III Gauss theorem

1) Need of Geometry

a. Notion of vector flux

b. Solid Angle

2) Statement of Gauss theorem

3) Direct applications

a. Electric field created by an infinite wire

b. Electric field created by an infinite plane

4) Electric field calculations

a. Empty and full charged cylinder

b. Empty and full sphere

4) Cylindrical Capacitors

5) Earth as a Capacitor

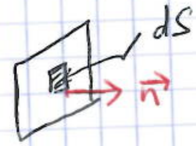
III Gauss theorem

1) Need of geometry

a- Notion of vector flux.

- Be a field of vectors: $\vec{V}$

- Be a surface S defined as $S = \iint dS$

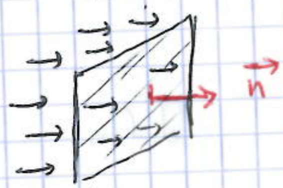


- Be a unit vector $\vec{n}$ normal to the surface

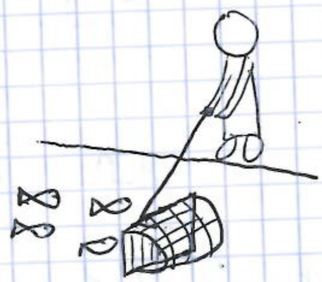
We define the flux of a vector through a surface as

$$\phi = \iint \vec{V} \cdot dS\,\vec{n} = \iint \vec{V} \cdot \vec{dS} \quad \text{with } \vec{dS} = dS\,\vec{n}$$

↑ scalar product.



Quantity of $\vec{V}$ that cross surface S.



Analogy with fishing salmons

Vector $\longrightarrow$ = 

Surface  =  Basket

$S_1 > S_2$

$$\phi_1 = \iint_{S_1} \vec{V} \cdot dS\,\vec{n}$$

$$\phi_2 = \iint_{S_2} \vec{V} \cdot dS\,\vec{n}$$

- Quantity depend on value of $\vec{V}$, value of S $\qquad \phi_1 > \phi_2$

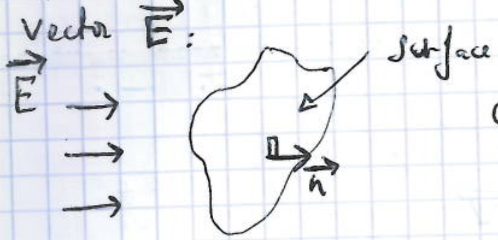
- ϕ depend on orientation of $\vec{V}$ and $\vec{n}$

$\vec{V} \,//\, \vec{n}$ $\quad \phi$ is maximum $\qquad\qquad \vec{V} \perp \vec{n}$ $\quad \phi$ is minimum

no flux through surface S

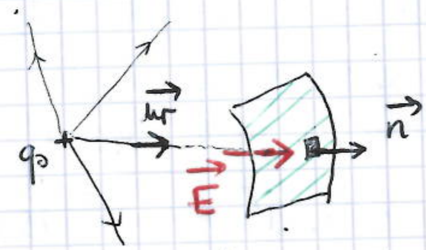
$\phi_{max} > \phi > \phi_{min}$

- In electrostatics we look the flux of the electric field vector $\vec{E}$:



$$\phi = \iint \vec{E} \cdot \vec{dS}$$

- charge q_0:

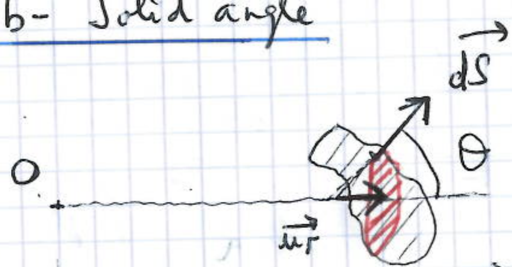


$$\phi = \iint \vec{E} \cdot \vec{dS} = \iint \frac{q}{4\pi\varepsilon_0 r^2} \vec{u_r} \cdot \vec{dS}$$

$$\phi = \frac{q}{4\pi\varepsilon_0} \iint \frac{\vec{u_r} \cdot \vec{dS}}{r^2} = \frac{q}{4\pi\varepsilon_0} \iint d\Omega$$

$$d\Omega = \frac{\vec{u_r} \cdot \vec{dS}}{r^2}$$

b- Solid angle



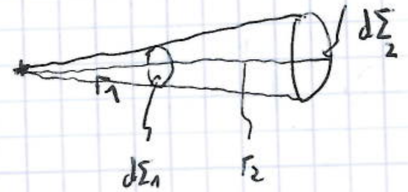
$\vec{u_r} \cdot \vec{dS} = dS \cos\theta =$ ▨ $d\Sigma$

part orthogonal projection of the surface $\vec{dS}$ onto $\vec{u_r}$

cone of center O

$$d\Omega = \frac{\vec{u_r} \cdot \vec{dS}}{r^2} = \frac{d\Sigma}{r^2}$$

is angle with which we see surface $d\Sigma$ from O



$$d\Omega = \frac{d\Sigma_2}{r_2^2} = \frac{d\Sigma_1}{r_1^2}$$

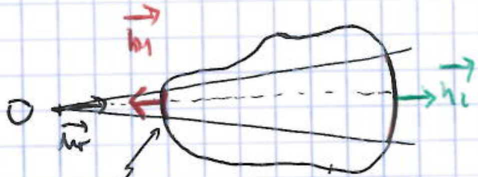
* Solid angle under which is seen a closed surface

