Lecture 5

Cezar Ionescu

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Administrative

- Homework from 08/06/2019 due now!
- Please complete and hand in the declarations of authorship.

Questions?

Solution to homework from lecture 2

Apply Bayes' theorem to answer the following question (Elmer Mode 1966, page 53):

> *A class in advanced mathematics contains 10 juniors, 30 seniors, and 10 graduate students. Three of the juniors, 10 of the seniors, and 5 of the graduate students received an A in the course. If a student is chosen at random from this class and is found to have earned an A, what is the probability that he is a graduate student?*

Solution to homework from lecture 2

$\Omega = \{\text{student}_1, \ldots, \text{student}_{50}\}$, $\text{Event} = \mathbb{P}(\Omega)$

$p(X) = \text{card}(X)/\text{card}(\Omega)$

Events of interest:

$G = \{x \mid x \in \Omega, x \text{ is a graduate student}\}$

$A = \{x \mid x \in \Omega, x \text{ has earned an } A\}$

Solution to homework from lecture 2

We want `p(G | A)`. By Bayes' theorem

```
p(G | A) = p(A | G) * p(G) / p(A)
```

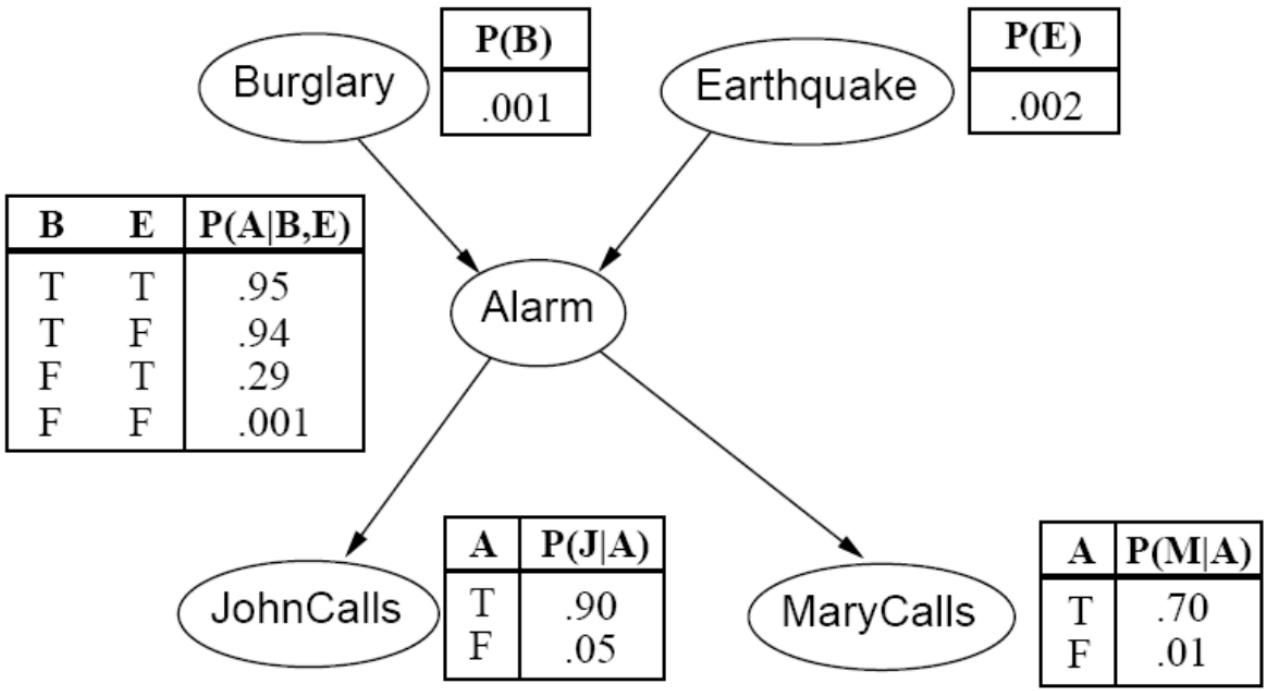
```
p(A) = card(A) / card(Ω) = 18/50
p(G) = card(G) / card(Ω) = 10/50
p(A | G) = p(A ∩ G) / p(G) = card(A ∩ G) / card(G) = 5/10
```

