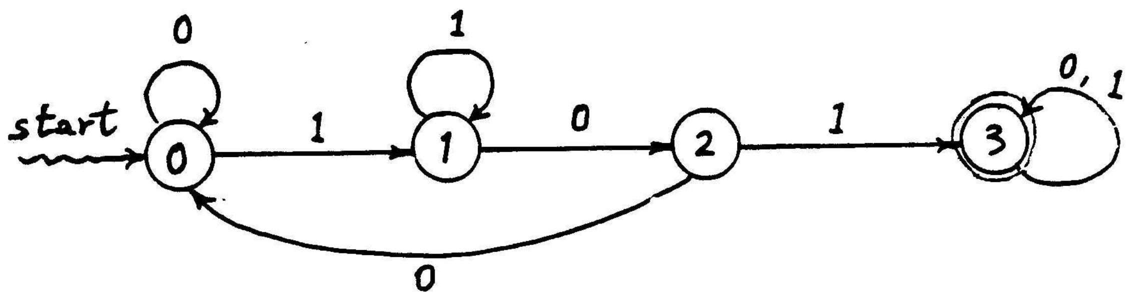
Chapter 2. FINITE AUTOMATA

<u>Example</u> Design a "sequential lock". The lock has 1-bit sequential input. Initially the lock is closed. If the lock is closed it will open when the last three input signals are "1", "0", "1", and then remains open.

— <u>state (transition) diagram</u>



— state (or transition) table

Present state \ Present symbol	0	1
0	0	1
1	2	1
2	0	3
3	3	3

state set : $\{0,\ 1,\ 2,\ 3\}$
input alphabet : $\{0,\ 1\}$
Transition function : $\delta(0,0)=0,\ \delta(0,1)=1,\ \ldots$
Start state : 0
Final state set : $\{3\}$

Deterministic Finite Automata (DFA)

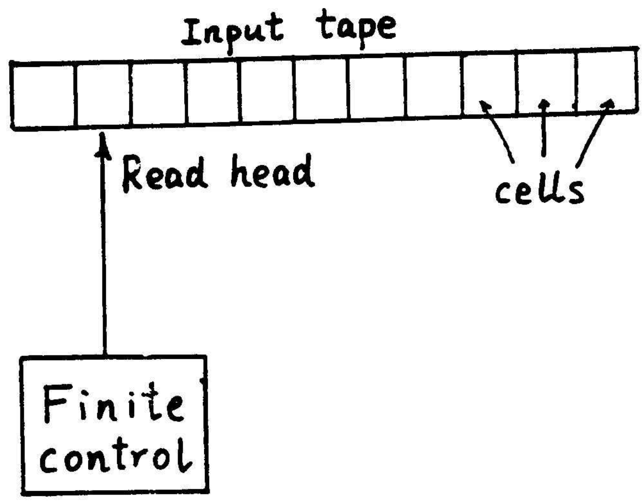
$M=(Q,\ \Sigma,\ \delta,\ s,\ F)$ **where**

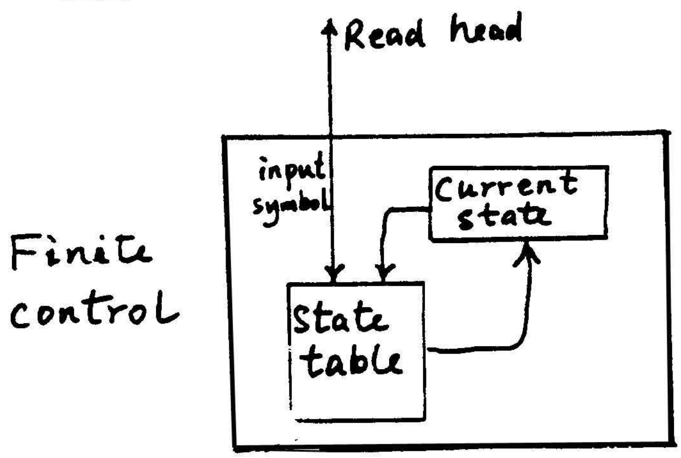
- Q **is a finite nonempty set of states**
- Σ **is the input alphabet**
- $\delta : Q\times\Sigma\rightarrow Q$ **transition function**
- **s** **start state**
- $F\subseteq Q$ **final state set**

A computer is a finite state system (i.e. FA) which has millions of states.

There are many examples of <u>FINITE STATE SYSTEMS</u>. A finite automaton is an <u>ABSTRACTION</u> of them.

