Artificial Intelligence 122 (2000) 137–187

Artificial Intelligence

www.elsevier.com/locate/artint

A new approach to cyclic ordering of 2D orientations using ternary relation algebras ☆

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Received 24 September 1998; received in revised form 3 February 1999

Abstract

In Tarski's formalisation, the universe of a relation algebra (RA) consists of a set of binary relations. A first contribution of this work is the introduction of RAs whose universe is a set of ternary relations: these support *rotation* as an operation in addition to those present in Tarski's formalisation. Then we propose two particular RAs: a binary RA, $\mathcal{CYC}_b$, whose universe is a set of (binary) relations on 2D orientations; and a ternary RA, $\mathcal{CYC}_t$, whose universe is a set of (ternary) relations on 2D orientations. The RA $\mathcal{CYC}_t$, more expressive than $\mathcal{CYC}_b$, constitutes a new approach to cyclic ordering of 2D orientations. An atom of $\mathcal{CYC}_t$ expresses for triples of orientations whether each of the three orientations is equal to, to the left of, opposite to, or to the right of each of the other two orientations. $\mathcal{CYC}_t$ has 24 atoms and the elements of its universe consist of all possible 2^{24} subsets of the set of all atoms. Amongst other results,

(1) we provide for $\mathcal{CYC}_t$ a constraint propagation procedure computing the closure of a problem under the different operations, and show that the procedure is polynomial, and complete for a subset including all atoms;
(2) we prove that another subset, expressing only information on parallel orientations, is NP-complete;
(3) we show that provided that a subset $\mathcal{S}$ of $\mathcal{CYC}_t$ includes two specific elements, deciding consistency for a problem expressed in the closure of $\mathcal{S}$ can be polynomially reduced to deciding consistency for a problem expressed in $\mathcal{S}$; and
(4) we derive from the previous result that for both RAs we "jump" from tractability to intractability if we add the universal relation to the set of all atoms.

☆ A preliminary version of this paper, prepared while the first author was at the University of Leeds, has appeared in the Proceedings of the Fifteenth American Conference on Artificial Intelligence, pp. 643–649, AAAI Press, 1998. This work was supported by the *Spatial Inference* project of the DFG priority program on Spatial Cognition, under grant Fr 806/7-2, and by EPSRC under grants GR/K65041 and GR/M56807.

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PII: S0004-3702(00)00044-8

A comparison to the most closely related work in the literature indicates that the approach is promising.

Keywords: Qualitative spatial reasoning; Relation algebra; Constraint satisfaction; Orientation; Computational complexity; Knowledge representation

1. Introduction

Qualitative spatial reasoning (QSR) has become an important and challenging research area of Artificial Intelligence. An important aspect of it is topological reasoning (see the survey in [5]). However, many applications (among which are robot navigation [26], reasoning about shape [42], route description [10,20]) require the representation and processing of orientation knowledge. A variety of approaches to this have been proposed: the CYCORD theory for cyclic ordering of 2D orientations [31,39,40], global reference system models for reasoning about cardinal directions [11,12], relative orientation models [14,15,21,49], and models for the representation of a panorama [41].

One may want to describe a configuration of points in the plane as viewed from a global point of view; this could, for instance, correspond to the situation when a robot has to be located with respect to a number of known landmarks: such a description may consist of specifying the cyclic order of triples of objects in the configuration with respect to the viewpoint at the robot's location. The CYCORD theory [39,40] and Schlieder's system of panorama representation [41] may be used for such a task. However, in addition to providing the cyclic order for triples of orientations, many applications may need the specification for pairs of orientations in the configuration of whether one orientation is to left of, to the right of, opposite to, or equal to, the other orientation. This feature is not captured by the system in [39,40] nor by the one in [41]: indeed, these neglect the *left/straight/right* partition of the plane determined by an observer placed at the point of view and looking in the direction of the reference object, which corresponds to:

(1) the left open half-plane delimited by the directed line point-of-view–reference-object;
(2) the directed line point-of-view–reference-object itself; and
(3) the right open half-plane delimited by the same line.

Such a partition allows, when captured by a model, for some kind of cognitively plausible reasoning (some aspects of cognitive plausibility of orientation models in qualitative spatial reasoning are discussed in [14,15]).

To illustrate, consider the simple natural language sentence "You see both the university and the hill on your left when you walk down to the station": the CYCORD theory fails to provide a representation to this description. Another limitation of the CYCORD theory appears when we consider the *same-direction/opposite-direction* partition determined by the same observer referred to above, which splits the directed line point-of-view–reference-object into the positive part, i.e., the part the observer is looking at, and the negative part, i.e., the part at the back of the observer. This partition is also important for qualitative spatial reasoning applications, as illustrated by the descriptions "The cinema is on the way to the university", or "To get to the cinema from the station, walk in the opposite direction to the university”. This motivates the need for a new, finer grained, approach to cyclic ordering of 2D orientations, which is what we propose in the paper. The new approach, which is an atomic relation algebra (RA) whose universe is a set of ternary relations on 2D orientations, overcomes the above limitations; furthermore, as it turns out, its atoms form a tractable subset, which is important for at least two reasons:

(1) Complete information can be checked for consistency in polynomial time.
(2) Deciding consistency for a general problem expressed in the RA, which we show is NP-complete, can be achieved using a backtracking search procedure, which refines at each node of the search tree the relation on a triple of ’variables’ to an atom.

The RA represents knowledge on cyclic ordering of 2D orientations as a ternary constraint satisfaction problem (ternary CSP) of which:

(1) the variables range over the set of 2D orientations, which, as we will see, is isomorphic to the set of points of a fixed circle, as well as to the set of directed lines containing a fixed point; and
(2) the constraints give for triples of the variables the relation of the RA they should satisfy.

We first define a binary RA and, based on that, develop our new approach to cyclic ordering. Among other things, we provide a composition table for the binary RA. One reason for doing this first is that it will then become easy to understand how the relations of the ternary RA are obtained.

The binary RA can model the qualitative configuration of two orientations. It is based on the *left/straight/right* partition of the plane referred to earlier, determined by the directed line point-of-view–reference-object, and on the *same-direction/opposite-direction* partition of the directed line point-of-view–reference-object itself. The point of view, say P, is global, and we make the realistic assumption that if a collection of point objects is to be qualitatively described relative to P then all objects in the collection are different from P. The point of view may, for instance, be a robot and the objects in the collection landmarks: equality of the position of the robot and that of one of the landmarks would correspond to a collision! In this way, given two objects A and B, it makes sense to consider the orientations z_1 and z_2 of the directed lines (PA) and (PB), respectively, which can be qualitatively compared with respect to the two partitions mentioned above: z_2 is e(qual) to, to the l(eft) of, o(pposite) to, or to the r(ight) of, z_1. To illustrate, consider the situation in Fig. 1 where a robot R has to be qualitatively located relative to four landmarks L_1, L_2, L_3, L_4. This can be achieved by considering the orientations Z_1, Z_2, Z_3, Z_4 of the directed lines (RL_1), (RL_2), (RL_3), (RL_4), respectively, joining the robot to the landmarks. We can then use the binary RA to represent the situation as a description specifying the relation holding on each pair of the four orientations. For instance, to “the robot is to the right of the directed line (L_1L_2)” corresponds the relation $r(Z_2, Z_1)$, stating that orientation Z_2 is to the right of orientation Z_1.

So far, constraint-based approaches to QSR have mainly used constraint propagation methods achieving path consistency [11,12,21,28,35]. These methods have been borrowed from qualitative temporal reasoning à la Allen [1], and make use of a composition table. It is, for instance, well-known from works of van Beek that path consistency achieves global consistency for CSPs of Allen’s convex relations. The proof of this result, given in [45,46], shows that it is mainly due to the 1-dimensional nature of the temporal domain. The proof

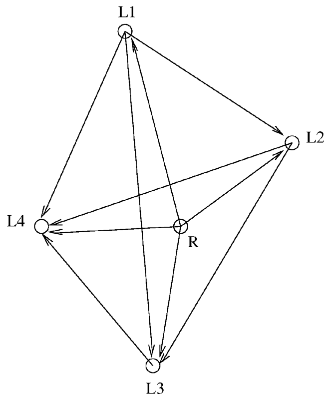

Fig. 1. Localisation of a robot R with respect to four landmarks $L1, L2, L3, L4$.

uses the specialisation of Helly's theorem [4] to $n = 1$: "If S is a set of convex regions of the n-dimensional space $\mathbb{R}^n$ such that every $n + 1$ elements in S have a nonempty intersection then the intersection of all elements in S is nonempty". For the 2-dimensional space $(n = 2)$, the application of the theorem gets a bit more complicated, since one has to check nonemptiness of the intersection of every three elements, instead of just every two; we will use this to show that a constraint propagation procedure to be given for the ternary RA, which achieves strong 4-consistency, has a similar behaviour for a subset including all atoms as path consistency for Allen's convex relations: the procedure achieves global consistency.

The rest of the paper is organised as follows. Section 3 provides some background on constraint satisfaction problems (CSPs) and constraint matrices. Section 4 deals with relation algebras (RAs): a background is presented on RAs as they appear in Tarski's formalisation [24,44], i.e., RAs whose universe is a set of binary relations (binary RAs); then ternary RAs, i.e., RAs whose universe is a set of ternary relations, are introduced. Section 5 presents a first particular RA: an RA of 2D orientations, $\mathcal{CYC}_b$, which is binary. A second particular RA, $\mathcal{CYC}_t$, which is ternary, and provides a new approach to cyclic ordering of 2D orientations, is presented in detail in Section 6. We then come back, in Section 7, to CSPs with a focus on CSPs of 2D orientations: $\mathcal{CYC}_b$-CSPs, i.e., CSPs of which the constraints are $\mathcal{CYC}_b$ relations on pairs of the variables; and $\mathcal{CYC}_t$-CSPs, i.e., CSPs of which the constraints are $\mathcal{CYC}_t$ relations on triples of the variables. The section provides an example showing that path consistency is not sufficient for deciding consistency for a CSP of $\mathcal{CYC}_b$ atoms; then a constraint propagation procedure achieving strong 4-consistency for $\mathcal{CYC}_t$-CSPs, which we show is polynomial; and finally, a procedure to search for a strongly 4-consistent, thus consistent, scenario of a general $\mathcal{CYC}_t$-CSP. In Section 8, we show that a subset of $\mathcal{CYC}_t$ including all atoms is tractable; specifically, we show that our strong 4-consistency procedure can decide consistency for a CSP expressed in the subset. In Section 9, we present some intractability results:

(1) From the NP-completeness of the CYCORD theory [18], we derive that $\mathcal{CYC}_t$ is NP-complete.

(2) We show that a subset of $\mathcal{CYC}_t$ expressing only information on parallel orientations is NP-complete.
(3) We show that provided that a subset $\mathcal{S}$ of $\mathcal{CYC}_t$ includes two specific elements, deciding consistency for a CSP expressed in the closure of $\mathcal{S}$ under the different $\mathcal{CYC}_t$ operations can be polynomially reduced to deciding consistency for a CSP expressed in $\mathcal{S}$.
(4) From the previous result, we derive that for both RAs the set obtained by adding the universal relation to the set of all atoms is NP-complete.

Section 10 compares our approach to the most closely related ones in the literature. Directions for future work are discussed in Section 11. Finally, Section 12 summarises the paper.

We first need to motivate the use of ternary relations.

2. Motivation of the use of ternary relations

The aim of this work, as stated in the introduction, is to provide a relation algebra (RA) as a new approach to the issue of qualitatively representing, and reasoning about, cyclic ordering of 2D orientations.

When the purpose is to order elements from a linear domain, such as the standard time line, the use of a binary relation is sufficient; an excellent illustration to this is Vilain and Kautz's point algebra [48], PA, presented by Ladkin and Maddux [24] as an RA. PA is based of the three binary relations $<$, $=$ and $>$: such relations represent for pairs (t_1, t_2) of linear time points the knowledge of whether t_2 precedes, coincides with, or follows, t_1; clearly, the three relations can be used to totally order any collection of linear time points.

When the purpose is to order elements from a cyclic domain, such as the domain of 2D orientations (which, as we will see, is isomorphic to the set of points of any fixed circle), the only useful knowledge similar relations can express for pairs of orientations is whether the two orientations are equal or different—in other words, switching from a linear domain to a cyclic domain makes the two relations $<$ and $>$ group together into the unique relation $\neq$. Such an expressiveness, however, is insufficient for the task of totally ordering any collection of 2D orientations. In order to get the expressiveness of Vilain and Kautz's linear time relations [48], $<$, $=$ and $>$, in the 2D orientation domain, a relation is needed which would express for triples (z_1, z_2, z_3) of 2D orientations, which of z_2 and z_3 is met first when we move, say, in an anticlockwise direction starting from z_1: the task of totally ordering any collection of 2D orientations would then become possible.

2.1. Cognitive plausibility

The above discussion shows the importance of ternary relations for our purpose. Furthermore, many applications, including those mentioned in the introduction, may require finer knowledge on triples (z_1, z_2, z_3) of orientations than just the cyclic order of z_1, z_2 and z_3: for instance, they may require for some or all pairs (z_i, z_j) of orientations in $\{z_1, z_2, z_3\}$ the additional knowledge of whether z_i is equal to, to the left of, opposite to, or to the right of, z_j. As alluded to briefly in the introduction, the partitioning of the universe of 2D orientations into the orientation that is equal to, the orientations that are to the left of, the orientation that is opposite to, and the orientations that are to the right of, a given orientation (parent orientation) is important because of its cognitive plausibility [14, 15]: the parent orientation might correspond to the orientation of the directed line (*PR*) joining a parent object P to a reference object R; the line can be used by an observer placed at the parent object and looking in the direction of the reference object to describe a primary object *S* relative to the reference object R in the following, cognitively plausible, way, where $z_{(PR)}$ and $z_{(PS)}$ stand for the orientations of the directed lines (*PR*) and (*PS*), respectively: *S* may be

(1) in front of the observer, colinear with P and R—*equal*$(z_{(PS)}, z_{(PR)})$;
(2) to the left of R—*left*$(z_{(PS)}, z_{(PR)})$;
(3) at the back of the observer, colinear with P and R—*opposite*$(z_{(PS)}, z_{(PR)})$; or
(4) to the right of R—*right*$(z_{(PS)}, z_{(PR)})$.

Thus what is needed is to combine a cyclic ordering, thus ternary, relation with these other, binary, relations, *equal*, *left*, *right*, and *opposite*, in order to offer the possibility of expressing finer grained knowledge than just cyclic ordering. This paper provides a calculus to satisfy this need.

2.2. Convexity and tractability

Combining a cyclic ordering relation with the relations *equal*, *left*, *opposite* and *right*, as discussed above, leads not only to a cognitively plausible calculus; it turns out that the atoms of the calculus we get are such that their regions are convex regions of the plane. The convexity property of the atoms, in turn, is used to prove a tractability result: a propagation procedure to be provided is complete for a set including all atoms.

3. Constraint satisfaction problems

A constraint satisfaction problem (CSP) of order n consists of:

(1) a finite set of n variables $x_1, \ldots, x_n$;
(2) a set U (called the universe of the problem); and
(3) a set of constraints on values from U which may be assigned to the variables.

The problem is solvable if the constraints can be satisfied by some assignement of values $a_1, \ldots, a_n \in U$ to the variables $x_1, \ldots, x_n$, in which case the sequence $(a_1, \ldots, a_n)$ is called a solution. Two problems are equivalent if they have the same set of solutions.

An m-ary constraint is of the form $R(x_{i_1}, \ldots, x_{i_m})$, and asserts that the m-tuple of values assigned to the variables $x_{i_1}, \ldots, x_{i_m}$ must lie in the m-ary relation R (an m-ary relation over the universe U is any subset of U^m). An m-ary CSP is one of which the constraints are m-ary constraints. We will be concerned exclusively with binary CSPs and ternary CSPs.

A unary relation, say R, is equivalent to the binary relation $\{(a, a)\colon a \in R\}$, and to the ternary relation $\{(a, a, a)\colon a \in R\}$. In turn, a binary relation R is equivalent to the ternary relation $\{(a, b, a)\colon (a, b) \in R\}$.

3.1. Operations on binary relations

A binary relation is a set of ordered pairs, denoted (a,b). For any two binary relations R and S, $R \cap S$ is the intersection of R and S, $R \cup S$ is the union of R and S, $R \circ S$ is the composition of R and S, and $R^{\smile}$ is the converse of R; these are defined as follows:

$$R \cap S = \{(a,b)\colon\ (a,b) \in R \text{ and } (a,b) \in S\},$$
$$R \cup S = \{(a,b)\colon\ (a,b) \in R \text{ or } (a,b) \in S\},$$
$$R \circ S = \{(a,b)\colon\ \text{for some } c,\ (a,c) \in R \text{ and } (c,b) \in S\},$$
$$R^{\smile} = \{(a,b)\colon\ (b,a) \in R\}.$$

Three special binary relations over a universe U are the empty relation $\emptyset$ which contains no pairs at all, the identity relation $\mathcal{I}_U^b = \{(a,a)\colon\ a \in U\}$, and the universal relation $\top_U^b = U \times U$.

Composition and converse for binary relations were introduced by De Morgan [6,7]. We define below these operations for ternary relations; furthermore, we introduce for ternary relations the operation of rotation, which is not needed for binary relations.

3.2. Operations on ternary relations

A ternary relation is a set of ordered triples, denoted (a,b,c). For any two ternary relations R and S, $R \cap S$ is the intersection of R and S, $R \cup S$ is the union of R and S, $R \circ S$ is the composition of R and S, $R^{\smile}$ is the converse of R, and $R^{\frown}$ is the rotation of R; these are defined as follows:

$$R \cap S = \{(a,b,c)\colon\ (a,b,c) \in R \text{ and } (a,b,c) \in S\},$$
$$R \cup S = \{(a,b,c)\colon\ (a,b,c) \in R \text{ or } (a,b,c) \in S\},$$
$$R \circ S = \{(a,b,c)\colon\ \text{for some } d,\ (a,b,d) \in R \text{ and } (a,d,c) \in S\},$$
$$R^{\smile} = \{(a,b,c)\colon\ (a,c,b) \in R\},$$
$$R^{\frown} = \{(a,b,c)\colon\ (c,a,b) \in R\}.$$

In terms of expressiveness, it should be said that the converse and the rotation of a relation R record the same information as R itself. For binary relations, a converse operation is sufficient because there are two possible ordered pairs involving two objects, say x and y: (x,y) and (y,x); the converse operation alone allows going from one of the two pairs to the other. For ternary relations, a converse operation is no longer sufficient because there are altogether six possible ordered triples involving three objects, say x, y and z: (x,y,z), (x,z,y), (y,x,z), (y,z,x), (z,x,y), (z,y,x). The converse operation allows going from an ordered triple (x,y,z) to the ordered triple (x,z,y), but does not allow going to the other four ordered triples. With the addition of the rotation operation, we can move as well to (y,z,x); then from (y,z,x) to (y,x,z) using converse, and to (z,x,y) using rotation; and from (x,z,y) to (z,y,x) using rotation.

Three special ternary relations over a universe U are the empty relation $\emptyset$ which contains no triples at all, the identity relation $\mathcal{I}_U^t = \{(a,a,a)\colon a \in U\}$, and the universal relation $\top_U^t = U \times U \times U$. Another special ternary relation, which expresses equality of the last two arguments and will be needed later, is $\mathcal{I}_U^{t23} = \{(a,b,b)\colon a, b \in U\}$.

The field of a binary relation R is $\mathit{field}(R) = \{a\colon \text{ for some } b, (a,b) \in R \text{ or } (b,a) \in R\}$; the field of a ternary relation R is $\mathit{field}(R) = \{a\colon \text{ for some } b \text{ and } c, (a,b,c) \in R \text{ or } (b,a,c) \in R \text{ or } (b,c,a) \in R\}$. The field of a set $\mathcal{A}$ of relations is the union of the fields of the relations in $\mathcal{A}$: $\mathit{field}(\mathcal{A}) = \bigcup_{R \in \mathcal{A}} \mathit{field}(R)$.

3.3. Constraint matrices

Let P be a CSP of order n, with variables $x_1, \ldots, x_n$ and universe U.

3.3.1. The case of a binary CSP

Let x_i, x_j be two variables. If a constraint of P is given on the ordered pair (x_j, x_i), specifying that (x_j, x_i) should belong to a relation R, this can be converted into a constraint on the ordered pair (x_i, x_j): $(x_i, x_j) \in R^{\smile}$. Therefore, we can assume that if m constraints involve the variables x_i and x_j then these constraints consist of binary relations $R_1, \ldots, R_m$ the ordered pair (x_i, x_j) is required to belong to. These m constraints are then converted into the single constraint $(x_i, x_j) \in R_1 \cap \cdots \cap R_m$. We can therefore, without loss of generality, make the assumption that for any two variables x_i and x_j, there is at most one constraint involving x_i and x_j.

A binary constraint matrix of order n over U is an $n \times n$-matrix, say $\mathcal{B}$, of binary relations over U verifying the following:

$$(\forall i \leqslant n)(\mathcal{B}_{ii} \subseteq \mathcal{I}_U^b) \qquad \text{(the diagonal property)},$$
$$(\forall i, j \leqslant n)\big(\mathcal{B}_{ij} = (\mathcal{B}_{ji})^{\smile}\big) \qquad \text{(the converse property)}.$$

A binary CSP P over a universe U can be associated with the following binary constraint matrix, denoted $\mathcal{B}^P$:

(1) Initialise all entries to the universal relation: $(\forall i, j \leqslant n)((\mathcal{B}^P)_{ij} \leftarrow \top_U^b)$.
(2) Initialise the diagonal elements to the identity relation: $(\forall i \leqslant n)((\mathcal{B}^P)_{ii} \leftarrow \mathcal{I}_U^b)$.
(3) For all pairs (x_i, x_j) of variables on which a constraint $(x_i, x_j) \in R$ is specified: $(\mathcal{B}^P)_{ij} \leftarrow (\mathcal{B}^P)_{ij} \cap R$, $(\mathcal{B}^P)_{ji} \leftarrow ((\mathcal{B}^P)_{ij})^{\smile}$.

