Bagging and blocking: Inference via particle filters for interacting dynamic systems

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Outline

The curse of dimensionality. Particle filter (PF) methods are effective for inference on low-dimensional nonlinear partially observed stochastic dynamic systems. They scale exponentially badly.

Bagged filters. Combining independent Monte Carlo filters.

- Unadapted bagged filter (UBF)
- adapted bagged filter (ABF)
- adapted bagged filter with intermediate resampling (ABF-IR)

Blocked particle filter (BPF). Independently developed by Rebeschini and van Handel (2015) (in theory) and Ng et al. (2002) (in practice)

From filtering to inference. Iterated filtering using stochastically perturbed parameters.

Metapopulation dynamics. Bagged and blocked filters work on collections of weakly coupled populations, in theory and practice.

What is a SpatPOMP?

POMP models are partially observed Markov processes, also known as state space models or hidden Markov models

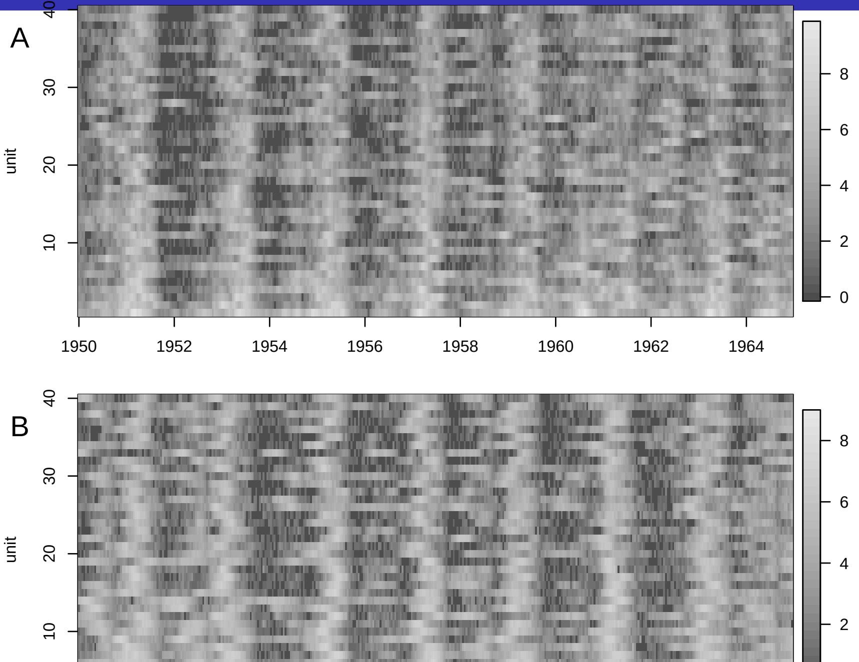
SpatPOMP models are POMP models with a unit structure

Latent Markov process: $X_{u,n} = X_u(t_n), \quad u \in 1\!:\!U, \quad n \in 1\!:\!N$

Observation process: $Y_{u,n}$ depends only on $X_{u,n}$

The units could be a metapopulation, say cities in an epidemic model

$U = 40$ units for a coupled measles SEIR model



Bagged filter inputs, outputs and implicit loops.

input:

simulator for $f_{\boldsymbol{X}_0}(\boldsymbol{x}_0)$ and $f_{\boldsymbol{X}_n|\boldsymbol{X}_{n-1}}(\boldsymbol{x}_n \,|\, \boldsymbol{x}_{n-1})$

evaluator for $f_{Y_{u,n}|X_{u,n}}(y_{u,n} \,|\, x_{u,n})$

number of replicates, $\mathcal{I}$

neighborhood structure, $B_{u,n}$

data, $\boldsymbol{y}^*_{1:N}$

ABF and ABF-IR: particles per replicate, J

ABF-IR: number of intermediate timesteps, S

ABF-IR: measurement variance parameterizations, $\overleftarrow{\mathrm{v}}_{u,n}$ and $\overrightarrow{\mathrm{v}}_{u,n}$

ABF-IR: approximate process and observation mean functions, $\boldsymbol{\mu}$ and $h_{u,n}$

output:

Log likelihood estimate, $\ell^{\mathsf{MC}} = \sum_{n=1}^{N} \sum_{u=1}^{U} \ell^{\mathsf{MC}}_{u,n}$

implicit loops:

u in $1\!:\!U$, n in $1\!:\!N$, i in $1\!:\!\mathcal{I}$, j in $1\!:\!J$

UBF. Unadapted bagged filter.

