

Island filters for inference on metapopulation dynamics

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Background

The curse of dimensionality. Particle filter (PF) methods are effective for inference on low-dimensional nonlinear partially observed stochastic dynamic systems. They scale exponentially badly.

Independent island filters. We study three algorithms that combine many non-interacting Monte Carlo processes.

- Basic island filter (BIF)
- Adapted simulation island filter (ASIF)
- Adapted simulation island filter with intermediate resampling (ASIF-IR)

Metapopulation dynamics. Island filters have theoretical and empirical scaling properties suited to collections of weakly coupled populations.

What is a SpatPOMP?

POMP models are partially observed Markov processes, also known as state space models or hidden Markov models

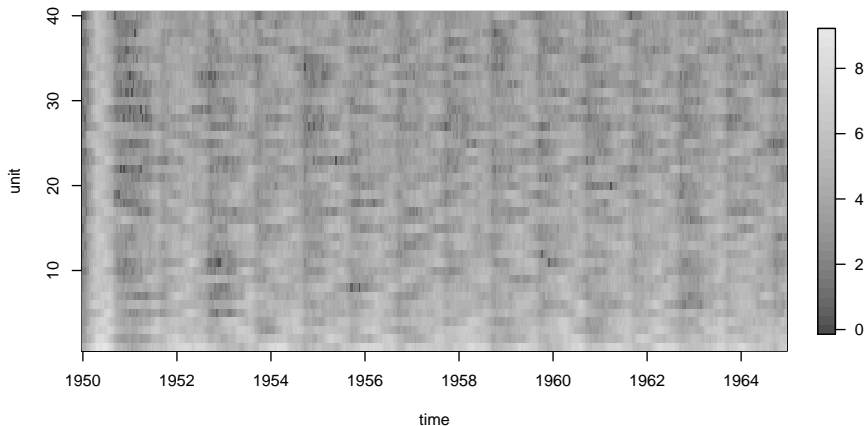
SpatPOMP models are POMP models with a unit structure

Latent Markov process: $X_{u,n} = X_u(t_n)$, $u \in 1:U$, $n \in 1:N$

Observation process: $Y_{u,n}$ depends only on $X_{u,n}$

The units could be a metapopulation, say cities in an epidemic model

$U = 40$ units for a coupled measles SEIR model



Simulated data using a gravity model with geography, demography and transmission parameters corresponding to UK pre-vaccination measles.

Island filter inputs, outputs and implicit loops.

input:

simulator for $f_{\mathbf{X}_0}(\mathbf{x}_0)$ and $f_{\mathbf{X}_n|\mathbf{X}_{n-1}}(\mathbf{x}_n | \mathbf{x}_{n-1})$

evaluator for $f_{Y_{u,n}|X_{u,n}}(y_{u,n} | x_{u,n})$

number of islands, \mathcal{I}

neighborhood structure, $B_{u,n}$

data, $\mathbf{y}_{1:N}^*$

ASIF and ASIF-IR: particles per island, J

ASIF-IR: number of intermediate timesteps, S

ASIF-IR: measurement variance parameterizations, $\overleftarrow{\mathbf{v}}_{u,n}$ and $\overrightarrow{\mathbf{v}}_{u,n}$

ASIF-IR: approximate process and observation mean functions, $\boldsymbol{\mu}$ and $h_{u,n}$

output:

Log likelihood estimate, $\ell^{\text{MC}} = \sum_{n=1}^N \sum_{u=1}^U \ell_{u,n}^{\text{MC}}$

implicit loops:

u in $1:U$, n in $1:N$, i in $1:\mathcal{I}$, j in $1:J$

BIF. Basic island filter.

Simulate $\mathbf{X}_{0:N,i} \sim f_{\mathbf{X}_{0:N}}(\mathbf{x}_{0:N})$

Measurement weights, $w_{u,n,i}^M = f_{Y_{u,n}|X_{u,n}}(y_{u,n}^* | X_{u,n,i})$

Prediction weights, $w_{u,n,i}^P = \prod_{(\tilde{u}, \tilde{n}) \in B_{u,n}} w_{\tilde{u}, \tilde{n}, i}^M$

$$\ell_{u,n}^{\text{MC}} = \log \left(\sum_{i=1}^{\mathcal{I}} w_{u,n,i}^M w_{u,n,i}^P \right) - \log \left(\sum_{i=1}^{\mathcal{I}} w_{u,n,i}^P \right)$$

The basic island filter is not as naive as it may first appear

- BIF seems naive. Particle filter (PF) methods are well known to scale better with N than unconditional simulations.
- BIF scales well with U for weakly coupled systems.
- With modern computers, large numbers of simulations are feasible even when U and N are not small.
- Initially we studied BIF as a theoretical toy. Then we found it is competitive in practice on some models of interest.