3.3.2. The case of a ternary CSP

Let x_i, x_j, x_k be three variables; there are altogether six possible ordered triples on them: (x_i, x_j, x_k), (x_i, x_k, x_j), (x_j, x_i, x_k), (x_j, x_k, x_i), (x_k, x_i, x_j), (x_k, x_j, x_i). If a constraint of P involving x_i, x_j and x_k is given on an ordered triple other than (x_i, x_j, x_k), this can be converted into a constraint on the ordered triple (x_i, x_j, x_k) by using a finite combination of the converse and rotation operations. For instance, a constraint of the form $(x_k, x_j, x_i) \in R$ is equivalent to $(x_i, x_j, x_k) \in (R^{\smile})^{\frown}$. We can therefore assume that if m constraints of P involve the variables x_i, x_j, x_k then these consist of ternary relations $R_1, \ldots, R_m$ the ordered triple (x_i, x_j, x_k) is required to belong to. These m constraints are then converted into the single constraint $(x_i, x_j, x_k) \in R_1 \cap \cdots \cap R_m$. We can therefore, without loss of generality, make the assumption that for any three variables x_i, x_j, x_k, there is at most one constraint involving them.

A ternary constraint matrix of order n over U is an $n \times n \times n$-matrix, say $\mathcal{T}$, of ternary relations over U verifying the following:

$$\begin{array}{ll}(\forall i \leqslant n)(\mathcal{T}_{iii} \subseteq \mathcal{I}_U^t) & \text{(the identity property),}\\ (\forall i, j, k \leqslant n)\big(\mathcal{T}_{ijk} = (\mathcal{T}_{ikj})^{\smile}\big) & \text{(the converse property),}\\ (\forall i, j, k \leqslant n)\big(\mathcal{T}_{ijk} = (\mathcal{T}_{kij})^{\frown}\big) & \text{(the rotation property).}\end{array}$$

A ternary CSP P over a universe U can be associated with the following ternary constraint matrix, denoted $\mathcal{T}^P$:

(1) Initialise all entries to the universal relation:

$$(\forall i, j, k \leqslant n)\big((\mathcal{T}^P)_{ijk} \leftarrow \top_U^t\big).$$

(2) Initialise the diagonal elements to the identity relation:

$$(\forall i \leqslant n)\big((\mathcal{T}^P)_{iii} \leftarrow \mathcal{I}_U^t\big).$$

(3) For all triples (x_i, x_j, x_k) of variables on which a constraint $(x_i, x_j, x_k) \in R$ is specified:

$$\begin{array}{ll}(\mathcal{T}^P)_{ijk} \leftarrow (\mathcal{T}^P)_{ijk} \cap R, & (\mathcal{T}^P)_{ikj} \leftarrow \big((\mathcal{T}^P)_{ijk}\big)^{\smile},\\ (\mathcal{T}^P)_{jki} \leftarrow \big((\mathcal{T}^P)_{ijk}\big)^{\frown}, & (\mathcal{T}^P)_{jik} \leftarrow \big((\mathcal{T}^P)_{jki}\big)^{\smile},\\ (\mathcal{T}^P)_{kij} \leftarrow \big((\mathcal{T}^P)_{jki}\big)^{\frown}, & (\mathcal{T}^P)_{kji} \leftarrow \big((\mathcal{T}^P)_{kij}\big)^{\smile}.\end{array}$$

We make the assumption that, unless explicitly specified otherwise, a CSP is given as a constraint matrix.

3.4. Strong k-consistency, refinement

Let P be a CSP of order n, V its set of variables and U its universe. An instantiation of P is any n-tuple $(a_1, a_2, \ldots, a_n)$ of U^n, representing an assignment of a value to each variable. A consistent instantiation is an instantiation $(a_1, a_2, \ldots, a_n)$ which is a solution:

- If P is a binary CSP: $(\forall i, j \leqslant n)((a_i, a_j) \in (\mathcal{B}^P)_{ij})$;
- If P is a ternary CSP: $(\forall i, j, k \leqslant n)((a_i, a_j, a_k) \in (\mathcal{T}^P)_{ijk})$.

P is consistent if it has at least one solution; it is inconsistent otherwise. The consistency problem of P is the problem of verifying whether P is consistent.

Let $V' = \{x_{i_1}, \ldots, x_{i_j}\}$ be a subset of V. The sub-CSP of P generated by V', denoted $P_{|V'}$, is the CSP with set of variables V' and whose constraint matrix is obtained by projecting the constraint matrix of P onto V':

- If P is a binary CSP then: $(\forall k, l \leqslant j)((\mathcal{B}^{P_{|V'}})_{kl} = (\mathcal{B}^P)_{i_k i_l})$;
- If P is a ternary CSP then: $(\forall k, l, m \leqslant j)((\mathcal{T}^{P_{|V'}})_{klm} = (\mathcal{T}^P)_{i_k i_l i_m})$.

P is k-consistent [16,17] if for any subset V' of V containing $k-1$ variables, and for any variable $X \in V$, every solution to $P_{|V'}$ can be extended to a solution to $P_{|V' \cup \{X\}}$. P is strongly k-consistent if it is j-consistent, for all $j \leqslant k$.

1-consistency, 2-consistency and 3-consistency correspond to node-consistency, arc-consistency and path-consistency, respectively [30,32]. Strong n-consistency of P corresponds to what is called global consistency in [8]. Global consistency facilitates the important task of searching for a solution, which can be done, when the property is met, without backtracking [17].

A refinement of P is a CSP P' with the same set of variables and such that

- $(\forall i, j)((\mathcal{B}^{P'})_{ij} \subseteq (\mathcal{B}^{P})_{ij})$, in the case of binary CSPs.
- $(\forall i, j, k)((\mathcal{T}^{P'})_{ijk} \subseteq (\mathcal{T}^{P})_{ijk})$, in the case of ternary CSPs.

4. Relation algebras

In Tarski's formalisation of relation algebras (RAs) [24,44], the universe of an RA is a set of binary relations; Tarski was mainly interested in formalising the theory of binary relations.

The section first provides a background on Tarski's formalisation of (binary) RAs [24, 44]. We then proceed to one of the main contribution of this work: the introduction of ternary RAs, i.e., RAs whose universe is a set of ternary relations.

We will be using unary operators ($^-$, $^\smile$ and $^\frown$) and binary operators ($\oplus$, $\odot$ and $\circ$). In expressions without full parentheses, unary operators should be computed first, followed by $\circ$, $\odot$, and $\oplus$, in that order.

4.1. Boolean algebras

A Boolean algebra with universe $\mathcal{A}$ is an algebra of the form $\langle \mathcal{A}, \oplus, \odot, ^-, \bot, \top \rangle$ which satisfies the following properties, for all $R, S, T \in \mathcal{A}$:

$$R \oplus (S \oplus T) = (R \oplus S) \oplus T,$$

$$R \oplus S = S \oplus R,$$

$$R \odot S \oplus R = R,$$

$$R \odot S \oplus T = (R \oplus T) \odot (S \oplus T),$$

$$R \oplus \overline{R} = \top.$$

Of particular interest to this work are Boolean algebras of the form $\langle 2^A, \cup, \cap, ^-, \emptyset, A \rangle$, where A is a nonempty finite set.

4.2. Binary RAs

$\mathcal{U}$ is a binary RA with universe $\mathcal{A}$ [24,44] if:

(1) $\mathcal{A}$ is a set of binary relations; and

(2) $\mathcal{U} = \langle \mathcal{A}, \oplus, \odot, ^-, \bot, \top, \circ, ^\smile, \mathcal{I} \rangle$ where $\langle \mathcal{A}, \oplus, \odot, ^-, \bot, \top \rangle$ is a Boolean algebra (called the Boolean part, or reduct, of $\mathcal{U}$), $\circ$ is a binary operation, $^\smile$ is a unary operation, $\mathcal{I} \in \mathcal{A}$, and the following identities hold for all $R, S, T \in \mathcal{A}$:

$$
\begin{aligned}
&(R \circ S) \circ T = R \circ (S \circ T),\\
&(R \oplus S) \circ T = R \circ T \oplus S \circ T,\\
&R \circ \mathcal{I} = \mathcal{I} \circ R = R,\\
&(R^{\smile})^{\smile} = R,\\
&(R \oplus S)^{\smile} = R^{\smile} \oplus S^{\smile},\\
&(R \circ S)^{\smile} = S^{\smile} \circ R^{\smile},\\
&R^{\smile} \circ \overline{R \circ S} \odot S = \bot.
\end{aligned}
$$

The properties that hold for a binary RA, i.e., the properties in Tarski's formalisation [24, 44], can be seen as the minimal properties that hold for an RA whose universe is a set of m-ary relations, with $m \geqslant 2$. When the universe is a set of m-ary relations, with $m \geqslant 3$, further properties arise, due to further operations.

We now introduce ternary RAs, which need an additional (unary) operation and therefore additional properties.

4.3. Ternary RAs

$\mathcal{U}$ is a ternary RA with universe $\mathcal{A}$ if:

(1) $\mathcal{A}$ is a set of ternary relations; and

(2) $\mathcal{U} = \langle \mathcal{A}, \oplus, \odot, ^-, \bot, \top, \circ, ^{\smile}, ^{\frown}, \mathcal{I} \rangle$ where $\langle \mathcal{A}, \oplus, \odot, ^-, \bot, \top \rangle$ is a Boolean algebra (called the Boolean part, or reduct, of $\mathcal{U}$), $\circ$ is a binary operation, $^{\smile}$ and $^{\frown}$ are unary operations, $\mathcal{I} \in \mathcal{A}$, and the following identities hold for all $R, S, T \in \mathcal{A}$:

$$
\begin{aligned}
&(R \circ S) \circ T = R \circ (S \circ T),\\
&(R \oplus S) \circ T = R \circ T \oplus S \circ T,\\
&R \circ \mathcal{I} = \mathcal{I} \circ R = R,\\
&(R^{\smile})^{\smile} = R,\\
&(R \oplus S)^{\smile} = R^{\smile} \oplus S^{\smile},\\
&(R \circ S)^{\smile} = S^{\smile} \circ R^{\smile}, \qquad \big((R^{\frown})^{\frown}\big)^{\frown} = R,\\
&R^{\smile} \circ \overline{R \circ S} \odot S = \bot, \qquad (R \oplus S)^{\frown} = R^{\frown} \oplus S^{\frown}.
\end{aligned}
$$

4.4. Atomic RA

An atom of an RA $\mathcal{U}$ is a minimal nonzero element, i.e., R is an atom if $R \neq \bot$ and for every $S \in \mathcal{A}$, either $R \odot S = \bot$ or $R \odot \overline{S} = \bot$. An RA is atomic if every nonzero element has an atom below it; i.e., if for all nonzero elements R, there exists an atom A such that $A \odot R = A$.

In the rest of the paper, we focus on atomic, finite RAs of which the Boolean part is of the form $\langle 2^{\top}, \cup, \cap, ^-, \emptyset, \top \rangle$:

(1) The top element $\top$ is a finite set of atoms; the bottom element $\bot$ is the empty set $\emptyset$; the universe is the set $2^{\top}$ of all subsets of $\top$; and

(2) the operations $\oplus$, $\odot$ and $^{-}$ are the usual set-theoretic operations of union ($\cup$), intersection ($\cap$) and complement ($^{-}$) with respect to $\top$ (i.e., $(\forall R \in 2^{\top})(\overline{R} = \top \setminus R)$).

A finite RA is atomic, and its Boolean part is completely determined by its atoms. Furthermore, in an atomic RA, the result of applying any of the operations of the RA to any elements can be obtained from the results of applying the different operations to the atoms. Specifying a finite, thus atomic, RA reduces thus to specifying the identity element and the results of applying the different operations to the different atoms.

5. An atomic binary RA of 2D orientations

We make the assumption that the 2D space is associated with a reference system (O, x, y), and refer to the circle centred at O and of unit radius as $\mathcal{C}_{O,1}$, and to the set of 2D orientations as 2D$\mathcal{O}$. Three natural isomorphisms will be of use in the rest of the paper. In order to facilitate their definitions,we introduce the following sets:

(1) $\mathcal{R}_{0,1}$ is the set of all radii of $\mathcal{C}_{O,1}$ excluding the centre O but closed at the other endpoint.
(2) $d\mathcal{L}_O$ is the set of all directed lines containing O.

Definition 1. The isomorphisms h_1, h_2 and h_3 are defined as follows:

$$h_1\colon \begin{cases} 2\mathrm{D}\mathcal{O} \to \mathcal{R}_{0,1} \\ h_1(z) \text{ is the radius } (OP_z] \in \mathcal{R}_{0,1} \text{ such that the orientation} \\ \quad \text{of the vector } \overrightarrow{OP_z} \text{ is } z \end{cases}$$

$$h_2\colon \begin{cases} 2\mathrm{D}\mathcal{O} \to \mathcal{C}_{O,1} \\ h_2(z) \text{ is the point } P_z \in \mathcal{C}_{O,1} \text{ such that the orientation of the vector } \overrightarrow{OP_z} \text{ is } z \end{cases}$$

$$h_3\colon \begin{cases} 2\mathrm{D}\mathcal{O} \to d\mathcal{L}_O \\ h_3(z) \text{ is the line } \ell_{O,z} \in d\mathcal{L}_O \text{ of orientation } z. \end{cases}$$

Definition 2. The angle determined by two directed lines D_1 and D_2, denoted (D_1, D_2), is the one corresponding to the move in an anticlockwise direction from D_1 to D_2 (see Fig. 2). The angle (z_1, z_2) determined by orientations z_1 and z_2 is the angle $(\ell_{O,z_1}, \ell_{O,z_2})$, where $\ell_{O,z_1} = h_3(z_1)$ and $\ell_{O,z_2} = h_3(z_2)$.

The set 2D$\mathcal{O}$ can thus be viewed as the set of radii of $\mathcal{C}_{O,1}$ (or, indeed, of any fixed circle), as the set of points of $\mathcal{C}_{O,1}$ (or of any fixed circle), or as the set of directed lines containing O (or any fixed point). We will not restrict ourselves to any of these sets; however:

(1) in order to illustrate the relation holding between 2D orientations, it seems more intuitive to look at orientations as directed lines containing a fixed point, for instance O (isomorphism h_3). For example, an orientation z_1 is to the left of an orientation z_2 if the angle (z_2, z_1) belongs to $(0, \pi)$; and

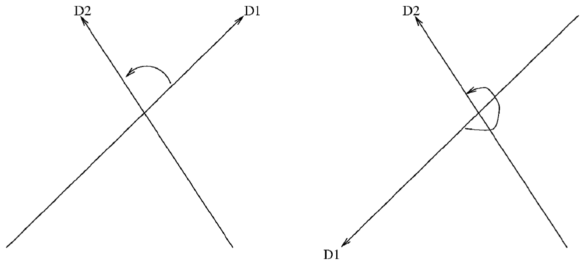

Fig. 2. The angle (D_1, D_2) determined by two directed lines D_1 and D_2 is the one corresponding to the move in an anticlockwise direction from D_1 to D_2.

(2) for the proof of Theorem 6, we will look at an orientation as a radius of $\mathcal{C}_{O,1}$ excluding the centre O (isomorphism h_1).

We are now in a position to introduce the first RA of 2D orientations.

5.1. The field

The field *field*$(\mathcal{U})$ of an RA $\mathcal{U}$ with universe $\mathcal{A}$ is the union of the fields of the relations in $\mathcal{A}$; i.e., *field*$(\mathcal{U}) = \bigcup_{R \in \mathcal{A}}$ *field*(R). The field of the RA to be introduced is the set 2D$\mathcal{O}$ of 2D orientations.

5.2. The universe

Given an orientation X of the plane, another orientation Y can form with X one of the following qualitative configurations:

(1) Y is equal to X: the angle (X, Y) is equal to 0.
(2) Y is to the left of X: the angle (X, Y) belongs to $(0, \pi)$.
(3) Y is opposite to X: the angle (X, Y) is equal to π.
(4) Y is to the right of X: the angle (X, Y) belongs to $(\pi, 2\pi)$.

We denote the four configurations by *e(Y,X)*, *l(Y,X)*, *o(Y,X)* and *r(Y,X)*, respectively. The configurations are Jointly Exhaustive and Pairwise Disjoint (JEPD): given any two orientations of the plane, they stand in one and only one of the configurations.

Definition 3 (*The atoms*). The RA contains four atoms: e, l, o, r. We will refer to the set of all atoms as *BIN*.

BIN is the universal binary relation over 2D$\mathcal{O}$: $\mathit{BIN} \equiv \top^b_{2\mathrm{D}\mathcal{O}} \equiv 2\mathrm{D}\mathcal{O} \times 2\mathrm{D}\mathcal{O}$.

Definition 4 (*The universe*). The universe of the RA, i.e., the set of all its relations, is the set of subsets of *BIN*. An element B of the universe is to be interpreted as follows: $(\forall X, Y \in 2\mathrm{D}\mathcal{O})(B(Y, X) \Leftrightarrow \bigvee_{b \in B} b(Y, X))$.

We refer to the set of singleton relations as $\mathcal{AT}_b$: $\mathcal{AT}_b = \{\{e\}, \{l\}, \{o\}, \{r\}\}$. We observe that $\mathcal{AT}_b$ is a set of relations, whereas *BIN* is a relation. When there is no risk of confusion, we omit the braces in the representation of a singleton relation.

b	e	l	o	r
$b^{\smile}$	e	r	o	l

$\circ$	e	l	o	r
e	e	l	o	r
l	l	$\{l,o,r\}$	r	$\{e,l,r\}$
o	o	r	e	l
r	r	$\{e,l,r\}$	l	$\{l,o,r\}$

Fig. 3. (Left) The converse $b^{\smile}$ of a $\mathcal{CYC}_b$ atom b; (Right) The composition for every pair of $\mathcal{CYC}_b$ atoms: the entry on row i, column j is the $\mathcal{CYC}_b$ relation consisting of the composition of the leftmost element of the row and the top element of the column.

5.3. *The operations applied to the atoms*

Fig. 3(Left) gives the converse for each of the atoms. Fig. 3(Right) gives the composition for every pair of atoms. Both tables can be formally derived as explained in Appendix A.

5.4. *The identity element*

The identity element is the atom e; the composition table of Fig. 3(Right) can be used to verify that: $(\forall R \in 2^{BIN})(R \circ e = e \circ R = R)$.

The RA so defined is an atomic binary RA, which we name $\mathcal{CYC}_b$: $\mathcal{CYC}_b = \langle 2^{BIN}, \cup, \cap, ^-, \emptyset, BIN, \circ, ^{\smile}, e\rangle$. BIN is the universal $\mathcal{CYC}_b$ relation: $(\forall X, Y \in 2\mathrm{D}\mathcal{O})$ $(BIN(Y, X))$.

The structure of $\mathcal{CYC}_b$ is very similar to Allen's algebra of temporal intervals [1], presented by Ladkin and Maddux as an atomic binary RA [24]. In Appendix A, we verify the RA properties for $\mathcal{CYC}_b$.

5.5. *Additional definitions*

We make use of the isomorphism h_1 (Definition 1).

Definition 5 (*Sector of a $\mathcal{CYC}_b$ relation*). The sector determined by an orientation z and a $\mathcal{CYC}_b$ relation B, denoted $sect(z, B)$, is the sector of circle $\mathcal{C}_{O,1}$, excluding the centre O, representing the set of orientations z' related to z by the relation B: $sect(z, B) = \{h_1(z') \mid B(z', z)\}$.

Remark 1. The sector determined by an orientation and a $\mathcal{CYC}_b$ relation does not include the centre O of circle $\mathcal{C}_{O,1}$. Therefore, given n orientations $z_1, \ldots, z_n$ and n $\mathcal{CYC}_b$ relations $B_1, \ldots, B_n$, the intersection $\bigcap_{i=1}^{n} sect(z_i, B_i)$ is either the empty set or a set of radii: [1] this cannot be equal to the centre O, which would be possible if the sector determined by an orientation and a $\mathcal{CYC}_b$ relation included O. This is important for the understanding of the proof of Theorem 6.

[1] A set of radii represents, according to our convention (Definition 1), a set of orientation values.

Definition 6. Let B be a $\mathcal{CYC}_b$ relation:
(1) B is *convex* if for all orientations z, $sect(z, B)$ is a convex part of the plane.
(2) The *dimension* of B is the dimension of the sector it determines with any orientation.
(3) B is *holed* if:
  (a) it is equal to *BIN*; or
  (b) the difference $BIN \setminus B$ is a $\mathcal{CYC}_b$ relation of dimension 1 (is equal to e, o or $\{e, o\}$).

The two atoms e and o of $\mathcal{CYC}_b$ are of dimension 1, the other two (l and r) of dimension 2. Moreover, the dimension of a $\mathcal{CYC}_b$ relation in general is the greatest of the dimensions of its atoms.

Intuitively, a $\mathcal{CYC}_b$ relation is holed if the sector it determines with any orientation is almost equal to the entire surface of circle $\mathcal{C}_{O,1}$; i.e., the topological closure of the sector is equal to the entire surface.

We will refer to the set of all $\mathcal{CYC}_b$ relations which are either convex or holed as *BCH*. *BCH* splits into:
(i) eight convex relations: $\{e\}, \{l\}, \{o\}, \{r\}, \{e, l\}, \{e, r\}, \{l, o\}, \{o, r\}$; and
(ii) four holed relations: $\{l, r\}, \{e, l, r\}, \{l, o, r\}, \{e, l, o, r\}$.
Note that neither of the $\mathcal{CYC}_b$ relations $\{e, l, o\}$ and $\{e, o, r\}$ is convex. For instance, the sector determined by an orientation, say z, and the former relation, $\{e, l, o\}$, is equal to π minus the centre of $C_{0,1}$.

