What Is a Mathematical Property?

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Natural numbers

$$\mathbb{N} = \{0\,,\ 1\,,\ 2\,,\ 3\,,\ \ldots\}$$

God created the natural numbers. Everything else is the work of man.

— Leopold Kronecker (1823-1891)

Examples of mathematical properties

$$\mathbb{N} = \{0\ ,\ \ 1\ ,\ \ 2\ ,\ \ 3\ ,\ \ ...\}$$

- even
- equals *2*
- smaller than *3*
- prime
- ...

Examples of non-mathematical properties

$$\mathbb{N} = \{0\,,\ \ 1\,,\ \ 2\,,\ \ 3\,,\ \ ...\}$$

- is big
- is pretty
- is interesting, friendly, heavy, ...

The extensional view

$\mathbb{N} = \{\mathit{0}\ ,\ \ \mathit{1}\ ,\ \ \mathit{2}\ ,\ \ \mathit{3}\ ,\ \ ...\}$

- even: $\mathit{Evens} = \{\mathit{0}\ ,\ \ \mathit{2}\ ,\ \ \mathit{4}\ ,\ \ \mathit{6}\ ,\ \ \mathit{8}\ ,\ \ ...\}$
- equals $\mathit{2}$: $\mathit{Eq2} = \{\mathit{2}\}$
- smaller than $\mathit{3}$: $\mathit{Smaller3} = \{\mathit{0}\ ,\ \ \mathit{1}\ ,\ \ \mathit{2}\}$
- prime: $\mathit{Primes} = \{\mathit{2}\ ,\ \ \mathit{3}\ ,\ \ \mathit{5}\ ,\ \ \mathit{7}\ ,\ \ \mathit{11}\ ,\ \ ...\}$
- ...

The extensional view

$$\mathbb{N} = \{0\,,\ 1\,,\ 2\,,\ 3\,,\ \ldots\}$$

- $Evens = \{n \mid n \text{ divisible by } 2\}$
- $Eq2 = \{n \mid n = 2\}$
- $Smaller3 = \{n \mid n < 3\}$
- $Primes = \{n \mid n \text{ has exactly two divisors}\}$
- ...

The extensional view

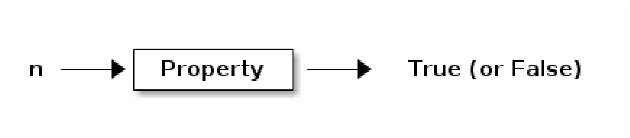
$$\mathbb{N} = \{\mathit{0}\ ,\ \ \mathit{1}\ ,\ \ \mathit{2}\ ,\ \ \mathit{3}\ ,\ \ ...\}$$

- $\mathit{Evens} = \{n \mid n \textit{ divisible by } 2\}$
- $\mathit{Eq2} = \{n \mid n = 2\}$
- $\mathit{Smaller3} = \{n \mid n < 3\}$
- $\mathit{Primes} = \{n \mid n \textit{ has exactly two divisors}\}$
- ...

The **intensional** view

$$\mathbb{N} = \{0\ ,\ \ 1\ ,\ \ 2\ ,\ \ 3\ ,\ \ ...\}$$

Properties are functions:



Function

Definition: Let A and B sets. A function $f : A \to B$ is a *correspondence* between the elements of A and those of B that associates to *each* element of A a *unique* element of B. The unique element associated with $a \in A$ is denoted $f(a)$.

Thus, **if** $a \in A$ and $f : A \to B$, **then** $f(a) \in B$.

Example:

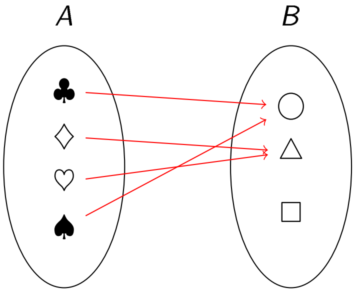
$$\begin{aligned} A &= \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\} \\ B &= \{\bigcirc, \triangle, \square\} \end{aligned}$$

$$\begin{aligned} f(\clubsuit) &= \bigcirc \\ f(\diamondsuit) &= \triangle \\ f(\heartsuit) &= \triangle \\ f(\spadesuit) &= \bigcirc \end{aligned}$$

