Supplementary Material

Ivan Oreshnikov[1], Oliver Melchert[2,3,4], Stephanie Willms[2,3], Surajit Bose[3], Ihar Babushkin[2,3], Ayhan Demircan[2,3,4], and Alexey Yulin[5]

[1]Max Planck Institute for Intelligent Systems, Max-Planck-Ring 4, 72076 Tübingen, Germany
[2]Cluster of Exellence PhoenixD, Welfengarten 1, 30167 Hannover, Germany
[3]Institute of Quantum Optics, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
[4]Hannover Centre for Optical Technologies, Neinburger Strasse 17, 30167 Hannover, Germany
[5]Department of Nanophotonics and Metamaterials, ITMO University, Kronverskiy pr. 49, 19701 St. Petersburg, Russia

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Small internal oscillations of the soliton

When analyzing the nonlinear scattering near an oscillatory mode we used an expression for the oscillation frequency. In this section we will derive this expression.

Let us return to equation (3) for coupled solitons. We can re-normalize the equations by performing the following transformation

$$U_n \to \gamma_n^{1/2} e^{i\beta_n z} \cdot U_n,$$

which will make the equations symmetric

$$i\partial_z U_n - \frac{1}{2}\beta_n'' \partial_t^2 U_n + \gamma_n^2 \left|U_n\right|^2 U_n + 2\gamma_n\gamma_m \left|U_m\right|^2 U_n = 0.$$

This in turn allows us to recognize the modified couple of equations as Euler-Lagrange equations for Lagrangian

$$\int_{-\infty}^{+\infty} \mathcal{L}(U_1, \partial_z U_1, \partial_t U_1, \ldots)\, dt,$$

where Lagrangian density $\mathcal{L}$ is defined as a sum of three components $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{\text{int}}$, with L_n being a single-soliton Lagrangian density

$$\mathcal{L}_n = \frac{i}{2}\left(\partial_z U_n \cdot U_n^* - \partial_z U_n^* \cdot U_n\right) + \frac{1}{2}\beta_n''\,\partial_t U_n \partial_t U_n^* + \frac{1}{2}\gamma_n^2\,|U_n|^4\,, \tag{S.1}$$

and $\mathcal{L}_{\text{int}}$ being the interaction term

$$\mathcal{L}_{\text{int}} = 2\gamma_1\gamma_2\,|U_1|^2\,|U_2|^2\,. \tag{S.2}$$

Let us assume that the soliton components U_n can be described by the following generic *ansatz*

$$U_n(z,t) = A_n(z) S\left(\frac{t - t_n(z)}{\sigma_n(z)}\right)\exp\left(-i\Omega_n(z)t + i\phi_n(z)\right). \tag{S.3}$$

In here A_n is the amplitude of the pulse, t_n is the central position, σ_n is the pulse width, Ω_n is the frequency detuning, ϕ_n is the phase, and $S(x)$ is function that defines the envelope shape. At the moment we will not specify the concrete form of $S(x)$, but will assume that it is an even function.

Before we continue let us stress one important thing: this ansatz cannot express all the possible internal oscillations of the soliton. One obvious example, as it was noted in the text, is the case of the pulse-width oscillation. In order to capture this dynamics, we need to add frequency chirp to the ansatz.

Substituting (S.3) into (S.1) and (S.2) and integrating over t we arrive at the expressions for the averaged Lagrangians

$$\begin{aligned} L_n = I_1\, t_n\sigma_n A_n^2 \frac{d\Omega_n}{dz} + I_1\,\sigma_n A_n^2\frac{d\phi_n}{dz} + I_2\,\frac{\beta_n''}{2}\frac{A_n^2}{\sigma_n} \\ +I_1\,\frac{\beta_n''}{2}\sigma_n A_n^2\Omega_n^2 + I_3\,\frac{\gamma_n^2}{2}\sigma_n A_n^4 \end{aligned} \tag{S.4}$$

$$L_{\text{int}} = 2\,\gamma_1\gamma_2\,A_1^2 A_2^2\,I_{\text{int}}(\sigma_1,\sigma_2,t_1,t_2), \tag{S.5}$$

where the following integrals have been defined

$$I_1 = \int_{-\infty}^{+\infty} S^2(x)\,dx \qquad\qquad I_2 = \int_{-\infty}^{+\infty} (S'(x))^2\,dx$$

$$I_3 = \int_{-\infty}^{+\infty} S^4(x)\,dx \qquad I_{\text{int}} = \int_{-\infty}^{+\infty} S^2\left(\frac{t-t_1}{\sigma_1}\right) S^2\left(\frac{t-t_2}{\sigma_2}\right)\,dt$$

Due to the time invariance in the problem, I_{int} depends only on the difference between t_1 and t_2

$$I_{\text{int}} = I_{\text{int}}(t_1 - t_2, \sigma_1, \sigma_2),$$

and it is an even function of that difference.

