Bagging and blocking: Inference via particle filters for interacting dynamic systems

Edward Ionides
University of Michigan, Department of Statistics

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Collaborators:
Kidus Asfaw, Ning Ning, Joonha Park and Aaron King

Background

The curse of dimensionality. Particle filter (PF) methods are effective for inference on low-dimensional nonlinear partially observed stochastic dynamic systems. They scale exponentially badly.

Bagged filters. We study three algorithms that combine many non-interacting Monte Carlo processes.

- Unadapted bagged filter (UBF)
- adapted bagged filter (ABF)
- adapted bagged filter with intermediate resampling (ABF-IR)

Metapopulation dynamics. Bagged filters have theoretical and empirical scaling properties suited to collections of weakly coupled populations.

So, what about COVID-19?

- Researchers have developed many models for disease spread.
- Most of these build on the SIR (Susceptible-Infected-Removed) model that divides a population into three homogeneous classes.
- Extensions can include a latent period after infection, age structure, spatial structure, temperature, control policies.
- We may observe some fraction of cases.
- These models are partially observed stochastic dynamic systems.
- Understanding of COVID-19 epidemiology draws on analysis of previous epidemics combined with assessment of limited available data.
- Methods that fit a general class of mechanistic models assist with formulating and testing scientific hypotheses.

What is a SpatPOMP?

POMP models are partially observed Markov processes, also known as state space models or hidden Markov models

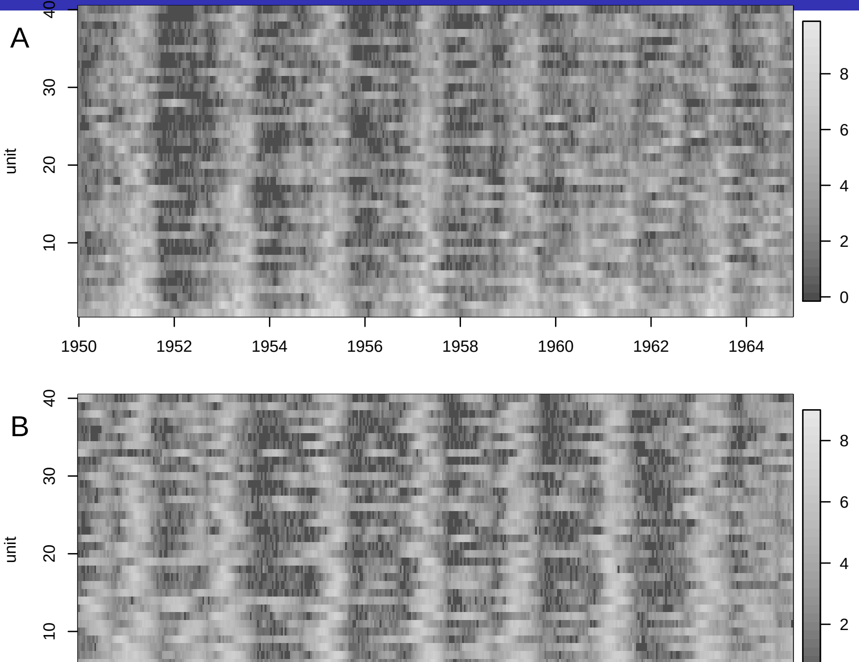
SpatPOMP models are POMP models with a unit structure

Latent Markov process: $X_{u,n} = X_u(t_n), \quad u \in 1\!:\!U, \quad n \in 1\!:\!N$

Observation process: $Y_{u,n}$ depends only on $X_{u,n}$

The units could be a metapopulation, say cities in an epidemic model

$U = 40$ units for a coupled measles SEIR model



Bagged filter inputs, outputs and implicit loops.

input:

simulator for $f_{\boldsymbol{X}_0}(\boldsymbol{x}_0)$ and $f_{\boldsymbol{X}_n|\boldsymbol{X}_{n-1}}(\boldsymbol{x}_n \,|\, \boldsymbol{x}_{n-1})$

evaluator for $f_{Y_{u,n}|X_{u,n}}(y_{u,n} \,|\, x_{u,n})$

number of islands, $\mathcal{I}$

neighborhood structure, $B_{u,n}$

data, $\boldsymbol{y}^*_{1:N}$

ABF and ABF-IR: particles per island, J

ABF-IR: number of intermediate timesteps, S

ABF-IR: measurement variance parameterizations, $\overleftarrow{\mathrm{v}}_{u,n}$ and $\overrightarrow{\mathrm{v}}_{u,n}$