Therefore

```
p(G | A) = 5/18 = 0.278
```

Solution to homework from lecture 3

(based on Nilsson, 19.4, page 340) Consider the following Bayesian network:



Compute `p(¬J, ¬M, A, B, E)`. This is the probability that there is both an earthquake and a burglary, the alarm rings, but neither John nor Mary call.

Solution to homework from lecture 3

```
   p(¬J, ¬M, A, B, E)
=
   p(¬J | ¬M ∩ A ∩ B ∩ E) * p(¬M | A ∩ B ∩ E) *
   p(A | B ∩ E) * p(B | E) * p(E)
=
   p(¬J | A) * p(¬M | A) * p(A | B ∩ E) * p(B) * p(E)
=
   0.1 * 0.3 * 0.95 * 0.001 * 0.002
=  5.7 * 10^(-8)
```

McCulloch & Pitts

- McCulloch & Pitts 1943 *A Logical Calculus of the Ideas Immanent in Nervous Activity*:
  "neural events and the relations among them can be treated by means of propositional logic"

- The MC & P neuron had a number of boolean inputs, some positive and others negative. The neuron was activated if sufficiently many inputs were activated (as specified by a "threshold" parameter), provided that *no* inhibiting connections were activated:

```
mc_p_neuron : ℕ -> ({0, 1}ⁿ, {0, 1}ᵐ) -> {0, 1}
mc_p_neuron θ (pos, neg) = if sum neg > 0 then 0
                           else if sum pos ≥ θ then 1
                                else 0
```

