General Collisions of Smooth Spheres

Ivo O. Verhoeven*

University College Roosevelt

(Dated: September 4, 2024)

Abstract

Collisions of spheres are a standard topic in any undergraduate classical mechanics class. While the collision along a line is a simple exercise, modelling motion in higher dimensions is considerably more difficult. Standard methods exist, but didactic quality remains to be desired. This paper builds off of a recent publication on a simple, clear and intuitive method for computation of final velocities for perfect spheres. This method is elaborated and more general by considering transformations to and from a collision-frame. The results are extended with a discussion on inelastic collisions, limited to compliance along the common line-of-action. The loss of conservation of kinetic energy is circumvented by defining an empirical law relating the change in relative velocities. This law has important implications for kinetic energy and relative velocity change due to collision, resulting in classic expressions for general collisions. This paper concludes by considering the generalised case of the billiard scatter angle problem. Given the described framework for simplifying collisions, the scatter angles for inelastic collisions follows quickly. This final, novel approach to the billiard collision clarifies the scattering behaviour greatly. A Mathematica notebook accompanies this paper, providing methods for simulating arbitrary two dimensional collisions of spheres.

I. INTRODUCTION

Collisions, while likely studied in all physics classrooms, remain a poorly understood and tremendously complex topic. Despite the precedent, more intuitive methods are still being published that elucidate various concepts[4]. A reason for the complexity of general collisions is the intersection of many differing theories regarding conservation laws, friction, deformation, material sciences, many of which are experimental in nature and aim to capture strongly nonlinear phenomena. Regardless, under some useful assumptions and reasonable constraints, largely theoretical results based on Newtonian and classical mechanics can get very close to realistically modelling collisions in nature.

Thus, it is prudent to set some initial conditions and parameters. A complete discussion regarding the necessity, motivation and use of assumptions in (rigid body) collision mechanics is presented in Chatterjee's thesis[5]. Many of these are also relevant for the discussion presented in this paper:

1. Rigid bodies. Generally, it is assumed that any deformation due to collision or motion is negligible and its form stays constant when comparing pre- and post-collision behaviour. During the collisions, all deformations do not alter the reactive behaviour.

2. Short collision duration. It is assumed that the actual contact between the bodies is of such short duration that, again, it may be assumed that the form of the masses stays constant.

3. Low velocities. The relative velocities of the masses prior to collision are low enough that during collision the assumption of rigid bodies remain intact.

Beyond these assumptions, this paper is limited to cases where the colliding masses are perfectly uniform spheres. The symmetry present in this geometry provides many simplifications for important quantities (e.g. a simplified inertia tensor and limiting the contact surface to a point between the two masses).

A. Paper Structure

The remaining sections of this paper will be structured as follows: Section II review a simple and intuitive method presented by Čepič[4] for collisions of spheres along a plane. A slight alteration is presented, such that simulations become easier to generate and hopefully also to understand. Section III will present an extension to inelastic collisions, limited to inelastic translatory motion. While a further extension to tangential adherence was considered, this merely obfuscates the intuition of the framework. This is left for future research.

II. ELASTIC COLLISIONS

A. Collisions along a Line

The method presented here summarises the work of Čepič, while also providing a general method for translating from the collision frame back to the laboratory frame.

All collisions observe the law of conservation of *total* energy, but for elastic collisions the kinetic energy is also conserved. Inelastic collisions instead see some energy transform due to dissipative or frictive forces. As such, the situation immediately prior to and after the elastic collision of two spheres, m and M, may be summarised as,

$$Mv_M + mv_m = Mv'_M + mv'_m \qquad \text{(Momentum)} \tag{1a}$$

$$\frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2 = \frac{1}{2}Mv_M^{2\,\prime} + \frac{1}{2}mv_m^{2\,\prime} \qquad \text{(Energy)} \tag{1b}$$

A standard result, see for example Morin's theorem 5.3[8] is the third conserved quantity for 1D collisions. Taking the quotient of the relative momenta and energy quantities indicates that the absolute relative velocity is conserved.