ABF-IR: approximate process and observation mean functions, $\boldsymbol{\mu}$ and $h_{u,n}$

output:

Log likelihood estimate, $\ell^{\,\mathsf{MC}} = \sum_{n=1}^{N} \sum_{u=1}^{U} \ell^{\,\mathsf{MC}}_{u,n}$

implicit loops:

u in $1\!:\!U$, n in $1\!:\!N$, i in $1\!:\!\mathcal{I}$, j in $1\!:\!J$

BIF. Unadapted bagged filter.

Simulate $\boldsymbol{X}_{0:N,i} \sim f_{\boldsymbol{X}_{0:N}}(\boldsymbol{x}_{0:N})$

. Measurement weights, $w^M_{u,n,i} = f_{Y_{u,n}|X_{u,n}}(y^*_{u,n} \,|\, X_{u,n,i})$

Prediction weights, $w^P_{u,n,i} = \prod_{(\tilde{u},\tilde{n}) \in B_{u,n}} w^M_{\tilde{u},\tilde{n},i}$

$\ell^{\mathsf{MC}}_{u,n} = \log\left(\sum_{i=1}^{\mathcal{I}} w^M_{u,n,i} w^P_{u,n,i}\right) - \log\left(\sum_{i=1}^{\mathcal{I}} w^P_{u,n,i}\right)$

The basic island filter is not as naive as it may first appear

- UBF seems naive. Particle filter (PF) method are well known to scale better with N than unconditional simulations.

- UBF scales well with U for weakly coupled systems.

- With modern computers, large numbers of simulations are feasible even when U and N are not small.

- Initially we studied UBF as a theoretical toy. Then we found it is competitive in practice on some models of interest.

Adapted simulation: An easier problem than filtering

- We aim to make each island track the data in a weak sense that does not involve a solution to the full filtering problem.

- The adapted simulation problem is to draw from $f_{\boldsymbol{X}_n|\boldsymbol{Y}_n,\boldsymbol{X}_{n-1}}\big(\boldsymbol{x}_n \,|\, \boldsymbol{y^*}_n, \boldsymbol{x}_{n-1}\big)$.

- The adapted bagged filter (ABF) algorithm uses importance sampling to carry out adapted simulation on each island.

- ABF calculates the likelihood using the proper weight restricted to a neighborhood.

ABF. Adapted bagged filter.

Initialize adapted simulation: $\boldsymbol{X}^{\mathrm{A}}_{0,i} \sim f_{\boldsymbol{X}_0}(\boldsymbol{x}_0)$

For n in $1\!:\!N$

Proposals: $\boldsymbol{X}^{\mathrm{P}}_{n,i,j} \sim f_{\boldsymbol{X}_n|\boldsymbol{X}_{n-1}}\big(\boldsymbol{x}_n \,|\, \boldsymbol{X}^{\mathrm{A}}_{n-1,i}\big)$

Measurement weights: $w^{M}_{u,n,i,j} = f_{Y_{u,n}|X_{u,n}}\big(y^*_{u,n} \,|\, X^{\mathrm{P}}_{u,n,i,j}\big)$

Adapted resampling weights: $w^{\mathrm{A}}_{n,i,j} = \prod_{u=1}^{U} w^{M}_{u,n,i,j}$

Resampling: $\mathbb{P}\big[r(i)=a\big] = w^{\mathrm{A}}_{n,i,a}\Big(\sum_{k=1}^{J} w^{\mathrm{A}}_{n,i,k}\Big)^{-1}$

.

$\boldsymbol{X}^{\mathrm{A}}_{n,i} = \boldsymbol{X}^{\mathrm{P}}_{n,i,r(i)}$

$$w^{\mathrm{P}}_{u,n,i,j} = \prod_{\tilde{n}=1}^{n-1}\Big[\frac{1}{J}\sum_{k=1}^{J}\prod_{(\tilde{u},\tilde{n})\in B^{[\tilde{n}]}_{u,n}} w^{M}_{\tilde{u},\tilde{n},i,k}\Big]\prod_{(\tilde{u},n)\in B^{[n]}_{u,n}} w^{M}_{\tilde{u},n,i,j}$$