Examples

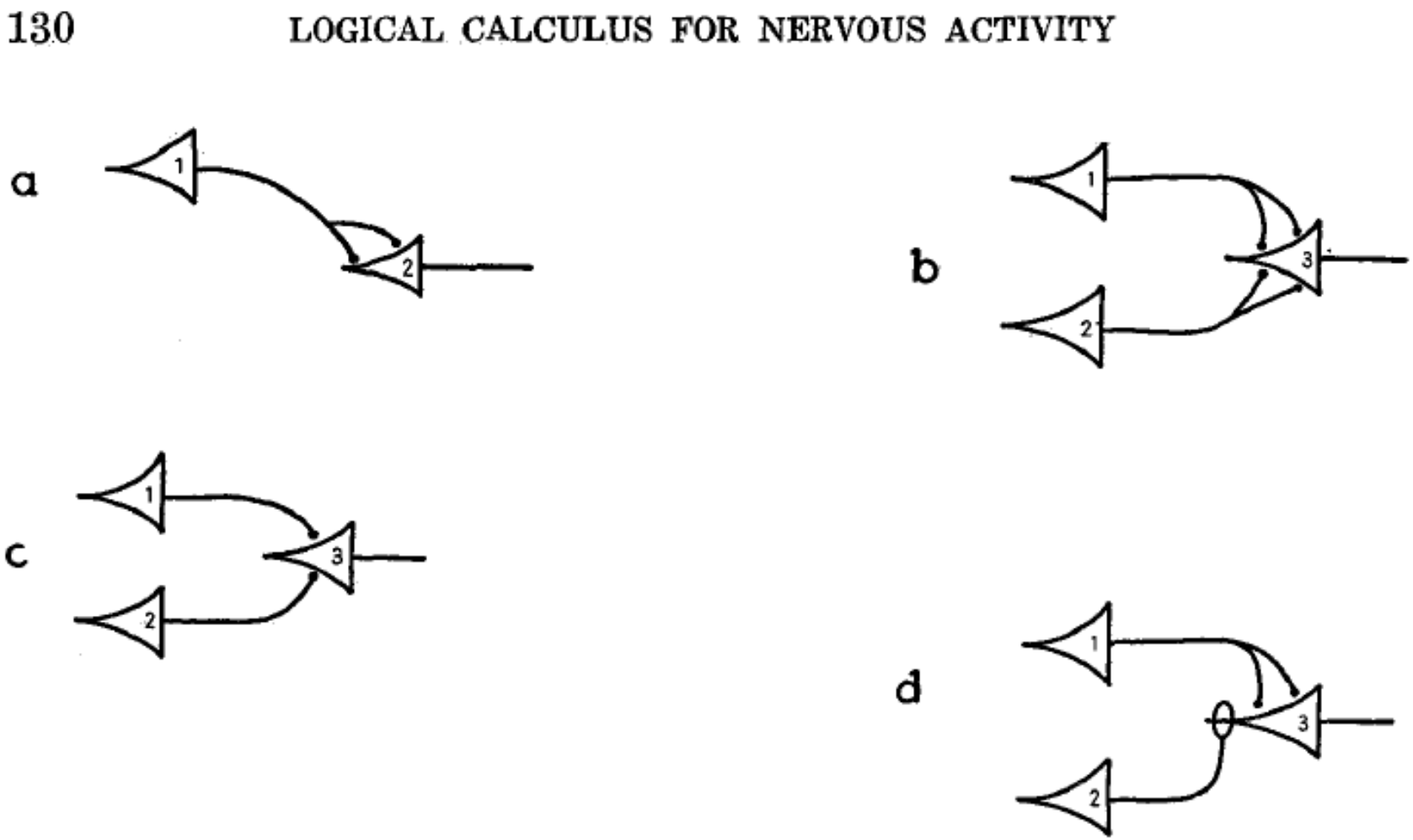


Figure 1: McCulloch & Pitts neuronal networks

Logic with McCulloch & Pitts neurons

Logical functions:

```
not : {0, 1} -> {0, 1}
not x = mc_p_neuron 0 ([], [x])

and : ({0, 1}, {0, 1}) -> {0, 1}
and (x, y) = mc_p_neuron 2 ([x, y],[])
```

“Exclusive or” with McCulloch & Pitts neurons

```
xor : ({0, 1}, {0, 1}) -> {0, 1}
xor (x, y) = mc_p_neuron 2 ([case₁, case₁, case₂, case₂], [])
 where
 case₁ = mc_p_neuron 2 ([x, x], [y])
 case₂ = mc_p_neuron 2 ([y, y], [x])
```