6. An atomic ternary RA of 2D orientations

As we will see, the CYCORD relation *cyc* [39,40] holds on a triple (z_1, z_2, z_3) of 2D orientations if the images P_{z_1}, P_{z_2} and P_{z_3} of z_1, z_2 and z_3, respectively, of the isomorphism h_2 (Definition 1) are:
(1) pairwise distinct, and
(2) such that P_{z_2} is met before P_{z_3} when we scan the circle $\mathcal{C}_{O,1}$ in a clockwise direction starting from P_{z_1}.

Definition 7 (*Induced ternary relation*). Given three $\mathcal{CYC}_b$ atoms b_1, b_2, b_3, we define the induced ternary relation $b_1b_2b_3$ as follows (see Fig. 4):

$$(\forall X, Y, Z)\big(b_1b_2b_3(X, Y, Z) \Leftrightarrow b_1(Y, X) \wedge b_2(Z, Y) \wedge b_3(Z, X)\big).$$

The $\mathcal{CYC}_b$ composition table (Fig. 3(Right)) has 12 entries consisting of atoms, the remaining four consisting of three-atom relations. Therefore any three 2D orientations stand in one of the following 24 JEPD configurations: *eee, ell, eoo, err, lel, lll, llo, llr, lor, lre, lrl, lrr, oeo, olr, ooe, orl, rer, rle, rll, rlr, rol, rrl,rro, rrr*. According to Definition 7, $rol(X, Y, Z)$, for instance, means $r(Y, X) \wedge o(Z, Y) \wedge l(Z, X)$.

The $\mathcal{CYC}_b$ composition table rules out the other, $(4 \times 4 \times 4) - 24$, induced ternary relations $b_1b_2b_3$; these are inconsistent: no triple (z_1, z_2, z_3) of orientations exists such that for such an induced relation one has $b_1(z_2, z_1) \wedge b_2(z_3, z_2) \wedge b_3(z_3, z_1)$.

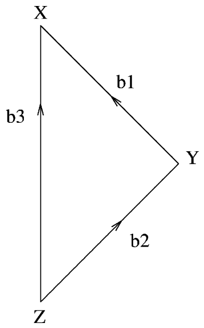

Fig. 4. The ternary relation induced from three $\mathcal{CYC}_b$ atoms: $b_1b_2b_3(X,Y,Z)$ iff $b_1(Y,X) \wedge b_2(Z,Y) \wedge b_3(Z,X)$.

The RA $\mathcal{CYC}_b$ cannot represent the relation *cyc*. However, we can define an atomic ternary RA of which the atoms are the "*induced ternary relations*" described above, which will have *cyc* as one of the elements of its universe.

6.1. The field

As for $\mathcal{CYC}_b$, the field of this new RA, which we name $\mathcal{CYC}_t$, is the set 2D$\mathcal{O}$ of 2D orientations.

6.2. The universe

Definition 8 (*The atoms*). An atom of $\mathcal{CYC}_t$ is any of the 24 JEPD configurations a triple of 2D orientations can stand in. We denote the set of all atoms by *TER*: *TER* $=$ {*eee*, *ell*, *eoo*, *err*, *lel*, *lll*, *llo*, *llr*, *lor*, *lre*, *lrl*, *lrr*, *oeo*, *olr*, *ooe*, *orl*, *rer*, *rle*, *rll*, *rlr*, *rol*, *rrl*, *rro*, *rrr*}.

TER is the universal ternary relation over 2D$\mathcal{O}$: $TER \equiv \top^t_{2D\mathcal{O}} \equiv 2D\mathcal{O} \times 2D\mathcal{O} \times 2D\mathcal{O}$.

Definition 9 (*The universe*). The universe of the ternary RA, i.e., the set of all its relations, is the set of subsets of *TER*. An element T of the universe is to be interpreted as follows: $(\forall X, Y, Z \in 2D\mathcal{O})(T(X,Y,Z) \Leftrightarrow \bigvee_{t \in T} t(X,Y,Z))$

We refer to the set of singleton relations as $\mathcal{AT}_t$: $\mathcal{AT}_t = \{\{t\}\colon t \in TER\}$. When there is no risk of confusion, we omit the braces in the representation of a singleton relation.

6.3. The operations applied to the atoms

Fig. 5 gives the converse and the rotation for each of the 24 atoms. Appendix B explains how to derive formally the table.

In order to give a simple way of writing the composition tables, we have to look closely at how composition is computed. Given four 2D orientations X, Y, Z, W and two atoms t_1

t	$t^{\smile}$	$t^{\frown}$	t	$t^{\smile}$	$t^{\frown}$	t	$t^{\smile}$	$t^{\frown}$	t	$t^{\smile}$	$t^{\frown}$
eee	eee	eee	llo	orl	lor	oeo	oeo	eoo	rll	lrr	lrl
ell	lre	lre	llr	rrl	llr	olr	rro	llo	rlr	rrr	lll
eoo	ooe	ooe	lor	rol	olr	ooe	eoo	oeo	rol	lor	orl
err	rle	rle	lre	ell	rer	orl	llo	rro	rrl	llr	rrl
lel	lel	err	lrl	lll	rrr	rer	rer	ell	rro	olr	rol
lll	lrl	lrr	lrr	rll	rlr	rle	err	lel	rrr	rlr	rll

Fig. 5. The converse $t^{\smile}$ and the rotation $t^{\frown}$ of a $\mathcal{CYC}_t$ atom t.

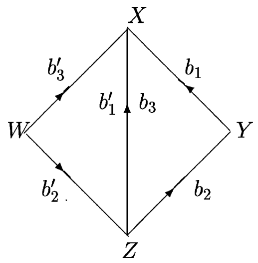

Fig. 6. The conjunction $b_1b_2b_3(X, Y, Z) \wedge b'_1b'_2b'_3(X, Z, W)$ is inconsistent if $b_3 \neq b'_1$.

and t_2, corresponding, respectively, to the induced ternary relations $b_1b_2b_3$ and $b'_1b'_2b'_3$, the conjunction $t_1(X, Y, Z) \wedge t_2(X, Z, W)$ is inconsistent if $b_3 \neq b'_1$ (see Fig. 6 for illustration); this is because the $\mathcal{CYC}_b$ atoms are JEPD. Stated otherwise, when $b_3 \neq b'_1$ we have $t_1 \circ t_2 = \emptyset$. Thus composition splits into four composition tables, corresponding to the following four cases:

(1) Case 1: $b_3 = b'_1 = e$. This corresponds to $t_1 \in \{eee, lre, ooe, rle\}$ and $t_2 \in \{eee, ell, eoo, err\}$.
(2) Case 2: $b_3 = b'_1 = l$. This corresponds to $t_1 \in \{ell, lel, lll, lrl, orl, rll, rol, rrl\}$ and $t_2 \in \{lel, lll, llo, llr, lor, lre, lrl, lrr\}$.
(3) Case 3: $b_3 = b'_1 = o$. This corresponds to $t_1 \in \{eoo, llo, oeo, rro\}$ and $t_2 \in \{oeo, olr, ooe, orl\}$.
(4) Case 4: $b_3 = b'_1 = r$. This corresponds to $t_1 \in \{err, llr, lor, lrr, olr, rer, rlr, rrr\}$ and $t_2 \in \{rer, rle, rll, rlr, rol, rrl, rro, rrr\}$.

Fig. 7 presents the four composition tables.[2] Again, the reader can find in Appendix B how to derive formally the tables.

[2] Alternatively, one could define one single composition table for $\mathcal{CYC}_t$. Such a table would have 24×24 entries, most of which (i.e., $24 \times 24 - (16 + 64 + 16 + 64)$) would be the empty relation.

$\circ$	eee	ell	eoo	err
eee	eee	ell	eoo	err
lre	lre	{lel, lll, lrl}	llo	{llr, lor, lrr}
ooe	ooe	orl	oeo	olr
rle	rle	{rll, rol, rrl}	rro	{rer, rlr, rrr}

$\circ$	lel	lll	llo	llr	lor	lre	lrl	lrr
ell	ell	ell	eoo	err	err	eee	ell	err
lel	lel	lll	llo	llr	lor	lre	lrl	lrr
lll	lll	lll	llo	{llr,lor,lrr}	lrr	lre	{lel,lll,lrl}	lrr
lrl	lrl	{lel,lll,lrl}	llo	llr	llr	lre	lrl	{llr,lor,lrr}
orl	orl	orl	oeo	olr	olr	ooe	orl	olr
rll	rll	{rll,rol,rrl}	rro	rrr	rrr	rle	rll	{rer,rlr,rrr}
rol	rol	rrl	rro	rrr	rer	rle	rll	rlr
rrl	rrl	rrl	rro	{rer,rlr,rrr}	rlr	rle	{rll,rol,rrl}	rlr

$\circ$	oeo	olr	ooe	orl
eoo	eoo	err	eee	ell
llo	llo	{llr, lor, lrr}	lre	{lel, lll, lrl}
oeo	oeo	olr	ooe	orl
rro	rro	{rer, rlr, rrr}	rle	{rll, rol, rrl}

$\circ$	rer	rle	rll	rlr	rol	rrl	rro	rrr
err	err	eee	ell	err	ell	ell	eoo	err
llr	llr	lre	lrl	{llr,lor,lrr}	lrl	{lel,lll,lrl}	llo	llr
lor	lor	lre	lrl	lrr	lel	lll	llo	llr
lrr	lrr	lre	{lel,lll,lrl}	lrr	lll	lll	llo	{llr,lor,lrr}
olr	olr	ooe	orl	olr	orl	orl	oeo	olr
rer	rer	rle	rll	rlr	rol	rrl	rro	rrr
rlr	rlr	rle	{rll,rol,rrl}	rlr	rrl	rrl	rro	{rer,rlr,rrr}
rrr	rrr	rle	rll	{rer,rlr,rrr}	rll	{rll,rol,rrl}	rro	rrr

Fig. 7. The $\mathcal{CYC}_t$ composition tables: case 1, case 2, case 3 and case 4, respectively, from top to bottom.

6.4. The identity element

Given a universe U, we have defined the relation $\mathcal{I}_U^{t_{23}}$ as $\{(a,b,b)\colon\ a,b \in U\}$. $\mathcal{I}_U^{t_{23}}$ expresses equality of the last two arguments, and leaves unspecified the relation between the first two. Since $b_1b_2b_3(X, Y, Z)$, where $b_1b_2b_3$ is a $\mathcal{CYC}_t$ atom, represents the conjunction $b_1(Y, X) \wedge b_2(Z, Y) \wedge b_3(Z, X)$, this means that if Y and Z are equal then for $b_1b_2b_3(X, Y, Z)$ to hold, b_2 must be e, and b_1 and b_3 must be identical. Thus, when U coincides with 2D$\mathcal{O}$, we get:

$$\mathcal{I}^{t_{23}}_{2\mathrm{D}\mathcal{O}} = \{eee, lel, oeo, rer\}.$$

Using the composition tables, we can verify that $\mathcal{I}^{t_{23}}_{2\mathrm{D}\mathcal{O}}$ is an identity element for $\mathcal{CYC}_t$:

$$(\forall R \in 2^{TER})\big(R \circ \mathcal{I}^{t_{23}}_{2\mathrm{D}\mathcal{O}} = \mathcal{I}^{t_{23}}_{2\mathrm{D}\mathcal{O}} \circ R = R\big).$$

This completes the presentation of $\mathcal{CYC}_t$:

$$\mathcal{CYC}_t = \big\langle 2^{TER}, \cup, \cap, ^-, \emptyset, TER, \circ, ^\smile, ^\frown, \mathcal{I}^{t_{23}}_{2\mathrm{D}\mathcal{O}}\big\rangle.$$

TER is the universal $\mathcal{CYC}_t$ relation: $(\forall X, Y, Z \in 2\mathrm{D}\mathcal{O})(TER(X, Y, Z))$.

In Appendix B, we verify the RA properties for an atomic ternary RA.

6.5. Examples

Example 1. For each $\mathcal{CYC}_t$ atom t, Fig. 8 presents a configuration of orientations X, Y and Z such that $t(X, Y, Z)$ holds:

- The top row illustrates, from left to right, the atoms lrl, orl, rll, rol, rrl, rro, rrr.
- The second row from the top illustrates, from left to right, the atoms lll, llo, lrr, lor, llr, olr, rlr.
- The third row from the top illustrates, from left to right, the atoms eee, ell, eoo, err, lel, oeo, rer.
- Finally, the bottom row illustrates, from left to right, the atoms lre, ooe, rle.

Example 2. Consider again Fig. 8:

(1) Each atom illustrated on the second and fourth rows from the top is the converse of the atom illustrated just above it, on the preceding row.

(2) The first and the last three illustrations of the third row from the top have nothing underneath them, on the bottom row: each of the corresponding atoms is its proper converse.

(3) Consider the relation cyc defined on the set 2D$\mathcal{O}$ as follows:

$$(\forall X, Y, Z \in 2\mathrm{D}\mathcal{O})\big(cyc(X, Y, Z) \Leftrightarrow Y \neq X \wedge Z \neq Y \wedge Z \neq X \wedge cw(X, Y, Z)\big).$$

The relation cw holds on a triple (X, Y, Z) of 2D orientations if and only if we first meet Y and then Z when we move in a clockwise direction starting from X. Thus the relation cyc expresses strict betweenness in a clockwise direction. This relation is the unique relation of the CYCORD theory [39,40], and is indeed an element of $\mathcal{CYC}_t$: $cyc = \{lrl, orl, rll, rol, rrl, rro, rrr\}$ (the set of all atoms illustrated on the top row).

(4) The converse of cyc is the set of all atoms illustrated on the second row from the top:

$$cyc^\smile = \{lll, llo, lrr, lor, llr, olr, rlr\}.$$

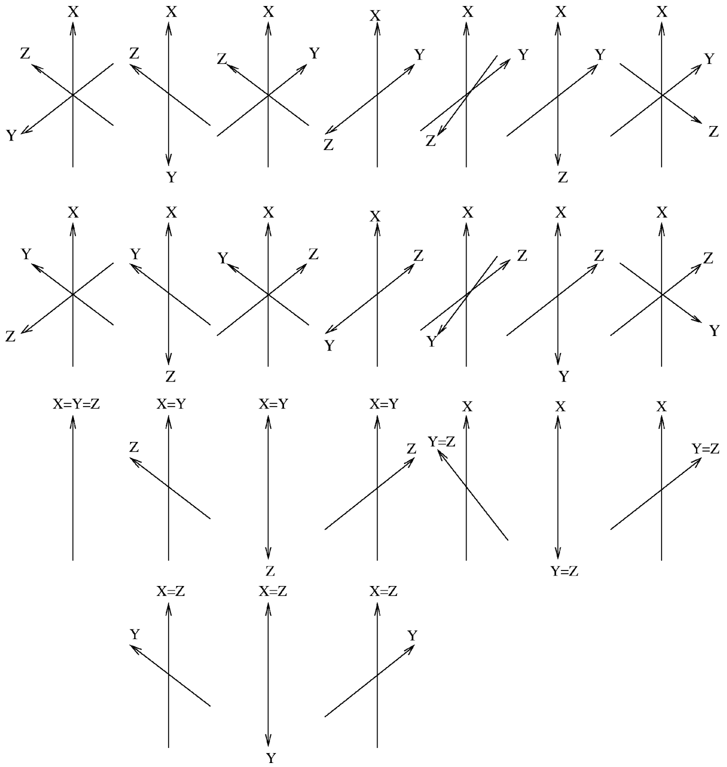

Fig. 8. Graphical illustration of the 24 $\mathcal{CYC}_t$ atoms: from top to bottom, left to right, the atoms are *lrl*, *orl*, *rll*, *rol*, *rrl*, *rro*, *rrr*, *lll*, *llo*, *lrr*, *lor*, *llr*, *olr*, *rlr*, *eee*, *ell*, *eoo*, *err*, *lel*, *oeo*, *rer*, *lre*, *ooe*, *rle*.

Example 3. The composition rule for the CYCORD theory is as follows, and can be verified using the $\mathcal{CYC}_t$ composition tables:

$$(\forall X, Y, Z, W)\big(cyc(X, Y, Z) \wedge cyc(X, Z, W) \Rightarrow cyc(X, Y, W)\big).$$

6.6. Additional definitions

Definition 10 (*Cross product of $\mathcal{CYC}_b$ relations*). The cross product of three $\mathcal{CYC}_b$ relations B_1, B_2, B_3, denoted $\Pi(B_1, B_2, B_3)$, is the $\mathcal{CYC}_t$ relation consisting of those atoms $b_1b_2b_3$ such that $b_1 \in B_1, b_2 \in B_2, b_3 \in B_3$:

$$\Pi(B_1, B_2, B_3) = \{b_1b_2b_3\colon\ b_1 \in B_1, b_2 \in B_2, b_3 \in B_3\} \cap TER.$$

Definition 11. Let R be a $\mathcal{CYC}_t$ relation:

(1) The first, second and third projections of R are the $\mathcal{CYC}_b$ relations $\nabla^1(R)$, $\nabla^2(R)$ and $\nabla^3(R)$, respectively, defined as follows:

$$\nabla^1(R) = \{b_1 \in BIN\colon\ (\exists b_2, b_3 \in BIN)(b_1 b_2 b_3 \in R)\},$$
$$\nabla^2(R) = \{b_2 \in BIN\colon\ (\exists b_1, b_3 \in BIN)(b_1 b_2 b_3 \in R)\},$$
$$\nabla^3(R) = \{b_3 \in BIN\colon\ (\exists b_1, b_2 \in BIN)(b_1 b_2 b_3 \in R)\}.$$

(2) R is projectable if $R = \Pi(\nabla^1(R), \nabla^2(R), \nabla^3(R))$.
(3) R is convex if it is projectable, and each of its projections is a convex $\mathcal{CYC}_b$ relation.
(4) R is said to be {convex, holed} (convex or holed) if it is projectable, and each of its projections is a $\mathcal{CYC}_b$ relation which is either convex or holed (belongs to BCH).

Note that, given a $\mathcal{CYC}_t$ relation R, $\nabla^1(R)$, $\nabla^2(R)$ and $\nabla^3(R)$ are the most specific $\mathcal{CYC}_b$ relations such that:

$$(\forall X, Y, Z)\big(R(X, Y, Z) \Rightarrow \nabla^1(R)(Y, X) \wedge \nabla^2(R)(Z, Y) \wedge \nabla^3(R)(Z, X)\big).$$

Example 4.
(1) $\Pi(\{e, o\}, \{l\}, \{l, r\}) = \{ell, olr\}$.
(2) Let $R = \{ell, llo\}$. We have the following: $\nabla^1(R) = \{e, l\}$, $\nabla^2(R) = \{l\}$ and $\nabla^3(R) = \{l, o\}$.
(3) The cross product of the three projections of the relation R above is $\Pi(\nabla^1(R), \nabla^2(R), \nabla^3(R)) = \Pi(\{e, l\}, \{l\}, \{l, o\}) = \{ell, lll, llo\}$. Thus $R \neq \Pi(\nabla^1(R), \nabla^2(R), \nabla^3(R))$, and R is not projectable.

The set of all projectable $\mathcal{CYC}_t$ relations can be enumerated by computing for every three $\mathcal{CYC}_b$ relations their cross product. The set contains 1518 elements, including the empty relation.

We will refer to the subset of all {convex, holed} $\mathcal{CYC}_t$ relations as TCH.

Definition 12 (*Closures*). Let $\mathcal{S}$ denote a subset of $\mathcal{CYC}_t$. The weak closure of $\mathcal{S}$ is the smallest subset $\mathcal{S}^{wc}$ of $\mathcal{CYC}_t$ verifying the following properties:

$$(P1)\quad \mathcal{S} \subseteq \mathcal{S}^{wc};\quad \text{and}$$
$$(P2)\quad (\forall R, S \in \mathcal{S}^{wc})(R^{\smile} \in \mathcal{S}^{wc}, R^{\frown} \in \mathcal{S}^{wc}, R \cap S \in \mathcal{S}^{wc}).$$

The closure of $\mathcal{S}$ under strong 4-consistency, or $s4c$-closure of $\mathcal{S}$, is the smallest subset $\mathcal{S}^{s4c}$ of $\mathcal{CYC}_t$ verifying the following properties:

$$(P3)\quad \mathcal{S} \subseteq \mathcal{S}^{s4c};\quad \text{and}$$
$$(P4)\quad \big(\forall R, S, T \in \mathcal{S}^{s4c}\big)\big(R^{\smile} \in \mathcal{S}^{s4c}, R^{\frown} \in \mathcal{S}^{s4c}, R \cap S \in \mathcal{S}^{s4c}, R \cap S \circ T \in \mathcal{S}^{s4c}\big).$$

The closure of $\mathcal{S}$ is the smallest subset $\mathcal{S}^c$ of $\mathcal{CYC}_t$ verifying the following properties:

$$(P5)\quad \mathcal{S} \subseteq \mathcal{S}^{c};\quad \text{and}$$
$$(P6)\quad (\forall R, S \in \mathcal{S}^{c})(R^{\smile} \in \mathcal{S}^{c}, R^{\frown} \in \mathcal{S}^{c}, R \cap S \in \mathcal{S}^{c}, R \circ S \in \mathcal{S}^{c}).$$

Given a subset $\mathcal{S}$ of $\mathcal{CYC}_t$, we have $\mathcal{S}^{wc} \subseteq \mathcal{S}^{s4c} \subseteq \mathcal{S}^c$. The relations in $\mathcal{S}^c$ can be viewed as resulting from the "execution" of (well-formed) expressions constructed from the alphabet $V_\mathcal{S} = \mathcal{S} \cup \{^\smile, ^\frown, \cap, \circ, (,)\}$; we refer to such expressions as $\mathcal{S}^c$-expressions, and to the set of all of them as $Xp(\mathcal{S}^c)$.