Adapted simulation: An easier problem than filtering

- We aim to make each island track the data in a weak sense that does not involve a solution to the full filtering problem.
- The adapted simulation problem is to draw from $f_{\mathbf{X}_n | \mathbf{Y}_n, \mathbf{X}_{n-1}}(\mathbf{x}_n | \mathbf{y}_n^*, \mathbf{x}_{n-1})$.
- The adapted simulation island filter (ASIF) algorithm uses importance sampling to carry out adapted simulation on each island.
- ASIF calculates the likelihood using the proper weight restricted to a neighborhood.

ASIF. Adapted simulation island filter.

Initialize adapted simulation: $\mathbf{X}_{0,i}^A \sim f_{\mathbf{X}_0}(\mathbf{x}_0)$

For n in $1:N$

Proposals: $\mathbf{X}_{n,i,j}^P \sim f_{\mathbf{X}_n|\mathbf{X}_{n-1}}(\mathbf{x}_n | \mathbf{X}_{n-1,i}^A)$

Measurement weights: $w_{u,n,i,j}^M = f_{Y_{u,n}|X_{u,n}}(y_{u,n}^* | X_{u,n,i,j}^P)$

Adapted resampling weights: $w_{n,i,j}^A = \prod_{u=1}^U w_{u,n,i,j}^M$

Resampling: $\mathbb{P}[r(i) = a] = w_{n,i,a}^A \left(\sum_{k=1}^J w_{n,i,k}^A \right)^{-1}$

$\mathbf{X}_{n,i}^A = \mathbf{X}_{n,i,r(i)}^P$

$$w_{u,n,i,j}^P = \prod_{\tilde{n}=1}^{n-1} \left[\frac{1}{J} \sum_{k=1}^J \prod_{(\tilde{u}, \tilde{n}) \in B_{u,\tilde{n}}^{[\tilde{n}]}} w_{\tilde{u}, \tilde{n}, i, k}^M \right] \prod_{(\tilde{u}, n) \in B_{u,n}^{[n]}} w_{\tilde{u}, n, i, j}^M$$

End for

$$\ell_{u,n}^{\text{MC}} = \log \left(\frac{\sum_{i=1}^{\mathcal{I}} \sum_{j=1}^J w_{u,n,i,j}^M w_{u,n,i,j}^P}{\sum_{i=1}^{\mathcal{I}} \sum_{j=1}^J w_{u,n,i,j}^P} \right)$$

Intermediate resampling

- **Intermediate resampling** splits the time interval between observations into S subintervals.
- Reweighting and/or sampling at each subinterval uses a revised estimate of the anticipated measurement density at the end of the interval called a **guide function**.
- This is applicable to continuous time models.
- Intermediate resampling has useful theoretical and empirical properties (Del Moral and Murray, 2015; Park and Ionides, 2019).
- Intermediate resampling for adapted simulation within ASIF gives the ASIF-IR algorithm.
- Intermediate resampling within PF gives the guided intermediate resampling filter (GIRF) of Park and Ionides (2019), a generalization of the auxiliary particle filter of Pitt and Shepard (1999).

ASIF-IR. ASIF with intermediate resampling.

Initialize adapted simulation: $X_{0,i}^A \sim f_{X_0}(x_0)$

For n in $1:N$

Guide simulations: $X_{n,i,j}^G \sim f_{X_n|X_{n-1}}(x_n | X_{n-1,i}^A)$

Guide variance: $V_{u,n,i} = \text{Var}\{h_{u,n}(X_{u,n,i,j}^G), j \text{ in } 1:J\}$

$g_{n,0,i,j}^R = 1$ and $X_{n,0,i,j}^{\text{IR}} = X_{n-1,i}^A$

For s in $1:S$

Intermediate proposals: $X_{n,s,i,j}^{\text{IP}} \sim f_{X_{n,s}|X_{n,s-1}}(\cdot | X_{n,s-1,i,j}^{\text{IR}})$

$\mu_{n,s,i,j}^{\text{IP}} = \mu(X_{n,s,i,j}^{\text{IP}}, t_{n,s}, t_n)$

$V_{u,n,s,i,j}^{\text{meas}} = \vec{v}_u(\theta, \mu_{u,n,s,i,j}^{\text{IP}})$, $V_{u,n,s,i}^{\text{proc}} = V_{u,n,i}(t_n - t_{n,s}) / (t_n - t_{n,0})$

$\theta_{u,n,s,i,j} = \overleftarrow{v}_u(V_{u,n,s,i,j}^{\text{meas}} + V_{u,n,s,i}^{\text{proc}}, \mu_{u,n,s,i,j}^{\text{IP}})$

