The Mazur Swindle

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Introduction

The Mazur Swindle is a surprising proof of the fact that the connected sum of two non-trivial knots is always non-trivial. In this paper, I will outline proof and the minutiae of how it works. In Part 1, I will state the argument of the Swindle. In Part 2, I will investigate the definition of a knot, as well as the definition of an infinite sum of knots. In Part 3, I will explore the fundamental definition of knot equality, and discuss why we cannot rely on Reidemeister moves. Then in Part 4 I will explain why the swindle is a valid argument, based on these definitions.

Part I

In class, we used the concept of genus to prove the fact that non-trivial knots do not have inverses. The Mazur swindle proves the same fact without appealing to genus. Here is the statement of the argument:

> Suppose K and L are knots, and $K\#L = \bigcirc$ (the unknot). Then let X be the infinite sum of knots $K\#L\#K\#L$...
>
> We can group the terms of this sum in two ways:
> $X = (K\#L)\#(K\#L)... = \bigcirc\# \bigcirc \# \bigcirc ... = \bigcirc$
> $X = K\#(L\#K)\#(L\#K)... = K\# \bigcirc \# \bigcirc ... = K$
>
> Therefore, $K \cong \bigcirc$. By symmetry, $L \cong \bigcirc$ as well.

This argument seems extremely unintuitive at first, since the corresponding statement for real numbers would lead to a proof that $1 = 0$. However, the Mazur swindle is actually completely mathematically valid. In order to show why, we have to examine the basic definitions at the root of knot theory.

Part II: What is a knot?

A knot is an embedding of the space S^1 (the unit circle) into $\mathbb{R}^3$. An embedding is simply an injective continuous map that yelds a homeomorphism between the domain and its image in the codomain space. One can prove easily that in fact any injective continuous map from S^1 to $\mathbb{R}^3$ is an embedding, and hence generates a valid knot (Kobayashi, 2020).

This definition allows for knots to have infinitely many crossings. This means that the knots that we have dealt with in our class are really only a subset of knots; The "tame knots." These are the knots that can be represented by a diagram with finitely many crossings. In fact, Livingston's textbook goes as far as to take this as the definition of a knot in order to avoid the trouble of wild knots: He defines a knot as a *polygonal* closed curve. That is, a curve consisting of finitely many straight line segments. Then any such curve must necessarily have only finitely many crossings, since each segment intersects each other segment at most once, so even if every segment somehow crossed every other segment, there would only be a finite number of crossings in total.

The polygonal perspective on knots is extremely useful, as it allows one to prove things like Reidemeister's theorem; Kobayashi explains that, "The usual proof of Reidemeister's theorem relies on using polygonalizations to reduce the problem of ambient isotopy to the elementary moves" (Kobayashi, 2021). However, despite its convenience, the polygonal definition does not allow knots with infinite crossings to exist, so we cannot use it for our proof.

We also need to choose a definition for the infinite sum of knots. In class, we defined it using spheres that intersect the knot in 2 places. In Fox's paper outlining the swindle, he instead thinks of it as tying multiple knots into the same piece of string. We didn't discuss this view in class, but it is equivalent, since any knot can be thought of as a series of crossings tied into a large loop. Then summing these loops together essentially ties the different knots onto the same string. Figure 1 depicts this process.

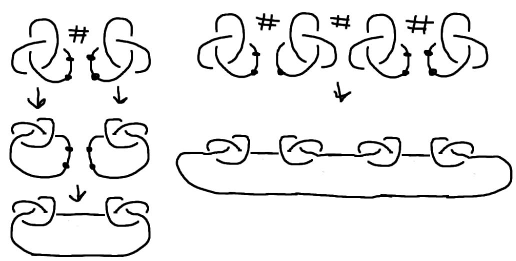

Figure 1: knot summation can be thought of as tying multiple knots into the same string.

Then an infinite sum of knots is formed by placing infinitely many knots end to end, each shrunk down to half the size of the last. Specifically, each knot lives in a box in $\mathbb{R}^3$, with the string starting in one corner and ending in the opposite corner. Essentially, we are creating a new embedding piecewise, by defining the function to follow the path of one of the summands inside each rectangle.

The resulting function is continuous by the pasting lemma and clearly injective since each knot is in a disjoint box, separate from all others. Then this is a valid embedding, hence a valid knot. The result is depicted in figure 2.

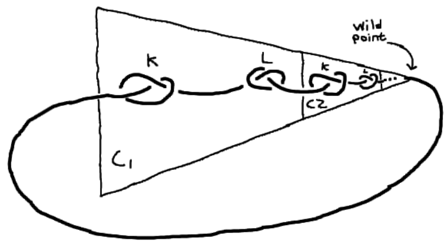

Figure 2: example of an infinite sum of knots. (traced from Fox's *a quick trip*)

Part III: What is knot equality?

