Bounded First-Class Universe Levels in Type Theory

Jonathan Chan
University of Pennsylvania, Philadelphia, USA

Stephanie Weirich
University of Pennsylvania, Philadelphia, USA

Abstract

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Supplementary Material *Software (source code)*: https://github.com/ionathanch/TT-model
archived at swh:1:dir:

Acknowledgements hi types.pl!

1 Introduction

1.1 Comparison to other work

2 A basic type theory with bounded first-class universe levels

3 Metatheory

3.1 Type safety

3.2 Consistency and canonicity

3.3 Attempts at proving normalization

4 Conclusion and future work

4.1 Extensions

[2, 1]

References

1 Marc Bezem, Thierry Coquand, Peter Dybjer, and Martín Escardó. Type Theory with Explicit Universe Polymorphism. In Delia Kesner and Pierre-Marie Pédrot, editors, *28th*

$$
\begin{aligned}
i, j &::= \texttt{<external universe levels>} \\
x, y, z &::= \texttt{<term variables>} \\
a, b, c, A, B, C, k, \ell &::= x \mid i \mid \Pi x : A.\, B \mid \lambda x.\, b \mid b\ a \mid \bot \mid \mathsf{absurd}\ b \mid \mathsf{U}\ k \mid \mathsf{Level}{<}\ \ell \\
\Gamma, \Delta &::= \cdot \mid \Gamma, x : A
\end{aligned}
$$

Figure 1 Syntax

$$\textsc{Var}\ \frac{\vdash \Gamma \qquad x : A \in \Gamma}{\Gamma \vdash x : A} \qquad \textsc{Pi}\ \frac{\Gamma \vdash A : \mathsf{U}\, k \qquad \Gamma, x : A \vdash B : \mathsf{U}\, k}{\Gamma \vdash \Pi x : A.\, B : \mathsf{U}\, k}$$

$$\textsc{Lam}\ \frac{\Gamma \vdash \Pi x : A.\, B : \mathsf{U}\, k \qquad \Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda x.\, b : \Pi x : A.\, B} \qquad \textsc{App}\ \frac{\Gamma \vdash b : \Pi x : A.\, B \qquad \Gamma \vdash a : A}{\Gamma \vdash b\ a : B[x \mapsto a]} \qquad \textsc{Mty}\ \frac{\Gamma \vdash \mathsf{U}\, k : \mathsf{U}\, \ell}{\Gamma \vdash \bot : \mathsf{U}\, k}$$

$$\textsc{Abs}\ \frac{\Gamma \vdash A : \mathsf{U}\, k \qquad \Gamma \vdash b : \bot}{\Gamma \vdash \mathsf{absurd}\ b : A} \qquad \textsc{Conv}\ \frac{\Gamma \vdash a : A \qquad \Gamma \vdash B : \mathsf{U}\, k \qquad A = B}{\Gamma \vdash a : B}$$

$$\textsc{E-Beta}\ \frac{}{(\lambda x.\, b)\ a = b[x \mapsto a]} \qquad \textsc{E-Refl}\ \frac{}{a = a} \qquad \textsc{E-Sym}\ \frac{a = b}{b = a} \qquad \textsc{E-Trans}\ \frac{a = b \qquad b = c}{a = c}$$

Figure 2 Typing and selected equality rules (no universes or levels)

$$\textsc{Univ}\ \frac{\Gamma \vdash k : \mathsf{Level}{<}\ \ell}{\Gamma \vdash \mathsf{U}\, k : \mathsf{U}\, \ell} \qquad \textsc{Level}{<}\ \frac{\Gamma \vdash \mathsf{U}\, k_1 : \mathsf{U}\, \ell_1 \qquad \Gamma \vdash k_0 : \mathsf{Level}{<}\ \ell_0}{\Gamma \vdash \mathsf{Level}{<}\ k_0 : \mathsf{U}\, k_1} \qquad \textsc{Lvl}\ \frac{\vdash \Gamma \qquad i < j}{\Gamma \vdash i : \mathsf{Level}{<}\ j}$$

$$\textsc{Trans}\ \frac{\Gamma \vdash k_1 : \mathsf{Level}{<}\ k_2 \qquad \Gamma \vdash k_2 : \mathsf{Level}{<}\ k_3}{\Gamma \vdash k_1 : \mathsf{Level}{<}\ k_3} \qquad \textsc{Sub}\ \frac{\Gamma \vdash A : \mathsf{U}\, k \qquad \Gamma \vdash k : \mathsf{Level}{<}\ \ell}{\Gamma \vdash A : \mathsf{U}\, \ell}$$

Figure 3 Typing rules (universes and levels)

$$\textsc{I-Univ}\ \frac{j < i}{[\![\mathsf{U}\ j]\!]_i \searrow \{z \mid \exists P.\, [\![z]\!]_j \searrow P\}}$$

$$\textsc{I-Pi}\ \frac{[\![A]\!]_i \searrow P_1 \qquad \forall y.\, y \in P_1 \to \exists P_2.\, R(y, P_2) \qquad \forall y.\, \forall P_2.\, R(y, P_2) \to [\![B[x \mapsto y]]\!]_i \searrow P_2}{[\![\Pi x : A.\, B]\!]_i \searrow \{f \mid \forall y.\, \forall P_2.\, R(y, P_2) \to y \in P_1 \to f\ y \in P_2\}}$$

$$\textsc{I-Pi'}\ \frac{[\![A]\!]_i \searrow P_1 \qquad \forall y.\, y \in P_1 \to \exists P_2.\, [\![B[x \mapsto y]]\!]_i \searrow P_2}{[\![\Pi x : A.\, B]\!]_i \searrow \{f \mid \forall y.\, \forall P_2.\, ([\![B[x \mapsto y]]\!]_i \searrow P_2) \to y \in P_1 \to f\ y \in P_2\}} \qquad \textsc{I-Mty}\ \frac{}{[\![\bot]\!]_i \searrow \varnothing}$$

$$\textsc{I-Level}{<}\ \frac{}{[\![\mathsf{Level}{<}\ j_1]\!]_i \searrow \{z \mid \exists j_2.\, z \Rightarrow^* j_2 \wedge j_1 < j_2\}} \qquad \textsc{I-Step}\ \frac{A \Rightarrow B \qquad [\![B]\!]_i \searrow P}{[\![A]\!]_i \searrow P}$$

Figure 4 Logical relation for closed types

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