Weyl Character Formula

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1 Algebraic Groups

Lie groups are a classical object of study in representation theory and at their core, they are "group objects" in the category of smooth manifolds. But what makes manifolds the "correct" choice of geometric objects? It turns out that we can select many different types of geometric objects that can be endowed with a compatible group structure. Algebraic groups are group objects in the category of affine algebraic schemes, and the representations of algebraic groups can be investigated through many powerful algebraic-geometric tools. For example, the Borel-Weil-Bott Theorem relates sheaf cohomology on a certain variety with certain representations of algebraic groups, and it can be used to prove the Weyl Character Formula, which provides a simple and beautiful formula for the characters of representations. We will be working over $\mathbb{C}$ unless otherwise specified.

1.1 Algebraic Groups

We first provide a general definition which can be specialised in many interesting cases.

Definition 1 (group object)**.** *Let $\mathcal{C}$ be a category with finite products. Denote $1 \in \mathcal{C}$ as the terminal object. Then a* ***group object*** *in $\mathcal{C}$ is the data of*

1. *An object $G \in \mathcal{C}$*
2. *An arrow $m : G \times G \to G$ ("multiplication")*
3. *An arrow $e : 1 \to G$ ("identity")*
4. *An arrow $i : G \to G$ ("inverse")*

such that the following diagrammes commute:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mathrm{id} \times m} & G \times G \\ \downarrow{\scriptstyle m \times \mathrm{id}} & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{\quad m \quad} & G \end{array}$$

$$\begin{tikzcd}
1 \times G \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "m"] & G \times 1 \arrow[l, "\mathrm{id} \times e"'] \arrow[dl, "\cong"] \\
 & G &
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "(\mathrm{id}, i)"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "(i, \mathrm{id})"'] \arrow[d] \\
1 \arrow[r, "e"] & G & 1 \arrow[l, "e"']
\end{tikzcd}$$

In the case $\mathcal{C} = \mathsf{Set}$, we recover the classical definition of a group, where $m(g, g') = gg'$, $e(1) = e$, $i(g) = g^{-1}$. The three diagrammes give us the axioms of groups: the first axiom is associativity of multiplication, the second is that $eg = ge = g$, and the third is that $gg^{-1} = g^{-1}g = e$.

In the classical case, a Lie group is a group object in the category of smooth manifolds. We will be working in the case $\mathcal{C}$ is the category of affine algebraic schemes over $\mathbb{C}$. A *algebraic group* is a group object in the category of affine algebraic schemes over $\mathbb{C}$ (omitting the word "algebraic" in the description of our category gives us *group schemes* which are more general, although we will not consider them).

Example 1. *All of the classical Lie groups can be considered algebraic groups. As an explicit example, we can define* $\mathrm{SL}_n := \mathrm{Spm}\,\mathbb{C}[T_{11}, T_{12}, \ldots, T_{nn}]/(\det(T_{ij}) - 1)$ *([Mil15] 18).*

(we use Spm to denote what is sometimes called MaxSpec, the spectrum of maximal ideals.) From now on, "group" means algebraic group (which we assume to be connected unless otherwise specified).

1.2 Semisimple and Reductive Groups

This section is based on previous work in [Ste23], which was given as a seminar talk following [Knu+98] Chapter 24 supported by [Mil15].

Definition 2 (solvable group)**.** *A group G is* ***solvable*** *if $G(\mathbb{C})$ is solvable.*

Definition 3 (semisimple group)**.** *A group G is* ***semisimple*** *if G has no nontrivial solvable normal subgroups.*

It is not immediately clear that a group G that we have defined as semisimple is actually a sum of simple parts. We will see shortly that this is in fact the case.

Definition 4 (split semisimple group)**.** *A semisimple group G called* ***split*** *if it contains a split maximal torus $T \subset G$.*

Recall that U_n is the group of strictly upper-triangular matrices: upper triangular matrices with diagonal entries equal to 1. We follow [CW20] primarily and [Knu+98] secondarily in our discussion of root systems.