$$\left.\begin{aligned} m(v_m - v'_m) &= M(v'_M - v_M) \\ m(v_m^2 - {v'_m}^2) &= M({v'_M}^2 - v_M^2) \end{aligned}\right\} (v_m - v_M) = -(v'_m - v'_M)$$
$$\implies |v_m - v_M| = |v'_m - v'_M| \tag{2}$$

Using Eqs. 1a and 2, one can solve this system of two unknowns (v'_m, v'_M) in two equations. Eq. 3 is the general solution for 1D elastic collisions.

$$\begin{aligned} v'_M &= \frac{(M-m)v_M + 2mv_m}{M+m} \\ v'_m &= \frac{(m-M)v_m + 2mv_M}{M+m} \end{aligned} \tag{3}$$

Rewriting Eq. 2 as $v_m + v'_m = v_M + v'_M$ reveals the core argument of Čepič's method. For collisions along a line, the change in velocities due to collisions are equal but opposite for the

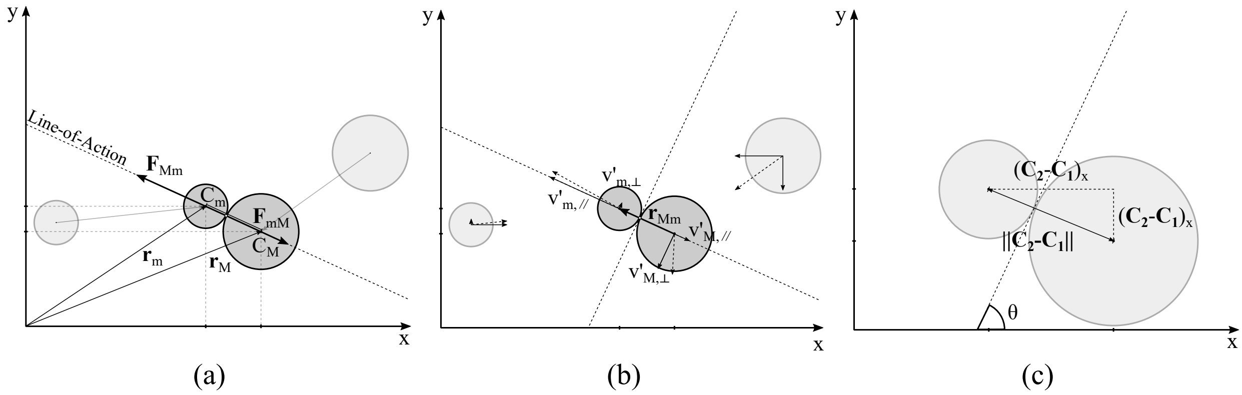

FIG. 1: A geometric depiction of the elastic collision of two spheres. Subfigure (a) depicts the initial conditions and the forces/displacement at collision. Subfigure (b) depicts the initial and final velocities of the spheres. Subfigure (c) shows the displacement as a function of the sine and cosine of the displacement between the spheres, itself dependent on the rotation of the line-of-action relative to the Cartesian coordinate frame.

colliding particles. This follows from the fact that the change in velocities must follow the line along which the impulses between the masses occur. Clear and immediate implications for higher dimension collisions are present.

B. Collisions along a Plane

The concept of exchange of momentum along a line-of-action may also be extended to 2D collisions, with a component along the parallel of the line-of-action that is affected according to Eq. 2, and a component along the perpendicular. A sketch and geometrical motivation for this argument is given by Fig. 1.

