

Cherenkov radiation and scattering of external dispersive waves by two-color solitons

Ivan Oreshnikov,^{1,*} Oliver Melchert,^{2,3,4} Stephanie Willms,^{2,3} Surajit Bose,^{2,5}
Ihar Babushkin,^{2,3} Ayhan Demircan,^{2,3,4} Uwe Morgner,^{2,3,4} and Alexey Yulin⁶

¹*Max Planck Institute for Intelligent Systems, Max-Planck-Ring 4, 72076 Tübingen, Germany*

²*Cluster of Excellence PhoenixD, Welfengarten 1, 30167 Hannover, Germany*

³*Leibniz Universität Hannover, Institute of Quantum Optics, Welfengarten 1, 30167 Hannover, Germany*

⁴*Hannover Centre for Optical Technologies, Nienburger Strasse 17, 30167 Hannover, Germany*

⁵*Leibniz Universität Hannover, Institute of Photonics,*

Nienburger Strasse 17, 30167 Hannover, Germany

⁶*Department of Physics, ITMO University, Kronverskiy pr. 49, 19701 St. Petersburg, Russia*

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For waveguides with two separate regions of anomalous dispersion, it is possible to create a quasi-stable two-color solitary wave. In this paper we consider how those waves interact with dispersive radiation, both generation of Cherenkov radiation and scattering of incident dispersive waves. We derive the analytic resonance conditions and verify them through numeric experiments. We also report incident radiation driving the internal oscillations of the soliton during the scattering process in case of an intense incident radiation. We generalize the resonance conditions for the case of an oscillating soliton and demonstrate how one can use the scattering process to probe and excite an internal mode of two-color soliton molecules.

I. INTRODUCTION

Solitary waves or solitons are localized nonlinear excitations that preserve their shapes during evolution. In the systems close to integrable ones solitons are proven to be robust and in many cases different interactions result only in the variation of the soliton parameters, such as its intensity or frequency. In a certain sense solitons can be called “eigenmodes of the nonlinear problem”, meaning that the dynamics of the system can be considered as a set of solitons interacting with quasi-linear dispersive waves. Due to their robustness, solitons are of fundamental as well as practical importance, for example, in the context of optical supercontinuum generation [1–3] or soliton fiber lasers [4–7].

The mutual interaction of optical solitons in fibers can enable the formation of bound states, often referred to as soliton molecules. They can be realized via dispersion engineering in the framework of the standard nonlinear Schrödinger equation (NSE) and consist of two pulses that maintain a fixed separation in time [8, 9]. Further, optical soliton clusters have been discovered and studied in a large variety of physical systems described by different equations such as the generalized nonlinear Schrödinger equation [10–12], coupled NLSs [13, 14], the Ginzburg-Landau equation [15, 16], Lugiato-Lefever equation [17] and many others [18–28]. The aforementioned bound states of solitons have a single central frequency and the whole spectrum is localized around this frequency.

Another possibility to observe soliton molecules is to provide interaction between solitons having their carrier frequencies well detuned from each other. To make the interaction efficient the velocities of the solitons must

be close. In the case of scalar solitons this means that we need higher order dispersion with two different frequency ranges where the solitons can be accommodated. These solitons were recently reported in [29–32] for pulses propagating in conservative fibers with higher order dispersion. Similar solitons were discovered also in mode-locked cavity lasers [33] and in coherently pumped ring resonators [34, 35].

In the case of nonintegrable systems, solitons can interact with dispersive waves (DWs) of low intensity and this interaction leads to interesting physical phenomena, as the efficient generation of new frequencies [36–40]. It should be noted that this effect is closely related to the optical push broom effect [41]. Such resonant scattering can be cascaded, enriching optical supercontinuum spectra significantly [42]. Similar effects were studied in Refs. [43–45], where the term “optical event horizon” was coined. It was also established that resonant dispersive waves affect the parameters of the solitons and can lead to dispersive wave mediated acceleration of solitons [37, 43, 46–50]. It should be mentioned that resonant scattering of dispersive waves is also studied for dark solitons [51–54], and oscillating solitons [55–60].

The present paper aims to study in detail the interaction of dispersive waves with two-color soliton molecules, consisting of two bound solitons having well separated frequencies. In the prior work [29] we have demonstrated how to create quasi-stable configurations of two tightly-coupled pulses in a dispersion landscape $\beta(\omega)$ with two regions of anomalous dispersion, separated by a region of normal dispersion. Each of these subpulses propagates on its own carrier frequency. This coupled state, especially in the process of initial evolution, sheds dispersive waves that resemble Cherenkov radiation that is observed for other types of solitary waves in a vast variety of settings [56–58, 60–62]. And since DW generation is demonstrated, it is reasonable to anticipate that two-color soliton molecules

*Electronic address: oreshnikov.ivan@gmail.com

colliding with DWs will produce resonant emission in the same way as it occurs in the case of conventional solitons. Thanks to a more complex structure of the two-color solitons, one can expect the scattering dynamics to be much richer compared to the conventional single soliton case. For instance, it is possible that the internal degrees of freedom can be excited due to the interaction with a DW and this should strongly affect the dynamics of the system. Subsequently we confirm this influence, providing a combined analytical and numerical view of the process.

