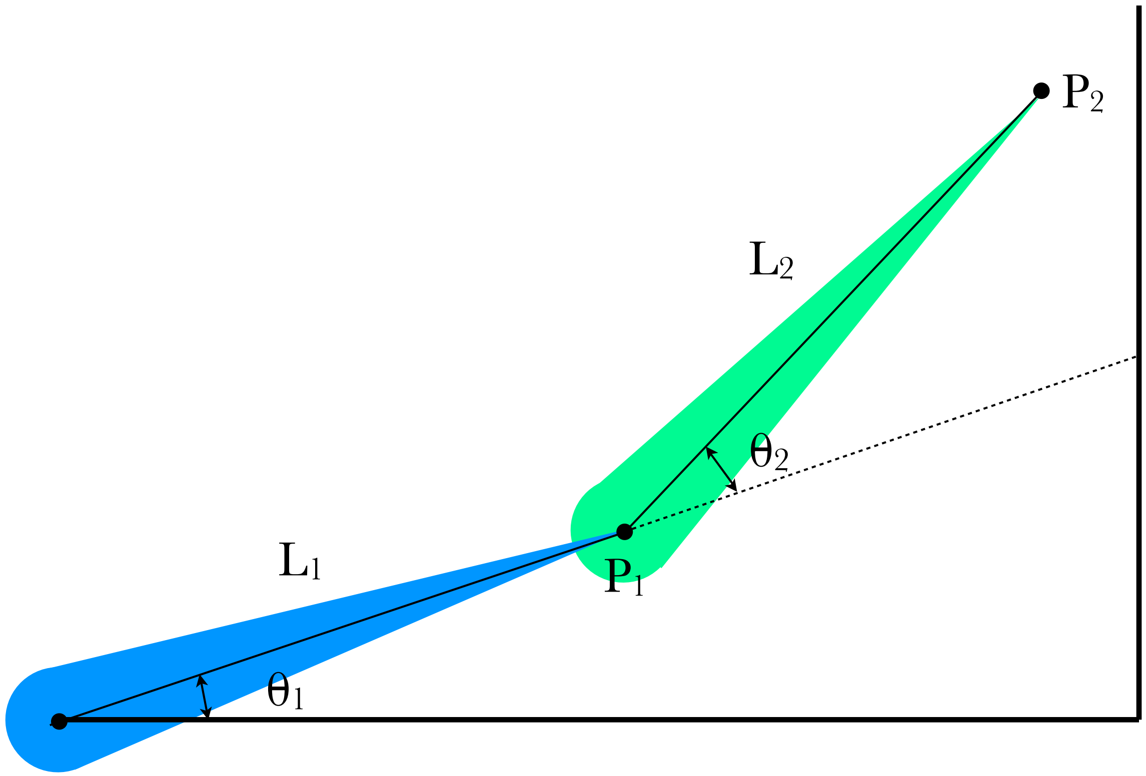
Game Architecture

4/4/14: Inverse Kinematics

Closed Form / Analytical Solution



$$Px_1 = L_1 * \cos(\theta_1)$$

$$Px_2 = (L_1 * \cos(\theta_1)) + (L_2 * \cos(\theta_1 + \theta_2))$$

$$Py_1 = L_1 * \sin(\theta_1)$$

$$Py_2 = (L_1 * \sin(\theta_1)) + (L_2 * \sin(\theta_1 + \theta_2))$$

$$Px_2 = Px_1 + (L_2 * \cos(\theta_1 + \theta_2))$$

$$Py_2 = Py_1 + (L_2 * \sin(\theta_1 + \theta_2))$$

Using:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$$

we get:

$$Px_2 = L_1 \cos\theta_1 + L_2 \cos\theta_1 \cos\theta_2 - L_2 \sin\theta_1 \sin\theta_2$$

$$Py_2 = L_1 \sin\theta_1 + L_2 \cos\theta_1 \sin\theta_2 + L_2 \sin\theta_1 \cos\theta_2$$