“Exclusive or” with McCulloch & Pitts neurons

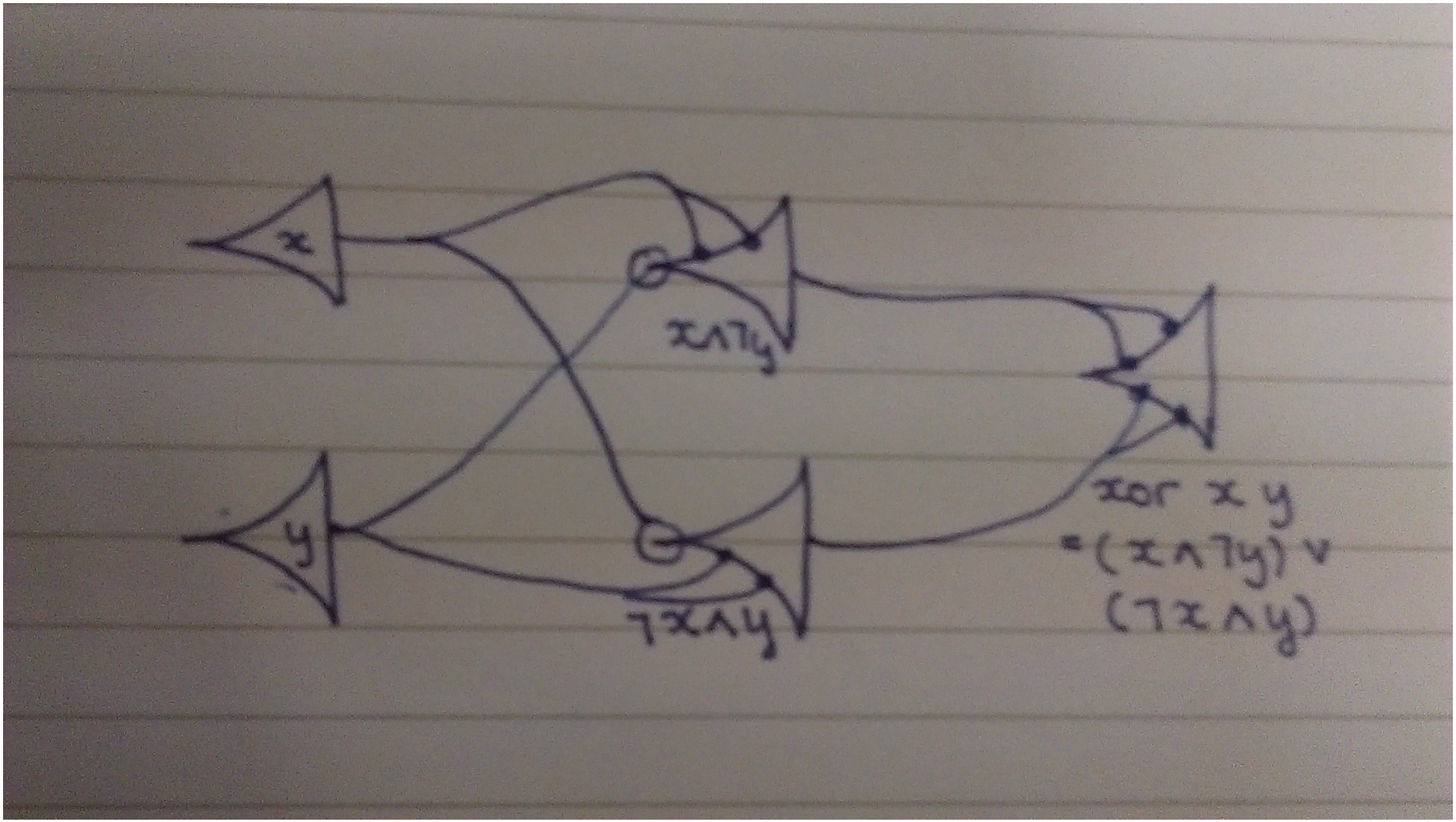


Figure 2: Logical “xor” implemented with McCulloch & Pitts neurons

Perceptrons

- Frank Rosenblatt 1957
- A perceptron had real-valued inputs and binary outputs. The output was a step function of the weighted sum of these inputs

```
perceptron : (ℝⁿ, ℝ) -> ℝⁿ -> {-1, 1}
perceptron ([w₁,..., wₙ], θ) [x₁, ..., xₙ] = if s ≥ θ then 1
                                              else -1
 where s = w₁*x₁ + ... + wₙ * xₙ
```

