Analytic Geometry Formulas

1. Lines in two dimensions

Line forms

Slope - intercept form:

$$y = mx + b$$

Two point form:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Point slope form:

$$y - y_1 = m(x - x_1)$$

Intercept form

$$\frac{x}{a} + \frac{y}{b} = 1 \quad (a, b \neq 0)$$

Normal form:

$$x \cdot \cos\sigma + y \sin\sigma = p$$

Parametric form:

$$x = x_1 + t\cos\alpha$$
$$y = y_1 + t\sin\alpha$$

Point direction form:

$$\frac{x - x_1}{A} = \frac{y - y_1}{B}$$

where (A,B) is the direction of the line and $P_1(x_1, y_1)$ lies on the line.

General form:

$$A \cdot x + B \cdot y + C = 0 \quad A \neq 0 \;\; or\, B \neq 0$$

Distance

The distance from $Ax + By + C = 0$ to $P_1(x_1, y_1)$ is

$$d = \frac{|A \cdot x_1 + B \cdot y_1 + C|}{\sqrt{A^2 + B^2}}$$

Concurrent lines

Three lines

$$A_1 x + B_1 y + C_1 = 0$$
$$A_2 x + B_2 y + C_2 = 0$$
$$A_3 x + B_3 y + C_3 = 0$$

are concurrent if and only if:

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0$$

Line segment

A line segment P_1P_2 can be represented in parametric form by

$$x = x_1 + (x_2 - x_1)t$$
$$y = y_1 + (y_2 - y_1)t$$
$$0 \le t \le 1$$

Two line segments P_1P_2 and P_3P_4 intersect if any only if the numbers

$$s = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}}{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_4 & y_3 - y_4 \end{vmatrix}} \quad and \quad t = \frac{\begin{vmatrix} x_3 - x_1 & y_3 - y_1 \\ x_3 - x_4 & y_3 - y_4 \end{vmatrix}}{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_4 & y_3 - y_4 \end{vmatrix}}$$

satisfy $0 \le s \le 1 \;\; and \;\; 0 \le t \le 1$

2. Triangles in two dimensions

Area

The area of the triangle formed by the three lines:

$$A_1 x + B_1 y + C_1 = 0$$
$$A_2 x + B_2 y + C_2 = 0$$
$$A_3 x + B_3 y + C_3 = 0$$

is given by

$$K = \frac{\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}^2}{2 \cdot \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \cdot \begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix} \cdot \begin{vmatrix} A_3 & B_3 \\ A_1 & B_1 \end{vmatrix}}$$

The area of a triangle whose vertices are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$:

$$K = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$K = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}.$$

Centroid

The centroid of a triangle whose vertices are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$:

$$(x, y) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Incenter

The incenter of a triangle whose vertices are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$:

$$(x, y) = \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$$

where a is the length of P_2P_3, b is the length of P_1P_3, and c is the length of P_1P_2.

Circumcenter

The circumcenter of a triangle whose vertices are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$:

$$(x, y) = \left(\frac{\begin{vmatrix} x_1^2 + y_1^2 & y_1 & 1 \\ x_2^2 + y_2^2 & y_2 & 1 \\ x_3^2 + y_3^2 & y_3 & 1 \end{vmatrix}}{2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \frac{\begin{vmatrix} x_1 & x_1^2 + y_1^2 & 1 \\ x_2 & x_2^2 + y_2^2 & 1 \\ x_3 & x_3^2 + y_3^2 & 1 \end{vmatrix}}{2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}} \right)$$

Orthocenter

The orthocenter of a triangle whose vertices are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$:

$$(x, y) = \left(\frac{\begin{vmatrix} y_1 & x_2x_3 + y_1^2 & 1 \\ y_2 & x_3x_1 + y_2^2 & 1 \\ y_3 & x_1x_2 + y_3^2 & 1 \end{vmatrix}}{2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \frac{\begin{vmatrix} x_1^2 + y_2y_3 & x_1 & 1 \\ x_2^2 + y_3y_1 & x_2 & 1 \\ x_3^2 + y_1y_2 & x_3 & 1 \end{vmatrix}}{2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}} \right)$$

3. Circle

Equation of a circle

In an x-y coordinate system, the circle with centre (a, b) and radius r is the set of all points (x, y) such that:

$$(x - a)^2 + (y - b)^2 = r^2$$

Circle is centred at the origin

$$x^2 + y^2 = r^2$$

Parametric equations

$$x = a + r\cos t$$
$$y = b + r\sin t$$

where t is a parametric variable.

In polar coordinates the equation of a circle is:

$$r^2 - 2rr_o\cos(\theta - \varphi) + r_o^2 = a^2$$

Area

$$A = r^2\pi$$

Circumference

$$c = \pi \cdot d = 2\pi \cdot r$$

Theorems:

(Chord theorem)

The chord theorem states that if two chords, CD and EF, intersect at G, then:

$$CD \cdot DG = EG \cdot FG$$

(Tangent-secant theorem)

If a tangent from an external point D meets the circle at C and a secant from the external point D meets the circle at G and E respectively, then

$$DC^2 = DG \cdot DE$$

(Secant - secant theorem)

If two secants, DG and DE, also cut the circle at H and F respectively, then:

$$DH \cdot DG = DF \cdot DE$$

(Tangent chord property)

The angle between a tangent and chord is equal to the subtended angle on the opposite side of the chord.