For planar motion, the collision may be summarised similarly to Eqs. 1a and 1b, however stressing the existence of additional components denoting horizontal and vertical directions.

$$Mv_{M,y} + mv_{m,y} = Mv'_{M,y} + m'_{m,y}, Mv_{M,y} + mv_{m,y} \qquad = Mv'_{M,y} + m'_{m,y}$$

$$\frac{1}{2}M(v_{M,x}^2 + v_{M,y}^2) + \frac{1}{2}m(v_{m,x}^2 + v_{m,y}^2) = \frac{1}{2}M({v'_{M,x}}^2 + {v'_{M,y}}^2) + \frac{1}{2}m({v_{m,x}\prime}^2 + {v'_{m,y}}^2) \qquad \text{(En}$$

Eventually Čepič achieved their result of a general equation for elastic collisions in two dimensions by introducing a unit vector $e_{\parallel}$ along the line-of-action. An equivalent solution may be achieved by noting that the orthogonal axes parallel to the line-of-action (the collision-frame) is a simple rotation by θ away from the standard laboratory coordinate frame. Given the displacement vector (the line connecting the CoMs of each sphere) $r_{mM} = C_M - C_m$, a rotation matrix can be constructed using the x- and y-components. See Fig. 1(c) for a geometric depiction of the transformation. The rotation matrix[9] follows as,

$$\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{r_{mM,x}}{\|r_{mM}\|} & \frac{r_{mM,y}}{\|r_{mM}\|} \\ -\frac{r_{mM,y}}{\|r_{mM}\|} & \frac{r_{mM,x}}{\|r_{mM}\|} \end{pmatrix}$$

The relationship between the velocity vectors in the lab-frame and the collision-frame follows as,

$$\begin{pmatrix} v_{\|} \\ v_{\perp} \end{pmatrix} = \mathbf{R} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \Leftrightarrow \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \mathbf{R}^T \begin{pmatrix} v_{\|} \\ v_{\perp} \end{pmatrix}$$

The core argument remains the same. As the spheres collide with point contact, the forces between them act along two perpendicular axes. As only collision forces are responsible for change in velocities, a problem with motion along a plane[10] collapses to one of forces along two lines. Furthermore, for elastic collisions of smooth spheres, only one of these 1D-axes sees change before and after contact with the other mass. The solution for planar collisions follows from Eq. 3 as,

$$v_M^{(\mathrm{C})}\prime = \begin{pmatrix} \frac{(M-m)v_{M,\|}+2mv_{m,\|}}{M+m} \\ v_{M,\perp} \end{pmatrix}, v_m^{(\mathrm{C})}\prime = \begin{pmatrix} \frac{(m-M)v_{m,\|}+2Mv_{M,\|}}{m+M} \\ \mathbf{v}_{m,\perp} \end{pmatrix} \tag{4}$$
$$v_M^{(\mathrm{L})\prime} = \mathbf{R}^T \mathbf{v}_M^{(\mathrm{C})\prime}, \mathbf{v}_m^{(\mathrm{L})\prime} = \mathbf{R}^T \mathbf{v}_m^{(\mathrm{C})\prime}$$

Thus, the collision in planar motion collapses into the relatively simple linear motion case. This result extends to higher dimensions also, however, only for spherical objects colliding.

III. INELASTIC COLLISIONS WITHOUT FRICTION

Realistic collisions are not as neatly summarised in a two-liner as Eq. 3. In fact, a universal, theoretically derived law for collisions remains out of grasp, except for the most limiting of cases. Again, many reasons for discrepancy between elastic and realistic collisions exist, some of which come from the violation of the assumptions made in the first part of this paper, some coming from dissipative forces at a variety of levels at the contact surface. Collision restitution laws closely approximating experiment are by necessity empirical.

The first, and most commonly used restitution law is one suggested by Newton himself, introduced as an experimental multiplicative correction to the conservation of relative velocity as Eq. 2:

$$\epsilon = -\frac{\mathbf{v_M}' - \mathbf{v_m}'}{\mathbf{v_M} - \mathbf{v_m}} \tag{5}$$

Since Newton's introduction of the coefficient, many different formulations of ϵ have been attempted; recent attempts being deceptively complex[3]. For Newton's ϵ, they key distinguishing factor is the materials of the colliding objects. Other relevant factors include the relative velocities pre-collision; the shapes, sizes, masses and elastic modulii of the colliding bodies; the density and temperature of the medium in which the bodies reside[1].