The paper is structured as follows. In Sect. II we introduce a mathematical model for the light propagating in a nonlinear fiber with higher order dispersion and derive the condition of resonant four-wave mixing of the soliton molecules with DWs. In Sect. III we report the results of the numerical simulations of the different propagation regimes of bichromatic soliton molecules. The results of the simulations are compared against analytical resonance conditions. Section IV discusses the weakly nonlinear case where the intensity of the dispersive waves is sufficiently large to modify the interaction between the soliton molecules and the waves. The excitation of the

internal mode is also discussed in this section. The paper concludes with a summary in Sect. V.

II. ANALYTICAL MODEL

In this section we derive a perturbation theory that explains the resonance conditions that were proposed before in [29], demonstrate additional Cherenkov radiation mechanisms due to four-wave mixing (FWM) between the frequency components of the soliton, and then extend this theory to describe the process of scattering of external waves on the soliton.

We start by considering a non-envelope version of a nonlinear Schrödinger equation [63]

$$i\partial_z \tilde{u} + \beta(\omega)\tilde{u} + \gamma(\omega)\mathcal{F}\left\{|u|^2 u\right\}_{(\omega>0)} = 0. \quad (1)$$

Here and further, $\tilde{u} \equiv \tilde{u}(z, \omega)$ indicates the Fourier image of the field $u \equiv u(z, t)$ and $\mathcal{F}\{\dots\}_{(\omega>0)}$ is the explicit Fourier transform taken for the positive frequencies only.

To build the perturbation theory let us introduce an ansatz that represents the solution as a sum of two single-frequency solitary waves $U_{1,2}(z, t)$ and a small residue radiation $\psi(z, t)$

$$u(z, t) = U_1(z, t) + U_2(z, t) + \psi(z, t), \quad (2)$$

$$|\psi| \ll |U_1| \sim |U_2|.$$

For the solitary waves $U_{1,2}(z, t)$ we assume that they satisfy a pair of coupled nonlinear Schrödinger equations below

$$i\partial_z U_n + \beta_n(i\partial_t)U_n(z, t) + \gamma(\omega_n)\left(|U_n|^2 U_n + 2|U_m|^2 U_n\right) = 0, \quad (3)$$

where $n = 1, 2$ and $m = 2, 1 \neq n$. Essentially, this is an assumption that the two-color soliton molecule consists of two pulses, which are incoherently coupled by inducing a refractive-index potential on each other [64]. Each subpulse exists in some sort of truncated dispersion landscape, defined by the operator $\beta_n(i\partial_t)$. Under specific conditions, the subpulses are given by fundamental solitons of a modified NLS [31]. A reasonable guess for the truncated operator is a parabolic approximation close to the carrier frequency

$$\beta_n(i\partial_t) = \beta(\omega_n) + i\beta'(\omega_n)\partial_t - \frac{1}{2}\beta''(\omega_n)\partial_t^2. \quad (4)$$

We additionally suppose that in the soliton's frame of reference the envelope evolves with wavenumber $k_n(\omega)$ which we approximate by the wavenumber of the fundamental soliton in the ordinary nonlinear Schrödinger equation and a correction from a secondary soliton as

$$k_n(\omega) \approx \frac{\gamma(\omega_n)A_n^2}{2} + \beta(\omega_n) + \beta'(\omega_n)(\omega - \omega_n) + \gamma(\omega_n)A_m^2, \quad (5)$$

where $n = 1, 2$, $m = 2, 1 \neq n$ and A_n is the soliton amplitude.

By substituting ansatz (2) into Eq. (1), linearizing with respect to perturbation $\psi(z, \omega)$, and discarding the terms corresponding to the soliton Eqs. (3), we get the equation for $\tilde{\psi}$

$$i\partial_z \tilde{\psi} + \beta(\omega)\tilde{\psi} + \gamma(\omega)\mathcal{F}\left\{2|U_1 + U_2|^2 \psi + (U_1 + U_2)^2 \psi^*\right\}_{(\omega>0)} =$$

$$- [\beta(\omega) - \beta_1(\omega)]\tilde{U}_1 - [\beta(\omega) - \beta_2(\omega)]\tilde{U}_2 - \gamma(\omega)\mathcal{F}\left\{U_2^2 U_1^* + U_1^2 U_2^*\right\}_{(\omega>0)}. \quad (6)$$

