EQUATIONS DE MAXWELL

1. Statement

$$\begin{cases} \mathrm{div}\, \vec{E} = \dfrac{\rho}{\varepsilon_0} & \mathrm{div}\, \vec{B} = 0 \\ \vec{\mathrm{rot}}\, \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \vec{\mathrm{rot}}\, \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \end{cases}$$

Maxwell-Gauss (Poisson) — Maxwell-flux (or Gauss)

or

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} & \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \end{cases}$$

Maxwell-Faraday — Maxwell-Ampere.

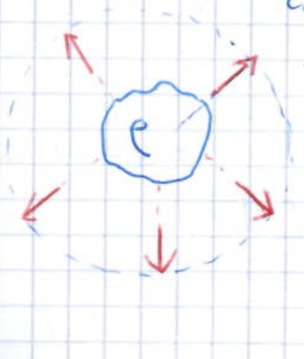
2. Limit cases

a) Electrostatics

- only electric charge; no current: $\vec{j} = \vec{0}$
- no time-dependance; steady-state regime $\partial_t = 0$.
- no magnetic field

Maxwell eq° $\Longrightarrow$ $\boxed{\begin{cases} \mathrm{div}\, \vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \vec{\mathrm{rot}}\, \vec{E} = 0 \end{cases}}$

$\boxed{i)}$ $\mathrm{div}\, \vec{E} = \dfrac{\rho}{\varepsilon_0} \iff \iiint_V \mathrm{div}\, \vec{E}\, dV = \iiint_V \dfrac{\rho\, dV}{\varepsilon_0} = \dfrac{1}{\varepsilon_0} \iiint \rho\, dV$



$$\oint\!\!\!\oint_S \vec{E} \cdot \vec{dS} = \frac{Q_{int}}{\varepsilon_0}$$ Gauss Theorem.

ii) Mathematical formula

$$\forall f \quad \boxed{\begin{array}{l} \overrightarrow{\text{rot}}\, \overrightarrow{\text{grad}}\, f = \vec{0} \\ \vec{\nabla} \times (\vec{\nabla} f) = \vec{0} \end{array}}$$

- $\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{e_x} + \frac{\partial f}{\partial y}\vec{e_y} + \frac{\partial f}{\partial z}\vec{e_z} = \overrightarrow{\text{grad}}\, f$

- $\overrightarrow{\text{rot}}\, \overrightarrow{\text{grad}}\, f = \vec{\nabla} \times (\vec{\nabla} f)$

$$= \begin{vmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{vmatrix} \times \begin{vmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} = 0 \\ \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} = 0 \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} = 0 \end{vmatrix}$$

If $\overrightarrow{\text{rot}}\, \vec{E} = \vec{0}$ it implies that $\vec{E}$ is related to a gradient...

$\exists\, V$ such as $\boxed{\vec{E} = -\overrightarrow{\text{grad}}\, V}$

V is the electric potential.

iii)

$$\begin{cases} \text{div}\, \vec{E} = \frac{\rho}{\varepsilon_0} \\ \vec{E} = -\overrightarrow{\text{grad}}\, V \end{cases} \Rightarrow \quad \text{div}(-\overrightarrow{\text{grad}}\, V) = \frac{\rho}{\varepsilon_0}$$

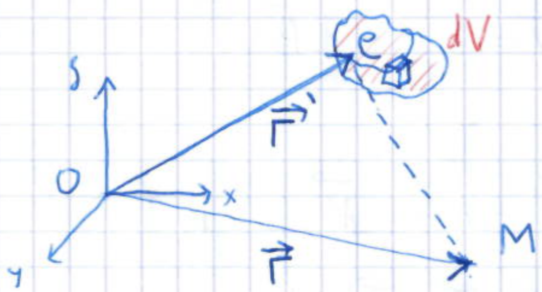
$$-\Delta V = \frac{\rho}{\varepsilon_0}$$

Δ : Laplacian : $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$\Delta = \vec{\nabla} \cdot \vec{\nabla}$