Definition 5 (unipotent group)**.** *A group G is called **unipotent** if it is isomorphic to a subgroup of U_n.*

Definition 6 (reductive group)**.** *A group G is **reductive** if G is connected and G has no nontrivial unipotent connected normal subgroups.*

Every unipotent group is solvable so every semisimple group is reductive (since we work over $\mathbb{C}$ which is algebraically closed). The reverse is not true: $\mathrm{GL}_n, n > 1$ is reductive but not semisimple. Fix a reductive group G and a maximal torus $T \subset G$. We have an action of G on itself by conjugation, and the map $G \to \mathrm{Aut}(G)$ can be differentiated to a map $G \to \mathrm{Aut}(\mathfrak{g})$. This is called the *adjoint action* of G on $\mathfrak{g}$. Recall a torus T is a group isomorphic to $\mathbb{G}_m^k$ for some $k > 0$. Since we are working over $\mathbb{C}$ which is algebraically closed, every torus is split and we will not worry about the split condition. Since our torus is split it is diagonalisable. Take the adjoint action of G, and restrict it to T. We have a decomposition

$$\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}(T) \oplus \bigoplus_{\alpha \in R(T)} V_\alpha.$$

Here, we write $R(T)$ for the *roots* of G (with respect to T), that is, $R(T) \subset X^\bullet(T)$ where $X^\bullet(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ is called the *character group* of T. We also write V_α for the α-eigenspace of T in $\mathfrak{g}$. We explicitly select $R(T)$ a finite subgroup of $X^\bullet(T)$ so that $V_\alpha \neq 0$ when $\alpha \in R$. The roots of G do not depend on the choice of T (up to isomorphism of root systems, which we will define shortly). We also have the dual data $X_\bullet(T) := \mathrm{Hom}(\mathbb{G}_m, T)$ (called the *cocharacter group*), $R^\vee \subset X_\bullet(T)$ called the *coroots*. This data is still extremely important but in the limited cases we will work with, we will not need it, so we will omit the definitions of this dual data.

Definition 7 (root datum)**.** *A **root datum** is a collection of four objects: $R \subset X, R^\vee \subset X^\vee$ along with the following data:*

1. *$X, X^\vee$ are free abelian groups of finite rank.*
2. *We have a perfect pairing $\langle -, - \rangle$ on $X \times X^\vee \to \mathbb{Z}$.*
3. *R, $R^\vee$ are finite and equipped with a bijection $R \leftrightarrow R^\vee$, $\alpha \leftrightarrow \alpha^\vee$.*
4. *For every $\alpha \in R$, $\langle \alpha, \alpha^\vee \rangle = 2$ and the map $s_\alpha : X \to X$ given by $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ permutes R and induces an action on $X^\vee$ which restricts to a permutation of $R^\vee$.*

*Elements of R are called **roots** and elements of $R^\vee$ are called **coroots**.*

The construction we previously gave for reductive groups defines a root system in $X^\bullet(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Define V as the span of R in $X^\bullet(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 8 (Weyl group)**.** *For any reductive group G, we have a finite group $W(G)$, called the **Weyl group** of G, generated by s_α for $\alpha \in R$.*

Definition 9 (simple and positive roots)**.** *There exists a (possibly non-unique) subset $\Sigma \subset R$ called **simple roots** that is a basis of V and so that every element in R is either a non-negative or non-positive sum of elements of Σ. We call the **positive roots** the roots that can be written as a non-negative sum of elements of Σ. The set of positive roots is denoted X_+.*

In fact, root data is sufficient to classify reductive groups.

Theorem 1 (Chevalley)**.** *There is a bijection between reductive groups over $\mathbb{C}$ up to isomorphism and root data up to isomorphism.*

2 Geometric Representation Theory

Much of this discussion is based on the expository notes in [CW20], although they work in general characteristic and here we limit ourselves to characteristic 0. As a result, we have the stronger Borel-Weil-Bott instead of the weaker Kempf Vanishing. Borel-Weil-Bott is an extremely deep theorem that relates purely algebraic-geometric data (sheaf cohomology on a variety) with the representations of G. It allows us to work with the rich algebraic and geometric structure of sheaves on varieties in order to deduce information about the representation theory of groups.

2.1 Flag Varieties

Fix a reductive group G.

Definition 10 (Borel subgroup)**.** *We say $B \subset G$ is **Borel** if it is a maximal solvable subgroup.*

Example 2. *In the case $G = \mathrm{SL}_n$, we have, for example, B^+ as the subgroup of upper-triangular matrices, which contains the unipotent subgroup U^+ of strictly upper-triangular matrices. (We write B^+ as this standard choice of Borel subgroup in contrast with B^- which refers to the other standard choice of Borel, the subgroup of lower-triangular matrices.)*

Definition 11 (flag variety of G)**.** *For any Borel subgroup B, the quotient G/B is a projective variety over $\mathbb{C}$ which we call the **flag variety of** G.*

This variety does not depend on a choice of B, up to isomorphism, so we will often omit the choice of B and simply refer to the "flag variety of G", or just the "flag variety" when G is clear.