On the right-hand side (RHS) of Eq. (6) we see two types of driving terms and each of those terms can be in resonance with the linear DWs that exist in the system if the particular wavenumber $k(\omega_*)$ of the driving term is equal to the

wavenumber of a DW $\beta(\omega_*)$ at some frequency ω_* [36, 61]. The first type is given by $[\beta(\omega) - \beta_n(\omega)] \tilde{U}_n(z, \omega)$ which drives the generation of Cherenkov radiation by an individual soliton U_n if the following resonance condition is satisfied

$$\beta(\omega) = k_n(\omega). \quad (7)$$

The second type of the terms are $\gamma(\omega) \mathcal{F} \{U_2^2 U_1^*\}$ and $\gamma(\omega) \mathcal{F} \{U_1^2 U_2^*\}$ and they correspond to the process of four-wave mixing that in our case results in the generation of dispersive radiation at some frequency where

$$\beta(\omega) = 2k_n(\omega) - k_m(\omega). \quad (8)$$

Let us move on to the problem of external dispersive wave scattering. For that, we split the perturbation into the incident and the scattered parts

$$\psi(z, t) = \psi_{\text{inc}}(z, t) + \psi_{\text{sc}}(z, t). \quad (9)$$

We explicitly define $\psi_{\text{inc}}(z, t)$ as a linear wave that is propagating in a soliton-free medium

$$i\partial_z \tilde{\psi}_{\text{inc}}(z, t) + \beta(\omega) \tilde{\psi}_{\text{inc}}(z, t) = 0. \quad (10)$$

Substituting Eq. (9) into (6) and eliminating the terms corresponding to Eq. (10) we are left with the equation for the scattered component

$$i\partial_z \tilde{\psi}_{\text{sc}} + \beta(\omega) \tilde{\psi}_{\text{sc}} + \gamma(\omega) \mathcal{F} \left\{ 2|U_1 + U_2|^2 \psi_{\text{sc}} + (U_1 + U_2)^2 \psi_{\text{sc}}^* \right\}_{(\omega>0)} = \dots \text{(omitted is the RHS of Eq. (6))} \\ - \gamma(\omega) \mathcal{F} \left\{ 2(|U_1|^2 + U_1 U_2^* + U_2 U_1^* + |U_2|^2) \psi_{\text{inc}} + (U_1^2 + 2U_1 U_2 + U_2^2) \psi_{\text{inc}}^* \right\}_{(\omega>0)}. \quad (11)$$

In addition to the resonance terms already discussed in Eq. (6), we see terms that arise due to interaction between the incident radiation and the soliton. Here, six new types of resonance behavior are possible. The first one is due to terms $|U_1|^2 \psi_{\text{inc}}$ and $|U_2|^2 \psi_{\text{inc}}$, both with the resonance condition

$$\beta(\omega_{\text{sc}}) = \beta(\omega_{\text{inc}}). \quad (12)$$

The next two are due to mixed terms $U_1 U_2^* \psi_{\text{inc}}$ and $U_2 U_1^* \psi_{\text{inc}}$, with the corresponding resonance condition being

$$\beta(\omega_{\text{sc}}) = \pm k_1 \mp k_2 + \beta(\omega_{\text{inc}}). \quad (13)$$

Another two are due to $U_n^2 \psi_{\text{inc}}^*$ and the resonance condition is

$$\beta(\omega_{\text{sc}}) = 2k_n(\omega_{\text{sc}}) - \beta(\omega_{\text{inc}}), \quad n = 1, 2. \quad (14)$$

The final one is due to $2U_1 U_2 \psi_{\text{inc}}^*$ with the resonances at

$$\beta(\omega_{\text{sc}}) = k_1(\omega_{\text{sc}}) + k_2(\omega_{\text{sc}}) - \beta(\omega_{\text{inc}}). \quad (15)$$

To verify the predictions given by resonance conditions (7), (8), (12), (13), (14), and (15) we proceed to numeric experiments.

III. NUMERICAL EXPERIMENTS

To numerically integrate Eq. (1) we use the integrating factor method and transform the equation into a non-stiff

version for a modified spectrum [3]. The modified equation can be handled by any standard ODE solver; we use a `scipy` interface to `ZVODE` solver from `ODEPACK` [65, 66] (the code necessary to reproduce the results in the paper can be found in [67]). All the computations are performed in a frame of reference co-moving with the soliton, which is achieved by a transformation

$$t \rightarrow t - \beta'(\omega_1)z,$$

which, in turn, results in

$$\beta(\omega) \rightarrow \beta(\omega) - \beta(\omega_1) - \beta'(\omega_1)(\omega - \omega_1) \\ k_1(\omega) \rightarrow \frac{\gamma(\omega_1)A_1^2}{2} + \gamma(\omega_1)A_2^2 \\ k_2(\omega) \rightarrow \frac{\gamma(\omega_2)A_2^2}{2} + \gamma(\omega_1)A_1^2 + \beta(\omega_2) - \beta(\omega_1).$$

Description of the specific dispersion profile model $\beta(\omega)$ used in the simulations can be found in the Supplemental Material [68].

