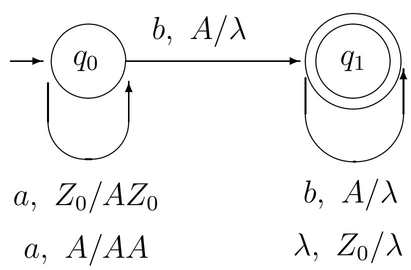
2. Pushdown Automata

Definition **A PDA** A **is a 7-tuple** $(Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ **where**

- Q **: a finite set of states;**
- Σ **: input alphabet;**
- Γ **: stack alphabet;**
- δ **:** $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^*$ **transition relation;**
- $q_0 \in Q$ **: the initial state;**
- $Z_0 \in \Gamma$ **: the bottom-of-stack symbol;**
- $F \subseteq Q$ **: set of final states;**



By Final State : $\{a^i b^j \mid i \geq j \geq 1\}$

Instantaneous descriptions (IDs)

$$(\underbrace{\quad q \quad}_{\text{current state}}, \overbrace{\quad x \quad}^{\text{remaining part of the input}}, \underbrace{\quad \alpha \quad}_{\text{current content of the stack}})$$

An ID describes a configuration of a PDA.

Example

$$\overbrace{(q_0, aab, Z_0)}^{\text{ID for the initial configuration}} \vdash (q_0, ab, AZ_0) \vdash (q_0, b, AAZ_0)$$
$$\underbrace{\vdash (q_1, \varepsilon, AZ_0)}_{\text{ID for an accepting configuration}}$$

Acceptance methods of PDA:

(1) by final state

$$T(A) = \{w \mid (q_0, w, Z_0) \vdash^* (q_f, \varepsilon, \alpha),\ \ q_f \in F\}$$

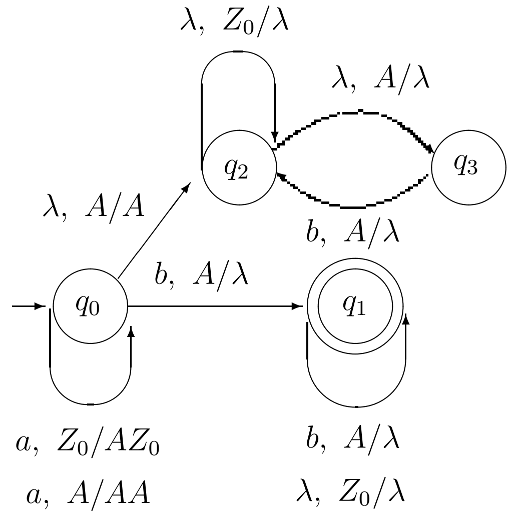
(2) by empty stack

$$N(A) = \{w \mid (q_0, w, Z_0) \vdash^* (q, \varepsilon, \varepsilon)\}$$

(3) by both final state and empty stack

$$L(A) = \{w \mid (q_0, w, Z_0,) \vdash^* (q_f, \varepsilon, \varepsilon),\ \ q_f \in F\}$$

Example



$$N(B) = \{a^i b^i \mid i > 0\} \cup \{a^{2i} b^i \mid i > 0\}$$

$$L(B) = \{a^i b^i \mid i > 0\}$$

$$T(B) = \{a^i b^j \mid i \geq j > 0\}$$

Deterministic Context-free Languages

Definition A PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is deterministic if

1) for each q in Q, a in $\Sigma \cup \{\varepsilon\}$, and $X \in \Gamma$, $\delta(q, a, X)$ contains at most one element,
2) whenever $\delta(q, a, X)$ is nonempty for some $a \in \Sigma$, then $\delta(q, \varepsilon, X)$ is empty.

Note that

— DPDA allow $\varepsilon - transitions$.

— Each transition is determined by the current state, the input symbol, and the top-of-stack symbol.

So, for each pair of a state and an input symbol, there can be several transitions, one for each stack symbol.