Definition 13. $Xp(\mathcal{S}^c)$ is the smallest set of expressions over $V_\mathcal{S}$ verifying the following two properties:
(1) a $\mathcal{CYC}_t$ relation belonging to $\mathcal{S}$ belongs to $Xp(\mathcal{S}^c)$; and
(2) if e_1 and e_2 belong to $Xp(\mathcal{S}^c)$ then so do $(e_1)^\smile$, $(e_1)^\frown$, $e_1 \cap e_2$, $e_1 \circ e_2$.

We suppose the reader to be familiar with (labelled) binary trees (each node of such a tree has at most two immediate successors). If a (binary) tree t reduces to a leaf labelled with R, we represent it as R; otherwise, let r be the root of t and α the label of r:
(1) if r has one immediate successor then we represent t as $\langle \alpha, t' \rangle$, where t' is (the representation of) the subtree rooted at the immediate successor of r;
(2) if r has two immediate successors then we represent t as $\langle t_1, \alpha, t_2 \rangle$, where t_1 and t_2 are (the representations of) the subtrees rooted, respectively, at the left immediate successor and at the right immediate successor of r.

Definition 14 (*Tree*). The tree, $t_\mathcal{S}(e)$, and the number of subtrees, $nst_\mathcal{S}(e)$, of an $\mathcal{S}^c$-expression e are defined recursively as follows:
(1) for all $R \in \mathcal{S}$, $t_\mathcal{S}(R) = R$ and $nst_\mathcal{S}(R) = 1$;
(2) $t_\mathcal{S}(e^\smile) = \langle ^\smile, t_\mathcal{S}(e) \rangle$ and $nst_\mathcal{S}(e^\smile) = 1 + nst_\mathcal{S}(e)$;
(3) $t_\mathcal{S}(e^\frown) = \langle ^\frown, t_\mathcal{S}(e) \rangle$ and $nst_\mathcal{S}(e^\frown) = 1 + nst_\mathcal{S}(e)$;
(4) $t_\mathcal{S}(e_1 \cap e_2) = \langle t_\mathcal{S}(e_1), \cap, t_\mathcal{S}(e_2) \rangle$ and $nst_\mathcal{S}(e_1 \cap e_2) = 1 + nst_\mathcal{S}(e_1) + nst_\mathcal{S}(e_2)$; and
(5) $t_\mathcal{S}(e_1 \circ e_2) = \langle t_\mathcal{S}(e_1), \circ, t_\mathcal{S}(e_2) \rangle$ and $nst_\mathcal{S}(e_1 \circ e_2) = 1 + nst_\mathcal{S}(e_1) + nst_\mathcal{S}(e_2)$.

Thus the leaves of the tree of an $\mathcal{S}^c$-expression are labelled with elements of $\mathcal{S}$, and the internal nodes with the operators $^\smile$, $^\frown$, $\cap$ and $\circ$. The number of subtrees, $nst_\mathcal{S}(e)$, of an $\mathcal{S}^c$-expression e is the sum of the number, $nl_\mathcal{S}(e)$, of leaves of $t_\mathcal{S}(e)$ and the number, $no_\mathcal{S}(e)$, of operators of e: $nst_\mathcal{S}(e) = nl_\mathcal{S}(e) + no_\mathcal{S}(e)$; $no_\mathcal{S}(e)$ indicates the number of internal nodes of $t_\mathcal{S}(e)$, i.e., the number of operators to apply in order to get the corresponding element of $\mathcal{S}^c$. The procedure *enumerate*() in Fig. 9 enumerates the closure of a subset $\mathcal{S}$ of $\mathcal{CYC}_t$: the elements of $\mathcal{S}$ are supposed ordered. When the procedure completes, the variable *size* indicates the number of relations in the closure of $\mathcal{S}$, the array c contains the elements of the closure of $\mathcal{S}$, and for each $i = 1 \dots size$, $t[i]$ is the tree of an $\mathcal{S}^c$-expression whose "execution" gives $c[i]$, $nst[i]$ is the number of subtrees of $t[i]$.

Remark 2. In the remainder of the paper, and particularly in the proof of Theorem 9, we refer to the tree $t[i]$ as the tree, $t_\mathcal{S}(c[i])$, of the relation $c[i]$ of $\mathcal{S}^c$, and to $nst[i]$ as the number of subtrees, $nst_\mathcal{S}(t_\mathcal{S}(c[i]))$, of $t_\mathcal{S}(c[i])$: $t[i] = t_\mathcal{S}(c[i])$, $nst[i] = nst_\mathcal{S}(t_\mathcal{S}(c[i]))$.

```
Input: a subset S = {R_1, ..., R_m} of CYC_t (S ⊆ 2^TER).
Output: enumeration of the closure S^c.
procedure enumerate(S, c, t, nst);
for i ← 1 to m {c[i] ← R_i; t[i] ← R_i; nst[i] = 1;}
size ← m;
i ← 1;
while(i ⩽ size){
  R ← (c[i])⌣;
  if(R ∉ c){size++; c[size] ← R; t[size] ← ⟨⌣, t[i]⟩; nst[size] ← 1 + nst[i];}
  R ← (c[i])⌢;
  if(R ∉ c){size++; c[size] ← R; t[size] ← ⟨⌢, t[i]⟩; nst[size] ← 1 + nst[i];}
  j ← 1;
  while(j ⩽ i){
    R ← c[i] ∩ c[j];
    if(R ∉ c){size++; c[size] ← R; t[size] ← ⟨t[i], ∩, t[j]⟩; nst[size] ← 1+nst[i]+nst[j];}
    R ← c[i] ∘ c[j];
    if(R ∉ c){size++; c[size] ← R; t[size] ← ⟨t[i], ∘, t[j]⟩; nst[size] ← 1+nst[i]+nst[j];}
    R ← c[j] ∘ c[i];
    if(R ∉ c){size++; c[size] ← R; t[size] ← ⟨t[j], ∘, t[i]⟩; nst[size] ← 1+nst[j]+nst[i];}
    j++;
    }
  i++;
  }
```

Fig. 9. Enumeration of the closure of a subset of $\mathcal{CYC}_t$.

7. CSPs on cyclic ordering of 2D orientations

We define a $\mathcal{CYC}_b$-CSP as a CSP of which the constraints are $\mathcal{CYC}_b$ relations on pairs of the variables; a $\mathcal{CYC}_t$-CSP as a CSP of which the constraints are $\mathcal{CYC}_t$ relations on triples of the variables. For both types of CSPs, the universe is the set $2\mathrm{D}\mathcal{O}$ of 2D orientations. We use the term $\mathcal{CYC}$-CSP to refer to a CSP which is either a $\mathcal{CYC}_b$-CSP or a $\mathcal{CYC}_t$-CSP.

A $\mathcal{CYC}_b$-matrix (respectively $\mathcal{CYC}_t$-matrix) of order n is a constraint matrix of order n of which the entries are $\mathcal{CYC}_b$ (respectively $\mathcal{CYC}_t$) relations. The constraint matrix associated with a $\mathcal{CYC}_b$-CSP (respectively $\mathcal{CYC}_t$-CSP) is a $\mathcal{CYC}_b$-matrix (respectively $\mathcal{CYC}_t$-matrix).

A scenario of a $\mathcal{CYC}$-CSP is a refinement P' such that all entries of the constraint matrix of P' are atoms. A consistent scenario is a scenario which is consistent.

If we make the assumption that a $\mathcal{CYC}$-CSP does not include the empty constraint, which indicates a trivial inconsistency, then a $\mathcal{CYC}_b$-CSP is strongly 2-consistent and a $\mathcal{CYC}_t$-CSP is strongly 3-consistent.

7.1. *Achieving path consistency for a $\mathcal{CYC}_b$-CSP*

A simple adaptation of Allen's constraint propagation algorithm [1] can be used to achieve path consistency (hence strong 3-consistency) for $\mathcal{CYC}_b$-CSPs. Applied to a $\mathcal{CYC}_b$-

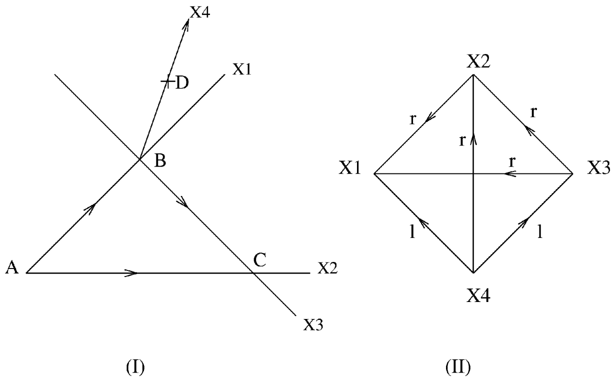

Fig. 10. (I) The ‘Indian tent’; and (II) its associated $\mathcal{CYC}_b$-CSP: the CSP is path consistent but not consistent (path consistency does not detect inconsistency even for $\mathcal{CYC}_b$-CSPs entirely labelled with atoms).

CSP P, such an adaptation would repeat the following steps until either stability is reached or the empty relation is detected (indicating inconsistency):

(1) Consider a triple (X_i, X_j, X_k) of variables verifying $(\mathcal{B}^P)_{ij} \not\subseteq ((\mathcal{B}^P)_{ik} \circ (\mathcal{B}^P)_{kj})$.
(2) $(\mathcal{B}^P)_{ij} \leftarrow (\mathcal{B}^P)_{ij} \cap (\mathcal{B}^P)_{ik} \circ (\mathcal{B}^P)_{kj}$.
(3) If $((\mathcal{B}^P)_{ij} = \emptyset)$ then exit (the CSP is inconsistent).

Example 5 (*The ‘Indian tent’*). The ‘Indian tent’ consists of a clockwise triangle (*ABC*), together with a fourth point *D* which is to the left of each of the directed lines (*AB*) and (*BC*) (see Fig. 10(I)).

The knowledge about the ‘Indian tent’ can be represented as a $\mathcal{CYC}_b$-CSP on four variables, X_1, X_2, X_3 and X_4, representing the orientations of the directed lines (*AB*), (*AC*), (*BC*) and (*BD*), respectively. From (*ABC*) being a clockwise triangle, we get a first set of constraints: $\{r(X_2, X_1), r(X_3, X_1), r(X_3, X_2)\}$. From D being to the left of each of the directed lines (*AB*) and (*BC*), we get a second set of constraints: $\{l(X_4, X_1), l(X_4, X_3)\}$.

If we add the constraint $r(X_4, X_2)$ to the CSP, which states that the point D should be to the right of the directed line (*AC*), this leads to an inconsistency. Röhrig [40] has shown that using the CYCORD theory one can detect such an inconsistency, whereas this cannot be detected using classical constraint-based approaches such as those in [11,12,21].

The $\mathcal{CYC}_b$-CSP is represented graphically in Fig. 10(II): a $\mathcal{CYC}_b$ constraint $R(X, Y)$ is represented as the directed edge (X, Y) labelled with R. The CSP is path-consistent: $(\forall i, j, k)(P_{ij} \subseteq P_{ik} \circ P_{kj})$.[3] However, as mentioned above, the CSP is inconsistent.

From Example 5, we get:

Theorem 1. *Path-consistency does not detect inconsistency even for $\mathcal{CYC}_b$-CSPs entirely labelled with atoms.*

[3] This can be easily verified using the $\mathcal{CYC}_b$ composition table.

7.2. Achieving strong 4-consistency for a $\mathcal{CYC}_t$-CSP

A constraint propagation procedure, $s4c()$, for $\mathcal{CYC}_t$-CSPs is given in Fig. 11; the procedure is an adaptation of Allen's algorithm [1] to ternary relations. The input is a $\mathcal{CYC}_t$-CSP P of order n. When the procedure completes, P verifies the following: $(\forall i, j, k, l \leqslant n)((\mathcal{T}^P)_{ijk} \subseteq (\mathcal{T}^P)_{ijl} \circ (\mathcal{T}^P)_{ilk})$.

The procedure makes use of a queue *Queue*. Initially, we can assume that all triples (X_i, X_j, X_k) such that $1 \leqslant i \leqslant j \leqslant k \leqslant n$ are entered into *Queue*. The procedure removes one triple from *Queue* at a time. When a triple (X_i, X_j, X_k) is removed from *Queue*, the procedure eventually updates the relations on the neighbouring triples (triples sharing two variables with (X_i, X_j, X_k)). If such a relation is successfully updated, the corresponding triple is sorted, in such a way to have the variable with the smallest index first and the variable with the greatest index last, and the sorted triple is placed in *Queue* (if it is not already there) since it may in turn constrain the relations on neighbouring triples: this is done by add-to-queue(). The process terminates when *Queue* becomes empty.

Input: a $\mathcal{CYC}_t$-CSP P.
Output: the CSP P made strongly 4-consistent.
procedure $s4c(P)$;
initialise *Queue*;
repeat{
    get (and remove) next triple (X_i, X_j, X_k) from $Queue$;
    for $m \leftarrow 1$ to n{
        Temp $\leftarrow (\mathcal{T}^P)_{ijm} \cap (\mathcal{T}^P)_{ijk} \circ (\mathcal{T}^P)_{ikm}$;
        If *Temp* $= \emptyset$ then exit (the CSP is inconsistent);
        if *Temp* $\neq (\mathcal{T}^P)_{ijm}$
            {add-to-queue(X_i, X_j, X_m); *update*$(P, i, j, m, Temp)$;}
        Temp $\leftarrow (\mathcal{T}^P)_{ikm} \cap (\mathcal{T}^P)_{ikj} \circ (\mathcal{T}^P)_{ijm}$;
        If *Temp* $= \emptyset$ then exit (the CSP is inconsistent);
        if *Temp* $\neq (\mathcal{T}^P)_{ikm}$
            {add-to-queue(X_i, X_k, X_m); *update*$(P, i, k, m, Temp)$;}
        Temp $\leftarrow (\mathcal{T}^P)_{jkm} \cap (\mathcal{T}^P)_{jki} \circ (\mathcal{T}^P)_{jim}$;
        If *Temp* $= \emptyset$ then exit (the CSP is inconsistent);
        if *Temp* $\neq (\mathcal{T}^P)_{jkm}$
            {add-to-queue(X_j, X_k, X_m); *update*$(P, j, k, m, Temp)$;}
        }
    }
until *Queue* is empty;
procedure update(P, i, j, k, T);
$(\mathcal{T}^P)_{ijk} \leftarrow T$; $(\mathcal{T}^P)_{ikj} \leftarrow T^{\smile}$; $(\mathcal{T}^P)_{jki} \leftarrow T^{\frown}$;
$(\mathcal{T}^P)_{jik} \leftarrow ((\mathcal{T}^P)_{jki})^{\smile}$; $(\mathcal{T}^P)_{kij} \leftarrow ((\mathcal{T}^P)_{jki})^{\frown}$; $(\mathcal{T}^P)_{kji} \leftarrow ((\mathcal{T}^P)_{kij})^{\smile}$;

Fig. 11. A constraint propagation procedure for $\mathcal{CYC}_t$-CSPs.

Theorem 2. *The constraint propagation procedure $s4c()$ achieves strong 4-consistency for the input $\mathcal{CYC}_t$-CSP, and runs into completion in $O(n^4)$ time, where n is the number of variables of the CSP.*

Proof. A $\mathcal{CYC}_t$-CSP is strongly 3-consistent. Procedure $s4c()$ achieves 4-consistency, therefore it achieves strong 4-consistency. The number of variable triples (X_i, X_j, X_k) is $O(n^3)$. A triple may be placed in Queue at most a constant number of times (24, which is the total number of $\mathcal{CYC}_t$ atoms). Every time a triple is removed from Queue for propagation, the procedure performs $O(n)$ operations. $\square$

Remark 3. Let $\mathcal{S}$ denote a subset of $\mathcal{CYC}_t$. If a $\mathcal{CYC}_t$-CSP P is expressed in $\mathcal{S}^{s4c}$, and in particular in $\mathcal{S}$, then the $\mathcal{CYC}_t$-CSP resulting from applying the constraint propagation procedure $s4c()$ to P is expressed in $\mathcal{S}^{s4c}$.

7.3. A consistent scenario search algorithm for $\mathcal{CYC}_t$-CSPs

We will show that the task of checking consistency for a general $\mathcal{CYC}_t$-CSP is NP-complete; thus, with the assumption $P \neq NP$, no polynomial algorithm can be found for such a task. On the other hand, we will show that the set of $\mathcal{CYC}_t$ atoms is tractable; specifically, we will show that if a $\mathcal{CYC}_t$-CSP is such that every three variables X, Y, Z are involved in a constraint of the form $t(X, Y, Z)$, where t is a $\mathcal{CYC}_t$ atom, it can be checked for consistency using the $s4c()$ procedure, which performs in polynomial time. In order to check consistency for a general $\mathcal{CYC}_t$-CSP, we can thus use a backtracking search procedure, which searches for a strongly 4-consistent, thus consistent, scenario, if any, or, otherwise, reports inconsistency, of the input $\mathcal{CYC}_t$-CSP. Such a search procedure is provided in Fig. 12, which is similar to the one of Ladkin and Reinefeld [25] for temporal interval networks [1], except that:

(1) it refines the relation on a triple of variables at each node of the search tree, instead of the relation on a pair of variables; and
(2) it makes use of the procedure $s4c()$, which achieves strong 4-consistency, in the preprocessing step and as the filtering method during the search, instead of a path consistency procedure.

The other details are similar to those of Ladkin and Reinefeld's algorithm.

Definition 15. Let P denote a $\mathcal{CYC}_t$-CSP of order n:

(1) P is projectable if for all i, j, k, $(\mathcal{T}^P)_{ijk}$ is a projectable $\mathcal{CYC}_t$ relation.
(2) The projection of P is the $\mathcal{CYC}_b$-CSP $\nabla(P)$ with the same set of variables, and such that:

$$(\forall i, j \leqslant n)\\ \left(\left(\mathcal{B}^{\nabla(P)}\right)_{ji} = \bigcap_{k \leqslant n} \left[\nabla^1\left((\mathcal{T}^P)_{ijk}\right) \cap \nabla^2\left((\mathcal{T}^P)_{kij}\right) \cap \nabla^3\left((\mathcal{T}^P)_{ikj}\right)\right]\right).$$

```
Input: A CYC_t-CSP P;
Output: true if and only if P is consistent;
function consistent(P);
  s4c(P);
  if(P contains the empty relation)return false;
  else
    if(P contains triples labelled with relations other than atoms){
      choose such a triple, say (X_i, X_j, X_k);
      T ← (T^P)_ijk; % save before branching %
      for each atom t in T{
        refine (T^P)_ijk to t (i.e., (T^P)_ijk ← t);
        if(consistent(P))return true;
        }
      (T^P)_ijk ← T; % restore before backtracking %
      return false;
      }
    else return true; % consistent scenario found %
```

Fig. 12. A consistent scenario search function for $\mathcal{CYC}_t$-CSPs.

The next two theorems will be needed in the next section, for the proof of Theorem 6.

Theorem 3. *A projectable $\mathcal{CYC}_t$-CSP is equivalent to its projection.*

Proof. Let P be a projectable $\mathcal{CYC}_t$-CSP; thus:

$$(\forall i,j,k)\big[(\mathcal{T}^P)_{ijk} \equiv \Pi\big(\nabla^1\big((\mathcal{T}^P)_{ijk}\big), \nabla^2\big((\mathcal{T}^P)_{ijk}\big), \nabla^3\big((\mathcal{T}^P)_{ijk}\big)\big)\big].$$

In other words, the constraint $(\mathcal{T}^P)_{ijk}(X_i,X_j,X_k)$ can be equivalently written as the following conjunction of binary constraints: $\nabla^1((\mathcal{T}^P)_{ijk})(X_j,X_i) \wedge \nabla^2((\mathcal{T}^P)_{ijk})(X_k,X_j) \wedge \nabla^3((\mathcal{T}^P)_{ijk})(X_k,X_i)$. P can be written as the conjunction $\bigwedge_{i,j,k\leqslant n}(\mathcal{T}^P)_{ijk}(X_i,X_j,X_k)$; replacing the constraint $(\mathcal{T}^P)_{ijk}(X_i,X_j,X_k)$ by the equivalent conjunction of binary constraints, we get:

$$P \equiv \bigwedge_{i,j,k\leqslant n}\big[\nabla^1\big((\mathcal{T}^P)_{ijk}\big)(X_j,X_i) \wedge \nabla^2\big((\mathcal{T}^P)_{ijk}\big)(X_k,X_j) \wedge \nabla^3\big((\mathcal{T}^P)_{ijk}\big)(X_k,X_i)\big].$$

Because the conjunction considers all possible triples (i,j,k), with $i,j,k \leqslant n$, it can be split into:

$$P \equiv \bigwedge_{i,j,k\leqslant n}\nabla^1\big((\mathcal{T}^P)_{ijk}\big)(X_j,X_i) \wedge \bigwedge_{i,j,k\leqslant n}\nabla^2\big((\mathcal{T}^P)_{ijk}\big)(X_k,X_j) \wedge \bigwedge_{i,j,k\leqslant n}\nabla^3\big((\mathcal{T}^P)_{ijk}\big)(X_k,X_i).$$

We consider now the main three subconjunctions and rename i,j,k as:

(1) k, i, j, respectively, in the second subconjunction; and
(2) as i, k, j, respectively, in the third subconjunction.