To study Cherenkov radiation of two color solitons we chain two separate simulations. First, following the prior work [29], we produce a two-color soliton by integrating an initial condition that is given by a sum of two fundamental solitons of standard NLSE

$$u_0(t) = A_1 \text{sech}(t/T_1) e^{-i\omega_1 t} + A_2 \text{sech}(t/T_2) e^{-i\omega_2 t}, \quad (16)$$

where frequencies ω_1 and ω_2 are both lying in the regions of anomalous dispersion. Frequency ω_1 is otherwise arbitrary; ω_2 is chosen so that group velocities of both the

frequency components match

$$\beta'(\omega_1) = \beta'(\omega_2).$$

In most of the simulations presented subsequently, we fix $T_1 = T_2 = 20$ fs and the amplitudes A_1 and A_2 are chosen as the fundamental soliton amplitudes at the corresponding frequencies. This configuration sheds a significant amount of radiation and relaxes to a quasi-stable solitary wave. We propagate up to $z = 10$ cm, take the output field of this seed simulation, and suppress the radiation tails by multiplying it by a super-Gaussian temporal window, centered on the peak of the soliton molecule. This isolated soliton molecule serves as an input to the second simulation that is carried with the same parameters as the original one.

Once we suppress the radiation that is shed by the seed solitons during the initial relaxation process the isolated two-color soliton molecule propagates generating only a narrow-spectrum Cherenkov radiation. An example is shown in Fig. 1. Figure 1(b) shows the normalized spectral densities at the input and output, i.e. $z = 0$ cm and 10 cm, respectively. For clarity, the difference between both spectra is shown on a logarithmic scale in Fig. 1(c). It is clearly evident that on top of the input spectrum [thin black line in Fig. 1(b)], two additional spectral lines appear in the output spectrum [gray line in Fig. 1(b)]. To clarify the origin of these pronounced spectral lines, Fig. 2(d) demonstrates resonance conditions (7) and (8): the black curve corresponds to $\beta(\omega)$, i.e. the left-hand side of both equations; the horizontal lines correspond to the right-hand sides. Intersections between the dispersive curve and the horizontal lines that contribute to Cherenkov radiation are marked separately: ① labels radiation due to the second component of the soliton as predicted by Eq. (7); ② labels the location of frequencies due to FWM between the frequency components, resulting in radiation with wavenumber $2k_2 - k_1$, as predicted by Eq. (8).

To study the scattering processes we perform the seed simulation and isolate the two-color soliton. To accomplish that we add an incident DW in the form of a Gaussian pulse

$$\psi_{\text{inc}}(t) = A_{\text{inc}} \exp \left(-\frac{(t - t_0)^2}{T_{\text{inc}}^2} - i\omega_{\text{inc}} t \right).$$

Below we set A_{inc} to 1% of the maximum amplitude of the isolated soliton and fix the width to $T_{\text{inc}} = 300$ fs. The initial pulse delay t_0 is chosen as ± 1000 fs from the soliton center, with the sign depending on the relative group velocity between the soliton and the DW and chosen so that both pulses engage in a collision. Upon propagation the incident radiation interacts with the two-color soliton to produce scattered radiation. This process can evolve in several different ways depending on the frequencies of the soliton and the incident radiation. Below we consider three concrete examples. In each of them the incident radiation is split between three different components.

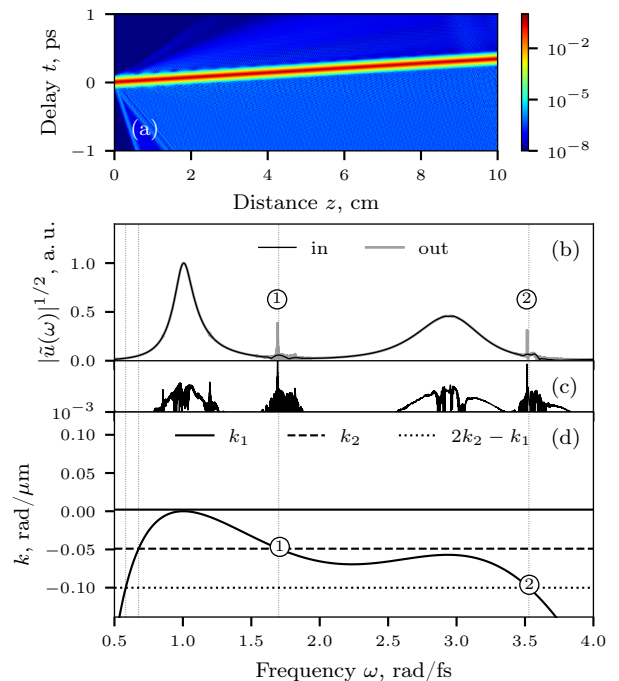


FIG. 1: (color online) Cherenkov radiation of an isolated two-color soliton molecule, produced by an initial condition with first-component carrier frequency $\omega_1 = 1.010$ rad/fs. (a) Time-domain view, (b) input and output spectra, (c) difference between both spectra on a logarithmic scale, and (d) diagram of the resonance conditions.