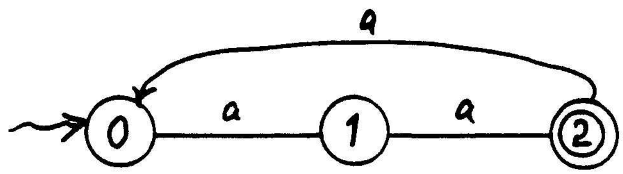
<u>View a DFA as a machine</u>



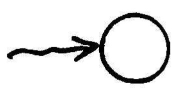


Specifying δ

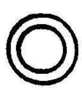
1)



State diagram
(Transition diagram)



Start state



Final state

2)

Present state \ Present symbol	a
0	1
1	2
2	0

Configurations

a word in $Q\Sigma^*$

$$px$$

where p is the present state, and
x is the remaining input

Example:

$0aa$ **. . .** $1a$ **. . .** 2
(start configuration) **(final configuration)**

Moves of a DFA

$0aa \vdash 1a$ $\qquad$ $px \vdash qy$
$1a \vdash 2$ $\qquad$ **if** $x = ay$ **and** $\delta(p, a) = q$

Configuration sequence

$0aa \vdash 1a \vdash 2$

$\vdash^+$ and $\vdash^*$

$\vdash$ **is a binary relation over $Q\Sigma^*$.**
$\vdash^+$: **transitive closure of $\vdash$.**
$\vdash^*$: **reflexive transitive closure of $\vdash$.**

$$0aa \vdash^{+} 2$$
$$0aa \vdash^{*} 2$$
$$0aa \vdash^{*} 0aa$$
$$0aa \vdash^{2} 2$$

$$px \vdash^{k} qy$$

if $px \underbrace{\vdash p_{i_1}x_{i_1} \vdash p_{i_2}x_{i_2} \vdash \ldots}_{k \textbf{ steps}} \vdash qy$

<u>Accepting Configuration Sequence</u>

$0aa \vdash 1a \vdash 2$

$\vdash$ **can also be viewed as a <u>function</u>**

$$\vdash : Q\Sigma^* \to Q\Sigma^*,$$

since the next configuration is determined uniquely for a given configuration.

The DFA stops <u>when</u>:

(i) we have no more input,

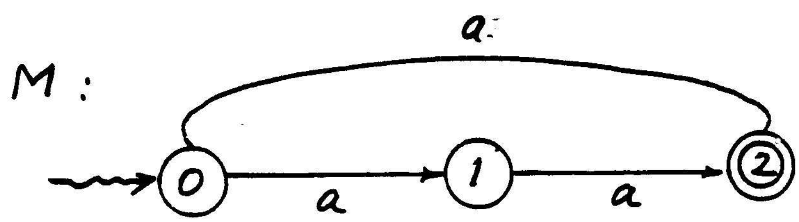
or (ii) the next configuration is undefined.

A <u>word</u> x is said to be <u>accepted</u> by a DFA M if $sx \ \vdash^* \ f, \ f \in F$.

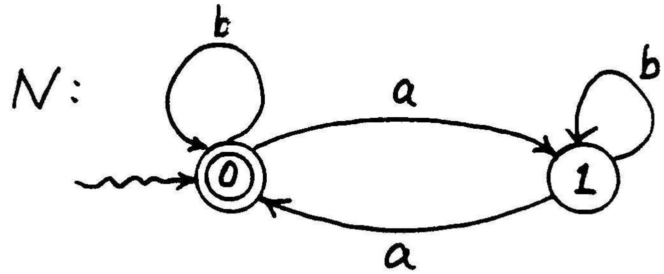
The <u>language</u> of a DFA M, $L(M)$, is defined as:

$$\underline{L(M) = \{x \mid sx \ \vdash^* \ f, \ \textbf{for some } f \in F\}}$$

<u>Examples</u>



$L(M) =$



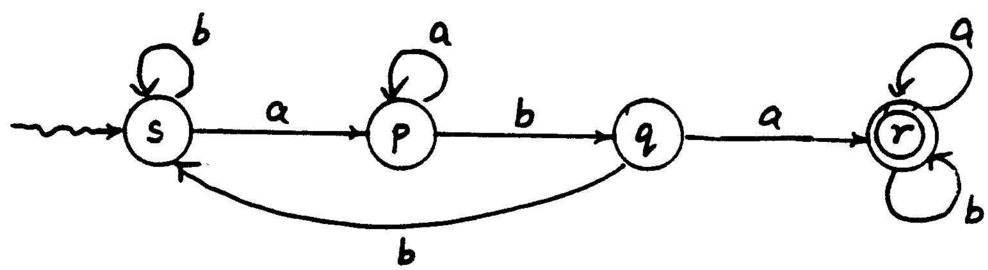
$L(N) =$

DFA membership problem

DFA MEMBERSHIP
INSTANCE: A DFA, $M = (Q, \Sigma, \delta, s, F)$
and a word $x \in \Sigma^*$.
QUESTION: Is x **in** $L(M)$ **?**

Run the DFA M **with input** x.
In at most $|x|$ **steps it accepts, rejects or aborts.**