We get:

$$P \equiv \bigwedge_{i,j,k \leqslant n} \nabla^1\big((\mathcal{T}^P)_{ijk}\big)(X_j, X_i) \wedge \bigwedge_{i,j,k \leqslant n} \nabla^2\big((\mathcal{T}^P)_{kij}\big)(X_j, X_i) \wedge \bigwedge_{i,j,k \leqslant n} \nabla^3\big((\mathcal{T}^P)_{ikj}\big)(X_j, X_i).$$

Putting back the three subconjunctions into one main conjunction, we get:

$$P \equiv \bigwedge_{i,j,k \leqslant n} \big[\nabla^1\big((\mathcal{T}^P)_{ijk}\big)(X_j, X_i) \wedge \nabla^2\big((\mathcal{T}^P)_{kij}\big)(X_j, X_i) \wedge \nabla^3\big((\mathcal{T}^P)_{ikj}\big)(X_j, X_i)\big],$$

which is equivalent to:

$$P \equiv \bigwedge_{i,j \leqslant n} \left[\bigwedge_{k \leqslant n} \big\{\nabla^1\big((\mathcal{T}^P)_{ijk}\big)(X_j, X_i) \wedge \nabla^2\big((\mathcal{T}^P)_{kij}\big)(X_j, X_i) \wedge \nabla^3\big((\mathcal{T}^P)_{ikj}\big)(X_j, X_i)\big\}\right],$$

which in turn is equivalent to:

$$P \equiv \bigwedge_{i,j \leqslant n} \left[\bigcap_{k \leqslant n} \big\{\nabla^1\big((\mathcal{T}^P)_{ijk}\big) \cap \nabla^2\big((\mathcal{T}^P)_{kij}\big) \cap \nabla^3\big((\mathcal{T}^P)_{ikj}\big)\big\}\right](X_j, X_i).$$

This corresponds exactly to the constraint matrix of the projection $\nabla(P)$ of P. □

Theorem 4. *Let P denote a projectable $\mathcal{CYC}_t$-CSP of order n. If P is strongly* 4-*consistent then its projection $\nabla(P)$ verifies the following*: $(\forall i, j, k_1, k_2 \leqslant n)[(\mathcal{B}^{\nabla(P)})_{ji} = \nabla^1((\mathcal{T}^P)_{ijk_1}) = \nabla^2((\mathcal{T}^P)_{k_1ij}) = \nabla^3((\mathcal{T}^P)_{ik_1j}) = \nabla^1((\mathcal{T}^P)_{ijk_2})]$.

Proof (*Sketch*). Strong 4-consistency of P implies its closure under the operations of converse and rotation, as well as under what we will refer to as the operation of strong 4-consistency, or $s4c$-operation for short:

$$(\mathcal{T}^P)_{ijk} \leftarrow (\mathcal{T}^P)_{ijk} \cap (\mathcal{T}^P)_{ijl} \circ (\mathcal{T}^P)_{ilk}.$$

From the closure under the operations of converse and rotation, we get:

$$(\forall b_1b_2b_3 \in \mathit{TER})$$
$$\big(b_1b_2b_3 \in (\mathcal{T}^P)_{ijk_1} \Leftrightarrow b_3(b_2)^{\smile}b_1 \in (\mathcal{T}^P)_{ik_1j} \Leftrightarrow (b_3)^{\smile}b_1(b_2)^{\smile} \in (\mathcal{T}^P)_{k_1ij}\big)$$

which implies the following:

$$(\forall b_1 \in \mathit{BIN})\big[b_1 \in \nabla^1\big((\mathcal{T}^P)_{ijk_1}\big) \Leftrightarrow b_1 \in \nabla^3\big((\mathcal{T}^P)_{ik_1j}\big) \Leftrightarrow b_1 \in \nabla^2\big((\mathcal{T}^P)_{k_1ij}\big)\big].$$

Thus $(\forall i, j, k_1)[\nabla^1((\mathcal{T}^P)_{ijk_1}) = \nabla^2((\mathcal{T}^P)_{k_1ij}) = \nabla^3((\mathcal{T}^P)_{ik_1j})]$.

Let $b_1 \in \nabla^1((\mathcal{T}^P)_{ijk_1})$, and suppose that for some $k_2 \neq k_1$, $b_1 \notin \nabla^1((\mathcal{T}^P)_{ijk_2})$. We use the fact that given two $\mathcal{CYC}_t$ atoms t_1 and t_2 and a $\mathcal{CYC}_b$ atom b, if $b \notin \nabla^1(t_1)$ then $b \notin \nabla^1(t_1 \circ t_2)$. We get that $b_1 \notin \nabla^1((\mathcal{T}^P)_{ijk_2} \circ (\mathcal{T}^P)_{ik_2k_1})$. Now closure under the $s4c$-operation implies that $(\mathcal{T}^P)_{ijk_1} \subseteq (\mathcal{T}^P)_{ijk_2} \circ (\mathcal{T}^P)_{ik_2k_1}$. From $b_1 \notin \nabla^1((\mathcal{T}^P)_{ijk_2} \circ (\mathcal{T}^P)_{ik_2k_1})$, we derive that $b_1 \notin \nabla^1((\mathcal{T}^P)_{ijk_1})$, which contradicts our supposition. □

A $\mathcal{CYC}_b$-CSP P can be transformed into an equivalent $\mathcal{CYC}_t$-CSP, say P', as follows:
(1) P' has the same set of variables as P; and
(2) $(\forall i, j, k)((\mathcal{T}^{P'})_{ijk} = \Pi((\mathcal{B}^P)_{ji}, (\mathcal{B}^P)_{kj}, (\mathcal{B}^P)_{ki}))$.

8. A tractability result

The aim of this section is to show that the closure under strong 4-consistency, $(\mathcal{CT}_t)^{s4c}$, of the set $\mathcal{CT}_t$ of all entries of the $\mathcal{CYC}_t$ composition tables is tractable; more specifically, using the terminology in [2], we show that the $\mathcal{CYC}_t$ composition tables are complete for $(\mathcal{CT}_t)^{s4c}$. We first prove that if a $\mathcal{CYC}_t$-CSP expressed in *TCH* is strongly 4-consistent then it is globally consistent, from which the result will follow. The proof will need Helly's convexity theorem:

Theorem 5 (Helly's Theorem [4]). *Let S be a set of convex regions of the n-dimensional space $\mathbb{R}^n$. If every $n+1$ elements in S have a nonempty intersection then the intersection of all elements of S is nonempty.*

For $n = 2$, the theorem states that if a set of convex planar regions is such that every three regions in the set have a nonempty intersection then the intersection of all regions in the set is nonempty.

Van Beek [45] has used the specialisation to $n = 1$ of Helly's theorem to prove a tractability result for path consistent CSPs of Allen's convex relations. We will need the specialisation to $n = 2$.

Theorem 6. *Let P be a $\mathcal{CYC}_t$-CSP expressed in TCH: $(\forall i, j, k)((\mathcal{T}^P)_{ijk} \in TCH)$. If P is strongly 4-consistent then it is globally consistent.*

Proof. Since P is expressed in *TCH* and is strongly 4-consistent, we have the following:
(1) P is equivalent to its projection $\nabla(P)$, which is a $\mathcal{CYC}_b$-CSP expressed in *BCH*: $(\forall i, j)((\mathcal{B}^{\nabla(P)})_{ij} \in BCH)$.
(2) The projection $\nabla(P)$ is strongly 4-consistent.

So the problem becomes that of showing that $\nabla(P)$ is globally consistent. For this purpose, we suppose that the instantiation $(X_{i_1}, X_{i_2}, \ldots, X_{i_k}) = (z_1, z_2, \ldots, z_k)$, $k \geqslant 4$, is a solution to the k-variable sub-CSP $(\nabla(P))_{|\{X_{i_1},\ldots,X_{i_k}\}}$ of $\nabla(P)$. We need to prove that the partial solution can be extended to any $(k+1)$st variable, say $X_{i_{k+1}}$, of $\nabla(P)$. [4] This is equivalent

[4] Since the $\mathcal{CYC}_t$-CSP P is projectable, any solution to any sub-CSP of the projection $\nabla(P)$ is solution to the corresponding sub-CSP of P. This would not be necessarily the case if P were not projectable.

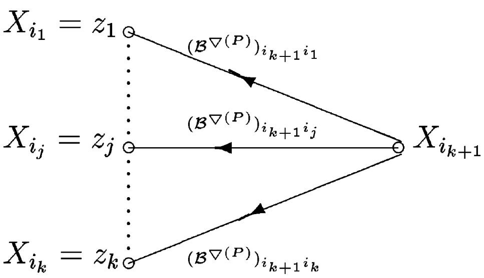

Fig. 13. Illustration of the proof of Theorem 6.

to showing that the following sectors have a nonempty intersection (see Fig. 13 for illustration): $sect(z_1, (\mathcal{B}^{\nabla(P)})_{i_{k+1}i_1})$, $sect(z_2, (\mathcal{B}^{\nabla(P)})_{i_{k+1}i_2}), \ldots, sect(z_k, (\mathcal{B}^{\nabla(P)})_{i_{k+1}i_k})$.

Since the $(\mathcal{B}^{\nabla(P)})_{i_{k+1}i_j}$, $j = 1 \ldots k$, belong to *BCH*, each of these sectors is:

(1) a convex subset of the plane; or
(2) almost equal to the surface of circle $\mathcal{C}_{O,1}$ (its topological closure is equal to that surface).

We split these sectors into those verifying condition (1) and those verifying condition (2). We assume, without loss of generality, that the first m verify condition (1), and the last $k - m$ verify condition (2). We write the intersection of the sectors as $I = I_1 \cap I_2$, with $I_1 = \bigcap_{j=1}^{m} sect(z_j, (\mathcal{B}^{\nabla(P)})_{i_{k+1}i_j})$, $I_2 = \bigcap_{j=m+1}^{k} sect(z_j, (\mathcal{B}^{\nabla(P)})_{i_{k+1}i_j})$.

Due to strong 4-consistency, every three of these sectors have a nonempty intersection. If any of the sectors is a radius (the corresponding relation is either e or o) then the entire intersection must be equal to that radius since the sector intersects with every other two.

We now need to show that when no sector reduces to a radius, the intersection is still nonempty:

Case 1: $m = k$.

This means that all sectors are convex. Since every three of them have a nonempty intersection, Helly's theorem immediately implies that the intersection of all sectors is nonempty.

Case 2: $m = 0$.

This means that no sector is convex; which in turn implies that each sector is such that its topological closure covers the entire surface of $\mathcal{C}_{O,1}$. Hence, for all $j = 1 \ldots k$:

(1) the sector $sect(z_j, (\mathcal{B}^{\nabla(P)})_{i_{k+1}i_j})$ is equal to the entire surface of $\mathcal{C}_{O,1}$ minus the centre (the relation $(\mathcal{B}^{P})_{i_{k+1}i_j}$ is equal to *BIN*); or
(2) the sector $sect(z_j, (\mathcal{B}^{\nabla(P)})_{i_{k+1}i_j})$ is equal to the entire surface of $\mathcal{C}_{O,1}$ minus the centre and one or two radii (the relation $(\mathcal{B}^{P})_{i_{k+1}i_j}$ is equal to $\{e, l, r\}$, $\{l, o, r\}$ or $\{l, r\}$).

So the intersection of all sectors is equal to the entire surface of $\mathcal{C}_{O,1}$ minus the centre and a finite number (at most $2k$) of radii. Since the surface is of dimension 2, a radius of dimension 1, and the centre of dimension 0, the intersection must be nonempty (of dimension 2).

Case 3: $0 < m < k$.

This means that some sectors (at least one) are convex, the others (at least one) are such that their topological closures cover the entire surface of $\mathcal{C}_{O,1}$. The intersection I_1 is nonempty due to Helly's theorem, since every three sectors appearing in it have a nonempty intersection. We need to consider two subcases.

Subcase 3.1: *I_1 is a single radius, say t.*

Since no sector reduces to a radius, and the sectors appearing in I_1 are less than π, there must exist two sectors, say s_1 and s_2, appearing in I_1 such that their intersection is t. Since, due to strong 4-consistency, s_1 and s_2 together with any sector appearing in I_2 form a nonempty intersection, the whole intersection, i.e., I, must be equal to t.

Subcase 3.2: *I_1 is a* 2*-dimensional* (*convex*) *sector*.

The intersection I_2 is the entire surface of $\mathcal{C}_{O,1}$ minus the centre and a finite number (at most $2(k-m)$) of radii. Since the centre is of dimension 0, a finite union of radii is of dimension 0 or 1, and the intersection I_1 is of dimension 2, the whole intersection I must be nonempty (of dimension 2).

The intersection of all sectors is nonempty in all cases. The partial solution can therefore be extended to the variable $X_{i_{k+1}}$ (which can be instantiated with any orientation in the intersection of the k sectors). $\square$

It follows from Theorems 2 and 6 that if the *TCH* subclass is closed under strong 4-consistency, it must be tractable. Unfortunately, as illustrated by the following example, *TCH* is not so closed.

Example 6 (*Nonclosure of TCH under strong* 4*-consistency*). The $\mathcal{CYC}_b$-CSP depicted in Fig. 14 can be represented as the projectable $\mathcal{CYC}_t$-CSP P verifying the following: $(\mathcal{T}^P)_{123} = lll$, $(\mathcal{T}^P)_{124} = \Pi(l, \{l,r\}, \{l,r\})$, $(\mathcal{T}^P)_{134} = (\mathcal{T}^P)_{234} = \Pi(l, l, \{l,r\})$. Applying the propagation procedure $s4c()$ to P leaves unchanged $(\mathcal{T}^P)_{123}$, $(\mathcal{T}^P)_{134}$, $(\mathcal{T}^P)_{234}$, but transforms $(\mathcal{T}^P)_{124}$ into the relation $\{lll, llr, lrr\}$, which is not projectable: this is done by the operation $(\mathcal{T}^P)_{124} \leftarrow (\mathcal{T}^P)_{124} \cap (\mathcal{T}^P)_{123} \circ (\mathcal{T}^P)_{134}$.

Indeed, as we will see, the subset *TCH* is not tractable. Even worse, we will prove that the strict subset $\mathcal{AT}_t \cup \{TER\}$ is already NP-complete (Corollary 4).

The set $(\mathcal{CT}_t)^{s4c}$ includes all 28 entries of the $\mathcal{CYC}_t$ composition tables: the 24 atoms together with the relations $\Pi(l, \{e,l,r\}, l)$, $\Pi(l, \{l,o,r\}, r)$, $\Pi(r, \{e,l,r\}, r)$, $\Pi(r, \{l,o,r\},$

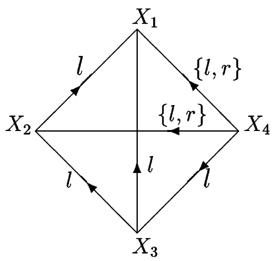

Fig. 14. Illustration of nonclosure of *TCH* under strong 4-consistency.

$\emptyset$	lre	rlr	$\Pi(\{l,o,r\},l,r)$	$\Pi(r,\{l,r\},r)$
eee	lrl	rol	$\Pi(\{e,l,r\},l,l)$	$\Pi(l,l,\{l,r\})$
ell	lrr	rrl	$\Pi(\{l,r\},l,r)$	$\Pi(l,r,\{e,l,r\})$
eoo	oeo	rro	$\Pi(\{l,r\},r,r)$	$\Pi(r,\{l,r\},l)$
err	olr	rrr	$\Pi(\{l,o,r\},r,l)$	$\Pi(l,\{l,r\},l)$
lel	ooe	$\Pi(l,\{e,l,r\},l)$	$\Pi(\{l,r\},r,l)$	$\Pi(r,r,\{l,r\})$
lll	orl	$\Pi(l,\{l,o,r\},r)$	$\Pi(\{l,r\},l,l)$	$\Pi(l,\{l,r\},r)$
llo	rer	$\Pi(r,\{e,l,r\},r)$	$\Pi(r,l,\{e,l,r\})$	$\Pi(r,l,\{l,r\})$
llr	rle	$\Pi(r,\{l,o,r\},l)$	$\Pi(r,r,\{l,o,r\})$	$\Pi(l,r,\{l,r\})$
lor	rll	$\Pi(\{e,l,r\},r,r)$	$\Pi(l,l,\{l,o,r\})$	

Fig. 15. Enumeration of $(\mathcal{CT}_t)^{s4c}$.

$l)$. Furthermore, enumerating $(\mathcal{CT}_t)^{s4c}$ leads to 49 relations (including the empty relation), all of which are {convex, holed} relations (belong to *TCH*). This immediately gives the following corollary, stating tractability of $(\mathcal{CT}_t)^{s4c}$.

Corollary 1 (Tractability of $(\mathcal{CT}_t)^{s4c}$). *Let P be a $\mathcal{CYC}_t$-CSP expressed in $(\mathcal{CT}_t)^{s4c}$: $(\forall i,j,k)((\mathcal{T}^P)_{ijk} \in (\mathcal{CT}_t)^{s4c})$. Deciding consistency for P is tractable.*

Proof. $(\mathcal{CT}_t)^{s4c}$ is a subset of *TCH*; furthermore, $(\mathcal{CT}_t)^{s4c}$ is, by definition, closed under strong 4-consistency. Thus applying the (polynomial) $s4c()$ procedure to a $\mathcal{CYC}_t$-CSP expressed in $(\mathcal{CT}_t)^{s4c}$:
(1) either detects the empty relation, indicating inconsistency of the input CSP; or
(2) does not detect any inconsistency, in which case the output of the procedure is a CSP which is strongly 4-consistent and expressed in *TCH*; from Theorem 6, the output CSP, therefore the input CSP, is consistent. □

The enumeration of $(\mathcal{CT}_t)^{s4c}$ is given in Fig. 15.

Example 7. Transforming the $\mathcal{CYC}_b$-CSP of the 'Indian tent' into a $\mathcal{CYC}_t$-CSP, say P', leads to $(\mathcal{T}^{P'})_{123} = rrr$, $(\mathcal{T}^{P'})_{124} = rrl$, $(\mathcal{T}^{P'})_{134} = rll$, $(\mathcal{T}^{P'})_{234} = rlr$. P' lies in $(\mathcal{CT}_t)^{s4c}$, hence the propagation procedure $s4c()$ must detect its inconsistency. Indeed, the operation $(\mathcal{T}^{P'})_{124} \leftarrow (\mathcal{T}^{P'})_{124} \cap (\mathcal{T}^{P'})_{123} \circ (\mathcal{T}^{P'})_{134}$ leads to the empty relation, since $rrr \circ rll = rll$.

Corollary 2 (Tractability of $\mathcal{AT}_b$). *Let P be a $\mathcal{CYC}_b$-CSP expressed in $\mathcal{AT}_b$: $(\forall i,j)((\mathcal{B}^P)_{ij} \in \mathcal{AT}_b)$. Deciding consistency for P is tractable.*

Proof (*Sketch*). Let P be a $\mathcal{CYC}_b$-CSP as stated in the corollary. Construct from P the $\mathcal{CYC}_t$-CSP P' of which P is the projection: $(\forall i,j,k)((\mathcal{T}^{P'})_{ijk} = \Pi((\mathcal{B}^P)_{ji}, (\mathcal{B}^P)_{kj},$

$(\mathcal{B}^P)_{ki}))$. All entries of $\mathcal{T}^{P'}$ belong $\mathcal{AT}_t$, thus to $(\mathcal{CT}_t)^{s4c}$. From Corollary 1, deciding consistency for P', thus for P, is tractable. □

Remark 4. Given a subset S of $\mathcal{CYC}_t$, we denote by $\nabla(S)$ the set of all projections of relations in S (see Definition 11):

$$\nabla(S) = \{\nabla^1(R):\ R \in S\} \cup \{\nabla^2(R):\ R \in S\} \cup \{\nabla^3(R):\ R \in S\}.$$

In turn, given a subset S of $\mathcal{CYC}_b$, we denote by $\Pi(S)$ the set of all cross products of relations in S (see Definition 10):

$$\Pi(S) = \{\Pi(R_1, R_2, R_3):\ R_1 \in S, R_2 \in S, R_3 \in S\}.$$

An interesting question is whether the subset

$$\nabla\big((\mathcal{CT}_t)^{s4c}\big) = \big\{\{e\}, \{l\}, \{o\}, \{r\}, \{l, r\}, \{e, l, r\}, \{l, o, r\}\big\}$$

of $\mathcal{CYC}_b$, i.e., the subset of $\mathcal{CYC}_b$ consisting of the projections of the relations in the $s4c$-closure of the (set of all) entries in the $\mathcal{CYC}_t$ composition tables, is tractable. The question can also be addressed as whether the subset $\Pi(\nabla((\mathcal{CT}_t)^{s4c}))$ is tractable. $\Pi(\nabla((\mathcal{CT}_t)^{s4c}))$ is a subset of *TCH*; however, the $s4c$-closure, $(\Pi(\nabla((\mathcal{CT}_t)^{s4c})))^{s4c}$, of $\Pi(\nabla((\mathcal{CT}_t)^{s4c}))$ is not a subset of *TCH* (see Example 6):

(1) $t_1 = \Pi(\{l\}, \{l, r\}, \{l, r\}) \in \Pi(\nabla((\mathcal{CT}_t)^{s4c}))$.
(2) $t_2 = \{lll\} \in \Pi(\nabla((\mathcal{CT}_t)^{s4c}))$.
(3) $t_3 = \Pi(\{l\}, \{l\}, \{l, r\}) \in \Pi(\nabla((\mathcal{CT}_t)^{s4c}))$.
(4) $t_4 = t_1 \cap t_2 \circ t_3 = \{lll, llr, lrr\} \notin TCH$.

Thus the $s4c$-closure, $(\Pi(\nabla((\mathcal{CT}_t)^{s4c})))^{s4c}$, of $\Pi(\nabla((\mathcal{CT}_t)^{s4c}))$ is not a subset of *TCH*. Therefore, if $\Pi(\nabla((\mathcal{CT}_t)^{s4c}))$ is tractable:

(1) the tractability cannot be derived from Theorem 6 (as in Corollary 1); and
(2) the tractability cannot be proved using a method similar to the one in the proof of Theorem 6 (which uses Helly's theorem [4]).