Function

Another way of picturing the same function:

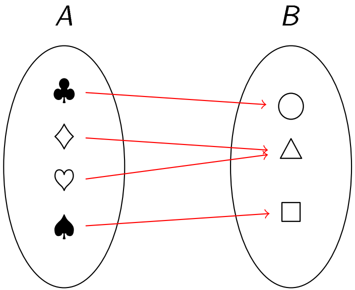
$$\begin{aligned} A &= \{\clubsuit\,,\ \diamondsuit\,,\ \heartsuit\,,\ \spadesuit\} \\ B &= \{\bigcirc\,,\ \triangle\,,\ \square\} \end{aligned}$$



Surjective function

An example of an *onto*, or surjective, function:

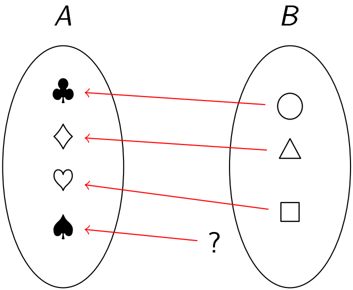
$$
\begin{aligned}
A &= \{\clubsuit\,,\ \diamondsuit\,,\ \heartsuit\,,\ \spadesuit\} \\
B &= \{\bigcirc\,,\ \triangle\,,\ \square\}
\end{aligned}
$$



Surjective function

We cannot have functions from a "smaller" set *onto* a "bigger" set.

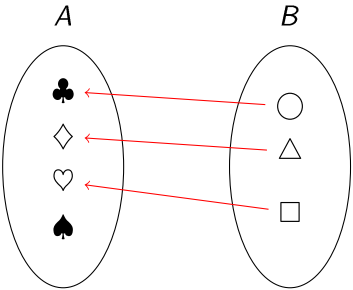
$$A = \{\clubsuit, \ \diamondsuit, \ \heartsuit, \ \spadesuit\}$$
$$B = \{\bigcirc, \ \triangle, \ \square\}$$



Injective function

An example of a *one-to-one*, or injective, function:

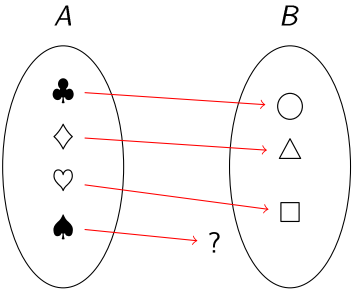
$$A = \{\clubsuit,\ \diamondsuit,\ \heartsuit,\ \spadesuit\}$$
$$B = \{\bigcirc,\ \triangle,\ \square\}$$



Injective function

We cannot have a *one-to-one* function from a "bigger" set to a "smaller" set:

$$\begin{aligned} A &= \{\clubsuit\,,\ \diamondsuit\,,\ \heartsuit\,,\ \spadesuit\} \\ B &= \{\bigcirc\,,\ \triangle\,,\ \square\} \end{aligned}$$



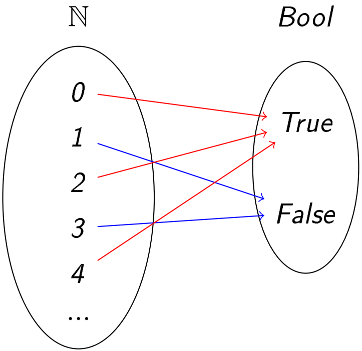
Functions and set cardinalities

If $f : A \to B$ is ...

- *onto*, then the cardinality of A is at least equal to that of B
- *one-to-one*, then the cardinality of A is at most equal to that of B
- *onto* and *one-to-one*, then the cardinality of A is equal to that of B

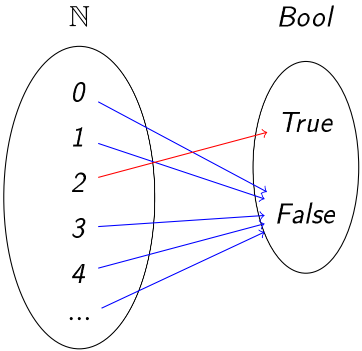
Properties as functions

$\mathbb{N} = \{\mathit{0}, \mathit{1}, \mathit{2}, \mathit{3}, \ldots\}, \quad \mathit{Bool} = \{\mathit{True}, \mathit{False}\}$
$\mathit{even} : \mathbb{N} \to \mathit{Bool}$