End for

$$\ell^{\mathsf{MC}}_{u,n} = \log\left(\frac{\sum_{i=1}^{\mathcal{I}}\sum_{j=1}^{J} w^{M}_{u,n,i,j} w^{P}_{u,n,i,j}}{\sum_{i=1}^{\mathcal{I}}\sum_{j=1}^{J} w^{P}_{u,n,i,j}}\right)$$

Intermediate resampling

- **Intermediate resampling** splits the time interval between observations into S subintervals.
- Reweighting and/or sampling at each subinterval uses a revised estimate of the anticipated measurement density at the end of the interval called a **guide function**.
- This is applicable to continuous time models.
- Intermediate resampling has useful theoretical and empirical properties (Del Moral and Murray, 2015; Park and Ionides, 2019).
- Intermediate resampling for adapted simulation within ABF gives the ABF-IR algorithm.
- Intermediate resampling within PF gives the guided intermediate resampling filter (GIRF) of Park and Ionides (2019), a generalization of the auxiliary particle filter of Pitt and Shepard (1999).

ABF-IR. ABF with intermediate resampling.

Initialize adapted simulation: $\boldsymbol{X}^{\mathrm{A}}_{0,i} \sim f_{\boldsymbol{X}_0}(\boldsymbol{x}_0)$

For n in $1\!:\!N$

Guide simulations: $\boldsymbol{X}^{G}_{n,i,j} \sim f_{\boldsymbol{X}_n|\boldsymbol{X}_{n-1}}\big(\boldsymbol{x}_n \,|\, \boldsymbol{X}^{\mathrm{A}}_{n-1,i}\big)$

Guide variance: $V_{u,n,i} = \mathrm{Var}\big\{h_{u,n}\big(X^{G}_{u,n,i,j}\big), j \text{ in } 1\!:\!J\big\}$

$g^{\mathrm{R}}_{n,0,i,j} = 1$ and $\boldsymbol{X}^{\mathrm{IR}}_{n,0,i,j} = \boldsymbol{X}^{\mathrm{A}}_{n-1,i}$

For s in $1\!:\!S$

Intermediate proposals: $\boldsymbol{X}^{\mathrm{IP}}_{n,s,i,j} \sim f_{\boldsymbol{X}_{n,s}|\boldsymbol{X}_{n,s-1}}\big(\,\cdot\,|\boldsymbol{X}^{\mathrm{IR}}_{n,s-1,i,j}\big)$

$\boldsymbol{\mu}^{\mathrm{IP}}_{n,s,i,j} = \boldsymbol{\mu}\big(\boldsymbol{X}^{\mathrm{IP}}_{n,s,i,j}, t_{n,s}, t_n\big)$

$V^{\mathrm{meas}}_{u,n,s,i,j} = \overrightarrow{\mathrm{v}}_u(\theta, \mu^{\mathrm{IP}}_{u,n,s,i,j})\,, \qquad V^{\mathrm{proc}}_{u,n,s,i} = V_{u,n,i}\,\big(t_n - t_{n,s}\big)\Big/\big(t_n - t_{n,0}\big)$

$\theta_{u,n,s,i,j} = \overleftarrow{\mathrm{v}}_u\big(V^{\mathrm{meas}}_{u,n,s,i,j} + V^{\mathrm{proc}}_{u,n,s,i},\, \mu^{\mathrm{IP}}_{u,n,s,i,j}\big)$

$g_{n,s,i,j} = \prod_{u=1}^{U} f_{Y_{u,n}|X_{u,n}}\big(y^*_{u,n} \,|\, \mu^{\mathrm{IP}}_{u,n,s,i,j}\,;\theta_{u,n,s,i,j}\big)$

Guide weights: $w^{G}_{n,s,i,j} = g_{n,s,i,j}\big/g^{\mathrm{R}}_{n,s-1,i,j}$

Resampling: $\mathbb{P}\big[r(i,j)=a\big] = w^{G}_{n,s,i,a}\Big(\sum_{k=1}^{J} w^{G}_{n,s,i,k}\Big)^{-1}$

$\boldsymbol{X}^{\mathrm{IR}}_{n,s,i,j} = \boldsymbol{X}^{\mathrm{IP}}_{n,s,i,r(i,j)}$ and $g^{\mathrm{R}}_{n,s,i,j} = g_{n,s,i,r(i,j)}$

