Sample Midterm 2

Math/Stats 425 (Instructor: Edward Ionides)

1. Suppose X has probability mass function $p(k) = k/10$ for $k = 1, 2, 3, 4$. Plot the probability mass function of $X^2 - 4X + 3$.

   Solution: Set $Y = X^2 - 4X + 3$, and let Y have p.m.f. $p_Y(y) = \mathbb{P}(Y = y)$. Then, possible values of Y are 0, -1, 3.

   $p_Y(0) = \mathbb{P}(Y = 0) = \mathbb{P}(X = 1 \text{ or } X = 3) = 1/10 + 3/10 = 2/5$.

   $p_Y(-1) = \mathbb{P}(X = 2) = 1/5$, $p_Y(3) = \mathbb{P}(X = 4) = 2/5$.

   You are asked to sketch this function.

2. There are 4 balls in a box, numbered $1, 2, 3$ and 4. I invite you to draw two balls, at random without replacement. I will then pay you an amount in dollars equal to the sum of the numbers on the two balls you picked. Calculate the expected value and variance of your winnings from playing. What would be the fair price you should pay to play? Explain.

   Solution: Let W be the winnings, with possible values 3,4,5,6,7. There are six ways to choose 2 balls from 4, with equally likely outcomes. So,

   $\mathbb{E}[W] = \frac{1}{6} \times 3 + \frac{1}{6} \times 4 + \frac{2}{6} \times 5 + \frac{1}{6} \times 6 + \frac{1}{6} \times 7 = 5$.

   $\mathbb{E}[W^2] = \frac{1}{6} \times 9 + \frac{1}{6} \times 16 + \frac{2}{6} \times 25 + \frac{1}{6} \times 36 + \frac{1}{6} \times 49 = 160/6$.

   $\text{Var}(W) = \mathbb{E}[W^2] - [\mathbb{E}(W)]^2 = 160/6 - 150/6 = 1.67$.

   The 'fair price' for playing is the long run average winnings, which is the expected value, i.e. 5 dollars.

3. Use a Poisson approximation to write an expression approximating the chance that in a university of $30,000$ students there are exactly k students who share the same birthday as both his/her mother and father. Explain your reasoning.

   Solution: Let X be the number of students sharing a birthday with both their mother and father. Suppose that each birthday is drawn independently and is equally likely to take any value in $1, 2, \ldots, 365$. Then, the probability that mother and father share the kid's birthday is $(1/365)^2$. The mean number of such occurrences among 30000 students is $\lambda = 30000/365^2$ (≈ 0.2). Making a Poisson approximation, X is approximately Poission(λ) and so

   $$\mathbb{P}(X = k) \approx \frac{\lambda^k}{k!} e^{-\lambda}.$$

4. If it is raining, students come to class on Friday independently with probability 0.8. If it is not raining this probability increases to 0.9. The chance of rain this Friday is 0.3. Find an expression for the chance that at least 19 students will show up to class on Friday out of a class of 20 students.

   Solution: let X be the number of students showing on Friday, and let R be the event that it rains on Friday. By the law of total probability,

   $$\mathbb{P}(X \geq 19) = \mathbb{P}(X \geq 19 \,|\, R)\mathbb{P}(R) + \mathbb{P}(X \geq 19 \,|\, R^c)\mathbb{P}(R^c).$$

Thus,

$$\mathbb{P}(X \geq 19) = 0.3 \left\{ 0.8^{20} + \binom{20}{1} 0.8^{19} \times 0.2 \right\} + 0.7 \left\{ 0.9^{20} + \binom{20}{1} 0.9^{19} \times 0.1 \right\}.$$

5. Suppose X has Geometric (1/3) distribution (i.e., X has probability mass function $p(k) = (1/3)(2/3)^{k-1}$, for $k = 1, 2, \ldots$). Find the expected value of e^{-X}. Explain your reasoning.

   Solution: Using the formula $\mathbb{E}[g(X)] = \sum_k g(k)p(k)$,

$$\begin{aligned} \mathbb{E}[e^{-X}] &= \sum_{k=1}^{\infty} e^{-k}(1/3)(2/3)^{k-1} \\ &= \frac{1}{3e} \sum_{k=0}^{\infty} (2/3e)^k \\ &= \frac{1}{3e} \frac{1}{[1 - (2/3e)]} = \frac{1}{3e - 2} \end{aligned}$$

6. Suppose X and Y are continuous random variables whose probability density functions and cumulative distribution functions are $f_X(\cdot)$, $f_Y(\cdot)$, $F_X(\cdot)$ and $F_Y(\cdot)$ respectively. Z is a random variable defined as follows: a biased coin is flipped, which lands on heads with probability p. If the coin lands on heads then Z is assigned to take the same value as X, otherwise Z is assigned to take the same value as Y. Find expressions for the probability density function and cumulative distribution function of Z. Explain your reasoning.

   Solution: Let $H = \{\text{coin lands heads}\}$. By the law of total probability,

$$\begin{aligned} F_Z(z) &= \mathbb{P}(Z \leq z) = \mathbb{P}(Z \leq z \mid H)\mathbb{P}(H) + \mathbb{P}(Z \leq z \mid H^c)\mathbb{P}(H^c) \\ &= p\, F_X(z) + (1-p)F_Y(z) \end{aligned}$$

   Differentating both sides with respect to z gives

$$f_Z(z) = p\, f_X(z) + (1-p) f_Y(z).$$