Generally, a coefficient of restitution only describes collisions between two specific objects; it does not generalise. Newton's coefficient of restitution is defined in the range of $\epsilon \in [0, 1]$, with 0 denoting a perfectly inelastic collision and 1 a perfectly elastic. A more intuitive result (the momentum transfer) will be derived below.

Newton's restitution law is relevant only for modelling frictive forces along the line of action. Therefore, Čepič's method remains relevant. Modelling dissapative forces along the perpendicular dimensions is considerably more difficult. For a recent attempt limiting itself to purely sliding of spheres' surfaces over each other, see Schwager et al.[12] Otherwise, standard works like Stronge's '*Impact Mechanics*'[13] provide some discussion for restitution laws with regards to rigid bodies.

The remaining part of this section focuses on deriving some results related to inelastic collisions using the framework set out in section I. It finishes with an application to scatter angles of colliding billiard balls. The standard result of right angle collisions given an elastic collision, also presented in Čepič's paper[4] is shown to be a special case.

A. Derivation of Final Velocities

Consider a situation similar to that of Fig. 1. The two spheres collide in such a manner, that for each sphere a force is applied from the point of contact, opposite to the direction of motion. By Newton's third law, a relationship between the change in momenta (before and after contact) and the force applied may be defined; the impulse J. From this, the expression for the velocity post-collision follows as,

$$
\begin{aligned}
&\mathbf{F} = \frac{d\mathbf{p}}{dt} \Longleftrightarrow \mathbf{J} = \int_{t_1}^{t_2} \mathbf{F} dt = \mathbf{p}' - \mathbf{p} \\
&\mathbf{p}' = \mathbf{p} - \mathbf{J} \\
&\mathbf{v}' = \mathbf{v} - \frac{1}{m}\mathbf{J}
\end{aligned} \tag{6}
$$

While the conservation of kinetic energy may no longer hold for inelastic collisions, the change of relative velocities is known to be mediated by an experimentally defined multiplicative correction: Newton's coefficient of restitution ϵ.

$$
(v_M - v_m) = -\epsilon(v_M - v_m) \tag{7}
$$

What remains is a system of three equations in three unknowns, v'_M, v'_m and $\mathbf{J}$. Note the alternating sign in the term for the impulses correction for masses M and m.

$$
\left.
\begin{aligned}
v'_{M,\|} &= v_{M,\|} - \tfrac{1}{M} J \\
v'_{m,\|} &= v_{m,\|} + \tfrac{1}{m} J \\
(v'_{M,\|} - v'_{m,\|}) &= -\epsilon(v_{M,\|} - v_{m,\|})
\end{aligned}
\right\}
\begin{aligned}
v'_{M,\|} &= \tfrac{M v_{M,\|} + m v_{m,\|} + \epsilon m (v_{m,\|} - v_{M,\|})}{M+m} \\
v'_{m,\|} &= \tfrac{m v_{m,\|} + M v_{M,\|} + \epsilon M (v_{M,\|} - v_{m,\|})}{M+m} \\
\mathbf{J} &= -m_{eff}^{-1}(1+\epsilon)(v_{m,\|} - v_{M,\|})
\end{aligned} \tag{8}
$$

Here the factor m_{eff} denotes the effective mass, defined as $m_{eff} = \frac{1}{m} + \frac{1}{M} = \frac{M+m}{mM}$. The final velocities in the elastic collision model, Eq. 4, follow as the special case $\epsilon = 1$.

Reminiscent of Eqs. 4, the velocities of the spheres post-collision may be summarised as,

$$
\mathbf{v}'_M = \mathbf{R}^T \begin{pmatrix} \frac{M v_{M,\|} + m v_{m,\|} + \epsilon m (v_{m,\|} - v_{M,\|})}{M+m} \\ v_{M,\perp} \end{pmatrix}, \mathbf{v}'_m = \mathbf{R}^T \begin{pmatrix} \frac{M v_{M,\|} + m v_{m,\|} + \epsilon M (v_{m,\|} - v_{M,\|})}{M+m} \\ v_{m,\perp} \end{pmatrix} \tag{9}
$$

where the superscript (L) denotes the lab frame and (C) the collision frame.