Closed surface: every volume is delimited by a surface :

Ex : cube  closed surface $S = 6a^2$

$V = a^3$

O is outside the volume :

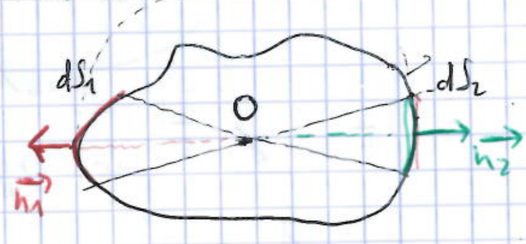


$d\Omega_1 = \frac{\vec{u_r} \cdot \vec{dS_1}}{r_1^2}$ $\quad d\Omega_2 = \frac{\vec{u_r} \cdot \vec{dS_2}}{r_2^2}$

$d\Omega_1 = - d\Omega_2$

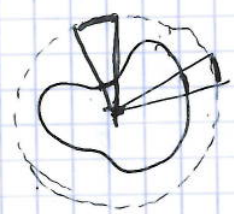
So $\iint d\Omega = 0$

O is inside the volume :



$$\iint d\Omega = \iint \frac{\vec{u_r} \cdot \vec{dS}}{r^2}$$

We assume $\vec{u_r} \cdot \vec{dS} = dS$ spherical

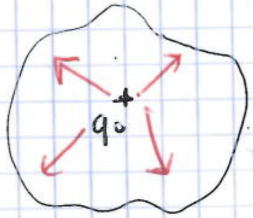


$dS = r^2 d\theta \sin\theta d\varphi$

$$\Omega = \iint d\Omega = \oiint \frac{r^2 d\theta \sin\theta d\varphi}{r^2} = \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi = 4\pi$$

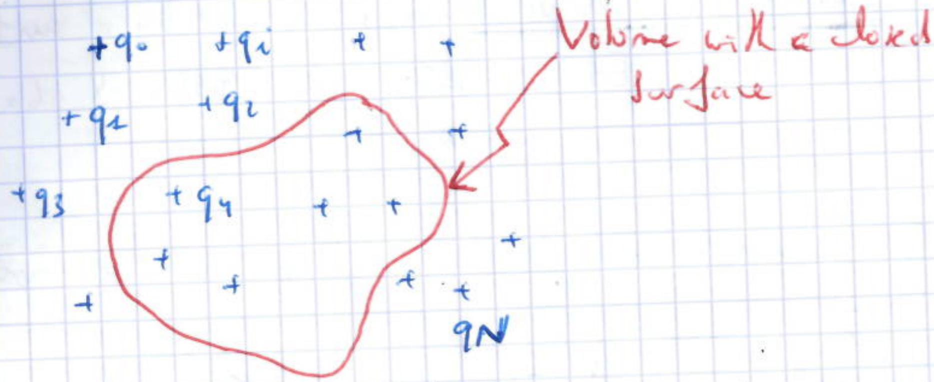
closed surface

$\Omega = 4\pi$



$$\phi = \frac{q_0}{4\pi\varepsilon_0} \iint d\Omega = \frac{q}{4\pi\varepsilon_0} \times \overbrace{4\pi}^{\Omega} = \frac{q_0}{\varepsilon_0}$$

2. Statement of Gauss theorem



Flux of $\vec{E}$ crossing the surface

$$\phi = \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_0}\Omega_i$$

$$= \sum_{i_{int}} \frac{q_i}{4\pi\varepsilon_0}\Omega_i + \underbrace{\sum_{i_{ext}} \frac{q_i \Omega_i}{4\pi\varepsilon_0}}_{=0}$$

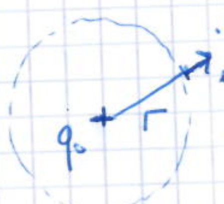
$= 0$; all flux with origine out of the volume is zero.