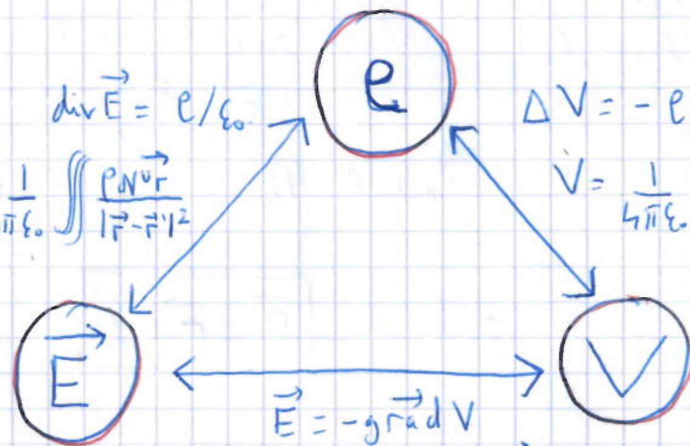
$$\boxed{\Delta V = -\frac{\rho}{\varepsilon_0}}$$

Poisson equation = equation de Laplace $\Delta V = 0$ but with a source term : $\left(-\frac{\rho}{\varepsilon_0}\right)$



$$V(M) = \frac{1}{4\pi\varepsilon_0} \iiint_{vol} \frac{\rho(\vec{r}\,')\, dV}{|\vec{r} - \vec{r}\,'|}$$ solution of Poisson equation

When ρ is not on the origin O.



$\operatorname{div} \vec{E} = \rho/\varepsilon_0$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho\, dV\, \vec{u}_r}{|\vec{r} - \vec{r}\,'|^2}$$

$\Delta V = -\rho/\varepsilon_0$

$$V = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho\, dV}{|r - r'|}$$

$\vec{E} = -\overrightarrow{\operatorname{grad}}\, V$

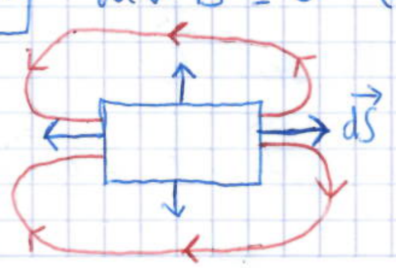
$[V] = -\int \vec{E} \cdot \vec{dr}$

b) Magnetostatics

- only current
- steady-state regime $\partial_t = 0$

Maxwell equations $\Rightarrow$ $\boxed{\begin{cases} \operatorname{div} \vec{B} = 0 \\ \overrightarrow{\operatorname{rot}}\, \vec{B} = \mu_0 \vec{j} \end{cases}}$

$\boxed{i)}$ $\operatorname{div} \vec{B} = 0 \iff \iiint \operatorname{div} \vec{B} = 0$



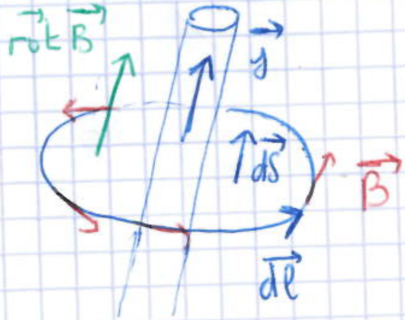
$\oiint \vec{B} \cdot \vec{dS} = 0$

- Flux conservation
- No magnetic charge

ii) $\vec{rot}\,\vec{B} = \mu_0 \vec{j} \iff \iint_S \vec{rot}\,\vec{B}\cdot\vec{dS} = \iint \mu_0 \vec{j}\cdot\vec{dS}$

$\iff$

$$\iint \vec{rot}\,\vec{B}\cdot\vec{dS} = \mu_0 \iint \vec{j}\cdot\vec{dS}$$



$$\oint \vec{B}\cdot\vec{dl} = \mu_0 i$$

Ampere theorem

iii) - Vector potential
- Mathematic formula

$$\mathrm{div}\,\vec{rot}\,\vec{A} = 0$$
$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0 \quad \forall \vec{A}$$

