Fundamentals of signals and systems

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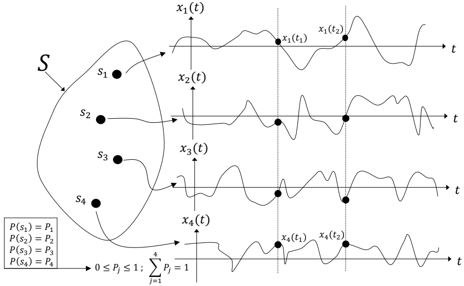
Unit 4

Random signals and noise

Random signals and noise
Random processes

- Consider a random experiment that generates a sample space S
- Let S contain the sample points s_1, s_2,. . . , s_n,. . .
- For every $s_j \in S$, let $x(s_j, t)$ or $x_j(t)$ be a real-valued function of time

Random signals and noise
Random processes



Random signals and noise
Random processes

- The Figure in the previous slide shows a simple random process containing a set of waveforms, $x_j(t)$, associated with the sample points in S: s_1, s_2, s_3, and s_4
- Each point s_j of S has $x_j(t)$ and probability P_j
- Let us observe the waveforms at particular time intervals, say t_1 and t_2
- For $j = 1, 2, 3, 4$, $x_j(t_1)$ and $x_j(t_2)$ give two random variables

Random signals and noise

Random processes

- Random process (RP), denoted by $X(t)$ contains the probability system with the following attributes:
  - Sample space
  - Ensemble (collection) of time functions
  - Probability measure
- RP denotes a function of two variables, namely s and $t \in (-\infty, \infty)$

Random signals and noise

Random processes

- Generally, $X(s,t)$ denotes a random process. For convenience, we will use $X(t)$
- $x_j(t)$ (the individual waveforms in the Figure) denote sample functions
- Probability measure assigns the probability to any meaningful event in $x_j(t)$

Random signals and noise

Random processes

Definition of quantities

- $X(t)$: Random process
- $x_j(t)$: Sample function associated with s_j
- $X(t_i)$: Random variable obtained by observing the process at $t = t_i$
- $x_j(t_i)$: Real-valued number, generating the value of $x_j(t)$ at $t = t_i$

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Random processes

Example

Let a fair coin be tossed, and consider a random process $X(t)$ with the following definition:

$$X(t) = \begin{cases} \sin \pi t, & \text{Head shows up} \\ 2t, & \text{Tail shows up} \end{cases}$$

- Sketch the sample functions
- Find PDFs of the RVs by sampling the process at $t = 0$ and $t = 1$

Random signals and noise
Random processes

Example

Let a fair die be thrown. A set of all possible outcomes (sample space) includes six sample points s_1, s_2,...,s_6 corresponding to six die faces. If the sample functions are given by $x_i(t) = \frac{1}{2}t + (i-1)$, $i = 1, \ldots, 6$ then find the mean of $X = X(t)$ at $t = 1$

Note: In the given examples, (1) Number of sample functions is known, and (2) Sample functions can mathematically be modeled. In some situations, these conditions cannot be achieved.

Random signals and noise

Random processes

Example

Let a large storage box contain N identical resistors. Consider an experiment of randomly picking a resistor from the box, which results in the probability P. Also, consider that sample functions are represented by voltages across terminals of the resistors.

1. What happens to P as $N \rightarrow \infty$?
2. Can we accurately describe time variation for any given waveform? Are the waveforms deterministic?
3. What makes this experiment random?
4. Sketch the ensemble for this experiment

Random signals and noise
Random processes

Stationarity

1. The random process $X(t)$ generates an infinite number of random variables in the range $-\infty < t < \infty$
2. Let $X(t)$ be observed at the time instants $t_1, t_2, \ldots, t_n$
3. Hence, we have the random variables $X(t_1), X(t_2), \ldots, X(t_n)$
4. The Joint Distribution Function of $X(t_i)$, $i = 1, 2, \ldots, n$, can be given in vector form as
$F_{\boldsymbol{X(t)}}(\boldsymbol{x}) = P\{X(t_1) \leq x_1, \ldots, X(t_2) \leq x_2\}$:
$\boldsymbol{X(t)} = \{X(t_1), \ldots, X(t_2)\}$, and $\boldsymbol{x} = \{x_1, \ldots, x_n\}$