B. Relative Velocity and Kinetic Energy Loss due to Collision

A key result for the evaluation of the final velocities in the elastic collision case was the conservation of velocity along the line-of-action. It was already given that this would no longer hold for inelastic collisions, however using result 8 the change in the relative velocities, $\Delta \mathbf{v}_\Delta$ may be found.

$$
\begin{aligned}
\Delta|\mathbf{v}_\Delta| = (\mathbf{v}'_M - \mathbf{v}'_m) - (\mathbf{v}_M - \mathbf{v}_m) &= -(\frac{1}{M} - \frac{1}{m})\mathbf{J} \\
&= -(1+\epsilon)(\mathbf{v}_m - \mathbf{v}_M)
\end{aligned} \tag{10}
$$

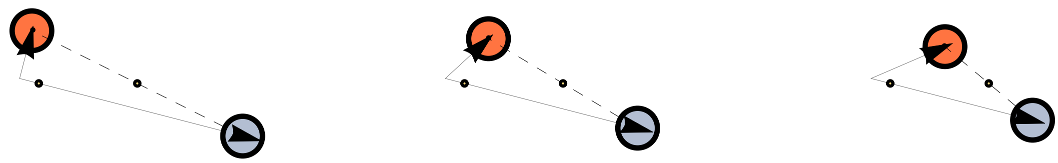

FIG. 2: Billiard ball collisions with various values for ϵ (l to r: $\frac{3}{3}, \frac{2}{3}, \frac{1}{3}$. The left case shows a perfect elastic collision, whereas the other have increasing values for the coefficient of restitution. These plots come from the accompanying Mathematica notebook.

The observation that elastic collisions are simply a specific case of inelastic collisions also becomes apparent when considering the change in kinetic energy before and after a collision. For elastic collisions, one needs only to invoke the law of conservation of energy to realise that this change is 0. The exact magnitude for inelastic collisions is considerably more complex. The derivation is provided in Appendix A, but the result for arbitrary values for ϵ follows below.

$$\Delta|E_K| = E'_{K,\|} - E_{K,\|} = -\frac{1}{2}(1-\epsilon^2)m_{eff}^{-1}(\mathbf{v}_m - \mathbf{v}_M)^2 \tag{11}$$

From the implication of Newton's $0 \leq \epsilon \leq 1$ implies that $\Delta E_K \leq 0$; energy is always lost in inelastic collisions. Again, the limiting case for $\epsilon = 1$ simply provides Eq. 1b, indicating elastic collisions. This result is a form of Carnot's 1803 theorem for rigid bodies in translator motion[2,6], although limiting cases may have been known much earlier already[1].

C. Generalised Collisions of Billiard Balls

Billiard ball collisions are a simplified application of the above described collision laws. Two equal balls are placed on a frictionless surface. The cue ball has moves towards the object ball, such that after collision the balls move away from each other at a constant angle. Otherwise, $m = M, r_m = r_M, ||v_M|| = 0$. Plots of billiard ball collisions for various ϵ are provided in Fig. 2.

While the definition of the coefficient of restitution is sufficient, considering the equal masses and radii of the balls, a more intuitive result is given by considering the proportion of momentum transferred. Let τ denote the transfer rate, such that

$$\begin{aligned} \tau &= \frac{v'_{1,||}}{v'_{1,||} + v'_{2,||}} \\ &= \frac{mv_{1,||} + m\epsilon v_{1,||}}{mv_{1,||} + m\epsilon v_{1,||} + mv_{1,||} - m\epsilon v_{1,||}} \\ &= \frac{v_{1,||} - \epsilon v_{1,||}}{2mv_{1,||}} \\ &= \frac{1}{2} - \frac{1}{2}\epsilon \end{aligned} \tag{12}$$

Again, it is clear collisions occur between one of two extremes. In the elastic case, 100% of the parallel velocity transfers from the cue ball to the object ball, leaving the first with only its perpendicular velocity. Juxtaposed to this is the perfectly inelastic case, where the minimum of 50% of the parallel velocity is transferred. This explains why collisions that have the bodies eventually 'stick' together require a ϵ value of 0; their final parallel velocities are identical[11]

Substituting, Eq. 12 provides a definition of the resultant parallel velocities $v'_{M,||}$ and $v'_{m,||}$ solely in terms of the incident parallel velocity $v_{m,||}$. Furthermore, the incident velocity of the object ball is set to be 0, such that the eventual final velocities of both balls is a system of two equation in two unknowns.