Simulate $\boldsymbol{X}_{0:N,i} \sim f_{\boldsymbol{X}_{0:N}}(\boldsymbol{x}_{0:N})$

. Measurement weights, $w^{M}_{u,n,i} = f_{Y_{u,n}|X_{u,n}}(y^{*}_{u,n} \mid X_{u,n,i})$

Prediction weights, $w^{P}_{u,n,i} = \prod_{(\tilde{u},\tilde{n})\in B_{u,n}} w^{M}_{\tilde{u},\tilde{n},i}$

$\ell^{\mathsf{MC}}_{u,n} = \log\left(\sum_{i=1}^{\mathcal{I}} w^{M}_{u,n,i} w^{P}_{u,n,i}\right) - \log\left(\sum_{i=1}^{\mathcal{I}} w^{P}_{u,n,i}\right)$

The unadapted bagged filter is not as naive as it may first appear

- UBF seems naive. Particle filter (PF) method are well known to scale better with N than unconditional simulations.

- UBF scales well with U for weakly coupled systems.

- With modern computers, large numbers of simulations are feasible even when U and N are not small.

- Initially we studied UBF as a theoretical toy. Then we found it is competitive in practice on some models of interest.

Adapted simulation: An easier problem than filtering

- We aim to make each replicate track the data in a weak sense that does not involve a solution to the full filtering problem.

- The adapted simulation problem is to draw from $f_{\boldsymbol{X}_n|\boldsymbol{Y}_n,\boldsymbol{X}_{n-1}}\big(\boldsymbol{x}_n \,|\, \boldsymbol{y}^*{}_n, \boldsymbol{x}_{n-1}\big)$.

- The adapted bagged filter (ABF) algorithm uses importance sampling to carry out adapted simulation on each replicate.

- ABF calculates the likelihood using the proper weight restricted to a neighborhood.

ABF. Adapted bagged filter.

Initialize adapted simulation: $\boldsymbol{X}^{\mathrm{A}}_{0,i} \sim f_{\boldsymbol{X}_0}(\boldsymbol{x}_0)$

For n in $1\!:\!N$

Proposals: $\boldsymbol{X}^{\mathrm{P}}_{n,i,j} \sim f_{\boldsymbol{X}_n|\boldsymbol{X}_{n-1}}\big(\boldsymbol{x}_n \,|\, \boldsymbol{X}^{\mathrm{A}}_{n-1,i}\big)$

Measurement weights: $w^{M}_{u,n,i,j} = f_{Y_{u,n}|X_{u,n}}\big(y^*_{u,n} \,|\, X^{\mathrm{P}}_{u,n,i,j}\big)$

Adapted resampling weights: $w^{\mathrm{A}}_{n,i,j} = \prod_{u=1}^{U} w^{M}_{u,n,i,j}$

Resampling: $\mathbb{P}\big[r(i)=a\big] = w^{\mathrm{A}}_{n,i,a}\Big(\sum_{k=1}^{J} w^{\mathrm{A}}_{n,i,k}\Big)^{-1}$

$\boldsymbol{X}^{\mathrm{A}}_{n,i} = \boldsymbol{X}^{\mathrm{P}}_{n,i,r(i)}$

$$w^{\mathrm{P}}_{u,n,i,j} = \prod_{\tilde{n}=1}^{n-1}\Big[\frac{1}{J}\sum_{k=1}^{J}\prod_{(\tilde{u},\tilde{n})\in B^{[\tilde{n}]}_{u,n}} w^{M}_{\tilde{u},\tilde{n},i,k}\Big]\prod_{(\tilde{u},n)\in B^{[n]}_{u,n}} w^{M}_{\tilde{u},n,i,j}$$

End for

$$\ell^{\mathsf{MC}}_{u,n} = \log\left(\frac{\sum_{i=1}^{\mathcal{I}}\sum_{j=1}^{J} w^{M}_{u,n,i,j} w^{P}_{u,n,i,j}}{\sum_{i=1}^{\mathcal{I}}\sum_{j=1}^{J} w^{P}_{u,n,i,j}}\right)$$

