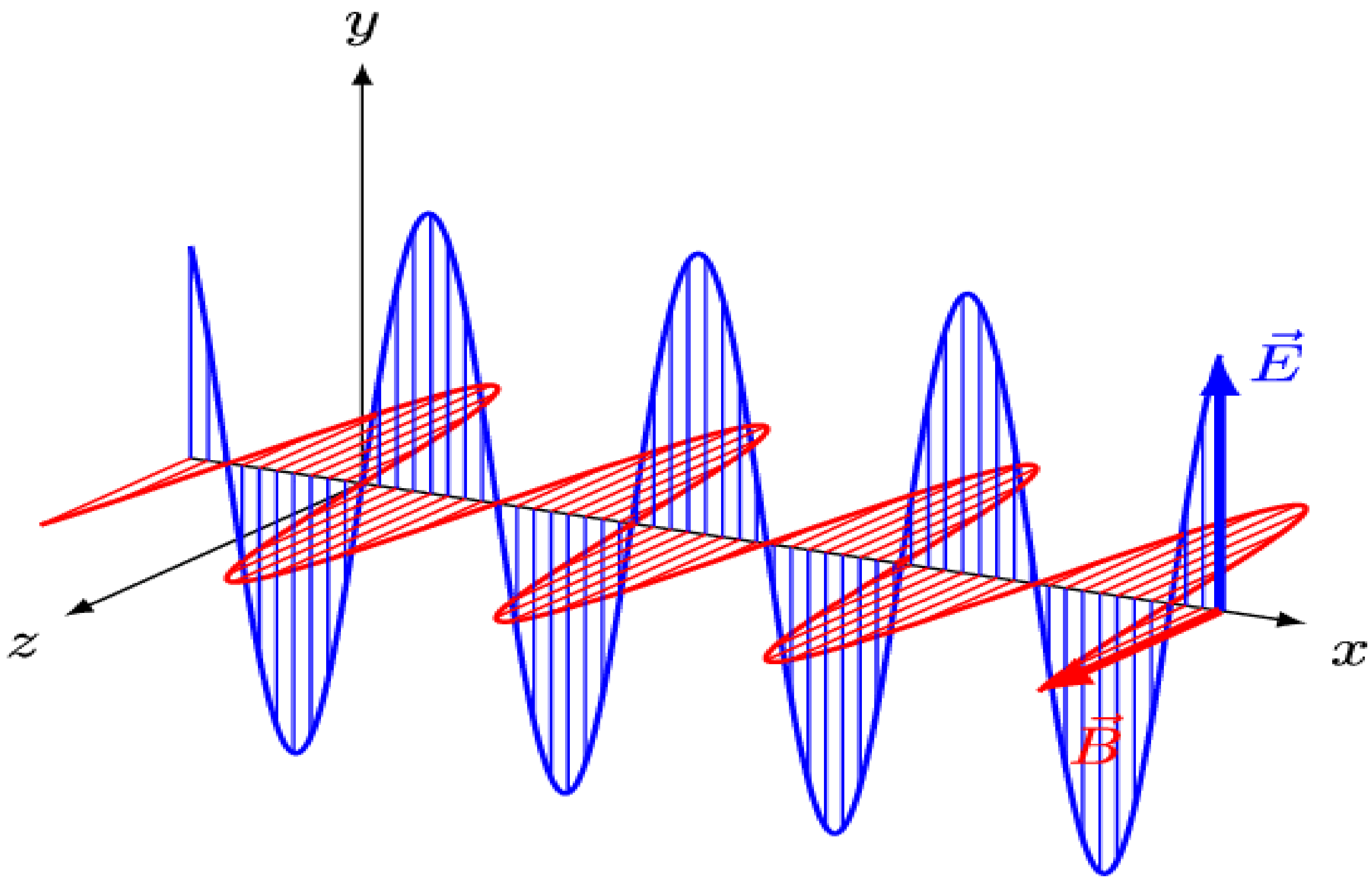
EE254/DP274

Engineering Electromagnetics II

Introduction to Time-Varying Fields

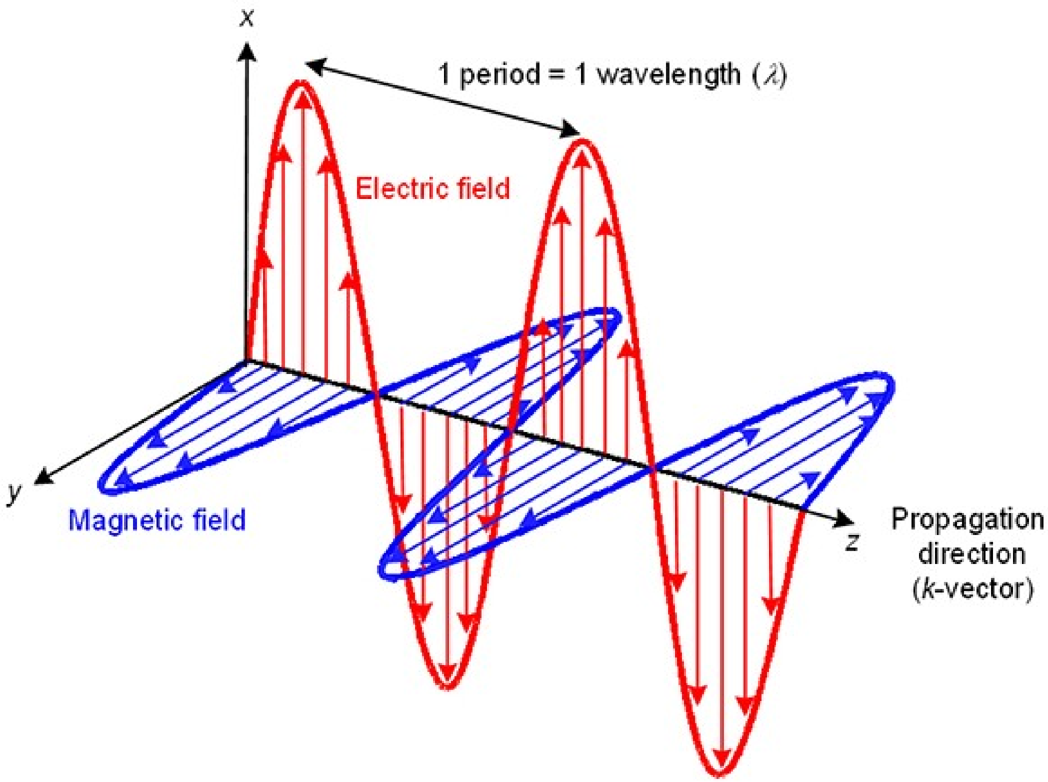
Wave Propagation

Wave Propagation



Wave Propagation

- Direction of wave propagation is given by: $\boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k$



$$\lambda = vT$$

$$v = f\lambda$$

$$\frac{dz}{dt} = \frac{\omega}{\beta} = v$$

Wave Propagation

- Waves are means of transporting energy or information.
- A wave is a function of both space and time.
- For practical purposed, EM fields are assumed to be **time-harmonic** fields.

$$\mathrm{E}(z,t) = \mathrm{E}_o \cos(\omega t - \beta z)\, \boldsymbol{a}_x$$

- **Time-harmonic** fields are easily expressed in **phasors**, which are more convenient to work with.
- Using complex number notation:

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

Wave Propagation

- The wave equation can then be expressed as:

$$\mathbf{E}(z,t) = \mathrm{E}_o \cos(\omega t - \beta z) = Re\left(\mathrm{E}_o e^{j\omega t} e^{-j\beta z}\right)$$

- The **phasor** function is obtained by dropping the time factor $e^{j\omega t}$ from the time-harmonic function. That is:

$$\mathbf{E}_s = Re\left(\mathrm{E}_o e^{-j\beta z}\right)$$

- Generally, the **time-harmonic** field and its **phasor** form are related as:

$$\mathbf{E}(z,t) = Re\left(\mathbf{E}_s e^{j\omega t}\right)$$

Time-Harmonic Maxwell’s Equations

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Wave Propagation in Different Media

Using Maxwell's equations we can analyze wave propagation in:

- **Free space**
- **Lossless media**
- **Lossy media**
- **Good conductors**

Wave Propagation in Lossy Media

- Assuming sinusoidal wave propagation in a charge free homogeneous lossy medium:

$$\nabla \cdot \mathbf{E}_s = 0 \qquad (1)$$

$$\nabla \cdot \mathbf{H}_s = 0 \qquad (2)$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \qquad (3)$$

$$\nabla \times \mathbf{H}_s = (\sigma + j\omega\varepsilon)\mathbf{E}_s \qquad (4)$$