End For

Set $\boldsymbol{X}^{\mathrm{A}}_{n,i} = \boldsymbol{X}^{\mathrm{IR}}_{n,S,i,1}$

Measurement weights: $w^{M}_{u,n,i,j} = f_{Y_{u,n}|X_{u,n}}\big(y^*_{u,n} \,|\, X^{G}_{u,n,i,j}\big)$

$$w^{\mathrm{P}}_{u,n,i,j} = \prod_{\tilde{n}=1}^{n-1}\Big[\frac{1}{J}\sum_{a=1}^{J}\prod_{(\tilde{u},\tilde{n})\in B^{[\tilde{n}]}_{u,n}} w^{M}_{\tilde{u},\tilde{n},i,a}\Big]\prod_{(\tilde{u},n)\in B^{[n]}_{u,n}} w^{M}_{\tilde{u},n,i,j}$$

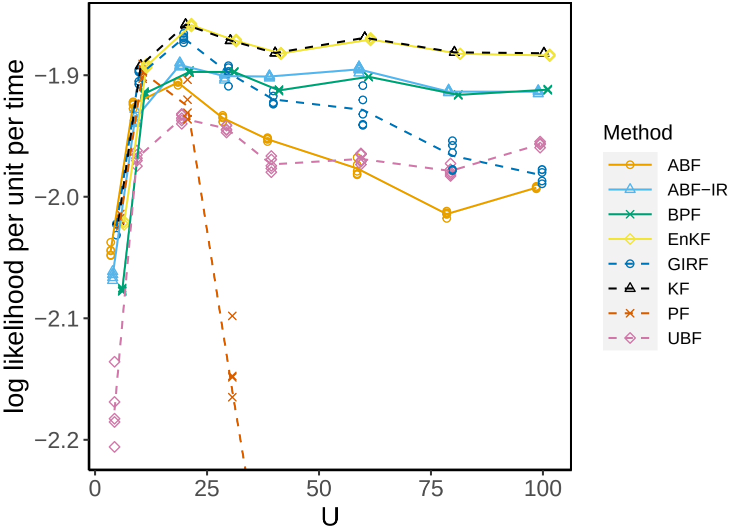
End for

$$\ell^{\mathsf{MC}}_{u,n} = \log\left(\frac{\sum_{i=1}^{\mathcal{I}}\sum_{j=1}^{J} w^{M}_{u,n,i,j} w^{P}_{u,n,i,j}}{\sum_{i=1}^{\mathcal{I}}\sum_{j=1}^{J} w^{P}_{u,n,i,j}}\right)$$

Software for SpatPOMP models

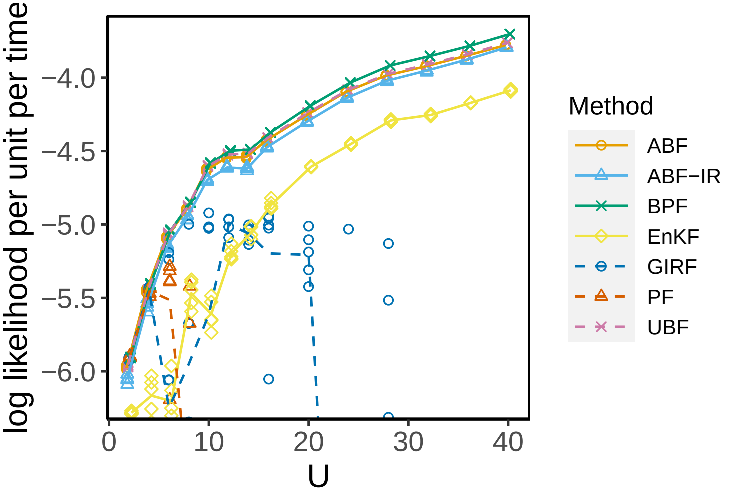
- We use the `asif`, `asifir` and `girf` implementations in the R package `spatPomp` (Asfaw et al., 2019).
- `spatPomp` offers a class '`spatPomp`' that extends the '`pomp`' class for POMP models in the R package `pomp` (King et al., 2016).
- All methods available in `pomp` can formally be applied to '`spatPomp`' objects, though they may not be practically effective for spatiotemporal POMPs.

