Unit 5

1 Nash Games

Consider a system in which there are multiple decision makers (or players). Assume that the choices of one influence the outcome of the others. In this general case standard optimization is not enough to model the system. But we can resort to Nash games.

Basic assumptions:

(i) the decision makers do not cooperate (**no cooperation**)

(ii) the decision makers must make their choices at the same time (**simultaneity**)

(iii) any decision maker knows the model of the others (**complete information**)

(iv) the decision makers are sensible (**rationality**).

The Nash equilibrium is a solution concept for the system under these assumptions. It can be described by the following definition. If each decision maker has chosen a strategy and no one can benefit by changing its own strategy while the others keep theirs unchanged, then the current set of strategy choices constitutes a Nash equilibrium.

Remarks:

- A set of strategy choices that is not a Nash equilibrium is not optimal for at least one decision maker. Therefore it is not a stable set of strategy choices, because such unsatisfied decision maker will move from it as soon as possible. Vice versa, a Nash equilibrium is stable.

- Under the complete information assumption, any decision maker can (independently) compute the Nash equilibria of the game. If there is a unique Nash equilibrium, or if there is a Nash equilibrium that is, in some sense, the most preferable one for all the decision makers, then, by the rationality assumption, it is the best possible set of strategy choices for all the decision makers.

1.1 Bimatrix games

A bimatrix game is a simultaneous game for two players in which each player has a finite number of possible actions. The name comes from the fact that the normal form of such a game can be described by two matrices: matrix A describing the payoffs of player 1 and matrix B describing the payoffs of player 2.

Player 1 is often called the "row player" and player 2 the "column player". If player 1 has m possible actions and player 2 has n possible actions, then each of the two matrices has m rows by n columns. When the row player selects the ith action and the column player selects the jth action, the payoff to the row player is A_{ij} and the payoff to the column player is B_{ij}.

The players can also play mixed strategies. A mixed strategy for the row player is a nonnegative vector x of length m such that: $\sum_{i=1}^m x_i = 1$. Similarly, a mixed strategy for the column player is a nonnegative vector y of length n such that: $\sum_{j=1}^n y_j = 1$. When the players play mixed strategies with vectors x and y, the expected payoff of the row player is $x^T Ay$, and of the column player is $x^T By$. The Nash equilibrium problem is the following

$$\begin{array}{llcll}
\underset{x}{\text{minimize}} & -\frac{1}{2}x^T Ay & \qquad & \underset{y}{\text{minimize}} & -\frac{1}{2}x^T By \\
\text{subject to} & \sum_{i=1}^m x_i = 1 & & \text{subject to} & \sum_{j=1}^n y_j = 1 \\
& x \geq 0 & & & y \geq 0.
\end{array}$$

This problem has at least one solution, that is, every bimatrix game has a Nash equilibrium in mixed strategies. A solution of the Nash game can be computed by solving together the KKT conditions of the two players' problems.

$$\begin{aligned}
& -(Ay)_i + \mu_1 \geq 0, \quad i = 1, \ldots, m \\
& \sum_{i=1}^m x_i = 1 \\
& x \geq 0 \\
& (-(Ay)_i + \mu_1)\, x_i = 0, \quad i = 1, \ldots, m \\
& -(B^T x)_j + \mu_2 \geq 0, \quad j = 1, \ldots, n \\
& \sum_{j=1}^n y_j = 1 \\
& y \geq 0 \\
& (-(B^T x)_j + \mu_2)\, y_j = 0, \quad j = 1, \ldots, n.
\end{aligned}$$

Remarks:

- These KKT conditions are obtained from the general ones described in unit 4.
- Observe that the problem of the first player must be derived with respect to x only, while that of the second player to y only.

As in unit 4, the KKT conditions of the Nash game can be equivalently rewritten by using some binary variables $\delta^1 \in \{0, 1\}^m$ and $\delta^2 \in \{0, 1\}^n$:

$$-(Ay)_i + \mu_1 \geq 0, \quad i = 1, \ldots, m$$

$$
\begin{aligned}
&\sum_{i=1}^{m} x_i = 1 \\
&x \geq 0 \\
&(-(Ay)_i + \mu_1) \leq M\delta_i^1, \quad x_i \leq M(1-\delta_i^1), \quad i = 1, \ldots, m \\
&-(B^T x)_j + \mu_2 \geq 0, \quad j = 1, \ldots, n \\
&\sum_{j=1}^{n} y_j = 1 \\
&y \geq 0 \\
&(-(B^T x)_j + \mu_2) \leq M\delta_j^2, \quad y_j \leq M(1-\delta_j^2), \quad j = 1, \ldots, n,
\end{aligned}
$$

where $M > 0$ is a huge number.