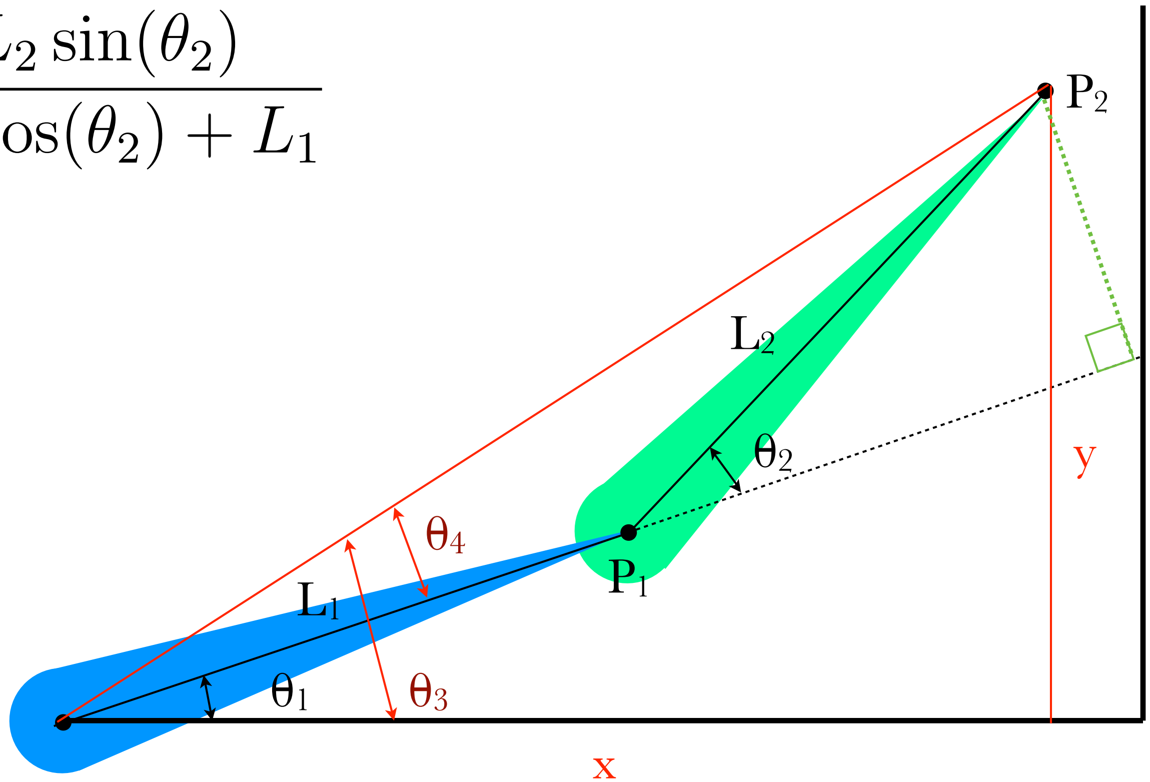
after rearranging this a whole bunch to get rid of the θ_1 term:

$$\theta_2 = \arccos\left(\frac{Px_2^2 + Py_2^2 - L_1^2 - L_2^2}{2L_1L_2}\right)$$

$$\tan(\theta_3) = y/x$$

$$\tan(\theta_4) = \frac{L_2 \sin(\theta_2)}{L_2 \cos(\theta_2) + L_1}$$

$$\theta_1 = \theta_3 - \theta_4$$



Using:

$$\tan(a-b) = \frac{\tan(a) - \tan(b)}{1 + \tan(a)\tan(b)}$$

and doing some substitution:

$$\tan(\theta_1) = \frac{\frac{y}{x} - \frac{L_2\sin(\theta_2)}{L_2\cos(\theta_2)+L_1}}{1 + \left(\frac{y}{x} * \frac{L_2\sin(\theta)}{L_2\cos(\theta_2)+L_1}\right)}$$