Perceptrons

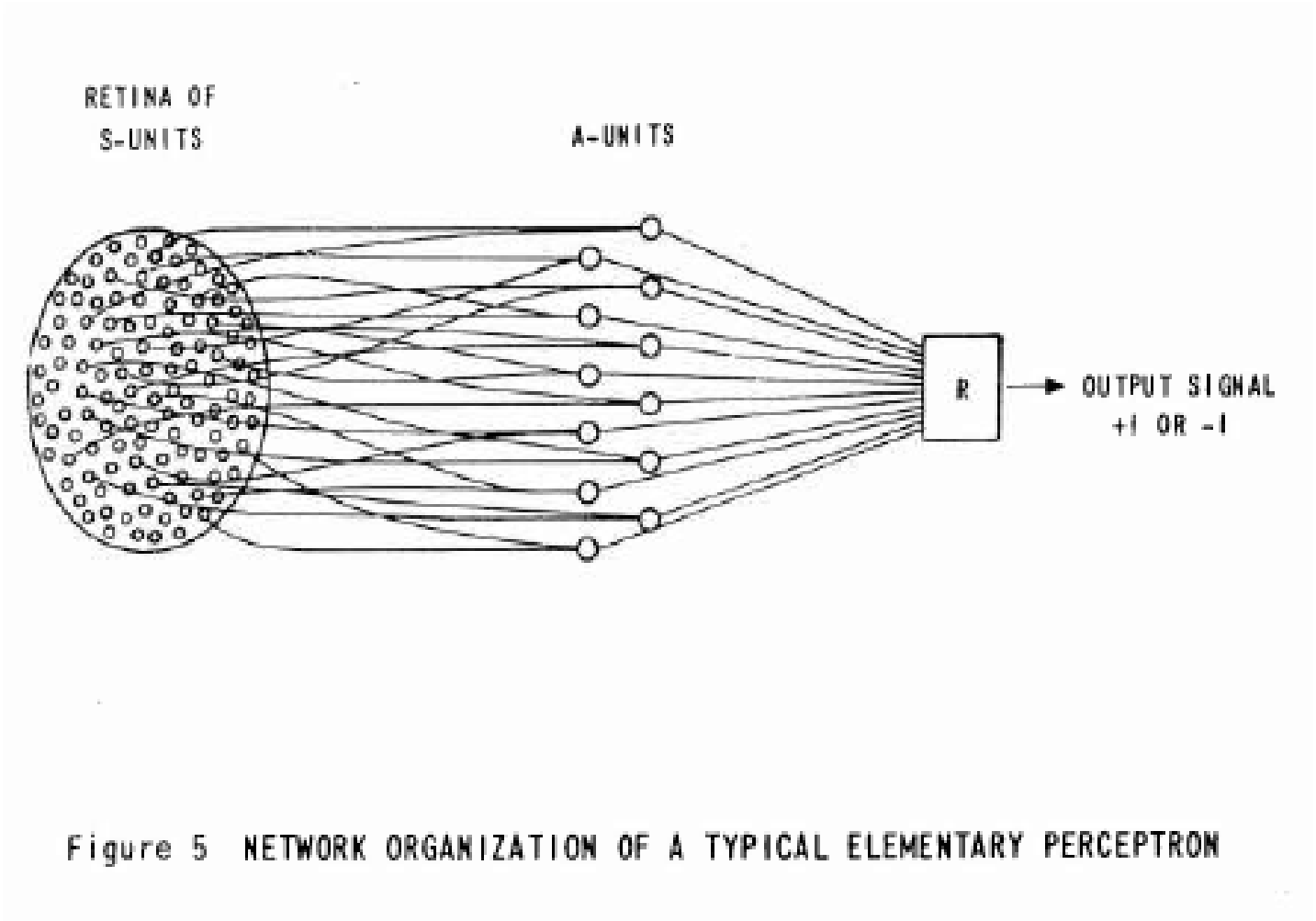


Figure 5 NETWORK ORGANIZATION OF A TYPICAL ELEMENTARY PERCEPTRON

Figure 3: Rosenblatt's perceptron (1961)

Logic with perceptrons

Logical functions:

```
not x       = perceptron ([-1], 0) [x]

and (x₁, x₂) = perceptron ([0.5, 0.5], 1) [x₁, x₂]
```

