Dynamic Model Based Control
Lab (State Space approach for Linear Systems)

Tuesday 19 March 2024 (Group 1) - Thursday 21 March 2024 (Group 2)

Deliverables	**Type :** group of 2 students	
Format : Printed Document	**Due date :** 2 April 2024	**Max. Page :** 10
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Note that your Matlab code will not be read.
Only the plots and the comments on the plots will be taken into account.

The Inverted Pendulum

The aim of this study is to control the angle of a pendulum on a cart as depicted on figure 1. The rotation of the pendulum is completely free. The motion of the cart is controlled by the force u.

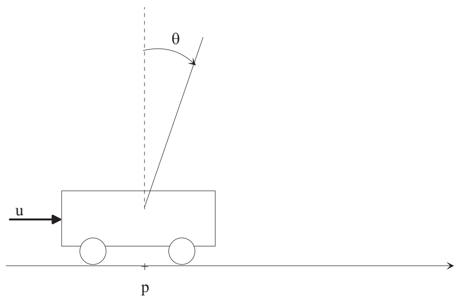

FIGURE 1 – The Inverted Pendulum.

One notes :
- θ, the angle between the vertical axis and the pendulum
- p, the position of the cart.
- m, the mass of the pendulum. It is supposed to be concentrated in its center of gravity, located at a distance L of the rotating point.
- M, the mass of the cart.

Friction forces are supposed to be negligible in the rotation motion of the cart and in the translation motion of the cart. The mechanical equations are

$$\ddot{\theta}\left[mLcos^2\theta - (M+m)\,L\right] - mL\dot{\theta}^2\sin\theta\cos\theta + (M+m)\,g\sin\theta = u\cos\theta \qquad (1)$$

$$\ddot{p}\left[mcos^2\theta - (M+m)\right] + mL\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta = -u \qquad (2)$$

Here, small variations are considered around the equilibrium point defined by $\theta = 0$ and any p. So a linearized version of this system is suitable to design a controller.

1 State Space model of the system

1. Show that the linearized model around this equilibrium point is

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{p} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \left(\frac{m+M}{ML}\right)g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \left(-\frac{mg}{M}\right) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{ML} \\ 0 \\ \frac{1}{M} \end{bmatrix} u$$

   This model will be the *design model* for a control that will try to maintain the pendulum vertical and the cart in an equilibrium position.

2. Numerical Application : $m = 0.5kg$, $M = 5kg$, $L = 1m$, $g = 9.8m/s^2$. Give the numerical values of the matrices A, and B of the state space representation.

3. What are the eigenvalues associated to this linear state space representation ?

2 State feedback Controller - Access to the whole state space

In this section we suppose that we have access to the whole state as if sensors could give the angle θ, its derivative $\dot{\theta}$, the position p and its derivative $\dot{p}$.

2.1 State feedback

1. Is it possible, with a static state feedback, to stabilize the pendulum and the cart around an equilibrium point ?
2. Can one place freely all the eigenvalues ?
3. If yes, compute (using Matlab) the state feedback F such that the eigenvalues are $-1 \pm j$ and $-2 \pm 2j$.
4. Construct the Simulink block diagram of the closed loop with the control law $u = Fx + v$ and plot the states response $x(t)$ and the control response $u(t)$ when $v(t) = 0$ for the following initial conditions, $\theta(0) = 0.1\ rd$, $\dot{\theta}(0) = 0\ rd/s$, $p(0) = 0.1\ m$ and $\dot{p}(0) = 0\ m/s$.
   Plot $x_1(t), x_2(t), x_3(t), x_4(t)$ and $u(t)$ on five different axis, on the same figure : use the subplot function of Matlab (see the appendix of section 5).

2.2 Observer

In this subsection, one introduces an observer even if it will not be used in the loop to apply estimated state feedback (this will be done in the next section 3).
For this observer, one will consider that only two sensors are available, one gives the angle θ and the other one the position of the cart p.
So, here, the idea is to use the previous control (static state feedback $u = Fx + v$) and plug in parallel the observer to check if it correctly estimates the states.

1. Is the system observable ?
2. If yes, add an observer in the previous Simulink block diagram.
   — Compute the two gains (Matlab function `place`) corresponding to :
      (a) "faster poles" than the ones used for the control : take $-100 \pm j$ and $-200 \pm 2j$,
      (b) "poles as fasts as" the ones used for the control : take $-1 \pm j$ and $-2 \pm 2j$.
   — Choose "zero Initial Conditions" for the observer.
   — Choose the previous state feedback (eigenvalues of $A + BF$ equal to $-1 \pm j$ and $-2 \pm 2j$).
   — Plot the responses of $x_i(t)$ and $\hat{x}_i(t)$ on the same axis for $i = 1 \ldots 4$ and the response of $u(t)$ for the two series of observer poles (look at the code of the appendix in section 5).
   — Remember that only plots and comments will be taken into account for the assessment !

