A comment on Definition 3 $\Rightarrow$ Definition 2

We supposed that the memoryless property of the exponential distribution held at a random time, and when conditioning on an additional event. Here we expand the argument.

$$\mathbb{P}\big[N(t+s) \geq n+m \mid N(t)=n\big]$$

$$= \int_0^t \mathbb{P}\big[N(t+s) \geq n+m \mid S_n = u, N(t)=n\big]\, dF_{S_n \mid N(t)=n}(u)$$

$$= \int_0^t \mathbb{P}\Big[\sum_{i=n+1}^{n+m} X_i \leq s+t-u \mid S_n = u, X_{n+1} > t-u\Big]\, dF_{S_n \mid N(t)=n}(u) \tag{1}$$

$$= \int_0^t \mathbb{P}\Big[\sum_{i=n+1}^{n+m} X_i \leq s+t-u \mid X_{n+1} > t-u\Big]\, dF_{S_n \mid N(t)=n}(u) \tag{2}$$

$$= \int_0^t \mathbb{P}\Big[\sum_{i=n+1}^{n+m} X_i \leq s\Big]\, dF_{S_n \mid N(t)=n}(u) \tag{3}$$

$$= \mathbb{P}\Big[\sum_{i=n+1}^{n+m} X_i \leq s\Big].$$

- Equation (2) follows from (1) since $\{S_n = u\}$ is independent of $\{X_{n+1} > t-u\}$ and $\{\sum_{i=n+1}^{n+m} X_i \leq s+t-u\}$.
- Here, we require $\mathbb{P}[A \mid B, C] = \mathbb{P}[A \mid C]$, i.e., conditional independence of A and B given C. This does not follow from the independence of A and B. However, it does follow if B is independent of A, C and $A \cap C$.
- Equation (3) follows from (2) by a fairly immediate application of the memoryless property, since X_{n+1} is independent of $X_{n+2}, \ldots, X_{n+m}$.
- We are supposing that $\{S_n = u\}$ can be treated like an event of positive probability whether or not this is the case. This is another thing to check!