# Type inference from scratch

Theory and implementation

Christoph Hegemann

#### **DISCLAIMER**



Hi, I'm Christoph.

Hi, I'm Christoph. I love types.

 $\mbox{\rm Hi, I'm}$  Christoph.  $\mbox{\rm I}$  love types.  $\mbox{\rm I}$  hate typing.

Hi, I'm Christoph. I love types. I hate typing. I hate typing types.

So let's learn about type inference!

#### Definition

#### Definition

#### Definition

#### Definition

#### OUR LANGUAGE

We'll be implementing a type inference algorithm for the Lambda Calculus, extended with let-bindings, integer literals, and boolean literals.

### **EXPRESSIONS**

1

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1

\x → x

#### **EXPRESSIONS**

```
1 \x \rightarrow x let const = \alpha \rightarrow \b \rightarrow a in const 1 true
```

```
E ::=

1 -- int literal

| true/false -- boolean literal
```

```
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1 -- int literal

| true/false -- boolean literal

| x -- Variable
```

```
1 -- int literal
| true/false -- boolean literal
| x -- Variable
| E E -- Application
| \x → E -- Lambda
| let x = E in E -- Let binding
```

```
data Lit
    = LInt Integer
    | LBool Bool

data Exp
    = ELit Lit
```

```
E ::=

| true/false
| x
| E E
| \x → E
| let x = E in E
```

```
data Lit
    = LInt Integer
    | LBool Bool

data Exp
    = ELit Lit
    | EVar Text
```

```
data Lit
    = LInt Integer
    | LBool Bool

data Exp
    = ELit Lit
    | EVar Text
    | EApp Exp Exp
```

```
data Lit
 = LInt Integer
  | LBool Bool
data Exp
  = ELit Lit
   EVar Text
  EApp Exp Exp
   ELam Text Exp
```

```
data Lit
  = LInt Integer
  | LBool Bool
data Exp
  = ELit Lit
   EVar Text
  EApp Exp Exp
  | ELam Text Exp
   ELet Text Exp Exp
```

Int

Int

 $Int \to Bool$ 

Int

 $Int \rightarrow Bool$ 

 $\forall a. a \rightarrow a$ 

#### Int

$$Int \rightarrow Bool$$

$$\forall a. a \rightarrow a$$

$$\forall a\; b.\; (a \to b) \to a \to b$$

T ::=

```
T ::=

Int -- primitive
| Bool -- primitive
```

```
T ::=

Int -- primitive
| Bool -- primitive
| a -- type variable
```

```
T ::=

Int — primitive

Bool — primitive

a — type variable

T → T — function type
```

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Scheme :=
```

### DEFINING OUR TYPE LANGUAGE

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```
T ::=

Int -- primitive

Bool -- primitive

a -- type variable

T -- T -- function type

Scheme :=

T

V a1 a2 an. T
```

```
data Type
= TInt
| TBool
| TVar Text
| TFun Type Type

data Scheme = Scheme [Text] Type

T ::=
Int
| Bool
| a
| T → T

Scheme := T
| ∀ a1 a2 an. T
```

Warmup Quiz

1:?

1 : *Int* 

true:?

true: Bool

 $\x \rightarrow x : ?$ 

 $\xspace \xspace \xsp$ 

 $\x \rightarrow \y \rightarrow x:?$ 

 $\xspace \xspace \xsp$ 

\x → x 1:?

 $\x \rightarrow x \ 1 : \forall b. (Int \rightarrow b) \rightarrow b$ 

let x = 5 in x:?

let x = 5 in x : Int

let id =  $\x \rightarrow x$  in id true:?

let id = \x → x in id true : Bool

let id =  $\x \rightarrow x$  in id:?

let id =  $\xspace x \rightarrow x$  in id:  $\forall a. a \rightarrow a$ 

# **JUDGEMENTS**

#### Definition

A typing judgement describes a relation between a piece of syntax and its type.

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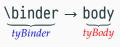
**Notation** 

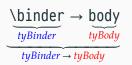
$$\frac{A}{C}$$

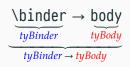
In order to show C, one needs to show A and B.

 $\verb|\binder| \to body$ 

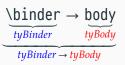
$$\underbrace{\text{binder}}_{\textit{tyBinder}} \rightarrow \text{body}$$



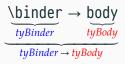


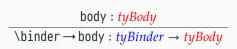


 $\begin{tabular}{l} \verb+\begin{tabular}{l} \verb+\begin{$ 



```
body: tyBody \\ binder \rightarrow body: tyBinder \rightarrow tyBody
```





#### Problem

We're missing the fact that when inferring we're the type of **body**, there is an additional variable **binder** in scope. We'll fix that by introducing *Contexts*.