Filtering U-dimensional correlated Brownian motion



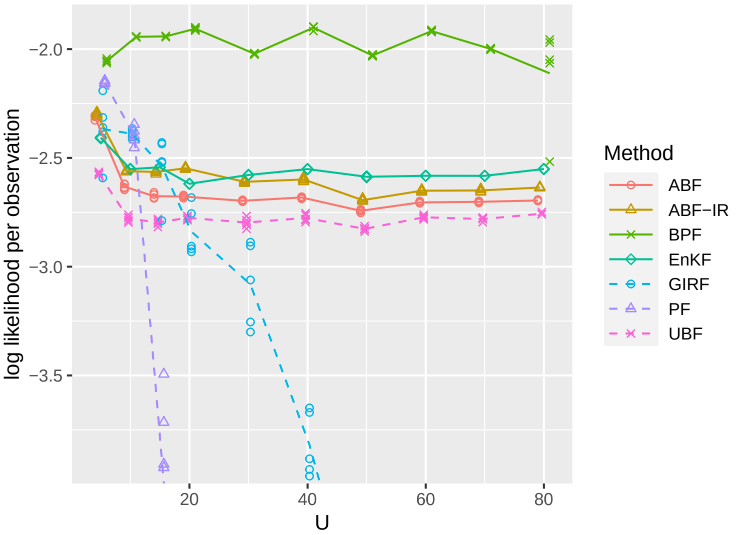
$$\mathrm{Cov}\big(X_{u,n} - X_{u,n-1}, X_{\tilde{u},n} - X_{\tilde{u},n-1}\big) \sim 0.4^{|u-\tilde{u}|}$$

Filtering U units of a coupled measles SEIR model



Simulated data using a gravity model with geography, demography and transmssion parameters corresponding to UK pre-vaccination measles.

Filtering U units of Lorenz 96 toy atmospheric model



$$dX_u(t) = \Big\{X_{u-1}(t)\big(X_{u+1}(t) - X_{u-2}(t)\big) - X_u(t) + F\Big\}dt + \sigma\, dB_u(t)$$

From filtering to parameter inference

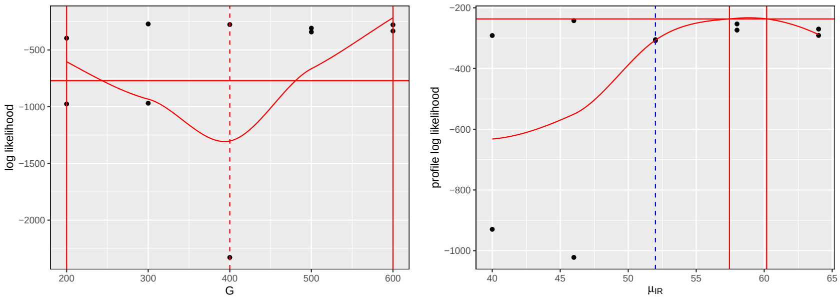
- Log likelihood evaluation in principle enables likelihood-based or Bayesian inference.

- Iterated filtering maximizes the likelihood for PF or GIRF (Ionides et al., 2015).

- Particle Markov chain Monte Carlo can be applied with any likelihood estimate (Andrieu et al., 2010). It is numerically intractable when Monte Carlo estimates are costly and noisy.

- Extending iterated filtering to island filters is future work.

Measles likelihood slices for G and μ_{IR} via ABF



- Simulating 15 year of data from $U = 40$ cities for the measles model.
- The gravitational coupling constant G is fairly weakly identified: a week of computing on a 30 core machine gives Monte Carlo error on the same scale as the statistical uncertainty.
- The recovery rate μ_{IR} is well identified.

Theorem

Let ℓ^{MC} denote the Monte Carlo likelihood approximation constructed by UBF, ABF or ABF-IR. Consider a limit with a growing number of islands, $\mathcal{I} \to \infty$. Suppose regularity assumptions listed in the arXiv preprint. There are quantities $\epsilon(U, N) = O(1)$ and $V(U, N) = O(U^2 N^2)$ such that

$$\mathcal{I}^{1/2}\big[\ell^{MC} - \ell - \epsilon U N\big] \xrightarrow[\mathcal{I}\to\infty]{d} \mathcal{N}\big[0, V\big],$$

where $\xrightarrow[\mathcal{I}\to\infty]{d}$ denotes convergence in distribution and $\mathcal{N}[\mu, \Sigma]$ is the normal distribution with mean μ and variance Σ. If an additional spatiotemporal mixing assumption holds, we obtain an improved variance bound

$$V(U, N) = O(UN)$$

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