Example 3. *When $G = \mathrm{SL}_2$, we have B^+ as the subgroup of upper-triangular matrices, and $G/B^+ \cong \mathbb{P}^1(\mathbb{C})$. We note that $\mathbb{P}^1(\mathbb{C})$ can be naturally thought of as lines through the origin in $\mathbb{C}^2$, and G acts transitively on this set of lines. Each line is stabilised by B^+ so the action map induces the isomorphism.*

2.2 Borel-Weil-Bott

Definition 12 (G-equivariant sheaf)**.** *Let G act on X via $\sigma : G \times X \to X$. A* ***$G$-equivariant sheaf on*** *X, $\mathcal{F}$, is a sheaf of $\mathcal{O}_X$-modules together with the data of an isomorphism $\phi : \sigma^*\mathcal{F} \cong p_2^*\mathcal{F}$, where p_2 is the projection on $G \times X \to X$, so that ϕ satisfies the "cocycle condition":*

$$p_{23}^*\phi \circ (1_G \times \sigma)^*\phi = (m \times 1_X)^*\phi$$

where p_{23} is the projection on $G \times G \times X \to G \times X$, and m is the multiplication arrow of G.

This leads to an extremely powerful tool: when $\mathcal{F}$ is a G-equivariant sheaf, $H^0(X, \mathcal{F})$ admits the structure of a G-representation, where g acts on $w \in H^0(X, \mathcal{F})$ via

$$gw = (\phi \circ \sigma^{\#})(w)(g^{-1})$$

where $\sigma^{\#}$ is the induced map $H^0(X, \mathcal{F}) \to H^0(G \times X, \sigma^*\mathcal{F})$. This becomes particularly powerful when we work with sheaves on the flag variety of G.

We now have developed enough technology to introduce the extremely powerful Borel-Weil-Bott theorem. Let G be a semisimple group over $\mathbb{C}$ and T a maximal torus in G. Fix B^+ a choice of a Borel subgroup containing T, and let U^+ be the unipotent radical of B^+. (Note that $T \cong B^+/U^+$.) We say a weight λ is **integral** if $2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ for any $\alpha \in R$. Then for any integral weight λ, we have a 1-dimensional representation of B^+ by pulling back along $B^+ \twoheadrightarrow B^+/U^+ \cong T$, which we call V_λ. We can treat the map $G \twoheadrightarrow G/B^+$ as a principal B^+-bundle, so we can pull back V_λ to a line bundle on G/B^+, which we denote $\mathcal{L}_{-\lambda}$ (a sign change occurs due to a swap from a left to a right action). By abuse of notation, we associate $\mathcal{L}_{-\lambda}$ with its sheaf of holomorphic sections. Then $\mathcal{L}_{-\lambda}$ is a G-equivariant sheaf on G/B^+.

Let $\rho = \frac{1}{2}\sum_{r \in X_+} r$. Then for any integral weight λ, one of two cases occurs.

1. There is no $w \in W$ so that $w(\lambda + \rho) - \rho$ is dominant.
2. There is a unique $w \in W$ so that $w(\lambda + \rho) - \rho$ is dominant.

(We say a weight λ is **dominant** if $\lambda(\alpha^\vee) \geq 0$ for $\alpha \in \Sigma$.)

Theorem 2 (Borel-Weil-Bott)**.** *Let G be a semisimple group over $\mathbb{C}$, B a Borel subgroup, and λ an integral weight. Then if there is no $w \in W$ so that $w(\lambda+\rho)-\rho$ is dominant, then $H^i(G/B, \mathcal{L}_\lambda) = 0$ for all $i \geq 0$. If there is a unique $w \in W$ so that $w(\lambda + \rho) - \rho$ is dominant, then $H^i(G/B, \mathcal{L}_\lambda) = 0$ for all $i \neq \ell(w)$ (where ℓ is the length function in W), and $H^{\ell(w)}(G/B, \mathcal{L}_\lambda)$ is the dual of the irreducible representation of G with highest weight $w(\lambda + \rho) - \rho$.*

A proof can be found in [Lur]. In particular, when λ is already dominant, then $w = e$ and we see $H^0(G/B, \mathcal{L}_\lambda)$ is the dual to an irreducible representation of G with highest weight λ. This theorem reflects a beautiful relation between a core concept of representation theory, the induced representation, and a purely geometric concept. Letting R^n denote the n-th right derived functor: when H_1 and H_2 are algberaic groups with H_1/H_2 a scheme, and M a representation of H_2, $R^n \operatorname{Ind}_{H_1}^{H_2} M \cong H^n(H_1/H_2, \mathcal{O}(M))$ ($\mathcal{O}(M)$ is some sheaf on H_1/H_2 that is constructed from M, we omit the details of its construction). In particular, we take the case $H_1 = G, H_2 = B$ here.