Wave Propagation in Lossy Media

- Taking the curl on both sides of eqn. (3):

$$\nabla \times \nabla \times \mathbf{E}_s = -j\omega\mu \nabla \times \mathbf{H}_s$$

- Applying the vector identity to the LHS:

$$\nabla \times \nabla \times \mathbf{E}_s = \nabla(\nabla \cdot \mathbf{E}_s) - \nabla^2 \mathbf{E}_s$$

- Upon substituting eqn. (1) and (4), we get:

$$-\nabla^2 \mathbf{E}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\mathbf{E}_s$$

Wave Propagation in Lossy Media

- Letting:

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

- We obtain:

$$\nabla^2\mathbf{E}_s - \gamma^2\mathbf{E}_s = 0 \qquad (5a)$$

- Using a similar procedure, it can be shown that:

$$\nabla^2\mathbf{H}_s - \gamma^2\mathbf{H}_s = 0 \qquad (5b)$$

- These are known as Helmholtz equations/ Wave equations

Wave Propagation in Lossy Media

- Assuming that the wave propagates along $+a_z$ and that E_s has only x-component:

Assuming that the wave propagates along and that has only x-component:

$$\mathbf{E}_s = \mathrm{E}_{\mathbf{x}s}(z)\boldsymbol{a}_x$$

- Equation becomes:

Equation becomes:

$$\frac{\partial^2 \mathrm{E}_{xs}(z)}{\partial z^2} - \gamma^2 \mathrm{E}_{\mathbf{x}s}(z) = 0$$

- A solution of this linear homogeneous 2nd DE leads to:

A solution of this linear homogeneous 2nd DE leads to:

$$\mathrm{E}_{\mathbf{x}s}(z) = \mathrm{E}_{\mathrm{o}} e^{-\gamma z} + \mathrm{E}'_{\mathrm{o}} e^{\gamma z} \qquad (6)$$

(6)

- γ = Propagation constant (/m)

= Propagation constant (/m)

Wave Propagation in Lossy Media

- Recall that:

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) \qquad (7)$$

- Since γ is a complex quantity, we may let

$$\gamma = \alpha + j\beta \qquad (8)$$

- Hence, inserting back the time factor $e^{j\omega t}$ into (6):

$$\mathbf{E}(z,t) = \mathrm{Re}\left[\mathrm{E}_{\mathbf{x}s}(z)e^{j\omega t}\boldsymbol{a}_x\right] = \mathrm{Re}\left(\mathrm{E}_o e^{-\alpha z} e^{j(\omega t - \beta z)}\,\boldsymbol{a}_x\right)$$

- That is:

$$\mathbf{E}(z,t) = \mathrm{E}_o e^{-\alpha z}\cos(\omega t - \beta z)\,\boldsymbol{a}_x \qquad (9)$$

Wave Propagation in Lossy Media

- Solving equations (7) and (8) simultaneously:

(10)

(11)

$$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]} \qquad (10)$$

- Attenuation constant (Np/m or dB/m)
- Phase constant (rad/m)

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]} \qquad (11)$$

- $\alpha =$ Attenuation constant (Np/m or dB/m)
- $\beta =$ Phase constant (rad/m)

Wave Propagation in Lossy Media

- Having obtained $\mathbf{E}(z,t)$, we can obtain $\mathbf{H}(z,t)$ by using a similar procedure, eventually we arrive at:

$$\mathbf{H}(z,t) = \mathrm{H_o} e^{-\alpha z} \cos(\omega t - \beta z)\, \boldsymbol{a}_y \qquad (12)$$

- Where:

$$\mathrm{H_o} = \frac{\mathrm{E_o}}{\eta} \qquad (13)$$

- **Intrinsic impedance of the medium (η)** is defined as the ratio of the electric field and magnetic field phasors (ohms).

