II Examples of electric field calculations

1) Electric field created by a charged electric wire

a. Calculation of the electric field
b. Calculation of the electric potential
c. Limit case of the infinite wire.
d. Analysis in terms of field lines

2) Some examples of 2D electric charged structures

a. Electric field and electric potential created by a crown and a disc
b. Limit case of the infinite charged plane
c. Analysis in terms of field lines

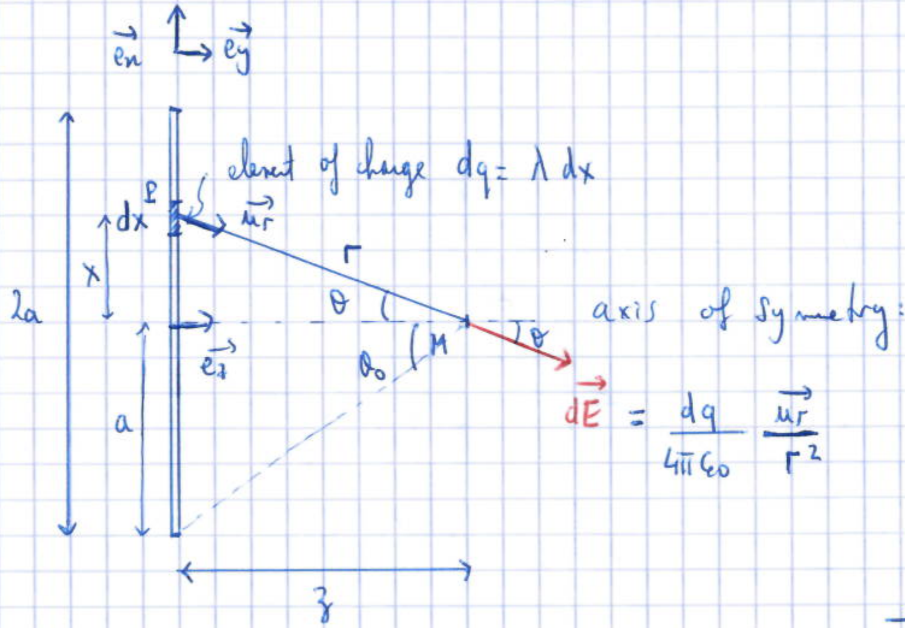
3) Application to the plane capacitor

a. Electric field and electric potential
b. Capacitance and energy.
c. Electron dynamics in a constant and uniform electric field

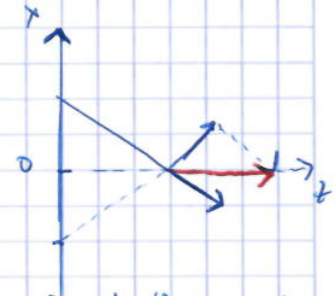
Chapter II. Examples of electric field calculations

1) Electric field created by a charged electric wire

a- Calculation of the electric field



$$\vec{dE} = \frac{dq}{4\pi\varepsilon_0}\,\frac{\vec{u_r}}{r^2}$$



M belong to the axis of symmetry so the field will also

Component along z will be $dE\cos\theta\,\vec{e_z} = d\vec{E_z}$

TOTAL FIELD is

$$\vec{E}(z) = \int_{\theta=-\theta_0}^{\theta=\theta_0} d\vec{E_z} = 2\int_0^{\theta_0} dE\cos\theta\,\vec{e_z} = 2\int_0^{\theta_0} \frac{\lambda\,dx}{4\pi\varepsilon_0}\,\frac{\cos\theta}{r^2}\,\vec{e_z}$$

By integrating: for all point P.

