

# Analysis of Time Series

## Chapter 6: Extending the ARMA model: Seasonality, integration and trend

Edward L. Ionides

# Outline

- 1 Seasonality
  - The SARMA model
- 2 Differencing and integration
  - The ARIMA model
  - The SARIMA model
- 3 Trend estimation: regression with ARMA errors

# Seasonal autoregressive moving average (SARMA) models

- A general  $\text{SARMA}(p, q) \times (P, Q)_{12}$  model for monthly data is

$$[\text{S1}] \quad \phi(B)\Phi(B^{12})(Y_n - \mu) = \psi(B)\Psi(B^{12})\epsilon_n,$$

where  $\{\epsilon_n\}$  is a white noise process and

$$\begin{aligned}\mu &= \mathbb{E}[Y_n] \\ \phi(x) &= 1 - \phi_1 x - \cdots - \phi_p x^p, \\ \psi(x) &= 1 + \psi_1 x + \cdots + \psi_q x^q, \\ \Phi(x) &= 1 - \Phi_1 x - \cdots - \Phi_P x^P, \\ \Psi(x) &= 1 + \Psi_1 x + \cdots + \Psi_Q x^Q.\end{aligned}$$

- SARMA is a special case of ARMA, where the AR and MA polynomials are factored into a **monthly** polynomial in  $B$  and an **annual polynomial** (also called **seasonal polynomial**) in  $B^{12}$ .
- Everything we learned about ARMA models (including assessing causality, invertibility and reducibility) also applies to SARMA.

## Choosing the period for a SARMA model

- For the  $\text{SARMA}(p, q) \times (P, Q)_{12}$  model, 12 is called the **period**.
- One could write a SARMA model for some period other than 12.
- A  $\text{SARMA}(p, q) \times (P, Q)_4$  model could be appropriate for quarterly data.
- In principle, a  $\text{SARMA}(p, q) \times (P, Q)_{52}$  model could be appropriate for weekly data, though in practice ARMA and SARMA may not work so well for higher frequency data.
- The seasonal period should be appropriate for the system being modeled. It is usually inappropriate to fit a  $\text{SARMA}(p, q) \times (P, Q)_9$  model just because you notice a high sample autocorrelation at lag 9.

Consider the following two models:

$$[S2] \quad Y_n = 0.5Y_{n-1} + 0.25Y_{n-12} + \epsilon_n,$$

$$[S3] \quad Y_n = 0.5Y_{n-1} + 0.25Y_{n-12} - 0.125Y_{n-13} + \epsilon_n,$$

**Question 6.1.** Which of [S2] and/or [S3] is a SARMA model?

**Question 6.2.** Why do we assume a multiplicative structure in the SARMA model, [S1]? What theoretical and practical advantages (or disadvantages) arise from requiring that an ARMA model for seasonal behavior has polynomials that can be factored as a product of a monthly polynomial and an annual polynomial?

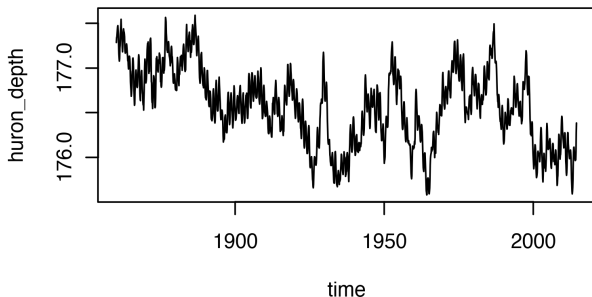
# Fitting a SARMA model

We fit the full, monthly, version of the Lake Huron depth data described earlier.

```
head(dat,3)
```

Date	Average	year	month
1860-01-01	177.285	1860	1
1860-02-01	177.339	1860	2
1860-03-01	177.349	1860	3

```
huron_depth <- dat$Average  
time <- dat$year + dat$month/12  
# Note: this treats December 2011 as time 2012.0  
plot(huron_depth~time,type="l")
```



Based on our previous analysis, we try fitting AR(1) for the annual polynomial. Let's try ARMA(1,1) for the monthly part. In other words, we consider the model

$$(1 - \Phi_1 B^{12})(1 - \phi_1 B)Y_n = (1 + \psi_1 B)\epsilon_n. \quad (1)$$

```
huron_sarma11x10 <- arima(huron_depth,  
  order=c(1,0,1),  
  seasonal=list(order=c(1,0,0),period=12)  
)
```



```
huron_sarma11x10
```

Call:

```
arima(x = huron_depth, order = c(1, 0, 1), seasonal = list(order = c(1, 0, 0),  
  period = 12))
```

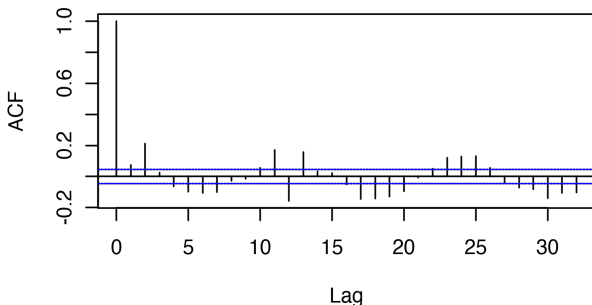
Coefficients:

	ar1	ma1	sar1	intercept
	0.9641	0.3782	0.5104	176.5714
s.e.	0.0063	0.0203	0.0218	0.0909

sigma^2 estimated as 0.002592: log likelihood = 2884.36, aic = -5758.72

## Residual analysis

- Residual analysis is similar to non-seasonal ARMA models.
- We look for residual correlations at lags corresponding to multiples of the period (here, 12, 24, 36, ...) for misspecified annual dependence.