Logic with perceptrons

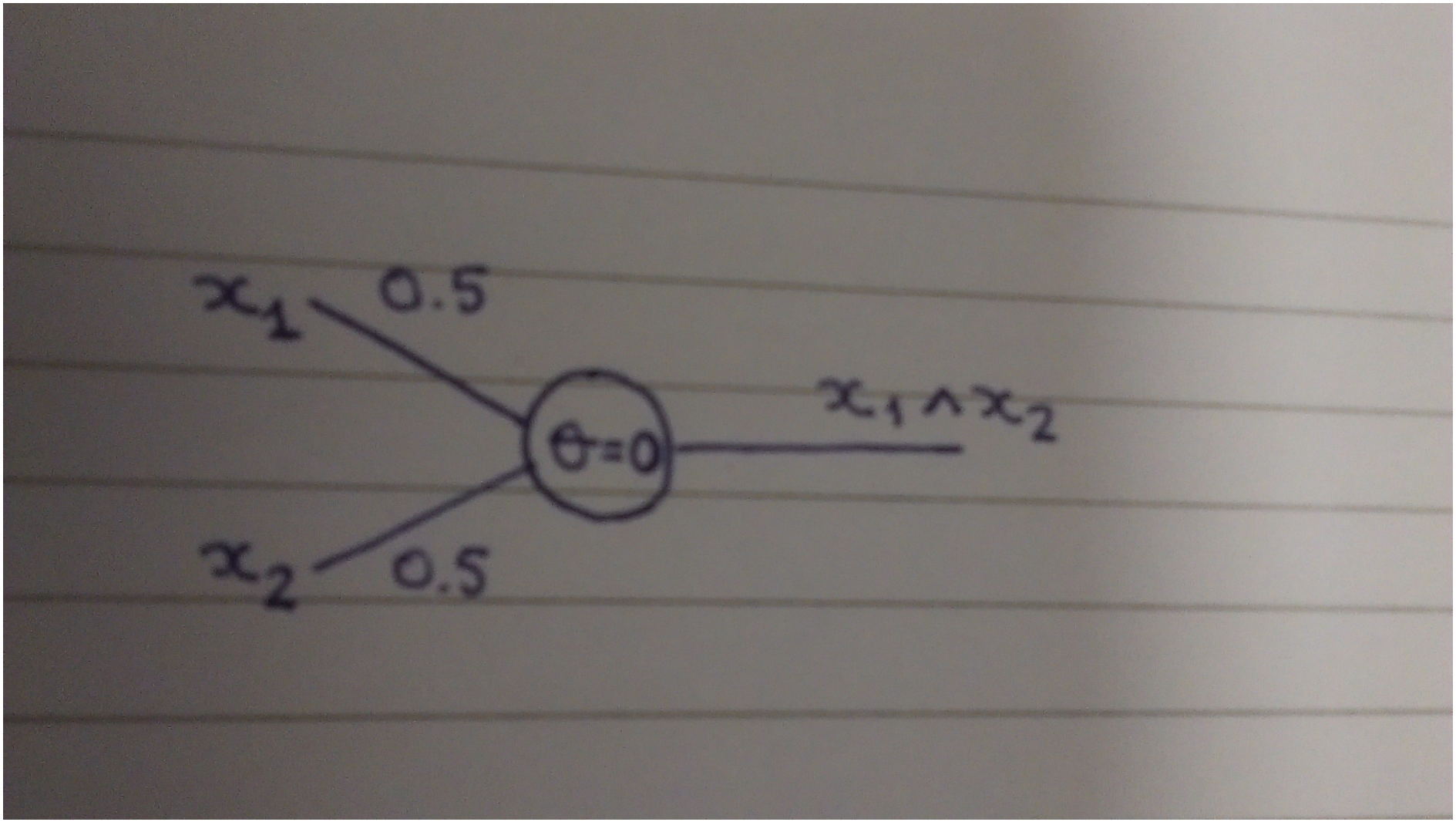


Figure 4: Logical “and” implemented with a perceptron

“Exclusive or” with perceptrons

```
xor (x₁, x₂) = perceptron ([1, 1], 0) [case₁, case₂]
 where
 case₁ = perceptron ([0.5, -0.5], 0) [x₁, x₂]
 case₂ = perceptron ([-0.5, 0.5], 0) [x₁, x₂]
```