Changing parameters	fixed parameters
r, θ, x	z

$$\begin{cases} \tan\theta = \dfrac{x}{z} \\ \dfrac{d\theta}{\cos^2\theta} = \dfrac{dx}{z} \end{cases} \Rightarrow dx = \frac{z\,d\theta}{\cos^2\theta} \quad \text{and} \quad \cos\theta = \frac{z}{r} \Leftrightarrow \frac{1}{r^2} = \frac{\cos^2\theta}{z^2}$$

$$\vec{E}(z) = 2\int_0^{\theta_0} \frac{\lambda\,\vec{e_z}}{4\pi\varepsilon_0}\,\frac{z\,d\theta}{\cancel{\cos^2\theta}} \times \cos\theta \times \frac{\cancel{\cos^2\theta}}{z^{\cancel{2}}} = \frac{2\lambda}{4\pi\varepsilon_0}\cdot\frac{\vec{e_z}}{z}\int_0^{\theta_0}\cos\theta\,d\theta$$

$$= \frac{2\lambda}{4\pi\varepsilon_0 z}\,\vec{e_z}\,\Big[\sin\theta\Big]_0^{\theta_0} = \frac{\lambda}{2\pi\varepsilon_0 z}\,\sin\theta_0\,\vec{e_z}$$

and $\sin\theta_0 = \dfrac{a}{\sqrt{a^2+z^2}}$

$$\boxed{\vec{E}(z) = \frac{\lambda}{2\pi\varepsilon_0 z}\frac{a}{\sqrt{a^2+z^2}}\vec{e_z}} \quad (*)$$

2) $[E] = \dfrac{\text{Volt}}{\text{meter}}$ or $\left[\dfrac{q}{4\pi\varepsilon_0 r^2}\right] = \dfrac{\text{Coulomb}}{[\varepsilon_0]\ m^2}$

$[\varepsilon_0] = F.m^{-1}$

$$\left[\frac{\lambda}{2\pi\varepsilon_0 z}\frac{a}{\sqrt{a^2+z^2}}\right] \approx \frac{[\lambda]}{[\varepsilon_0]\,[z]}\frac{[a]}{[\sqrt{a^2+z^2}]}$$

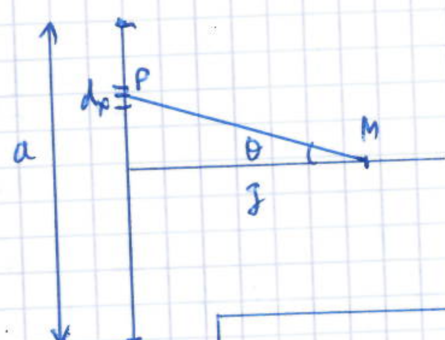
$$\approx \frac{C.m^{-1}}{[\varepsilon_0]\ m.}\frac{m}{m} = \frac{C}{[\varepsilon_0]\ m^2} \text{ , ok.}$$

3) $\vec{E} = -\vec{\text{grad}}\,V$

$[V] = -\int \vec{E}.\vec{dr}$

$[V] = -\int \dfrac{\lambda}{2\pi\varepsilon_0 z}\dfrac{a\ dz}{\sqrt{a^2+z^2}}$ ..... not easy ....

if we calculate by integration ....



$$V(z) = \int_{\theta=\theta_0}^{\theta=\theta_0} \frac{\lambda\, dx}{4\pi\varepsilon_0\, PM}$$

not easy neither

$$\boxed{V(z) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{\sqrt{a^2+z^2}+2a}{\sqrt{a^2+z^2}-2a}\right]} \text{ F.Y.I}$$

4) Wire is infinite: it means $a \gg z$

$$\lim_{a \to \infty} \frac{a}{\sqrt{a^2+z^2}} = \lim_{a \to \infty} \frac{a}{\sqrt{a^2}} = 1$$

$$\boxed{\underset{\text{infinite wire}}{\vec{E}(z)} = \frac{\lambda}{2\pi\varepsilon_0 z}\vec{e_z}}$$

Potential .... $[V(z)] = -\int \vec{E} \cdot dz\,\vec{e_z} = -\int \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{z}\,dz$

$$V(z) = -\frac{\lambda}{2\pi\varepsilon_0}\ln z + cte.$$

$\downarrow$

But .... problem ... $\ln z$ ..... should be $\ln\left(\frac{z}{\text{distance}}\right)$

