# 7. Random Walks

## The Duality Principle for Random Walks

- For $S_n = \sum_{i=1}^{n} X_i$ with $X_1, X_2, \ldots$ iid, we note that $(X_1, \ldots, X_n)$ has the same joint distribution as $(X_n, \ldots, X_1)$
- This obvious property has surprising consequences!

Example 1. Show that

$$\mathbb{P}\big[\text{random walk doesn't exceed 0 by time } n\big]$$
$$= \mathbb{P}\big[\text{random walk hits a new low at time } n\big].$$

- Now, notice that the times at which a random walk hits a new low are arrival times for a renewal process (possibly a defective renewal process, with positive probability of infinite arrival times). Why?

Example 2. Use Example 1 to show that, for a random walk with **positive drift** (i.e., $\mathbb{E}[X_1] > 0$) $N = \min\{n : S_n > 0\}$ has $\mathbb{E}[N] < \infty$.

Example 3. Let $S_n$ be a random walk on the integers (i.e., $X_1$ takes integer values). Show that

$$\mathbb{P}\big[S_n = k,\ \text{no return to zero before time } n\big]$$
$$= \mathbb{P}\big[\text{random walk first hits } k \text{ at time } n\big].$$

- Duality is related to time reversal. Sample paths of the dual process can be obtained by:
(i) Look backwards in time, starting at time $n$.
(ii) Shift the trajectory so its initial value is $0$.
(iii) Reflect the trajectory about the $x$-axis.

For Example 1

For Example 3

Example 4: Use Example 3 to show that

$\mathbb{E}\big[\#$ of visits to $k$ before returning to $0\big]$

$= \mathbb{P}\big[$random walk ever hits $k\big]$

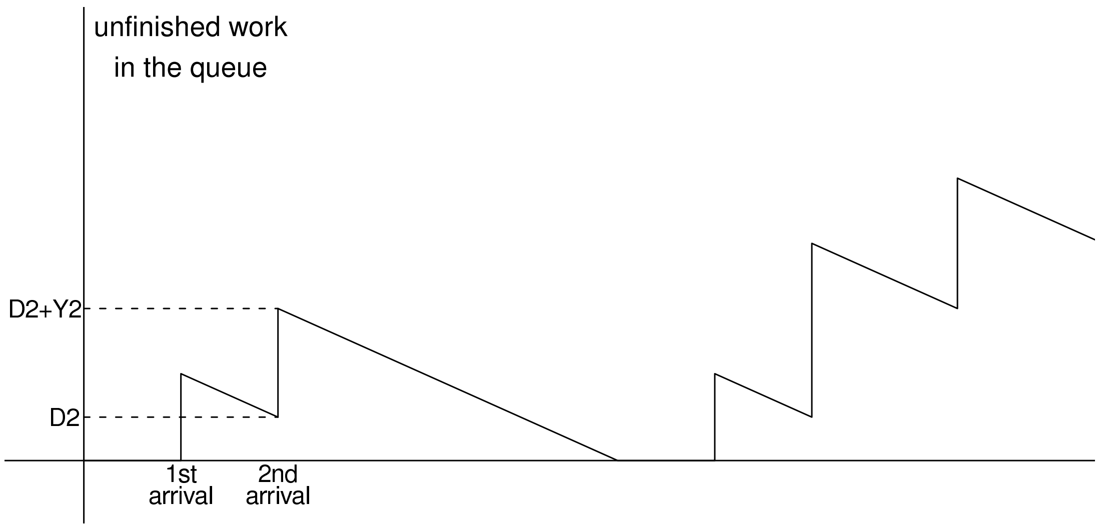
($= 1$ for a recurrent random walk).

# Duality of Ruin and G/G/1 Queue Models

- Suppose claims arrive at an insurance company as a renewal process $N(t)$ and claims $Y_1, Y_2, \ldots$ are iid $F_Y$. In the absence of claims, the insurance company receives income from premiums at rate $c$ per unit time. What is the chance that the company will eventually go bankrupt?

- In a G/G/1 queue, arrivals occur as a renewal process $N(t)$ and have iid service times $\{Y_i, i = 1, 2, \ldots\}$. Let $D_n$ be the time that the $n^{th}$ customer must wait for service (the **delay**). What is the distribution of the limiting delay, $D_\infty$?