While the fully general case of scattering angles is rather daunting, using the above defined quantities, it can be derived for inelastic billiard collisions. Recall the definition of the dot product as the product of the magnitudes of two vectors and the cosine of the angle separating them. Rewritten, the angle between the incident velocities follows as

$$\mathbf{v}'_1 \cdot \mathbf{v}'_2 = |v'_1||v'_2| \cos\theta_{\mathbf{v}'_1,\mathbf{v}'_2} \Longleftrightarrow \theta_{\mathbf{v}'_1,\mathbf{v}'_2} = \arccos\frac{\mathbf{v}'_1 \cdot \mathbf{v}'_2}{|v'_1||v'_2|} \tag{13}$$

The dot-product of the velocities post-collision is simply the product of the parallel components, given the lack of a perpendicular velocity of the object ball. For the object ball this implies that the final velocity magnitude is entirely dependent on the momentum transferred from the incident cue ball.

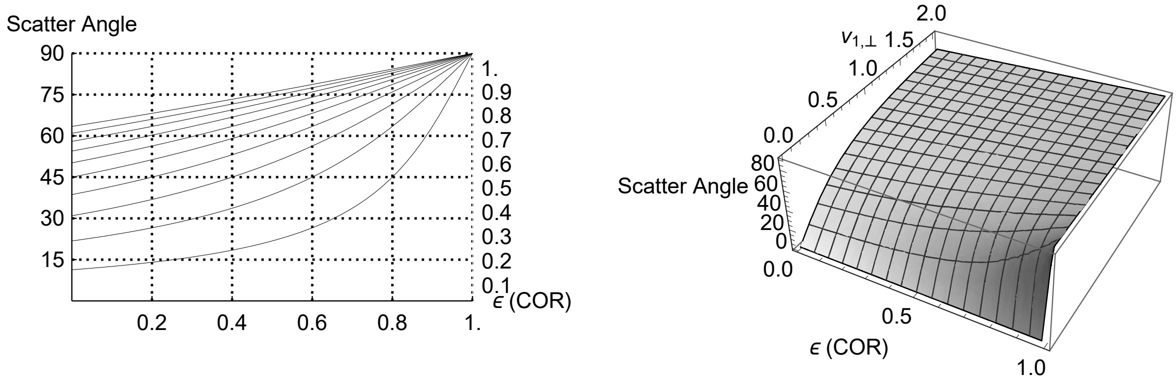

FIG. 3: Plot of the final scatter angle as a function of the pre-collision perpendicular collision and the ϵ. The parallel incident velocity of the cue ball is set to 1. The left graph has lines in order of the incident perpendicular velocity of the cue ball, whose values are to the right of the plot.

$$\begin{aligned}\theta_{\mathbf{v}'_1,\mathbf{v}'_2} &= \frac{v'_{1,\|}v'_{2,\|}}{\sqrt{v'^2_{1,\|}+v'^2_{1,\perp}}\sqrt{v'^2_{2,\|}}} = \frac{(\frac{1}{2}-\frac{1}{2}\epsilon)v_{1,\|}\cdot(\frac{1}{2}-\frac{1}{2}\epsilon)v_{1,\|}}{\sqrt{((\frac{1}{2}-\frac{1}{2}\epsilon)v_{1,\|})^2+v^2_{1,\perp}}\cdot(\frac{1}{2}+\frac{1}{2}\epsilon)v_{1,\|}} \\ &= \frac{\frac{1}{2}(1-\epsilon)v_{1,\|}}{\frac{1}{2}\sqrt{((1-\epsilon)v_{1,\|})^2+4v_{1,\perp}}} = \frac{(1-\epsilon)v_{1,\|}}{\sqrt{(1-\epsilon)^2v^2_{1,\|}+4v_{1,\perp}}}\end{aligned} \tag{14}$$