3 Weyl Character Formula

3.1 Introduction

Fix G a semisimple group. Let M be a finite-dimensional representation of G.

Definition 13 (character of representation)**.** *We define the **character** of M to be*

$$\operatorname{ch}(M) := \sum_{\lambda \in X^\bullet(T)} (\dim M_\lambda) e^\lambda$$

as an element of $\mathbb{Z}[X^\bullet(T)]$. We write M_λ to denote the (potentially trivial) λ-eigenspace.

When λ is a dominant weight, denote $\nabla_\lambda := H^0(G/B, \mathcal{L}_\lambda)^*$. This is an irreducible representation of G with highest weight λ. Every irreducible representation of G has a highest weight, and since these representations cover every such highest weight, every irreducible representation of G is exactly (isomorphic to) one of these ∇_λ for a dominant λ (this is a theorem of Chevalley, dominant weights are in bijection with irreducible G-representations via this association). So classifying characters of representations of G is equivalent to classifying characters of ∇_λ. But these representations have a beautiful and simple formula for their characters:

Theorem 3 (Weyl Character Formula)**.** *When λ is a dominant weight,*

$$\operatorname{ch}(\nabla_\lambda) = \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)}}{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho)}}$$

Corollary 1. *Equivalently, when V is a G-representation with highest weight λ,*

$$\operatorname{ch}(V) = \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)}}{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho)}}$$

A priori, it is not even clear that the right-hand side of this expression even lies in $\mathbb{Z}[X^\bullet(T)]$, but remarkably, it does. This statement looks completely algebraic in nature, but it admits a beautiful and simple geometric proof that truly captures the essence of the power and strength of geometric representation theory.

3.2 Proof

This proof follows a proof of the Weyl Character Formula due to Demazure in [Dem69] (translated by me).

Proof. Let G be a simply connected semisimple group, T a maximal torus (which is split since we are over $\mathbb{C}$), and B a Borel subgroup of G containing T.

Let G act on an algebraic $\mathbb{C}$-scheme X. Let $\mathcal{K}_G(X)$ be the category of coherent G-modules on X. The Grothendieck group of this abelian category is denoted $K_G(X)$ and the image of the object $V \in \mathcal{K}_G(X)$ is denoted $\mathrm{cl}(V) \in K_G(X)$. Let P be a parabolic subgroup of G containing B. Let $\mathcal{R}(P)$ be the ccategory of finite-dimensional representations of P and $R(P)$ its Grothendieck group. Again, let the image of $V \in \mathcal{R}(P)$ in $R(P)$ as $\mathrm{cl}(V)$. Consider the two additive functors

$$\Phi : \mathcal{R}(P) \to \mathcal{K}_G(G/P)$$

$$\Psi : \mathcal{K}_G(G/P) \to \mathcal{R}(P)$$

defined as follows: to the representation $\rho : G \to \mathrm{GL}(V)$, Φ gives the corresponding bundle $G \times^P V$ which is a locally free $\mathcal{O}_{G/P}$-module (as the fibration $G \to G/P$ is locally trivial), and to the G-module S on G/P, Ψ gives the representation of P in the fiber of S of the marked point of G/P, deduced from the action of G on S. It is easy to verify that these functors are quasi-inverses. It follows in particular that any object of $\mathcal{K}_G(G/P)$ is a locally free $\mathcal{O}_{G/P}$-module and that Φ induces an isomorphism of groups

$$\phi_P : R(P) \to K_G(G/P)$$

Moreover, if we endow $R(P)$ and $K_G(G/P)$ a ring structure under the tensor product, ϕ_P is an isomorphism of rings. Now, let G act on two algebraic $\mathbb{C}$-schemes, X and Y. If $f : X \to Y$ is a morphism (respectively a proper morphism) we have the map $f^* : K_G(Y) \to K_G(X)$ (respectively $f_! : K_G(X) \to K_G(Y)$) given by

$$f^*(\mathrm{cl}(\mathcal{F})) = \mathrm{cl}(f^*\mathcal{F})$$

$$f_!(\mathrm{cl}(S)) = \sum (-1)^n \,\mathrm{cl}(R^n f_*(S))$$

where $f^*\mathcal{F}$ is the usual inverse image and $R^n f_* S$ is the n-th direct image, both equipped with the G-module structure deduced from the G-module structure of $\mathcal{F}$ and S respectively. If f is proper, then $y \in K_G(Y)$ is the class of a locally free $\mathcal{O}_Y$-module and we have

$$f_!(f^*(y)x) = y f_!(x) \forall x \in K_G(X)$$

In particular, consider two parabolic subgroups P and Q of G so that $P \subset Q$. Let $i : P \to Q$ be the inclusion and $f : G/P \to G/Q$ be the canonical projection. Let