$$= \sum_{i_{int}} \frac{q_i}{4\pi\varepsilon_0} 4\pi$$

$$= \frac{1}{\varepsilon_0} \sum_{i_{int}} q_i = \frac{Q_{int}}{\varepsilon_0}$$

charges internal (inside the volume)

$$\boxed{\phi = \oint \vec{E}.\vec{dS} = \frac{Q_{int}}{\varepsilon_0}}$$



. What is field at distance r?
amplitude

. We take a sphere of radius r

$$\oint \vec{E}.\vec{dS} = \frac{Q_{int}}{\varepsilon_0}$$

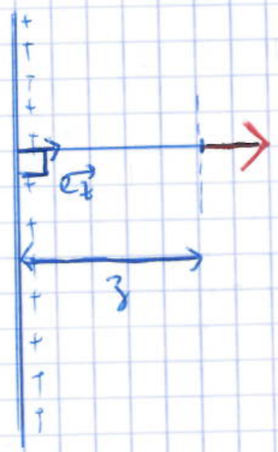
$$E(r). \oint dS = \frac{Q_{int}}{\varepsilon_0} = \frac{q_0}{\varepsilon_0}$$

$$E(r)\, 4\pi r^2 = \frac{q_0}{\varepsilon_0}$$

$$E = \frac{q_0}{4\pi\varepsilon_0 r^2} \quad \text{and} \quad \vec{E}(r) = \frac{q_0}{4\pi\varepsilon_0 r^2}\,\vec{u_r}.$$

3. Direct application.

a- Infinite wire

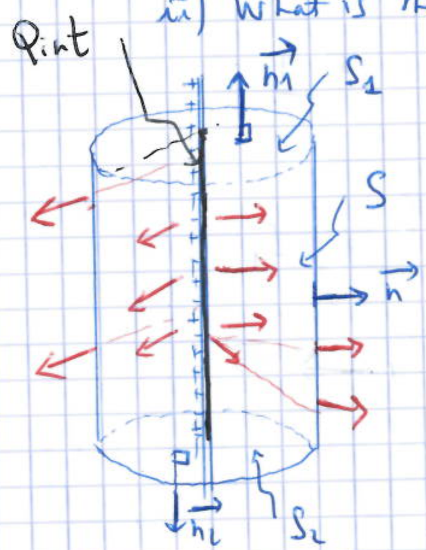


see II.1) $\vec{E}(z) = \dfrac{\lambda}{2\pi\varepsilon_0 z}\,\vec{e_z}$

With Gauss theorem

i) Symetry say $\vec{E}$ should be $\perp$ to the wire $\sim \vec{e_z}$

ii) What is the value at distance z?



$$\oiint \vec{E}.\vec{dS} = \frac{Q_{int}}{\varepsilon_0}$$

- A cylindric volume of radius z and height h
- Flux of $\vec{E}$ that crosses closed surface of cylinder:

$$\oiint \vec{E}.\vec{dS} = \iint_{S_1} \vec{E}.dS\,\vec{n_1} + \iint_{S_2} \vec{E}.dS\,\vec{n_2} + \iint_{S} \vec{E}.dS.\vec{n}$$

$$= \Phi_1 + \Phi_2 + \Phi$$

$$= 0 + 0 + E(z).\iint dS$$

$$= E(z)\, 2\pi r h$$

$\vec{E} \perp \vec{n_1}$

$\perp \vec{n_2}$

$\vec{E} \parallel \vec{n}$

$$\oiint \vec{E}.\vec{dS} = E(z)\, 2\pi r h$$

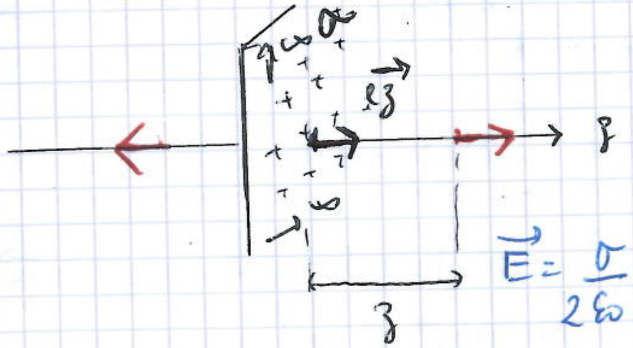
and $Q_{int} = \int_0^h \lambda \, dx = \lambda h$