The first configuration, shown in Fig. 2, is very close to the degenerate case where the wavenumbers of the individual soliton components k_1 and k_2 coincide. Equation (13) then turns into Eq. (12). This case resembles the case of fundamental single-component solitons [36], however, due to the shape of the dispersive curve $\beta(\omega)$ there is now more than one nontrivial solution to Eq. (12) and a single incident frequency yields up to four possible resonances. In practice, only some of them contribute to the scattered radiation. Those solutions are separately marked on a diagram on panel 2(d): ② corresponds both to the incident and the partially transmitted radiation, ① is the reflected component and ② is the additional transmitted component.

The second configuration, shown in Fig. 3, is a case with a significant difference between k_1 and k_2 . Since Eq. (13) is no longer degenerate, the resonance diagram in Fig. 3(d) is more complex. However, as one can notice, in that case only the solution marked with ②, corresponding to the upper (i.e. “−, +”) branch of Eq. (13), contributes to the scattered radiation. Component ①, corresponding to the scattered radiation, comes from resonance condition (12). The unmarked spectral line close to $\omega \approx 2.4$ rad/fs is the Cherenkov radiation of the soliton itself.

The third configuration, shown in Fig. 4, is in a sense symmetric to the previous case. Again, the difference between k_1 and k_2 is significant and Eq. (13) is far from

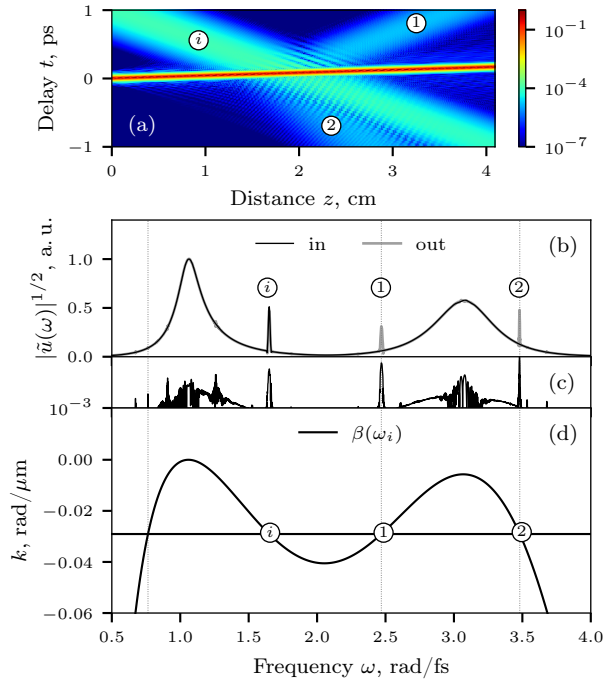


FIG. 2: (color online) Scattering of a weak DW with carrier frequency $\omega_{\text{inc}} = 1.650$ rad/fs on a two-color soliton with first-component carrier frequency $\omega_1 = 1.070$ rad/fs. (a) Time-domain view, (b) input and output spectra, (c) difference between both spectra on a logarithmic scale, and (d) diagram of the resonance conditions.

being degenerate, but now the resonances from the lower (i.e. “+,” “−”) branch of the resonance condition (13) play a significant role. This is achieved by choosing the incident frequency greater than the carrier frequency of the second soliton component and enhanced by using an asymmetric seed soliton with $T_1 = 30$ fs and $T_2 = 10$ fs. This creates a solitary wave with the amplitude of the second component almost equal to the amplitude of the first one.

Overall, in all of our experiments, only the scattered components corresponding to Eqs. (12) and (13) turn out to be significant, while the components predicted by Eqs. (14) and (15) do not seem to contribute to the resulting radiation. In other words, the terms proportional to ψ_{inc}^* on the RHS of Eq. (11) can be safely neglected.

IV. NONLINEAR EFFECTS

When deriving resonance conditions (12) – (15) we assumed that the parameters of the solitons themselves stay constant throughout propagation and scattering processes. This can be considered a reasonable approximation when the amplitude of the incident wave is negligible compared to the amplitudes of the individual soliton components. However, in the general case of more intensive incident radiation this assumption does not hold. In this section we will briefly discuss two specific examples, where the