Properties as functions

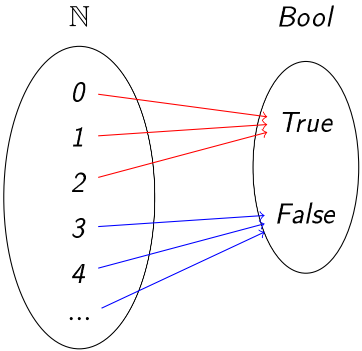
$\mathbb{N} = \{ 0 , 1 , 2 , 3 , ... \} ,\ Bool = \{ True, False \}$
$eq2 : \mathbb{N} \to Bool$



Properties as functions

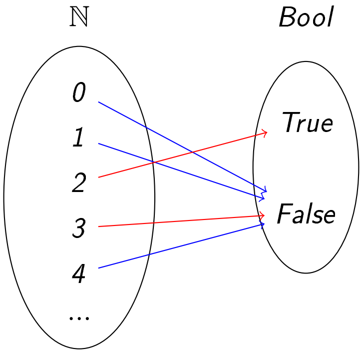
$\mathbb{N} = \{\mathit{0}, \mathit{1}, \mathit{2}, \mathit{3}, \ldots\}, \quad \mathit{Bool} = \{\mathit{True}, \mathit{False}\}$

$\mathit{smaller3} : \mathbb{N} \to \mathit{Bool}$



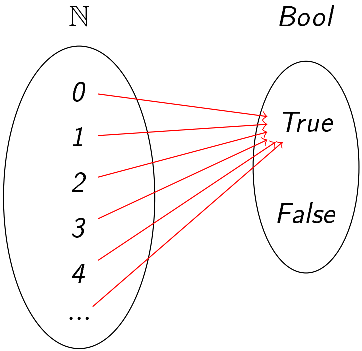
Properties as functions

$\mathbb{N} = \{\mathit{0}, \mathit{1}, \mathit{2}, \mathit{3}, \ldots\}$, $\mathit{Bool} = \{\mathit{True}, \mathit{False}\}$
$\mathit{prime} : \mathbb{N} \to \mathit{Bool}$



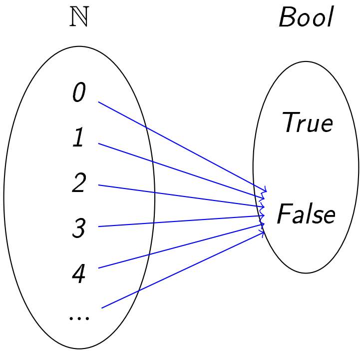
Properties as functions

$\mathbb{N} = \{\mathit{0}\ ,\ \mathit{1}\ ,\ \mathit{2}\ ,\ \mathit{3}\ ,\ ...\}\ ,\ \mathit{Bool} = \{\mathit{True}, \mathit{False}\}$
$\mathit{always} : \mathbb{N} \rightarrow \mathit{Bool}$



Properties as functions

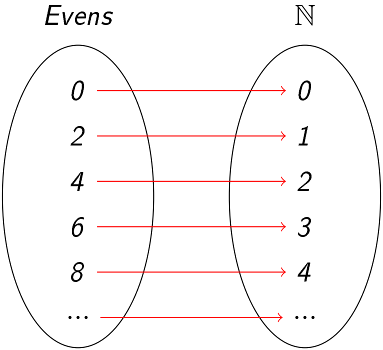
$\mathbb{N} = \{ \mathit{0}\ ,\ \mathit{1}\ ,\ \mathit{2}\ ,\ \mathit{3}\ ,\ \dots \},\quad \mathit{Bool} = \{\mathit{True}, \mathit{False}\}$
$\mathit{never} : \mathbb{N} \to \mathit{Bool}$



Bijections

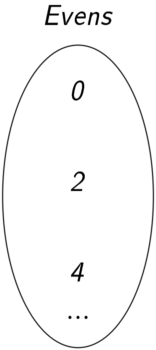
A function that is both onto and one-to-one:

$$f : \{n ; n \in Nat, even\ n = True\} \rightarrow \mathbb{N}$$
$$f\ n = n\ /\ 2$$