$$\oint \vec{E} \cdot \vec{dS} = \frac{Q_{int}}{\varepsilon_0}$$

$$E(z)\, 2\pi z h = \frac{\lambda h}{\varepsilon_0}$$

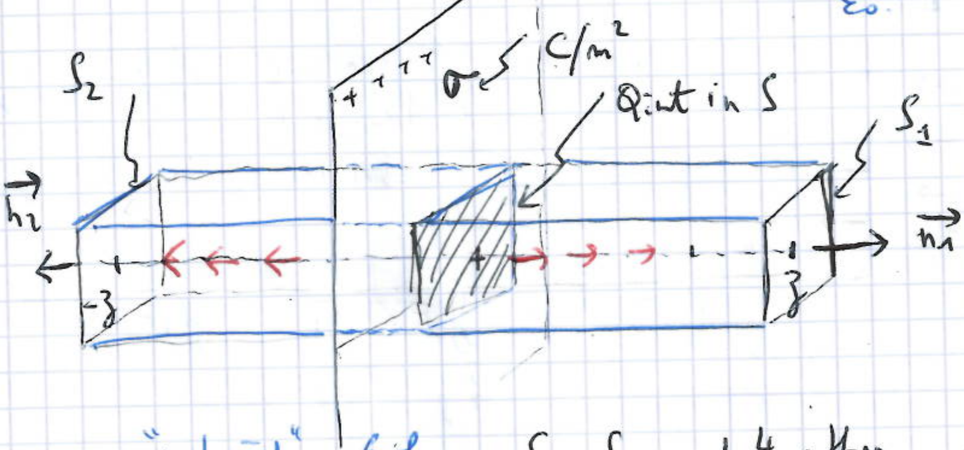
$$E(z) = \frac{\lambda}{2\pi\varepsilon_0 z} \quad \text{and} \quad \boxed{\vec{E}(z) = \frac{\lambda}{2\pi\varepsilon_0 z} \vec{e_z}}$$

b- Infinite plane



$\vec{E} = \frac{\sigma}{2\varepsilon_0} \vec{e_z}$ UNIFORM $\forall z$!!!

With Gauss: $\oint \vec{E} \cdot \vec{dS} = \frac{Q_{int}}{\varepsilon_0}$

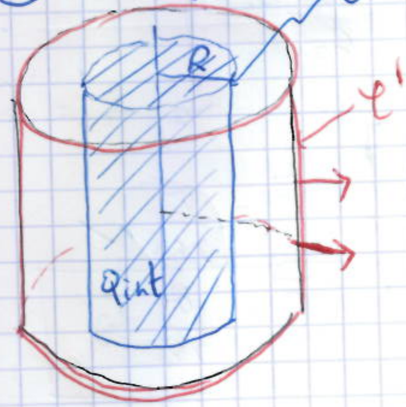


"cuboid": 6 faces S_1, S_2 and 4 others.

$$\underset{\text{Surface of cuboid}}{\oint \vec{E} \cdot \vec{dS}} = \iint \vec{E}(z) \cdot \vec{n_1} + \iint \vec{E}(z) \cdot \vec{n_2} + \underset{\vec{E} \,/\!/\, \text{other surfaces}}{0 + 0 + 0 + 0}$$

$$= E(z) S_1 + E(z) S_2$$

(ii) if $z > R$



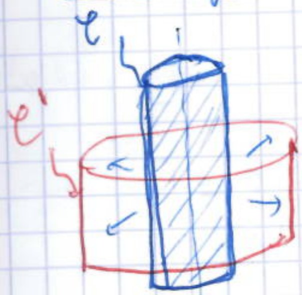
$$\oint \vec{E}\cdot d\vec{S} = \frac{Q_{int}}{\varepsilon_0} \Big\{ \text{ limited by } \mathcal{C}$$

$$\iint \vec{E}\cdot d\vec{S} + 0 + 0 = \frac{Q_{int}}{\varepsilon_0}$$

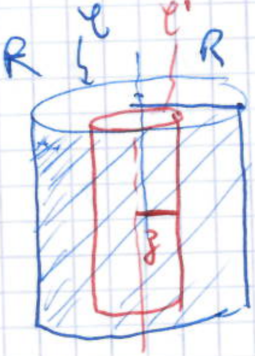
$$E\, 2\pi z h = \frac{\rho \pi R^2 h}{\varepsilon_0}$$

$$E = \frac{\rho R^2}{2\varepsilon_0 z} \quad \text{and} \quad \vec{E}(z) = \frac{\rho R^2}{2\varepsilon_0 z}\vec{e_z}$$

Then cylinder charged in surface only (no charge inside)



$z < R$



$$\iint \vec{E}\cdot d\vec{S} = \frac{Q_{int}}{\varepsilon_0}$$

$$E \cdot 2\pi z h = 0$$

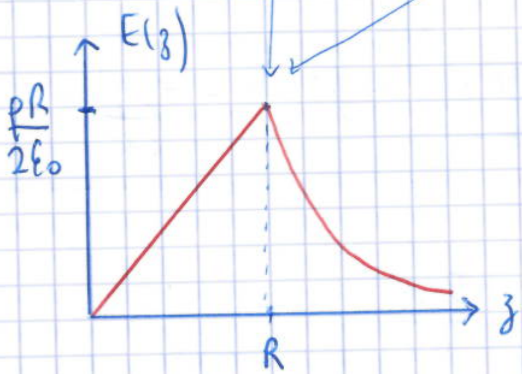
$$\boxed{\vec{E}(z) = \vec{0}}$$

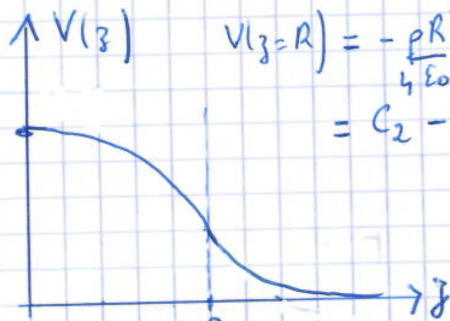
$z > R$

$$E(z) \cdot 2\pi z h = \frac{Q_{int}}{\varepsilon_0} = \frac{\sigma\, 2\pi R h}{\varepsilon_0}$$