Wave Propagation in Lossy Media

- η is a complex quantity. It can be shown that:

is a complex quantity. It can be shown that:

(14)

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta|\angle\theta_\eta = |\eta|e^{j\theta_\eta} \qquad (14)$$

Where the magnitudes of and can be found from:

- Where the magnitudes of $\boldsymbol{\eta}$ and $\boldsymbol{\theta_\eta}$ can be found from:

and

Substituting equation (13) and (14) into (12) gives:

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}} \quad \text{and} \quad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}$$

(15)

- Substituting equation (13) and (14) into (12) gives:

$$\mathbf{H}(z,t) = \frac{\mathrm{E_o}}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta\eta)\, \boldsymbol{a}_y \qquad (15)$$

Wave Propagation in Lossy Media

- Finally we notice that, the ratio of the magnitude of conduction to displacement current densities is given by:

$$\frac{|J_s|}{|J_{ds}|} = \frac{|\sigma E_s|}{|j\omega\varepsilon E_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta$$

- $\tan\theta$ is known as loss tangent of the medium.
- It is used to determine how **lossy** a medium is:
  - Perfect dielectric/Lossless media $(\sigma \ll \omega\varepsilon)$
  - Good conductor $(\sigma \gg \omega\varepsilon)$

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Wave Propagation in **Lossless media**

- In a lossless medium $(\sigma \ll \omega\varepsilon)$, therefore:

$$\sigma \approx 0, \qquad \varepsilon = \varepsilon_o \varepsilon_r, \qquad \mu = \mu_o \mu_r$$

- Substituting into equations (11) and (12) gives:

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu\varepsilon}$$

$$\nu = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad \lambda = \frac{2\pi}{\beta}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\angle 0°$$

Wave Propagation in **Free space**

- In a free space medium ($\sigma = 0$), therefore:

$$\sigma = 0, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o$$

- Substituting into equations (10) and (11) gives:

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu_o \varepsilon_o} = \frac{\omega}{c}$$

$$v = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c, \qquad \lambda = \frac{2\pi}{\beta}$$

$$\eta = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi = 377\Omega$$

Wave Propagation in **Good conductors**

- In a good conductors ($\sigma \gg \omega\varepsilon$) therefore:

  $$\sigma \approx \infty, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o\mu_r$$

- Substituting these into eqn. (11) and (12) gives:

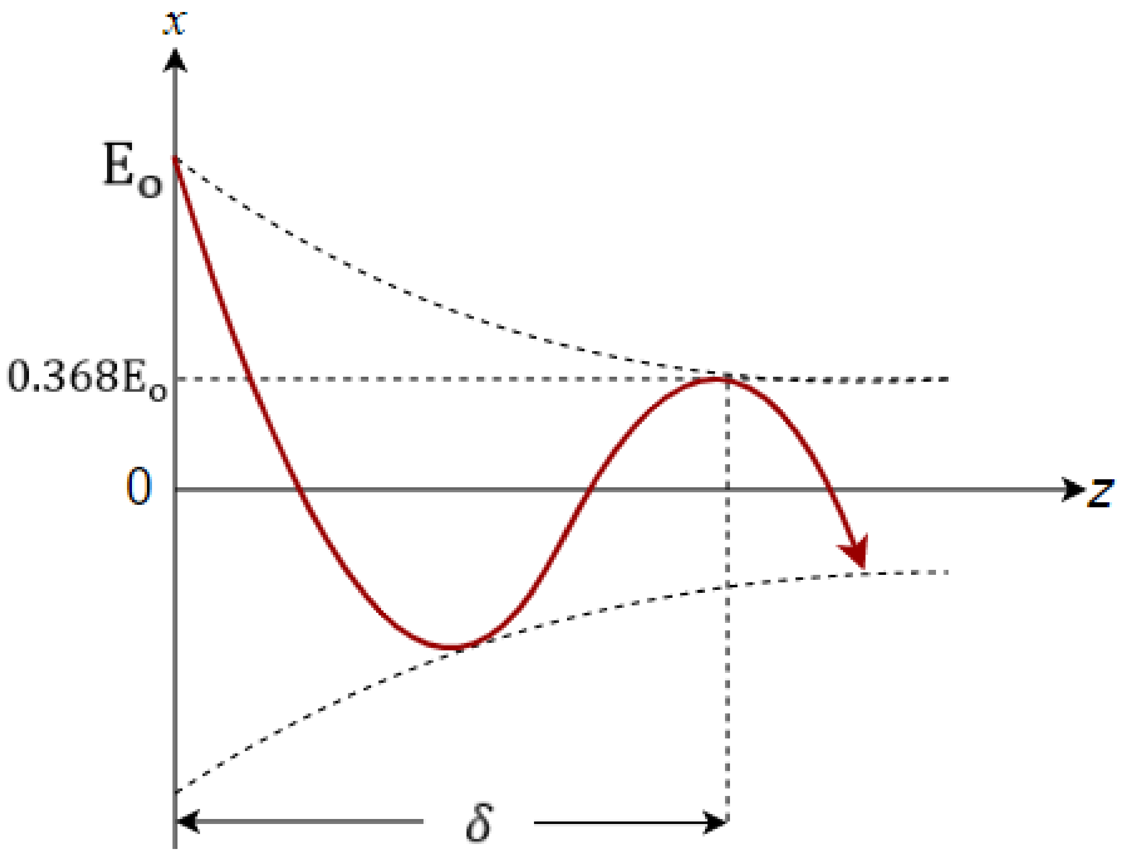
$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma}$$