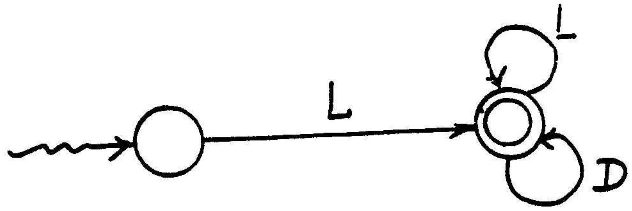
<u>Examples</u>



Checking for words that contain <u>aba</u> **as subword.**

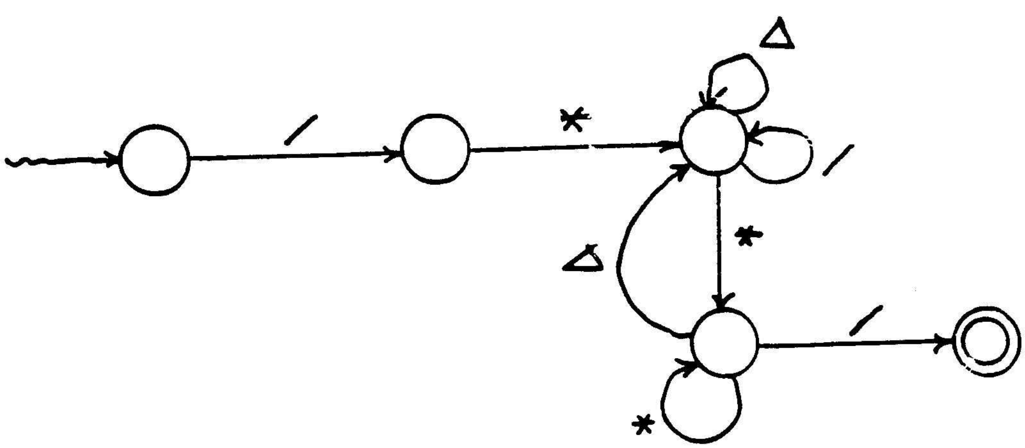
Check: $ababba$
$abbaabbbab$

Let L denote any letter of English alphabet and D any decimal digit; the form of PASCAL IDENTIFIERS can be specified by



Recognizing comments that may go over several lines.

/*................*/

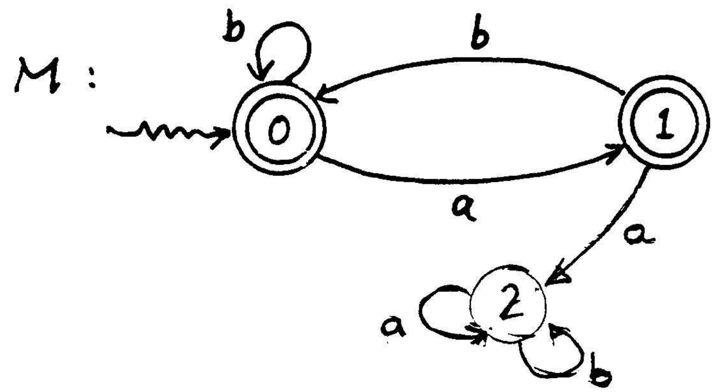


Δ: symbols other than "$*$" and "/"

A DFA which has a total δ is said to be complete; if δ is nontotal it is incomplete.

Theorem. Every incomplete DFA M can be "completed" by adding one new state ("sink") to give DFA M' such that $L(M') = L(M)$.

Example:



$L(M)$ is the set of all words that do not contain two consecutive a's.

$\triangle$ **Two DFA M_1 and M_2 are equivalent if $L(M_1) = L(M_2)$.**

$\triangle$ **The collection of languages accepted by DFA's is denoted by**

$$\underline{\mathcal{L}_{DFA}}.$$

It is called the family of DFA languages and it is defined as:

$\mathcal{L}_{DFA} = \{L \mid L = L(M)$ **for some DFA** $M\ \}$

$\triangle$
$K = \{a^i b^i \mid i \geq 1\}$ **is not accepted by any DFA.**

Proof: Use contradiction and Pigeonhole principle.

Assume $K = L(M)$, for some DFA

$$M = (Q, \{a, b\}, \delta, s, F).$$

Let $n = \#Q$. Consider the accepting configuration sequence for $a^n b^n$,

$$s_0 a^n b^n \vdash s_1 a^{n-1} b^n \vdash \ldots \vdash s_n b^n \vdash \ldots \vdash s_{2n}$$

where $s_0 = s$ and $s_{2n} \in F$. Now $n+1$ states appear during the reading of a^n, but there are only n distinct states in Q. By Pigeonhole principle at least one state must appear at least twice during the reading of a's.

Assume $s_i = s_j, 0 \leq i < j \leq n$.
Then

$$s_0 a^{n-(j-i)} b^n \vdash \ldots \vdash s_i a^{n-j} b^n$$
$$s_j a^{n-j} b^n \vdash \ldots \vdash s_n b^n \vdash \ldots \vdash$$
$$\vdash s_{2n}$$

Therefore $a^{n-(j-i)} b^n \in K$.

This is a contradiction.

$\triangle$ $L_i = \{a^i b^i\}, i \geq 1.$

For any $i \geq 1$, is L_i a DFA language?

$\triangle$ $K_j = \{a^i b^i : 0 \leq i \leq j\}, j \geq 1.$

For any $j \geq 1$, is K_j a DFA language ?

Nondeterministic Finite Automata (NFA)

$M = (Q, \Sigma, \delta, s, F)$

same as a DFA except

$\delta \subseteq Q \times \Sigma \times Q$.