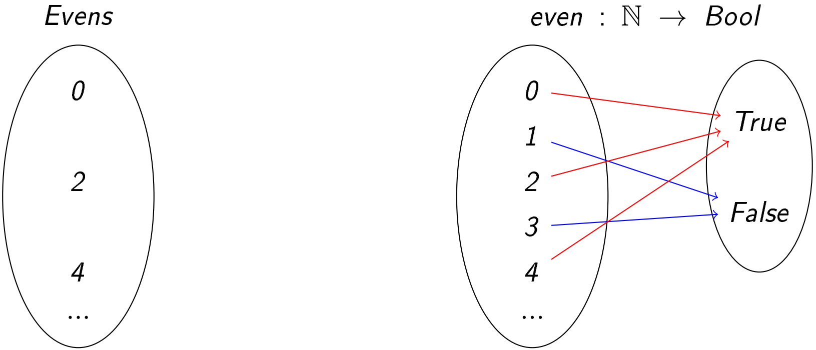
Bijections

From extensional to intensional:



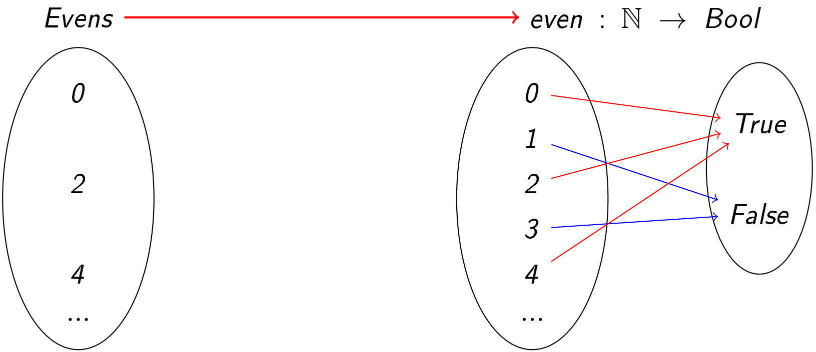
Bijections

From extensional to intensional:



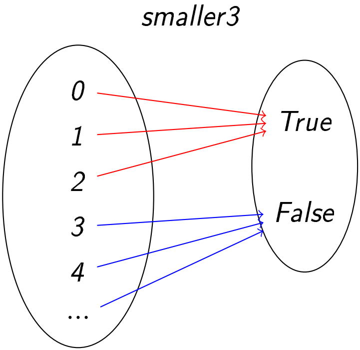
Bijections

From extensional to intensional:



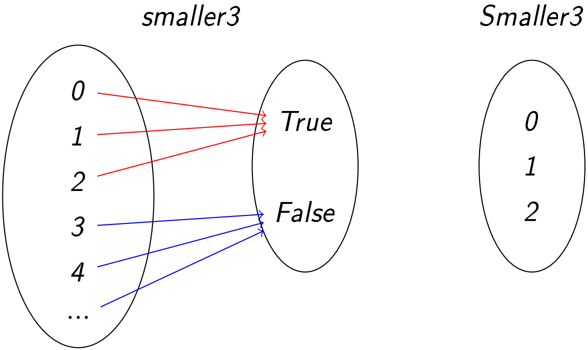
Bijections

From intensional to extensional:



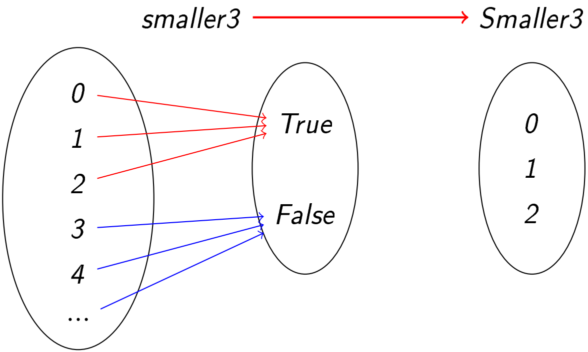
Bijections

From intensional to extensional:



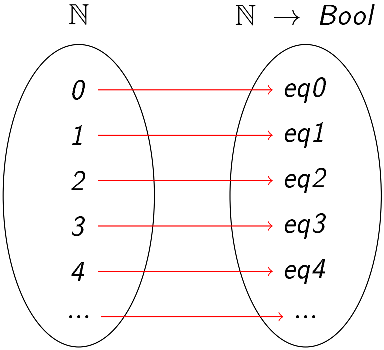
Bijections

From intensional to extensional:



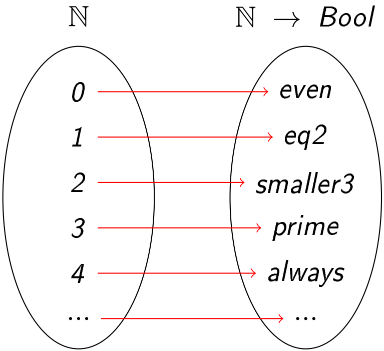
Enumerating properties

Does there exist a one-to-one and unto function $f : \mathbb{N} \to (\mathbb{N} \to \mathit{Bool})$?
It's easy to find one-to-one functions:



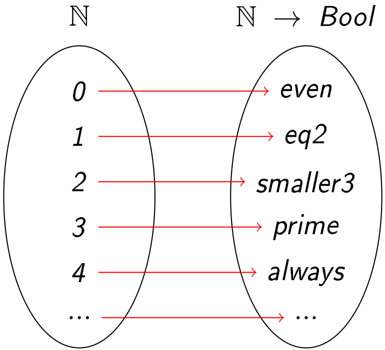
Enumerating properties

There are many one-to-one functions $f : \mathbb{N} \to (\mathbb{N} \to \mathit{Bool})$?
But can we find one which is unto?



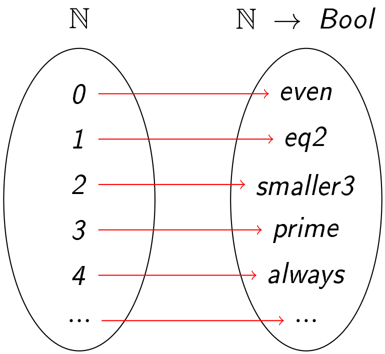
Enumerating properties

Suppose we have $f : \mathbb{N} \to (\mathbb{N} \to Bool)$ unto.



Enumerating properties

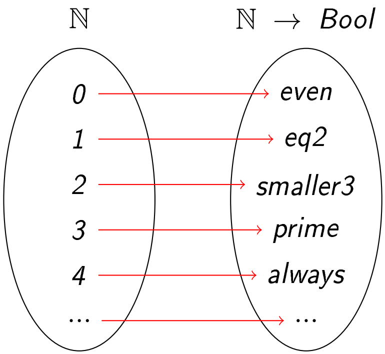
Suppose we have $f : \mathbb{N} \to (\mathbb{N} \to \mathit{Bool})$ unto.



$$
\begin{aligned}
f\ 0 &= \mathit{even} \\
f\ 1 &= \mathit{eq2} \\
f\ 2 &= \mathit{smaller3} \\
f\ 3 &= \mathit{prime} \\
f\ 4 &= \mathit{always}
\end{aligned}
$$

Enumerating properties

Suppose we have $f : \mathbb{N} \rightarrow (\mathbb{N} \rightarrow Bool)$ unto.

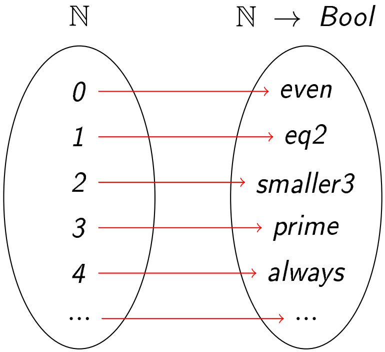


$$
\begin{aligned}
C &: \mathbb{N} \rightarrow Bool \\
C\ n &= not\ ((f\ n)\ n)
\end{aligned}
$$

Example:

Enumerating properties

Suppose we have $f : \mathbb{N} \to (\mathbb{N} \to Bool)$ unto.