$$\boxed{\vec{E}(z) = \frac{\sigma R}{\varepsilon_0 z}\vec{e_z}}$$

	INSIDE	OUTSIDE
FULL CHARGED cylinder	$\vec{E}^{in}(z) = \frac{\rho z}{2\varepsilon_0}\vec{e_z}$	$\vec{E}^{out}(z) = \frac{\rho R^2}{2\varepsilon_0 z}\vec{e_z}$
	$V^{in}(z) = -\frac{\rho}{4\varepsilon_0}z^2 + C_1$	$V^{out}(z) = C_2 - \frac{\rho R^2}{2\varepsilon_0}\ln z$
	$E(z=R) = \frac{\rho R}{2\varepsilon_0}$ and	$E(z=R) = \frac{\rho R^2}{2\varepsilon_0 R} = \frac{\rho R}{2\varepsilon_0}$



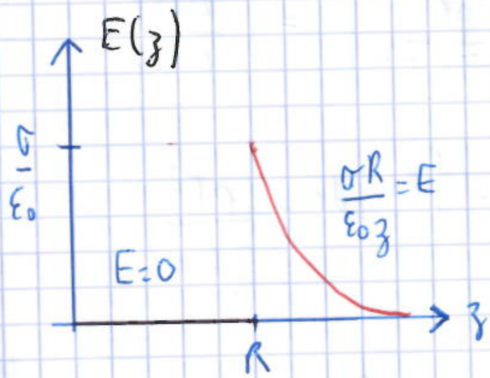


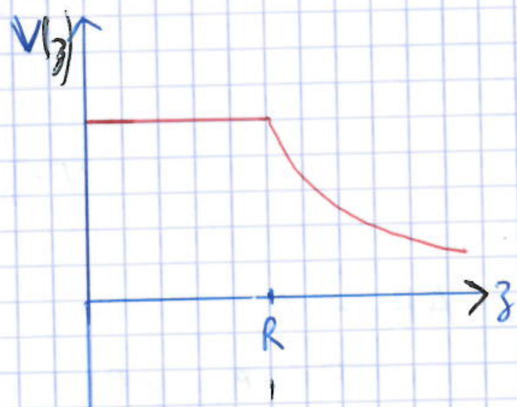
$$V(z=R) = -\frac{\rho R^2}{4\varepsilon_0} + C_1 = C_2 - \frac{\rho R^2}{2\varepsilon_0}\ln z$$

So: $C_2 = C_1 - \frac{\rho R^2}{4\varepsilon_0} + \frac{\rho R^2}{2\varepsilon_0}\ln R$

$$\boxed{V^{out}(z) = C_1 - \frac{\rho R^2}{4\varepsilon_0}\ln\frac{z}{R} - \frac{\rho R^2}{4\varepsilon_0}}$$

Charged only on surface





$E(z=R) = \frac{\sigma}{\varepsilon_0}$

$V(z) = C_1$

$V(z) = C_1$ | $V(z) = C_2 - \frac{\sigma R}{\varepsilon_0}\ln z$

In $z=R$

$$C_1 = C_2 - \frac{\sigma R}{\varepsilon_0}\ln R$$

$$\boxed{V^{out}(z) = C_1 - \frac{\sigma R}{\varepsilon_0}\ln\frac{z}{R}}$$

b - empty and full charged sphere

full sphere

$\boxed{r < R}$

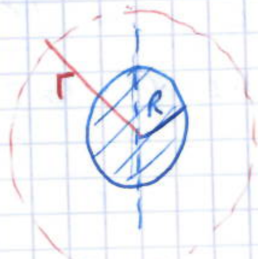
$$\oiint \vec{E} \cdot \vec{dS} = \frac{Q_{int}}{\varepsilon_0}$$

$$E(r)\, 4\pi r^2 = \rho \frac{4\pi r^3}{3\varepsilon_0}$$

$$\boxed{E^{in}(r) = \frac{\rho r}{3\varepsilon_0}}$$

and $\underline{V(r)} = -\int \vec{E} \cdot \vec{dr} = \underline{-\frac{\rho r^2}{6\varepsilon_0} + C_1}$