“Exclusive or” with perceptrons

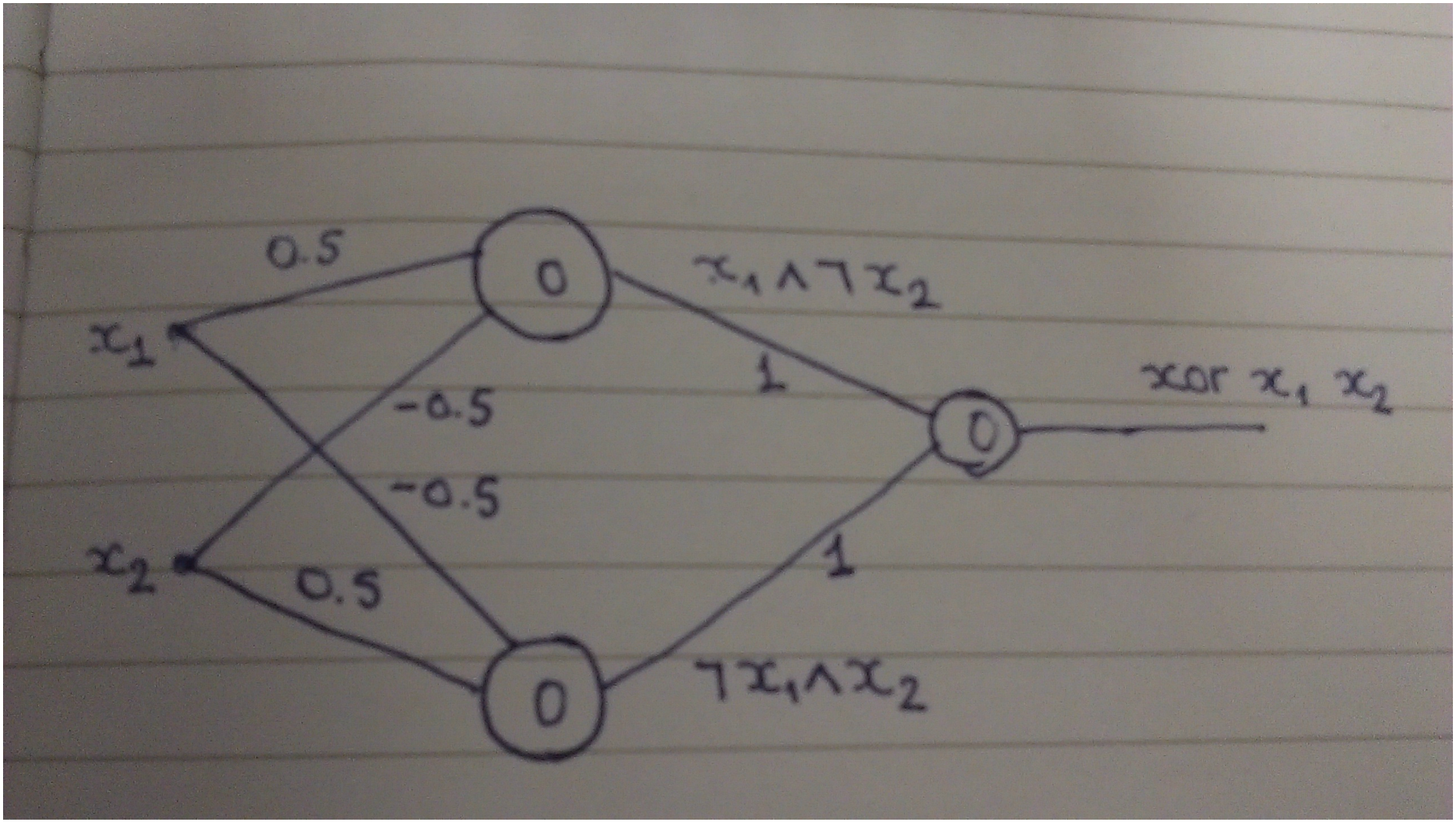


Figure 5: Logical “xor” implemented with perceptrons

Homework

Implement the "two out of three" function defined by:

```
two_of_three : ({0, 1}, {0, 1}, {0, 1}) -> {0, 1}
two_of_three (x, y, z)
 | (1, 1, 0)  =  True
 | (1, 0, 1)  =  True
 | (0, 1, 1)  =  True
 | otherwise  =  False
```