# **CONTEXTS**

# Definition

A Context is a mapping from (value level) variables to schemes.

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#### **Notation**

We use capital greek letters to denote contexts, for our type system we'll only need one  $\Gamma$  (Gamma). The  $\vdash$  symbol means "in the context"

$$\Gamma \vdash x : ty_1$$

In the context  $\Gamma$ , x has type  $ty_1$ .

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$$\Gamma \vdash x : ty_1$$

In the context  $\Gamma$ , x has type  $ty_1$ .

### **Implementation**

type Context = Map Text Scheme

### INFERRING LAMBDAS

$$\begin{array}{c} \text{binder} \rightarrow \text{body} \\ \hline tyBinder \rightarrow tyBody \\ \end{array}$$

body: tyBody\binder  $\rightarrow$  body:  $tyBinder \rightarrow tyBody$ 

### INFERRING LAMBDAS

$$\begin{array}{c}
\text{binder} \to \text{body} \\
 \underline{tyBinder} \to \underline{tyBody} \\
\hline
 \underline{tyBinder} \to \underline{tyBody}
\end{array}$$

body: tyBody  $\Gamma \vdash \land binder \rightarrow body: tyBinder \rightarrow tyBody$ 

### INFERRING LAMBDAS

$$\begin{array}{c}
\text{binder} \to \text{body} \\
\underline{tyBinder} \to \underline{tyBody} \\
\underline{tyBinder} \to \underline{tyBody}
\end{array}$$

```
\Gamma, binder : tyBinder \vdash body : tyBody

\Gamma \vdash \land binder \rightarrow body : tyBinder \rightarrow tyBody
```

## INFER WITH CONTEXT

infer :: Expr  $\rightarrow$  Type

### INFER WITH CONTEXT

infer :: Context → Expr → Type

### INFER WITH CONTEXT

infer :: Context  $\rightarrow$  Expr  $\rightarrow$  TI Type

#### FRESH TYPE VARIABLES

```
type TI a = State Int a

newTyVar :: TI Type
newTyVar = do
   s <- get
   put (s + 1)
   pure (TVar ("u" <> showT s))
```

Don't worry about this too much. Just remember we can call **newTyVar** whenever we want a type variable with a unique name.

 $\Gamma \vdash \land binder \rightarrow body : tyBinder \rightarrow tyBody$ 

```
\Gamma \vdash \backslash binder \rightarrow body : tyBinder \rightarrow tyBody
```

```
infer :: Context → Expr → TI Type
infer ctx (ELam binder body) = do
```

pure (TFun tyBinder tyBody)

 $\Gamma$ , binder :  $tyBinder \vdash body : tyBody$ 

| \f → const 1 (f true)

```
const 1 (f true) : ? | f \rightarrow const 1 (f true)
```

```
const 1 (f true) : Int | \f \rightarrow const 1 (f true)
```

```
| \f → const 1 (f true)
```

Things we learn about **f**:

```
\f → const 1 (f true)
```

# Things we learn about **f**:

• **f** is a function

```
\f → const 1 (f true)
```

## Things we learn about **f**:

- **f** is a function
- f accepts a Bool as its first argument

```
\footnote{$\setminus$} f \Rightarrow const 1 (f true)
```

 $\five f \Rightarrow const 1 (f true):?$ 

\f  $\rightarrow$  const 1 (f true):  $\forall a. (Bool \rightarrow a) \rightarrow Int$ 

#### INFER WITH SUBSTITUTIONS

We need to propagate additional information when inferring.

infer :: Context  $\to$  Expression  $\to$  TI Type

#### INFER WITH SUBSTITUTIONS

We need to propagate additional information when inferring. infer  $:: Context \rightarrow Expression \rightarrow TI (Substitution, Type)$ 

## **SUBSTITUTION**

# Definition

A substitution is a mapping from type variables to types.

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#### **Notation**

We write the substitution S, mapping type variables  $var_1, ..., var_n$  to the types  $ty_1, ..., ty_n$  like so:

$$S = [var_1 \mapsto ty_1, \, var_2 \mapsto ty_2, \, \dots, \, var_n \mapsto ty_n]$$

#### Substitution

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$$S = [var_1 \mapsto ty_1, var_2 \mapsto ty_2, \dots, var_n \mapsto ty_n]$$

### **Implementation**

type Substitution = Map Text Type

#### APPLYING A SUBSTITUTION

When applying a substitution to a type, we substitute all occurrences of variables in that type that also occur in the substitution.