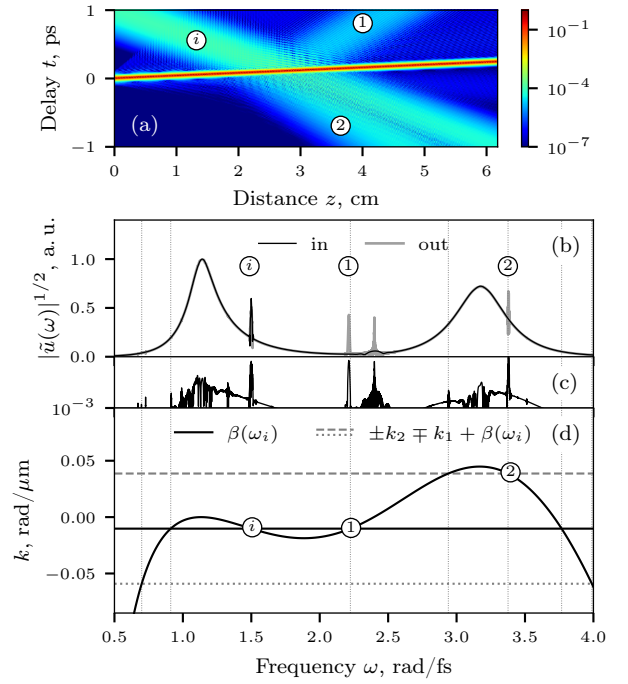


FIG. 3: (color online) Scattering of a weak DW with carrier frequency $\omega_{\text{inc}} = 1.500$ rad/fs on a two-color soliton with first-component carrier frequency $\omega_1 = 1.150$ rad/fs. (a) Time-domain view, (b) input and output spectra, (c) difference between both spectra on a logarithmic scale, and (d) diagram of the resonance conditions.

process of scattering noticeably affects the parameters of the soliton. The general setup of the numerical experiments remains as in the preceding section, i.e. we consider scattering of a Gaussian pulse on an isolated two-color soliton, but this time we increase the amplitude of the DW to 5% of the soliton’s maximum amplitude. This change might seem subtle, but it is sufficient to make the scattering dynamics much more involved.

In the first example, shown in Fig. 5, we consider scattering of a DW with incident frequency $\omega_i = 2.100$ rad/fs on a two-color soliton with $\omega_1 = 1.010$ rad/fs. From panel 5(a) we can immediately notice, that interaction with the DW significantly decelerates the two-color soliton. This is connected to the change of the soliton’s carrier frequency, as demonstrated in panel 5(b.2). This effect has been demonstrated before for the conventional solitons of nonlinear Schrödinger equation [46, 69]. What is remarkable about this interaction in case of a two-color soliton is the fact that the soliton appears to be stable during this process. Granted, the frequency offset gained by the soliton during the scattering is not especially prominent [panel 5(b.2)], but at the same time the amplitude difference in both the frequency components stays under 1% [panel 5(b.1)], so almost no power loss occurs.

In the second example, shown in Fig. 6, we consider the scattering of a DW with the incident frequency $\omega_i = 1.100$ rad/fs on a two-color soliton with $\omega_1 = 1.010$ rad/fs.

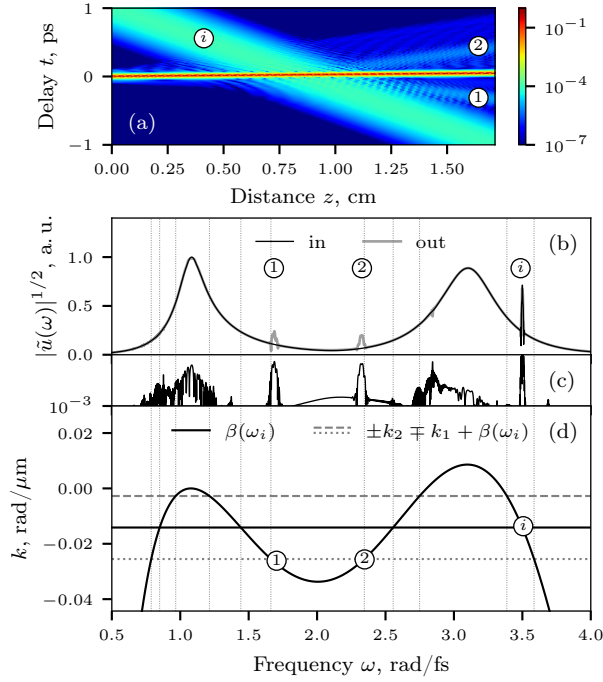


FIG. 4: (color online) Scattering of a weak DW with carrier frequency $\omega_{\text{inc}} = 3.500$ rad/fs on a two-color soliton with first-component carrier frequency $\omega_1 = 1.150$ rad/fs. (a) Time-domain view, (b) input and output spectra, (c) difference between both spectra on a logarithmic scale, and (d) diagram of the resonance conditions.

In the plots in panels 6(b.1) and 6(b.2) one can notice how the interaction with the DW generates oscillations in the soliton parameters; this is especially prominent in 6(b.2) that displays the absolute change of the soliton's central frequencies ω_1 and ω_2 . From the latter plot the period of those oscillations can be estimated as $Z_0 = 2$ mm. The interaction of freely oscillating nonlinear waves, both the radiation and the scattering processes, is a well studied problem [56–58, 60, 70] and the common trait in this setting, independent of the nature of the soliton oscillations, is that the dispersive radiation produced (generated or scattered) by the soliton is polychromatic, i.e. it consists of several isolated spectral components. This is indeed what we see in the output spectrum on panel 6(c).