δ **is a finite transition relation.**

In a DFA

δ **is a transition function:**

$\delta : Q \times \Sigma \rightarrow Q$

It can be viewed as a relation

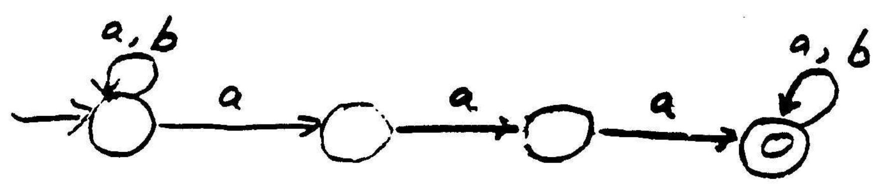
$\delta : Q \times \Sigma \times Q$

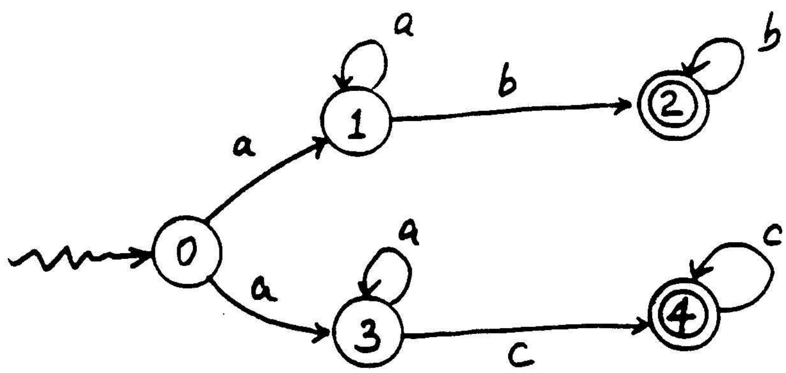
In a NFA, δ **can be be viewed as a function:**

$\delta : Q \times \Sigma \rightarrow 2^Q$

Examples:

NFA for words in $\{a, b\}^*$ **that contain three consecutive a's.**





Both $(0, a, 1)$ **and** $(0, a, 3)$ **are in** δ**.**

We define acceptance by existence of a computation that leads to a final state.

Conversely, we define rejection by the nonexistence of any computation that leads to a final state.

The language of an NFA $M = (Q, \Sigma, \delta, s, F)$ **is defined by**

$L(M) = \{x \mid sx \vdash^* f$**, for some** f **in** F $\}$**.**

The family of NFA languages $\mathcal{L}_{NFA}$

is defined by:

$\mathcal{L}_{NFA} = \{L \mid L = L(M)$**, for some NFA** M $\}$.

Two NFAs M_1, and M_2 are equivalent if $L(M_1) = L(M_2)$.

Why NFA?

(i) easy to construct;
(ii) useful theoretically;
(iii) are of same power as DFA.

Note:

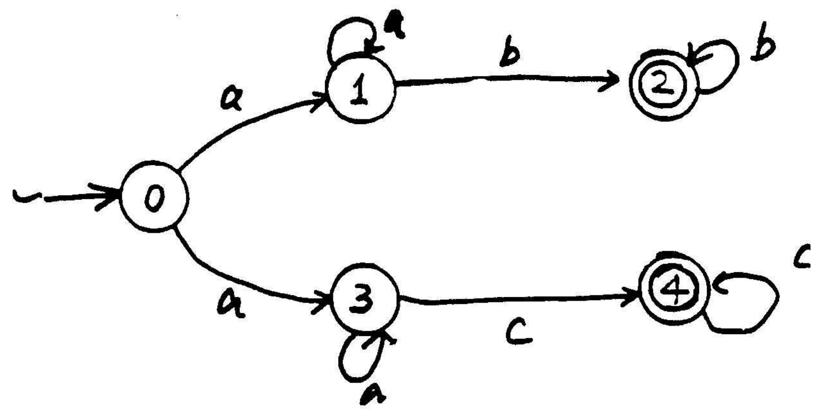
configurations are defined in the same way
Transition (move)

$$px \vdash qy$$

if $x = ay$, **for some** $a \in \Sigma$, **and** $(p, a, q) \in \delta$.

Transforming NFA to DFA

Consider the NFA M_1 again



There are only limited number of choices. For example:

$0\underline{a}ab \vdash 1ab \vdash 1b \vdash 2$

$0aab \vdash 3ab \vdash 3b$

$\{0\}aab \vdash \{1,3\}ab \vdash \{1,3\}b \vdash \{2\}$

Why <u>limited</u> number of choices?

The state set is finite.

We summarize the choices at each step by combining all configuration sequences into one "<u>super-conf. sequence</u>".

$\{0\}aab \vdash \{1,3\}ab \vdash \{1,3\}b \vdash \{2\}$.

We now have a set of all possible states at each step. From this point of view the computation of the NFA on an input word is <u>deterministic</u>.

A <u>super-configuration</u> has the form

$$Kx$$

where $K \subseteq Q$ and $x \in \Sigma^*$.

Note that $\underline{\emptyset x}$ is a super-conf.,
it means that the NFA cannot be in any state at that point, i.e., an abort has occured.