 $[a \mapsto Int] \ a = ?$ 

 $[a \mapsto Int] \ a = Int$ 

 $[a \mapsto Int] (a \rightarrow a) = ?$ 

 $[a \mapsto Int] (a \rightarrow a) = Int \rightarrow Int$ 

 $[a \mapsto Int] (a \rightarrow b) = ?$ 

 $[a\mapsto Int]\ (a\to b)=Int\to b$ 

 $[a \mapsto Int] (\forall a. a) = ?$ 

 $[a \mapsto Int] (\forall a. a) = \forall a. a$ 

 $[a \mapsto Int, b \mapsto Bool] (\forall a. a \rightarrow b) = ?$ 

 $[a \mapsto Int, b \mapsto Bool] (\forall a. a \rightarrow b) = \forall a. a \rightarrow Bool$ 

```
applySubst :: Substitution → Type → Type applySubst subst ty = case ty of
```

```
applySubst :: Substitution → Type → Type
applySubst subst ty = case ty of
  TVar var →
    fromMaybe (TVar var) (Map.lookup var subst)
```

```
applySubst :: Substitution → Type → Type
applySubst subst ty = case ty of
  TVar var →
    fromMaybe (TVar var) (Map.lookup var subst)
  TFun arg res →
    TFun (applySubst subst arg) (applySubst subst res)
```

```
applySubst :: Substitution → Type → Type
applySubst subst ty = case ty of
  TVar var →
    fromMaybe (TVar var) (Map.lookup var subst)
  TFun arg res →
    TFun (applySubst subst arg) (applySubst subst res)
  TInt → TInt
  TBool → TBool
```

```
applySubstScheme :: Substitution → Scheme → Scheme
applySubstScheme subst (Scheme vars t) =
   Scheme vars (applySubst (foldr Map.delete subst vars) t)
```

When applying a substitution to a scheme, we substitute all occurrences of *free* variables in that type that also occur in the substitution.

```
applySubstScheme :: Substitution → Scheme → Scheme
applySubstScheme subst (Scheme vars t) =
   Scheme vars (applySubst (foldr Map.delete subst vars) t)
```

We delete all variables that are bound by the scheme from the substitution before applying it to the inner type.

When applying a substitution to a context, we apply the substitution to every scheme in the context.

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```
applySubstCtx :: Substitution → Context → Context
applySubstCtx subst ctx =
  Map.map (applySubstScheme subst) ctx
```

### **COMPOSING SUBSTITUTIONS**

#### Definition

Let S and U be substitutions then their composition for all types t is defined as:

$$(S \circ U) \ t = S \ (U \ t)$$

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#### Definition

Let *S* and *U* be substitutions then their composition for all types *t* is defined as:

$$(S \circ U) t = S (U t)$$

### **Implementation**

composeSubst :: Substitution  $\rightarrow$  Substitution  $\rightarrow$  Substitution composeSubst s1 s2 = Map.union (Map.map (applySubst s1) s2) s1

## INFERRING LAMBDAS WITH SUBSTITUTIONS

```
infer :: Context → Expr → TI Type
infer ctx (ELam binder body) = do
   tyBinder ← newTyVar
   let tmpCtx = Map.insert binder (Scheme [] tyBinder) ctx
   tyBody ← infer tmpCtx body
   pure (TFun tyBinder tyBody)
```

## INFERRING LAMBDAS WITH SUBSTITUTIONS

```
infer :: Context → Expr → TI (Substitution, Type)
infer ctx (ELam binder body) = do
   tyBinder ← newTyVar
   let tmpCtx = Map.insert binder (Scheme [] tyBinder) ctx
   (s1, tyBody) ← infer tmpCtx body
   pure (s1, TFun (applySubst s1 tyBinder) tyBody)
```

#### INFERRING LITERALS

When inferring literals we don't care about substitution, nor context.