Random signals and noise
Random processes

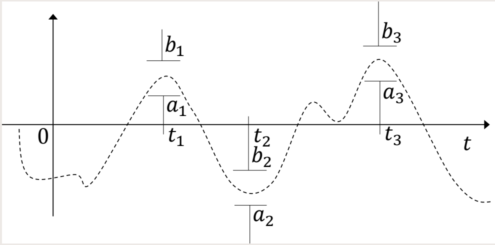
Stationarity

1. Hence, joint PDF of $F_{\boldsymbol{X(t)}}(\boldsymbol{x})$ is defined by

$$f_{\boldsymbol{X(t)}}(\boldsymbol{x}) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} F_{\boldsymbol{X(t)}}(\boldsymbol{x})$$

Random signals and noise
Random processes

Stationarity



- Let a random process $X(t)$ pass through a set of windows (stationary points)

Random signals and noise
Random processes

Stationarity

- If $f_{\boldsymbol{X(t)}}(\boldsymbol{x})$ is known, how can we compute the probability of an event
$A = \{s : a_1 \leq X(t_1) \leq b_1, a_2 \leq X(t_2) \leq b_2, a_3 \leq X(t_3) \leq b_3\}$?
- Intuitively,

$$P(A) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f_{\boldsymbol{X(t)}}(\mathbf{x})\ \boldsymbol{dx}$$

Where $\boldsymbol{X(t)} = \{X(t_1), X(t_2), X(t_3)\}$ and $\mathbf{dx} = dx_1 dx_2 dx_3$

Random signals and noise
Random processes

Stationarity

- A random process is said to be *Strictly Stationary* or *Stationary in Strict Sense* (SSS) if the joint PDF $f_{\boldsymbol{X(t)}}(\boldsymbol{x})$ is invariant to translation of the time origin
- $X(t)$ is SSS if

$$f_{\boldsymbol{X(t+T)}}(\boldsymbol{x}) = f_{\boldsymbol{X(t)}}(\boldsymbol{x})$$

  Where $\boldsymbol{(t+T)} = \{t_1 + T, t_2 + T, \ldots, t_n + T\}$

Random signals and noise
Random processes

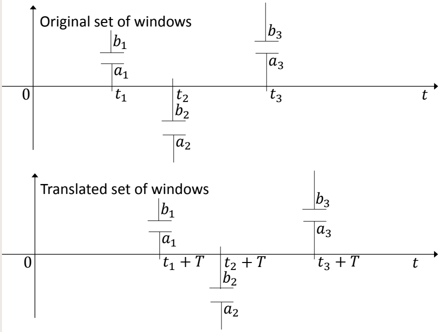
Stationarity

Implication of stationarity:

- Let $X(t)$ be SSS
- Then, probability of a set of sample functions of $X(t)$ passing through the selected windows equals probability of a set of sample functions passing through the corresponding time shifted windows. The two pairs of sample functions are unnecessarily the same.

Random signals and noise
Random processes

Stationarity



Random signals and noise

Random processes

Stationarity

Example:

- Let $X(t) = \sin 2\pi F t$, where F denotes a random variable with PDF
$$f_F(f) = \begin{cases} \frac{1}{100}, & 100 \leq f \leq 200Hz \\ 0, & \text{Elsewhere} \end{cases}$$
Show that $X(t)$ is non-stationary.
- Hint: Plot a few members of the ensemble, say with $f = 100, 150,$ and 200. Observe behavior of the ensemble at $t = 0$, $0 < t < 2.5ms$, and $-2.5 < t < 0$. Compute $X(t_1)$ for $t_1 = 1ms$ and $X(t_2)$ for $t_2 = -1ms$. Note that SSS process is independent of the observation instant.