In our class, the only definition of knot equality that we worked with was that of Reidemeister moves. We said that two knots are equivalent if and only if there exists a finite sequence of Reidemeister moves that transforms one knot into another. However, this is not the fundamental definition of knot equality. The more basic definition, that can be used for any knot, even those with infinitely many crossings, is that two knots are said to be equal if there is an ambient isotopy of $\mathbb{R}^3$ that transforms one knot into the other. Here, an isotopy is similar to the idea of a homotopy except that the intermediate steps are not just continuous functions, but in fact homeomorphisms, which have to be bijective, with a continuous inverse. The reason why we require an *ambient* isotopy (which deforms the space around the knot) rather than simply an isotopy of the curves themselves is because of a trick called "bachelor's unknotting," where you shrink all of the crossings of a knot into an infinitely small point. This is a valid isotopy between any knot and the unknot, and means that all knots are isotopic to each other (Kobayashi, 2020).

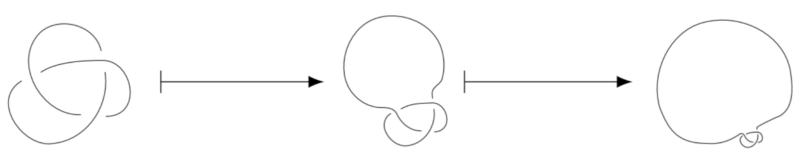

Figure 3: Bachelor's unknotting (from Kobayashi, 2020)

Ambient isotopies do not allow this. In fact, they partition the knots into equivalence classes in the desired way. Therefore, it was chosen as the fudamental definition of knot equality. Reidemeister's theorem builds off this definition by showing that if an ambient isotopy exists between two polygonal/tame knots, then there is a sequence of Reidemeister moves that transform one into the other, and vice versa.

A natural question is whether Reidemeister's theorem extends to wild knots. Unfortunately, the answer seems to be "sometimes". Kobayashi shows in his 2021 paper an example of a wild knot that seems to be equivalent to the unknot through Reidemeister moves, but isn't. It is depicted in figure 4.

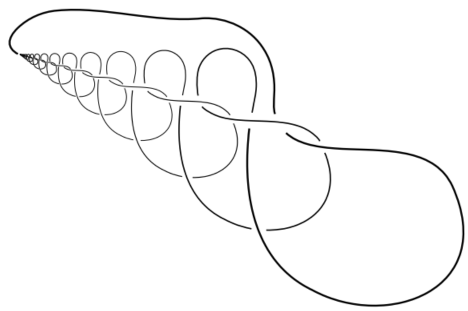

Figure 4: This curve seems to be unknotted by Reidemeister moves, but it is actually wild. (from Kobayashi, 2021)

You can use Reidemeister moves to push the loops through each other, untying any finite number of loops. It may seem that you could construct an ambient isotopy to perform an infinite number of these moves to untie the whole knot. One might attempt to define an ambient isotopy as a limit of ambient isotopies f_n, each of which unties the first n loops. But in fact, the resulting map in the limit will *not* be an ambient isotopy . The reason is shown in Figure 5.

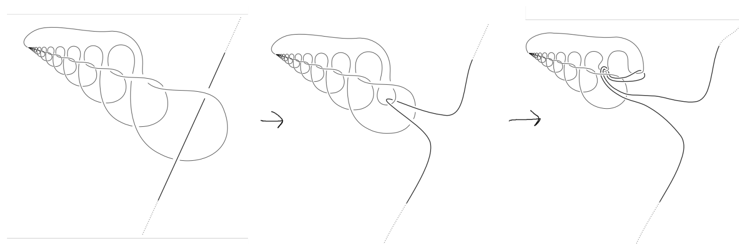

Figure 5: Line of points in the ambient space as two loops are removed

If you consider a line of points from the ambient space, then the Reidemeister moves essentially drag parts of this line towards the wild point. In the limit, an infinite subset of points will be mapped to the wild point. Hence, the resulting map is not bijective, so we do not have an ambient isotopy from this method. In fact, we know that there is *no* ambient isotopy that can tame the knot in figure 4; the knot was proven to be wild by Fox and Artin in 1949, using a fact about the neighborhoods of the wild point.

However, there are some other knots where the existence of a countable sequence of Reidemeister moves *does* give you an ambient isotopy. This is a simple example from Kobayashi's 2021 paper:

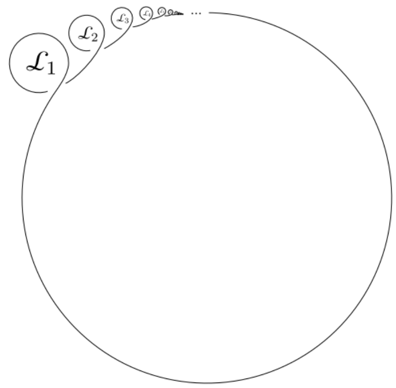

Figure 6: Unknot with infinite crossings

In this case, the limit of the sequence of ambient isotopies given by R1 moves is itself a valid ambient isotopy, and the knot is in fact equal to the unknot.