So we define a given z_0 for which

$$V(z_0) \equiv 0 = -\frac{\lambda}{2\pi\varepsilon_0}\ln z_0 + cte$$

$$cte = \frac{\lambda}{2\pi\varepsilon_0}\ln z_0 = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{1}{z_0}\right)$$

$$\boxed{V(z) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{z}{z_0}}$$

$[V]$ is Volt or $\frac{C}{[\varepsilon_0]\,m}$

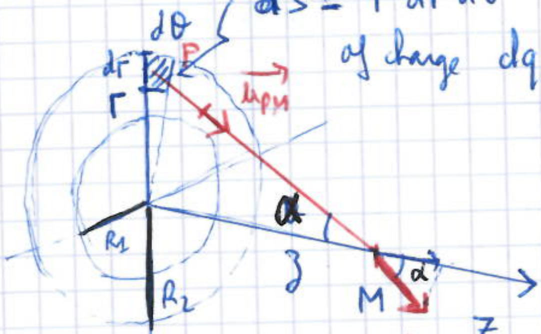
it is ok with $\left[\frac{\lambda}{2\pi\varepsilon_0}\right] = \frac{[\lambda]}{[\varepsilon_0]} = \frac{C}{[\varepsilon_0].m}$

5) <u>Orientation of field lines.</u>

See Slides II.

2 - Some exampls of 2D electric charged structures

a- Electric field and electric potential created by a crown and a disc



$dS = r\,dr\,d\theta$ of charge $dq = \sigma\, dS$

$$\vec{dE} = \frac{\sigma\, dS}{4\pi\varepsilon_0 PM^2}\vec{u_{PM}}$$

For symetry reasons .... we look the contribution on Oz axis

$d\vec{E}_z = dE\cos\alpha\,\vec{e_z}$ and total field

$$\vec{E}(z) = \int_{r=R_1}^{r=R_2}\int_{\theta=0}^{2\pi}\frac{\sigma\, r\,dr\,d\theta\cos\theta}{4\pi\varepsilon_0\, PM^2}\vec{e_z}$$

$$\begin{cases} PM^2 = z^2 + r^2 \\ \cos\theta = \dfrac{z}{PM} = \dfrac{z}{\sqrt{z^2+r^2}} \end{cases}$$

$$\vec{E}(z) = \frac{\sigma}{4\pi\varepsilon_0}\vec{e_z}\int_{r=R_1}^{r=R_2}\int_{\theta=0}^{2\pi}\frac{r\,dr\,d\theta\, z}{(z^2+r^2)^{3/2}}$$

$$= \frac{\sigma z\vec{e_z}}{4\pi\varepsilon_0}\underbrace{\int_{\theta=0}^{2\pi}d\theta}_{[\theta]_0^{2\pi}}\int_{r=R_1}^{r=R_2}\frac{r\,dr}{\underbrace{(z^2+r^2)^{3/2}}_{\text{Find primitive...}}}$$

$$f(r) = u(r)^n$$
$$f'(r) = n\, u'(r)\, u(r)^{n-1}$$

$$\frac{r}{(z^2+r^2)^{3/2}} = r \times (r^2+z^2)^{-3/2}$$
$$= \frac{\overset{u'(r)}{2r}}{2} \underbrace{(r^2+z^2)}_{u(r)}^{\overset{n-1}{-3/2}}$$

$$\begin{cases} u(r) = r^2+z^2 \\ u'(r) = 2r \end{cases}$$

$$= \frac{(-1)}{(-1)} \underbrace{\frac{1}{2}}_{n} u'(r) \times u(r)^{n-1}$$

$$n-1 = -3/2$$

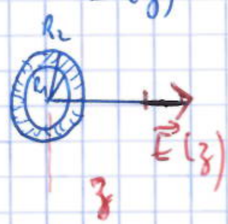
$$n = -\frac{1}{2}$$

$$= -\, n\, u'(r)\, u(r)^{n-1} = f'(r)$$

So $f(r) = -(r^2+z^2)^{-1/2} = -\dfrac{1}{\sqrt{r^2+z^2}}$

$$\vec{E}(z) = \frac{\sigma z}{4\pi\varepsilon_0} \times 2\pi \times \left[\frac{-1}{\sqrt{r^2+z^2}}\right]_{R_1}^{R_2} \vec{e_z}$$

$$\boxed{\vec{E}(z) = \frac{\sigma z}{2\varepsilon_0} \vec{e_z} \left(\frac{1}{\sqrt{R_1^2+z^2}} - \frac{1}{\sqrt{R_2^2+z^2}}\right)}$$