We say that

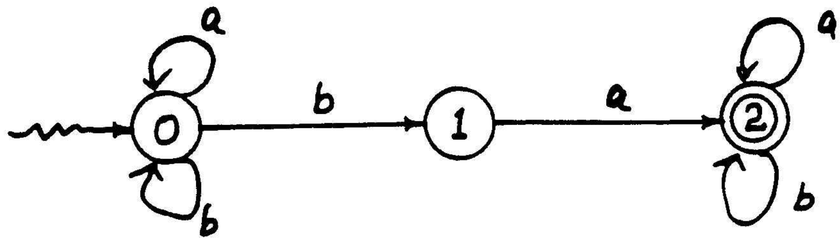
$$Kx \vdash Ny$$

if $x = ay$, for some $a \in \Sigma$, and

$N = \{q \mid (p, a, q) \in \delta$**, for some** $p \in K\}$

More examples on super-configurations

M: $L(M)$ is the set of all words that have "ba" as a subword.



The super-configuration sequence on input word "$abbaa$" is as follows:

$\{0\}abbaa \vdash \{0\}bbaa \vdash \{0,1\}baa \vdash \{0,1\}aa$
$\vdash \{0,2\}a \vdash \{0,2\}$

Notice that given a set $K \subseteq Q$ and an input symbol $a \in \Sigma$, the set $N \subseteq Q$ s.t. $Ka \vdash N$ is uniquely determined.

<u>**Lemma**</u> **(2.3.1) (Determinism Lemma) Let $M = (Q, \Sigma, \delta, s, F)$ be an NFA. Then for all words $\underline{x}$ in Σ^* and for all $\underline{K} \subseteq Q$.**

$\underline{Kx \vdash^* N \text{ and } Kx \vdash^* P}$

implies

$\underline{P = N}$.

<u>**Lemma**</u> **(2.3.2) Let $M = (Q, \Sigma, \delta, s, F)$ be an NFA. Then for all words $\underline{x}$ in Σ^* and for all $\underline{q}$ in Q,**

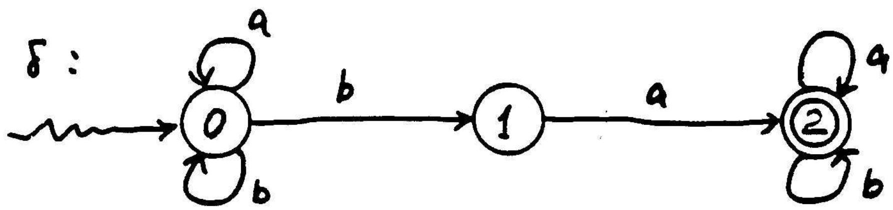
$\underline{qx \vdash^* p}$

iff $\underline{\{q\}x \vdash^* P}$, for some P with p in P.

<u>Example</u> (Transformation of an NFA to a DFA)

$M = (Q, \Sigma, \delta, s, F)$ **where**

$Q = 0, 1, 2, \qquad \Sigma = a, b$
$s = 0, \qquad F = \{2\}$



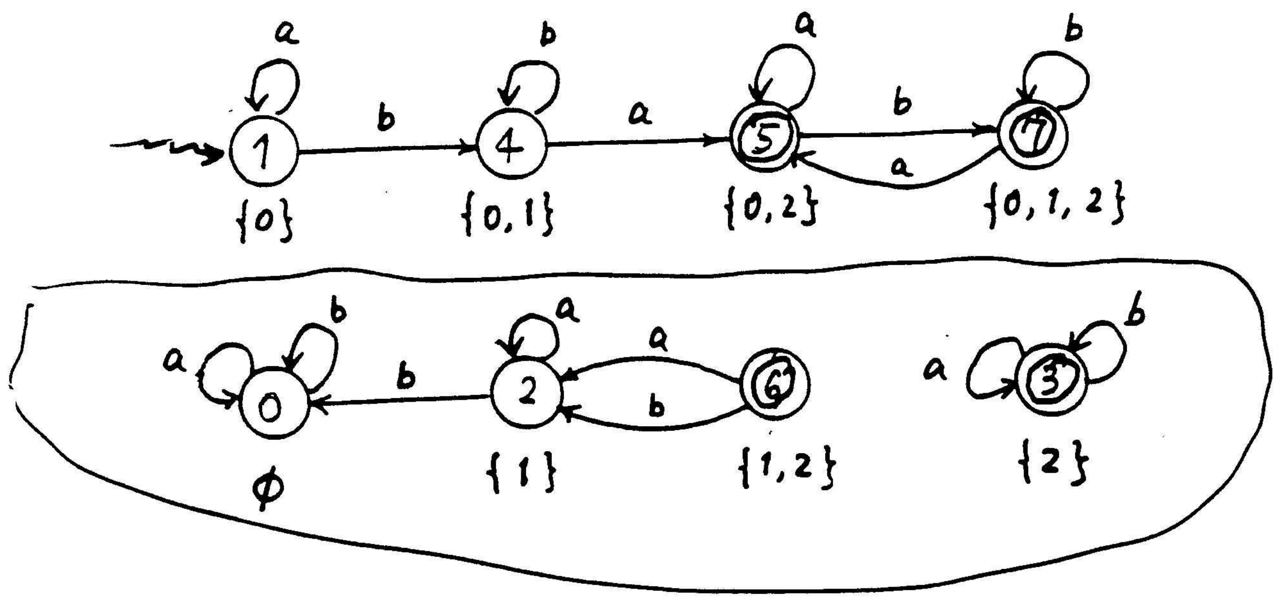
$M' = (Q', \Sigma, \delta', s', F')$ **where**