Eq. 14 gives the scatter angle for a general billiard collision problem as a function of the coefficient of restitution and the incident perpendicular velocity of the first ball. The standard result of the scatter angle being 90° for all elastic collisions again is a limiting case for $\epsilon = 0$. Fig. 3 gives the scatter angle of billiard collisions for an incident parallel velocity of $v_{m,\|} = 1$. Each curve is limited between 0° and 90° degrees, with the steepness being determined by incident perpendicular velocity. For situations where $v_{m,\|} >> v_{m,\perp}$ the scatter angle tends towards 90°, whereas $v_{m,\|} << v_{m,\perp}$ implies tendency towards the minimum of 0°.

IV. CONCLUSION

Investigated and discussed throughout this paper are the general collisions of smooth spheres. While exhaustively studied, extending the collisions along a line to higher dimensions remains a difficult exercise for many students. Rather than using transformations to centre-of-mass frames or using convoluted algebraic manipulations in the lab frame, a novel collision frame approach is taken. Inspired by Čepič's recent simple 4 step method, spherical collisions remain simple, even in higher dimensions. While they employed unit-vectors along tangential lines to the collision, here a rotation matrix is used to highlight the proximity to evaluating in the lab-frame. This method proves tremendously useful for modelling efforts, further allowing derivation of classic results without tremendous effort.

One such result is the application of billiard collisions. With the inclusion of Newton's coefficient of restitution, the transferred velocity in the parallel collision axis may be considered. From this, the resultant velocities of both spheres are entirely dependent on the incident velocity of the cue ball. The eventual angle between the balls' paths follows as an elegant expression.

Another example of the intuition of this method is the constant reminder of special and limiting cases being the well understood elastic and perfectly inelastic collisions. For spheres, highlighting the fact that higher-dimensional motion is not necessarily more difficult than motion along a line, didactic discussions of collisions need not limit themselves to these limiting cases.

Not discussed in this paper is the complex topic of compliance along the other orthogonal collision axes. These collisions not only see dissipative forces along the line through the collision point, and can thus consider rotations of the spheres and sliding over each other. While models exist for such collisions, the natural and intuitive measures of a coefficient of restitution and application of conservation laws fall away. Instead, the composition of materials need to be considered, as well friction regimes and bounded functions.

Čepič concludes by noting that this method is potentially useful for animations and simulations. This was found to be the case, evidenced by the accompanying Mathematica notebook. Consideration of a collision frame allowed for easy construction of functions that model arbitray collisions.

