MT271: STATISTICS FOR MATHEMATICS NON-MAJORS

Course Contents & Probability Theory

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Outline

Course contents

1. **Descriptive Statistics and measures of central and dispersion:** Representation of data, frequency distributions, histograms, frequency polygon, cumulative frequency polygon. Arithmetic mean, mode and median for ungrouped and grouped data; mean absolute deviation, variance, standard deviation and quartile deviations for ungrouped and grouped data
2. **Elementary Probability Theory:** Probability experiment, classical and non-theoretic models, sample spaces and events, sure and impossible events, conditional probability, the binomial experiment; independent and mutually exclusive events, use of tree and Venn diagrams, Baye's theorem.

course contents continues...

3. **Probability Distributions:** Random variables, discrete probability distributions, distribution functions, continuous probability distribution (density function) and cumulative density functions, joint distributions, marginal and conditional distributions, covariance, correlation, linear independence. Binomial, Poisson and normal distributions and their applications
4. **Sampling distributions and Estimations:** Sampling distribution of the arithmetic mean, central limit theorem, distribution of sample proportion, difference between two means, chi square, t distribution and F distribution. Confidence interval estimates of the population concerning mean, proportion and differences between means and proportions. Estimation of population variance, estimation of sample sizes.
5. **Hypotheses Testing:** Testing of hypotheses concerning; population mean, proportion and differences between means and proportions.

Course contents continues...

6. **Regression Analysis:** Regression Analysis and Correlation: The Least Squares Line, Coefficient of correlation and determination; multiple linear regression; Examples from science and engineering.
7. **Experimental Design and Analysis of Variance:** One-way ANOVA, comparing means, the ANOVA model, the ANOVA table and the F - test; multiple comparisons; two-way ANOVA.
8. **Quality control:** Detecting process change; control charts: x chart, R-chart, Runs analysis, p-chart, c-chart; tolerant limits, acceptance sampling for defectives.
9. **System reliability:** Definition, failure time distributions, hazard rates, life testing censored sampling, estimating parameters of an exponential failure time distribution.

Course contents continues...

Delivery: 45 Lecture hours and 15 Tutorials sessions

Assessment: Course work 40%, Final Examination 60%

Textbooks

1. Walpole, R. E., Myers, R. H., Myers S. L. and Ye K. Probability and statistics for Engineers and Scientists Prentice Hall, 7th Ed., 2002.
2. Kreyzig, E. Advanced Engineering Mathematics, Wiley, 8th Ed., 1998

Assignment:

Make a review on Descriptive statistics and Measures of central tendency and dispersion

Some questions on descriptive statistics can be

1. A student has grades of 65, 70, 95 and 62 on four quizzes. If the student is to have an average of at least 75 after the next quiz, what is the lowest grade that can be received for the next quiz?
2. Average cost when four units are produced is \$10 per unit. What must be the value of fifth unit if the average cost is to decline to \$9 per unit?

3. Using the table below, calculate the mean salary

Salary class ($ per week)	No. of workers
205.00 – 214.99	5
215.00 – 224.99	30
225.00 – 234.99	40
235.00 – 244.99	20
245.00 – 254.99	5

4. Business experiences period of expansion and contraction called business cycles. The national Bureau of Economic Research found that the duration of 10 cycles were 28, 36, 40, 64, 63, 88, 48, 58, 44 and 34 months. Compute the standard deviation of these cycle durations.

5. The data below were compiled from the 1975 issue of the statistical abstract of the united states. Compute the mean μ and standard deviation σ.

113	138	160	164	177	185	192	195	202	209	222
129	140	161	165	178	187	192	195	202	211	222
130	142	163	167	179	187	192	196	205	211	227
131	143	163	171	179	187	193	198	206	213	228
133	149	163	172	179	189	193	199	206	216	229
133	150	163	172	180	190	193	199	206	219	235
133	151	163	174	180	190	193	199	207	219	252
133	154	163	174	182	190	193	199	208	220	252
135	155	164	176	182	191	194	200	208	220	253
137	158	164	177	184	191	194	200	209	221	254

6. Salaries paid to supervisors had a mean of $\$25,000$ with a standard deviation of $\$2000$. What will be the new mean and standard deviation if all salaries are increased by $\$2500$?

Introduction to Probability Theory

Probability theory is the study of random or unpredictable experiment.

Although all possible outcomes of an experiment may be known in advance, the outcomes of a particular performance of an experiment cannot be predicted owing to a number of unknown causes.

For instance, Although the number of telephone calls received in a board in a 5 minutes interval is non-negative integer, we cannot predict exactly the number of calls received in the next 5 minutes.

In such situations we talk of the chance or probability of occurrence of a particular outcome, which is the quantitative measure of the likelihood of occurrence of the outcome.