$g_{n,s,i,j} = \prod_{u=1}^U f_{Y_{u,n}|X_{u,n}}(y_{u,n}^* | \mu_{u,n,s,i,j}^{\text{IP}}; \theta_{u,n,s,i,j})$

Guide weights: $w_{n,s,i,j}^G = g_{n,s,i,j} / g_{n,s-1,i,j}^R$

Resampling: $\mathbb{P}[r(i,j) = a] = w_{n,s,i,a}^G \left(\sum_{k=1}^J w_{n,s,i,k}^G \right)^{-1}$

$X_{n,s,i,j}^{\text{IR}} = X_{n,s,i,r(i,j)}^{\text{IP}}$ and $g_{n,s,i,j}^R = g_{n,s,i,r(i,j)}$

End For

Set $X_{n,i}^A = X_{n,S,i,1}^{\text{IR}}$

Measurement weights: $w_{u,n,i,j}^M = f_{Y_{u,n}|X_{u,n}}(y_{u,n}^* | X_{u,n,i,j}^G)$

$$w_{u,n,i,j}^P = \prod_{\tilde{n}=1}^{n-1} \left[\frac{1}{J} \sum_{a=1}^J \prod_{(\tilde{u}, \tilde{n}) \in B_{u,\tilde{n}}^{[\tilde{n}]}} w_{\tilde{u},\tilde{n},i,a}^M \right] \prod_{(\tilde{u}, \tilde{n}) \in B_{u,n}^{[n]}} w_{\tilde{u},\tilde{n},i,j}^M$$

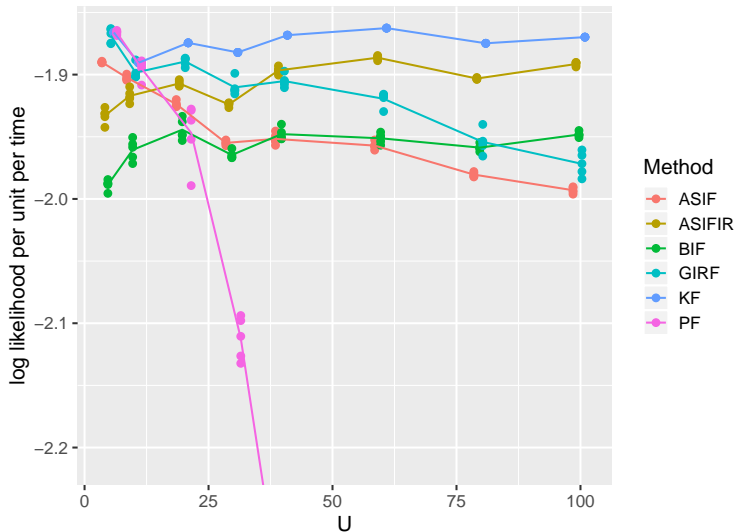
End for

$$\ell_{u,n}^{\text{MC}} = \log \left(\frac{\sum_{i=1}^{\mathcal{I}} \sum_{j=1}^J w_{u,n,i,j}^M w_{u,n,i,j}^P}{\sum_{i=1}^{\mathcal{I}} \sum_{j=1}^J w_{u,n,i,j}^P} \right)$$

Software for SpatPOMP models

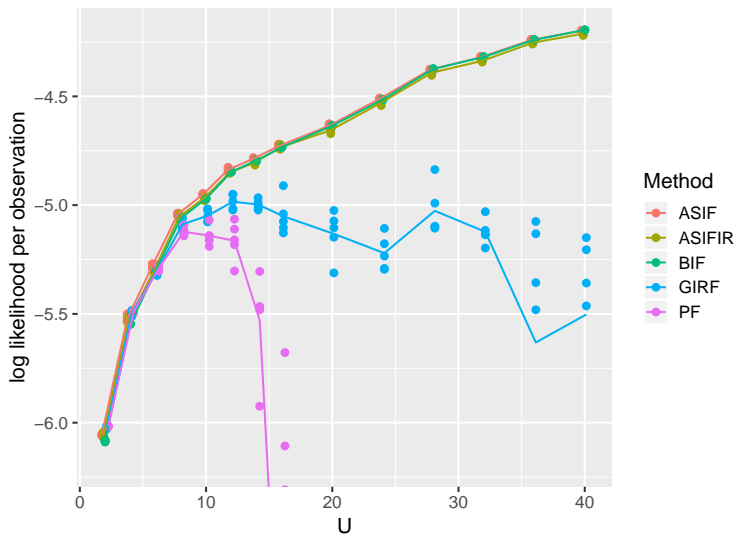
- We use the `asif`, `asifir` and `girf` implementations in the R package `spatPomp` (Asfaw et al., 2019).
- `spatPomp` offers a class `'spatPomp'` that extends the `'pomp'` class for POMP models in the R package `pomp` (King et al., 2016).
- All methods available in `pomp` can formally be applied to `'spatPomp'` objects, though they may not be practically effective for spatiotemporal POMP_s.