```
infer :: Context → Expr → TI (Substitution, Type)
infer _ (ELit (LInt _)) = pure (emptySubst, TInt)
infer _ (ELit (LBool _)) = pure (emptySubst, TBool)
```

$$\underbrace{tyExpr in body}_{tyBody}$$

$$\underbrace{tyExpr in body}_{tyBody}$$

 $\Gamma \vdash \mathbf{let} \ \mathbf{x} = \mathbf{exprin} \ \mathbf{body} : tyBody$ 

$$\underbrace{tyExpr}_{tyBody} \underbrace{tyBody}_{tyBody}$$

$$\frac{\Gamma \vdash \mathsf{expr} : \mathit{tyExpr}}{\Gamma \vdash \mathsf{let} \; \mathsf{x} = \mathsf{expr} \; \mathsf{in} \; \mathsf{body} : \mathit{tyBody}}$$

$$\underbrace{tyExpr}_{tyBody} \underbrace{tyBody}_{tyBody}$$

```
\frac{\Gamma \vdash \mathsf{expr} : \mathit{tyExpr} \qquad \Gamma, \, \mathsf{x} : \mathit{tyExpr} \vdash \mathsf{body} : \mathit{tyBody}}{\Gamma \vdash \mathsf{let} \, \mathsf{x} = \mathsf{exprin} \, \mathsf{body} : \mathit{tyBody}}
```

```
\frac{\Gamma \vdash \mathsf{expr} : \mathit{tyExpr} \quad \Gamma, \, \mathsf{x} : \mathit{tyExpr} \vdash \mathsf{body} : \mathit{tyBody}}{\Gamma \vdash \mathsf{let} \, \mathsf{x} = \mathsf{expr} \, \mathsf{in} \, \mathsf{body} : \mathit{tyBody}}
```

```
\frac{\Gamma \vdash \mathsf{expr} : \mathit{tyExpr} \qquad \Gamma, \, \mathsf{x} : \mathit{tyExpr} \vdash \mathsf{body} : \mathit{tyBody}}{\Gamma \vdash \mathsf{let} \, \mathsf{x} = \mathsf{exprin} \, \mathsf{body} : \mathit{tyBody}}
```

```
\frac{\Gamma \vdash \mathsf{expr} : \mathit{tyExpr} \qquad \Gamma, \, \mathsf{x} : \mathit{tyExpr} \vdash \mathsf{body} : \mathit{tyBody}}{\Gamma \vdash \mathsf{let} \, \mathsf{x} = \mathsf{expr} \, \mathsf{in} \, \mathsf{body} : \mathit{tyBody}} infer ctx (Let x expr body) = do
```

```
\frac{\Gamma \vdash \mathsf{expr} : \mathit{tyExpr} \qquad \Gamma, \, \mathsf{x} : \mathit{tyExpr} \vdash \mathsf{body} : \mathit{tyBody}}{\Gamma \vdash \mathsf{let} \, \mathsf{x} = \mathsf{exprin} \, \mathsf{body} : \mathit{tyBody}} infer ctx (Let x expr body) = do
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```

pure tyBody

```
\frac{\Gamma \vdash \mathsf{expr} : \mathit{tyExpr} \qquad \Gamma, \, \mathsf{x} : \mathit{tyExpr} \vdash \mathsf{body} : \mathit{tyBody}}{\Gamma \vdash \mathsf{let} \, \mathsf{x} = \mathsf{expr} \, \mathsf{in} \, \mathsf{body} : \mathit{tyBody}} \mathsf{infer} \, \mathsf{ctx} \, \, (\mathsf{Let} \, \, \mathsf{x} \, \, \mathsf{expr} \, \, \mathsf{body}) \, = \, \mathsf{do}
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```

```
\frac{\Gamma \vdash \mathsf{expr} : ty Expr}{\Gamma \vdash \mathsf{let} \ \mathsf{x} = \mathsf{exprin} \ \mathsf{body} : ty Body}
\Gamma \vdash \mathsf{let} \ \mathsf{x} = \mathsf{exprin} \ \mathsf{body} : ty Body
\mathsf{infer} \ \mathsf{ctx} \ (\mathsf{Let} \ \mathsf{x} \ \mathsf{expr} \ \mathsf{body}) = \mathsf{do}
\mathsf{tyExpr} \leftarrow \mathsf{infer} \ \mathsf{ctx} \ \mathsf{expr}
\mathsf{let} \ \mathsf{tmpCtx} = \mathsf{Map.insert} \ \mathsf{x} \ (\mathsf{Scheme} \ [] \ \mathsf{tyExpr}) \ \mathsf{ctx}
\mathsf{pure} \ \mathsf{tyBody}
```