1.2 Examples from game theory

Prisoner's dilemma The prisoner's dilemma is a standard example of a game analyzed in game theory that shows why two completely rational individuals might not cooperate, even if it appears that it is in their best interests to do so.

Two members of a criminal gang, Tom and Harry, are arrested and imprisoned. Each prisoner is in solitary confinement with no means of communicating with the other. The prosecutors lack sufficient evidence to convict the pair on the principal charge, but they have enough to convict both on a lesser charge. Simultaneously, the prosecutors offer each prisoner a bargain. Each prisoner is given the opportunity either to betray the other by testifying that the other committed the crime, or to cooperate with the other by remaining silent. The offer is:

- If Tom and Harry each betray the other, each of them serves two years in prison
- If Tom betrays Harry but Harry remains silent, Tom will be set free and Harry will serve three years in prison (and vice versa)
- If Tom and Harry both remain silent, both of them will only serve one year in prison (on the lesser charge).

	Harry stays silent	Harry betrays
Tom stays silent	T: -1, H: -1	T: -3, H: 0
Tom betrays	T: 0, H: -3	T: -2, H: -2

$$
A = \begin{pmatrix} -1 & -3 \\ 0 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 \\ -3 & -2 \end{pmatrix}.
$$

It is implied that the prisoners will have no opportunity to reward or punish their partner other than the prison sentences they get and that their decision will not affect their reputation in the future. Because betraying a partner offers a greater reward than cooperating with them, all purely rational self-interested prisoners will betray the other, meaning the only possible outcome for two purely rational prisoners is for them to betray each other. The interesting part of this result is that pursuing individual reward logically leads both of the prisoners to betray when they would get a better reward if they both kept silent. In reality, humans display a systemic bias towards cooperative behavior in this and similar games despite what is predicted by simple models of "rational" self-interested action.

This model written in GNU MathProg is reported in the file `nash1.mod`:

```
param m;
param n;
param A{1..m,1..n};
param B{1..m,1..n};
param M;

var x1{1..m} >= 0;
var x2{1..n} >= 0;
var mu1;
var mu2;
var delta1{1..m} binary;
var delta2{1..n} binary;


s.t. grad_obj1 {i in 1..m}:
  sum{j in 1..n} - A[i,j]*x2[j] + mu1 >= 0;

s.t. equality1:
  sum{i in 1..m} x1[i] = 1;

s.t. complementarity1_1 {i in 1..m}:
  sum{j in 1..n} - A[i,j]*x2[j] + mu1 <= M*delta1[i];

s.t. complementarity1_2 {i in 1..m}:
  x1[i] <= M*(1-delta1[i]);


s.t. grad_obj2 {i in 1..n}:
  sum{j in 1..m} - B[j,i]*x1[j] + mu2 >= 0;

s.t. equality2:
  sum{i in 1..n} x2[i] = 1;
```

```
s.t. complementarity2_1 {i in 1..n}:
  sum{j in 1..m} - B[j,i]*x1[j] + mu2 <= M*delta2[i];

s.t. complementarity2_2 {i in 1..n}:
  x2[i] <= M*(1-delta2[i]);

end;
```

And in the file `prisoners.dat`:

```
param m := 2;
param n := 2;

param A: 1 2 :=
1 -1 -3
2 0 -2;

param B: 1 2 :=
1 -1 0
2 -3 -2;

param M := 1000;

end;
```

Rock-paper-scissors Rock-paper-scissors is a hand game usually played between two people, in which each player simultaneously forms one of three shapes with an outstretched hand. These shapes are "rock" (a closed fist), "paper" (a flat hand), and "scissors" (a fist with the index finger and middle finger extended, forming a V). A player who decides to play rock will beat another player who has chosen scissors, but will lose to one who has played paper; a play of paper will lose to a play of scissors. If both players choose the same shape, the game is tied and is usually immediately replayed to break the tie.

	Harry plays rock	Harry plays paper	Harry plays scissors
Tom plays rock	T: 0, H: 0	T: -1, H: 1	T: 1, H: -1
Tom plays paper	T: 1, H: -1	T: 0, H: 0	T: -1, H: 1
Tom plays scissors	T: -1, H: 1	T: 1, H: -1	T: 0, H: 0

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Rockpaperscissors is often used as a fair choosing method between two people, similar to coin flipping or throwing dice, in order to settle a dispute or make an unbiased group decision. Unlike truly random selection methods, however, rockpaperscissors can be played with a degree of skill by recognizing and exploiting non-random behavior in opponents.

This model written in GNU MathProg is reported in the file `nash1.mod`, which is reported above, and in the file `rockpaperscissors.dat`:

```
param m := 3;
param n := 3;

param A: 1 2 3 :=
1 0 -1 1
2 1 0 -1
3 -1 1 0;

param B: 1 2 3 :=
1 0 1 -1
2 -1 0 1
3 1 -1 0;

param M := 1000;

end;
```