Intermediate resampling

- **Intermediate resampling** splits the time interval between observations into S subintervals.
- Reweighting and/or sampling at each subinterval uses a revised estimate of the anticipated measurement density at the end of the interval called a **guide function**.
- This is applicable to continuous time models.
- Intermediate resampling has useful theoretical and empirical properties (Del Moral and Murray, 2015; Park and Ionides, 2019).
- Intermediate resampling for adapted simulation within ABF gives the ABF-IR algorithm.
- Intermediate resampling within PF gives the guided intermediate resampling filter (GIRF) of Park and Ionides (2019), a generalization of the auxiliary particle filter of Pitt and Shepard (1999).

ABF-IR. ABF with intermediate resampling.

Initialize adapted simulation: $\boldsymbol{X}^{\mathrm{A}}_{0,i} \sim f_{\boldsymbol{X}_0}(\boldsymbol{x}_0)$
For n in $1\!:\!N$
  Guide simulations: $\boldsymbol{X}^{G}_{n,i,j} \sim f_{\boldsymbol{X}_n|\boldsymbol{X}_{n-1}}\big(\boldsymbol{x}_n \,|\, \boldsymbol{X}^{\mathrm{A}}_{n-1,i}\big)$
  Guide variance: $V_{u,n,i} = \mathrm{Var}\big\{h_{u,n}\big(X^{G}_{u,n,i,j}\big), j \text{ in } 1\!:\!J\big\}$
  $g^{\mathrm{R}}_{n,0,i,j} = 1$ and $\boldsymbol{X}^{\mathrm{IR}}_{n,0,i,j} = \boldsymbol{X}^{\mathrm{A}}_{n-1,i}$
  For s in $1\!:\!S$
    Intermediate proposals: $\boldsymbol{X}^{\mathrm{IP}}_{n,s,i,j} \sim f_{\boldsymbol{X}_{n,s}|\boldsymbol{X}_{n,s-1}}\big(\,\cdot\,|\boldsymbol{X}^{\mathrm{IR}}_{n,s-1,i,j}\big)$
    $\boldsymbol{\mu}^{\mathrm{IP}}_{n,s,i,j} = \boldsymbol{\mu}\big(\boldsymbol{X}^{\mathrm{IP}}_{n,s,i,j}, t_{n,s}, t_n\big)$
    $V^{\mathrm{meas}}_{u,n,s,i,j} = \overrightarrow{\mathrm{v}}_u(\theta, \mu^{\mathrm{IP}}_{u,n,s,i,j})\ ,\qquad V^{\mathrm{proc}}_{u,n,s,i} = V_{u,n,i}\,\big(t_n - t_{n,s}\big)\Big/\big(t_n - t_{n,0}\big)$
    $\theta_{u,n,s,i,j} = \overleftarrow{\mathrm{v}}_u\big(V^{\mathrm{meas}}_{u,n,s,i,j} + V^{\mathrm{proc}}_{u,n,s,i},\, \mu^{\mathrm{IP}}_{u,n,s,i,j}\big)$
    $g_{n,s,i,j} = \prod_{u=1}^{U} f_{Y_{u,n}|X_{u,n}}\big(y^*_{u,n} \,|\, \mu^{\mathrm{IP}}_{u,n,s,i,j}\,; \theta_{u,n,s,i,j}\big)$
    Guide weights: $w^{G}_{n,s,i,j} = g_{n,s,i,j}\big/g^{\mathrm{R}}_{n,s-1,i,j}$
    Resampling: $\mathbb{P}\big[r(i,j) = a\big] = w^{G}_{n,s,i,a}\Big(\sum_{k=1}^{J} w^{G}_{n,s,i,k}\Big)^{-1}$
    $\boldsymbol{X}^{\mathrm{IR}}_{n,s,i,j} = \boldsymbol{X}^{\mathrm{IP}}_{n,s,i,r(i,j)}$ and $g^{\mathrm{R}}_{n,s,i,j} = g_{n,s,i,r(i,j)}$
  End For
  Set $\boldsymbol{X}^{\mathrm{A}}_{n,i} = \boldsymbol{X}^{\mathrm{IR}}_{n,S,i,1}$
  Measurement weights: $w^{M}_{u,n,i,j} = f_{Y_{u,n}|X_{u,n}}\big(y^*_{u,n} \,|\, X^{G}_{u,n,i,j}\big)$