APPENDIX A: DERIVATION OF ΔE_K FOR INELASTIC COLLISIONS

$$
\begin{aligned}
E'_K &= \frac{1}{2}mv'_m + \frac{1}{2}mv'_M \\
&= \frac{1}{2}m(\mathbf{v}_m + \frac{1}{m}\mathbf{J})^2 + \frac{1}{2}M(\mathbf{v}_M - \frac{1}{M}\mathbf{J})^2 \\
&= \frac{1}{2}m(\mathbf{v}_m^2 + 2\frac{1}{m}\mathbf{v}_m\mathbf{J} + \frac{1}{m^2}\mathbf{J}^2) + \frac{1}{2}M(\mathbf{v}_M^2 - 2\frac{1}{M}\mathbf{v}_M\mathbf{J} + \frac{1}{M^2}\mathbf{J}^2) \\
&= \frac{1}{2}m\mathbf{v}_m^2 + \mathbf{v}_m\mathbf{J} + \frac{1}{2m}\mathbf{J}^2 + \frac{1}{2}M\mathbf{v}_M^2 - \mathbf{v}_M\mathbf{J} + \frac{1}{2M}\mathbf{J}^2 \\
&= \frac{1}{2}m\mathbf{v}_m^2 + \frac{1}{2}M\mathbf{v}_M^2 - (\mathbf{v}_m - \mathbf{v}_M)\mathbf{J} + (\frac{1}{2m} + \frac{1}{2M})\mathbf{J}^2 \\
&\rightarrow (\frac{1}{2m} + \frac{1}{2M}) = \frac{2M + 2m}{4mM} = \frac{1}{2}\frac{M+m}{mM} \\
&\rightarrow \frac{M+m}{mM}\mathbf{J}^2 = \frac{M+m}{mM}\frac{(mM)^2(1+\epsilon)^2(\mathbf{v}_m - \mathbf{v}_M)^2}{(M+m)^2} = \frac{mM(1+\epsilon)^2(\mathbf{v}_m - \mathbf{v}_M)^2}{M+m} \\
&\rightarrow E_K = \frac{1}{2}m\mathbf{v}_m^2 + \frac{1}{2}M\mathbf{v}_M^2 \\
&= E_K - \frac{mM(1+\epsilon)(\mathbf{v}_m - \mathbf{v}_M)^2}{M+m} + \frac{1}{2}\frac{mM(1+\epsilon)^2(\mathbf{v}_m - \mathbf{v}_M)^2}{M+m} \\
&= E_K - \frac{2mM(1+\epsilon)(\mathbf{v}_m - \mathbf{v}_M)^2}{2(M+m)} + \frac{mM(1+\epsilon)^2(\mathbf{v}_m - \mathbf{v}_M)^2}{2(M+m)} \\
&\rightarrow -2 - 2\epsilon + (1+\epsilon)^2 = -1 + \epsilon^2 = -(1-\epsilon^2) \\
&= E_K - (1-\epsilon^2)\frac{mM(\mathbf{v}_m - \mathbf{v}_M)^2}{2(M+m)} \\
\Delta E_K &= E'_K - E_K = -(1-\epsilon^2)\frac{mM(\mathbf{v}_m - \mathbf{v}_M)^2}{2(M+m)}
\end{aligned}
$$

---

* i.verhoeve@ucr.nl

[1] Bernard Brogliato and B Brogliato. *Nonsmooth mechanics.* Springer, 1999.

[2] Victor BURACU and Aurel ALECU. An extension of carnot's theorem for rigid body collisions. *energy*, 500:10.

[3] Edson Cataldo and Rubens Sampaio. A brief review and a new treatment for rigid bodies collision models. *Journal of the Brazilian Society of Mechanical Sciences*, 23(1):63–78, 2001.

[4] Mojca Čepič. Elastic collisions of smooth spherical objects: Finding final velocities in four simple steps. *American Journal of Physics*, 87(3):200–207, 2019.

[5] Anindya Chatterjee. *Rigid body collisions: some general considerations, new collision laws, and some experimental data.* Cornell University Ithaca, NY, 1997.

[6] Katica R Stevanović Hedrih. Central collision of two rolling balls: theory and examples. *Advances in Theoretical and Applied Mechanics*, 10(1):33–79, 2017.

[7] Bo Lan. A brief guide to the fundamentals of passive and active rotations in material science, 08 2015.

[8] David Morin. *Introduction to classical mechanics: with problems and solutions.* Cambridge University Press, 2008.

[9] Note that a rotation of axes is a *passive* transformation, and as such the presented rotation matrix is the transpose (and coincidentally) the inverse of the standard rotation matrix. See[7].

[10] The generalisation to collisions in three dimensions is a simple one.

[11] Note however that the contrapositive does not hold; collisions where objects do not stick together may still be inelastic, just with some perpendicular velocity remaining. The spheres then continue to move apart.

[12] T Schwager, V Becker, and T Pöschel. Coefficient of tangential restitution for viscoelastic spheres. *The European Physical Journal E*, 27(1):107–114, 2008.

[13] William James Stronge. *Impact mechanics.* Cambridge university press, 2018.