See "Magnetostatics" Part A. 4)

If $\mathrm{div}\,\vec{B} = 0$

$\iff \exists\, \vec{A}$ such as $\boxed{\vec{B} = \vec{rot}\,\vec{A}}$ $\quad \vec{A}$ is vector potential

iv) $\begin{cases} \vec{rot}\,\vec{B} = \mu_0 \vec{j} \\ \mathrm{div}\,\vec{B} = 0 \end{cases} \iff \begin{cases} \vec{rot}\,\vec{B} = \mu_0 \vec{j} \\ \vec{B} = \vec{rot}\,\vec{A} \end{cases}$

$$\vec{rot}(\vec{rot}\,\vec{A}) = \mu_0 \vec{j}$$

see $\rightarrow$ $\vec{grad}\,\mathrm{div}\,\vec{A} - \Delta\vec{A} = \mu_0 \vec{j}$

Formulary.

We can have a Poisson equation <u>if</u> $\vec{grad}\,\mathrm{div}\,\vec{A} = 0$.

$\Delta\vec{A} = -\mu_0\vec{j}$ $\quad$ or $\quad \mathrm{div}\,\vec{A} = 0$.

<u>Property of vector potential $\vec{A}$:</u>

Many $\vec{A}$ can be solution for the same $\vec{B}$:

$\vec{B} = \vec{rot}\,\vec{A}$ if we write $\vec{A} = \vec{A}' + \vec{grad}\,f$

$$\vec{rot}\,\vec{A} = \vec{rot}(\vec{A}' + \vec{grad} f)$$
$$= \vec{rot}\,\vec{A}' + \underbrace{\vec{rot}(\vec{grad} f)}_{0}$$
$$= \vec{B}$$

We can chose a $\vec{A}$ such as $div\,\vec{A} = 0$ so.

$$\vec{grad}\,\underbrace{div\,\vec{A}}_{0} - \Delta\vec{A} = \mu_0 \vec{j}$$

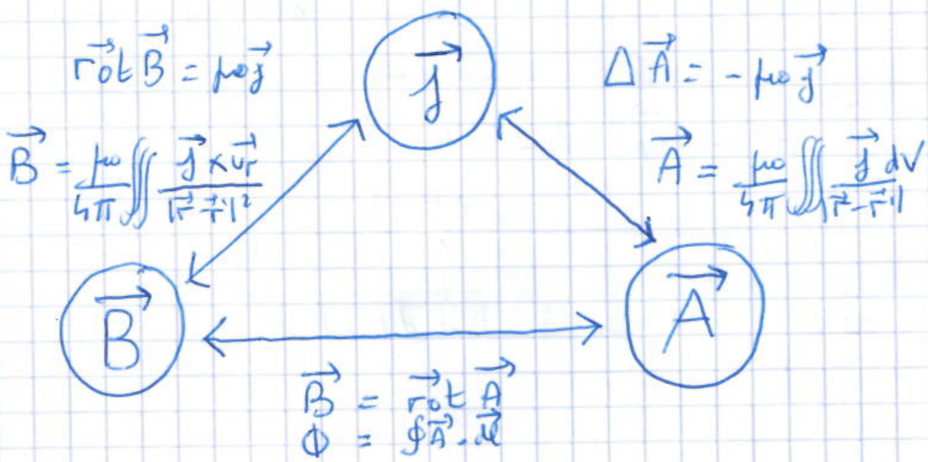
$$\boxed{\Delta\vec{A} = -\mu_0 \vec{j}}$$ Poisson equation.

. This choice $div\,\vec{A} = 0$ is called the Coulomb gauge.