$$
\begin{aligned}
C &: \mathbb{N} \to Bool \\
C\ n &= not\,((f\ n)\ n)
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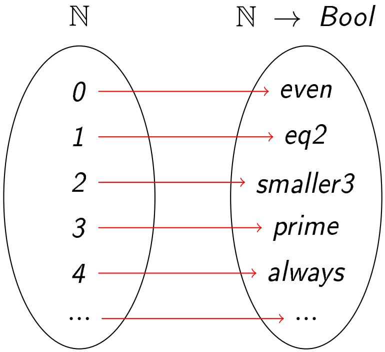
$$

Example:

$$C\ 0 = not\,(even\ 0) = False$$

Enumerating properties

Suppose we have $f : \mathbb{N} \to (\mathbb{N} \to Bool)$ unto.



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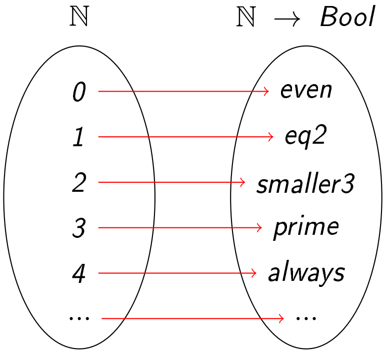
Example:

$$
\begin{aligned}
C\ 0 &= not\ (even\ 0) = False \\
C\ 1 &= not\ (eq2\ 1) = True
\end{aligned}
$$

Enumerating properties

Suppose we have $f : \mathbb{N} \to (\mathbb{N} \to \mathit{Bool})$ unto.



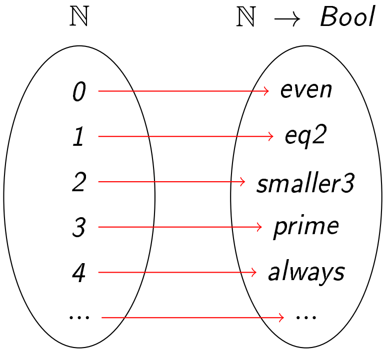
$$\begin{aligned} C &: \mathbb{N} \to \mathit{Bool} \\ C\ n &= \mathit{not}\ ((f\ n)\ n) \end{aligned}$$

Example:

$$\begin{aligned} C\ 0 &= \mathit{not}\ (\mathit{even}\ 0) = \mathit{False} \\ C\ 1 &= \mathit{not}\ (\mathit{eq2}\ 1) = \mathit{True} \\ C\ 2 &= \mathit{not}\ (\mathit{smaller3}\ 2) = \mathit{False} \end{aligned}$$

Enumerating properties

Suppose we have $f : \mathbb{N} \to (\mathbb{N} \to Bool)$ unto.



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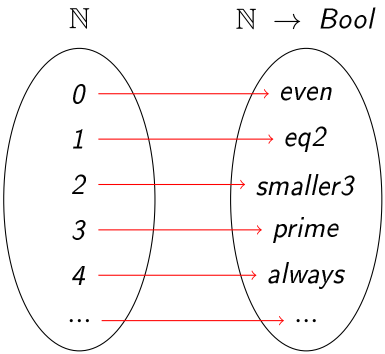

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C\ 4 &= not\ (always\ 4) = False
\end{aligned}
$$

Enumerating properties

Suppose we have $f : \mathbb{N} \to (\mathbb{N} \to \mathit{Bool})$ unto.

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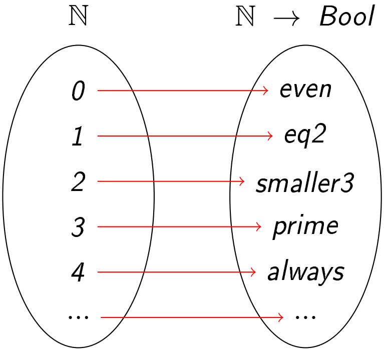