Case of a disc:  crown with $R_1 \longrightarrow 0$

$$\boxed{\vec{E}(z) = \frac{\sigma z}{2\varepsilon_0} \left(\frac{1}{z} - \frac{1}{\sqrt{R_2^2+z^2}}\right) \vec{e_z}}$$

b- Limit case of an infinite plan

$$\lim_{R_2 \to \infty} \left[\frac{1}{z} - \frac{1}{\sqrt{R_2 + z^2}} \right] = \frac{1}{z}$$

$$\left[\vec{E}(z) = \frac{\sigma}{2\varepsilon_0} z \, \vec{e_z} \cdot \frac{1}{z} = \frac{\sigma}{2\varepsilon_0} \vec{e_z} \right]$$

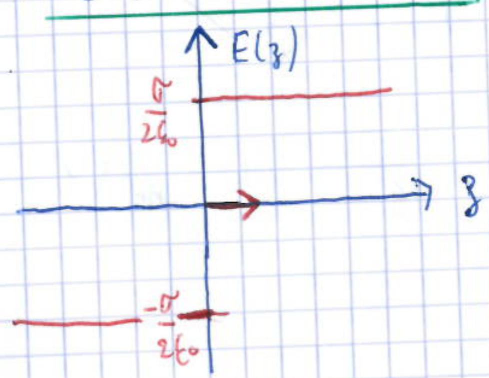
$$\| \vec{E} \| = \text{constante} = \frac{\sigma}{2\varepsilon_0}.$$

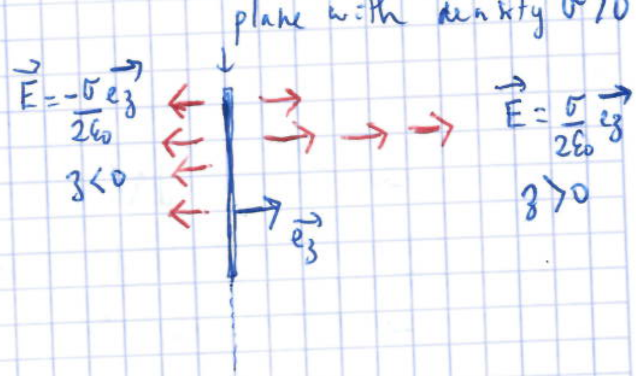
Potential : $V(z) = - \int \vec{E} \cdot dz \, \vec{e_z} = - \int \frac{\sigma}{2\varepsilon_0} dz$

$$\boxed{V(z) = - \frac{\sigma z}{2\varepsilon_0} + cte}$$

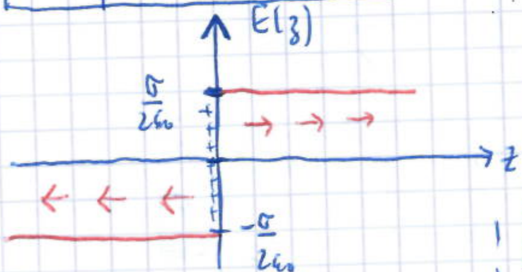
c- Analysis in terms of Field Lines

See slids II.2



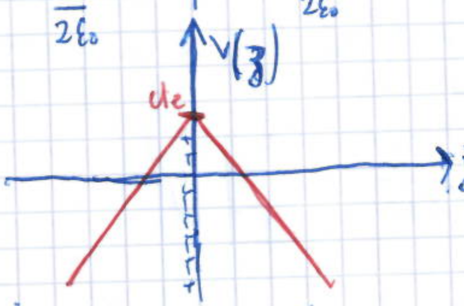


• $\boxed{\text{One plate with } \sigma > 0}$ and $\vec{E} = \frac{\sigma}{2\varepsilon_0}\vec{e_z} \rightarrow \begin{cases} E = \sigma/2\varepsilon_0 > 0 & z > 0 \\ E = -\sigma/2\varepsilon_0 < 0 & z < 0 \end{cases}$



• $V^- = -\int -\frac{\sigma}{2\varepsilon_0} dz = \frac{\sigma z}{2\varepsilon_0} + cte$