$$Q' = 2^Q = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

δ':

	current state \ input symbol	a	b
0	$\emptyset$	$\emptyset$	$\emptyset$
1	$\{0\}$	$\{0\}$	$\{0,1\}$
2	$\{1\}$	$\{2\}$	$\emptyset$
3	$\{2\}$	$\{2\}$	$\{2\}$
4	$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
5	$\{0,2\}$	$\{0,2\}$	$\{0,1,2\}$
6	$\{1,2\}$	$\{2\}$	$\{2\}$
7	$\{0,1,2\}$	$\{0,2\}$	$\{0,1,2\}$

$\underline{\delta'(P, a) = \{q \mid (p, a, q) \in \delta \text{ and } p \in P\}}$



$s' = \{0\}$
$F' = \{$

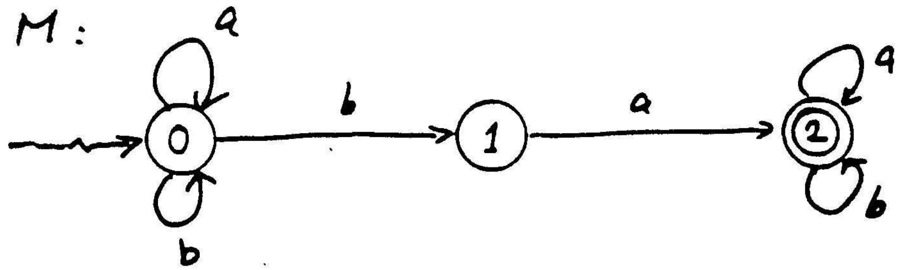
Algorithm NFA to DFA
—The Subset Construction

On entry: An NFA $M = (Q, \Sigma, \delta, s, F)$.

On exit: A DFA $M' = (Q', \Sigma, \delta', s', F')$
satisfying $L(M) = L(M')$**.**

begin Let $Q' = 2^Q, s' = \{s\}$ **and**
$F' = \{K \mid K \in Q',$ **and** $K \cap F \neq \emptyset\}$
We define $\delta' : Q' \times \Sigma \to Q'$ **by**
For all $K \in Q'$ **and for all** $a \in \Sigma$**,**
$\delta'(K, a) = N$**, if** $Ka \vdash N$ **in** M**.**
end of Algorithm

if $N = \{q \mid (p, a, q) \in \delta$ **and** $p \in K\}$

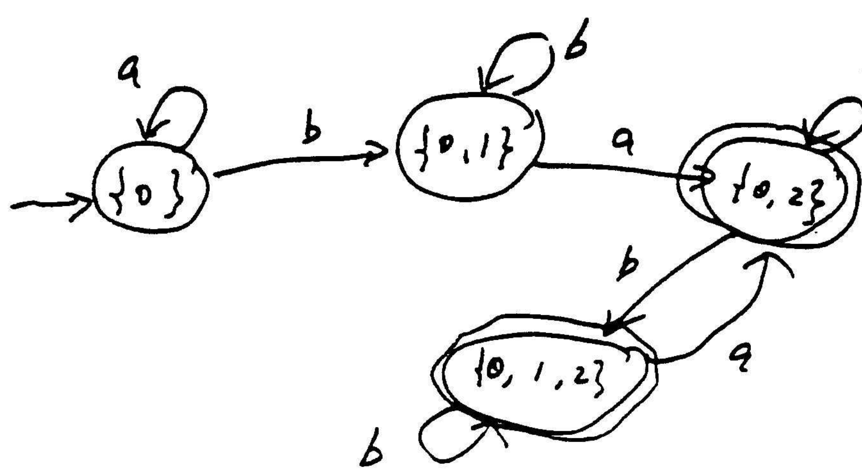


$s' = \{0\}$

current state \ input symbol	a	b
$\{0\}$	$\{0\}$	$\{0, 1\}$
$\{0,1\}$	$\{0,2\}$	$\{0,1\}$
$\{0,2\}$	$\{0,2\}$	$\{0,1,2\}$
$\{0,1,2\}$	$\{0,2\}$	$\{0,1,2\}$

	a	b
0	$\{0\}$	$\{0, 1\}$
1	$\{2\}$	ϕ
2	$\{2\}$	$\{2\}$

Algorithm NFA to DFA 2 —The Iterative Subset Construction



Theorem Given an NFA $M = (Q, \Sigma, \delta, s, F)$, then the DFA $M' = (Q', \Sigma', \delta', s', F')$ obtained by either subset construction satisfies $L(M') = L(M)$.