3 Estimated state feedback controller

In this section, one considers that only the angle θ and the position of the cart p are available for the control.

1. Give the new output equation of the state space representation corresponding to the new design model.
2. Construct the new Simulink block diagram which take into account the fact that the output of the system is now just θ and p and the state feedback connected to the observer.
3. Choose the previous state feedback (with desired eigenvalues of $A + BF$ equal to $-1 \pm j$ et $-2 \pm 2j$).
4. Plot and compare $x_i(t)$ and $\hat{x}_i(t)$ $(i = 1 \ldots 4)$ on four subplot and plot $u(t)$ on a separate subplot (see the appendix in section 5).
5. Compare these responses with the ones of the previous section (with the two series of observer poles, and with the same poles of $A + BF$).
6. Remember that only plots and comments will be taken into account for the assessment !

4 Animation

There exists a Matlab demo which gives the possibility to create an animation of the Inverted Pendulum.
With the help of the files "*lab1_dybac_L.slx*" and "*lab1_dybac_NL.slx*" (and also the file "*lab1_pendan_g.m*" see the server Hippocampus), you can create such an animation if you add your personal observer and state feedback. For this, just fill the block $u = Fxhat + v$ in theses two files.

5 Appendix

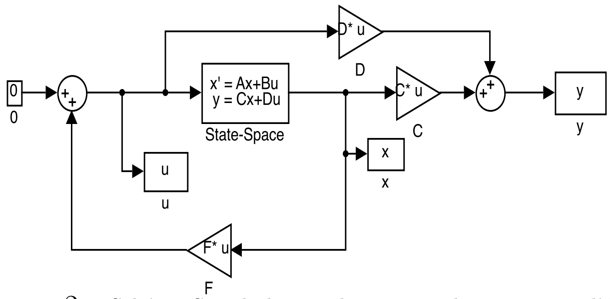

FIGURE 2 – Schéma Simulink pour la commande par retour d'état.

If for the above block diagram, the block "ToWorkspace x" has been configured to save the data in a variable denoted "x" and with the format array, then the code below is a solution to plot the following figure (be careful with a Scope the use of array is different, the first column of "x" would be the time ...).
Also note that the variable "tout" (read time out) should be selected in the "Data Import/Export" of the tool "Model Settings" in "MODELING"

```
f1=figure(1) ;
set(f1,'position',[1 305 672 500])
subplot(321),plot(tout,x( :,1)),title('angle'),grid,
subplot(323),plot(tout,x( :,2)),title('derivative of angle'),grid
subplot(322),plot(tout,x( :,3)),title('position'),grid,
subplot(324),plot(tout,x( :,4)),title('derivative of position'),grid
subplot(325),plot(tout,u),title('control'),grid
```

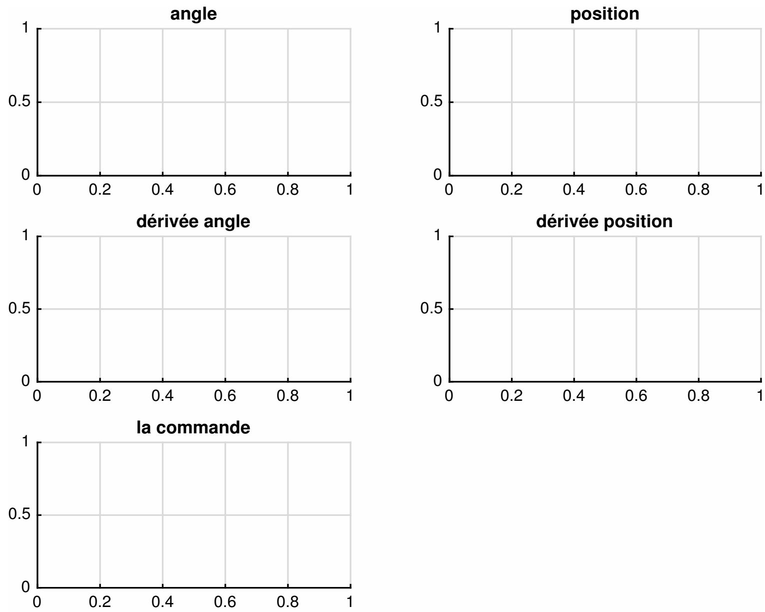

FIGURE 3 – Display of the plot of each state and of the control signal.

To compare the first component of the state, x_1, and its estimate, $\hat{x}_1$, one can draw multiple graphs on the same plot with the following code (here x_1 would be the first component of the signal in the scope "x" and $\hat{x}_1$ the first component of the signal of a scope "xhat" associated to the observer that will be introduced) :

```
f1=figure(1) ;
set(f1,'position',[1 305 672 500])
subplot(321),plot(tout,x( :,1),tout,xhat( :,1),'r :'),title('angle and its estimate'),grid,
```