. Similar to: $\vec{E} = -\vec{grad}\,V = -\vec{grad}\,V'$

if $V' = V + const.$

$$\vec{A} = \frac{\mu_0}{4\pi}\iiint \frac{\vec{j}(r')\,dV}{|\vec{r}-\vec{r}'|}$$ solution of Poisson equation



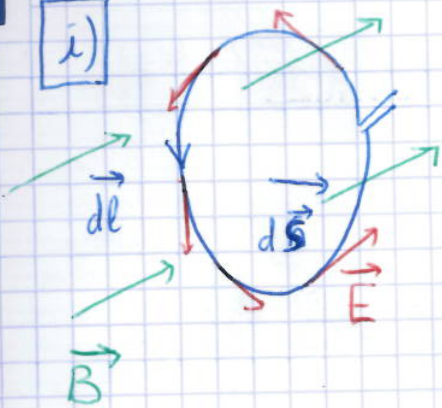
C. Time-dependent regime - Electromagnetic induction

Maxwell - Faraday equation

$$\vec{rot}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$\boxed{i)}$



$$\iint_S \vec{rot}\,\vec{E}\,.\,\vec{dS} = -\iint_S \frac{\partial \vec{B}}{\partial t}\,.\,\vec{dS}$$

$$\oint_{\mathcal{C}} \vec{E}\,.\,\vec{dl} = -\frac{d}{dt}\iint_S \vec{B}\,.\,\vec{dS}$$

$$\boxed{e = -\frac{d\phi}{dt}}$$ Faraday law

$\boxed{ii)}$ $\vec{rot}\,\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ $\qquad \vec{B} = \vec{rot}\,\vec{A}$

$$= -\frac{\partial}{\partial t}\vec{rot}\,\vec{A} = \vec{rot}\left(-\frac{\partial \vec{A}}{\partial t}\right)$$

$$\Leftrightarrow \quad \vec{rot}\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = \vec{0}$$

$\exists$ V such as $\quad \vec{E} + \dfrac{\partial \vec{A}}{\partial t} = -\vec{grad}\,V$

$$\boxed{\vec{E} = -\vec{grad}\,V - \frac{\partial \vec{A}}{\partial t}}$$

complete expression of the electric field

static

time-dependent

3) Propagation of E-M waves in vaccum

$$\begin{cases} \operatorname{div} \vec{E} = \dfrac{\rho}{\varepsilon_0} & \operatorname{div} \vec{B} \\ \vec{\operatorname{rot}}\, \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \vec{\operatorname{rot}}\, \vec{B} = \mu_0 \vec{j} + \underbrace{\mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}} \end{cases}$$

genious idea of Maxwell to "invent" that term For symmetry reasons.

In the vaccum:

$\rho = 0$ and $\vec{j} = 0$.

$$\begin{cases} \operatorname{div} \vec{E} = 0 \quad (3) & \operatorname{div} \vec{B} = 0 \quad (4) \\ \vec{\operatorname{rot}}\, \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \quad (1) & \vec{\operatorname{rot}}\, \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \quad (2) \end{cases}$$

- $\vec{\operatorname{rot}}(1)$: $\vec{\operatorname{rot}}\, \vec{\operatorname{rot}}\, \vec{E} = \vec{\operatorname{grad}}\, \overbrace{\operatorname{div} \vec{E}}^{0} - \Delta \vec{E} = \vec{\operatorname{rot}}\left(-\dfrac{\partial \vec{B}}{\partial t}\right)$
- $\dfrac{\partial}{\partial t}(2)$: $\dfrac{\partial}{\partial t} \vec{\operatorname{rot}}\, \vec{B} = \vec{\operatorname{rot}}\, \dfrac{\partial \vec{B}}{\partial t} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{E}}{\partial t^2}$

$$\Longleftrightarrow \quad \Delta \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\boxed{\Delta \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \qquad \boxed{c^2 = \frac{1}{\mu_0 \varepsilon_0}}$$

Wave equation for the electric field.

if $\vec{\operatorname{rot}}(2)$ and $\dfrac{\partial}{\partial t}(1)$ we obtain $\Delta \vec{B} - \dfrac{1}{c^2} \dfrac{\partial^2 \vec{B}}{\partial t^2} = 0$.