Proof:

By Lemma 2.3.2, for all $x \in \Sigma^*$ in M

$sx \vdash^* p,$ iff $\{s\}x \vdash^* P$ for some P with $p \in P$

By the construction of M',
$\{s\}x \vdash^* P$ in M iff
$\{s\}x \vdash^* P$ in M'.

$$\begin{aligned}
x \in L(M) &\Leftrightarrow sx \vdash^* f, \text{ for some } f \in F \\
&\Leftrightarrow \{s\}x \vdash^* P,\ f \in P, \text{ in } M \\
&\Leftrightarrow \{s\}x \vdash^* P, \text{ in } M' \text{ and } P \cap F \neq \emptyset \\
&\Leftrightarrow s'x \vdash^* P,\ P \in F \\
&\Leftrightarrow x \in L(M')
\end{aligned}$$

Theorem

Every NFA Language is a DFA language and conversely.
$(\mathcal{L}_{NFA} = \mathcal{L}_{DFA})$

Example

Every finite language is accepted by a DFA.

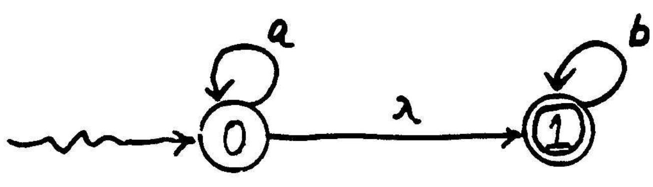
λ–NFA

It is useful to loosen the definition of NFA even more by allowing the <u>read head</u> to remain over the same symbol of the input and <u>read nothing</u>.

Example

$L_1 = \{a^i b^j \mid i \geq 0,\ j \geq 0\}$

M:



$$L(M) = L_1$$

$$\begin{array}{rl} \delta: & (0, a, 0) \\ & \underline{(0, \lambda, 1)} \quad \lambda - transition \\ & (1, b, 1) \end{array}$$

$$\begin{array}{rl} 0aab & \vdash\ 0ab\ \vdash\ 0b\ \vdash\ 1b\ \vdash\ 1 \\ 0bb & \vdash\ 1bb\ \vdash\ 1b\ \vdash\ 1 \\ 0a & \vdash\ 0\ \vdash\ 1 \end{array}$$

Formally, a λ-NFA $M = (Q, \Sigma, \delta, s, F)$ where $Q,\ \Sigma,\ s,\ F$ are as before, but δ is a finite transition relation for which

$$\delta \subseteq Q \times (\Sigma \cup \{\lambda\}) \times Q$$

Configurations are as before.
$\underline{\vdash}$ is defined by

$$px \vdash qy$$

if either $\underline{x = ay}$ for $a \in \Sigma$ and $(p, a, q) \in \delta$
or $\underline{x = y}$ and $(p, \lambda, q) \in \delta$

<u>Example</u>

Given FA M_1 and M_2, construct a FA M_3 such that
$L(M_3) = L(M_1) \cup L(M_2)$

<u>Transforming λ-NFA to NFA</u>

Two steps:
Step I: $\lambda-$ completion

Step II: $\lambda-$ transition removal

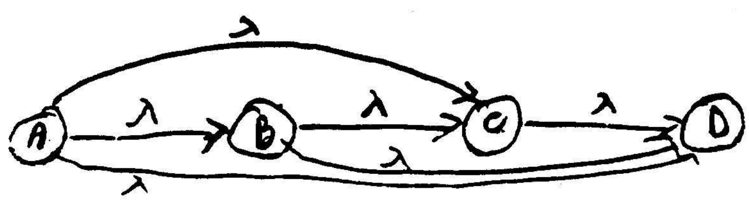
(I). <u>λ-Completion</u>

Given a λ-NFA $M = (Q, \Sigma, \delta, s, F)$
perform the following process:

<u>**For**</u> **all** $p, q, r \in Q$**:**
<u>**whenever**</u> $(p, \lambda, q), (q, \lambda, r)$ **are in** δ
<u>**add**</u> (p, λ, r) <u>**to**</u> δ
<u>**until**</u> **no new transitions are added to** δ
and let this be $\underline{\delta'}$**.**

Let the new $\lambda-$NFA be
$M' = (Q, \Sigma, \delta', s, F')$
where $F' = F \cup \{p \mid (p, \lambda, f) \in \delta$ **and** $f \in F\}$
and $\delta' = \delta \cup \{(p, \lambda, q) \mid p \vdash^{+} q\}$