Random signals and noise
Random processes

Stationarity

Example:

- Hint: Plot a few members of the ensemble, say with $f = 100, 150,$ and 200. Observe behavior of the ensemble at $t = 0$, $0 < t < 2.5ms$, and $-2.5 < t < 0$. Compute $X(t_1)$ for $t_1 = 1ms$ and $X(t_2)$ for $t_2 = -1ms$. Note that SSS process is independent of the observation instant.

Random signals and noise
Ensemble averages

- Characterization of a random process depends on the availability of $f_{\boldsymbol{X}(t)}(\boldsymbol{x})$
- If $f_{\boldsymbol{X}(t)}(\boldsymbol{x})$ is unknown, we may need to use *ensemble averages* to characterize the random process
    - Mean
    - Auto-correlation
    - Auto-covariance

Random signals and noise
Ensemble averages

Mean

$$m_X(t) = \overline{X(t)} = \int_{-\infty}^{\infty} x f_{X(t)}(x)\ dx$$

Let X_i be a random variable generated by sampling the random process at $t = t_i$. Then,

$$m_X(t_i) = \overline{X_i} = \int_{-\infty}^{\infty} x f_{X_i(t)}(x)\ dx$$

Random signals and noise

Ensemble averages

Auto-Correlation Function (ACF)

- Function of two variables, t_i and t_j
- Let $R_X(t_i, t_j)$ denote ACF of $X(t)$. Then, $R_X(t_i, t_j) = E[X(t_i)X(t_j)]$
- Let the Joint PDF between $X(t_i)$ and $X(t_j)$ be $f_{X_i,X_j}(x, y)$. Then,

$$R_X(t_i, t_j) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X_i,X_j}(x, y)\, dxdy$$

Random signals and noise

Ensemble averages

Auto-Correlation Function (ACF)

Properties: Let $R_X(\tau) = E[X(t+\tau)X(t)]$, where τ denotes the sampling time difference, be ACF of a wide sense stationary process $X(t)$. Then,

1. Mean square value of the process: $R_X(0) = E[X^2(t)]$
2. ACF is an even function of τ: $R_X(-\tau) = R_X(\tau)$
3. The origin gives the maximum ACF: $R_X(0) \geq |R_X(\tau)|$
4. If the sample functions of $X(t)$ are periodic, then ACF is also periodic with the same period

Random signals and noise
Ensemble averages

Auto-Covariance Function

- Function of two variables, t_i and t_j
- Let $C_X(t_i, t_j)$ denote the auto-covariance function of $X(t)$. Then, $C_X(t_i, t_j) = E[(X(t_i) - m_X(t_i))(X(t_j) - m_X(t_j))]$
- Assignment: Show that

$$C_X(t_i, t_j) = R_X(t_i, t_j) - m_X(t_i)m_X(t_j)$$

- A random process with zero mean value ($m_X(t) = 0$) has $C_X(t_i, t_j) = R_X(t_i, t_j)$

Random signals and noise
Ensemble averages

Auto-Covariance Function

- Mean value of a stationary process is constant
- That is

$$m_X(t) = m_X$$

Show that if $X(t)$ denotes a stationary process, then

- $R_X(t_i, t_j) = E[X(t_i - t_j)X(0)] = R_X(t_i - t_j, 0)$
- $C_X(t_i, t_j) = C_X(t_i - t_j, 0)$

Random signals and noise
Ensemble averages

Auto-Covariance Function

Wise Sense Stationary (WSS) processes satisfy the following conditions:

1. Constant mean value:
$$m_X(t) = m_X$$
2. Time-difference dependence of ACF:
$$R_X(t_i, t_j) = R_X(t_i - t_j, 0)$$

Random signals and noise

Ensemble averages

Example

Consider an experiment of tossing a fair die. Assume that the sample functions corresponding to every sample point s_i, $i = 1, 2, \ldots, 6$, are given by $x_i(t) = \frac{1}{2}t + (i - 1)$. Find $R_X(2, 4)$. (Ans: 18.66)