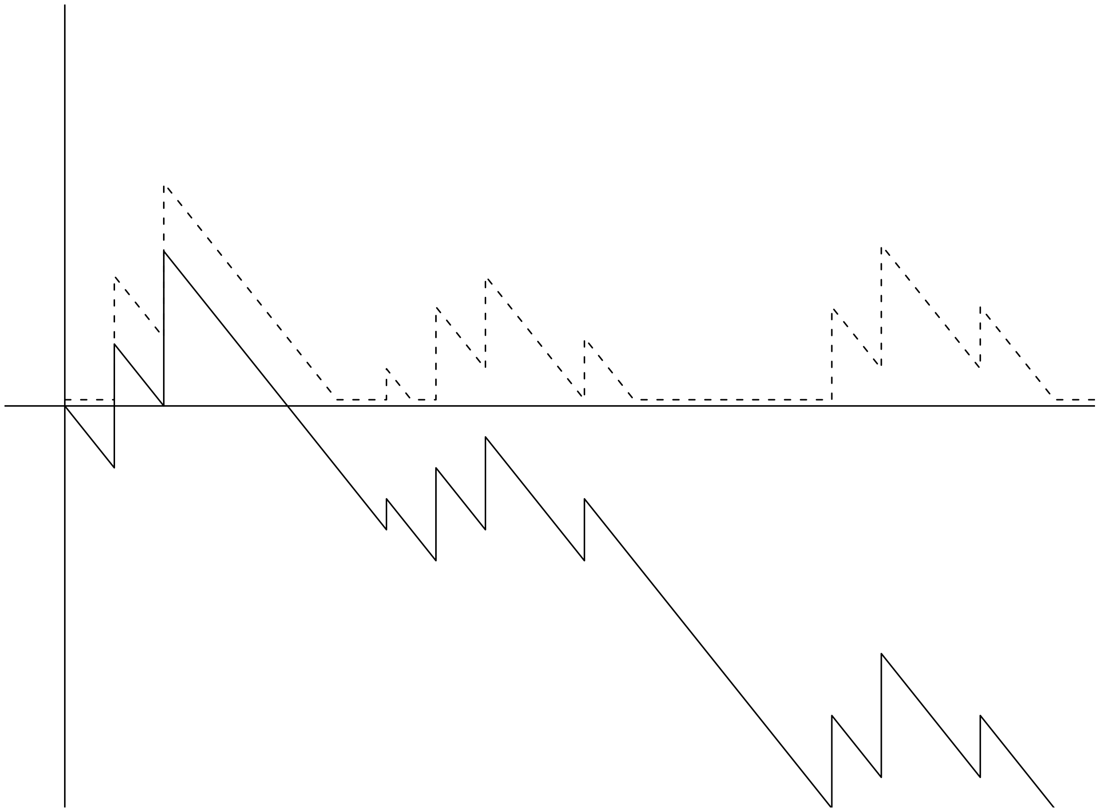
- To see the relationship between the queue and ruin model, we first consider the amount of unfinished work in the queue at time $t$:



- For the queue, let $U_n = Y_n - X_{n+1}$, the total amount of work added to the queue at the $n^{th}$ arrival which is still undone by the time of the $n+1^{th}$ arrival.

- From the diagram, we see that $D_{n+1} = \max\{0, D_n + U_n\}$.
  Now iterate:

# Another way to see this duality for queues



- Let $S_n = \sum_{i=1}^{n} U_i$ for $U_i = Y_i - X_{i+1}$.
- Notice that new minima of $S_n$ correspond to times when the queue is empty, thus $D_n = S_n - S_{[n]}$ where $[n]$ is the (random) time such that $[n] \leq n$ and $S_{[n]} \leq S_j$ for $j = 1, \ldots, n$.

- Now use duality to show that $S_n - S_{[n]}$ has the same distribution as $\max(0, S_1, \dots, S_n)$.

- This duality allows the waiting time for service in a queue to be addressed by martingale methods:

Example: Find $\mathbb{P}\big[D_\infty > A\big]$

- Note that this is the same as the probability of eventual bankruptcy in the ruin model.