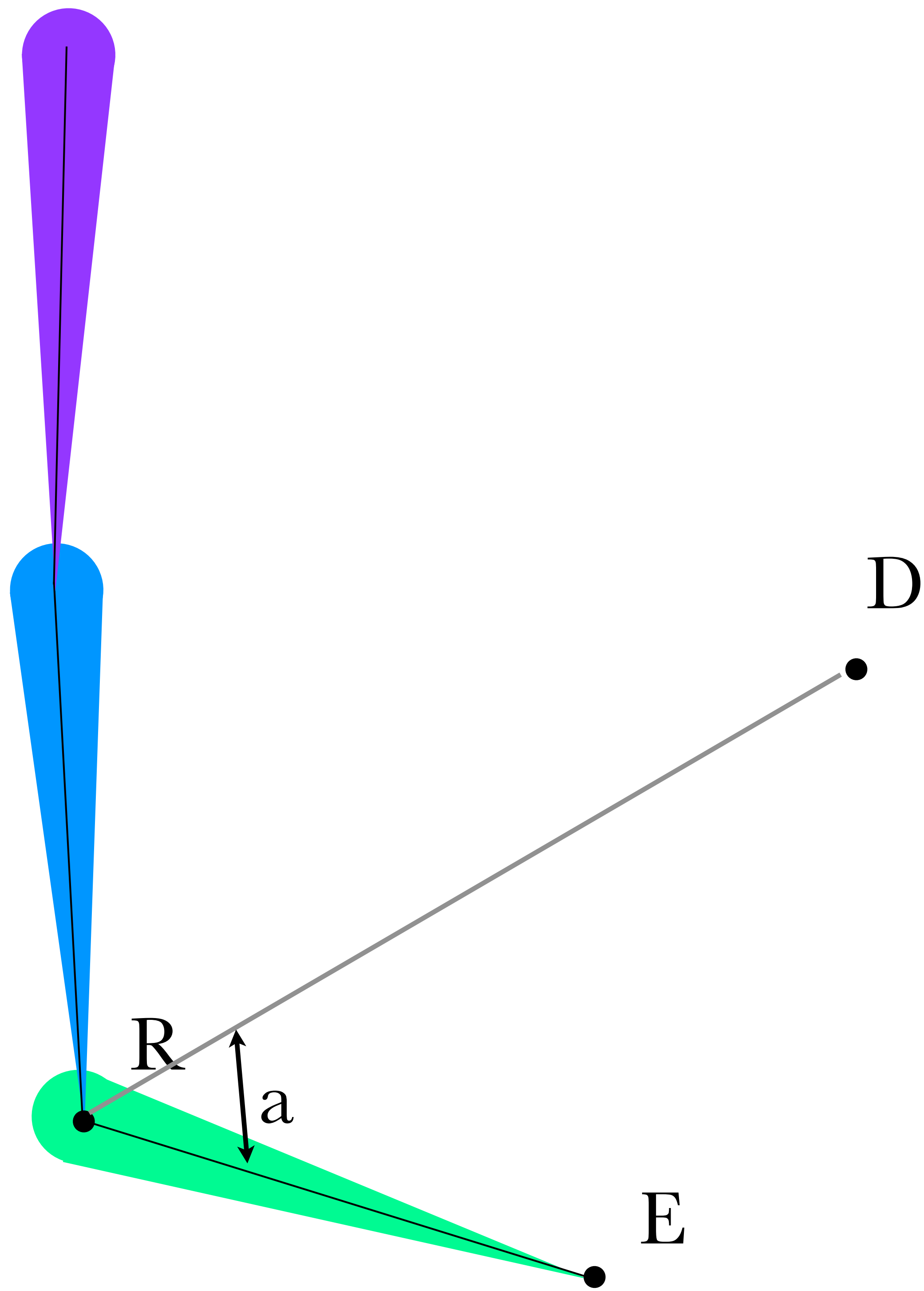
simplifying:

$$\theta_1 = \arctan\left(\frac{y(L_2\cos(\theta_2) + L_1) - x(L_2\sin(\theta_2))}{x(L_2\cos(\theta_2) + L_1) + y(L_2\sin(\theta_2))}\right)$$