Random signals and noise

Ensemble averages

Question

Consider a random process $X(t)$ with six equally likely sample functions, defined by $x_i(t) = it$, $i = 1, 2, \dots, 6$. Let X and Y be the random variables obtained by sampling the process at $t = 1$ and $t = 2$, respectively. Find

1. $E[X]$ and $E[Y]$
2. $f_{X,Y}(x, y)$
3. $R_X(1, 2)$

Random signals and noise

Ensemble averages

Question

A random process $X(t)$ consists of five sample functions, each with an occurrence chance of $\frac{1}{5}$. Four of these sample functions are $x_1(t) = \cos 2\pi t - \sin 2\pi t$, $x_2(t) = \sin 2\pi t - \cos 2\pi t$, $x_3(t) = -\sqrt{2}\cos t$, and $x_4(t) = -\sqrt{2}\sin t$. Find the fifth sample function $x_5(t)$ such that $X(t)$ is (i) zero mean, and $R_X(t_1, t_2) = R_X(|t_1 - t_2|)$. Let V be the random variable of $X(t)$ at $t = 0$ and let W be the random variable at $t = \frac{\pi}{4}$. Show that the process is WSS, and that, despite this condition, $f_V(v) \neq f_W(v)$.

Random signals and noise

Ensemble averages

Correlation

- Consider two random processes, $X(t)$ and $Y(t)$
- Let t_1 and t_2 be the time instants that $X(t)$ and $Y(t)$ were observed, respectively
- The cross-correlation between these processes is given by

$$R_{X,Y}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

Random signals and noise
Ensemble averages

Correlation

- Alternatively,

$$R_{X,Y}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x, y; t_1, t_2)\, dx\, dy$$

- The auto-correlation function of $X(t)$ contains two variables, t_k and t_i, and is given by

$$R_X(t_k, t_i) = E[X(t_k)X(t_i)]$$

Random signals and noise

Ensemble averages

Correlation

- Alternatively,

$$R_{X,Y}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x, y; t_1, t_2)\; dx\; dy$$

- The auto-correlation function of $X(t)$ contains two variables, t_k and t_i, and is given by

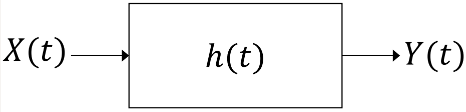
$$R_X(t_k, t_i) = E[X(t_k)X(t_i)]$$

Random signals and noise
Ensemble averages

Questions

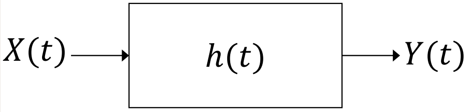
- Consider the previous example of throwing a die. Find $R_X(2,4)$. (Ans: 18.66)
- Consider six equally likely sample functions, given by $x_i(t) = it$ $(i = 1, \ldots, 6)$, of a random process $X(t)$. Let X and Y be random variables obtained by sampling the process at $t = 1$ and $t = 2$, respectively. Find
  - $E[X]$ and $E[Y]$
  - $F_{X,Y}(x, y)$
  - $R_X(1, 2)$

Random signals and noise
Transmission of random processes through LTI systems



- LTI systems can be excited by random signals
- Let $h(t)$ denote the known impulse response of the LTI system being excited by a random process $X(t)$ to produce an output Process $Y(t)$
- Objective: To characterize $Y(t)$ in terms of $X(t)$ and $h(t)$

Random signals and noise
Transmission of random processes through LTI systems



- Let $x_j(t) \in X(t)$ be the sample function of $X(t)$
- Let $y_j(t) \in Y(t)$ be the corresponding output
- Then,

$$y_j(t) = \int_{-\infty}^{\infty} h(\tau) x_j(t-\tau)\, d\tau$$

Random signals and noise
Transmission of random processes through LTI systems

- Because the equation for $y_j(t)$ is true for all sample functions of $X(t)$, then

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)\ d\tau$$

- Computing mean of the output process $Y(t)$, we have

$$m_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau)X(t-\tau)\ d\tau\right]$$

