# CHAPTER 5
# INTEGRATION

# INTEGRATION

Introduction - Antidifferentiation

In this section we will learn how to reverse the process of differentiation. The reverse of *differentiating* is *antidifferentiating*, and the result of *antidifferentiating* is an *antiderivative*. An *antiderivative* of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$ for all $x$.

In some contexts, the process of antidifferentiation is called integration and the general form of the antiderivatives is referred to as an indefinite integral. A common notation for the indefinite integral, from Leibneiz is

$$\int f(x)dx$$

integral sign

integrand

Example 1:

Three antiderivatives of $f(x) = 2x$ are $F(x) = x^2, F(x) = x^2 + 1$ and $F(x) = x^2 - 10$, because $F'(x) = 2x = f(x)$ in each case.

Integration of Basic Functions

**1)** $\int 0dx = C$

**2)** $\int dx = x + C$ @ $\int 1\,dx = x + C$

**3)** $\int kdx = kx + C$, k is a constant / coefficient

Example 2: Find each integral.

a) $\int 5dx$

b) $\int e^4 dx$

c) $\int \ln 1\, dx$

**Power Rule for Integrals**

For any rational power $n \neq -1$

**4)** $\int x^n dx = \dfrac{x^{n+1}}{(n+1)} + C$

**5)** $\int kx^n dx = k\int x^n dx = (k)\left(\dfrac{x^{n+1}}{n+1}\right) + C = \dfrac{k}{n+1}x^{n+1} + C$, k is a coefficient

Example 3: Find each integral.

a) $\int x^2 dx$

b) $\int x^{-4} dx$

c) $\int x^{-\frac{2}{3}} dx$

d) $\int 2x^8 dx$

e) $\int -\dfrac{1}{5x^3} dx$

f) $\int \dfrac{1}{8\sqrt[3]{x}} dx$

**6)** $\int \frac{1}{x}dx = \ln|x| + C$, when $x \neq 0$

**7)** $\int \frac{k}{x}dx = k\ln|x| + C$, when $x \neq 0$

**8)** $\int e^x dx = e^x + C$

**9)** $\int ke^x dx = ke^x + C$

**10)** $\int \sin x dx = -\cos x + C$

**11)** $\int \cos x dx = \sin x + C$

Example 4: Find each integral.

a) $\int \frac{3}{x}dx$

b) $\int \frac{1}{2x}dx$

c) $\int -7e^x dx$

d) $\int \frac{1}{3}e^x dx$

e) $\int 7\sin x dx$

f) $\int \frac{1}{5}\cos x dx$

**An Indefinite Integral of a sum/difference**

**12)** $\int [af(x) \pm bg(x)]dx = a\int f(x)dx \pm b\int g(x)dx$, a and b are coefficients

<u>Example 5:</u> Find each integral.

a) $\int (2x-5)dx$

b) $\int \left(2x^3 - 3x^{-2} + 5x^{-\frac{2}{3}}\right)dx$

c) $\int x(3x^2+2)dx$

d) $\int (5x^3 + x^2 + x)^2\, dx$

e) $\int (x+3)(-x^4+2x^2)dx$

The Definite Integral

**13)** $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$

Example 6: Evaluate.

a) $\int_{-3}^{2} x^3 dx$

b) $\int_{0}^{3} (3x - x^2) dx$

c) $\int_{0}^{1} (e^x + \sqrt{x}) dx$

**On Your Own**

1. Find each integral

a) $\int \frac{2}{3}x^3 dx$

b) $\int x^{\frac{1}{2}} dx$

c) $\int 5dx$

d) $\int 12e^x dx$

e) $\int \sqrt[3]{x} dx$

f) $\int \frac{1}{\sqrt[3]{x^2}} dx$

g) $\int \left(2t^3 - 3t^2 + 5\right) dt$

h) $\int \left(5x^{-2} + x^{-1/2} + x^{5/3}\right) dx$

2. Evaluate the definite integral

a) $\int_2^5 (3x+7) dx$

b) $\int_0^1 (e^x + \sqrt{x}) dx$

c) $\int_1^5 \left(8x^3 + 4x^2 + 2x - 1\right) dx$

d) $\int_{-3}^3 \left(4 - x^2\right) dx$

Integration of Composite Function,

For any rational power $n \neq -1$

**14)** $\int (ax+b)^n dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$

**15)** $\int e^{kx} dx = \dfrac{1}{k} e^{kx} + C$, for any nonzero constant k

**16)** $\int \dfrac{1}{ax+b} dx = \dfrac{1}{a} \ln|ax+b| + c$

**17)** $\int \sin ax \, dx = -\frac{1}{a} \cos ax + c$

**18)** $\int \cos ax \, dx = \frac{1}{a} \sin ax + c$

Example 7: Find each integral.