$$\boxed{c^2 \mu_0 \varepsilon_0 = 1}$$

optics electricity magnetism

4) Some consequences

a) Conservation of electric charge

$$\begin{cases} \operatorname{div} \vec{E} = \dfrac{\rho}{\varepsilon_0} & \operatorname{div} \vec{B} = 0 \\ \vec{\operatorname{rot}}\, \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \vec{\operatorname{rot}}\, \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \end{cases}$$

→ j current density : Max-Amp

→ ρ charge density Max-Gauss

Eliminate $\operatorname{div} \vec{E}$:

We apply $\underbrace{\operatorname{div}(\vec{\operatorname{rot}}\, \vec{B})}_{0} = \operatorname{div}\left(\mu_0 \vec{j} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \right)$

$$\operatorname{div}(\vec{\operatorname{rot}}\, \vec{B}) = 0 = \mu_0 \operatorname{div} \vec{j} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \operatorname{div} \vec{E} = 0$$

$\forall \vec{B}$

$$\operatorname{div} \vec{j} = -\varepsilon_0 \frac{\partial}{\partial t} \operatorname{div} \vec{E} = -\varepsilon_0 \frac{\partial}{\partial t}\left[\frac{\rho}{\varepsilon_0}\right] = -\frac{\partial}{\partial t}\rho$$

$$\Leftrightarrow \boxed{\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0}$$

Continuity equation

b) Conservation of energy

$$\vec{\Pi} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \text{Poynting vector}$$

$$\mathrm{div}\,\vec{\Pi} = \mathrm{div}\left(\frac{\vec{E} \times \vec{B}}{\mu_0}\right)$$

$$= \frac{\vec{B} \cdot \vec{\mathrm{rot}}\,\vec{E}}{\mu_0} - \frac{\vec{E} \cdot \vec{\mathrm{rot}}\,\vec{B}}{\mu_0}$$

$$= -\frac{1}{\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{j} - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

$$= -\frac{\partial}{\partial t} \underbrace{\left[\frac{1}{2\mu_0} B^2 + \frac{\varepsilon_0 E^2}{2} \right]}_{e \,=\, \text{energy density}} - \vec{E} \cdot \vec{j}$$

$$\boxed{\mathrm{div}\,\vec{\Pi} + \frac{\partial e}{\partial t} = - \vec{E} \cdot \vec{j}}$$

$\vec{E} \cdot \vec{j}$:
- = 0 in the vaccum
- Joule effect in material dissipation of energy.

5) Scalar and Vector potential

FYI: fundamental study of e-m is done with V and $\vec{A}$

i Electromagnetic gauge

$$\begin{cases} \vec{B} = \vec{rot}\,\vec{A} \\ \vec{E} = -\vec{grad}\,V - \dfrac{\partial \vec{A}}{\partial t} \end{cases}$$

- We have seen $\vec{A} = \vec{A}' + \vec{grad}\,f$ defined with any function f:

$$\vec{rot}\,\vec{A} = \vec{rot}\,\vec{A}' \quad \text{because} \quad \vec{rot}\,\vec{grad}\,f = 0 \quad \forall f.$$

- $$\vec{E} = -\vec{grad}\,V - \frac{\partial}{\partial t}\vec{A}' - \frac{\partial}{\partial t}\vec{grad}\,f$$

$$= -\vec{grad}\left[V + \frac{\partial f}{\partial t}\right] - \frac{\partial \vec{A}'}{\partial t}$$

$$= -\vec{grad}\,V' - \frac{\partial \vec{A}'}{\partial t}$$

For any $(\vec{A}, V)$ we can have a function f leading to $(\vec{A}', V')$ defined as:

$$\begin{cases} \vec{A} = \vec{A}' + \vec{grad}\,f \\ V = V' - \dfrac{\partial f}{\partial t} \end{cases} \quad \text{leading to the same } (\vec{E}, \vec{B})$$

- The choice of $(\vec{A}, V)$ is a <u>choice of gauge.</u>

- In magnetostatics : $\text{div}\,\vec{A} = 0$. Coulomb gauge.

