# One-Sample Hypothesis Testing: Case 1/2 (Z) 11/7

## Test Statistic

- Population is Normal and $\sigma^2$ is known (similar to Chapter 7.1)

Null hypothesis: $H_0 : \mu = \mu_0$

Case 1: $\sigma$ is known

Test statistic: $Z = \frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}$

Case 2: $\sigma$ is unknown (replace with $S$)

Test statistic value: $z = \frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}}$

| Alternative Hypothesis | $P$-value Determination |
|---|---|
| $H_a : \mu > \mu_0$ | Area under the standard normal curve to the right of $z$ |
| $H_a : \mu < \mu_0$ | Area under the standard normal curve to the left of $z$ |
| $H_a : \mu \neq \mu_0$ | 2(area under the standard normal curve to the right of $\lvert z \rvert$) |

- A **type I error** consists of rejecting the null hypothesis $H_0$ when it is true.
- A **type II error** involves not rejecting the null hypothesis $H_0$ when it is false.

| Decision | $H_0$ is True | $H_0$ is False |
|---|---|---|
| Fail to Reject $H_0$ | No Error | Type II Error |
| Reject $H_0$ | Type I Error | No Error |

| Decision | $H_0$ is True | $H_0$ is False |
|---|---|---|
| Fail to Reject $H_0$ | $1-\alpha$ | $\beta$ |
| Reject $H_0$ | $\alpha$ | $1-\beta$ |

Power of a test

## Type II Error

| Alternative Hypothesis | Type II Error Probability $\beta(\mu')$ for a Level $\alpha$ Test |
|---|---|
| $H_a : \mu > \mu_0$ | $\Phi\left(z_\alpha + \frac{\mu_0-\mu'}{\sigma/\sqrt{n}}\right)$ |
| $H_a : \mu < \mu_0$ | $1-\Phi\left(-z_\alpha + \frac{\mu_0-\mu'}{\sigma/\sqrt{n}}\right)$ |
| $H_a : \mu \neq \mu_0$ | $\Phi\left(z_{\frac{\alpha}{2}} + \frac{\mu_0-\mu'}{\sigma/\sqrt{n}}\right) - \Phi\left(-z_{\frac{\alpha}{2}} + \frac{\mu_0-\mu'}{\sigma/\sqrt{n}}\right)$ |

## Sample Size

The sample size $n$ for which a level $\alpha$ test also has $\beta(\mu') = \beta$ at the alternative value $\mu'$

$$n = \begin{cases} \left[\frac{\sigma(z_\alpha+z_\beta)}{\mu_0-\mu'}\right]^2 & \text{for a one-tailed (upper or lower test),} \\ \left[\frac{\sigma(z_{\alpha/2}+z_\beta)}{\mu_0-\mu'}\right]^2 & \text{for a two-tailed test.} \end{cases}$$

# Case 3 (t)

## Chapter 8.3: The One-Sample $t$ Tests for Hypotheses about a Population Mean

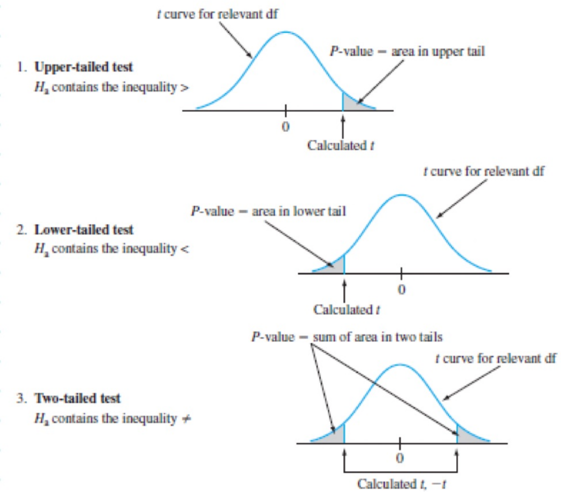
- Population is Normal and $\sigma^2$ is not known (similar to Chapter 7.3)