a) $\int (2x+1)^2 dx$

b) $\int 3(x+1)^{-4} dx$

c) $\int 2 \cos 5x \, dx$

d) $\int \sqrt{(3x-1)} \, dx$

e) $\int 4e^{3x} dx$

f) $\int \dfrac{1}{e^{3x}} dx$

Integration by substitution

The method of substitution is the integration analogue of the chain rule.

$$\int f(g(x))g'(x)dx = \int f(m)dm \text{ where } m = g(x)$$

This formula is called integration by substitution because it can be obtained formally by substituting $m = g(x)$ and $dm = g'(x)dx$ in the integral on the left.

**Guideline**

1. Make a choice for $m$, say $m = g(x)$
2. Compute $\dfrac{dm}{dx} = g'(x)$
3. Make the substitution $m = g(x)$, $dm = g'(x)dx$
   At this stage, the entire integral must be in terms of $m$, no $x$'s should remain. If this is not the case, try a different choice of $m$.
4. Evaluate the resulting integral.
5. Replace $m$ by $g(x)$, so that the final answer is in terms of $x$.

Example 8:

Find $\int (x^2 + 5)^{12} x\, dx$

Solution 8:

Let $m = x^2 + 5$, then $\dfrac{dm}{dx} = 2x$, which implies that $\dfrac{dm}{2} = xdx$.
Thus, the given integral can be written as:

$$\int (x^2 + 5)^{12} xdx = \int m^{12}\left(\frac{dm}{2}\right)$$

$$= \int \frac{1}{2} m^{12} dm$$

$$= \frac{1}{2}\left(\frac{m^{13}}{13}\right) + C$$

$$= \frac{1}{26}(x^2 + 5)^{13} + C$$

replace m to $x^2 + 5$

Example 9:

$$\int x^2 \sqrt{x-1}\, dx$$

Example 10:

$$\int_2^5 4\left(x^2 + 3x\right)^3 (2x+3)\, dx$$

**On Your Own**

1. Find each integral

a) $\int (5x+7)^7 dx$

b) $\int (5-3x)^{\frac{1}{3}} dx$

c) $\int \frac{1}{(4x+6)^4} dx$

d) $\int \frac{3}{4(2-x)^3} dx$

e) $\int 8e^{\frac{1}{4}} dx$

f) $\int \frac{1}{e^{4x}} dx$

g) $\int (5x^2+7)^{13}(10x)dx$

h) $\int (t^2+4t+6)^{11}(t+2)dt$

2. Evaluate the definite integral

a) $\int_2^5 (3x+7)^4 dx$

b) $\int_0^1 e^{-3x} dx$

c) $\int_1^4 2x(x^2+3)^3 dx$

d) $\int_0^3 4x^2(x^3-19)^{\frac{2}{3}} dx$

e) $\int_7^{26} (x+3)(x+1)^{\frac{1}{3}} dx$

Integration by parts

The methods of by parts is the integration analogue of the product rule.

Integration by parts formula for ***indefinite integral***

If $u = f(x)$ and $v = g(x)$ and if $f'(x)$ and $g'(x)$ are continuous, then

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Integration by parts formula for ***definite integral***

If $u = f(x)$ and $v = g(x)$ and if $f'(x)$ and $g'(x)$ are differentiable on $[a,b]$, then

$$\boxed{\int_a^b u\,dv = \left[uv - \int v\,du\right]_a^b}$$

To use integration by parts successfully, the choice of *u* and *dv* must be made so that the new integral is easier than the original.

For this choice of *u* and *dv*, the new integral is actually more complicated than the original. In the general there are no hard and fast rules for choosing *u* and *dv*, it is mainly a matter of experience that comes from lots of practice.

For the case in which the integrand is the product of different types of functions, an interesting mnemonic device was suggested by Herbert Kasube. The Author suggests the use of the acronym LIATE, which is short for **l**ogarithmic, **i**nverse trigonometric, **a**lgebraic, **t**rigonometric and **e**xponential. According to the author, when the integrand of an integration by parts problem consists of the product of two different types of function, we should let *u* designate the function that appears first in LIATE, and let *dv* denote the rest.