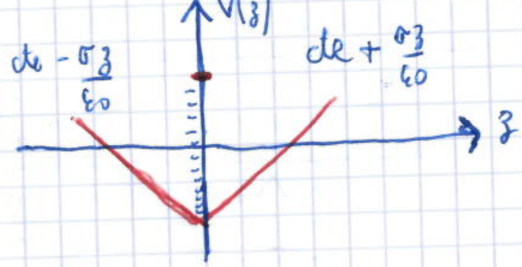
$V^+ = -\int \frac{\sigma}{2\varepsilon_0} dz = -\frac{\sigma z}{2\varepsilon_0} + cte$



$cte = V^+(z=0) = V(z=0)$

$\boxed{\text{One plate with } \sigma < 0}$

$\vec{E} = \frac{\sigma}{2\varepsilon_0}\vec{e_z} \rightarrow \begin{cases} E = \sigma/2\varepsilon_0 < 0 & z > 0 \\ E = -\frac{\sigma}{2\varepsilon_0} > 0 & z < 0 \end{cases}$



→ $\vec{E} = -\vec{grad}\, V$

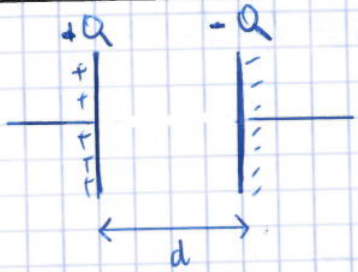
↳ electric field is going throug the decreasing V