Null hypothesis: $H_0 : \mu = \mu_0$

Test statistic: $T = \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}}$ $\qquad T = \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} \sim T_{n-1}$

Test statistic value: $t = \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}}$ $\qquad t = \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}}$

| Alternative Hypothesis | $P$-value Determination |
|---|---|
| $H_a : \mu > \mu_0$ | Area under the $t_{n-1}$ curve to the right of $t$ |
| $H_a : \mu < \mu_0$ | Area under the $t_{n-1}$ curve to the left of $t$ |
| $H_a : \mu \neq \mu_0$ | 2(area under the $t_{n-1}$ curve to the right of $\lvert t \rvert$) |



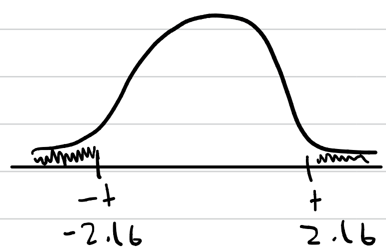
P-values for t tests

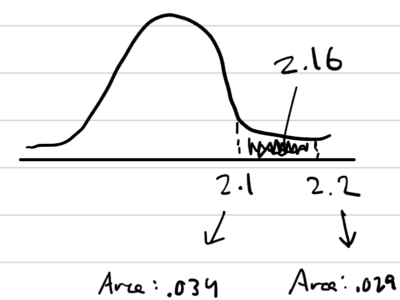
$$t = \frac{\overline{x} - \mu_0}{s/\sqrt{n}} = \frac{\overline{x} - 130}{s/\sqrt{9}}$$

$$= 2.16$$

$\Phi(2.16)$ ← Table A.8

$\approx 0.031 \cdot 2 = \boxed{0.062}$




$p = .062 > .01 \therefore$ Fail to reject $H_0$

# Case 4 ($\hat{p}$)

## Chapter 8.4: Tests Concerning a Population Proportion

- We have a large sample (similar to Chapter 7.2). These test procedures are valid provided that $np_0 \geq 10$ and $n(1-p_0) \geq 10$.

Null hypothesis: $H_0 : p = p_0$

Test statistic value: $z = \frac{\hat{p}-p_0}{\sqrt{p_0(1-p_0)/n}}$

| Alternative Hypothesis | $P$-value Determination |
|---|---|
| $H_a : p > p_0$ | Area under the standard normal curve to the right of $z$ |
| $H_a : p < p_0$ | Area under the standard normal curve to the left of $z$ |
| $H_a : p \neq p_0$ | 2(area under the standard normal curve to the right of $\lvert z \rvert$) |

## Type II Error

| Alternative Hypothesis | Type II Error Probability $\beta(p')$ for a Level $\alpha$ Test |
|---|---|
| $H_a : p > p_0$ | $\Phi\left(\frac{p_0-p'+z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right)$ |
| $H_a : p < p_0$ | $1-\Phi\left(\frac{p_0-p'-z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right)$ |
| $H_a : p \neq p_0$ | $\Phi\left(\frac{p_0-p'+z_{\frac{\alpha}{2}}\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right) - \Phi\left(\frac{p_0-p'-z_{\frac{\alpha}{2}}\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right)$ |

### Sample Size

The sample size $n$ for which a level $\alpha$ test also has $\beta(p') = \beta$ at the alternative value $p'$

$$n = \begin{cases} \left[\frac{z_\alpha\sqrt{p_0(1-p_0)}+z_\beta\sqrt{p'(1-p')}}{p'-p_0}\right]^2 & \text{for a one-tailed test,} \\ \left[\frac{z_{\frac{\alpha}{2}}\sqrt{p_0(1-p_0)}+z_\beta\sqrt{p'(1-p')}}{p'-p_0}\right]^2 & \text{for a two-tailed test.} \end{cases}$$

