FINAL EXAM

I. Coaxial cable

We consider a coaxial electric cable made of two metallic infinite cylinders of radius R_A and R_B sharing the same axis Δ ($R_A < R_B$). We attached an orthonormal cylindric direct basis $(\vec{e_r}, \vec{e_\theta}, \vec{e_z})$ whose vector $\vec{e_z}$ is along axis Δ. The region of the space going from $r = 0$ to $r = R_A$ is called region I and the **region going from $r = R_A$ to $r = R_B$ is called region II**. Both of them are empty of electric charges and we assume that the electric charges are only located on the surface of both cylinders whose thicknesses are supposed to be negligible. The surface density of charge (in C/m^2) is written σ_A for the small cylinder located at distance R_A. This charge density can be moved by applying a voltage difference along the Oz axis of the cable leading to the creation of an electric current $\vec{j}$ on the surface of the cylinders. We will study the electric and magnetic fields created by the sources distribution

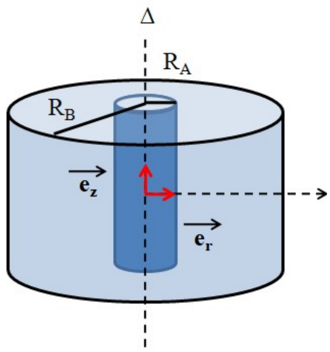

Figure 1.

A. Study of the Electric field in region II

1) Give the statement of the Gauss theorem.
2) Justify the direction of the electric field in the region II.
3) Use the Gauss theorem to determine the electric field amplitude $E(r)$ in the region II. Specify clearly the Gauss volume (or surface) used and the details of the flux calculations.
4) Give the relation between the electric field $E(r)$ and the electric potential $V(r)$ and deduce the expression of the electric potential in region II. We will assume that $V(R_A) = V_A$.
5) The coaxial cable can behave as a cylindrical capacitor. For a coaxial cable of total length l, it is possible to show that the capacitance is written as

$$C = \frac{2\pi l \epsilon_0}{ln\left(\frac{R_B}{R_A}\right)}$$

Determine the capacitance per length unit C_X and give its value for $R_A = 0.15$ cm, $R_B = 0.5$ cm and $\epsilon_0 = 8.85 \times 10^{-12}$ F/m.

B. Study of the Magnetic field in region II

We assume now that the electric charges can move on the surface of the cylinders leading to surface current densities $\vec{J_A}$ and $\vec{J_B}$ (in A/m). The vectors propagate along Oz only at the circumference of the two cylinders.
6) Find the relation between the norm of $|\vec{J_A}|$ and the current intensity i_A on the small cylinder.
7) Show using the Ampere theorem that the magnetic field in the region II can be written as

$$\vec{B}(r) = \frac{\mu_0 i_A}{2\pi\, r}\,\vec{e_\theta}.$$

8) The coaxial cable can behave also as a coil. Its inductance L can be calculated with the relation $\Phi = Li$ where Φ is the magnetic flux $\Phi = \iint \vec{\mathrm{B}}(\mathrm{r}) \cdot \vec{\mathrm{dS}}$ where the surface element is $\vec{\mathrm{dS}} = \mathrm{drdz}\vec{e_\theta}$. Figure out with a picture the elementary surface dS between the two cylinders.
9) The integration of parameters r and z will be taken respectively between R_A and R_B for r and between 0 and l for z. Calculate the flux and give the expression of the inductance L.
10) Deduce the inductance per length unit L_X and calculate its value. We give $\mu_0 = 4\pi \times 10^{-7}$ H/m.
11) It has been seen that the wave velocity for voltage and current was given by $v = 1/\sqrt{C_X L_X}$. Determine the expression of v.