$$w^{\mathrm{P}}_{u,n,i,j} = \prod_{\tilde{n}=1}^{n-1}\Big[\frac{1}{J}\sum_{a=1}^{J}\prod_{(\tilde{u},\tilde{n})\in B^{[\tilde{n}]}_{u,n}} w^{M}_{\tilde{u},\tilde{n},i,a}\Big]\prod_{(\tilde{u},n)\in B^{[n]}_{u,n}} w^{M}_{\tilde{u},n,i,j}$$

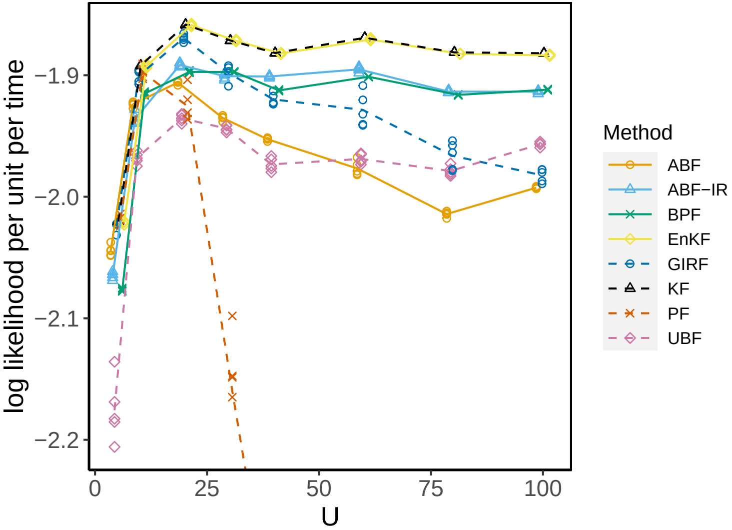
End for

$$\ell^{\mathsf{MC}}_{u,n} = \log\left(\frac{\sum_{i=1}^{\mathcal{I}}\sum_{j=1}^{J} w^{M}_{u,n,i,j} w^{P}_{u,n,i,j}}{\sum_{i=1}^{\mathcal{I}}\sum_{j=1}^{J} w^{P}_{u,n,i,j}}\right)$$

Software for SpatPOMP models

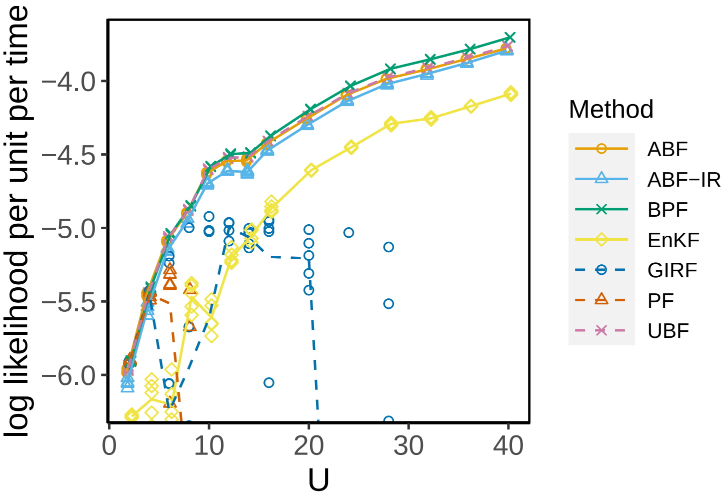
- We use the `asif`, `asifir` and `girf` implementations in the R package `spatPomp` (Asfaw et al., 2019).
- `spatPomp` offers a class '`spatPomp`' that extends the '`pomp`' class for POMP models in the R package `pomp` (King et al., 2016).
- All methods available in `pomp` can formally be applied to '`spatPomp`' objects, though they may not be practically effective for spatiotemporal POMPs.