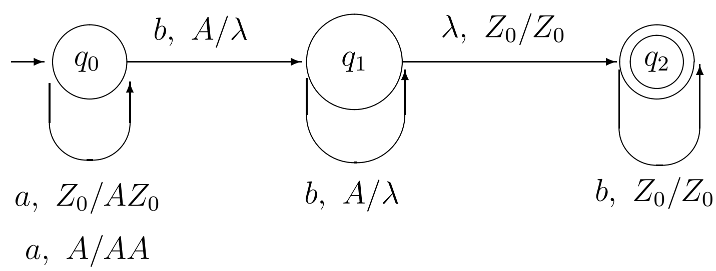
— $\delta(q, \varepsilon, X)$ should not be defined if $\delta(q, a, X)$ is defined for any $a \in \Sigma$.

Let $\mathcal{T}_{DPDA}$, $\mathcal{N}_{DPDA}$, and $\mathcal{L}_{DPDA}$ denote the sets of languages accepted by DPDA with acceptance by "final state", "empty stack", and "final state and empty stack", respectively.

Then $\mathcal{N}_{DPDA} = \mathcal{L}_{DPDA} \subset \mathcal{T}_{DPDA}$

<u>Example</u>

$L = \{a^m b^n | m \leq n,$ **and** $m, n > 0\}$
Then $L = T(A)$ **where** A**:**



But $L \notin \mathcal{N}_{DPDA}$

<u>Definition</u> The family of <u>deterministic context-free languages</u> is the set of all languages accepted by DPDA with acceptance by final state.

The family of DCFLs is a proper subset of the family of CFLs.

Examples

The following CFLs are not DCFLs

1. $\{a^n b^n \mid n \geq 0\} \cup \{a^n b^{2n} \mid n \geq 0\}$

2. $\{ww^R \mid w \in Z^*\}$

3. $\overline{\{ww \mid w \in Z^*\}}$

The family of DCFLs is closed under

(1) complementation,
(2) intersection with regular sets,

not closed under

(1) union
(2) intersection.

$\triangle$ Closure Properties

We show that $\mathcal{L}_{CF}$ is closed under $\cup$, $\bullet$, $*$, but not under $\cap$ and $^{-}$.

1. Union

$L_1, L_2 \in \mathcal{L}_{CF}$ **(i.e L_1, L_2 are CFLs).**
Show that $L = L_1 \cup L_2$ is CF.

Proof:

$G_1 = (N_1, \Sigma_1, P_1, S_1)$, $G_2 = (N_2, \Sigma_2, P_2, S_2)$
Assume $N_1 \cap N_2 = \emptyset$. Construct

$G =$
$(N_1 \cup N_2 \cup \{S\}), \Sigma_1 \cup \Sigma_2, P_1 \cup P_2 \cup \{S \rightarrow S_1 | S_2\}, S)$.

Then $L(G) = L(G_1) \cup L(G_2)$

2. Catenation

$$L_1, L_2 \in \mathcal{L}_{CF} \Rightarrow L_1L_2 \in \mathcal{L}_{CF}$$

$G_1 = (N_1, \Sigma_1, P_1, S_1)$ **,** $G_2 = (N_2, \Sigma_2, P_2, S_2)$

$$G = (N_1 \cup N_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2,$$
$$P_1 \cup P_2 \cup \{S \rightarrow S_1S_2\}, S))$$

$$L(G) = L(G_1) \bullet L(G_2)$$

3. *

$L_1 \in \mathcal{L}_{CF} \Rightarrow L_1^* \in \mathcal{L}_{CF}$ $\qquad (L(G_1))^*$

$$S \rightarrow S_1S|\varepsilon$$

4. Intersection

$$L_1, L_2 \in \mathcal{L}_{CF} \not\Rightarrow L_1 \cap L_2 \in \mathcal{L}_{CF}$$

$L = \{a^ib^ic^i \mid i \geq 0\}$ **is not in CF**

$$L_1 = \{a^ib^jc^k \mid i = j, i, j, k \geq 0\}$$
$$L_2 = \{a^ib^jc^k \mid j = k, i, j, k \geq 0\}$$

$$L_1 \cap L_2 = L$$

5. Complementation

$L_1 \in \mathcal{L}_{CF} \not\Rightarrow \overline{L_1} \in \mathcal{L}_{CF}$

Proof:

Assume $\mathcal{L}_{CF}$ **is closed under** $^{-}$**.**
Consider two arbitrary CFLs L_1, L_2**.**
$L = L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$**.**

L **is CF**

$\mathcal{L}_{CF}$ **is closed under** $\cap$**.**
This is a contradition.