**Question 6.3.** What do you conclude from this residual analysis? What would you do next?

# ARMA models for differenced data

- Applying a difference operation to the data can make it look more stationary and therefore more appropriate for ARMA modeling.
- This can be viewed as a **transformation to stationarity**
- We can transform the data  $y_{1:N}^*$  to  $z_{2:N}^*$

$$z_n^* = \Delta y_n^* = y_n^* - y_{n-1}^*. \quad (2)$$

- Then, an ARMA(p,q) model  $Z_{2:N}$  for the differenced data  $z_{2:N}^*$  is called an **integrated autoregressive moving average** model for  $y_{1:N}^*$  and is written as ARIMA(p,1,q).
- Formally, the ARIMA(p,d,q) model with intercept  $\mu$  for  $Y_{1:N}$  is

$$[S4] \quad \phi(B)((1-B)^d Y_n - \mu) = \psi(B)\epsilon_n,$$

where  $\{\epsilon_n\}$  is a white noise process;  $\phi(x)$  and  $\psi(x)$  are ARMA polynomials.

- It is unusual to fit an ARIMA model with  $d > 1$ .

- We see that an  $ARIMA(p,1,q)$  model is almost a special case of an  $ARMA(p+1,q)$  model with a **unit root** to the  $AR(p+1)$  polynomial.

**Question 6.4.** Why “almost” not “exactly” in the previous statement?

## Two reasons to fit an ARIMA( $p,d,q$ ) model with $d > 0$

1. You may really think that modeling the differences is a natural approach for your data. The S&P 500 stock market index analysis in Chapter 3 is an example of this, as long as you remember to first apply a logarithmic transform to the data.
2. Differencing often makes data look “more stationary” and perhaps it will then look stationary enough to justify applying the ARMA machinery.
  - We should be cautious about this second reason. It can lead to poor model specifications and hence poor forecasts or other conclusions.
  - The second reason was more compelling in the 1970s and 1980s. Limited computing power resulted in limited alternatives, so it was practical to force as many data analyses as possible into the ARMA framework and use method of moments estimators.

# Practical advice on using ARIMA models

- ARIMA analysis is relatively simple to do. It has been a foundational component of time series analysis since the publication of the influential book “Time Series Analysis” by Box and Jenkins (1st edition, 1970) which developed and popularized ARIMA modeling.
- A practical approach is:
  1. Do a competent ARIMA analysis.
  2. Identify potential limitations in this analysis and remedy them using more advanced methods.
  3. Assess whether you have in fact learned anything from (2) that goes beyond (1).

## The $\text{SARIMA}(p, d, q) \times (P, D, Q)$ model

Combining integrated ARMA models with seasonality, we can write a general  $\text{SARIMA}(p, d, q) \times (P, D, Q)_{12}$  model for nonstationary monthly data, given by

$$[S5] \quad \phi(B)\Phi(B^{12})((1-B)^d(1-B^{12})^DY_n - \mu) = \psi(B)\Psi(B^{12})\epsilon_n,$$

where  $\{\epsilon_n\}$  is a white noise process, the intercept  $\mu$  is the mean of the differenced process  $\{(1-B)^d(1-B^{12})^DY_n\}$ , and we have ARMA polynomials  $\phi(x)$ ,  $\Phi(x)$ ,  $\psi(x)$ ,  $\Psi(x)$  as in model [S1].

- The  $\text{SARIMA}(0, 1, 1) \times (0, 1, 1)_{12}$  model has often been used for forecasting monthly time series in economics and business. It is sometimes called the **airline model** after a data analysis by Box and Jenkins (1970).

# Modeling trend with ARMA noise

- A general **signal plus noise** model is

$$[S6] \quad Y_n = \mu_n + \eta_n,$$

where  $\{\eta_n\}$  is a stationary, mean zero stochastic process, and  $\mu_n$  is the mean function.

- If, in addition,  $\{\eta_n\}$  is uncorrelated, then we have a **signal plus white noise** model. The usual linear trend regression model fitted by least squares in Chapter 2 corresponds to a signal plus white noise model.
- We can say **signal plus colored noise** if we wish to emphasize that we're not assuming white noise.
- Here, **signal** and **trend** are used interchangeably. In other words, we are assuming a deterministic signal.
- At this point, it is natural for us to consider a signal plus ARMA(p,q) noise model, where  $\{\eta_n\}$  is a stationary, causal, invertible ARMA(p,q) process with mean zero.
- As well as the  $p + q + 1$  parameters in the ARMA(p,q) model, there will usually be unknown parameters in the mean function.