→ $\|\vec{E}\|$ = constante

→ V = is linear

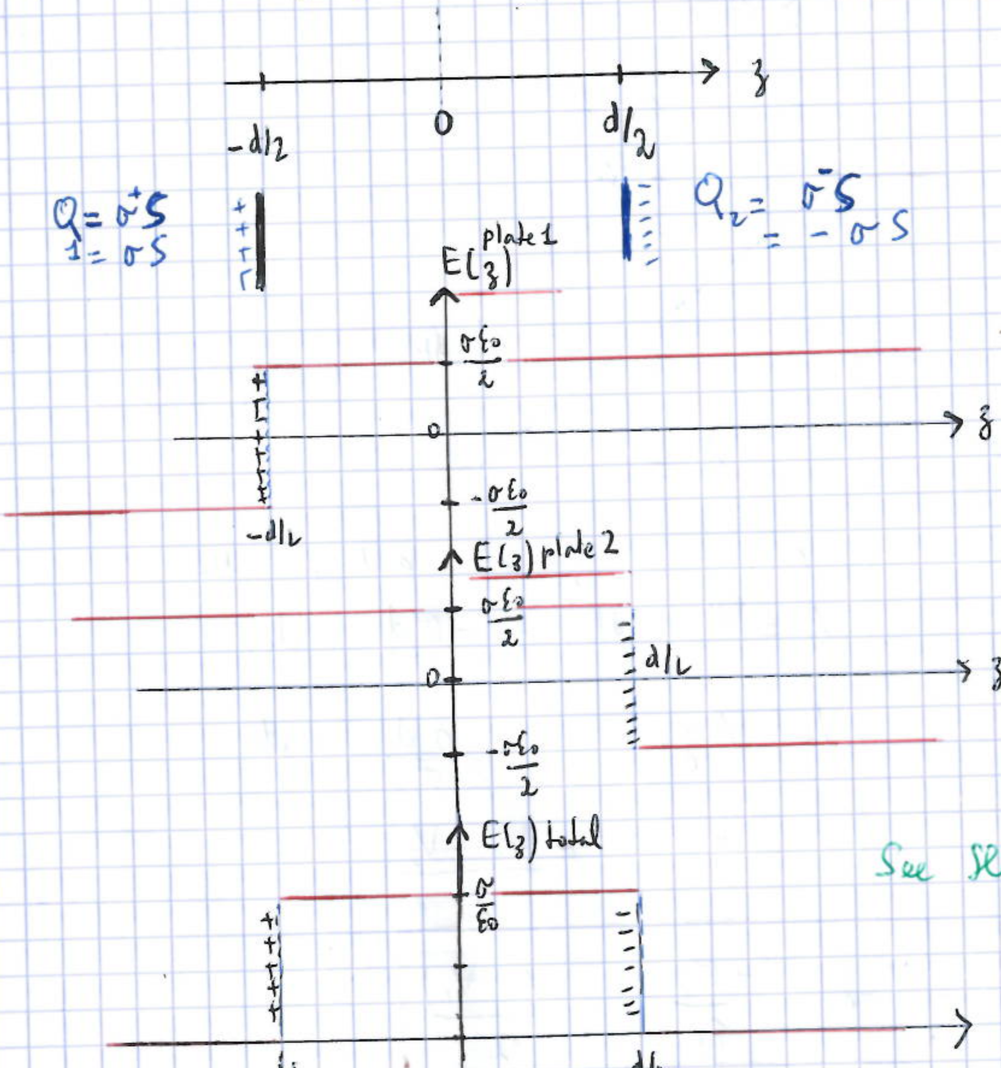
3) Application to the plane capacitor

Plane capacitor



a- Electric field and Electric potential

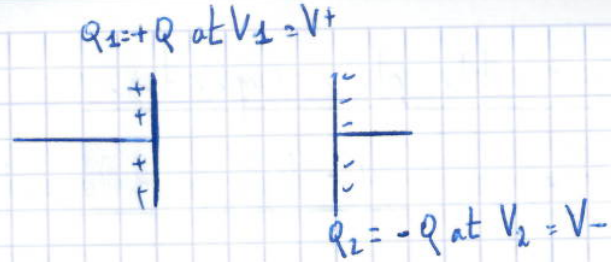
- Two metallic plates separated by insulator
- Plates of surface S are assumed to be infinite plane



See sli

$\vec{E} = \vec{0}$ $\qquad$ $\vec{E} = \frac{\sigma}{\varepsilon_0}\vec{e_z}$ $\qquad$ $\vec{E} = \vec{0}$

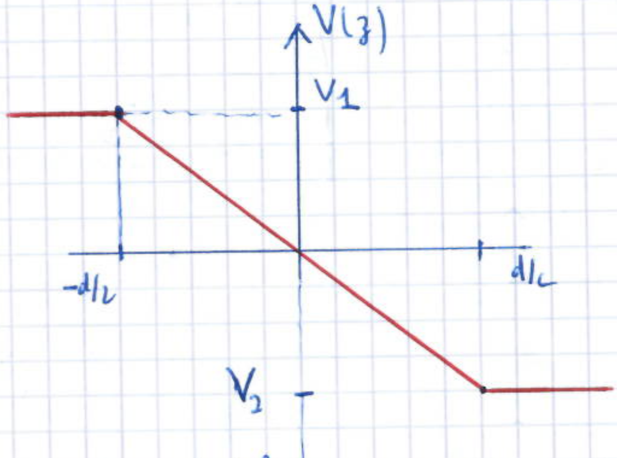
- The electric field exists only inside the plates and it is constant
- see slides $\boxed{\text{II.3.a}}$



Electric potential

$$V = -\int \vec{E} \cdot \vec{dr}$$

$$\vec{E} : \begin{cases} 0 & z < -d/2 \\ \frac{\sigma}{\varepsilon_0}\vec{e_z} & -d/2 < z < d/2 \\ 0 & z > d/2 \end{cases} \longrightarrow V(z) \begin{cases} V(z) = \text{cont } 1 = V_1 \\ V(z) = -\frac{\sigma z}{\varepsilon_0} + cte \\ V(z) = \text{cont } 2 = V_2 \end{cases}$$



Continuity of potential at $z = -d/2$ and $z = +d/2$

$$V_1 = V(z = -d/2) = -\frac{\sigma d}{\varepsilon_0 2} + cte.$$

$$V_2 = V(z = +d/2) = \frac{\sigma d}{\varepsilon_0 2} + cte$$

We find that $cte = \frac{V_1 + V_2}{2}$

$$\boxed{V(z) = -\frac{\sigma z}{\varepsilon_0} + \frac{(V_1 + V_2)}{2}}$$

<u>CCL</u> - In a capacitor (or two metallic plate separated from distance d and charged oppositely)

- The electric field is constant and going from the positive plate to the negative plate (or from the highest potential to the lowest potential)

b- Capacitance and energy.