9. Intractability results

This section presents some intractability results:

(1) We first show that the RA $\mathcal{CYC}_t$ is NP-complete; this directly follows from the NP-completeness of the CYCORD theory [18].
(2) We show that the weak closure, $(PAR)^{wc}$, of the subset $PAR = \{\{eee\}, \{eoo, ooe\}, \{eee, eoo, ooe\}, \{eee, eoo, oeo, ooe\}\}$ of $\mathcal{CYC}_t$, which expresses only information on parallel orientations, is NP-complete. This gives an idea of how hard reasoning within $\mathcal{CYC}_t$ is: even if we restrict ourselves to a world of parallel orientations, reasoning within that world is already NP-complete!
(3) We show that provided that a subset $\mathcal{S}$ of $\mathcal{CYC}_t$ includes the relations $\{eee\}$ and *TER*, deciding consistency for a CSP expressed in $\mathcal{S}^c$ can be polynomially reduced to deciding consistency for a CSP expressed in $\mathcal{S}$.
(4) We use the previous result to prove that the set $\mathcal{AT}_t^+ = \mathcal{AT}_t \cup \{TER\}$, i.e., the set obtained by adding the universal relation to the set of all $\mathcal{CYC}_t$ atoms, is NP-complete.

(5) From NP-completeness of $\mathcal{AT}_t^+$, we derive NP-completeness of $\mathcal{AT}_b^+ = \mathcal{AT}_b \cup \{BIN\}$, thus of the RA $\mathcal{CYC}_b$.

Theorem 7. *Deciding consistency for a $\mathcal{CYC}_t$-CSP is* NP-*complete.*

Proof. The set $\mathcal{AT}_t$ of all $\mathcal{CYC}_t$ atoms is tractable (Corollary 1); thus, if a $\mathcal{CYC}_t$-CSP is such that on every triple (X, Y, Z) there is a constraint of the form $t(X, Y, Z)$, where t is an atom, deciding its consistency is polynomial, and can be achieved using the $s4c()$ procedure. Therefore, all we need to show is that there exists a deterministic polynomial transformation of an instance of an NP-complete problem to a $\mathcal{CYC}_t$-CSP [19].

The CYCORD theory is NP-complete [18]. The transformation of a problem expressed in the CYCORD theory (a conjunction of CYCORD relations) into a problem expressed in $\mathcal{CYC}_t$ (i.e., into a $\mathcal{CYC}_t$-CSP) is immediate from the rule illustrated in Fig. 8(top) (see Example 2(2)) transforming a CYCORD relation into a $\mathcal{CYC}_t$ relation. Specifically, such a problem, say P, can be transformed into a $\mathcal{CYC}_t$-CSP, say P', in the following way:

(1) Initialise all entries of $\mathcal{T}^{P'}$ to the universal $\mathcal{CYC}_t$ relation *TER*: $(\forall i, j, k)((\mathcal{T}^{P'})_{ijk} \leftarrow \mathit{TER})$;
(2) Initialise the diagonal elements to *eee*: $(\forall i)((\mathcal{T}^{P'})_{iii} \leftarrow \mathit{eee})$;
(3) For all CYCORD relation X_i-X_j-X_k of P, stating that orientations X_i, X_j, X_k are distinct from each other and encountered in that order when we turn in a clockwise direction starting from X_i, perform the following: $T \leftarrow (\mathcal{T}^{P'})_{ijk} \cap \mathit{cyc}$; $\mathit{update}(P', i, j, k, T)$.

The procedure *update*() is defined in Fig. 11, just after the procedure $s4c()$. By construction, $\mathcal{T}^{P'}$ is a constraint matrix over $\mathcal{CYC}_t$. The transformation is deterministic and polynomial, and P is satisfiable if and only if P' is consistent. □

Corollary 3. *Let P be a $\mathcal{CYC}_t$-CSP expressed in* CYCORD$^+ = \{\mathit{cyc}, \mathit{cyc}^\smile, \{\mathit{eee}\}, \mathit{TER}\}$: $(\forall i, j, k)((\mathcal{T}^P)_{ijk} \in \text{CYCORD}^+)$. *Deciding consistency for P is* NP-*complete.*

Proof (*Sketch*). In the proof of Theorem 7, the $\mathcal{CYC}_t$-CSP P' associated with a problem P expressed in the CYCORD theory is such that its constraint matrix $\mathcal{T}^{P'}$ is entirely expressed in $\{\mathit{cyc}, \mathit{cyc}^\smile, \{\mathit{eee}\}, \mathit{TER}\}$ (the set $\{\mathit{cyc}, \mathit{cyc}^\smile, \{\mathit{eee}\}, \mathit{TER}, \emptyset\}$ is closed under intersection, rotation, and converse). □

The weak closure of the set $\mathit{PAR} = \{\{\mathit{eee}\}, \{\mathit{eoo}, \mathit{ooe}\}, \{\mathit{eee}, \mathit{eoo}, \mathit{ooe}\}, \{\mathit{eee}, \mathit{eoo}, \mathit{oeo}, \mathit{ooe}\}\}$ contains 15 of the 16 elements of $2^{\{\mathit{eee},\mathit{eoo},\mathit{oeo},\mathit{ooe}\}}$; it can be easily enumerated:

$$\begin{aligned}(\mathit{PAR})^{wc} = \big\{&\{\}, \{\mathit{eee}\}, \{\mathit{eoo}\}, \{\mathit{oeo}\}, \{\mathit{ooe}\}, \{\mathit{eee}, \mathit{eoo}\}, \{\mathit{eee}, \mathit{oeo}\}, \{\mathit{eee}, \mathit{ooe}\},\\ &\{\mathit{eoo}, \mathit{oeo}\}, \{\mathit{eoo}, \mathit{ooe}\}, \{\mathit{oeo}, \mathit{ooe}\}, \{\mathit{eee}, \mathit{eoo}, \mathit{oeo}\}, \{\mathit{eee}, \mathit{eoo}, \mathit{ooe}\},\\ &\{\mathit{eee}, \mathit{oeo}, \mathit{ooe}\}, \{\mathit{eee}, \mathit{eoo}, \mathit{oeo}, \mathit{ooe}\}\big\}.\end{aligned}$$

Theorem 8 (NP-completeness of $(\mathit{PAR})^{wc}$). *Let P be a $\mathcal{CYC}_t$-CSP expressed in $(\mathit{PAR})^{wc}$: $(\forall i, j, k)((\mathcal{T}^P)_{ijk} \in (\mathit{PAR})^{wc})$. Deciding consistency for P is* NP-*complete.*

Proof. The subset $(PAR)^{wc}$ belongs to NP, since solving a $\mathcal{CYC}_t$-CSP of atoms is polynomial (Corollary 1). We need to prove that there exists a (deterministic) polynomial transformation of an instance of an NP-complete problem (we consider an instance of 3-SAT: an instance of SAT of which every clause contains exactly three literals) into a $\mathcal{CYC}_t$-CSP expressed in $(PAR)^{wc}$ in such a way that the former is satisfiable (has a model) if and only if the latter is consistent.

Suppose that S is an instance of 3-SAT, and denote by:

(1) $Lit(S) = \{\ell_1, \ldots, \ell_n\}$ the set of literals appearing in S;

(2) $Cl(S)$ the set of clauses of S; and

(3) $BinCl(S)$ the set of binary clauses which are subclauses of clauses in $Cl(S)$.

The $\mathcal{CYC}_t$-CSP, P_S, we associate with S is as follows. Its set of variables is $V = \{X_c \mid c \in Lit(S) \cup BinCl(S)\} \cup \{X_0\}$. X_0 is a truth determining variable: all orientations which are equal to X_0 correspond to elements of $Lit(S) \cup BinCl(S)$ that are true, the others (those which are opposite to X_0) to elements of $Lit(S) \cup BinCl(S)$ that are false. The constraint matrix of P_S, $\mathcal{T}^{P_S}$, is regarded as being indexed with elements from $\{0\} \cup Lit(S) \cup BinCl(S)$, and the entry $(\mathcal{T}^{P_S})_{abc}$ stands for the relation on triple (X_a, X_b, X_c):

(1) Initialise all entries of $\mathcal{T}^{P_S}$ to $\{eee, eoo, oeo, ooe\}$:
$(\forall a, b, c)((\mathcal{T}^{P_S})_{abc} \leftarrow \{eee, eoo, oeo, ooe\})$;

(2) Initialise the diagonal elements to eee: $(\forall a)((\mathcal{T}^{P_S})_{aaa} \leftarrow eee)$;

(3) for all pairs $(X_p, X_{\overline{p}})$ of variables such that $\{p, \overline{p}\} \subseteq Lit(S)$, p and $\overline{p}$ should have complementary truth values; hence X_p and $X_{\overline{p}}$ should be opposite to each other in P_S: $T \leftarrow (\mathcal{T}^{P_S})_{0\overline{p}p} \cap \{eoo, ooe\}$; $update(P_S, 0, \overline{p}, p, T)$;

(4) for all variables X_{c_1}, X_{c_2} such that $(c_1 \vee c_2)$ is a clause of S, c_1 and c_2 cannot be simultaneously false; translated into P_S, X_{c_1} and X_{c_2} should not be both opposite to X_0: $T \leftarrow (\mathcal{T}^{P_S})_{0c_1c_2} \cap \{eee, eoo, ooe\}$; $update(P_S, 0, c_1, c_2, T)$;

(5) for all variables $X_{(\ell_1 \vee \ell_2)}, X_{\ell_1}$, if ℓ_1 is true then so is $(\ell_1 \vee \ell_2)$; translated into P_S, X_0 and X_{ℓ_1} should not be both opposite to $X_{(\ell_1 \vee \ell_2)}$:
$T \leftarrow (\mathcal{T}^{P_S})_{(\ell_1 \vee \ell_2)\ell_1 0} \cap \{eee, eoo, ooe\}$; $update(P_S, \ell_1 \vee \ell_2, \ell_1, 0, T)$.

Again, the procedure *update*() is defined in Fig. 11, just after the procedure $s4c()$. The transformation is deterministic and polynomial. Moreover, since $(PAR)^{wc}$ is closed under intersection, converse and rotation, the final matrix $\mathcal{T}^{P_S}$ is a constraint matrix over $(PAR)^{wc}$. If M is a model of S, it is mapped to a solution of P_S as follows. X_0 is assigned any value of $[0, 2\pi)$. For all $\ell \in Lit(S)$, X_ℓ is assigned the same value as X_0 if M assigns the value *true* to literal ℓ, the value opposite to that of X_0 otherwise. For all $(\ell_1 \vee \ell_2) \in BinCl(S)$, $X_{(\ell_1 \vee \ell_2)}$ is assigned the same value as X_0 if either X_{ℓ_1} or X_{ℓ_2} is assigned the same value as X_0, the opposite value otherwise. On the other hand, any solution to P_S can be mapped to a model of S by assigning to every literal ℓ the value *true* if and only if the variable X_ℓ is assigned the same value as X_0. $\square$

Before going further in the presentation of our intractability results, we want to be clear with respect to the issue of representing a $\mathcal{CYC}_t$-CSP. The most convenient way for representing such a CSP is certainly the use of an $n \times n \times n$-matrix, where n is the order of the CSP; one reason to this is that the standard way for constraint propagation algorithms and for solution search algorithms, which constitute the main reasoning tools for constraint-based frameworks, to deal with a CSP is to have it represented as a matrix.

We have assumed so far that the matrix associated with a $\mathcal{CYC}_t$-CSP was a constraint matrix; i.e., it verifies the diagonal property, the converse property, and the rotation property. However, in terms of solutions, if we associate with a $\mathcal{CYC}_t$-CSP P the $n \times n \times n$-matrix $\mathcal{T}^{P,2}$ defined as follows:

(1) Initialise all entries to the universal relation *TER*: $(\forall i, j, k)((\mathcal{T}^{P,2})_{ijk} \leftarrow \mathit{TER})$;
(2) Initialise all diagonal elements to *eee*: $(\forall i)((\mathcal{T}^{P,2})_{iii} \leftarrow \mathit{eee})$;
(3) For all triples (X_i, X_j, X_k) of variables such that a constraint $R(X_i, X_j, X_k)$ is specified: $(\mathcal{T}^{P,2})_{ijk} \leftarrow (\mathcal{T}^{P,2})_{ijk} \cap R$.

Then the matrices $\mathcal{T}^P$ and $\mathcal{T}^{P,2}$ are equivalent, i.e., they have the same set of solutions.

Binary CSPs of Allen's relations on pairs of interval variables in which every two variables are involved in exactly one constraint are called *normalised sets* of interval formulas in [34].

Definition 16. An orientation formula is a $\mathcal{CYC}_t$ relation on a triple of variables, i.e., a constraint of the form $R(X, Y, Z)$, where R is a $\mathcal{CYC}_t$ relation. A normalised set of orientation formulas is a $\mathcal{CYC}_t$-CSP given as a set of constraints in which every three variables are involved in exactly one constraint.

Given a $\mathcal{CYC}_t$-CSP P, the matrix $\mathcal{T}^P$ is closed under the operations of converse and rotation; this is not necessarily the case for the matrix $\mathcal{T}^{P,2}$: in particular, if P is a normalised set of orientation formulas then for any three variables X_i, X_j and X_k, at most one element in the set $\{(\mathcal{T}^{P,2})_{lmn}\colon \{l, m, n\} = \{i, j, k\}\}$ is not the universal relation.

Remark 5. If a subset $\mathcal{S}$ of $\mathcal{CYC}_t$ includes the relations *eee* and *TER* then a normalised set of orientation formulas, say P, which is entirely expressed in $\mathcal{S}$ is such that its associated matrix $\mathcal{T}^{P,2}$ is also entirely expressed in $\mathcal{S}$.

Theorem 9. *Let $\mathcal{S}$ be a subset of $\mathcal{CYC}_t$ such that $\mathit{eee} \in \mathcal{S}$ and $\mathit{TER} \in \mathcal{S}$. Deciding consistency for a normalised set of orientation formulas expressed in $\mathcal{S}^c$ can be polynomially reduced to deciding consistency for a normalised set of orientation formulas expressed in $\mathcal{S}$.*

Proof. We have seen how, given a subset $\mathcal{S}$ of $\mathcal{CYC}_t$, to associate with each relation R in the closure $\mathcal{S}^c$ a tree $t_{\mathcal{S}}(R)$ in such a way that the "execution" of $t_{\mathcal{S}}(R)$ gives R itself (see Definition 14 and Remark 2). We use the tree of a relation in $\mathcal{S}^c$ to transform a normalised set, P, of orientation formulas expressed in $\mathcal{S}^c$ into an equivalent set of orientation formulas, $g(P)$, expressed in $\mathcal{S}$, and in which every three variables are involved in at most one constraint:

(1) $g(\{R(X, Y, Z)\}) = h(t_{\mathcal{S}}(R)(X, Y, Z))$, for all $R \in \mathcal{S}^c$.
(2) $g(\{R(X, Y, Z)\} \cup P') = g(\{R(X, Y, Z)\}) \cup g(P')$, where $R \in \mathcal{S}^c$ and P' is a non-empty set of orientation formulas expressed in $\mathcal{S}^c$.

The mapping h is defined as follows:

(1) $h(R(X, Y, Z)) = \{R(X, Y, Z)\}$, for all $R \in \mathcal{S}$.
(2) $h(\langle {}^{\smile}, t\rangle(X, Y, Z)) = h(t(X, Z, Y))$.
(3) $h(\langle {}^{\frown}, t\rangle(X, Y, Z)) = h(t(Z, X, Y))$.

(4) $h(\langle t_1, \cap, t_2\rangle(X, Y, Z)) = h(t_1(X, Y, Z)) \cup h(t_2(X, Y, Z')) \cup \{eee(Z, Z', Z)\}$, where Z' is a fresh variable.
(5) $h(\langle t_1, \circ, t_2\rangle(X, Y, Z)) = h(t_1(X, Y, W)) \cup h(t_2(X, W, Z))$, where W is a fresh variable.

By construction, $g(P)$ is a set of orientation formulas expressed in $\mathcal{S}$ with the property that every three variables are involved in at most one formula. $g(P)$ is transformed into an equivalent normalised set of orientation formulas by creating for every three variables X, Y, Z not already involved in any formula the formula $TER(X, Y, Z)$.

Let m denote the number of orientation formulas in P, and refer to the orientation formulas as $f_1(X_{1_1}, X_{1_2}, X_{1_3}), \ldots, f_i(X_{i_1}, X_{i_2}, X_{i_3}), \ldots, f_m(X_{m_1}, X_{m_2}, X_{m_3})$. For each $i = 1, \ldots, m$, let s_i denote the size of f_i, i.e., the number, $nst_{\mathcal{S}}(f_i)$, of subtrees of f_i (see Remark 2): $s_i = nst_{\mathcal{S}}(f_i)$. If s is the greatest of the s_i's then the construction takes $O(ms)$ time; the transformation is thus polynomial. □

We are now in a position to derive that for both of the presented RAs, $\mathcal{CYC}_b$ and $\mathcal{CYC}_t$, we "jump" from tractability to intractability if add the universal relation to the set of all atoms.

Corollary 4. *The subset $\mathcal{AT}_t^+ = \mathcal{AT}_t \cup \{TER\}$ of $\mathcal{CYC}_t$ is* NP-*complete.*

Proof. Any set of orientation formulas over $\mathcal{AT}_t^+$ can be converted into an equivalent normalised set of orientation formulas over $\mathcal{AT}_t^+$. The subset $\mathcal{AT}_t^+$ includes the relations *eee* and *TER*. From Corollary 3 and Theorem 9, and the closure of $(\mathcal{AT}_t^+)^c$ under converse, it is sufficient to show that the relation *cyc* belongs to $(\mathcal{AT}_t^+)^c$. The following sequence shows that this is indeed the case:
(1) $R_1 = lll$, $R_2 = llr$, $R_3 = rll$.
(2) $R_4 = R_1 \circ R_2 = \{llr, lor, lrr\}$.
(3) $R_5 = (R_4)^\frown = \{llr, olr, rlr\}$.
(4) $R_6 = R_5 \circ R_3 = \{lrl, orl, rll, rol, rrl\}$.
(5) $R_7 = (R_6)^\frown = \{lrl, orl, rrl, rro, rrr\}$.
(6) $R_8 = R_7 \circ R_7 = \{lrl, orl, rll, rol, rrl, rro, rrr\} = cyc$. □

Corollary 5. *The subset $\mathcal{AT}_b^+ = \mathcal{AT}_b \cup \{BIN\}$ of $\mathcal{CYC}_b$ is* NP-*complete.*

Proof. A $\mathcal{CYC}_b$-CSP of atoms can be solved in polynomial time (Corollary 2). Thus we need to show that there is a polynomial deterministic transformation of an instance of an NP-complete problem into a problem expressed in $\mathcal{AT}_b^+$. We consider a normalised set, P, of orientation formulas expressed in $\mathcal{AT}_t^+$. According to Corollary 4, deciding consistency for P is NP-complete. The set $\mathcal{AT}_t^+ \cup \{\emptyset\}$ is closed under converse, rotation and intersection (in other words, $(\mathcal{AT}_t^+ \cup \{\emptyset\})^{wc} = \mathcal{AT}_t^+ \cup \{\emptyset\}$); therefore, the constraint matrix $\mathcal{T}^P$ is entirely expressed in $\mathcal{AT}_t^+$. Finally, P is projectable, and is therefore equivalent to its projection $\nabla(P)$. $\nabla(P)$, by definition, verifies the following: $(\forall i, j \leqslant n)((\mathcal{B}^{\nabla(P)})_{ji} = \bigcap_{k \leqslant n}[\nabla^1((\mathcal{T}^P)_{ijk}) \cap \nabla^2((\mathcal{T}^P)_{kij}) \cap \nabla^3((\mathcal{T}^P)_{ikj})])$. Because each of the projections $\nabla^1(R)$, $\nabla^2(R)$ and $\nabla^3(R)$ of any $\mathcal{CYC}_t$ relation R in $\mathcal{AT}_t^+$ is either a $\mathcal{CYC}_b$ atom or the relation *BIN*, the $\mathcal{CYC}_b$-CSP $\nabla(P)$ is entirely expressed in $\mathcal{AT}_b^+$. □

10. Related work

We compare our approach to cyclic ordering of 2D orientations with the most closely related research in the literature.

10.1. The CYCORD theory

The CYCORD theory [31,39,40] expresses cyclic ordering of 2D orientations; it contains only one relation, namely the relation *cyc* we have already mentioned and translated into the RA $\mathcal{CYC}_t$ (see Example 2(2)). The main disadvantage of the theory is that real applications generally need to represent finer knowledge than just what could be called, as we saw in Example 2(2), strict betweenness in a clockwise direction.

10.2. Representation of a panorama

In [26], Levitt and Lawton discussed QUALNAV, a qualitative landmark navigation system for mobile robots. One feature of the system is the representation of the information about the order of landmarks as seen by the visual sensor of a mobile robot. Such information provides the panorama of the robot with respect to the visible landmarks.

Fig. 16 illustrates the panorama of an object S with respect to five reference objects (landmarks) A, B, C, D, E in Schlieder's system [41, p. 527]. The panorama is described by the total cyclic order, in a clockwise direction, of the five directed lines $(SA), (SB), (SC), (SD), (SE)$, and the directed lines which are opposite to them, namely $(Sa), (Sb), (Sc), (Sd), (Se)$: (SA)-(Sc)-(Sd)-(SB)-(Se)-(Sa)-(SC)-(SD)-(Sb)-(SE). By using the RA $\mathcal{CYC}_b$, only the five lines joining S to the landmarks are needed to describe the panorama: $\{r((SB),(SA)), r((SC),(SB)), r((SD),(SB)), r((SD),(SC)), l((SE),(SB)), l((SE),(SA))\}$; using the RA $\mathcal{CYC}_t$, the description can be given as a 2-relation set: $\{rll((SA),(SB),(SE)), rrr((SB),(SC),(SD))\}$.