using

a) McCulloch & Pitts neurons

b) perceptrons

Perceptrons

The perceptron training rule:

$$w_i \leftarrow w_i + \eta * (t - o) * x_i$$

The threshold θ is treated as the weight of an always active input.

Gradient descent

Why does the perceptron training rule work?

"Naive" gradient descent:

```
x <- x - η * D f (x)
```

What is `f` in our case?

Error functions

$$t - o,\ |t - o|,\ (t - o)^2,\ (t - o)^4,\ \ldots$$

Bayesian learning and error minimisation

data: $(x_1, t_1), \ldots, (x_n, t_n)$, $x_i \in \mathbb{R}^m$, $t_i \in \mathbb{R}$
hypothesis: $w \in \mathbb{R}^m$

$$
\begin{aligned}
h_{map} &= \operatorname{argmax} p(h \mid d) \\
&= \operatorname{argmax} p(d \mid h) * p(h) \,/\, p(d) \\
&= \operatorname{argmax} p(d \mid h) * p(h) \\
&\quad \textit{-- assume } p(h) = \textit{const} \\
&= \operatorname{argmax} p(d \mid h) \\
&= h_{ml}
\end{aligned}
$$

Bayesian learning and error minimisation

$$p(d \mid h) = p((x_1, t_1), \ldots, (x_n, t_n) \mid w)$$
$$= p(x_1, t_1 \mid w) * \ldots * p(x_n, t_n \mid w)$$

$$h_{ml} = \operatorname{argmax} p(d \mid h)$$
$$= \operatorname{argmax} \ln p(d \mid h)$$
$$= \operatorname{argmax} \ln p(x_1, t_1 \mid w) + \ldots + \ln p(x_n, t_n \mid w)$$

Bayesian learning and error minimisation

Assume the $f(x_i, w)$ are normally distributed around the t_i, with the **same** σ:

$$p(x_i, t_i \mid w) = 1/\sqrt{(2 * \pi * \sigma^2)} \exp(-(t_i - f(x_i, w))^2/(2 * \sigma^2))$$

Therefore

$$\begin{aligned} \ln p(x_i, t_i \mid w) &= \ln 1/\sqrt{(2 * \pi * \sigma^2)} - (t_i - f(x_i, w))^2/(2 * \sigma^2) \\ &= k - (t_i - f(x_i, w))^2/(2 * \sigma^2) \end{aligned}$$

Bayesian learning and error minimisation

$$
\begin{aligned}
h_{ml} &= \text{argmax } \ln p(x_1, t_1 \mid w) + \ldots + \ln p(x_n, t_n \mid w) \\
&= \text{argmax } n * k - \Sigma\, (t_i - f(x_i, w))^2/(2 * \sigma^2) \\
&= \text{argmax } - \Sigma\, (t_i - f(x_i, w))^2 \\
&= \text{argmin } \Sigma\, (t_i - f(x_i, w))^2
\end{aligned}
$$

Bayesian learning and error minimisation

Therefore, the correct error to choose is the sum of squared errors…

…at least if:

- all hypothesis are equally likely a-priori,
- the errors are independent, and
- the errors have identical normal distributions.

Perceptrons and learning

- Using the learning rule, we can iteratively find the weights and threshold for any perceptron (such as the one implementing `and`).
- However, the learning rule does not apply to multi-layer perceptrons, such as the one needed for `xor`.

Linear separability

- Perceptrons can only classify linearly separable sets, but `xor` is not linearly separable.
- The 1969 book *Perceptrons* by Minsky and Pappert pointed out these facts, killing funding for neural networks for a decade or more.