```
\frac{\Gamma \vdash \mathsf{expr} : ty Expr}{\Gamma \vdash \mathsf{let} \; \mathsf{x} = \mathsf{expr} \; \mathsf{in} \; \mathsf{body} : ty Body}
\Gamma \vdash \mathsf{let} \; \mathsf{x} = \mathsf{expr} \; \mathsf{in} \; \mathsf{body} : ty Body}
\mathsf{infer} \; \mathsf{ctx} \; (\mathsf{Let} \; \mathsf{x} \; \mathsf{expr} \; \mathsf{body}) = \mathsf{do}
\mathsf{tyExpr} \; \mathsf{\leftarrow} \; \mathsf{infer} \; \mathsf{ctx} \; \mathsf{expr}
\mathsf{let} \; \mathsf{tmpCtx} = \mathsf{Map.insert} \; \mathsf{x} \; (\mathsf{Scheme} \; [] \; \mathsf{tyExpr}) \; \mathsf{ctx}
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```

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\frac{\Gamma \vdash \mathsf{expr} : \mathit{tyExpr} \qquad \Gamma, \, \mathsf{x} : \mathit{tyExpr} \vdash \mathsf{body} : \mathit{tyBody}}{\Gamma \vdash \mathsf{let} \, \mathsf{x} = \mathsf{expr} \, \mathsf{in} \, \mathsf{body} : \mathit{tyBody}} \mathsf{infer} \, \mathsf{ctx} \, (\mathsf{Let} \, \mathsf{x} \, \mathsf{expr} \, \mathsf{body}) = \mathsf{do} \mathsf{tyExpr} \, \hookleftarrow \, \mathsf{infer} \, \mathsf{ctx} \, \mathsf{expr} \mathsf{let} \, \mathsf{tmpCtx} = \mathsf{Map.insert} \, \mathsf{x} \, (\mathsf{Scheme} \, [] \, \mathsf{tyExpr}) \, \mathsf{ctx} \mathsf{tyBody} \, \hookleftarrow \, \mathsf{infer} \, \mathsf{tmpCtx} \, \mathsf{body} \mathsf{pure} \, \, \mathsf{tyBody}
```

```
\frac{\Gamma \vdash \mathsf{expr} : \mathit{tyExpr} \qquad \Gamma, \, \mathsf{x} : \mathit{tyExpr} \vdash \mathsf{body} : \mathit{tyBody}}{\Gamma \vdash \mathsf{let} \, \mathsf{x} = \mathsf{expr} \, \mathsf{in} \, \mathsf{body} : \mathit{tyBody}} infer ctx (Let x expr body) = do (s1, tyExpr) \leftarrow infer ctx expr let tmpCtx = Map.insert x (Scheme [] tyExpr) ctx tyBody \leftarrow infer tmpCtx body pure tyBody
```

```
\frac{\Gamma \vdash \mathsf{expr} : \mathit{tyExpr} \qquad \Gamma, \, \mathsf{x} : \mathit{tyExpr} \vdash \mathsf{body} : \mathit{tyBody}}{\Gamma \vdash \mathsf{let} \, \mathsf{x} = \mathsf{expr} \, \mathsf{in} \, \mathsf{body} : \mathit{tyBody}} infer ctx (Let x expr body) = do (s1, tyExpr) \leftarrow infer ctx expr let tmpCtx = Map.insert x (Scheme [] tyExpr) ctx (s2, tyBody) \leftarrow infer (applySubstCtx s1 tmpCtx) body pure tyBody
```

```
\frac{\Gamma \vdash \mathsf{expr} : \mathit{tyExpr} \qquad \Gamma, \, \mathsf{x} : \mathit{tyExpr} \vdash \mathsf{body} : \mathit{tyBody}}{\Gamma \vdash \mathsf{let} \, \mathsf{x} = \mathsf{expr} \, \mathsf{in} \, \mathsf{body} : \mathit{tyBody}} infer ctx (Let x expr body) = do (s1, tyExpr) \leftarrow infer ctx expr let tmpCtx = Map.insert x (Scheme [] tyExpr) ctx (s2, tyBody) \leftarrow infer (applySubstCtx s1 tmpCtx) body pure (composeSubst s1 s2, tyBody)
```

# INFERRING VAR

 $\frac{\text{var}}{\Gamma \vdash \text{var}: tyVar}$ 

infer ctx (EVar var) =

var tyVar  $\frac{\text{var}: tyVar \in \Gamma}{\Gamma \vdash \text{var}: tyVar}$ 

infer ctx (EVar var) =
 case Map.lookup var ctx of