Random signals and noise

Transmission of random processes through LTI systems

- If $E[X(t)]$ is finite $\forall t$ and the system is stable then orders of expectation and integration can be interchanged with respect to τ
- Hence,

$$m_Y(t) = E[Y(t)] = \int_{-\infty}^{\infty} h(\tau) E[X(t-\tau)]\, d\tau$$

- Note that $h(\tau)$ is deterministic, and can be taken outside the expectation operator $E[\cdot]$

Random signals and noise

Transmission of random processes through LTI systems

- Given that $X(t)$ is WSS, then $E[X(t)] = m_X$ (constant value)
- Hence,

$$m_Y(t) = m_X \int_{-\infty}^{\infty} h(\tau)\, d\tau = m_X H(0)$$

  Where $H(0)$ denotes the transfer function $H(f)$ of the system at $f = 0$
- $m_Y(t)$ is constant

Random signals and noise

Transmission of random processes through LTI systems

Question

- Show that

$$R_Y(t,u) = \int_{-\infty}^{\infty} h(\tau_1)\ d\tau_1 \int_{-\infty}^{\infty} h(\tau_2) R_X(\tau - \tau_1 + \tau_2)\ d\tau_2$$

  Where $\tau = t - u$. From this equation, explain why $X(t)$ is WSS.
- $R_Y(t,u)$ is a function of $t - u$ only, and thus $R_Y(t,u) = R_Y(\tau)$

Random signals and noise

Power spectral density (PSD) of random processes

- Provides frequency domain description of stationary random processes
- Given by

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} \, d\tau$$

- From the PSD formula,

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} \, df$$

Random signals and noise

Power spectral density (PSD) of random processes

Properties

- PSD of a WSS process at $f = 0$ equals the total area under the ACF curve:

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\ d\tau$$

- Mean square value of a WSS process equals the total area under the PSD curve:

$$E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f)\ df$$

- $S_X(-f) = S_X(f)$
- $S_X(f) \geq 0$

Random signals and noise

Power spectral density (PSD) of random processes

Example

Let $X(t) = A\cos(\omega_c t + \Theta)$, where A and ω_c are constants, and Θ denotes a random variable with PDF

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{Otherwise} \end{cases}$$

Find

1. $m_X(t)$ and $R_X(t_1, t_2)$
2. PSD

Random signals and noise
Gaussian process

- Contains great practical and mathematical significances
  - Most real-world noise processes interfering with communication systems can be described using Gaussian processes
  - Mathematical analysis of Gaussian processes is feasible

Random signals and noise

Gaussian process

- Let $X(t)$ be sampled at time instants $t_1, t_2, \ldots, t_n$
- Let $X(t_1) = x_1, X(t_2) = x_2, \ldots, X(t_n) = x_n$ and $\boldsymbol{x}$ represent a row vector $\{x_1, x_2, \ldots, x_n\}$
- For $X(t)$ to be Gaussian, $f_X(\boldsymbol{x})$ must be an n-dimensional joint Gaussian density for every $n \geq 1$ and $(t_1, t_2, \ldots, t_n) \in (-\infty, \infty)$

Random signals and noise
Gaussian process

- The n-dimensional Gaussian PDF is given by

$$f_X(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}|C_X|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{m}_X)C_X^{-1}(\boldsymbol{x} - \boldsymbol{m}_X)^T\right]$$

where $|C_X|$ denotes determinant of the covariance matrix C_X, whose inverse is given by C_X^{-1}; m_X represents the mean vector: $\{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}\}$; and T denotes transpose.

Random signals and noise
Gaussian process

- The covariance matrix is given by

$$C_X = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \dots & \lambda_{nn} \end{bmatrix}$$

  where $\lambda_{ij} = \mathrm{cov}[x_i, x_j] = E[(x_i - \overline{x_i})(x_j - \overline{x_j})]$
- $(\boldsymbol{x} - \boldsymbol{m}_X) = (x_1 - \overline{x_1}, x_2 - \overline{x_2}, \dots, x_n - \overline{x_n})$

Random signals and noise

Gaussian process

Example

X_1 and X_2 are two joint Gaussian variables with $\overline{X_1} = \overline{X_2} = 0$ and $\sigma_1 = \sigma_2 = \sigma$. The correlation coefficient between X_1 and X_2 is ρ. Write the joint PDF of X_1 and X_2 in matrix and expanded forms.