So to recap, Reidemeister moves don't generalize nicely to wild knots. Sometimes they imply the existence of an isotopy, but other times they do not. Unfortunately, we cannot rely on Reidemeister moves as a definition of equality for our proof. We need to directly use the ambient isotopy definition of equivalence.

Part IV: Putting the pieces together

Now that we have defined our terms, we are ready to explain the swindle. The supposition that $K\#L = \bigcirc$ tells us that in fact there is an ambient isotopy that deforms the rectangle C in figure 5, transforming it into the unknot in this region while keeping the ends fixed.

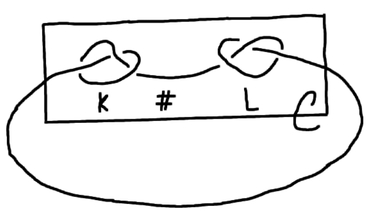

Figure 7: An ambient isotopy exists that unties the knot only in this box.

We proved in class that # is commutative so we know there is also an ambient isotopy that unties $L\#K$. Then given the knot $X = K\#L\#K...$, we can create a new ambient isotopy which is the sum of countably many ambient isotopies f_n, each of which unties only the nth instance of $K\#L$ and acts as the identity on the rest of $\mathbb{R}^3$. The result is a valid ambient isotopy, which essentially unties each rectangle of the sum all at once. This shows that the infinite sum knot is actually equal to the unknot. Next, we can construct another ambient isotopy that unties each instance of $L\#K$ in parallel, showing that the infinite sum is actually equal to the first summand, K. Therefore, we conclude that K is equal to the unknot. This process is depicted in Figure 8.

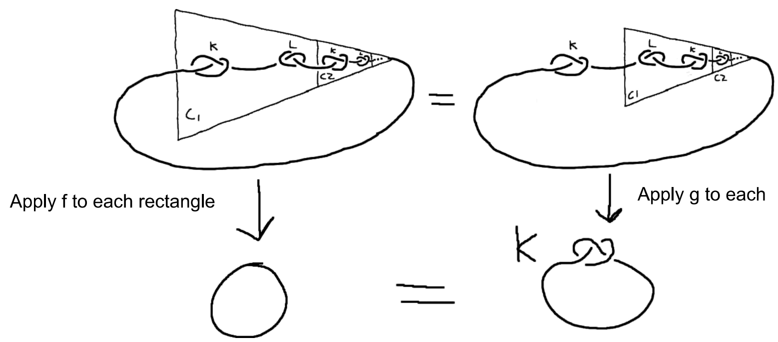

Figure 8: visual explanation of the swindle.

By symmetry, L is also equal to the unknot, since we could have started with the sequence $L\#K\#L...$ and applied the same argument.

Now we can fully understand why the Mazur swindle works. It relies on facts about combining ambient isotopies to form a new ones, as well as the fact that every infinite sum of knots is a valid knot.

The reason that the real version $1-1+1...$ does not work is because for real numbers, every reasonable definition of the infinite sum has to leave some sums undefined/divergent, including those of the form $x-x+x....$

Works Cited

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3. R. Fox, A quick trip through knot theory, in *Topology of 3-Manifolds*, M. K Fort (editor) Prentice-Hall, 1962

4. Livingston, C. (1993). WHAT IS A KNOT? In Knot Theory (pp. 11-28). Mathematical Association of America. doi:10.5948/UPO9781614440239.003

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Reflection

My main takeaway from this project is that we, as mathematicians, cannot rely on our intuitions. The Mazur Swindle seems like a naive mistake at first. Intuitively, it feels impossible that this method is actually valid. It's only once you fully grasp the fundamental definitions of knot theory that you believe that it actually works. This same phenomenon comes up everywhere in mathematics. For example, consider the Weierstrauss function, which is continuous everywhere but differentiable nowhere, and the Peano Curve, which is a 1-D curve that fills 2-D space. Both seemed to violate every basic intuition about curves. However, they are undeniably real once you look at the definitions. I am also reminded of the video, "How to lie using visual proofs" (https://www.youtube.com/watch?v=VYQVlVoWoPY) by 3blue1brown, which talks about how visual proofs can take advantage of your fallible intuitions to mislead you. In math, intuition is extremely important for leading us to different paths of study, but we should always be prepared to be surprised and confused by what we find. The Mazur swindle is a perfect example of this.