Mathematical Definition of Probability

Let S be the sample space (outcomes assumed to be equally likely) and A be an event (subset of S) associated with a random experiment.

Then the probability of event A occurring denoted as $P(A)$, is defined by

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to} A}{\text{Exhaustive number of cases in} S}. \tag{1}$$

For example, the probability of getting an even number in the die tossing experiment is 0.5, since $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 4, 6\}$, $n(S) = 6$ and $n(A) = 3$.

Statistical Definition of Probability

Let a random experiment be repeated n times and let an event A occur n_A times out of the n trials.

The ratio $\frac{n_A}{n}$ is called the relative frequency of the event A.

As n increases, $\frac{n_A}{n}$ shows a tendency to stabilize and approach a constant value, denoted by $P(A)$ called the probability of an event A, i.e.

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n}. \tag{2}$$

For example, if we want to find the probability that a spare part produced by a machine is defective, we study the record of defective items produced by the machine over a considered period of time. If out of 10000 items produced, 500 are defective it is assumed that the probability of a defective item is 0.05.

Axiomatic Definition of Probability

Let S be the sample space and A be an event associated with a random experiment. Then the probability of the event A, denoted by $P(A)$ satisfies the following axioms

(i) $0 \leq P(A) \leq 1$

(ii) $P(S) = 1$

(iii) If A is an impossible event, $P(A) = 0$

(iv) If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$

(v) If $A_1, A_2, \cdots, A_n, \cdots$ are sets of mutually exclusive events, then
$$P(A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots) = P(A_1) + P(A_2) + \cdots + P(A_n) + \cdots$$

Note: A set of events are said to be mutually exclusive if the occurrence of any one of them excludes the occurrence of the others, i.e. $P(A_1 \cap A_2 \cap \cdots) = 0$.

Laws of Probability

In the development of the probability theory, all results are derived directly/indirectly using the axioms of probability.

The following are some of the basic results.

(i) The probability of an impossible event is zero i.e. if ϕ is the event containing no sample point, $P(\phi) = 0$.

(ii) If $\bar{A}$ is the complement of event A, then $P(\bar{A}) = 1 - P(A) \leq 1$.

(iii) If A and B are any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B \leq P(A) + P(B)$.

(iv) If $B \subset A$, $P(B) \leq P(A)$.

Conditional Probability

Conditional probability of an event B, assuming that the event A has happened, is denoted by $P(B/A)$ and defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, P(A) \neq 0. \tag{3}$$

For example, when a fair dice is tossed, the conditional probability of getting 1, given that an odd number has been obtained, is given as follows: $S = \{1, 2, 3, 3, 4, 5, 6\}$, $A = \{1, 3, 5\}$, $B = \{1\}$. Then $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3}$.

Independent Events

A set of events is said to be independent if the occurrence any one of them does not depend on the occurrence/non-occurrence of the others.

If two events A and B are independent, then

(i) $P(B/A) = P(B)$,

(ii) $P(A \cap B) = P(A) \times P(B)$, this can be extended to any number of independent events

(iii) events $\bar{A}$ and B (similarly A and $\bar{B}$) are also independent,

(iv) events $\bar{A}$ and $\bar{B}$ are also independent

Example

Two defective tubes get mixed up with two good ones. The tubes are tested, one by one, until both defective are found. What is the probability that the last defective tube is obtained on (i) the second test, (ii) third test and (iii) the fourth test?

Answer

Let D represent defective and N represent non-defective tube.

$$\begin{aligned}
(i) & P(\text{Second } D \text{ in the second test}) \\
&= P(D \text{ in the first test and } D \text{ in the second test}) \\
&= P(D_1 \cap D_2) \\
&= P(D_1) \times P(D_2), (\text{by independence}) \\
&= \frac{2}{4} \times \frac{1}{3} = \frac{1}{6}
\end{aligned}$$

$$
\begin{aligned}
(ii)P(\text{Second } D \text{ in the third test}) &= P(D_1 \cap N_2 \cap D_3 \text{ or } N_1 \cap D_2 \cap D_3) \\
&= P(D_1 \cap N_2 \cap T_3) + P(N_1 \cap D_2 \cap T_3) \\
&= \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \\
&= \frac{1}{3}.
\end{aligned}
$$

$(iii)P(\text{Second } D \text{ in the fourth test})$

$$
\begin{aligned}
&= P(D_1 \cap N_2 \cap N_3 \cap D_4 \text{ or } N_1 \cap D_2 \cap N_3 \cap D_4 \text{ or } N_1 \cap N_2 \cap D_3 \cap D_4) \\
&= P(D_1 \cap N_2 \cap N_3 \cap D_4) + P(N_1 \cap D_2 \cap N_3 \cap D_4) + P(N_1 \cap N_2 \cap D_3 \cap D_4) \\
&= \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1 + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1 + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1 = \frac{1}{2}.
\end{aligned}
$$