```
\frac{\text{var}: tyVar}{\Gamma} \frac{\text{var}: tyVar}{\Gamma \vdash \text{var}: tyVar} \text{infer ctx (EVar var) =} \text{case Map.lookup var ctx of} \text{Nothing} \Rightarrow
```

throwError ("unbound variable: " <> showT var)

```
\begin{array}{c|c} & \text{var}: tyVar \in \Gamma \\ \hline \Gamma \vdash \text{var}: tyVar \end{array} infer ctx (EVar var) =  \begin{array}{c} \text{case Map.lookup var ctx of} \\ \text{Nothing} \Rightarrow \\ \text{throwError ("unbound variable: "} \Leftrightarrow \text{showT var}) \\ \text{Just scheme} \Rightarrow \text{do} \\ \text{pure (emptySubst, scheme)} \end{array}
```

```
\frac{\text{var}: tyVar}{\Gamma} \frac{\text{var}: tyVar}{\Gamma \vdash \text{var}: tyVar} \text{infer ctx (EVar var)} = \text{case Map.lookup var ctx of}
```

throwError ("unbound variable: " ⇔ showT var)

```
Just scheme → do
pure (emptySubst, scheme)
```

#### Problem

Nothing →

The context contains *schemes* not types. Getting a type from a scheme is called <u>instantiation</u>.

#### Instantiation

#### Definition

We instantiate a scheme by replacing all bound variables with fresh type variables.

# Example

instantiate 
$$(\forall a\ b.\ a \rightarrow b \rightarrow a) = u1 \rightarrow u2 \rightarrow u1$$

#### Definition

We instantiate a scheme by replacing all bound variables with fresh type variables.

```
instantiate :: Scheme → TI Type
instantiate (Scheme vars ty) = do
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```
instantiate :: Scheme → TI Type
instantiate (Scheme vars ty) = do
  newVars ← traverse (const newTyVar) vars
  let subst = Map.fromList (zip vars newVars)
  pure (applySubst subst ty)
```

### INFERRING VAR WITH INSTANTIATON

```
\begin{array}{c} \text{var} : \textit{tyVar} \in \Gamma \\ \hline \Gamma \vdash \text{var} : \textit{tyVar} \end{array} infer ctx (EVar var) = case Map.lookup var ctx of Nothing \Rightarrow throwError ("unbound variable: " \Leftrightarrow showT var) Just scheme \Rightarrow do pure (emptySubst, scheme)
```

#### INFERRING VAR WITH INSTANTIATON

```
var: tyVar \in \Gamma
             var
             tyVar
                                              \Gamma \vdash \mathsf{var} : tyVar
infer ctx (EVar var) =
  case Map.lookup var ctx of
    Nothing →
      throwError ("unbound variable: " 	♦ showT var)
    Just scheme → do
      ty ← instantiate scheme
      pure (emptySubst, ty)
```



 $\Gamma \vdash \mathsf{fun} \ \mathsf{arg} : \mathit{tyRes}$ 

 $\frac{\Gamma \vdash \mathsf{fun} : tyFun}{\Gamma \vdash \mathsf{fun} \ \mathsf{arg} : tyRes}$ 

$$\frac{\Gamma \vdash \mathsf{fun} : tyFun}{\Gamma \vdash \mathsf{fun} \ \mathsf{arg} : tyArg}$$

$$\frac{\Gamma \vdash \mathsf{fun} \ \mathsf{arg} : tyRes}{}$$

$$\frac{\Gamma \vdash \mathsf{fun} : \mathit{tyFun} \qquad \Gamma \vdash \mathsf{arg} : \mathit{tyArg}}{\Gamma \vdash \mathsf{fun} \ \mathsf{arg} : \mathit{tyRes}}$$

Relation:  $tyFun = tyArg \rightarrow tyRes$ 

$$\frac{\Gamma \vdash \mathsf{fun} : tyFun \qquad \Gamma \vdash \mathsf{arg} : tyArg}{\Gamma \vdash \mathsf{fun} \ \mathsf{arg} : tyRes}$$
$$tyFun = tyArg \rightarrow tyRes$$

infer ctx (App fun arg) = do

$$\frac{\Gamma \vdash \mathsf{fun} : tyFun \qquad \Gamma \vdash \mathsf{arg} : tyArg}{\Gamma \vdash \mathsf{fun} \ \mathsf{arg} : tyRes}$$

$$tyFun = tyArg \rightarrow tyRes$$

$$infer ctx (App fun arg) = do$$

pure tyRes

$$\frac{\Gamma \vdash \mathsf{fun} : tyFun \qquad \Gamma \vdash \mathsf{arg} : tyArg}{\Gamma \vdash \mathsf{fun} \ \mathsf{arg} : tyRes}$$

$$tyFun = tyArg \rightarrow tyRes$$

pure tyRes

$$\frac{\Gamma \vdash \mathsf{fun} : tyFun}{\Gamma \vdash \mathsf{fun} \ \mathsf{arg} : tyRes}$$

$$tyFun = tyArg \rightarrow tyRes$$

pure tyRes

$$\frac{\Gamma \vdash \mathsf{fun} : tyFun \qquad \Gamma \vdash \mathsf{arg} : tyArg}{\Gamma \vdash \mathsf{fun} \ \mathsf{arg} : tyRes}$$
 
$$tyFun = tyArg \to tyRes$$