Enumerating properties

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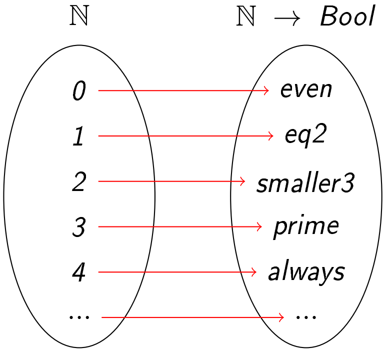
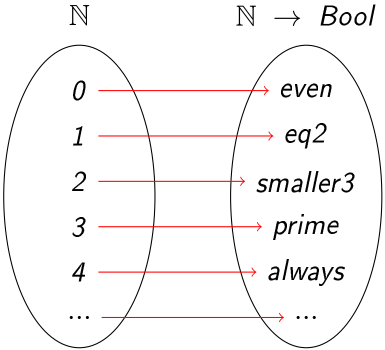


$$
\begin{aligned}
C &: \mathbb{N} \to Bool \\
C\ n &= not\,((f\ n)\ n)
\end{aligned}
$$

Consider N such that
$f\ N = C$.

Enumerating properties

Suppose we have $f : \mathbb{N} \to (\mathbb{N} \to \mathit{Bool})$ unto.



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Consider N such that
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$C\ N$

Enumerating properties

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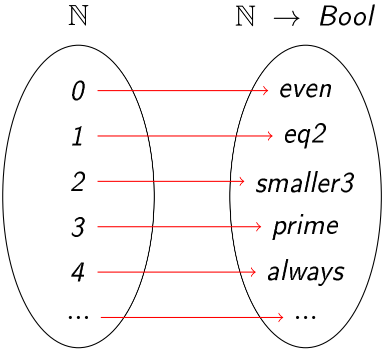


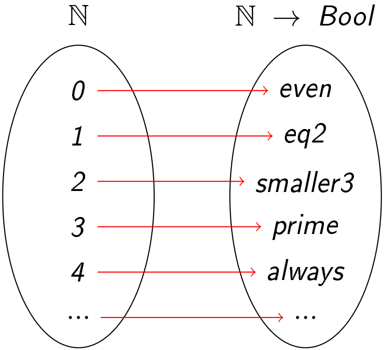
$$\begin{aligned} C &: \mathbb{N} \to \mathit{Bool} \\ C\ n &= \mathit{not}\ ((f\ n)\ n) \end{aligned}$$

Consider N such that
$f\ N = C$.

$$\begin{aligned} & C\ N \\ = & \quad \{\ \text{Definition } C\ \} \\ & \mathit{not}\ ((f\ N)\ N) \end{aligned}$$

Enumerating properties

Suppose we have $f \ : \ \mathbb{N} \ \to \ (\mathbb{N} \ \to \ \mathit{Bool})$ unto.



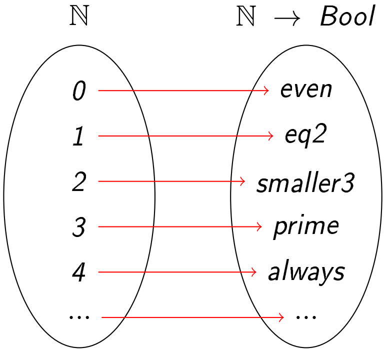
$$
\begin{aligned}
C \ &: \ \mathbb{N} \to \mathit{Bool} \\
C\ n \ &= \ \mathit{not}\,((f\ \ n)\ n)
\end{aligned}
$$

Consider N such that
$f \ N \ = \ C$.

$$
\begin{aligned}
& C\ N \\
= \quad & \{ \text{ Definition } C \ \} \\
& \mathit{not}\,((f\ \ N)\ N) \\
= \quad & \{ \ f\ N \ = \ C \ \} \\
& \mathit{not}\,(C\ \ N)
\end{aligned}
$$

Enumerating properties

Suppose we have $f : \mathbb{N} \to (\mathbb{N} \to Bool)$ unto.



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Consider N such that
$f\ N = C$.

$$
\begin{aligned}
& C\ N \\
= & \quad \{\ \text{Definition } C\ \} \\
& not\ ((f\ N)\ N) \\
= & \quad \{\ f\ N = C\ \} \\
& not\ (C\ N)
\end{aligned}
$$

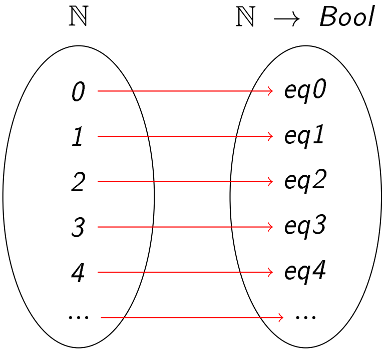
(G. Cantor, 1845-1918)

Consequences

- There are "more" properties of natural numbers than there are natural numbers.
- There are "more" properties of a set than there are elements of that set (e.g., properties of properties of natural numbers).
- There exist uncomputable functions!

