Near-Optimal Reinforcement Learning in Factored MDPs

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Abstract

Any reinforcement learning algorithm that applies to all MDPs will suffer $\Omega(\sqrt{SAT})$ regret on some MDP, where T is the elapsed time and S is the number of states and A is the number of actions. In many problems S and A are so huge that any regret bounds are totally impractical.

We show that, if the system is known to be a *factored* MDP, it is possible to achieve regret that scales polynomially in the number of *parameters* encoding the factored MDP, which may be exponentially smaller than S or A. We provide two algorithms that satisfy near-optimal regret bounds in this context: PSRL and UCRL-Factored.

Problem formulation

Learn to optimize a random finite horizon MDP M in repeated finite episodes of interaction.

Figure 1: classic reinforcement learning setting

- State space $\mathcal{S}$, action space $\mathcal{A}$
- Rewards $r_t \sim R^M(s_t, a_t)$
- Transitions $s_{t+1} \sim P^M(s_t, a_t)$
- Epsiode length τ, define $t_k := (k-1)\tau + 1$

For MDP M and policy μ, define a value function

$$V^M_{\mu,i}(s) := \mathbb{E}_{M,\mu}\left[\sum_{j=i}^{\tau} \overline{R}^M(s_j, a_j) \middle| s_i = s\right],$$

Define the regret in episode k using μ_k on M^*

$$\Delta_k := \sum_{\mathcal{S}} \rho(s)\left(\underbrace{V^{M^*}_{\mu^*,1}(s)}_{\text{optimal value}} - \underbrace{V^{M^*}_{\mu_k,1}(s)}_{\text{actual value}} \right)$$

And finally $\text{Regret}(T, \pi, M^*) := \sum_{k=1}^{\lceil T/\tau \rceil} \Delta_k$.

Naive exploration such as Boltzman or ϵ-greedy can lead to exponential regret. Good performance requires balancing **exploration vs exploitation**. Carefully designed optimism or posterior sampling can learn quickly in factored MDPs.

Factored MDPs

MDP with conditional independence structure.

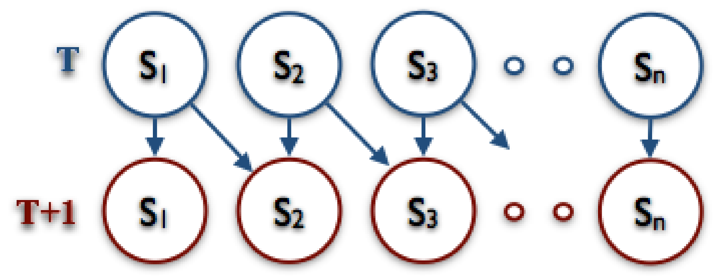

Figure 2: a graphical model for transitions.

Definition 1 (Scope operation for factored sets).
For any $\mathcal{X} = \mathcal{X}_1 \times .. \times \mathcal{X}_n$ and $Z \subseteq \{1, 2, .., n\}$ define $\mathcal{X}[Z] := \bigotimes_{i \in Z} \mathcal{X}_i$ and elements $x[Z] \in \mathcal{X}[Z]$.

Definition 2 (Factored reward functions).
The reward function r is factored over $\mathcal{S} \times \mathcal{A} = \mathcal{X} = \mathcal{X}_1 \times .. \times \mathcal{X}_n$ with scopes $Z_1, ..Z_l \iff$

$$\mathbb{E}[r(x)] = \sum_{i=1}^{l} \mathbb{E}\big[r_i(x[Z_i])\big] \quad \text{and each } r_i \text{ observed}$$

Definition 3 (Factored transition functions).
The transition function P is factored over $\mathcal{S} \times \mathcal{A} = \mathcal{X} = \mathcal{X}_1 \times .. \times \mathcal{X}_n$ and $\mathcal{S} = \mathcal{S}_1 \times .. \times \mathcal{S}_m$ with scopes $Z_1, ..Z_m \iff$

$$P(s|x) = \prod_{i=1}^{m} P_i\left(s[i] \,\middle|\, x[Z_i]\right)$$

Main results

For M^* factored with known graphical structure as above then for PSRL and UCRL-Factored

$$\text{Regret}(T, M^*) = \tilde{O}\left(\Xi \sum_{j=1}^{m} \sqrt{|\mathcal{X}[Z^P_j]|\,|\mathcal{S}_j|\,T}\right).$$

Here Ξ is a measure of MDP connectedness for each algorithm, expected span $\mathbb{E}[\Psi]$ for PSRL and diameter D for UCRL-Factored.