Example 11:

Find $\int xe^{x}dx$

Solution 11:

In this case the integrand is the product of the algebraic function $x$ with the exponential function $e^{x}$. According to the LIATE we should let

$$\begin{array}{lll} u = x & & dv = e^{x}\,dx \\ \dfrac{du}{dx} = 1 & \text{and} & \int dv = \int e^{x}dx \text{ so that } du = dx \text{ and } v = e^{x}. \\ du = dx & & v = e^{x} \end{array}$$

Thus, $\int x e^{x} dx = uv - \int vdu = xe^{x} - \int e^{x}dx = xe^{x} - e^{x} + c$

Example 12:

Find $\int \ln x\,dx$

Solution 12:

One choice is to let $u = \ln x$ and $dv = dx$, so that $du = \dfrac{1}{x}dx$ and $v = x$

Thus, $\int \ln x\,dx = x\ln x - \int x\left(\dfrac{1}{x}\right)dx = x\ln x - \int dx = x\ln x - x + C$

Example 13:

Evaluate $\int_{1}^{2} x\ln x dx$

**On Your Own**

1. Find each integral

a) $\int x^2 \ln x \, dx$

b) $\int (x-1) \ln x dx$

c) $xe^{3x} \, dx$

2. Evaluate the definite integral

a) $\int_0^1 xe^{-x} dx$

b) $\int_1^2 x^3 \ln x \, dx$

c) $\int_0^5 \ln(x+1) dx$

Integration of rational functions

A rational function is basically a division of two polynomial functions. That is, it is a polynomial divided by another polynomial. In formal notation, a rational function would be symbolized like this: $f(x)=\dfrac{g(x)}{h(x)}$.

**CASE 1**

Using the basic formula of integral:

Example 16: Find the integral.

a) $\displaystyle\int \frac{x^5 - x^{-1}}{x^2}\,dx$

b) $\displaystyle\int \frac{3}{2x}dx$

c) $\displaystyle\int \frac{3}{2x+1}\,dx$

**CASE 2**

The Derivative Of The Log Of An Absolute Value ,

Consider the chain rule : $\frac{d}{dx}\left[\ln\left|f(x)\right|\right]=\left[\frac{1}{f(x)}\right]\left[f'(x)\right]=\frac{f'(x)}{f(x)}$