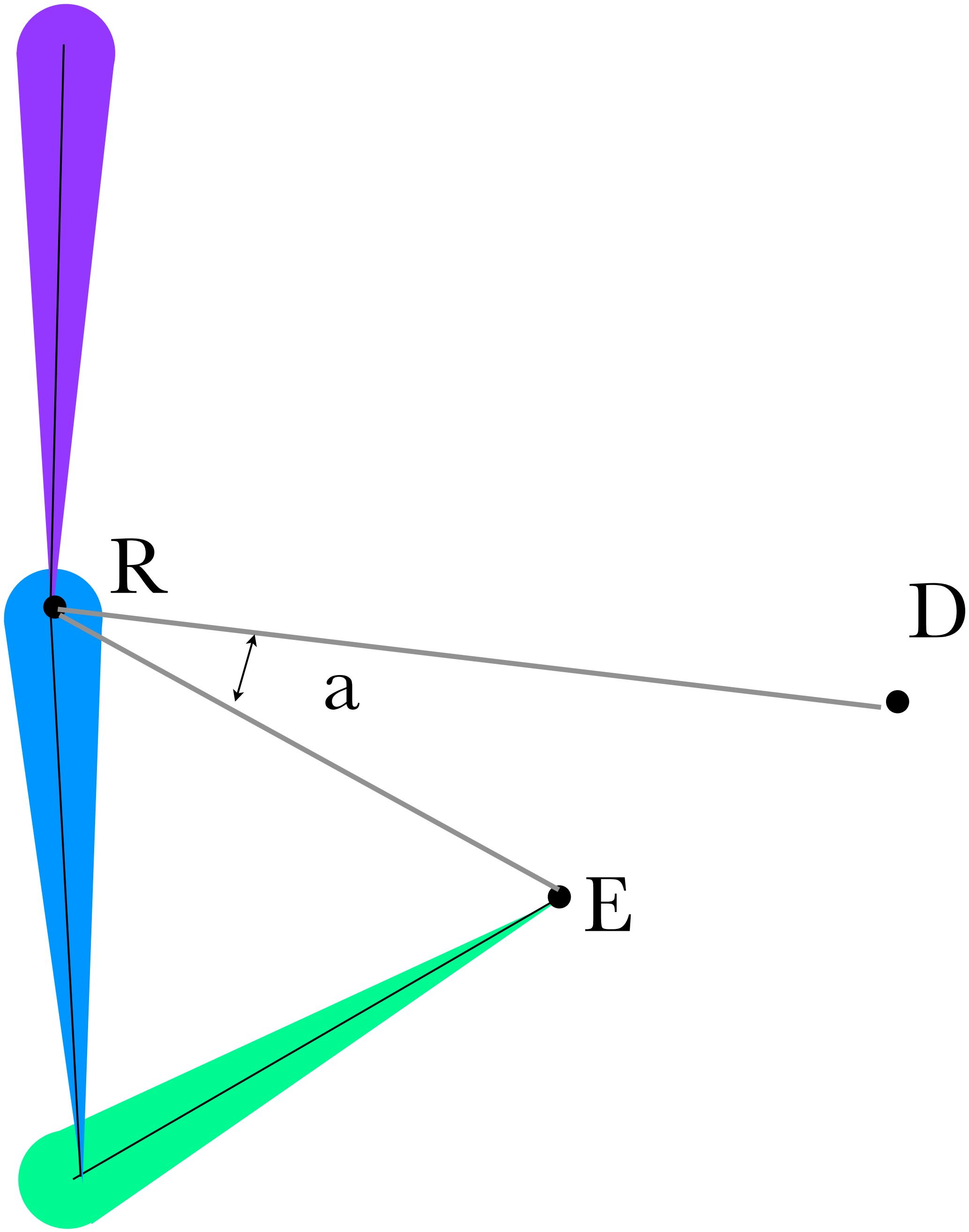
Problems

- Gets complicated as n increases
- Humans have more DOFs than needed to reach goal
- Sets of nonlinear equations often have no closed form
    - Iterate numerically

