Claim: $\lim_{m\to\infty} \mathbb{P}\left\{\sup_{k>0}|Z_{m+k} - Z_m| > \epsilon\right\} = 0$ implies that $\{Z_m\}$ is almost surely Cauchy, i.e.

$\lim\limits_{m\to\infty} \sup\limits_{k>0} |Z_{m+k} - Z_m| = 0$ with probability 1.

Proof: Let $G_m(\varepsilon) = \{\sup_{k>0}|Z_{m+k} - Z_m| > \varepsilon\}$ and $G(\varepsilon) = \bigcap_{m>0} G_m(\varepsilon)$

$$= \{\lim_{m\to\infty} \sup_{k>0}|Z_{m+k} - Z_m| > \varepsilon\}.$$

Since $G_m(\varepsilon)$ is a decreasing sequence of events for any $\varepsilon > 0$,

$$\mathbb{P}[G(\varepsilon)] = \mathbb{P}[\lim_{m\to\infty} G_m(\varepsilon)] = \lim_{m\to\infty} \mathbb{P}[G_m(\varepsilon)] = 0.$$

Now, let $H_n = \left\{G(\frac{1}{n})\right\}^c$. Then H_n is decreasing, and

$$H = \bigcap_{n>0} H_n = \left\{\lim_{m\to\infty} \sup_{k>0} |Z_{m+k} - Z_m| = 0\right\}.$$

It follows that

$$\mathbb{P}(H) = \mathbb{P}[\lim_{n\to\infty} H_n] = \lim_{n\to\infty} \mathbb{P}[H_n] = 1.$$