Programs

A *computable* function is a *program*.
Programs are stored as strings of ones and zeros, i.e., as natural numbers in binary representation.
So there cannot be "more" programs than natural numbers.
But *eq0*, *eq1*, *eq2*, . . . are programs, so we cannot have "more" natural numbers than programs:



Constructivism

There are "more" properties of $\mathbb{N}$ than elements of $\mathbb{N}$, but only as many programs as elements of $\mathbb{N}$.

Therefore, there exists non-computable properties, and therefore non-computable functions.

Constructivists reject non-computable functions.

Constructivism

Constructivists reject non-computable functions.

Therefore, there are as many properties of natural numbers as there are natural numbers.

And, in fact, there are as many functions as there are natural numbers (as many properties of properties etc.).

The constructive universe is very small!

Constructivism

Constructivists reject non-computable functions.

Therefore, there are as many properties of natural numbers as there are natural numbers.

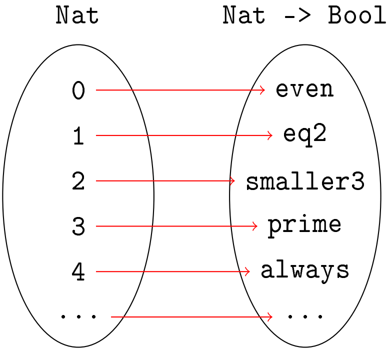
And, in fact, there are as many functions as there are natural numbers (as many properties of properties etc.).

The constructive universe is very small!

Or is it?

Enumerating **constructive** properties

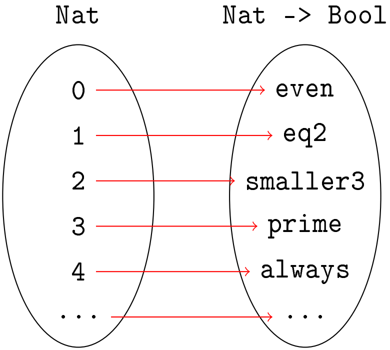
Suppose we have `f : Nat -> (Nat -> Bool)` unto.



Enumerating **constructive** properties

Suppose we have `f : Nat -> (Nat -> Bool)` unto.

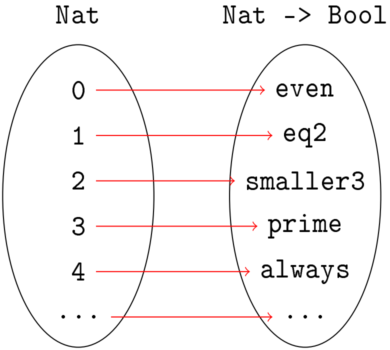
```
C : Nat -> Bool
C n = not ((f n) n)
```



Enumerating **constructive** properties

Suppose we have `f : Nat -> (Nat -> Bool)` unto.

```
C : Nat -> Bool
C n = not ((f n) n)
```



Consider `N` such that `f N = C`.

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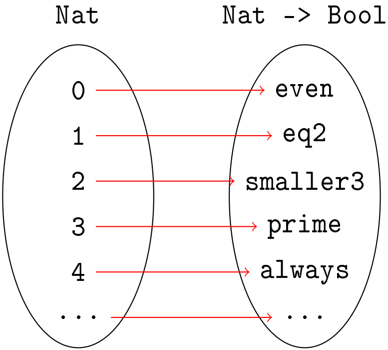

Consider `N` such that `f N = C`.

```
  C N
```
= { Definition `C` }
```
  not ((f N) N)
```
= { `f N = C` }
```
  not (C N)
```

(G. Cantor, 1845-1918)

Conclusions

The function that associates $\mathbb{N}$ to computable properties is not computable.

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Consequence: there is no program that can distinguish "good" programs from "bad" programs!