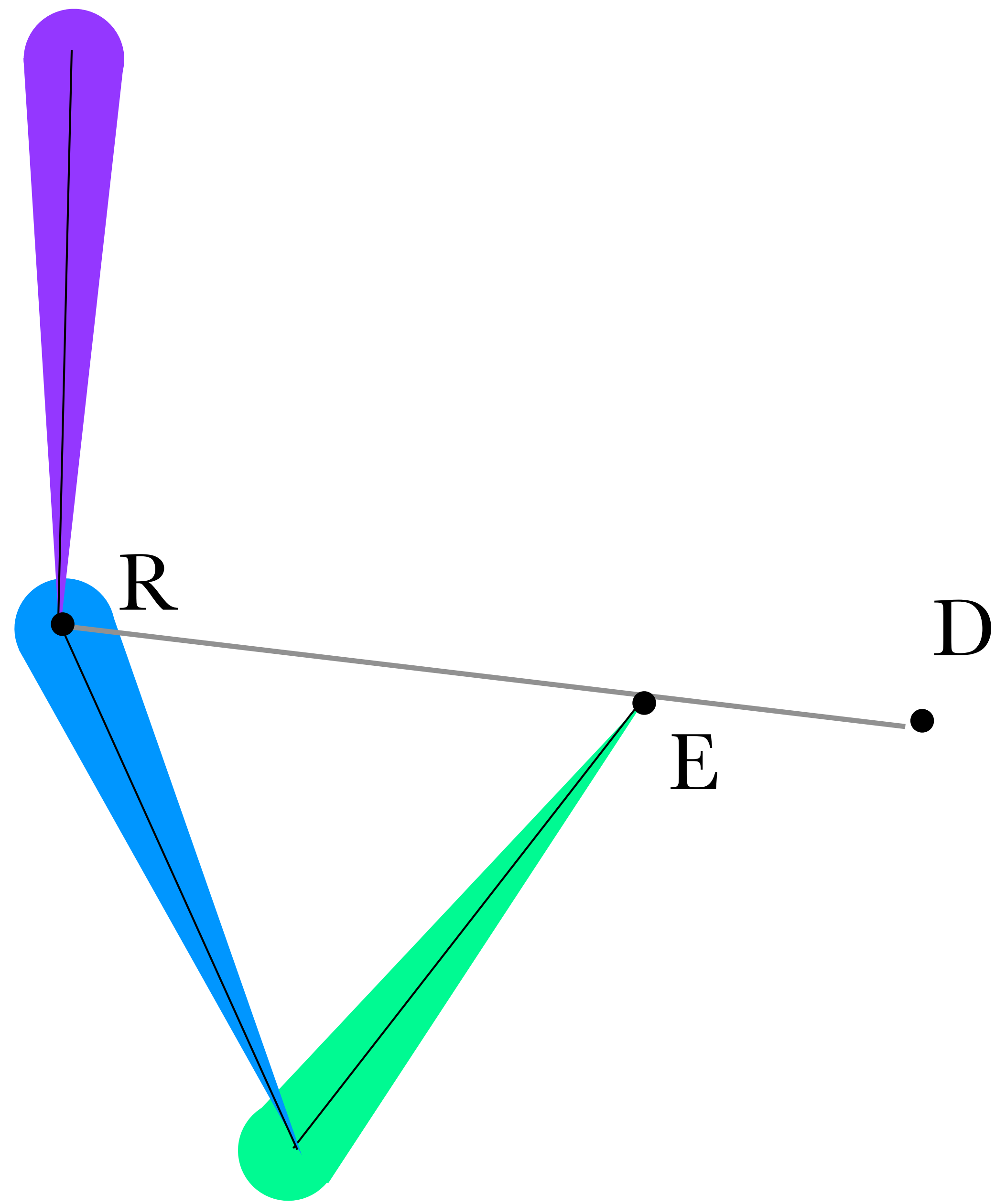
Cyclic-Coordinate Descent



$$cos(a) = \overrightarrow{RD} \cdot \overrightarrow{RE}$$



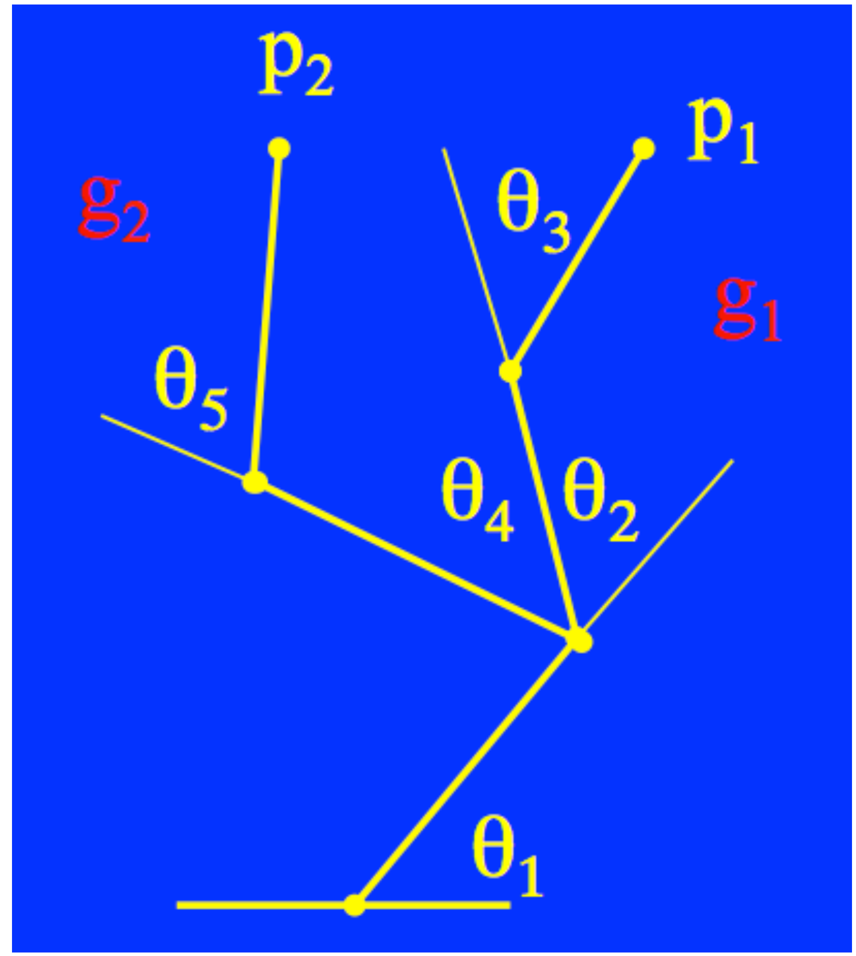
$$cos(a) = \overrightarrow{RD} \cdot \overrightarrow{RE}$$



$$cos(a) = \overrightarrow{RD} \cdot \overrightarrow{RE}$$

Problems

- Only handles serial chains
- Multiple goals needed for humans
- How to distribute desired angle behavior?
  - Can just average, but won't satisfy/prioritize all goals



Jacobian Inverse

we try to solve:

$$y_1 = f(x_0 + \Delta x)$$

where y_1 is a desired end effector position, x_0 is a current parameter vector, and Δx is the unknown we're trying to find that modifies x_0 to reach y_1