4. Conic Sections

The Parabola

The set of all points in the plane whose distances from a fixed point, called the focus, and a fixed line, called the directrix, are always equal.

The standard formula of a parabola:

$$y^2 = 2px$$

Parametric equations of the parabola:

$$x = 2pt^2$$
$$y = 2pt$$

Tangent line

Tangent line in a point $D(x_0, y_0)$ of a parabola $y^2 = 2px$

$$y_0 y = p(x + x_0)$$

Tangent line with a given slope (m)

$$y = mx + \frac{p}{2m}$$

Tangent lines from a given point

Take a fixed point $P(x_0, y_0)$.The equations of the tangent lines are

$$y - y_0 = m_1(x - x_0) \ and$$
$$y - y_0 = m_2(x - x_0) \ where$$
$$m_1 = \frac{y_0 + \sqrt{y_0^2 - 2px_0}}{2x_0} \ and$$
$$m_1 = \frac{y_0 - \sqrt{y_0^2 - 2px_0}}{2x_0}$$

The Ellipse

The set of all points in the plane, the sum of whose distances from two fixed points, called the foci, is a constant.

The standard formula of a ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parametric equations of the ellipse

$$x = a\cos t$$
$$y = b\sin t$$

Tangent line in a point $D(x_0, y_0)$ of a ellipse:

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

Eccentricity:

$$e = \frac{\sqrt{a^2 - b^2}}{a}$$

Foci:

$$if\ a > b \Rightarrow F_1(-\sqrt{a^2 - b^2}, 0)\ F_2(\sqrt{a^2 - b^2}, 0)$$
$$if\ a < b \Rightarrow F_1(0, -\sqrt{b^2 - a^2})\ F_2(0, \sqrt{b^2 - a^2})$$

Area:

$$K = \pi \cdot a \cdot b$$

The Hyperbola

The set of all points in the plane, the difference of whose distances from two fixed points, called the foci, remains constant.

The standard formula of a hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Parametric equations of the Hyperbola

$$x = \frac{a}{\sin t}$$
$$y = \frac{b \sin t}{\cos t}$$

Tangent line in a point $D(x_0, y_0)$ of a hyperbola:

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$$

Foci:

$$if\ a > b \Rightarrow F_1(-\sqrt{a^2 + b^2}, 0)\ F_2(\sqrt{a^2 + b^2}, 0)$$
$$if\ a < b \Rightarrow F_1(0, -\sqrt{b^2 + a^2})\ F_2(0, \sqrt{b^2 + a^2})$$

Asymptotes:

$$if\ a > b \Rightarrow y = \frac{b}{a}x\ and\ y = -\frac{b}{a}x$$
$$if\ a < b \Rightarrow y = \frac{a}{b}x\ and\ y = -\frac{a}{b}x$$

5. Planes in three dimensions

Plane forms

Point direction form:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

where P1(x1,y1,z1) lies in the plane, and the direction (a,b,c) is normal to the plane.

General form:

$$Ax + By + Cz + D = 0$$

where direction (A,B,C) is normal to the plane.

Intercept form:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

this plane passes through the points (a,0,0), (0,b,0), and (0,0,c).

Three point form

$$\begin{vmatrix} x - x_3 & y - y_3 & z - z_3 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \end{vmatrix} = 0$$

Normal form:

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p$$

Parametric form:

$$x = x_1 + a_1 s + a_2 t$$
$$y = y_1 + b_1 s + b_2 t$$
$$z = z_1 + c_1 s + c_2 t$$

where the directions (a1,b1,c1) and (a2,b2,c2) are parallel to the plane.

Angle between two planes:

The angle between two planes:

$$A_1 x + B_1 y + C_1 z + D_1 = 0$$
$$A_2 x + B_2 y + C_2 z + D_2 = 0$$

is

$$\arccos \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

The planes are parallel if and only if

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$$

The planes are perpendicular if and only if

$$A_1 A_2 + B_1 B_2 + C_1 C_2 = 0$$

Equation of a plane

The equation of a plane through $P_1(x_1,y_1,z_1)$ and parallel to directions (a_1,b_1,c_1) and (a_2,b_2,c_2) has equation

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

The equation of a plane through $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$, and parallel to direction (a,b,c), has equation

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a & b & c \end{vmatrix} = 0$$

Distance

The distance of P1(x1,y1,z1) from the plane Ax + By + Cz + D = 0 is

$$d = \frac{Ax_1 + By_1 + Cz_1}{\sqrt{A^2 + B^2 + C^2}}$$

Intersection

The intersection of two planes

$$A_1 x + B_1 y + C_1 z + D_1 = 0,$$
$$A_2 x + B_2 y + C_2 z + D_2 = 0,$$

is the line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

where

$$a = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}$$

$$b = \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}$$

$$c = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$$

$$x_1 = \frac{b\begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix} - c\begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix}}{a^2 + b^2 + c^2}$$

$$y_1 = \frac{c\begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix} - c\begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix}}{a^2 + b^2 + c^2}$$

$$z_1 = \frac{a\begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix} - b\begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix}}{a^2 + b^2 + c^2}$$

If a = b = c = 0, then the planes are parallel.