This proves the following integration rules

**28)** $\int\frac{f'(x)}{f(x)}dx=\ln\left|f(x)\right|+c$

<u>Example 17:</u> Find the integral

a) $\int\frac{3x^2}{x^3+2}dx$

b) $\int\frac{2x+5}{x^2+5x}dx$

c) $\int\frac{6x}{3x^2+2}dx$

d) $\int\frac{3x^2}{x^3-1}dx$

**CASE 3**

Using substitution method

Example 18: Find the integral

a) $\int \frac{x^2}{\left(4x^3+9\right)^{12}}\,dx$

b) $\int_2^4 \frac{6x^2}{x^3-1}\,dx$

**On Your Own**

1. Find each integral

a) $\int \frac{1}{4x^2}\,dx$

b) $\int \frac{x^2+1}{x^2}\,dx$

c) $\int \frac{t+2t^2}{\sqrt{t}}\,dt$

d) $\int \frac{x^2}{\left(1+x^3\right)^2}\,dx$

e) $\int \frac{6x}{\left(1+x^2\right)}\,dx$

f) $\int \frac{x}{\sqrt{1+2x^2}}\,dx$

2. Evaluate the definite integral

a) $\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx$

b) $\int_0^2 \frac{2x}{1+x^2}\,dx$

## TUTORIAL 1

1. Find the integral

a) $\int x^8 dx$

b) $\int x^{\frac{5}{7}} dx$

c) $\int \sqrt[3]{x^2}\, dx$

d) $\int \frac{1}{x^6} dx$

e) $\int x^3 \sqrt{x}\, dx$

f) $\int x^{\frac{-7}{8}} dx$

g) $\int \frac{1}{2x^3} dx$

h) $\int \left( x^{\frac{2}{3}} - 4x^{\frac{-1}{5}} + 4 \right) dx$

i) $\int \left( u^3 - 2u + 7 \right) du$

j) $\int \left( x^{-3} + \sqrt{x} - 3x^{\frac{1}{4}} + x^2 \right) dx$

k) $\int \left( \frac{7}{y^{\frac{3}{4}}} - \sqrt[3]{y} + 4\sqrt{y} \right) dy$

l) $\int x\left(1 + x^3\right) dx$

m) $\int \left(1 + x^2\right)\left(2 - x\right) dx$

n) $\int \left[ \frac{2}{x} + 3e^x \right] dx$

o) $\int \left[ \frac{1}{2t} - \sqrt{2}e^t \right] dt$

p) $\int x^2 \sqrt{x^3}\, dx$

q) $\int \left[ 4\sin x + 2\cos x \right] dx$

**Ans**:

1a) $\frac{1}{9}x^9 + C$ 1b) $\frac{7}{12}x^{\frac{12}{7}} + C$ 1c) $\frac{3}{5}x^{\frac{5}{3}} + C$ 1d) $-\frac{1}{5x^5} + C$ 1e) $\frac{2}{9}x^{\frac{9}{2}} + C$

1f) $8x^{\frac{1}{8}} + C$ 1g) $-\frac{1}{4x^2} + C$ 1h) $\frac{3}{5}x^{\frac{5}{3}} - 5x^{\frac{4}{5}} + 4x + C$ 1i) $\frac{1}{4}u^4 - u^2 + 7u + C$

1j) $-\frac{1}{2x^2} + \frac{2}{3}x^{\frac{3}{2}} - \frac{12}{5}x^{\frac{5}{4}} + \frac{1}{3}x^3 + C$ 1k) $28y^{\frac{1}{4}} - \frac{3}{4}y^{\frac{4}{3}} + \frac{8}{3}y^{\frac{3}{2}} + c$

1l) $\frac{1}{2}x^2 + \frac{1}{5}x^5 + C$ 1m) $2x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 + C$ 1n) $2\ln x + 3e^x + C$

1o) $\frac{1}{2}\ln t - \sqrt{2}e^t + C$ 1p) $\frac{2}{9}x^{\frac{9}{2}} + C$ 1q) $-4\cos x + 2\sin x + C$

# TUTORIAL 2

1. Find the integral

a) $\int_{2}^{3} x^2 dx$ b) $\int_{-2}^{3} x^3 dx$

c) $\int_{1}^{4} \sqrt{x} dx$ d) $\int_{0}^{2} e^x dx$

e) $\int_{1}^{32} x^{\frac{-2}{5}} dx$ f) $\int_{4}^{9} x^{\frac{3}{2}} dx$

g) $\int_{3}^{4} \left(\sqrt{x} - 3x^2\right) dx$ h) $\int_{1}^{2} \left(x^2 + 2x + \frac{5}{x} - \sqrt{x} + 8\right) dx$

i) $\int_{0}^{1} x^3 \left(1 + x^4\right) dx$ j) $\int_{-1}^{2} x\left(1 + x^4\right) dx$

**Ans**:

1a) $\frac{19}{3}$ 1b) $\frac{65}{4}$ 1c) $\frac{14}{3}$ 1d) $6.3891$ 1e) $\frac{35}{3}$

1f) $\frac{422}{5}$ 1g) $-35.1308$ 1h) $15.5801$ 1i) $\frac{3}{8}$ 1j) $12$

# TUTORIAL 3

1. Find the integrals by using appropriate substitution:

a) $\int x(x^2-1)^{99}\,dx$

b) $\int x^3\sqrt{2+x^4}\,dx$

c) $\int x\sqrt{1-x^2}\,dx$

d) $\int 4x\ \sqrt{x^2+3}\,dx$

e) $\int (x^4+x)^5(4x^3+1)\,dx$

f) $\int 6x(3x^2+1)^{-6}\,dx$

g) $\int 2x(4-5x^2)^3\,dx$

h) $\int \frac{2}{3}x^4(1-x^5)^3\,dx$

i) $\int x^3(1+x^4)\,dx$

j) $\int \frac{1}{x^2}\left(1+\frac{1}{x}\right)^3 dx$

k) $\int x^3\cos(x^4+2)\,dx$

l) $\int \sin(2x+3)\,dx$

m) $\int x\sin(2x^2)\,dx$

n) $\int \frac{1}{(2x+1)^3}\,dx$

o) $\int \frac{x+1}{x^2+2x}\,dx$

p) $\int \frac{2x+6}{(x^2+6x)^2}\,dx$

q) $\int x^2 e^{3x^3}\,dx$

r) $\int \frac{x}{e^{x^2}}\,dx$

2. Find the integrals by using

i) formula: $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c$.

ii) appropriate substitution.