Schlieder's system makes an implicit assumption, which is that the object to be located (i.e., S) is not on any of the lines joining pairs of the reference objects—such a fact cannot be represented within his system. This assumption can be made explicit (or indeed could be explicitly contradicted) in the RA $\mathcal{CYC}_b$ representation of the problem (the relations

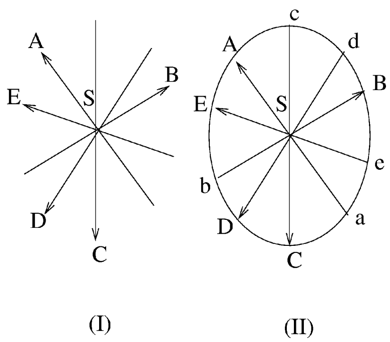

Fig. 16. The panorama of a location.

e(qual) and o(pposite) can be used to describe object S being on a line joining two reference objects). Note that Schlieder does not describe the important task of how to reason about a panorama description.

10.3. Cardinal direction models

Frank's models of cardinal directions in 2D [11,12] are illustrated in Fig. 17. They use a partition of the plane into regions determined by lines passing through a reference object, say S. Depending on the region a point P belongs to, the position of P relative to S is *North*, *North-East*, *East*, *South-East*, *South*, *South-West*, *West*, *North-West*, or *Equal*. Each of the two models can thus be seen as a binary RA, with nine atoms. Both use a global, *West-East/South-North*, reference system. The projection-based model has been assessed as being cognitively more plausible [11,12] (cognitive plausibility of spatial orientation models are discussed in [14,15]), and its computational properties have been studied by Ligozat [28]. In particular, Ligozat made use of tractability results known for Allen's interval algebra [1] and Vilain and Kautz's point algebra [48] to find a maximal tractable subset including all atoms (maximal in the sense that adding any other relation to the subset leads to an NP-hard subset). The drawback of Frank's models is that they use a global reference system.

The RA $\mathcal{CYC}_b$ we have presented can be used for the representation of relative orientation knowledge about a configuration of 2D points. Such knowledge would contain for pairs (A, B) of objects in the configuration the position of (the primary object) B relative to (the reference object) A, as viewed from a global point of view, say S: B is on line (SA) on the same side of S as A, to the left of A, on line (SA) on the side of S opposite to that of A, or to the right of A. The drawback here is that the point of view is global.

Thus the common points of Frank's models and our RA $\mathcal{CYC}_b$ are:

(1) the use of a global concept (a global reference system in the former case, a global viewpoint in the latter); and
(2) the representation of knowledge as binary relations describing a primary object relative to a reference object.

Their respective expressive powers are however incomparable.

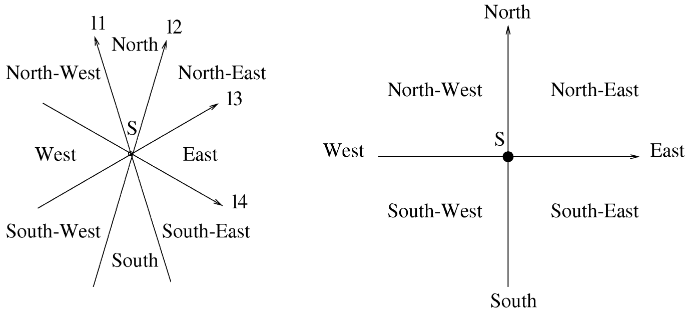

Fig. 17. Frank's cone-shaped (left) and projection-based (right) models of cardinal directions.

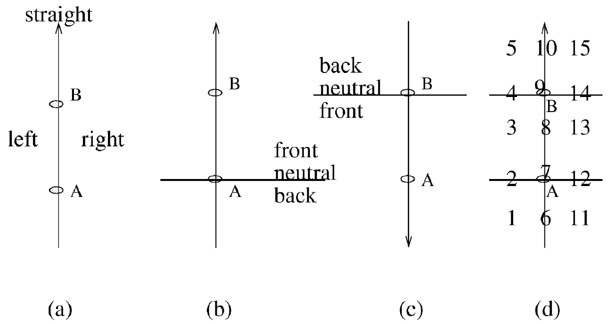

Fig. 18. The partition of the universe of 2D positions on which is based the Double-Cross calculus in [14,49].

10.4. Relative orientation models

A well-known model of relative orientation of 2D points is the Double-Cross calculus defined by Freksa [14], and developed further by Zimmermann and Freksa [49]. The calculus corresponds to a specific partition, into 15 regions, of the plane determined by a parent object, say A, and a reference object, say B (Fig. 18(d)). The partition is based on the following:

(1) the *left/straight/right* partition of the plane determined by an observer placed at the parent object and looking in the direction of the reference object (Fig. 18(a));
(2) the *front/neutral/back* partition of the plane determined by the same observer (Fig. 18(b)); and
(3) the similar *front/neutral/back* partition of the plane obtained when we swap the roles of the parent object and the reference object (Fig. 18(c)).

Combining the three partitions (a), (b) and (c) of Fig. 18 leads to the partition of the universe of 2D positions on which is based the calculus in [14,49] (Fig. 18(d)).

Our RA $\mathcal{CYC}_t$ can be used for the description of a configuration of 2D points as viewed from a global viewpoint; the model is thus more suited for a panorama-like description. Freksa's calculus, on the other hand, is more suited for the description of a configuration of 2D points (a spatial scene) relative to one another. The two calculi are thus incomparable in terms of expressive power. We have shown that the atoms of $\mathcal{CYC}_t$ form a tractable subset, from which derives a complete solution search procedure for a general problem expressed in the RA. For the Double-Cross calculus, however, no tractable procedure for the subset of all atomic relations is known.

10.5. Reasoning about 2D segments

In his paper "Reasoning About Ordering" [43], Schlieder presented a set of line segment relations. These relations are based on the cyclic ordering of endpoints of the segments involved. We believe that reasoning about 2D segments should combine at least orientational and topological information. Orientational information would be information about cyclic ordering of the orientations of the directed lines supporting the segments; on the other hand, topological information would be the description of the relative positions of the segments' endpoints. For instance, using the algebra of binary relations on 2D orientations, as defined in this work, we could define an algebra of 2D segments, which would have the following segment relations (given a segment s, we denote by s_l and s_r its left and its right endpoints, respectively, i.e., s is the segment $[s_l s_r]$; by z_s the orientation of the directed line $(s_l s_r)$ supporting segment s):

(1) If the orientations z_{s_1} and z_{s_2} are equal, the endpoints of s_2 are:
   (a) both to the left of the directed line supporting segment s_1 (one relation);
   (b) both on the line supporting segment s_1 (13 relations: see Allen's temporal interval algebra [1]); or
   (c) both to the right of the line supporting segment s_1 (one relation).
(2) If z_{s_2} is to the left of z_{s_1}, this leads to 25 segment relations, which are obtained as follows:
   (a) The endpoints of s_1 partition the directed line supporting the segment into five regions: the region strictly to the left of the left endpoint, the region consisting of the left endpoint, the region strictly between the two endpoints, the region consisting of the right endpoint, and the region strictly to the right of the right endpoint. Similarly, the endpoints of s_2 partition the directed line supporting the segment into five regions.
   (b) The lines supporting the segments s_1 and s_2 are intersecting, and the intersecting point is in either of the five regions of the first line, and in either of the five regions of the second line. This gives the 25 segment relations.
(3) if z_{s_1} and z_{s_2} are opposite to each other, we get 15 relations in a similar manner as in point (1) above.
(4) if z_{s_2} is to the right of z_{s_1}, we get 25 relations in a similar manner as in point (2) above.

Therefore the total number of JEPD segment relations would be 80.

11. Future work

In the spatial and temporal domains, there has already been much work based on Allen-like algebras [1]. For instance:

(1) In the temporal domain, Allen's algebra [1] has been shown to be NP-complete [48]. This gave rise to considerable work on tractable subsets of the algebra (see, for instance, [3,9,23,34,45]); the most important is certainly Nebel and Bürckert's ORD-Horn subclass [34], shown by the authors to be the unique maximal subset among all tractable subsets containing all 13 atomic relations.
(2) In the spatial domain, similar work has been achieved by Renz and Nebel [36,38] for the well known Region Connection Calculus (RCC) [35].
(3) Much work [13,14,27] has investigated the concept, closely related to tractable reasoning, of a conceptual neighbourhood.

Most of this work could be adapted to the two RAs of 2D orientations we have defined. We should say, however, that from corollaries 4 and 5 it follows that a tractable subset of either RA including all atoms cannot include the universal relation, and vice-versa. In terms of expressiveness, the minimal condition for a subset of an RA to be useful is to include all atoms as well as the universal relation; this can be justified thus:

(1) it is important for real applications to be offered the possibility of expressing complete information, which is made possible only if all atoms are present; and
(2) it is important as well for real applications to be offered the possibility of providing no information on some tuples of the manipulated objects, which is made possible only if the universal relation is present.

In the light of these comments, we are committed to face intractability if what we want is to get expressively useful subsets of either RA.

We have provided for the ternary RA a polynomial constraint propagation procedure, which is incomplete in the general case (the RA has been shown to be NP-complete), but still complete for a subset including all atoms. Problems corresponding to actual data (or most randomly generated data) may not lie in the subset. As a consequence, it would be interesting to study the behaviour of a general solution search algorithm, such as the one we have provided (which is exponential in the general case, but solves any problem expressed in the RA), on actual or most randomly generated instances. In the temporal domain as well as in the spatial domain, much work on this issue is known [22,25,33,37,47].

The RAs we have presented do not take into account the *front/neutral/back* partition of the plane determined by an observer placed at the point of view and looking in the direction of the reference object; i.e., the partition of the plane into

(1) the open half-plane consisting of the front of the observer;
(2) the open half-plane consisting of the back of the observer; and
(3) the borderline between the two half-planes.

Augmenting the binary RA with this feature would lead to eight atoms (*equal*, *left-front*, *left*, *left-back*, *opposite*, *right-back*, *right* and *right-front*). The corresponding ternary RA has 80 atoms, which can be enumerated by appropriately refining the illustrations of the $\mathcal{CYC}_t$ atoms depicted in Fig. 8. For instance, refining the leftmost configuration of the top row in Fig. 8 leads to five configurations (see Fig. 19). We plan to investigate the computational properties of this finer-grained calculus.

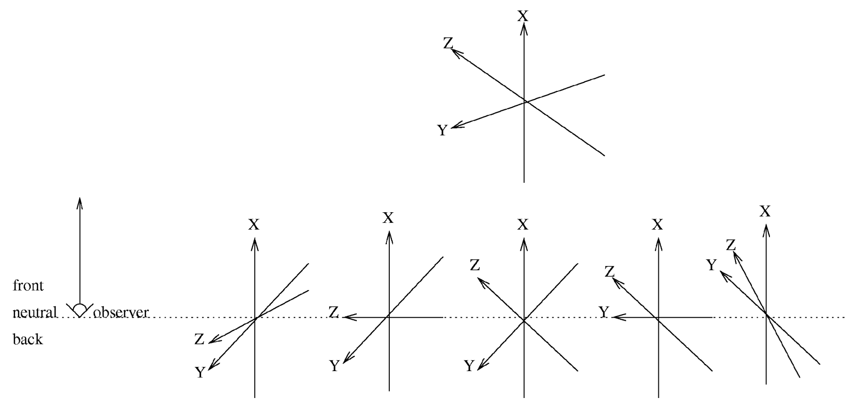

Fig. 19. (Top) The $\mathcal{CYC}_t$ atom *lrl*, and (bottom) its refinement resulting from adding the *front/neutral/back* partition of the plane to the relation algebras we have presented.

One of the biggest challenges for qualitative spatial reasoning is the integration of qualitative distance and qualitative orientation. A formalism with such a characteristic would, for instance, allow for the representation of natural language descriptions such as "B is closer than, and to left of, A" (B lies within an appropriate sector of the disc centred at the speaker's location, say S, and of radius SA). This challenge has been discussed by many authors [11,12,15], and one recent and promising work addressing the issue is [29].

An extension of our work worth looking at is the propositional calculus of which the literals are of the form $b(x,y)$ or $\neg b(x,y)$, where b is an atom of the RA $\mathcal{CYC}_b$, and x and y are (2D orientation) variables. A problem expressed in such an extension, after its transformation into a Conjunctive Normal Form (CNF), would be a conjunction of clauses of literals of the form $b(x,y)$ or $\neg b(x,y)$. Since $\neg b(x,y)$ is equivalent to $(\mathit{BIN} \setminus \{b\})(x,y)$, where *BIN* is the universal relation of the RA $\mathcal{CYC}_b$, the clauses can indeed be equivalently written as disjunctions of literals of the form $b(x,y)$. The ternary RA $\mathcal{CYC}_t$ is a sublanguage of this extension of the binary RA.

Finally, a calculus of 3D orientations, similar to the ternary RA of 2D orientations we have presented, might be developed.

12. Summary

We have provided a new approach to cyclic ordering of 2D orientations, consisting of a relation algebra (RA) whose universe is a set of ternary relations. We have investigated for the RA several algorithmic and computational properties; in particular:

(1) We have provided a constraint propagation procedure achieving strong 4-consistency for a CSP expressed in the RA; and shown that the procedure is polynomial, and complete for a subset including all atoms.
(2) We have shown that a subset expressing only information on parallel orientations is NP-complete.
(3) We have shown that provided that a subset $\mathcal{S}$ of $\mathcal{CYC}_t$ includes two specific elements, deciding consistency for a CSP expressed in the closure of $\mathcal{S}$ under the different operations of the RA can be polynomially reduced to deciding consistency for a CSP expressed in $\mathcal{S}$.
(4) From the previous result, we have derived that the set obtained by adding the universal relation to the set of all atoms of the RA is NP-complete.
(5) From the previous result, we have derived that a much less expressive RA, whose universe is a set of binary relations on 2D orientations, is already NP-complete.

We have discussed briefly how this work could be extended, and pointed out to work on other formalisms, such as tractable subsets of Allen's algebra of temporal intervals [1], that could be adapted to the two relation algebras we have presented.

Acknowledgements

Our thanks are due to helpful comments from the anonymous reviewers.

Appendix A. Verifying the RA properties for $\mathcal{CYC}_b$

For the RA $\mathcal{CYC}_t$, presented in Section 6, which is strictly more expressive than $\mathcal{CYC}_b$, we will verify all the RA properties. Thus, it is not necessary to verify them for $\mathcal{CYC}_b$. Note, however, that one has to check, at the level of atoms, that the results of applying the operations of converse and composition to the different atoms are correct; in other words, we have to check for $\mathcal{CYC}_b$ the following:

(1) the entries of the converse table: the entry $\mathit{Conv}(b)$ on row b of the converse table must be equal to the converse $b^{\smile}$ of b, i.e., we must have $\mathit{Conv}(b) = b^{\smile} = \{(x, y)\colon (y, x) \in b\}$; and

(2) the entries of the composition table: the entry $T(b_1, b_2)$ on row b_1 and column b_2 of the composition table must be equal to the composition, $b_1 \circ b_2$, of atoms b_1 and b_2, i.e., we must have $T(b_1, b_2) = b_1 \circ b_2 = \{(x, y)\colon (\exists z)(b_1(x, z) \wedge b_2(z, y))\}$.

We will show that this is indeed the case for $\mathcal{CYC}_b$. As we saw before, given two orientations X and Y:

$$e(Y, X) \quad \text{iff} \quad (X, Y) \in \{0\}, \tag{A.1}$$

$$l(Y, X) \quad \text{iff} \quad (X, Y) \in (0, \pi), \tag{A.2}$$

$$o(Y, X) \quad \text{iff} \quad (X, Y) \in \{\pi\}, \tag{A.3}$$

$$r(Y, X) \quad \text{iff} \quad (X, Y) \in (\pi, 2\pi). \tag{A.4}$$

In other words, the atoms e, l, o, r correspond, respectively, to the convex subsets $\{0\}, (0, \pi), \{\pi\}, (\pi, 2\pi)$ of $[0, 2\pi)$.

Checking the entries of the converse table: By definition, $e^{\smile} = \{(x, y)\colon (y, x) \in e\}, l^{\smile} = \{(x, y)\colon (y, x) \in l\}, o^{\smile} = \{(x, y)\colon (y, x) \in o\}, r^{\smile} = \{(x, y)\colon (y, x) \in r\}$. Using the four equivalences (A.1)–(A.4), we get the following: $e^{\smile} = \{(x, y)\colon (x, y) \in \{0\}\}, l^{\smile} = \{(x, y)\colon (x, y) \in (0, \pi)\}, o^{\smile} = \{(x, y)\colon (x, y) \in \{\pi\}\}, r^{\smile} = \{(x, y)\colon (x, y) \in (\pi, 2\pi)\}$. The assertions $(x, y) \in \{0\}, (x, y) \in (0, \pi), (x, y) \in \{\pi\}, (x, y) \in (\pi, 2\pi)$ being equivalent, respectively, to $(y, x) \in \{0\}, (y, x) \in (\pi, 2\pi), (y, x) \in \{\pi\}, (y, x) \in (0, \pi)$, we get: $e^{\smile} = \{(x, y)\colon (y, x) \in \{0\}\}, l^{\smile} = \{(x, y)\colon (y, x) \in (\pi, 2\pi)\}, o^{\smile} = \{(x, y)\colon (y, x) \in \{\pi\}\}, r^{\smile} = \{(x, y)\colon (y, x) \in (0, \pi)\}$. Using again the equivalences (A.1)–(A.4), we get that the converse table records the exact converses of the atoms: $e^{\smile} = \{(x, y)\colon e(x, y)\} = e, l^{\smile} = \{(x, y)\colon r(x, y)\} = r, o^{\smile} = \{(x, y)\colon o(x, y)\} = o, r^{\smile} = \{(x, y)\colon l(x, y)\} = l$.

Checking the entries of the composition table: In order to check that $T(b_1, b_2) = b_1 \circ b_2$, it is sufficient to use the following sound inference rule, in which A and B denote convex subsets of $[0, 2\pi)$, and $\mathit{size}(X)$ is the maximum of all $y - x$ for $x, y \in X$:

$$\begin{aligned} &\bigl[(X, Z) \in A \wedge (Z, Y) \in B \wedge \mathit{size}(A) < \pi \wedge \mathit{size}(B) < \pi\bigr] \\ &\quad \Rightarrow (X, Y) \in A +_s B, \end{aligned} \tag{A.5}$$

where $+_s$ is set addition (composition): $A +_s B = \{c\colon (\exists a \in A, \exists b \in B)(c = a + b)\}$. We claim that the inference rule is 3-complete for $A, B \in \{\{0\}, (0, \pi), \{\pi\}, (\pi, 2\pi)\}$; i.e., for any such A and B, we have the following:

$$(\forall X, Y)\bigl[(X, Y) \in A +_s B \Rightarrow (\exists Z)\bigl((X, Z) \in A \wedge (Z, Y) \in B\bigr)\bigr].$$

Theorem A.1. *The inference rule* (A.5) *is* 3-*complete for* $A, B \in \{\{0\}, (0, \pi), \{\pi\}, (\pi, 2\pi)\}$.

Proof. We proceed by enumerating all possible cases. Cases (1) and (6) in the enumeration are illustrated in Fig. A.1:

(1) If $A = \{0\}$ then $A +_s B = B$. For all X, Y such that $(X, Y) \in B$, if we take $Z = X$ then $(X, Z) = (X, X) = 0 \in A$ and $(Z, Y) = (X, Y) \in B$ (see Fig. A.1(a)).

(2) If $A = \{\pi\}$ then $A +_s B = \{\pi + \beta\colon\ \beta \in B\}$. Let X, Y be such that $(X, Y) \in A +_s B$: $(\exists \beta \in B)((X, Y) = \pi + \beta)$. We take $Z = X + \pi$; then from $(X, Y) = \pi + \beta$ and $Z = \pi + X$, we infer that $(X, Z) = \pi \in A$ and $(Z, Y) = \beta \in B$.

(3) If $B = \{0\}$ then $A +_s B = A$. For all X, Y such that $(X, Y) \in A$, if we take $Z = Y$ then $(X, Z) = (X, Y) \in A$ and $(Z, Y) = (Y, Y) = 0 \in B$.

(4) If $B = \{\pi\}$ then $A +_s B = \{\alpha + \pi\colon\ \alpha \in A\}$. Let X, Y be such that $(X, Y) \in A +_s B$: $(\exists \alpha \in A)((X, Y) = \alpha + \pi)$. We take $Z = Y + \pi$; thus $Y = Z + \pi$, from which we can infer $(Z, Y) = \pi \in B$. From $(X, Y) = \alpha + \pi$ and $(Y, Z) = \pi$, we infer $(X, Z) = \alpha \in A$.

(5) If $A = (0, \pi)$ and $B = (0, \pi)$ then $A +_s B = \{\alpha + \beta\colon\ \alpha \in (0, \pi), \beta \in (0, \pi)\} = (0, 2\pi)$. For all X and Y such that $(X, Y) \in (0, 2\pi)$, we can find Z such that $(X, Z) \in (0, \pi)$ and $(Z, Y) \in (0, \pi)$: take Z in such a way that $(X, Z) = (Z, Y)$ (i.e., Z is the bisector of (X, Y)).

(6) If $A = (0, \pi)$ and $B = (\pi, 2\pi)$ then $A +_s B = \{\alpha + \beta\colon\ \alpha \in (0, \pi), \beta \in (\pi, 2\pi)\} = (\pi, 3\pi) \equiv [0, \pi) \cup (\pi, 2\pi)$. For all X and Y such that $(X, Y) \in [0, \pi) \cup (\pi, 2\pi)$, we can find Z such that $(X, Z) \in (0, \pi)$ and $(Z, Y) \in (\pi, 2\pi)$:
   (a) if $(X, Y) \in (\pi, 2\pi)$ (see Fig. A.1(b)), take Z in such a way that $(X, Z) = (Z, Y')$, where Y' is the orientation opposite to Y (i.e., $(Y, Y') = \pi$);
   (b) if $(X, Y) = 0$ (see Fig. A.1(c)), take Z in such a way that $(X, Z) = \pi/2$; and
   (c) if $(X, Y) \in (0, \pi)$ (see Fig. A.1(d)), take Z in such a way that $(Y, Z) = (Z, X')$, where X' is the orientation opposite to X (i.e., $(X, X') = \pi$).