# Linear regression with ARMA errors

- When the mean function (also known as the trend) has a linear specification,

$$\mu_n = \sum_{k=1}^K Z_{n,k} \beta_k,$$

the signal plus ARMA noise model is known as **linear regression with ARMA errors**.

- Writing  $Y$  for a column vector of  $Y_{1:N}$ ,  $\mu$  for a column vector of  $\mu_{1:N}$ ,  $\eta$  for a column vector of  $\eta_{1:N}$ , and  $Z$  for the  $N \times K$  matrix with  $(n, k)$  entry  $Z_{n,k}$ , we have a general linear regression model with correlated ARMA errors,

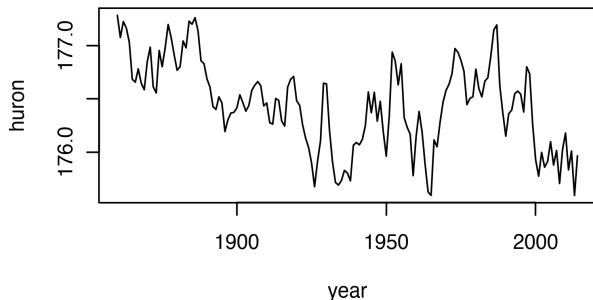
$$Y = Z\beta + \eta.$$

# Inference for the linear regression model with ARMA errors

- Maximum likelihood estimation of  $\theta = (\phi_{1:p}, \psi_{1:q}, \sigma^2, \beta)$  is a nonlinear optimization problem.
- Fortunately, `arima` in R can do it for us.
- As usual, we should look out for signs of numerical problems.
- Data analysis for a linear regression with ARMA errors model, using the framework of likelihood-based inference, is procedurally similar to fitting an ARMA model.
- This is a powerful technique, since the covariate matrix  $Z$  can include other time series. We can evaluate associations between different time series.
- With appropriate care (since **association is not causation**) we can draw inferences about mechanistic relationships between dynamic processes.

# Evidence for systematic trend in Lake Huron depth?

Let's go back to annual data, say the January depth, to avoid seasonality.



- Visually, there seems some evidence for a decreasing trend, but there are also considerable fluctuations.
- Let's test for a trend, using a regression model with Gaussian AR(1) errors. We have previously found that this is a reasonable model for these data.
- First, let's fit a null model.

```
fit0 <- arima(huron,order=c(1,0,0))
```

Call:

```
arima(x = huron, order = c(1, 0, 0))
```

Coefficients:

	ar1	intercept
	0.8694	176.4588
s.e.	0.0407	0.1234

sigma<sup>2</sup> estimated as 0.04368: log likelihood = 22, aic = -38

- We compare `fit0` with a linear trend model, coded as `fit1`.

```
fit1 <- arima(huron,order=c(1,0,0),xreg=year)
```

Call:

```
arima(x = huron, order = c(1, 0, 0), xreg = year)
```

Coefficients:

	ar1	intercept	year
	0.8240	186.0146	-0.0049
s.e.	0.0451	3.7417	0.0019

sigma<sup>2</sup> estimated as 0.0423: log likelihood = 24.62, aic = -41.25

# Setting up a formal hypothesis test

- To talk formally about these results, we must down a model and some hypotheses.
- Writing the data as  $y_{1:N}^*$ , collected at years  $t_{1:N}$ , the model we have fitted is

$$(1 - \phi_1 B)(Y_n - \mu - \beta t_n) = \epsilon_n,$$

where  $\{\epsilon_n\}$  is Gaussian white noise with variance  $\sigma^2$ . Our null model is

$$H^{(0)} : \beta = 0,$$

and our alternative hypothesis is

$$H^{(1)} : \beta \neq 0.$$

**Question 6.5.** How do we test  $H^{\langle 0 \rangle}$  against  $H^{\langle 1 \rangle}$ ?

- Construct two different tests using the R output above.
- Which test do you prefer, and why?
- How would you check whether your preferred test is indeed better?

**Question 6.6.** What other supplementary analysis could you do to strengthen your conclusions?



## Further reading

- Section 3.5 of Shumway and Stoffer (2017) gives a complementary discussion of parameter estimation for ARMA models.
- Section 3.7 of Shumway and Stoffer (2017) takes a different perspective on selecting ARMA models, putting less emphasis on likelihood. Both perspectives can be valuable.

# License, acknowledgments, and links

- Licensed under the [Creative Commons Attribution-NonCommercial license](#). Please share and remix non-commercially, mentioning its origin.
- The materials builds on [previous courses](#).
- Compiled on February 11, 2021 using R version 4.0.3.



[Back to course homepage](#)

# References

Shumway RH, Stoffer DS (2017). *Time series analysis and its applications: With R examples*. Springer. URL <http://www.stat.pitt.edu/stoffer/tsa4/tsa4.pdf>.