$\boxed{r > R}$



$$\oint \vec{E} \cdot \vec{dS} = \frac{Q_{int}}{\varepsilon_0}$$

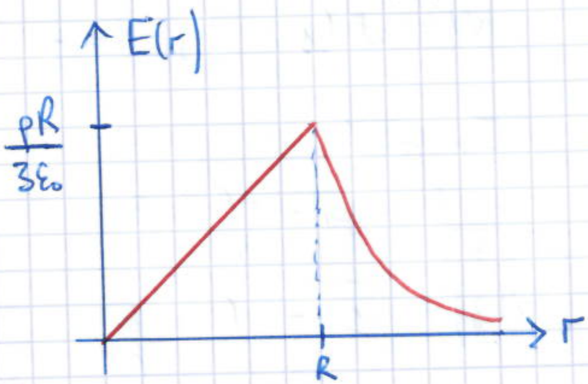
$$E(r)\, 4\pi r^2 = \frac{4\pi R^3}{3\varepsilon_0}\rho$$

$$\boxed{E(r) = \frac{\rho R^3}{3\varepsilon_0 r^2}} = \frac{4\pi \rho R^3}{4\pi \varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$V(r) = -\int E(r)\, dr = \frac{\rho R^3}{3\varepsilon_0 r} + C_2$$

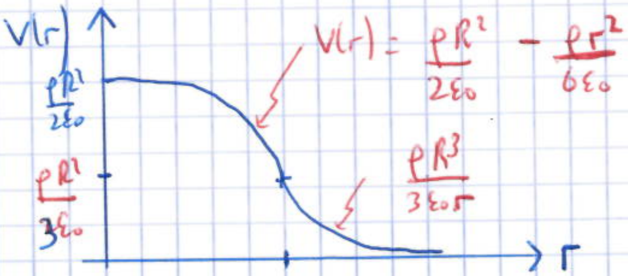
At $r = R$:

$E^{in}(R) = \frac{\rho R}{3\varepsilon_0}$ ok with $E^{out}(R) = \frac{\rho R^3}{3\varepsilon_0 R^2} = \frac{\rho R}{3\varepsilon_0}$

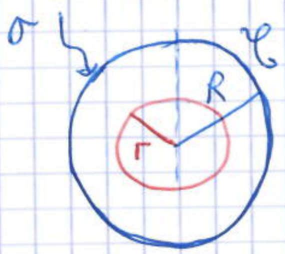


- No charge at ∞, so we have $C_2 = 0$
- at $r = R$ $\quad V_{int}(r=R) = -\dfrac{\rho R^2}{6\varepsilon_0} + C_1 = V_{ext} = \dfrac{\rho R^3}{3\varepsilon_0 R} = \dfrac{\rho R^2}{3\varepsilon_0}$

$$C_1 = \frac{\rho R^2}{3\varepsilon_0} + \frac{\rho R^2}{6\varepsilon_0} = \frac{\rho R^2}{2\varepsilon_0}$$



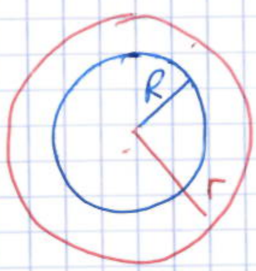
<u>Sphere charged in surface.</u>



<u>$r < R$</u>

$$\oint \vec{E}\cdot\vec{dS} = \frac{Q_{int}}{\varepsilon_0} = 0$$

$$E = 0$$



<u>$r > R$</u>

$$\underbrace{\oint \vec{E}\cdot\vec{dS}}_{4\pi r^2 E} = \frac{Q_{int}}{\varepsilon_0} = \frac{4\pi R^2 \sigma}{\varepsilon_0}$$

$$\boxed{E(r) = \frac{4\pi R^2 \sigma}{4\pi r^2} = \frac{R^2 \sigma}{\varepsilon_0 r^2}}$$

Potential:

$r < R$: $V = cte = C_1$

$r > R$ $\quad V = \dfrac{R^2 \sigma}{\varepsilon_0 r} + \underbrace{C_2}_{= 0}$

4- Cylindrical Capacitor:

See slides

5- Earth as a capacitor

see slides

IV Some Applications

1) Electrostatic dipole

- a- Potential and electric fields in the dipolar approximation
- b- Molecules
- c- Dipole-dipole interactions