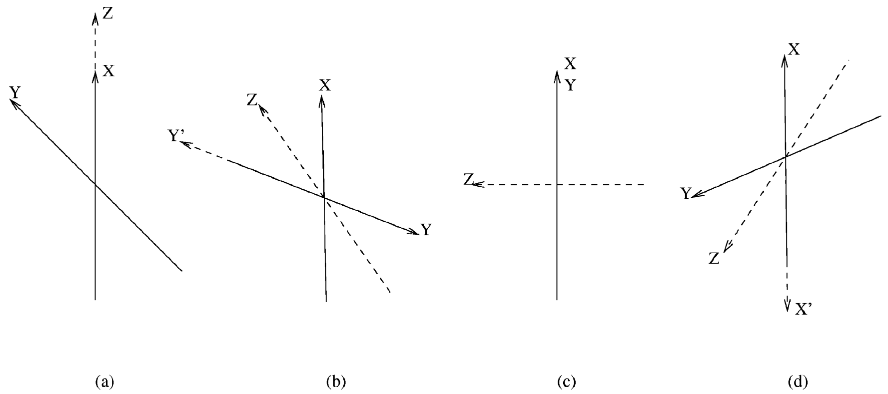

Fig. A.1. Illustration of the proof of Theorem A.1.

(7) If $A = (\pi, 2\pi)$ and $B = (0, \pi)$ then $A +_s B = \{\alpha + \beta\colon\ \alpha \in (\pi, 2\pi), \beta \in (0, \pi)\} = (\pi, 3\pi) \equiv [0, \pi) \cup (\pi, 2\pi)$. For all X and Y such that $(X, Y) \in (\pi, 3\pi)$, we can find Z such that $(X, Z) \in (\pi, 2\pi)$ and $(Z, Y) \in (0, \pi)$:
   (a) if $(X, Y) \in (\pi, 2\pi)$, take Z in such a way that $(X', Z) = (Z, Y)$ (i.e., Z is the bisector of (X', Y)), where X' is the orientation opposite to X (i.e., X' is such that $(X, X') = \pi$);
   (b) if $(X, Y) = 0$, take Z in such a way that $(X, Z) = 3\pi/2$; and
   (c) if $(X, Y) \in (0, \pi)$, take Z in such a way that $(Y', Z) = (Z, X)$ (i.e., Z is the bisector of (Y', X)), where Y' is the orientation opposite to Y (i.e., Y' is such that $(Y, Y') = \pi$).

(8) If $A = (\pi, 2\pi)$ and $B = (\pi, 2\pi)$ then $A +_s B = \{\alpha + \beta\colon\ \alpha \in (\pi, 2\pi), \beta \in (\pi, 2\pi)\} = (2\pi, 4\pi) \equiv (0, 2\pi)$. For all X and Y such that $(X, Y) \in (0, 2\pi)$, we can find Z such that $(X, Z) \in (\pi, 2\pi)$ and $(Z, Y) \in (\pi, 2\pi)$:
   (a) if $(X, Y) \in (\pi, 2\pi)$, take Z in such a way that $(Y, Z) = (Z, X)$ (i.e., Z is the bisector of (Y, X));
   (b) if $(X, Y) = \pi$, take Z in such a way that $(X, Z) = 3\pi/2$; and
   (c) if $(X, Y) \in (0, \pi)$, take Z in such a way that $(X', Z) = (Z, Y')$ (i.e., Z is the bisector of (X', Y')), where X' is the orientation opposite to X (i.e., $(X, X') = \pi$) and Y' is the orientation opposite to Y (i.e., $(Y, Y') = \pi$). $\square$

Appendix B. Verifying the RA properties for an atomic ternary RA

(1) $(R \circ S) \circ T = R \circ (S \circ T)$?
   (a) Let $(a, b, c) \in (R \circ S) \circ T$. Thus $(\exists d)((a, b, d) \in (R \circ S) \wedge (a, d, c) \in T)$. $(a, b, d) \in (R \circ S)$ implies $(\exists e)((a, b, e) \in R \wedge (a, e, d) \in S)$. From $(a, e, d) \in S$ and $(a, d, c) \in T$, we infer $(a, e, c) \in (S \circ T)$. From $(a, b, e) \in R$ and $(a, e, c) \in (S \circ T)$, we infer $(a, b, c) \in R \circ (S \circ T)$. Therefore $(R \circ S) \circ T \subseteq R \circ (S \circ T)$.
   (b) Now let $(a, b, c) \in R \circ (S \circ T)$. Thus $(\exists d)((a, b, d) \in R \wedge (a, d, c) \in (S \circ T))$. $(a, d, c) \in (S \circ T)$ implies $(\exists e)((a, d, e) \in S \wedge (a, e, c) \in T)$. From $(a, b, d) \in R$ and $(a, d, e) \in S$, we infer $(a, b, e) \in (R \circ S)$. From $(a, b, e) \in (R \circ S)$ and $(a, e, c) \in T$, we infer $(a, b, c) \in (R \circ S) \circ T$. Therefore $(R \circ S) \circ T \supseteq R \circ (S \circ T)$.

(2) $(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$?
   $(R \cup S) \circ T = \{(a, b, c)\colon\ (\exists d)((a, b, d) \in (R \cup S) \wedge (a, d, c) \in T)\} = \{(a, b, c)\colon\ (\exists d)([(a, b, d) \in R \vee (a, b, d) \in S] \wedge (a, d, c) \in T)\} = \{(a, b, c)\colon\ (\exists d)((a, b, d) \in R \wedge (a, d, c) \in T)\} \cup \{(a, b, c)\colon\ (\exists d)((a, b, d) \in S \wedge (a, d, c) \in T)\} = (R \circ T) \cup (S \circ T)$.

(3) $R \circ \mathcal{I} = \mathcal{I} \circ R = R$?
   We prove this for $\mathcal{I} = \mathcal{I}_U^{t_{23}} = \{(a, b, b)\colon\ a, b \in U\}$. The reason for this is that the identity element of the atomic ternary RA of our interest, $\mathcal{CYC}_t$, is $\mathcal{I}_{2\mathrm{D}\mathcal{O}}^{t_{23}}$ $(U = 2\mathrm{D}\mathcal{O})$.
   $R \circ \mathcal{I} = \{(a, b, c)\colon\ (\exists d)((a, b, d) \in R \wedge (a, d, c) \in \mathcal{I})\}$. But $(a, d, c) \in \mathcal{I}$ implies $(d = c)$; thus $R \circ \mathcal{I} = \{(a, b, c)\colon\ (a, b, c) \in R \wedge (a, c, c) \in \mathcal{I}\}$. Since

$(\forall a,c)((a,c,c)\in\mathcal{I})$, we infer $R\circ\mathcal{I}=\{(a,b,c)\colon (a,b,c)\in R\}=R$. On the other hand, $\mathcal{I}\circ R=\{(a,b,c)\colon (\exists d)((a,b,d)\in\mathcal{I}\wedge(a,d,c)\in R)\}$. But $(a,b,d)\in\mathcal{I}$ implies $(d=b)$; thus $\mathcal{I}\circ R=\{(a,b,c)\colon (a,b,b)\in\mathcal{I}\wedge(a,b,c)\in R\}$. Since $(\forall a,b)((a,b,b)\in\mathcal{I})$, we infer $\mathcal{I}\circ R=\{(a,b,c)\colon (a,b,c)\in R\}=R$.

(4) $(R^{\smile})^{\smile}=R$?
$(R^{\smile})^{\smile}=\{(a,b,c)\colon (a,c,b)\in R^{\smile}\}$. But $(a,c,b)\in R^{\smile}$ is equivalent to $(a,b,c)\in R$. Therefore $(R^{\smile})^{\smile}=\{(a,b,c)\colon (a,b,c)\in R\}=R$.

(5) $(R\cup S)^{\smile}=R^{\smile}\cup S^{\smile}$?
$(R\cup S)^{\smile}=\{(a,b,c)\colon (a,c,b)\in R\cup S\}=\{(a,b,c)\colon (a,c,b)\in R\vee(a,c,b)\in S\}=\{(a,b,c)\colon (a,c,b)\in R\}\cup\{(a,b,c)\colon (a,c,b)\in S\}=R^{\smile}\cup S^{\smile}$.

(6) $(R\circ S)^{\smile}=S^{\smile}\circ R^{\smile}$?
$(R\circ S)^{\smile}=\{(a,b,c)\colon (a,c,b)\in R\circ S\}=\{(a,b,c)\colon (\exists d)((a,c,d)\in R\wedge(a,d,b)\in S)\}=\{(a,b,c)\colon (\exists d)((a,d,c)\in R^{\smile}\wedge(a,b,d)\in S^{\smile})\}=\{(a,b,c)\colon (\exists d)((a,b,d)\in S^{\smile}\wedge(a,d,c)\in R^{\smile})\}=S^{\smile}\circ R^{\smile}$.

(7) $R^{\smile}\circ\overline{R\circ S}\cap S=\emptyset$?
Let $(a,b,c)\in R^{\smile}\circ\overline{R\circ S}$. Thus $(\exists d)((a,b,d)\in R^{\smile}\wedge(a,d,c)\in\overline{R\circ S})$. $(a,d,c)\in\overline{R\circ S}$ is equivalent to $(a,d,c)\notin R\circ S$, which in turn implies $(\forall e)((a,d,e)\notin R\vee(a,e,c)\notin S)$. Now consider the special case $e=b$: from $(a,b,d)\in R^{\smile}$, we derive $(a,d,b)\in R$; thus $(a,b,c)\notin S$.

(8) $((R^{\frown})^{\frown})^{\frown}=R$?
$((R^{\frown})^{\frown})^{\frown}=\{(a,b,c)\colon (c,a,b)\in(R^{\frown})^{\frown}\}$. But $(c,a,b)\in(R^{\frown})^{\frown}$ is equivalent to $(b,c,a)\in R^{\frown}$, which in turn is equivalent to $(a,b,c)\in R$. Therefore $((R^{\frown})^{\frown})^{\frown}=\{(a,b,c)\colon (a,b,c)\in R\}=R$.

(9) $(R\cup S)^{\frown}=R^{\frown}\cup S^{\frown}$?
$(R\cup S)^{\frown}=\{(a,b,c)\colon (c,a,b)\in R\cup S\}=\{(a,b,c)\colon (c,a,b)\in R\vee(c,a,b)\in S\}=\{(a,b,c)\colon (c,a,b)\in R\}\cup\{(a,b,c)\colon (c,a,b)\in S\}=R^{\frown}\cup S^{\frown}$.

(10) *Checking the entries of the different tables*: Similarly to $\mathcal{CYC}_b$, we have to check that the converse table, the rotation table and the composition tables of $\mathcal{CYC}_t$ record the exact converses, the exact rotations and the exact compositions of the atoms.
The converse table and the rotation table: From the fact that the $\mathcal{CYC}_b$ converse table records the exact converses of the atoms, we derive straightforwardly that the converse table and the rotation table of $\mathcal{CYC}_t$ record the exact converses and the exact rotations of the atoms. We illustrate this with the atom *lrr*. By definition, $(lrr)^{\smile}=\{(x,y,z)\colon (x,z,y)\in lrr\}$. Applying the definition of a $\mathcal{CYC}_t$ atom, we get: $(lrr)^{\smile}=\{(x,y,z)\colon (z,x)\in l\wedge(y,z)\in r\wedge(y,x)\in r\}$. Reordering the elements of the conjunction $(z,x)\in l\wedge(y,z)\in r\wedge(y,x)\in r$, we get: $(lrr)^{\smile}=\{(x,y,z)\colon (y,x)\in r\wedge(y,z)\in r\wedge(z,x)\in l\}$. Thanks to the fact that the $\mathcal{CYC}_b$ converse table records the exact converses of the atoms, we derive that $(y,z)\in r$ iff $(z,y)\in l$, from which we get: $(lrr)^{\smile}=\{(x,y,z)\colon (y,x)\in r\wedge(z,y)\in l\wedge(z,x)\in l\}$. Now the set $\{(x,y,z)\colon (y,x)\in r\wedge(z,y)\in l\wedge(z,x)\in l\}$ corresponds exactly the $\mathcal{CYC}_t$ atom *rll*, which implies that $(lrr)^{\smile}=rll$. By definition of the rotation operation, we get: $(lrr)^{\frown}=\{(x,y,z)\colon (z,x,y)\in lrr\}$. Using the definition of a $\mathcal{CYC}_t$ atom, we get: $(lrr)^{\frown}=\{(x,y,z)\colon (x,z)\in l\wedge(y,x)\in r\wedge(y,z)\in r\}$. Reordering the elements of the conjunction $(x,z)\in l\wedge(y,x)\in r\wedge(y,z)\in r$, we get: $(lrr)^{\frown}=\{(x,y,z)\colon (y,x)\in r\wedge(y,z)\in r\wedge(x,z)\in l\}$. Thanks, again, to the

fact that the $\mathcal{CYC}_b$ converse table records the exact converses of the atoms, we get that the assertions $(y,z) \in r$ and $(x,z) \in l$ are equivalent, respectively, to $(z,y) \in l$ and $(z,x) \in r$, which implies: $(lrr)^\frown = \{(x,y,z)\colon (y,x) \in r \wedge (z,y) \in l \wedge (z,x) \in r\}$. Now the set $\{(x,y,z)\colon (y,x) \in r \wedge (z,y) \in l \wedge (z,x) \in r\}$ corresponds exactly to the $\mathcal{CYC}_t$ atom rlr; thus $(lrr)^\frown = rlr$.

The composition tables: We say that the $\mathcal{CYC}_t$ composition tables are sound if for any two atoms t_1 and t_2, it is the case that $T(t_1,t_2) \supseteq t_1 \circ t_2$, where $T(t_1,t_2)$ is the entry on the row labelled with t_1 and the column labelled with t_2; if the tables are sound, we say that they are 4-complete if for any two atoms t_1 and t_2, it is the case that $T(t_1,t_2) \subseteq t_1 \circ t_2$. Soundness implies that if we know that a triple (x,y,w) belongs to $t_1 \circ t_2$, which, by definition, means that we can find z such that $(x,y,z) \in t_1$ and $(x,z,w) \in t_2$, then it must be the case that the triple (x,y,w) also belongs to the entry $T(t_1,t_2)$; if 4-completeness also holds then the triples (x,y,w) in the relation recorded by an entry correspond exactly to the actual composition of the corresponding atoms. We show how to compute the entries of the composition tables; this will at the same time show 4-completeness of the tables. For this purpose, we consider two atoms $t_1 = b_1b_2b_3$ and $t_2 = b'_1b'_2b'_3$. As we saw before, due to the fact that the $\mathcal{CYC}_b$ atoms are JEPD, $t_1 \circ t_2 = \emptyset$ if $b_3 \neq b'_1$ (again, refer to Fig. 6 for illustration); so we suppose $b_3 = b'_1 = b$, which leads to $t_1 = b_1b_2b$, $t_2 = bb'_2b'_3$, and $t_1 \circ t_2 = \{(x,y,w)\colon (\exists z)((y,x) \in b_1 \wedge (z,y) \in b_2 \wedge (z,x) \in b \wedge (w,z) \in b'_2 \wedge (w,x) \in b'_3)\}$. We will need the isomorphim ϕ from $2\mathrm{D}\mathcal{O} \times 2\mathrm{D}\mathcal{O}$ onto $2\mathrm{D}\mathcal{O} \times 2\mathrm{D}\mathcal{O}$, defined as follows: $\phi((x,y)) = (x',y)$, where x' the orientation opposite to x, i.e., x' is such that $o(x,x')$; the isomorphism is extended to subsets of $2\mathrm{D}\mathcal{O} \times 2\mathrm{D}\mathcal{O}$ in the following natural way: $\phi(S) = \{\phi((x,y))\colon (x,y) \in S\}$; for the $\mathcal{CYC}_b$ atoms, which are particular subsets of $2\mathrm{D}\mathcal{O} \times 2\mathrm{D}\mathcal{O}$, we get $\phi(e) = o, \phi(l) = r, \phi(o) = e, \phi(r) = l$. We proceed by enumerating all possible cases:

(a) if $b_1 = e$ then $t_1 \circ t_2 = eb'_3b'_3$;
(b) if $b_2 = e$ then $t_1 \circ t_2 = t_2$;
(c) if $b'_2 = e$ then $t_1 \circ t_2 = t_1$;
(d) if $b'_3 = e$ then $t_1 \circ t_2 = b_1(b_1)^\smile e$;
(e) if $b_1 = o$ then from $(y,x) \in o \wedge (w,x) \in b'_3$ we get $(w,y) \in \phi(b'_3)$; thus $t_1 \circ t_2 = \{(x,y,w)\colon (y,x) \in b_1 \wedge (w,y) \in \phi(b'_3) \wedge (w,x) \in b'_3\} = b_1\phi(b'_3)b'_3$;
(f) if $b'_3 = o$ then from $(x,y) \in (b_1)^\smile \wedge (w,x) \in o$ we get $(w,y) \in \phi((b_1)^\smile)$; thus $t_1 \circ t_2 = \{(x,y,w)\colon (y,x) \in b_1 \wedge (w,y) \in \phi((b_1)^\smile) \wedge (w,x) \in b'_3\} = b_1\phi((b_1)^\smile)b'_3$;
(g) if $b_2 = o$ then from $(w,z) \in b'_2 \wedge (z,y) \in o$ we get $(w,y) \in \phi(b'_2)$; thus $t_1 \circ t_2 = \{(x,y,w)\colon (y,x) \in b_1 \wedge (w,y) \in \phi(b'_2) \wedge (w,x) \in b'_3\} = b_1\phi(b'_2)b'_3$;
(h) if $b'_2 = o$ then from $(z,y) \in b_2 \wedge (w,z) \in o$ we get $(w,y) \in \phi(b_2)$; thus $t_1 \circ t_2 = \{(x,y,w)\colon (y,x) \in b_1 \wedge (w,y) \in \phi(b_2) \wedge (w,x) \in b'_3\} = b_1\phi(b_2)b'_3$;
(i) if $b = e$ then JEPDness of the $\mathcal{CYC}_b$ atoms gives $b_2 = (b_1)^\smile$ and $b'_2 = b'_3$; this leads to $t_1 \circ t_2 = \Pi(b_1, b'_3 \circ (b_1)^\smile, b'_3)$ (4-completeness comes from 3-completess of the $\mathcal{CYC}_b$ composition table: each entry records the exact composition of the corresponding $\mathcal{CYC}_b$ atoms);

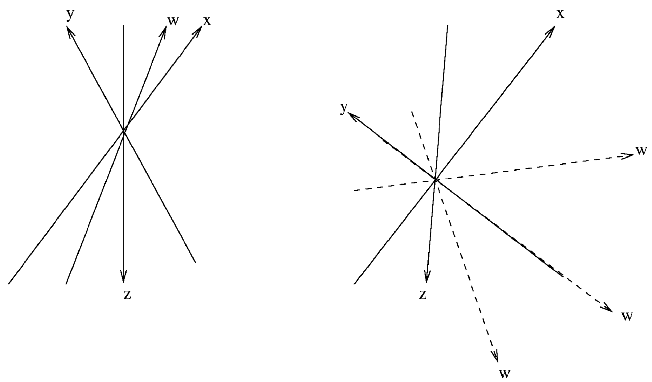

Fig. B.1. Illustration of 4-completeness of the $\mathcal{CYC}_t$ composition tables.

(j) In a similar way, if $b = o$ then we infer that $b'_3 = \phi(b'_2)$ and $b_1 = \phi((b_2)^{\smile})$; thus $t_1 \circ t_2 = \Pi(b_1, b'_3 \circ (b_1)^{\smile}, b'_3)$. Again, 4-completeness stems from 3-completeness of the $\mathcal{CYC}_b$ composition table.

The remaining cases are those when each of b_1, b_2, b, b'_2, b'_3 belongs to $\{l, r\}$. These cover altogether 32 entries of the composition tables: 16 of these consist of atoms, the other 16 of 3-atom relations. We prove 4-completeness for one 1-atom entry and for one 3-atom entry; the 4-completeness proof for the other entries is similar. We consider the entries $T(llr, rll) = lrl$ and $T(llr, rlr) = \{llr, lor, lrr\}$.

$T(llr, rll) = lrl$?

Consider four orientations x, y, z, w such that $llr(x, y, z) \wedge rll(x, z, w)$. This is illustrated in Fig. B.1(left). Orientation w is forced to be between-in-a-clockwise-direction the orientation opposite to z and orientation x. The illustration clearly indicates that the relation on triple (x, y, w) is lrl. Conversely, consider a configuration of three orientations x, y and w such that $lrl(x, y, w)$. We can always find z such that $llr(x, y, z) \wedge rll(x, z, w)$: for instance, we can take z such that $o(z, z')$, where z' in turn is such that $(w, z') = (z', y)$ (z' is the bisector of (w, y)).

$T(llr, rlr) = \{llr, lor, lrr\}$?

Consider four orientations x, y, z, w such that $llr(x, y, z) \wedge rlr(x, z, w)$. This is illustrated in Fig. B.1(right). Orientation w is forced to be to the left of, opposite to, or to the right of, y; thus the relation on triple (x, y, w) is $\Pi(l, \{l, o, r\}, r) = \{llr, lor, lrr\}$. Conversely, consider a configuration of three orientations x, y and w such that $\{llr, lor, lrr\}(x, y, w)$. We can always find z such that $llr(x, y, z) \wedge rlr(x, z, w)$: if $llr(x, y, w)$ or $lor(x, y, w)$ then take z such that $(x', z) = (z, w)$, where x' is such that $o(x, x')$; otherwise, take z such that $(x', z) = (z, y')$, where x' and y' are such that $o(x, x')$ and $o(y, y')$.

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