# Two-Sample Hypothesis Testing

11/7/22

## Case 1: Two Population Means (known $\sigma$)

$$Z = \frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}$$

$$\sigma_{\bar{x}-\bar{y}} = \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}$$

Square result for variance

| Type II Error | Formula |
|---|---|
| $H_a: \mu_1 - \mu_2 > \Delta_0$ | $\Phi\left(z_\alpha + \frac{\Delta_0 - \Delta'}{\sigma}\right)$ |
| $H_a: \mu_1 - \mu_2 < \Delta_0$ | $1 - \Phi\left(-z_\alpha + \frac{\Delta_0 - \Delta'}{\sigma}\right)$ |
| $H_a: \mu_1 - \mu_2 \neq \Delta_0$ | $\Phi\left(z_{\alpha/2} + \frac{\Delta_0 - \Delta'}{\sigma}\right) - \Phi\left(-z_{\alpha/2} + \frac{\Delta_0 - \Delta'}{\sigma}\right)$ |

| Alt. Hypothesis | p-value |
|---|---|
| $H_a: \mu_1 - \mu_2 > \Delta_0$ | Area to right |
| $H_a: \mu_1 - \mu_2 < \Delta_0$ | Area to left |
| $H_a: \mu_1 - \mu_2 \neq \Delta_0$ | 2(area to right) |

| Sample Size ($m=n$) | Formula |
|---|---|
| One-Tail | $\frac{(\sigma_1^2 + \sigma_2^2)(z_\alpha + z_\beta)^2}{(\Delta' - \Delta_0)^2}$ |
| Two-Tail | $\frac{(\sigma_1^2 + \sigma_2^2)(z_{\alpha/2} + z_\beta)^2}{(\Delta' - \Delta_0)^2}$ |

## Case 2: Non-normal, Unknown Variance, Large Sample $(m > 40,\ n > 40)$

$$z = \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}}$$

* Sub in $\sigma$ if pop SD is known *

Confidence Interval: $\bar{X} - \bar{Y} \pm z_{\alpha/2} \sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}$

Ha: $\mu > \#$ : Lower Confidence Bound

Ha: $\mu < \#$ : Upper Confidence Bound

| Alt. Hypothesis | p-value |
| --- | --- |
| Ha: $\mu_1 - \mu_2 > \Delta_0$ | Area to right of $z$ |
| Ha: $\mu_1 - \mu_2 < \Delta_0$ | Area to left of $z$ |
| Ha: $\mu_1 - \mu_2 \neq \Delta_0$ | 2(area to right) of $z$ |

# Case 3: Two - Sample t - test

$$t = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}}$$

CI: $\bar{X} - \bar{Y} + t_{\alpha/2, v} \sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}$

Ha: $\mu > \#$ : Lower Confidence Bound
Ha: $\mu < \#$ : Upper Confidence Bound

$$v = \frac{\left(\frac{S_1^2}{m} + \frac{S_2^2}{n}\right)^2}{\frac{(S_1^2/m)^2}{m-1} + \frac{(S_2^2/n)^2}{n-1}}$$

| Alt. Hypothesis | p-value |
|---|---|
| Ha: $\mu_1 - \mu_2 > \Delta_0$ | Area to right of $t_{\alpha,v}$ |
| Ha: $\mu_1 - \mu_2 < \Delta_0$ | Area to left of $t_{\alpha,v}$ |
| Ha: $\mu_1 - \mu_2 \neq \Delta_0$ | 2 (area to right) of $t_{\alpha,v}$ |

Case 4: Pooled t-test (both populations have the same variance)

$$t = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{S_p^2 \left(\frac{1}{m} + \frac{1}{n}\right)}}$$

$\sigma^2$ estimator: $S_p^2 = \frac{m-1}{m+n-2} S_1^2 + \frac{n-1}{m+n-2} S_2^2$

CI: $\bar{X} - \bar{Y} \pm t_{\alpha/2,\, n+m-2} \sqrt{S_p^2 \left(\frac{1}{m} + \frac{1}{n}\right)}$