Filtering U -dimensional correlated Brownian motion



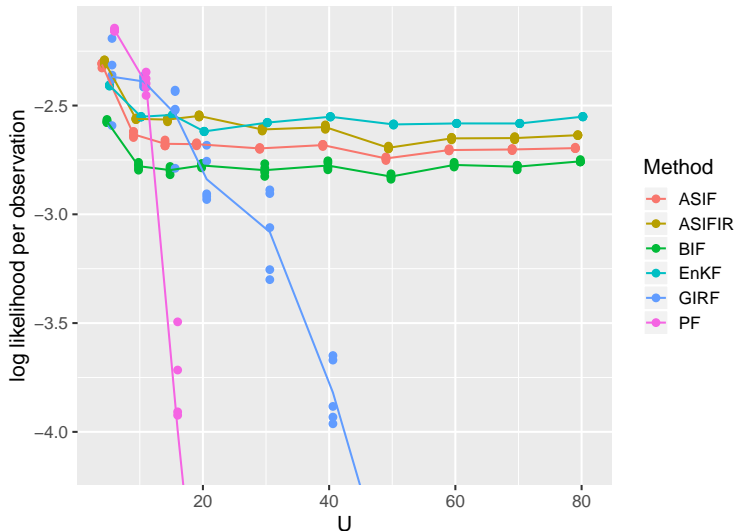
$$\text{Cov}(X_{u,n} - X_{u,n-1}, X_{\tilde{u},n} - X_{\tilde{u},n-1}) \sim 0.4^{|u-\tilde{u}|}$$

Filtering U units of a coupled measles SEIR model



Simulated data using a gravity model with geography, demography and transmission parameters corresponding to UK pre-vaccination measles.

Filtering U units of Lorenz 96 toy atmospheric model

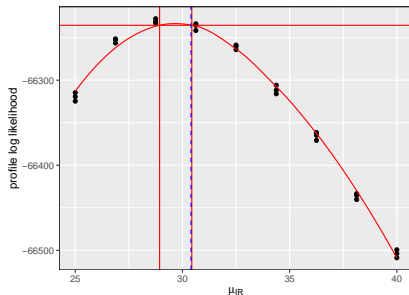
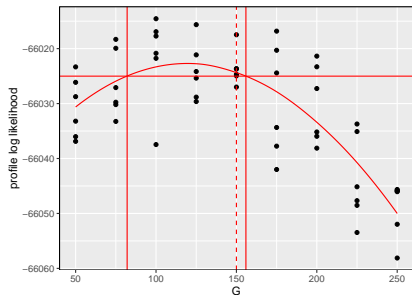


$$dX_u(t) = \{X_{u-1}(t)(X_{u+1}(t) - X_{u-2}(t)) - X_u(t) + F\}dt + \sigma dB_u(t)$$

From filtering to parameter inference

- Log likelihood evaluation in principle enables likelihood-based or Bayesian inference.
- Iterated filtering maximizes the likelihood for PF or GIRF (Ionides et al., 2015).
- Particle Markov chain Monte Carlo can be applied with any likelihood estimate (Andrieu et al., 2010). It is numerically intractable when Monte Carlo estimates are costly and noisy.
- Extending iterated filtering to island filters is future work.

Measles likelihood slices for G and μ_{IR} via ASIF



- Simulating 15 year of data from $U = 40$ cities for the measles model.
- The gravitational coupling constant G is fairly weakly identified: a week of computing on a 30 core machine gives Monte Carlo error on the same scale as the statistical uncertainty.
- The recovery rate μ_{IR} is well identified.

Theorem

Let ℓ^{MC} denote the Monte Carlo likelihood approximation constructed by BIF, ASIF or ASIF-IR. Consider a limit with a growing number of islands, $\mathcal{I} \rightarrow \infty$. Suppose regularity assumptions listed in the arXiv preprint. There are quantities $\epsilon(U, N) = O(1)$ and $V(U, N) = O(U^2 N^2)$ such that

$$\mathcal{I}^{1/2} [\ell^{\text{MC}} - \ell - \epsilon U N] \xrightarrow[\mathcal{I} \rightarrow \infty]{d} \mathcal{N}[0, V],$$

where $\xrightarrow[\mathcal{I} \rightarrow \infty]{d}$ denotes convergence in distribution and $\mathcal{N}[\mu, \Sigma]$ is the normal distribution with mean μ and variance Σ . If an additional spatiotemporal mixing assumption holds, we obtain an improved variance bound

$$V(U, N) = O(UN)$$

References I

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