Filtering U-dimensional correlated Brownian motion



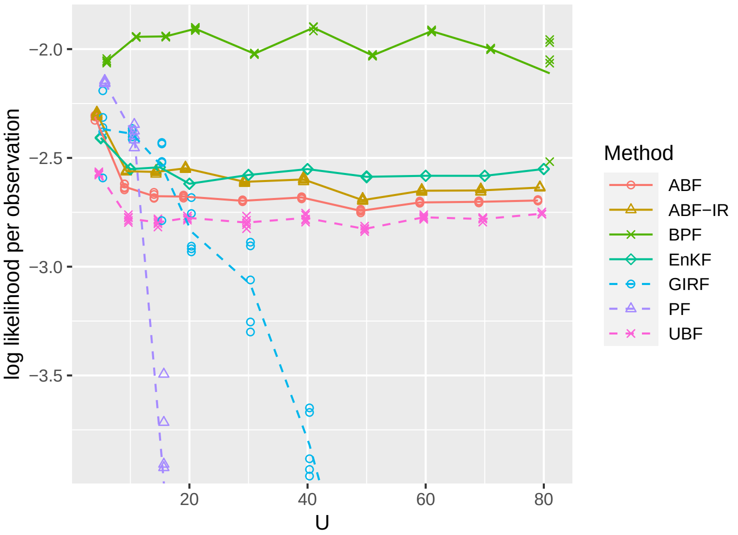
$$\mathrm{Cov}\big(X_{u,n} - X_{u,n-1}, X_{\tilde{u},n} - X_{\tilde{u},n-1}\big) \sim 0.4^{|u-\tilde{u}|}$$

Filtering U units of a coupled measles SEIR model



Simulated data using a gravity model with geography, demography and transmssion parameters corresponding to UK pre-vaccination measles.

Filtering U units of Lorenz 96 toy atmospheric model



$$dX_u(t) = \left\{X_{u-1}(t)\big(X_{u+1}(t) - X_{u-2}(t)\big) - X_u(t) + F\right\}dt + \sigma\, dB_u(t)$$

From filtering to parameter inference

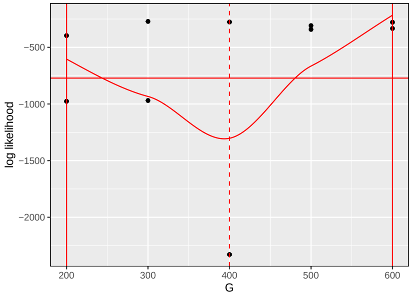
- Log likelihood evaluation in principle enables likelihood-based or Bayesian inference.

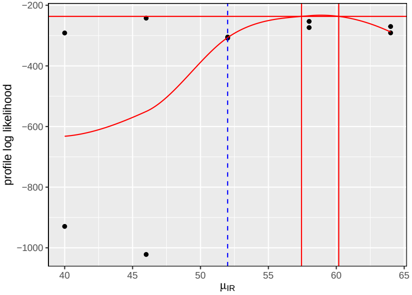
- Iterated filtering maximizes the likelihood for PF or GIRF (Ionides et al., 2015).

- Particle Markov chain Monte Carlo can be applied with any likelihood estimate (Andrieu et al., 2010). It is numerically intractable when Monte Carlo estimates are costly and noisy.

- Extending iterated filtering to bagged filters is future work.

Measles likelihood slices for G and μ_{IR} via ABF





- Simulating 15 year of data from $U = 40$ cities for the measles model.
- The gravitational coupling constant G is fairly weakly identified: a week of computing on a 30 core machine gives Monte Carlo error on the same scale as the statistical uncertainty.
- The recovery rate μ_{IR} is well identified.

Theorem

Let ℓ^{MC} denote the Monte Carlo likelihood approximation constructed by UBF, ABF or ABF-IR. Consider a limit with a growing number of replicates, $\mathcal{I} \to \infty$. Suppose regularity assumptions listed in the paper. There are quantities $\epsilon(U, N) = O(1)$ and $V(U, N) = O(U^2 N^2)$ such that

$$\mathcal{I}^{1/2}\left[\ell^{MC} - \ell - \epsilon U N\right] \xrightarrow[\mathcal{I}\to\infty]{d} \mathcal{N}\left[0, V\right],$$

where $\xrightarrow[\mathcal{I}\to\infty]{d}$ denotes convergence in distribution and $\mathcal{N}[\mu, \Sigma]$ is the normal distribution with mean μ and variance Σ. If an additional spatiotemporal mixing assumption holds, we obtain an improved variance bound

$$V(U, N) = O(UN)$$

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