$i^* : R(Q) \to R(P)$ be the restriction morphism. Then we have the commutative diagramme:

$$\begin{array}{ccc} R(Q) & \xrightarrow{\phi_Q} & K_G(G/Q) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle f^*} \\ R(P) & \xrightarrow{\phi_P} & K_G(G/P) \end{array}$$

We define a homomorphism of groups $i_! : R(P) \to R(Q)$ so that the following diagramme commutes:

$$\begin{array}{ccc} R(P) & \xrightarrow{\phi_P} & K_G(G/P) \\ \downarrow{\scriptstyle i_!} & & \downarrow{\scriptstyle f_!} \\ R(Q) & \xrightarrow{\phi_Q} & K_G(G/Q) \end{array}$$

So we have

$$i_!(i^*(\beta)\alpha) = \beta i_!(\alpha) \forall \alpha \in R(P), \beta \in R(Q)$$

For example, if we have $P = B$ and $G = Q$, we have $R(P) = R(B) = R(T)$, $G/Q = \operatorname{Spec} \mathbb{C} = *$, and thus we have the commutative diagramme

$$\begin{array}{ccc} R(G) & \xrightarrow{\phi_G} & K_G(*) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle f^*} \\ R(T) & \xrightarrow{\phi_B} & K_G(G/B) \\ \downarrow{\scriptstyle i_!} & & \downarrow{\scriptstyle f_!} \\ R(G) & \xrightarrow{\phi_G} & K_G(*) \end{array}$$

where i^* is the restriction morphism, and where, if $\rho : T \to \mathrm{GL}(V)$ is a representation of T, we have

$$i_!(\mathrm{cl}(\rho)) = \sum (-1)^n \,\mathrm{cl}(H^n(G/B, G \times^B V))$$

Now, recall $X^\bullet(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ is the character group of T. For $\chi \in X^\bullet(T)$, denote e^χ as the element of $R(T)$ where T acts on $\mathbb{C}$ via χ. Under this association, $R(T) \cong \mathbb{Z}[X^\bullet(T)]$ ([Ser68] 3.4). Let W denote the Weyl group of G with respect to T. Then the map $i_* : R(G) \to R(T)$ is injective ([Ser68] 3.6) and its image is $R(T)^W := \{x \in R(T) | wx = x \forall w \in W\}$. To calculate $i_! : R(T) \to R(G)$, it suffices to calculate $i^* i_! : R(T) \to R(T)$. Let J be the endomorphism of $R(T)$ given by

$$J(x) = \sum_{w \in W} \mathrm{sgn}(w) w(x)$$

Recall $\rho = \dfrac{1}{2}\sum_{r \in X_+} r$. By [Bou68] 3.2, $J(xe^\rho)$ is divisible by $J(e^\rho)$ for any $x \in R(T)$, and $J(xe^\rho)/J(e^\rho) \in R(T)^W$. Thus, for $x \in R(T)$, $i^* i_!(x) = J(xe^\rho)/J(e^\rho)$.

In particular, let $x = e^\lambda$ where λ is a dominant weight. Then $xe^\rho = e^{\rho+\lambda}$. By Borel-Weil-Bott, $H^n(G/B, \mathcal{L}_\lambda) = 0$ for $n > 0$ and $H^0(G/B, \mathcal{L}_\lambda)$ is an irreducible G-representation with highest weight λ which we denoted as ∇_λ. Thus $i_!(x) = \mathrm{cl}(\nabla_\lambda)$. Thus, we immediately arrive at the formula

$$\mathrm{ch}(\nabla_\lambda) = i^*(\mathrm{cl}(\nabla_\lambda)) = J(e^{\lambda+\rho})/J(e^\rho) = \frac{\sum_{w \in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)}}{\sum_{w \in W} \mathrm{sgn}(w) e^{w(\rho)}}.$$

(recalling that w acts on e^x by $we^x = e^{wx}$ and that i^* is the map that sends a G-representation to its character in $\mathbb{Z}[X^\bullet(T)]$.) $\square$

4 References

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