Taylor expansion:

$$y_1 = f(x_0) + \frac{\partial f}{\partial x}(x_o)\Delta x + O\left\|\Delta x\right\|^2$$

since f(x) is a vector function, we now have a matrix of first-order partial derivatives, often denoted J

$$y_1 = f(x_0) + J(x_o)\Delta x$$

$$\Delta x = J(x_0)^{-1}(y_1 - y_0)$$

Jacobian Entries

- The entries in the Jacobian matrix are usually very easy to calculate.
- If the j^{th} joint is a rotational joint with a single degree of freedom, the joint angle is a single scalar θ_j . Let $\mathbf{p}_j$ be the position of the joint and let $\mathbf{v}_j$ be a unit vector pointing along the current axis of rotation for the joint
- In this case, if angles are measured in radians with the direction of rotation given by the right-hand rule and if the i^{th} end effector is affected by the joint, then the corresponding entry in the Jacobian is:

$$\frac{\partial \mathbf{s}_i}{\partial \theta_j} = \mathbf{v}_j \times (\mathbf{s}_i - \mathbf{p}_j)$$

Broyden–Fletcher–Goldfarb–Shanno (BFGS) method

search direction is found by solving:

$$p_k^N = -\nabla^2 f_k^{-1} \nabla f_k$$

where $\nabla^2 f^k$ is the Hessian matrix and ∇f_k is the gradient of the objective function, given by

$$\nabla f_k = J(e - g)$$

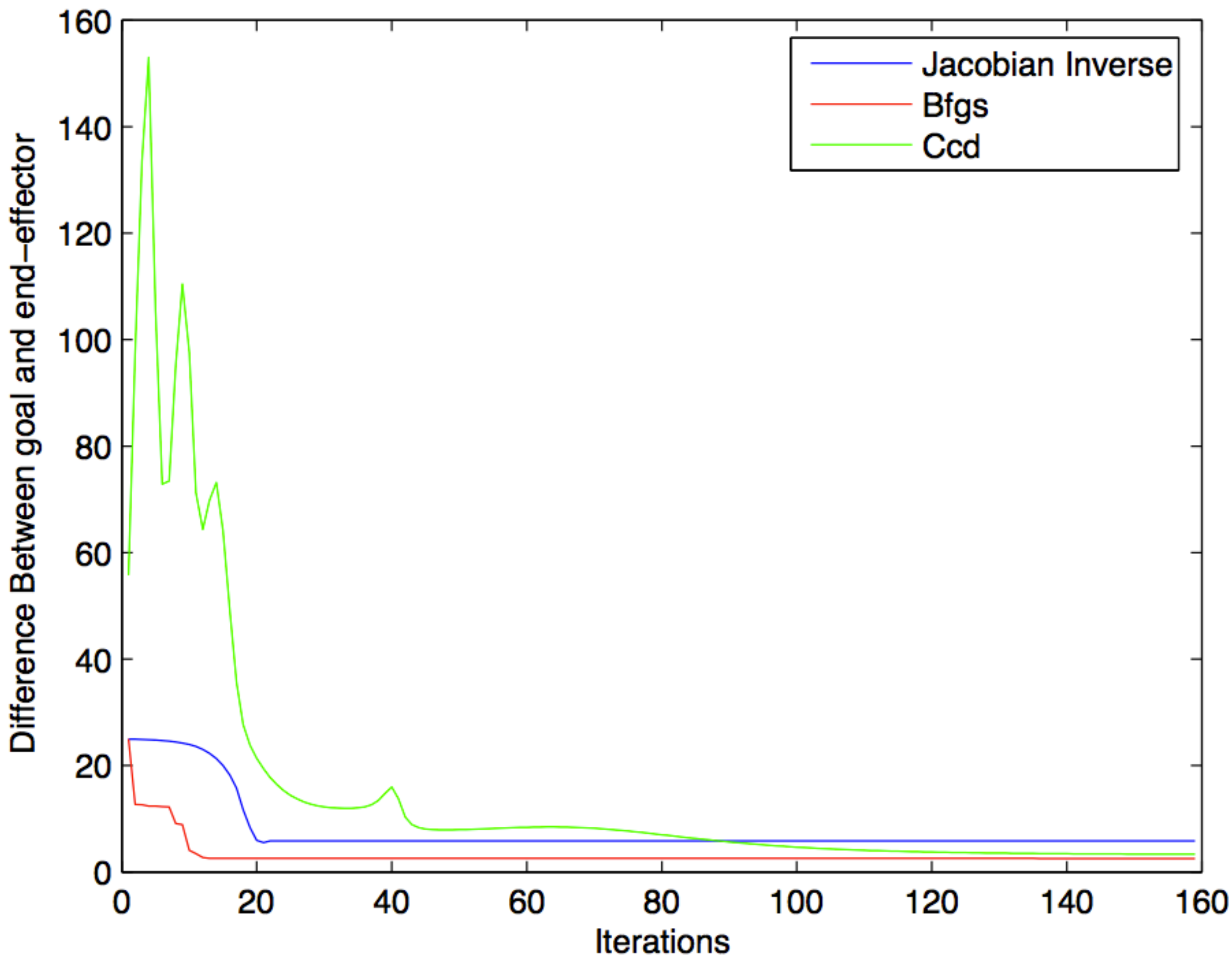
where e is the end-effector position and g is the goal position.