$$\nu = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \qquad \lambda = \frac{2\pi}{\beta}$$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$$

Skin Effect

- As the wave travels, its amplitude is decreased by the factor $e^{-\alpha z}$.



Skin Effect

- **Skin effect** is the tendency of an AC current to become distributed (confined) to a very thin layer (skin) of the conductor surface.
- The distance δ through which the amplitude of the wave decreases by a factor 1/e (36.7%) is known as **Skin depth**.
- It is a measure of the depth to which an EM wave can penetrate the medium.
- That is:

$$E_o e^{-\alpha\delta} = E_o e^{-1}$$

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha}$$

Causes of Skin Effect

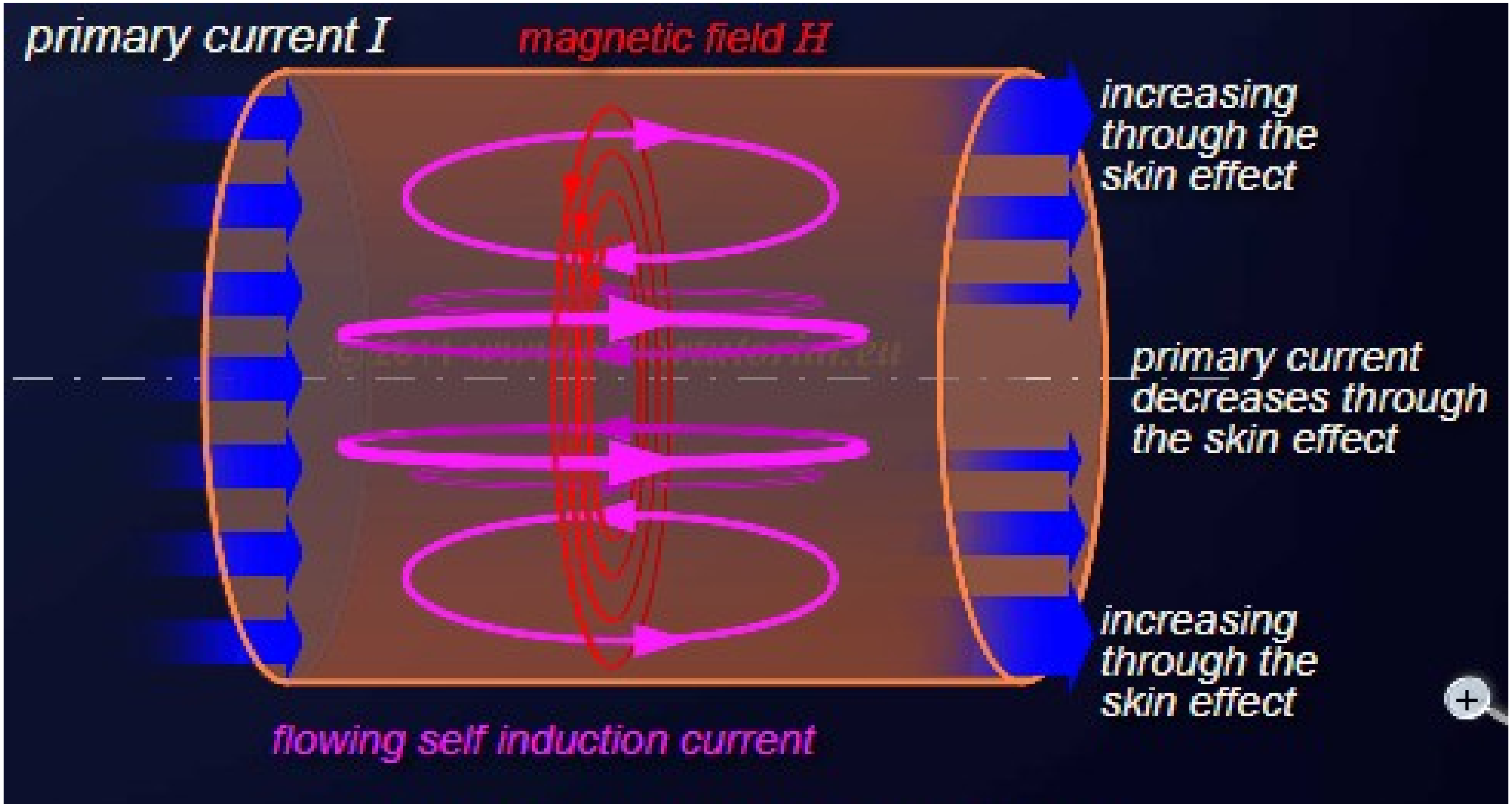
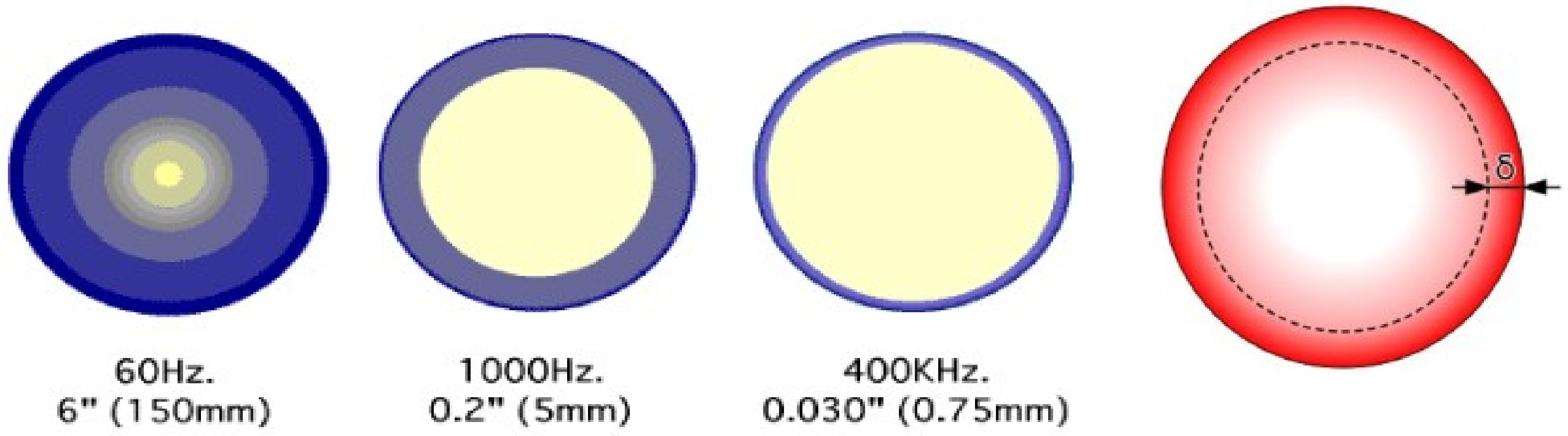


Figure 1: Visualisation of the skin effect

Effects of Skin Effect



- At low frequencies, current is uniformly distributed across the conductor cross section area.
- At high frequencies, the current tends to flow only in the conductor surface, effective cross-section area decreases.

Effects of Skin Effect

- DC resistance is given by:

$$R_{\mathrm{dc}} = \frac{\ell}{\sigma S}$$

- For a given width w and length ℓ, the AC resistance is:

$$R_{\mathrm{ac}} = \frac{\ell}{\sigma \delta w}$$

- For a conductor of radius a, $w = 2\pi a$, so that:

