GPT helps with Lemma 2.3

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PROMPT: what is this identity? can you prove it?

$$\sum_{N=n}^{\infty} \binom{N}{n} a^N = \frac{a^n}{(1-a)^{n+1}},$$

UM-GPT: Let's state the identity and then prove it:

The Identity

$$\sum_{N=n}^{\infty} \binom{N}{n} a^N = \frac{a^n}{(1-a)^{n+1}}$$

where (|a| < 1$.

Proof

Let's break the sum down and then manipulate it to get the right-hand side.

Step 1: Rewrite the Binomial Coefficient

Recall that:

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}$$

So,

$$\sum_{N=n}^{\infty} \binom{N}{n} a^N$$

But let's find a more convenient way—**Shift index:** Let $M = N - n$, so as N goes from n to ∞, M goes from 0 to ∞.

So,

$$\sum_{N=n}^{\infty} \binom{N}{n} a^N = \sum_{M=0}^{\infty} \binom{M+n}{n} a^{M+n}$$

$$= a^n \sum_{M=0}^{\infty} \binom{M+n}{n} a^M$$

Step 2: Recognize the Generating Function

Recall the generating function:

$$\sum_{M=0}^{\infty} \binom{M+n}{n} x^M = \frac{1}{(1-x)^{n+1}}, \quad |x| < 1$$

Set $x = a$:

$$\sum_{M=0}^{\infty} \binom{M+n}{n} a^M = \frac{1}{(1-a)^{n+1}}$$

Step 3: Multiply by a^n

$$a^n \sum_{M=0}^{\infty} \binom{M+n}{n} a^M = a^n \frac{1}{(1-a)^{n+1}}$$

Final Step: Write the Full Identity

$$\sum_{N=n}^{\infty} \binom{N}{n} a^N = \frac{a^n}{(1-a)^{n+1}}$$

Summary

This is a binomial sum identity, closely related to the generating function for the sequence $\binom{N+n}{n}$. The proof makes use of index shifting and properties of generating functions.

If you want more context or a different proof (e.g., combinatorial meaning or induction), just let me know!