$$B_{i+1} = B_i - \frac{B_i s_i s_i^T B_i}{s_i^T B_i s_i} + \frac{g_i g_i^T}{g_i^T s_i}$$

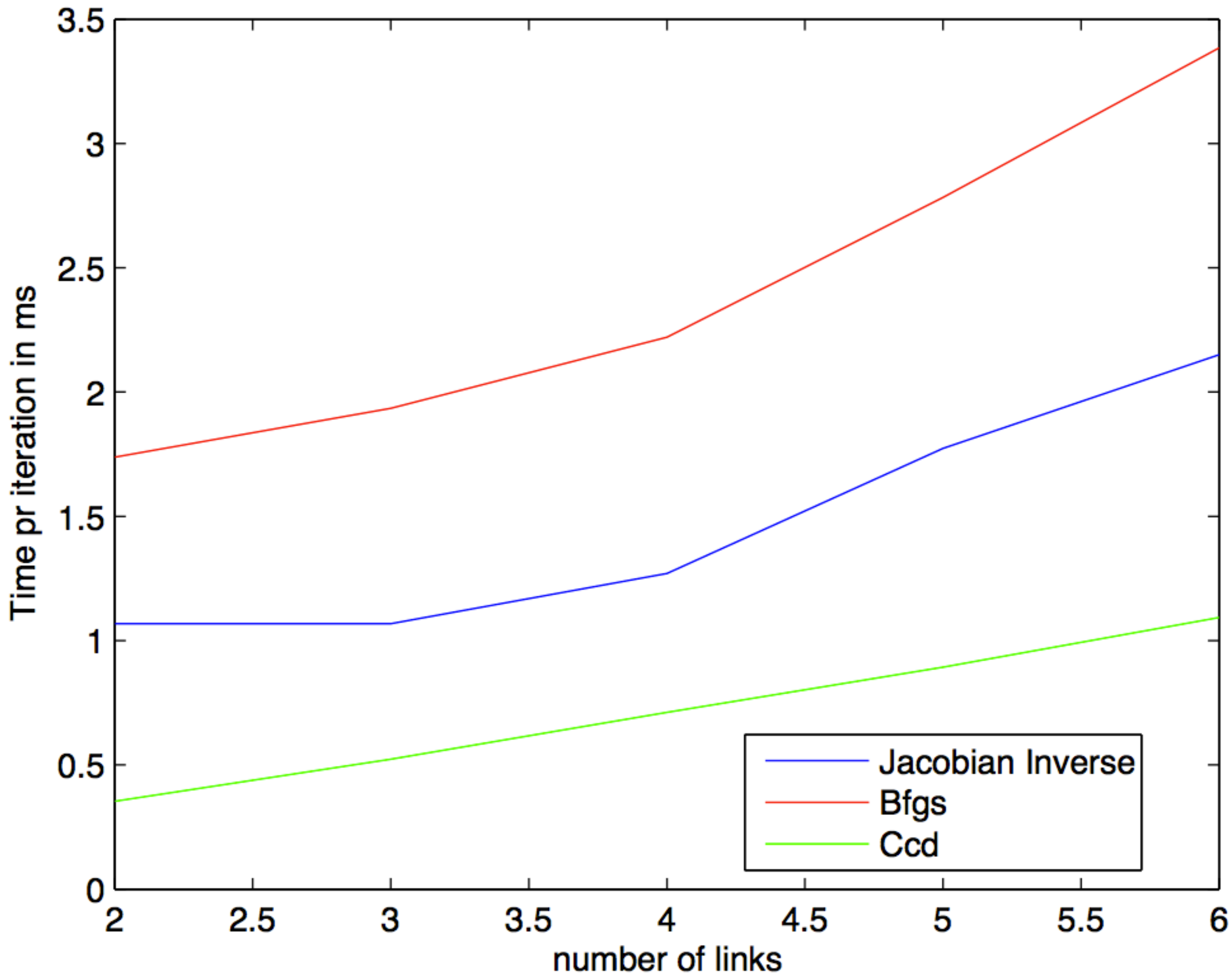
$$s_i = x_{i+1} - x_i$$

$$g = \nabla f(x_{i+1}) - \nabla f(x_i)$$

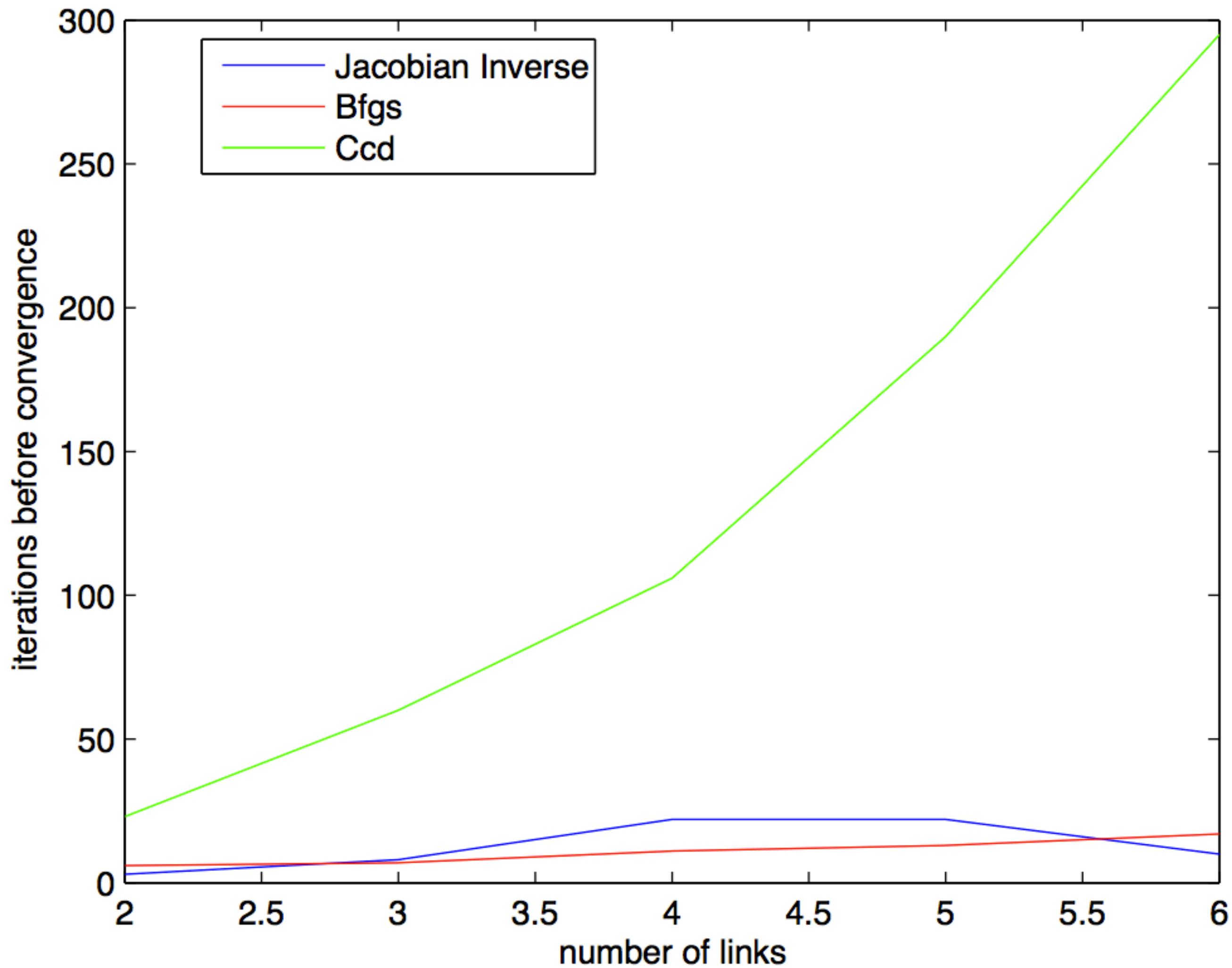
Convergence



Milliseconds per Iteration for Increasing Number of Joints



Number of Iterations as a Function of Chain Length



Precision

Method	Distance to Goal
JI	5.853667
CCD	3.368891
BFGS	2.534875
Optimal	2

Qualitatively…

Method	Convergence	Speed per Iteration	Number of Iterations	Precision
CCD	Meh	Pretty good	Awful	Not Bad
JI	Good	Right in the middle	Good	Pretty Bad
BFGS	Awesome	Not great	Good	Good