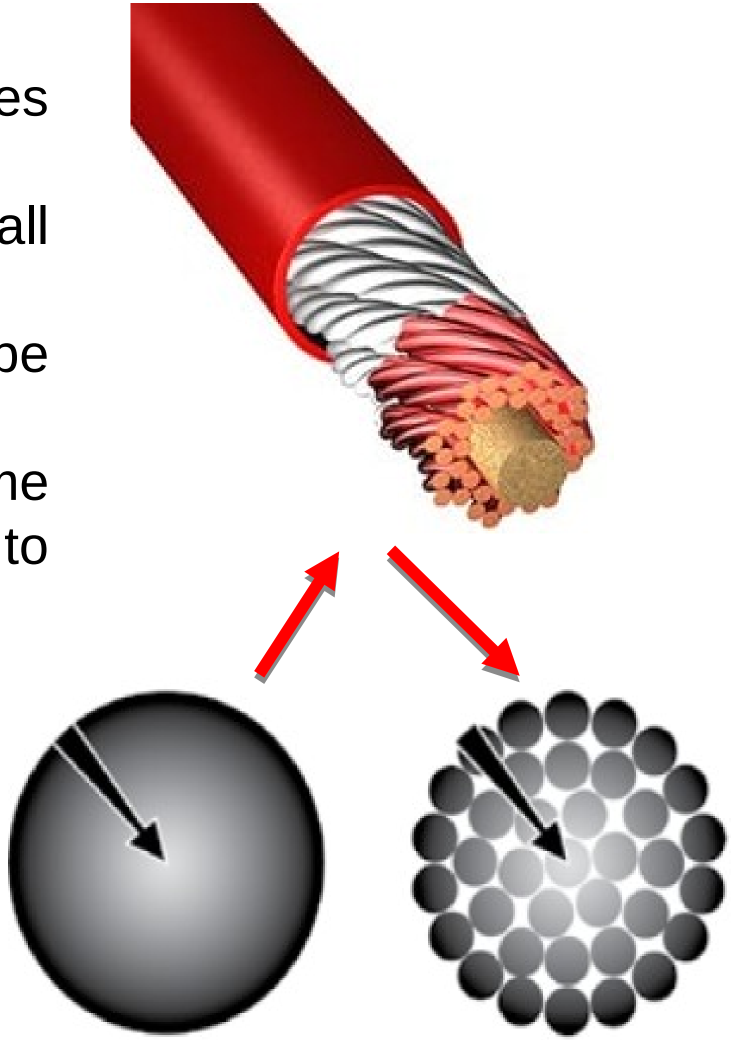
$$\frac{R_{\mathrm{ac}}}{R_{\mathrm{dc}}} = \frac{\frac{\ell}{\sigma 2\pi a \delta}}{\frac{\ell}{\sigma \pi a^2}} = \frac{a}{2\delta}$$

- Since $a \gg \delta$ at high frequencies, this shows that R_{ac} is far greater than R_{dc}.

How to Mitigate Skin Effect

Litz wire

- ❖ Consists of a number of insulated wires stranded together.
- ❖ The overall EM field acts equally on all the wires
- ❖ This causes the total current to be distributed equally among them.
- ❖ The bundle does not suffer the same increase in AC resistance compared to a solid conductor of the same C.S.A



How to Mitigate Skin Effect

Using multiple conductors

- Commonly used in high voltage power transmission.
- Each of the wire acts a single conductor.
- A single wire using the same amount of metal per kilometer would have higher losses due to skin effect.

Use of hollow tubular conductors

- Solid conductors are usually replaced by tubular conductors.
- Because the interior of a large conductor carries little current, tubular conductors can be use to save weight and cost

Example 1:

In a lossless medium for which $\eta = 60\pi$, $\mu_r = 1$ and $\mathbf{H} = 0.5\sin(\omega t - z)\boldsymbol{a}_y$ A/m. Calculate ε_r, ω and $\mathbf{E}$.

Answer:

Answer:

$\varepsilon_r = 4$

$\omega = 1.5 \times 10^8$ rad/sec

$\mathbf{E} = 94.25\sin(\omega t - z)\,\boldsymbol{a}_x$ V/m

Example 2:

A uniform plane wave in a medium has $\mathbf{E} = 2e^{-\alpha z}\sin(10^8 t - \beta z)\,\boldsymbol{a}_y$ V/m. If the medium is characterized by $\epsilon_r = 1$, $\mu_r = 20$ and $\sigma = 3$ mhos/m. Find α, β and H.

Answer:

$\alpha = 61.4$ Np/m

$\beta = 61.4$ rad/m

$\mathbf{H} = -69.1e^{-61.4z}\sin\left(10^8 t - 61.4z - \frac{\pi}{4}\right)\boldsymbol{a}_x$ mA/m