<u>Example:</u>



Claim 1: **For any** $p,\ q \in Q$**,**

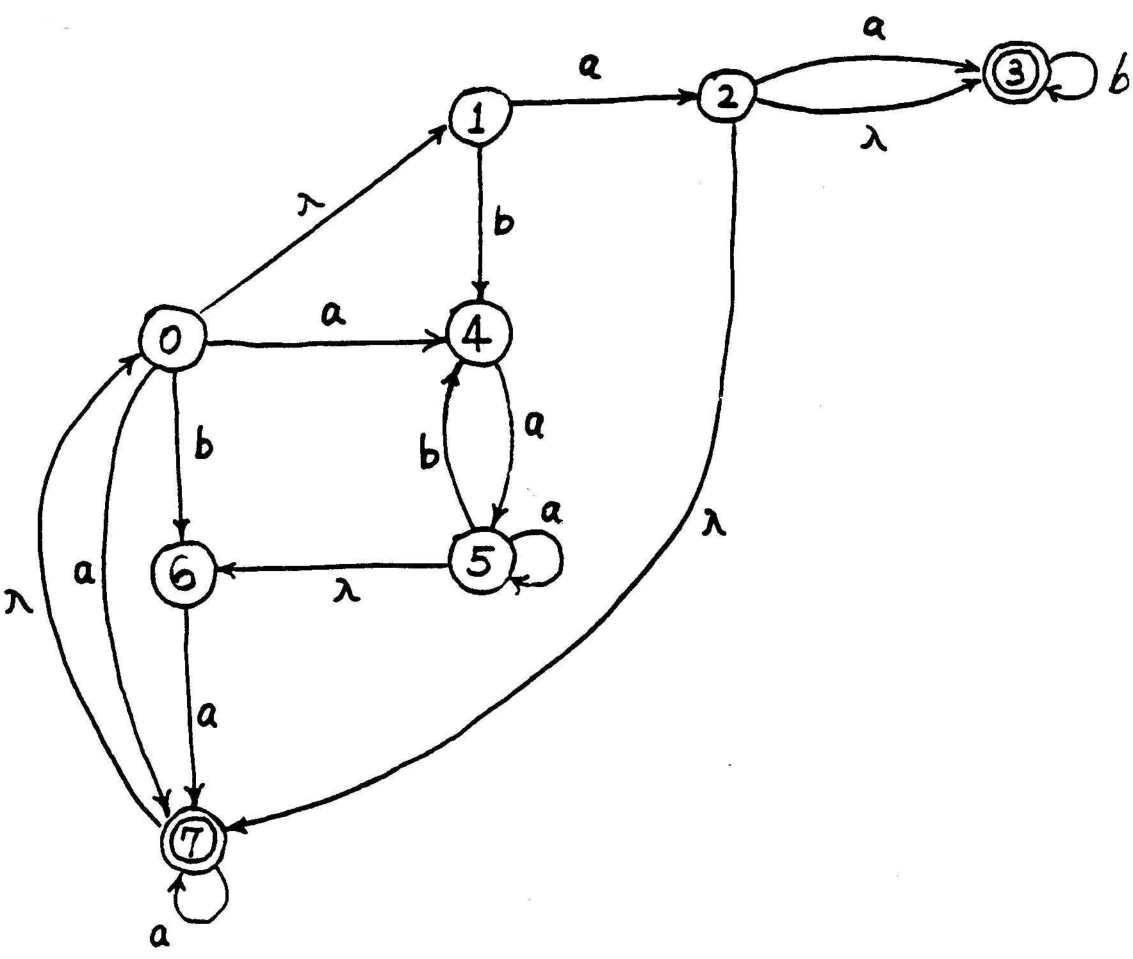
$$p \ \vdash_M^+ \ q \text{ if and only if } p \ \vdash_{M'} \ q$$

Claim 2: **For any** $p,\ q \in Q,\ x \in \Sigma^*$**,**

$$px \ \vdash_M^* \ q \text{ if and only if } px \ \vdash_{M'}^* \ q$$

Theorem: $L(M') = L(M)$

Example:



(II) λ–Transition Removal

Given a λ-completed λ-NFA

$$M = (Q,\ \Sigma,\ \delta,\ s,\ F),$$

perform the following process:

(0) $\delta' = \delta;$

(i) **For all** $p, q, r \in Q$,

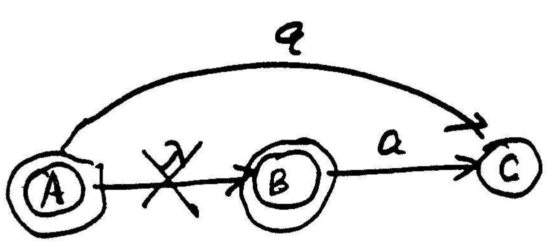
  if $\underline{(p, \lambda, q)}$ **and** $\underline{(q, a, r)}$ **in** δ

  then **<u>add</u>** $\underline{(p, a, r)}$ **to** δ';

(ii) **<u>Delete</u>** **all** **<u>λ-transitions</u>** **from** δ'.

Now we got $M' = (Q,\ \Sigma,\ \delta',\ s,\ F)$
where $\delta' = (\delta \cup \{(p, a, r) \mid (p, \lambda, q), (q, a, r) \in \delta\})$
$- \{(p, \lambda, q) \mid p, q \in Q\}$

<u>Example</u>



<u>Claim</u> Whenever

$$sx \ \vdash_M^* \ f$$

for some $f \in F$, we have

$$sx \ \vdash_{M'}^* \ f$$

and vice versa.

<u>Claim</u> $L(M') = L(M)$

<u>Theorem</u>

$$\mathcal{L}_{\lambda-NFA} = \mathcal{L}_{NFA}$$