Ha: $\mu > \#$: Lower Confidence Bound
Ha: $\mu < \#$: Upper Confidence Bound

| Alt. Hypothesis | p-value |
|---|---|
| Ha: $\mu_1 - \mu_2 > \Delta_0$ | Area to right of $t_{m+n-2}$ |
| Ha: $\mu_1 - \mu_2 < \Delta_0$ | Area to left of $t_{m+n-2}$ |
| Ha: $\mu_1 - \mu_2 \neq \Delta_0$ | 2(area to right of $t_{m+n-2}$) |

# Case 5: Paired t-test Procedure

$$t = \frac{\bar{d} - \Delta_0}{S_D / \sqrt{n}}$$

$$\bar{d} = \bar{x} - \bar{y}$$

$$S_D = \frac{1}{n-1} \sum_{i=1}^{n} (d_i - \bar{d})^2$$

CI: $\bar{d} \pm t_{\alpha/2,\, n-1}\, S_D / \sqrt{n}$

$H_a: \mu > \#$ : Lower Confidence Bound

$H_a: \mu < \#$ : Upper Confidence Bound

| Alt. Hypothesis | p-value |
| --- | --- |
| $H_a: \mu_1 - \mu_2 > \Delta_0$ | Area to right of $t_{n-1}$ |
| $H_a: \mu_1 - \mu_2 < \Delta_0$ | Area to left of $t_{n-1}$ |
| $H_a: \mu_1 - \mu_2 \neq \Delta_0$ | 2(area to right of $t_{n-1}$ |

# Case 6: Difference in Proportions (Large Sample)

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{m}+\frac{1}{n}\right)}}$$

$$\hat{p} = \frac{m}{m+n}\hat{p}_1 + \frac{n}{m+n}\hat{p}_2$$

$$\text{CI}: \hat{p}_1 - \hat{p}_2 \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{m} + \frac{\hat{p}_2\hat{q}_2}{n}}$$

$$\text{Sample Size}: n = \frac{\left(z_\alpha\sqrt{(p_1+p_2)(q_1+q_2)/2} + z_\beta\sqrt{p_1q_1 + p_2q_2}\right)^2}{d^2}$$

| Alt. Hypothesis | p-value |
|---|---|
| $H_a: \mu_1 - \mu_2 > \Delta_0$ | Area to right of $z$ |
| $H_a: \mu_1 - \mu_2 < \Delta_0$ | Area to left of $z$ |
| $H_a: \mu_1 - \mu_2 \neq \Delta_0$ | 2 (area to right) of $z$ |

| Type II Error | Formula |
|---|---|
| $H_a: p_1 - p_2 > 0$ | $\Phi\left(\frac{z_\alpha\sqrt{\bar{p}\bar{q}\left(\frac{1}{m}+\frac{1}{n}\right)} - (p_1 - p_2)}{\sigma}\right)$ |
| $H_a: p_1 - p_2 < 0$ | $1 - \Phi\left(\frac{-z_\alpha\sqrt{\bar{p}\bar{q}\left(\frac{1}{m}+\frac{1}{n}\right)} - (p_1 - p_2)}{\sigma}\right)$ |
| $H_a: p_1 - p_2 \neq 0$ | $\Phi\left(\frac{z_{\alpha/2}\sqrt{\bar{p}\bar{q}\left(\frac{1}{m}+\frac{1}{n}\right)} - (p_1 - p_2)}{\sigma}\right)$ $- \Phi\left(\frac{-z_{\alpha/2}\sqrt{\bar{p}\bar{q}\left(\frac{1}{m}+\frac{1}{n}\right)} - (p_1 - p_2)}{\sigma}\right)$ |

$$\bar{p} = \frac{mp_1 + np_2}{m+n} \qquad \sigma = \sqrt{\frac{p_1q_1}{m} + \frac{p_2q_2}{n}}$$

$$\bar{q} = \frac{mq_1 + nq_2}{m+n}$$