

# Unpublished technical notes for “A new iterated filtering algorithm”

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These notes describe some unpublished material, such as extra steps in proofs and discussion points that didn't get placed into the paper or supplement. They are relatively unpolished and are not part of the peer-reviewed materials.

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## T1 A comment on Equation [4] in the statement of Theorem 1

Our theory does not address how the bound on the Monte Carlo error in equation [4] depends on  $\sigma$ . The constant  $C$  in [4] could grow quickly as  $\sigma$  becomes small. However, this does not appear to result in numerical problems for the methodology. This may be because a weaker result than [4] would be adequate to explain the numerical performance of the algorithm, replacing

$$\mathbb{E} \left\{ \left| \frac{1}{J} \sum_{j=1}^J \phi(\Theta_j^M) - \int \phi(\theta) f_\sigma(\theta) d\theta \right| \right\} \leq \frac{\sup_\theta |\phi(\theta)|}{C \sqrt{J}}. \quad (\text{T1})$$

by

$$\left| \mathbb{E} \left\{ \frac{1}{J} \sum_{j=1}^J \phi(\Theta_j^M) - \int \phi(\theta) f_\sigma(\theta) d\theta \right\} \right| \leq \frac{\sup_\theta |\phi(\theta)|}{C \sqrt{J}}. \quad (\text{T2})$$

Whereas (T1) requires the empirical distribution of all  $J$  particles to approximate  $f_\sigma$  at a fixed time  $M$ , (T2) requires a single particle to approximate a draw from  $f_\sigma$  (e.g., equations 3.3.14 and 3.3.15 of Del Moral and Jacod, 2001). A bound such as (T2) is sufficient to study the performance of IF2, since (i) it implies that the particles have a high probability of being in a region in which  $f_\sigma$  concentrates, which is our immediate goal; (ii) assuming mixing, it allows us to use a long iterated filtering sequence to give a good approximation to the full distribution  $f_\sigma$ , even if the approximation based on any single filtering iteration is poor. One might expect that (T2) performs much better than (T1) as sigma becomes small.

We note that the SMC convergence result in Theorem 1 plays no role in our Theorem 2, and practical convergence of the algorithm is assessed empirically rather than relying on theoretical results. Further, theoretical results such as (T1) are more commonly considered in the literature than (T2). Finally, since we have proved the stronger result (T1), proper clarification of why the weaker result (T2) may be preferable would require some delicate reasoning that is outside the scope of this paper.

## T2 Checking $e_1 < e_2$ in the proof of Theorem 2

Write  $D = \lambda_0 - \lambda_2$ . We compute

$$\begin{aligned} e_1 &\leq (1 - \epsilon_1) \left[ \log(\lambda_0) + \frac{\epsilon_2}{\lambda_0} \right] + \epsilon_1 \left[ \log(\lambda_0) - \frac{D}{\lambda_0} + \frac{\epsilon_2}{\lambda_0} \right] \\ &= \log(\lambda_0) - \frac{(\epsilon_1 D - \epsilon_2)}{\lambda_0}. \end{aligned} \quad (\text{T3})$$

As long as  $\eta_2 < \lambda_3/2$ , we have

$$e_2 > (1 - \eta_1) \log(\lambda_1 - \eta_2) + \eta_1 \log(\lambda_3/2) \quad (\text{T4})$$

Pick  $\epsilon_2 < D\epsilon_1$ , so that  $e_1 < \log(\lambda_0)$  from (T3). This, together with the continuity of the log function, allows us to pick  $\lambda_1 < \lambda_0$  and  $\eta_2 > 0$  such that

$$\log(\lambda_1 - \eta_2) > e_1. \quad (\text{T5})$$

Therefore, the continuity of the right hand side of (T4) as a function of  $\eta_1$  implies the existence of an  $\eta_1 > 0$  such that  $e_2 > e_1$ .

### T3 Checking the derivation of Equation [7] in the proof of Theorem 2

To help readers validate the proof of Theorem 2, here are some extra steps in the bound on  $\mathbb{P}_{\check{Z}}[F_1]$  in the argument leading to equation [7], considering only the numerator since the denominator follows a similar argument.

$$\begin{aligned}
\mathbb{E}_{\check{\Theta}}[\check{\ell}_{1:M} 1_{F_1}] &= \mathbb{E}_{\check{\Theta}} \left[ \prod_{m=1}^M \check{\ell}(\theta_{0:N}^m) 1_{F_1} \right] \\
&\leq \mathbb{E}_{\check{\Theta}} \left[ \prod_{m=1}^M \{ \ell(\theta_N^m) + C_2 \sigma \} 1_{F_1} \right] \\
&= \mathbb{E}_{\check{\Theta}} \left[ \prod_{m: \ell(\theta_N^m) \leq \lambda_2} (\lambda_2 + C_2 \sigma) \prod_{m: \ell(\theta_N^m) > \lambda_2} (\lambda_0 + C_2 \sigma) 1_{F_1} \right] \\
&= \mathbb{E}_{\check{\Theta}} \left[ \exp \left\{ \sum_{m: \ell(\theta_N^m) \leq \lambda_2} \log(\lambda_2 + C_2 \sigma) + \sum_{m: \ell(\theta_N^m) > \lambda_2} \log(\lambda_0 + C_2 \sigma) \right\} 1_{F_1} \right] \\
&\leq \mathbb{E}_{\check{\Theta}} \left[ \exp \{ M \epsilon_1 \log(\lambda_2 + C_2 \sigma) + M(1 - \epsilon_1) \log(\lambda_0 + C_2 \sigma) \} 1_{F_1} \right] \\
&\leq \mathbb{E}_{\check{\Theta}} [\exp \{ M e_1 \} 1_{F_1}] \\
&= \exp \{ M e_1 \} \mathbb{P}_{\check{\Theta}}[F_1].
\end{aligned}$$

### References

Del Moral, P. and Jacod, J. (2001). Interacting particle filtering with discrete observations. In Doucet, A., de Freitas, N., and Gordon, N. J., editors, *Sequential Monte Carlo Methods in Practice*, pages 43–75. Springer, New York.