Theorem of Total Probability

If $B_1, B_2, \cdots, B_n$ is a set of exhaustive and mutually exclusive events, and A is another event associated with (caused by) B_i, then

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i) \tag{4}$$

Proof

The inner circle represent the event A, A can occur along with (due to) B_1, $B_2, \cdots, B_n$ that are exhaustive and mutually exclusive.
$AB_1, AB_2, \cdots, AB_n$ are also mutually exclusive,
It follows that $A = AB_1 + AB_2 + \cdots + AB_n$
Thus, $P(A) = P\left(\sum_i AB_i\right) = \sum_i P(AB_i) = \sum_{i=1}^{n} P(B_i)P(A/B_i)$.

Baye's Theorem

If $B_1, B_2, \cdots, B_n$ is a set of exhaustive and mutually exclusive events associated with a random experiment and A is another event associated with (or caused by) B_i, then

$$P(B_i/A) = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}, i = 1, 2, \ldots, n. \tag{5}$$

Proof

$$\begin{aligned} P(B_i \cap A) &= P(B_i) \times P(A/B_i) = P(A) \times P(B_i/A) \\ \therefore P(B_i/A) &= \frac{P(B_i) \times P(A/B_i)}{P(A)} \\ &= \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}, i = 1, 2, \ldots, n \end{aligned}$$

Example

One-half percent of the population has AIDS. There is a test to detect AIDS. A positive test result is supposed to mean that you have AIDS but the test is not perfect. For people with AIDS, the test misses the diagnosis 2% of the times. And for the people without AIDS, the test incorrectly tells 3% of them that they have AIDS.

(a) What is the probability that a person picked at random will test positive?

(b) What is the probability that you have AIDS given that your test comes back positive?

Answer:

Let A denote the event of one who has AIDS and B denote the event that the test comes out positive.

(a) The probability that a person picked at random will test positive is given by

$$\begin{aligned} P(\text{test positive}) &= (0.005)(0.98) + (0.995)(0.03) \\ &= 0.0049 + 0.0298 = 0.035. \end{aligned}$$

(b) The probability that you have AIDS given that your test comes back positive is given by

$$\begin{aligned} P(A/B) &= \frac{\text{favorable positive branches}}{\text{total positive branches}} \\ &= \frac{(0.005)(0.98)}{(0.005)(0.98) + (0.995)(0.03)} \\ &= \frac{0.0049}{0.035} = 0.14. \end{aligned}$$

Counting Technique

There are three basic counting techniques: multiplication rule, permutation and combination.

Multiplication Rule
If E_1 is an experiment with n_1 outcomes and E_2 is an experiment with n_2 possible outcomes,
then the experiment which consists of performing E_1 first and then E_2 consists of $n_1 n_2$ possible outcomes.

Example

Find the possible number of outcomes in a sequence of two tosses of a fair coin.

Solution
The number of possible outcomes is $2 \times 2 = 4$. This can easily be shown by tree diagram.

Example

Find the number of possible outcomes of the rolling of a die and then tossing a coin.

Example

How many different license plates are possible if the country numbering system uses three letters followed by three digits.

Solution

Here we have letters A to Z for three positions in the plate number. Also, we have digits 0 to 9 for three positions.
Then,

$$\begin{aligned}\text{the number of different license plates} &= (26)^3(10)^3\\ &= (17576)(1000)\\ &= 17,576,000.\end{aligned}$$

Permutation

Permutation is an ordered arrangement of objects (letters or numbers).

These n objects could be distinct or not distinct.

The number of permutations of n distinct objects taken r at a time denoted by nP_r is given by

$$nP_r = \frac{n!}{(n-r)!},$$

where $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ and $0! = 1$.

Example

How many numbers with three distinct digits are possible using the digits $3, 4, 5, 6, 7, 8$?

Solution

We need to find nP_r where $n = 6$ and $r = 3$

$$nP_r = \frac{n!}{(n-r)!} = \frac{6!}{(6-3)!} = 120.$$

Then, Five of these numbers are 345, 356, 347, 378, 567

The arrangement of n objects such that q of them resemble is given by $\frac{n!}{q!}$.

Example

In how many ways can the letters of the word ESSENTIAL be arranged?

Combination

Unlike in permutation, in the case of combination, the order is not important.

We can define a combination as a selection of r objects in a group of n objects. It is denoted by nC_r or $\binom{n}{r}$, and defined by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

Example

How many committees of two chemists and one physicist can be formed from 4 chemists and 3 physicists?