2) Electrostatics of conductors

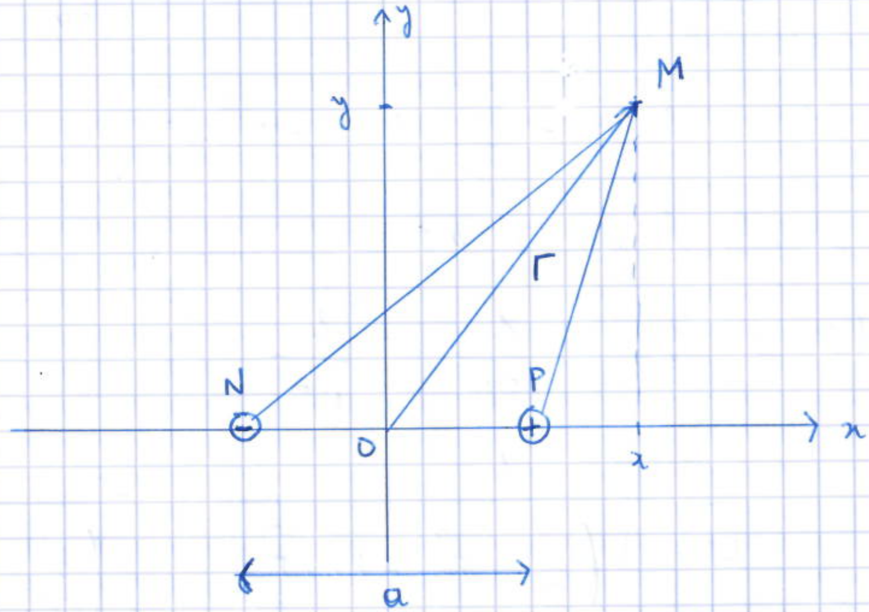
3) High Voltage breakdown

4) Electricity in the atmosphere (from Feynman lecture)

IV Some Applications

1- Electrostatic dipole

a- Potential and electric field in the dipolar approximation



- Potential $V(M) = \dfrac{q}{4\pi\varepsilon_0 PM} - \dfrac{q}{4\pi\varepsilon_0 NM}$

$$PM = \sqrt{(x_M - x_P)^2 + (y_M - y_P)^2} = \sqrt{(x - a/2)^2 + y^2}$$

$$NM = \sqrt{(x + a/2)^2 + y^2}$$

Complicated to solve.

- We work in dipolar approximation $\|\vec{OM}\| = r \gg a$
  distance of observation is larger than size of dipole a:

$$\vec{NM} = \vec{NO} + \vec{OM}$$

$$\|\vec{NM}\| = \sqrt{(\vec{NM} \cdot \vec{NM})} = \left(\|\vec{NO}\|^2 + \|\vec{OM}\|^2 + 2\vec{NO} \cdot \vec{OM}\right)^{1/2}$$

with $\|\vec{NO}\|^2 = (a/2)^2$ $\qquad \vec{NO} = \dfrac{1}{2}\vec{NP}$

$\|\vec{OM}\|^2 = r^2$

$$\|\vec{NM}\| = \left((a/2)^2 + r^2 + 2\vec{NO} \cdot \vec{OM}\right)^{1/2}$$

$$= \left(a^2/4 + r^2 + \vec{NP} \cdot \vec{r}\right)^{1/2}$$

In the same manner

$$\|\vec{PM}\| = \|\vec{PO} + \vec{OM}\| = \sqrt{\|\vec{PO} + \vec{OM}\|^2}$$
$$= \left(PO^2 + OM^2 + 2\,\vec{PO}\cdot\vec{OM}\right)^{1/2}$$

$$\left.\begin{aligned} PO^2 &= (-a/2)^2 = a^2/4 \\ OM^2 &= r^2 \\ 2\,\vec{PO}\cdot\vec{OM} &= 2\,\underbrace{\vec{PN}}_{2}\cdot\vec{OM} = -\vec{NP}\cdot\vec{r} \end{aligned}\right\} \|\vec{PM}\| = a^2/4 + r^2 - \vec{NP}\cdot\vec{r}$$

We rewrite the potential as:

$$V(M) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{PM} - \frac{1}{NP}\right.$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\left(\frac{a^2}{4} + r^2 - \vec{NP}\cdot\vec{r}\right)^{1/2}} - \frac{1}{\left(\frac{a^2}{4} + r^2 + \vec{NP}\cdot\vec{r}\right)^{1/2}}\right)$$

. Dipolar moment $\vec{p} = q\,\vec{NP}$

$$\frac{1}{(1+x)^{1/2}} = (1+x)^{-1/2} \;;$$

$$V(M) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\left(r^2\left(1 + \frac{a^2}{4r^2} - \frac{\vec{NP}\cdot\vec{r}}{r^2}\right)\right)^{1/2}} - \frac{1}{\left(r^2\left(1 + \frac{a^2}{4r^2} + \frac{\vec{NP}\cdot\vec{r}}{r^2}\right)\right)^{1/2}}\right]$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[\left(1 - \frac{\vec{NP}\cdot\vec{r}}{r^2} + \frac{a^2}{4r^2}\right)^{1/2} - \left(1 + \frac{\vec{NP}\cdot\vec{r}}{r^2} + \frac{a^2}{4r^2}\right)^{1/2}\right]$$