a) $\int (2x-1)^{100}\,dx$

b) $\int (x-2)^{-3}\,dx$

c) $\int \left(\frac{1}{3}x-8\right)^{-5} dx$

d) $\int \frac{1}{4}(1+3x)^3\,dx$

e) $\int \sqrt{1-2x}\,dx$

f) $\int \sqrt[3]{5x+7}\,dx$

g) $\int (3-5y)^4\,dy$

h) $\int (1-x)^9\,dx$

i) $\int \dfrac{2}{4x+2}\,dx$

j) $\int \dfrac{1}{7-3x}\,dx$

3. Evaluate

a) $\int_{-1}^{0} 3x^4(1-x^5)^3\,dx$

b) $\int_{-1}^{0} 5x^2(x^3+1)^{10}\,dx$

c) $\int_{-1}^{1} (1+x)^{\frac{1}{2}}\,dx$

d) $\int_{0}^{2} x(1+3x^2)^4\,dx$

e) $\int_{2}^{3} \dfrac{3x^2-1}{(x^3-x)^2}\,dx$

f) $\int_{0}^{3} \dfrac{1}{2x+3}\,dx$

g) $\int_{0}^{4} \dfrac{1}{(x-2)}\,dx$

**Ans**:

1a) $\frac{1}{200}\left(x^2-1\right)^{100}+C$ 1b) $\frac{1}{6}\left(2+x^4\right)^{\frac{3}{2}}+C$ 1c) $-\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}+C$

1d) $\frac{4}{3}\left(x^2+3\right)^{\frac{3}{2}}+C$ 1e) $\frac{1}{6}\left(x^4+x\right)^6+C$ 1f) $-\frac{1}{5}\left(3x^2+1\right)^{-5}+C$

1g) $-\frac{1}{20}\left(4-5x^2\right)^4+C$ 1h) $-\frac{1}{30}\left(1-x^5\right)^4+C$ 1i) $\frac{1}{8}\left(1+x^4\right)^2+C$

1j) $-\frac{1}{4}\left(1+\frac{1}{x}\right)^4+C$ 1k) $\frac{1}{4}\sin\left(x^4+2\right)+C$ 1l) $-\frac{1}{2}\cos\left(2x+3\right)+C$

1m) $-\frac{1}{4}\cos\left(2x^2\right)+C$ 1n) $-\frac{1}{4\left(2x+1\right)^2}+C$ 1o) $\frac{1}{2}\ln\left|x^2+2x\right|+C$

1p) $-\frac{1}{\left(x^2+6x\right)}+C$ 1q) $\frac{1}{9}e^{3x^3}+C$ 1r) $-\frac{1}{2}e^{-x^2}+C$

2a) $\frac{1}{202}\left(2x-1\right)^{101}+C$ 2b) $-\frac{1}{2}\left(x-2\right)^{-2}+C$ 2c) $-\frac{3}{4}\left(\frac{1}{3}x-8\right)^{-4}+C$

2d) $\frac{1}{48}\left(1+3x\right)^4+C$ 2e) $-\frac{1}{3}\left(1-2x\right)^{\frac{3}{2}}+C$ 2f) $\frac{3}{20}\left(5x+7\right)^{\frac{4}{3}}+C$

2g) $-\frac{1}{25}\left(3-5y\right)^5+C$ 2h) $-\frac{1}{10}\left(1-x\right)^{10}+C$ 2i) $\frac{1}{2}\ln\left|4x+2\right|+C$

2j) $-\frac{1}{3}\ln\left|7-3x\right|+C$

3a) $\frac{9}{4}$ 3b) $\frac{5}{33}$ 3c) 1.8856 3d) $\frac{61882}{5}$ 3e) $\frac{1}{8}$ 3f) 0.5493 3g) 0

## TUTORIAL 4

1. Find the integral using integration by parts:

a) $\int 12xe^{3x}dx$

b) $\int x^2 \ln x\,dx$

c) $\int \ln(x+1)dx$

d) $\int xe^{4x}dx$

e) $\int_0^{\frac{1}{2}} 9xe^{-2x}dx$

f) $\int_1^2 \ln 3x dx$

g) $\int_e^{e^2} \sqrt{x}\ln x dx$

h) $\int_0^1 xe^x dx$

**Ans**:

1a) $4xe^{3x} - \frac{4}{3}e^{3x} + C$

1b) $\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

1c) $x\ln|x+1| - (x+1) + \ln|x+1| + C$

1d) $\frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C$

1e) 0.5945 1f) 1.4849 1g) 16.8579 1h) 1