STATS 700-002 Class 5, extra material. Why are Kolmogorov's forward and backward equations adjoint?

Aaron King and Edward Ionides

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Setup

Let $\{X_t\}$ be a Markov process with transition density $f(x,s,y,t)$ determining the probbility density of X_t at y given $X_s = x$.

The forward operator, L, also known as the generator, satisfies

$$\frac{\partial}{\partial t} f(x,s,y,t) = Lf(x,s,y,t).$$

The backward operator, L^*, satisfies

$$\frac{\partial}{\partial s} f(x,s,y,t) = -L^* f(x,s,y,t).$$

We will see that the minus sign here is needed to give an adjoint relationship between L and L^*, so that, with $\langle p, q\rangle = \int p(x)q(x), .\, dx$,

$$\langle Lp, q\rangle = \langle p, L^* q\rangle.$$

Note. For a diffusion process, L and L^* have a $\partial/\partial y$ term describing drift and a $\partial^2/\partial y^2$ term describing noise. For a jump process, the equations have sums over transitions.

An integrated form of the forward and backward equations

Assuming enough regularity to enable us to move derivatives through integrals, we can rewrite the forward equation in terms of

$$P_t = \int p(x) f(x, s, y, t)\, dx,$$

the density at t given $X_s \sim p(x)$. The forward equation gives

$$\frac{\partial}{\partial t} P_t = L P_t.$$

Similarly, for the backward equation, define

$$Q_s = \int q(y) f(x, s, y, t)\, dy = E[q(X_t) | X_s = x].$$

The integral form of the backward equation is

$$\frac{\partial}{\partial s} Q_s = -L^* Q_s.$$

Why are L and L^* adjoint? An informal argument

Start with

$$E\Big[q(X_t)\Big|X_s \sim p\Big] = \int\int p(x)f(x,s,y,t)q(y)\,dx\,dy.$$

We can further integrate out over the value of X_u to get

$$E\Big[q(X_t)\Big|X_s \sim p\Big] = \int\int P_u Q_u\,dx\,dy = \langle P_u, Q_u\rangle. \qquad (1)$$

Note that $P_s = p$ and $Q_t = q$. Now,

$$P_{u+\delta} = P_u + \delta L P_u + o(\delta), \quad Q_{u+\delta} = Q_u - \delta L^* Q_u + o(\delta).$$

Since Equation 1 holds for all u, equating the expressions for u and $u+\delta$ gives

$$\langle P_u, Q_u\rangle = \langle P_u + \delta L P_u + o(\delta), Q_u - \delta L^* Q_u + o(\delta)\rangle.$$

Distributing and taking a limit as $\delta \to 0$, it follows that

$$\langle L P_u, Q_u\rangle = \langle P_u, L^* Q_u\rangle. \qquad (2)$$

The adjoint property in Equation 2 holds for arbitrary p and q.