```
infer ctx (App fun arg) = do
  tyRes ← newTyVar
  tyFun ← infer ctx fun
  tyArg ← infer ctx arg

pure tyRes
```

$$\frac{\Gamma \vdash \mathsf{fun} : \mathit{tyFun} \qquad \Gamma \vdash \mathsf{arg} : \mathit{tyArg}}{\Gamma \vdash \mathsf{fun} \ \mathsf{arg} : \mathit{tyRes}}$$

 $tyFun = tyArg \rightarrow tyRes$ 

```
infer ctx (App fun arg) = do
  tyRes ← newTyVar
  tyFun ← infer ctx fun
  tyArg ← infer ctx arg
?
  pure tyRes
```

## UNIFICATION

#### Definition

Unifying two types yields the most general substitution that when applied to both types makes them equal.

Notation

$$a \sqcup b = S$$

 $a \sqcup Int = ?$ 

 $a \sqcup Int = [a \mapsto Int]$ 

 $(a \to Int) \sqcup (Bool \to Int) = ?$ 

$$(a \rightarrow Int) \sqcup (Bool \rightarrow Int) = [a \mapsto Bool]$$

 $(Bool \rightarrow Int) \sqcup (a \rightarrow Int) = ?$ 

$$(Bool \rightarrow Int) \sqcup (a \rightarrow Int) = [a \mapsto Bool]$$

 $(a \rightarrow b) \sqcup (b \rightarrow a) = ?$ 

$$(a \rightarrow b) \sqcup (b \rightarrow a) = [a \mapsto b]$$

```
varBind :: Text \rightarrow Type \rightarrow TI Substitution varBind var ty
```

$$a \sqcup a = \lceil \rceil$$

varBind :: Text  $\rightarrow$  Type  $\rightarrow$  TI Substitution varBind var ty

$$a \sqcup a = \lceil \rceil$$

$$a \sqcup (a \rightarrow a) = \mathsf{ERROR}$$

$$a \sqcup (a \rightarrow a) = \mathsf{ERROR}$$

$$a \sqcup b = [a \mapsto b]$$

$$a \sqcup b = [a \mapsto b]$$

unify :: Type → Type → TI Substitution

```
unify :: Type \rightarrow Type \rightarrow TI Substitution unify TInt TInt = pure emptySubst
```

```
unify :: Type → Type → TI Substitution
unify TInt TInt = pure emptySubst
unify TBool TBool = pure emptySubst
```

```
unify :: Type → Type → TI Substitution
unify TInt TInt = pure emptySubst
unify TBool TBool = pure emptySubst
unify (TFun arg1 res1) (TFun arg2 res2) = do
   s1 ← unify arg1 arg2
   s2 ← unify (applySubst s1 res1) (applySubst s1 res2)
   pure (composeSubst s1 s2)
```

```
unify :: Type → Type → TI Substitution
unify TInt TInt = pure emptySubst
unify TBool TBool = pure emptySubst
unify (TFun arg1 res1) (TFun arg2 res2) = do
    s1 ← unify arg1 arg2
    s2 ← unify (applySubst s1 res1) (applySubst s1 res2)
    pure (composeSubst s1 s2)
unify (TVar u) t = varBind u t
unify t (TVar u) = varBind u t
```

```
unify :: Type → Type → TI Substitution
unify TInt TInt = pure emptySubst
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unify (TFun arg1 res1) (TFun arg2 res2) = do
  s1 ← unify arg1 arg2
  s2 ← unify (applySubst s1 res1) (applySubst s1 res2)
  pure (composeSubst s1 s2)
unify (TVar u) t = varBind u t
unify t (TVar u) = varBind u t
unify t1 t2 =
  throwError
   ("types do not unify: " 	> showT t1 	> " vs. " 	> showT t2)
```