Random signals and noise
Electrical noise

- Electronic communication systems consist of various circuit components: inductors (L), resistors (R), capacitors (C), transistors, and diodes, among others
- These components, even when in non-operational state, generate internal circuit noise because of random movement of electrons caused by temperature
- This noise limits quality of communication systems

Random signals and noise
Electrical noise

- Major noise types in communication systems are
  1. Thermal noise
  2. Shot noise
- Thermal noise in metallic resistors was first studied by Johnson and Nyquist. Hence, this noise type is also referred to as Johnson (or resistance) noise
- Assignment: Describe the origin of thermal noise in electronic circuits

Random signals and noise
Electrical noise

White noise

- Has noise quantity with flat power spectrum
- Contains PSD with frequency components in equal proportions between $-\infty < f < \infty$
- Has PSD given as

$$S_W(f) = \frac{N_0}{2}$$

(Unit: Watts/Hz)

Random signals and noise

Electrical noise

White noise

- The constant $\frac{1}{2}$ in $S_W(f)$ means 50% of the power is associated with positive frequencies, and the remaining 50% corresponds to negative frequencies
- ACF of a white noise is given by

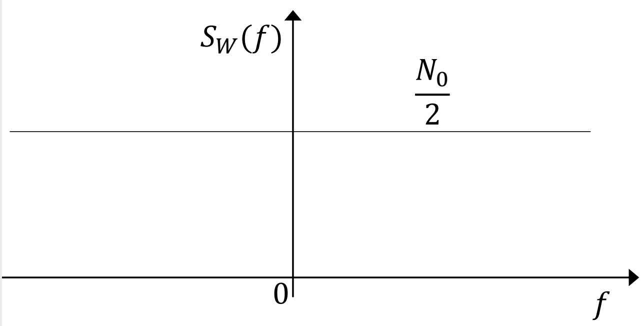
$$R_W(\tau) = \frac{N_0}{2}\delta(\tau)$$

- WGN (White Gaussian Noise) refers to a Gaussian process that contains white noise

Random signals and noise

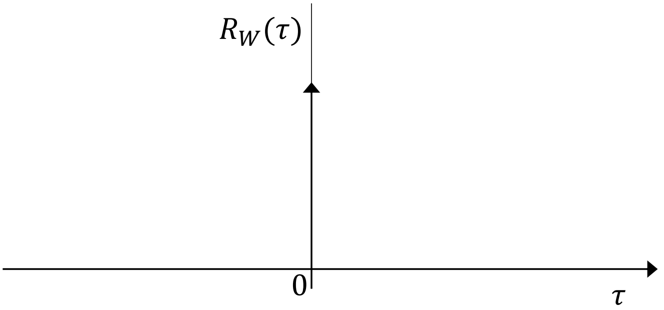
Electrical noise

White noise



Random signals and noise
Electrical noise

White noise



Random signals and noise
Electrical noise

Narrowband noise

- In communication theory, we are usually concerned with NarrowBand BandPass (NBBP) signals
- Spectrum of NBBP signals is concentrated around a (nominal) center frequency f_c
- Bandwidth of NBBP signals is smaller than f_c

Random signals and noise

Electrical noise

Narrowband noise

- Let $N(t)$ represent the noise process at the output of a narrowband filter generated in response to a white noise process $W(t)$
- $S_W(f) = \frac{N_0}{2}$; if $H(f)$ denotes the filter transfer function, then

$$S_N(f) = \frac{N_0}{2} |H(f)|^2$$

Random signals and noise
Reference

The materials in this Lecture have been based largely on Chapter 3 of the book *Principles of Communication*, "Random signals and noise," by V. Venkata Rao

End of Unit 4