II. Electromagnetic Waves propagation

Formulary: Mathematical relations

$\overrightarrow{grad}\, f = \vec{\nabla} f$

$\Delta f = \vec{\nabla} \cdot \vec{\nabla} f$

$div\, \vec{A} = \vec{\nabla} \cdot \vec{A}$

$\overrightarrow{rot}\, \vec{A} = \vec{\nabla} \times \vec{A}$

$\overrightarrow{rot}\left(\overrightarrow{rot}\, \vec{A}\right) = \overrightarrow{grad}\left(div \vec{A}\right) - \Delta \vec{A}$

Maxwell equations

$div\, \vec{E} = \frac{\rho}{\varepsilon_0}$ Maxwell-Gauss

$div\, \vec{B} = 0$ Maxwell-Flux

$\overrightarrow{rot}\, \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Maxwell-Faraday

$\overrightarrow{rot}\, \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ Maxwell-Ampere

A. Wave propagation in the vacuum

12) How are transformed the Maxwell equations in the vacuum?
13) Show how to obtain the D'Alembert Wave equation in the vacuum for the electric field.
14) Considering a plane wave solution $\vec{E} = \overrightarrow{E_0} e^{i(kr-\omega t)}$, deduce the dispersion relation and give the expression of the phase velocity $v_\varphi = \omega/k$.

B. Wave propagation in dielectric material

The vacuum is replaced by a dielectric material. The charge density ρ (and the current density $\vec{j}$) are splitted into ρ_F and ρ_B (and into $\overrightarrow{j_F}$ and $\vec{j}_B$) that denote respectively the free and the bound charges. We remember that $\rho_F = 0, \overrightarrow{j_F} = \vec{0},\ \rho_B = -div\, \vec{P}$ and $\vec{j}_B = \frac{\partial \vec{P}}{\partial t}$ where $\vec{P} = \chi \epsilon_0 \vec{E}$ is the volumic polarization and χ the dielectric susceptibility.
15) We define $\epsilon_r = 1 + \chi$ as the dielectric permittivity. Transform the Maxwell equations and obtain the new wave equation and show that the wave velocity is written as $v = 1/\sqrt{\epsilon_r \epsilon_0 \mu_0}$.
16) Deduce the expression of the phase velocity $v_\varphi = \omega/k$ and give the expression of the optical index of the medium.

C. Wave propagation in metallic material

17) Write the local Ohm law involving the electric conductivity γ.
It has been shown in lecture that inside a metallic material, the equation governing the electric field is written as $\Delta \vec{E} = \mu_0 \gamma \frac{\partial \vec{E}}{\partial t}$. Consequently, when obtaining the dispersion relation we can show that the wave is attenuated over a distance of 5δ where the expression of δ is given by $\delta = 2/\sqrt{\mu_0 \omega \gamma}$
18) Calculate the depth penetration δ for copper using a 100 kHz signal. We give $\gamma_{Cu} = 6 \times 10^7$ S/m.
19) The cylinders of coaxial cable are made of copper. Regarding the values of the radii of the cylinders given in **5)**, was the assumption of a surface charge location a correct hypothesis?

III. Laplace forces and induction

We consider the device in Fig 2 set in a magnetic field along $-\overrightarrow{e_z}$, made of a mobile stick of length L which permits to close an electric circuit where flows a current i (i>0).
20) Calculate the Laplace force acting on the stick (amplitude, direction).
21) We assume the stick moves with a velocity of amplitude V. Give the general integral expression of the Lorentz electromotive force e appearing in the circuit as a function of the vectors $\vec{V}$ and $\vec{B}$. Calculate its scalar expression as a function of V, B and L.
22) Is the value of e positive or negative? Justify.

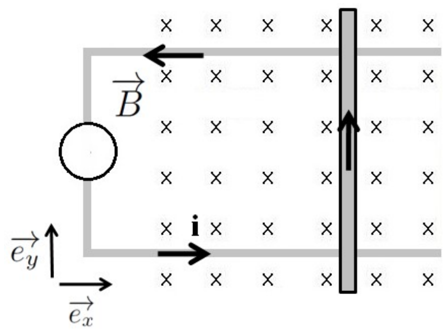

Figure 2.