The averaged Lagrangian $L = L_1 + L_2 + L_{\text{int}}$ is now a function defined in terms of soliton parameters $\{ A_n, \sigma_n, t_n, \Omega_n, \phi_n \}$ and only them. Therefore, the Euler-Lagrange equations for the new Lagrangian have to be defined in terms of variations over the soliton parameters

$$\frac{\delta L}{\delta P_n} = \frac{\partial L}{\partial P_n} - \frac{d}{dz}\frac{\partial L}{\partial \dot{P}_n} = 0,$$

where P_n stands for either A_n, σ_n, t_n, Ω_n or ϕ_n. The latter case — variation with respect to the phase ϕ_n — immediately yields the conservation of mass

$$N_n = \sigma_n(z) A_n^2(z) = const. \tag{S.6}$$

Variation with respect to the detuning Ω_n fixes the group velocity of individual solitons

$$\frac{dt_n}{dz} = \beta_n'' \Omega_n(z). \tag{S.7}$$

Variation with respect to the soliton position t_n gives us an equation for the frequency

$$\frac{d\Omega_n}{dz} = 2 \cdot \frac{N_m \gamma_1 \gamma_2}{I_1 \sigma_1(z) \sigma_2(z)} \cdot \frac{\partial I_{\text{int}}}{\partial t_n}. \tag{S.8}$$

The symmetry in the overlap integral I_{int} with respect to the soliton positions t_1 and t_2 leads to conservation of momentum

$$N_1 \Omega_1(z) + N_2 \Omega_2(z) = const. \tag{S.9}$$

Finally, the difference between the variations with respect to A_n and σ_n gives us

$$I_2 \beta_n'' + \frac{I_3 \gamma_n}{2} N_n \sigma_n(z) + 2 N_m \gamma_1 \gamma_2 \frac{\sigma_n(z)}{\sigma_m(z)} \left(I_{\text{int}} + \sigma_n \frac{\partial I_{\text{int}}}{\sigma_n} \right) = 0. \tag{S.10}$$

The very last equation — omitted here — is the evolution equation for the phase ϕ_n. The right-hand's side of the equation is quite complicated, but since the phase does not occur anywhere in (S.7), (S.8) or (S.10), it is not important for the remaining analysis.

Let us switch from the the individual soliton positions to the mean position and the relative delay instead

$$t_0 = \frac{1}{2}(t_1 + t_2) \qquad\qquad \Delta t = t_1 - t_2$$

Equation for the relative delay Δt

$$\frac{d\Delta t}{dz} = \beta_1'' \Omega_1(z) + \beta_2'' \Omega_2(z) \tag{S.11}$$

and equations (S.8) and (S.10) form a closed system, with equations for $d\Delta t/dz$, $d\Omega_n/dz$ acting as equations of motion and equations (S.10) fixing the widths $\sigma_n(z)$ as functions of Δt. By differentiating (S.11) one more time and using (S.8) we get

$$\frac{d^2\Delta t}{dz^2} + 2\frac{\gamma_1\gamma_2\left(\beta_1'' N_1 + \beta_2'' N_2\right)}{I_1\sigma_1(\Delta t)\sigma_2(\Delta t)}\frac{\partial}{\partial \Delta t} I_{\text{int}}(\Delta t, \sigma_1, \sigma_2) = 0$$

To transform this into a harmonic oscillator equation we need to linearize the second term around the equilibrium point $\Delta t = 0$. Since I_{int} is an even function, the derivative $\partial I_{\text{int}}/\partial \Delta t$ is odd and it vanishes at $\Delta t = 0$. This means we can ignore Δt dependency in σ_1 and σ_2 — only the term proportional to $\partial^2 I_{\text{int}}/\partial \Delta t^2$ will survive. Thus we finally arrive at

$$\frac{d^2\Delta t}{dz^2} + K_0^2 \Delta t = 0,$$

where the resonance frequency K_0 is

$$K_0^2 = 2\frac{\gamma_1\gamma_2\left(\beta_1'' N_1 + \beta_2'' N_2\right)}{I_1\sigma_1(0)\sigma_2(0)} I_{\text{int}}''(0; \sigma_1(0), \sigma_2(0)). \tag{S.12}$$

For a more concrete estimate let us finally consider a Gaussian envelope, i.e. let us set $S(x) = \exp(-x^2)$. Such a choice of the envelope shape fixes the integrals $I_1 = \sqrt{\pi/2}$ and

$$I_{\text{int}}(\Delta t, \sigma_1, \sigma_2) = \sqrt{\frac{\pi}{2}} \frac{\sigma_1\sigma_2}{\sqrt{\left(\sigma_1^2 + \sigma_2^2\right)}} \cdot \exp\left(\frac{-2\Delta t^2}{\sigma_1^2 + \sigma_2^2}\right),$$

which finally gives us the following expression for the resonance frequency

$$K_0^2 = -\frac{8\,\gamma(\omega_1)\gamma(\omega_2)}{\left(\sigma_1^2 + \sigma_2^2\right)^{3/2}} \cdot \left(\beta''(\omega_1)\sigma_1 A_1^2 + \beta''(\omega_2)\sigma_2 A_2^2\right). \tag{S.13}$$