To explain this behavior let us return to equations (6) and (11). In the oscillating case the individual solitons U_1 and U_2 are no longer represented by a single spatial frequency $\propto e^{ik_1 z}$ and $\propto e^{ik_2 z}$ and they rather correspond to Fourier series'

$$U_n(z, t) = \sum_{N \in \mathbb{Z}} C_{nN}(t) \exp \left(ik_1 z + i \frac{2\pi N}{Z_0} z \right),$$

where Z_0 is the oscillation period. This leads to a split in resonance conditions, and for the oscillating case equations

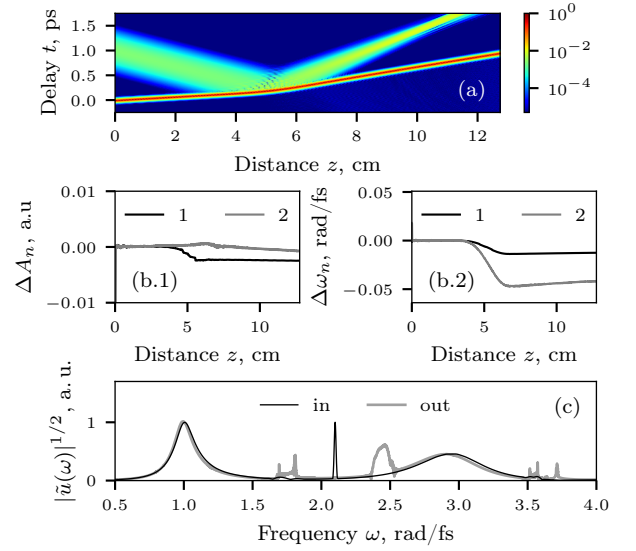


FIG. 5: (color online) Scattering of an intensive DW with carrier frequency $\omega_{\text{inc}} = 2.100$ rad/fs on a two-color soliton with first-component carrier frequency $\omega_1 = 1.010$ rad/fs. (a) Time-domain view. (b.1) Oscillation of the two-color soliton subpulse amplitudes, and, (b.2) oscillation of the corresponding center frequencies. (c) Output spectrum.

(7), (8), (12) and (13) read

$$\beta(\omega) = k_n(\omega) + \frac{2\pi N}{Z_0} \quad (7^*)$$

$$\beta(\omega) = 2k_n(\omega) - k_m(\omega) + \frac{2\pi N}{Z_0} \quad (8^*)$$

$$\beta(\omega_{\text{sc}}) = \beta(\omega_{\text{inc}}) + \frac{2\pi N}{Z_0} \quad (12^*)$$

$$\beta(\omega_{\text{sc}}) = \pm k_1 \mp k_2 + \beta(\omega_{\text{inc}}) + \frac{2\pi N}{Z_0}, \quad (13^*)$$

with $N \in \mathbb{Z}$ being the spatial harmonic number.

The resonance conditions for the scattering process in the last simulation are displayed on panel 6(d). In this process only the last two equations, i.e. Eqs. (12*) and (13*) are relevant. The solid black lines correspond to equation (12) with harmonics $N = -3, \dots, 0$. This harmonic split not only leads to the reflected part of the radiation ① becoming polychromatic, but also has the same effect on the incident component ①. Resonance condition (13) is represented by the gray dashed lines for harmonics $N = -3, \dots, +3$. It contributes to the scattered radiation twice: with a wide band ② that otherwise would degenerate into a sharp spectral line in a non-oscillating case, and with three wide spectral lines in the vicinity of ③ which is the contribution of the lower harmonics $N = -3, -2, -1$. One can notice that there are only two vertical lines corresponding to the numeric solution of Eq. (13) displayed around ③. This is merely due to a small numeric error in estimating the soliton wavenumber k_2 in the plotting procedure, which causes