ii Wave equations for $(\vec{A}, V)$

$$\begin{cases} \vec{B} = \vec{rot}\,\vec{A} & \longrightarrow \text{apply } \vec{rot} \\ \vec{E} = -\vec{grad}\,V - \dfrac{\partial \vec{A}}{\partial t} & \longrightarrow \text{plug into } \text{div}\,\vec{E} = \dfrac{\rho}{\varepsilon_0} \end{cases}$$

$*$ $\quad \vec{rot}\,\vec{B} = \vec{rot}(\vec{rot}\,\vec{A}) = \vec{grad}\,\mathrm{div}\,\vec{A} - \Delta\vec{A}$

$$\mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \vec{grad}\,\mathrm{div}\,\vec{A} - \Delta\vec{A}$$

$$\mu_0\vec{J} + \mu_0\varepsilon_0\left[\frac{\partial}{\partial t}(-\vec{grad}V) - \frac{\partial^2\vec{A}}{\partial t^2}\right] = \vec{grad}\,\mathrm{div}\,\vec{A} - \Delta\vec{A}$$

$$\mu_0\vec{J} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \underbrace{\left[\vec{grad}\left[-\mu_0\varepsilon_0\frac{\partial V}{\partial t} - \mathrm{div}\,\vec{A}\right]\right]}_{\text{BORING TERM}} + \Delta\vec{A}$$

$**$ $\quad \mathrm{div}\left[-\vec{grad}\,V - \frac{\partial\vec{A}}{\partial t}\right] = \frac{\rho}{\varepsilon_0}.$

$$-\Delta V - \underbrace{\mathrm{div}\left[\frac{\partial\vec{A}}{\partial t}\right]}_{\text{BORING TERM}} = \frac{\rho}{\varepsilon_0}.$$

We apply Lorentz gauge: $\mathrm{div}\,\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0.$

$$c^2 = \frac{1}{\mu_0\varepsilon_0}.$$

In $*$: eq° becomes: $\quad \boxed{\Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}}$

BORING TERM $= 0$

$$\frac{\partial}{\partial t}\left(\mathrm{div}\,\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = 0$$

$\Leftrightarrow \quad \mathrm{div}\frac{\partial\vec{A}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$ and $**$ becoms:

$$\boxed{\Delta V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}}$$

$(\vec{A}, V)$ are involved in D'Alembert equation <u>with sources</u>

6) Sources of charges and current densities in Maxwell equations

ME: $\times$ $\operatorname{div} \vec{E} = \dfrac{\boxed{\rho}}{\varepsilon_0}$ $\qquad$ $\times$ $\operatorname{div} \vec{B} = 0$

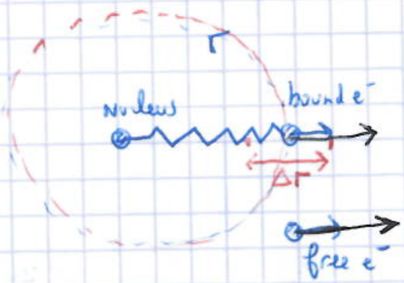
$\times$ $\overrightarrow{\operatorname{rot}} \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ $\qquad$ $\times$ $\overrightarrow{\operatorname{rot}} \vec{B} = \mu_0 \boxed{\vec{j}} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

Possibilities for sources

$\rho = \rho_{free} + \rho_{bound}$

$\vec{j} = \vec{j}_{free} + \vec{j}_{bound} + \vec{j}_{mag}$

Picture



insulator / dielectric } no free ρ_{bound}

conductor : ρ_{free}

• polarization

$\vec{p} = -e\,\Delta\vec{r}$: $\vec{P} = \dfrac{1}{V} \sum \vec{p}_i = \varepsilon_0 \chi \vec{E}$

. $\dfrac{\partial \vec{p}}{\partial t} = -e \dfrac{d\Delta\vec{r}}{dt} = -e\vec{v} = \boxed{\vec{j}_{bound} = \vec{j}_{pol}}$