```
\Gamma \vdash \mathsf{fun} : tyArg \rightarrow tyRes \qquad \Gamma \vdash \mathsf{arg} : tyArg
                                   \Gamma \vdash \mathsf{fun} \ \mathsf{arg} : \mathit{tyRes}
                                tyFun = tyArg \rightarrow tyRes
infer ctx (App fun arg) = do
   tyRes ← newTyVar
   tyFun ← infer ctx fun
   tyArg ← infer ctx arg
   pure tyRes
```

```
\Gamma \vdash \mathsf{fun} : tyArg \rightarrow tyRes \qquad \Gamma \vdash \mathsf{arg} : tyArg
                                  \Gamma \vdash \mathsf{fun} \ \mathsf{arg} : \mathit{tyRes}
                               tyFun = tyArg \rightarrow tyRes
infer ctx (App fun arg) = do
   tyRes ← newTyVar
   tyFun ← infer ctx fun
   tyArg ← infer ctx arg
   unify tyFun (TFun tyArg tyRes)
   pure tyRes
```

$$\frac{\Gamma \vdash \mathsf{fun} : \mathit{tyArg} \to \mathit{tyRes}}{\Gamma \vdash \mathsf{fun} \ \mathsf{arg} : \mathit{tyRes}}$$

$$tyFun = \mathit{tyArg} \to \mathit{tyRes}$$

infer ctx (App fun arg) = do
 tyRes ← newTyVar
 (s1, tyFun) ← infer ctx fun
 tyArg ← infer ctx arg
 unify tyFun (TFun tyArg tyRes)
 pure tyRes

```
\Gamma \vdash \mathsf{fun} : tyArg \rightarrow tyRes \qquad \Gamma \vdash \mathsf{arg} : tyArg
                                \Gamma \vdash \mathsf{fun} \ \mathsf{arg} : \mathit{tyRes}
                             tyFun = tyArg \rightarrow tyRes
infer ctx (App fun arg) = do
   tyRes ← newTyVar
   (s1, tyFun) ← infer ctx fun
   (s2, tyArg) ← infer (applySubstCtx s1 ctx) arg
   unify tyFun (TFun tyArg tyRes)
   pure tyRes
```

```
\Gamma \vdash \mathsf{fun} : tyArg \rightarrow tyRes \qquad \Gamma \vdash \mathsf{arg} : tyArg
                               \Gamma \vdash \mathsf{fun} \ \mathsf{arg} : \mathit{tyRes}
                             tyFun = tyArg \rightarrow tyRes
infer ctx (App fun arg) = do
  tyRes ← newTyVar
  (s1, tyFun) ← infer ctx fun
  (s2, tyArg) ← infer (applySubstCtx s1 ctx) arg
  s3 ← unify (applySubst s2 tyFun) (TFun tyArg tyRes)
  pure tyRes
```

```
\Gamma \vdash \mathsf{fun} : tyArg \to tyRes \qquad \Gamma \vdash \mathsf{arg} : tyArg
                            \Gamma \vdash \text{fun arg} : tyRes
                         tyFun = tyArg \rightarrow tyRes
infer ctx (App fun arg) = do
  tyRes ← newTyVar
  (s1, tyFun) ← infer ctx fun
  (s2, tyArg) ← infer (applySubstCtx s1 ctx) arg
  s3 ← unify (applySubst s2 tyFun) (TFun tyArg tyRes)
  let subst = composeSubst s3 (composeSubst s2 s1)
  pure tyRes
```

```
\Gamma \vdash \mathsf{fun} : tyArg \to tyRes \qquad \Gamma \vdash \mathsf{arg} : tyArg
                           \Gamma \vdash \text{fun arg} : tyRes
                         tyFun = tyArg \rightarrow tyRes
infer ctx (App fun arg) = do
  tyRes ← newTyVar
  (s1, tyFun) ← infer ctx fun
  (s2, tyArg) ← infer (applySubstCtx s1 ctx) arg
  s3 ← unify (applySubst s2 tyFun) (TFun tyArg tyRes)
  let subst = composeSubst s3 (composeSubst s2 s1)
  pure (subst, applySubst s3 tyRes)
```

#### CONCLUSION

We've implemented a fully functional type inferencer for a functional programming language in around 120 lines of Haskell.

This was just an introduction, but after this talk you should have enough context to pick up a paper on type systems on your own and have a go at implementing it.

# Demo?

# Demo?

All materials can be found at github.com/kRITZCREEK/fby19