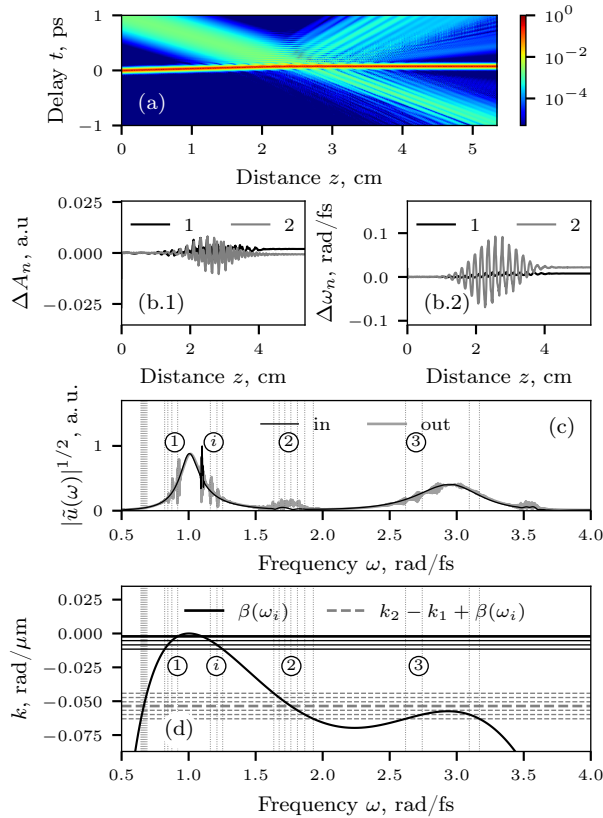


FIG. 6: (color online) Scattering of an intensive DW with carrier frequency $\omega_{\text{inc}} = 1.100$ rad/fs on a two-color soliton with first-component carrier frequency $\omega_1 = 1.010$ rad/fs. (a) Time-domain view. (b.1) Oscillation of the two-color soliton subpulse amplitudes, and, (b.2) oscillation of the corresponding center frequencies. (c) Output spectrum. (d) Diagram of the resonance conditions.

harmonic $N = -1$ to touch the dispersive rather than intersect it.

The specific oscillation mode we see in Fig. 6 appears to be heavily damped, since launching the initial soliton with an additional frequency detuning does not lead to free frequency oscillations during the propagation. Such internal dynamics, reminiscent of molecular vibrations, were reported also previously [29, 32], and the emission of DWs by the two-color soliton was found to explain the dampening of the oscillation mode [71]. Here, the incident radiation drives this oscillation, and one could expect to observe a resonance behavior with respect to the frequency of the incident DW. And indeed, as demonstrated by the parameter sweep in Fig. 7(a), by adjusting the incident frequency ω_i one can affect, to a certain degree, the amplitude of the frequency oscillations of the second component $\Delta\omega_2$. For this specific mode, where the frequency oscillations dominate, it is straightforward to get an estimate for the period Z_0 of the mode and the corresponding wavenumber K_0 in terms of a variational approach (see Supplemental Material [68]). As shown in Fig. 7(b), superimposing the resulting resonance wavenumber K_0 on

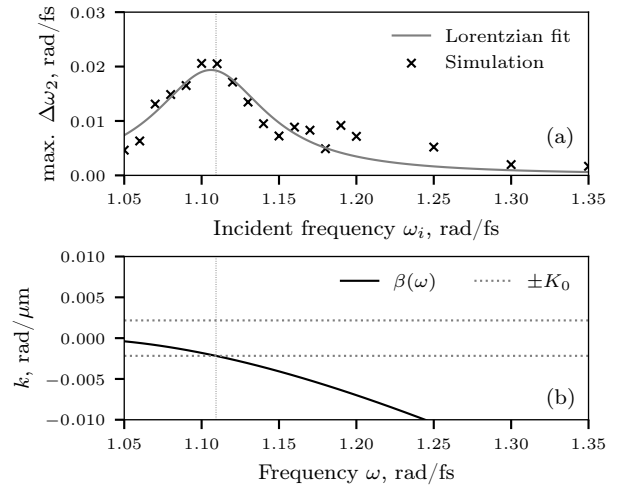


FIG. 7: Resonance behavior of the internal mode of the two-color soliton molecule during intensive scattering. (a) Oscillation of the center frequency of the second component of the two-color soliton as function of the frequency of the incident DW. (b) Diagram of the resonance condition for the internal mode.

the dispersion curve allows to graphically estimate the frequency of the incident DW that corresponds to the internal oscillation mode. The vertical dashed lines in Figs. 7(a,b) indicate this resonance frequency, confirming the excellent agreement of the numerical simulations and the approximate analytic estimate.

V. CONCLUSION

In this paper, we considered the resonant interaction of two-color soliton molecules with DWs. The resonance condition for Cherenkov radiation is derived and analyzed. The comparison with the results of numerical simulations shows that these conditions correctly predict the positions of the Cherenkov radiation. We also study the process of the collision of the two-color solitons with the incident DWs. The resonance conditions predict well all newly generated frequencies observed in the numerical experiments. Some of the predicted resonances are not seen in the numerically calculated radiation spectra. This can be explained by the low efficiency of the corresponding scattering processes. The theory developed for a simpler case in [37] shows that indeed for some parameters the generated radiation can be extremely weak. The theory giving the analytical estimations for four waves mixing between the DWs and two-color solitons is of interest but is out of the scope of the present paper. We also studied how the intensity of the incident dispersive waves affects the scattering. In particular, it is shown that the scattering can affect the soliton trajectory without affecting its integrity. Another important finding is that in the nonlinear case the DWs can resonantly excite internal

oscillations of the soliton. This results in polychromatic emission of DWs by oscillating two-color solitons.

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