PSRL's bounds are tighter since $\Psi(M) \le D(M)$ and may be exponentially smaller. However, UCRL-Factored holds with high probability for any M^* not just in expectation over the prior.

Key point: For m independent components with S states and A actions $= \tilde{O}(mS\sqrt{AT})$ and close to

$$\underbrace{m\sqrt{SAT}}_{\text{factored MDP lower bound}} \ll \underbrace{\sqrt{(SA)^m T}}_{\text{general MDP lower bound}}.$$

Posterior Sampling

For each episode k:

1. Sample an MDP from the posterior distribution for the true MDP: $M_k \sim \phi(\cdot|H_t)$.
2. Use policy $\mu_k \in \arg\max_{\mu, M} V^{M_k}_{\mu}$.

Proof sketch:

$$\begin{aligned}\Delta_k &= V^*_{*,1}(s) - V^*_{k,1}(s) \\ &= \underbrace{\left(V^k_{k,1}(s) - V^*_{k,1}(s)\right)}_{\text{Imagined - Actual}} + \underbrace{\left(V^*_{*,1}(s) - V^k_{k,1}(s)\right)}_{\mathbb{E}[\cdot]=0}\end{aligned}$$

We can decompose this into Bellman error:

$$V^k_{k,1} - V^*_{k,1} = \underbrace{\sum_{i=1}^{\tau} \left(\mathcal{T}^k_{k,i} - \mathcal{T}^*_{k,i}\right) V^k_{k,i+1}}_{B:=\text{Bellman error}} + \underbrace{\sum_{i=1}^{\tau} d_{t_k+1}}_{\mathbb{E}=0 \text{ martingale}}.$$

We can now use the Hölder inequality to bound:

$$B \le \sum_{i=1}^{\tau} \left\{ \underbrace{|\overline{R}^k - \overline{R}^*|}_{\text{reward error}} + \frac{1}{2} \underbrace{\Psi_k}_{\text{MDP span}} \underbrace{\|P^k - P^*\|_1}_{\text{transition error}} \right\}$$

We conclude the proof by upper bounding these deviations by maximum possible within $\mathcal{M}_k$. Concentration inequalities allows us to build tight $\mathcal{M}_k$ that contain M^* with high probability.

Key lemma

For any $P, \tilde{P}$ factored transition functions we may bound their L1 distance by the sum of the differences of their factorizations:

$$\|P(x) - \tilde{P}(x)\|_1 \le \sum_{i=1}^{m} \|P_i(x[Z_i]) - \tilde{P}_i(x[Z_i])\|_1$$

Proof sketch:
For any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$:

$$|\alpha_1\alpha_2 - \beta_1\beta_2| \le \alpha_2\,|\alpha_1 - \beta_1| + \beta_1\,|\alpha_2 - \beta_2|.$$

Repeat this argument for desired result.

References

Please see the full paper:
http://arxiv.org/abs/1403.3741

Optimsim

For each episode k:

1. Form $\mathcal{M}_k$ subset of MDPs M that are statistically plausible given the data.
2. Use policy $\mu_k \in \arg\max_{\mu} \left\{ \max_{M \in \mathcal{M}_k} V^M_{\mu}(s) \right\}$.

Proof sketch:

$$\begin{aligned}\Delta_k &= V^*_{*,1}(s) - V^*_{k,1}(s) \\ &= \underbrace{\left(V^k_{k,1}(s) - V^*_{k,1}(s)\right)}_{\text{Imagined - Actual}} + \underbrace{\left(V^*_{*,1}(s) - V^k_{k,1}(s)\right)}_{\le 0 \text{ by optimism}}\end{aligned}$$

Then follow the analysis per posterior sampling.

Example

Production line with 100 machines, each with 3 states and 3 actions. Each machine generates some revenue we want to maximize jointly.

Figure 3: automated production line

This MDP has state $s = (s_1, .., s_{100})$ and action $a = (a_1, .., a_{100})$. Here $S = A = 3^{100} \simeq 10^{50}$, so even a maximally efficient general-purpose learner would have regret $\Omega(\sqrt{SAT}) \simeq 10^{50}\sqrt{T}$.

If over a single timestep, each machine depends directly only upon its neighbours then this becomes a factored MDP. Now $|\mathcal{X}[Z^P_j]| \le 3^3$ and $|S_j| \le 3$ for each machine j.

We exploit this graphical structure for exponentially smaller regret $\simeq 100\sqrt{3^3 \times 3 \times T} \simeq 10^3\sqrt{T}$.

Key takeaway

Our regret bounds scale with the number of parameters, not the number of states.

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