. How to get $Q = C\Delta V = CU$ and $E = \frac{1}{2}CU^2$

(i) The voltage of the capacitor is $U = V_1 - V_2 = \Delta V$ with

$$V_1 = V\left(z = -\frac{d}{2}\right) = +\frac{\sigma d}{\varepsilon_0 2} + \frac{(V_1+V_2)}{2}$$

$$V_2 = V\left(z = \frac{d}{2}\right) = -\frac{\sigma d}{\varepsilon_0 2} + \frac{(V_1+V_2)}{2}$$

• difference of potential is voltage.

$$\Delta V = V_1 - V_2 = \frac{\sigma d}{2\varepsilon_0} - -\frac{\sigma d}{2\varepsilon_0} = \frac{\sigma d}{\varepsilon_0}.$$

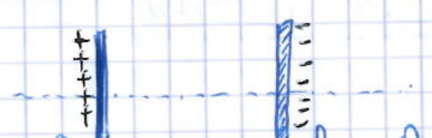
$$\Delta V = \frac{\sigma d}{\varepsilon_0} = \sigma S \frac{d}{\varepsilon_0 S} \underset{Q = \sigma S}{=} \frac{Q d}{\varepsilon_0 S}$$

and $Q = \frac{\varepsilon_0 S}{d} \Delta V = \frac{\varepsilon_0 S}{d} U = CU$

with $\boxed{C = \frac{\varepsilon_0 S}{d}}$ Capacitance ↗ if S ↗ (more charges) ↘ if d ↗ (less influence with other plate

$$[C] = [\varepsilon_0] \frac{[S]}{[d]} = \quad \frac{m^2}{m} = F \text{ (Farad)}.$$

(ii) Energy of the Capacitor.



$\Leftrightarrow$

Q_1 + - - - - - - + Q_2

Charge $Q_1 = Q$ at potential V_1

Charge $Q_2 = -Q$ at potential V_2

$$E_p = \frac{1}{2}(Q_1 V_2 + Q_2 V_1)$$

$$V_1 = \frac{\sigma d}{2\varepsilon_0} + \frac{(V_2+V_1)}{2} \quad ; \quad V_2 = -\frac{\sigma d}{2\varepsilon_0} + \frac{(V_2+V_1)}{2} \rightarrow$$

with $Q_1 = -Q_2 = Q$

$$E_p = \frac{1}{2}\left[Q\left(\frac{\sigma d}{2\varepsilon_0} + \left(\frac{V_1+V_2}{2}\right)\right) + (-Q)\left(-\frac{\sigma d}{2\varepsilon_0} + \frac{(V_1+V_2)}{2}\right)\right]$$

$$= \frac{1}{2}\left[Q\frac{\sigma d}{2\varepsilon_0} + Q\frac{\sigma d}{2\varepsilon_0}\right] = \frac{1}{2} Q \frac{\sigma d}{\varepsilon_0}$$

$$E_p = \frac{1}{2} \cdot \underbrace{Q}_{} \cdot (\sigma S) \cdot \frac{d}{\varepsilon_0 S}$$

$$= \frac{1}{2} \times C.U \times \frac{Q}{C} = \frac{1}{2} C.U.U$$

$$\boxed{E_p = \frac{1}{2} C U^2}$$

Numerical value.

- $Q = C\Delta V = 0{,}6\ F \times 4\ V = 6 \times 10^{-13} \times 4 = 2{,}4 \times 10^{-12}\ C$

  $N_{e^-} = \frac{-Q}{-e} = \frac{-2{,}4\ 10^{-12}}{-1{,}6 \times 10^{-19}} = 1{,}5\ 10^{7}$ electrons

- We have $S = 10^4 d^2$

$$C = \frac{\varepsilon_0 S}{d} = \varepsilon_0 \frac{d^2}{d} = \varepsilon_0 d$$

$$d = \frac{C}{\varepsilon_0} = \frac{0{,}6\ 10^{-13}}{8{,}85\ 10^{-12}} = 6{,}77\ 10^{-3}\ m \approx 6{,}8\ mm$$

C – Motion in uniform electric field

- see slides II.3.c)
- Newton law in a constant electric field
  see exercise 7 in Tutorial of "Dynamics".