. $\operatorname{div} \vec{j} + \dfrac{\partial \rho}{\partial t} = 0 \Rightarrow \boxed{\rho_{bound} = -\operatorname{div} \vec{P}}$

• conduction e-

$U = Ri = RSj$

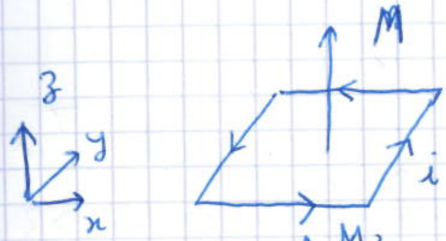
$\int \vec{E} \cdot \vec{d\ell} = RS\vec{j} = \vec{E}\ell$ $\qquad$ $j = \underbrace{\dfrac{\ell}{RS}}_{\sigma} E$

$$\boxed{\vec{j} = \sigma \vec{E}}$$

$\sigma = \dfrac{\ell}{RS} = \dfrac{\rho\ell}{S}$

6) Sources of charges and current densities in Maxwell equations

Matter divided into cubes

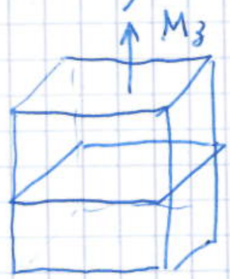


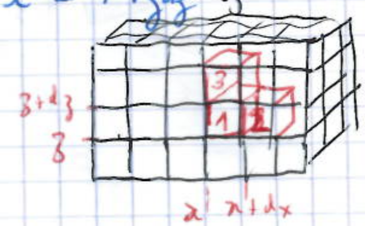
$\vec{m} = i S \vec{n}$

$\vec{m} = i\, dx\, dy\, \vec{e_z} = M_z\, dx\, dy\, dz\, \vec{e_z}$

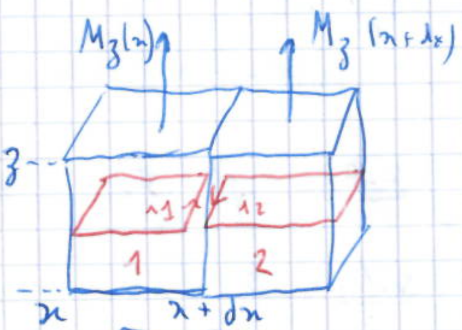
volumic magnetization

$i = M_z dz\, \vec{e_z}$



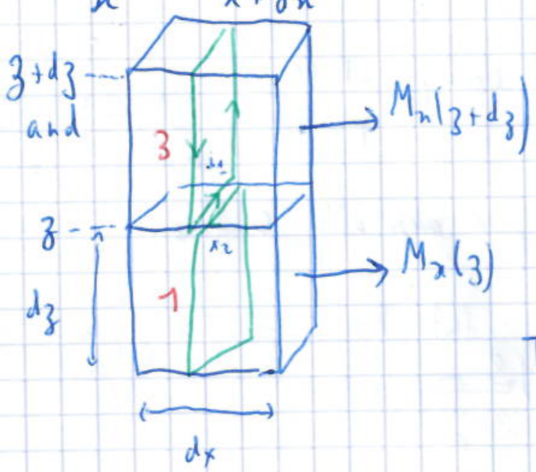


We look i_y!



Current along y

$$i_y = i_1 - i_2 = \left(M_z(x) - M_z(x+dx)\right) dz = -\frac{\partial M_z}{\partial x}\, dx\, dz$$



Current along y

$$i_y = i_1 - i_2 = M_x(z+dz) - M_x(z)\, dx = \frac{\partial M_x}{\partial z}\, dz\, dx$$

Total current:

$$i_y = \left(\frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}\right) dx\, dy$$

$$j_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x} = (\vec{\nabla} \times \vec{M}) \cdot \vec{e_y}$$

Total magnetization current will be

$$\boxed{\vec{j}_{mag} = \vec{j}_x + \vec{j}_y + \vec{j}_z = \vec{\nabla} \times \vec{M}}$$