$\vec{NP} = a\vec{e_x}$

$\vec{NP}\cdot\vec{r} = a\cos\theta\, r$

$$\frac{\vec{NP}\cdot\vec{r}}{r^2} = \frac{a\cos\theta\, r}{r^2} = \frac{a}{r}\cos\theta$$

$$V(M) = \frac{q}{4\pi\varepsilon_0 r}\left[\left(1 - \frac{a\cos\theta}{r} + \frac{a^2}{4r^2}\right)^{-1/2} - \left(1 + \frac{a\cos\theta}{r} + \frac{a^2}{4r^2}\right)^{-1/2}\right]$$

We keep the therm at first order in: $\frac{a}{r}$; we neglect $\frac{a^2}{r^2}$

$\frac{a}{r} \ll 1$ so $\frac{a^2}{r^2}$ really negligible

$$V(M) = \frac{q}{4\pi\varepsilon_0 r}\left[\left(1 - \frac{a\cos\theta}{r}\right)^{1/2} - \left(1 + \frac{a\cos\theta}{r}\right)^{1/2}\right]$$

$$\left.\begin{aligned} &(1+x)^{\alpha} = 1 - \alpha x \\ &x = \frac{a\cos\theta}{r} \\ &\alpha = 1/2 \end{aligned}\right\} \quad \begin{aligned} &\left(1 - \frac{a\cos\theta}{r}\right)^{1/2} = 1 + \frac{1}{2}\frac{a\cos\theta}{r} \\ &\left(1 + \frac{a\cos\theta}{r}\right)^{1/2} = 1 - \frac{1}{2}\frac{a\cos\theta}{r} \end{aligned}$$

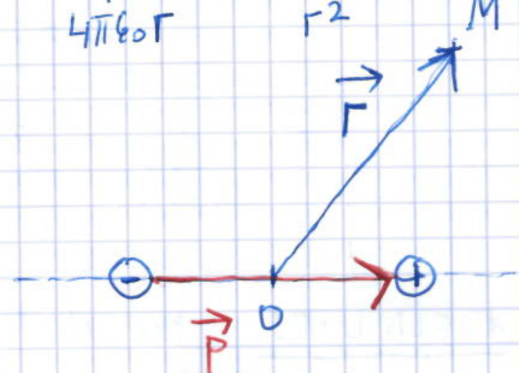
$$V(M) = \frac{q}{4\pi\varepsilon_0 r}\left[1 + \frac{1}{2}\frac{a\cos\theta}{r} - \left(1 - \frac{1}{2}\frac{a\cos\theta}{r}\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[\cancel{1} + \frac{a\cos\theta}{2r} - \cancel{1} + \frac{a\cos\theta}{2r}\right]$$

$$= \frac{q}{4\pi\varepsilon_0 r}\,\frac{\cancel{2}\, a\cos\theta}{r} = \frac{q}{4\pi\varepsilon_0 r}\,\frac{a\, r\cos\theta}{r^2}$$

$$= \frac{q\, a\, \vec{e_x}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$$

$$\boxed{V(M) = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}}$$



approximation of calculation:

Electric field calculations

$\vec{E} = -\vec{\text{grad}}\, V$ POLAR COORDINATES

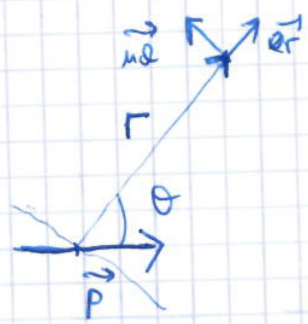
$$= -\frac{\partial V}{\partial r} - \frac{1}{r}\frac{\partial V}{\partial \theta}$$

$$= -\frac{\partial}{\partial r}\left(\frac{q\, p r \cos\theta}{4\pi\varepsilon_0 r^3}\right)\vec{e_r} - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{q\, p r\cos\theta}{4\pi\varepsilon_0 r^3}\right)$$

$$= \frac{-qp\cos\theta}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\vec{e_r} - \frac{q\,p}{4\pi\varepsilon_0 r^3}\left(\frac{\partial}{\partial\theta}\cos\theta\right)\vec{e_\theta}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[+\frac{2p\cos\theta}{r^3}\vec{e_r} + \frac{p}{r^3}\sin\theta\, \vec{e_\theta}\right]$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\left[2p\cos\theta\,\vec{e_r} + p\sin\theta\,\vec{e_\theta}\right]$$



$\vec{p} = p\cos\theta\,\vec{e_r} - p\sin\theta\,\vec{e_\theta}$

$2p\cos\theta\,\vec{e_r} + p\sin\theta\,\vec{e_\theta} + p\cos\theta\,\vec{e_r} - p\cos\theta\,\vec{e_r}$

$= 3p\cos\theta\,\vec{e_r} - p(\cos\theta\,\vec{e_r} - p\sin\theta\,\vec{e_\theta})$

$= 3\vec{p}\cdot\vec{e_r} - \vec{p}$

$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\left(3\vec{p}\cdot\vec{e_r} - \vec{p}\right)}$$

FIELD LINES: see slides

not $\frac{r}{a} \gg 1$

not defined by previous formula