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ipartzix

LINEAR ALGEBRA

Vector

Vector Space

Internal Composition

Let V be any set then the mapping $f: V\times V \to V$ is ~~sal~~ said to be internal Composition and also it called vector Addition.

Ex:- $f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

$$f(a,b) = a\cdot b$$

External Composition

Let V and F be two non-empty set. Then the mapping $f: V\times F \to V$ is said to be external Composition in V over F also called Scalar multiplication.

Vector Space.

Let $(F, +, \cdot)$ be a field. The element of F will be called Scalars. Let V be a non-empty set whose elements will called vectors. then V is a Vector Space Over the field F. if

(i) There is defined an internal Composition in V called addition of vectors and denoted by '+', Also for this Composition $\boxed{\text{V is an Abelian Group}}$

ⓐ $\alpha+\beta \in V$ for all $\alpha,\beta \in V$

ⓑ $\alpha+\beta = \beta+\alpha$ for all $\alpha,\beta \in V$

ⓒ $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$ for all $\alpha,\beta,\gamma \in V$

ⓓ $\exists$ an element $0 \in V$ such that $\alpha+0=\alpha$ for all $\alpha \in V$.

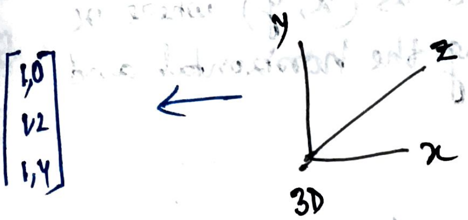
This element $0\in V$ will be called the zero vector.

ⓔ To every vector $\alpha \in V$, there exists a vector $-\alpha \in V$ such that $\alpha+(-\alpha)=0$

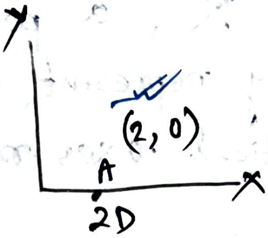
Real Number and Vector Space :

$\mathbb{R}$ the set of all real numbers.

$\mathbb{R}^2, \mathbb{R}^3, \ldots \mathbb{R}^n$ - Represents n-dimentional Euclidean spaces. For, instance, $\mathbb{R}^2$ is the 2D plane, while $\mathbb{R}^3$ is the 3D space we are familiar with



$\boxed{(1,0)}$



Norms (Refreshment for ~~exam~~ Trigonometry).

vector signature

The Norm of a vector, denoted as $|\vec{v}|$ or $\|\vec{v}\|$, represented the vector magnitude (Length)

- For a vector $\vec{v} \in \mathbb{R}^n$, its norm is calculated as

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots v_n^2}$$

$\|\vec{v}\| \rightarrow L_2$ norm

Norms provide

$\boxed{L_1 \text{ Norm}}$

$$\|x\|_1 = \sum |x_i|$$

$\boxed{L_2 \text{ Norm}}$

$$\|x\|_2 = \sqrt{\sum x_i^2}$$

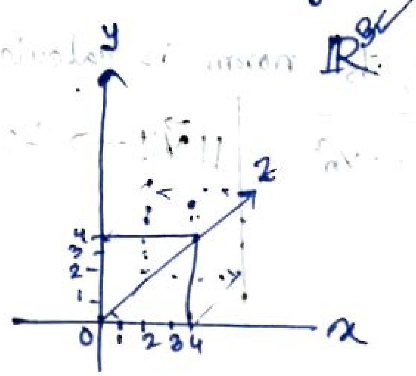
(*) Norm need for Regularization.

The Cartesian Coordinate System

The Cartesian Coordinate System is a framework for specifing points in a plane or space using ordered lists of number.

- In $\mathbb{R}^2$, a point is denoted as (x, y) where x and y represent distances along the horizontal and vertical axes respectively.
- In $\mathbb{R}^3$, points are denoted as (x, y, z), extending to three dimensions (3D).

This system is fundamental for visulizing and working with vectors geomatrically.



Angel. Angles and Their Measurement

Angles can be measured in degrees or radians.

$360°$ in a circle equals 2π radians.

Ex:- $180° = \pi$ radians.

* Trigonometric Identities & Formula

$\frac{\pi}{2}$, π, 2π, $\frac{\pi}{4}$ etcs

- $\boxed{\text{Pythagorean Identity} = \sin^2\theta + \cos^2\theta = 1}$
- Sum and Difference Formulas : ~~Sin(2θ)~~ $\sin(a \pm b)$; $\cos(a \pm b)$
- Double-Angle formula : $\sin(2\theta)$, $\cos(2\theta)$

Solving Basic Trigonometric Equn :

It can be often be solved by using identities or by referring to the unit Circle.

Ex:- $\sin\theta = \frac{1}{2}$

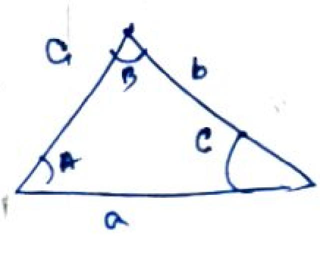
sol:- $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

Low of Sines and Cosines

The Law of Sines and Cosines are used to solve for unknown in non-right triangles.

Low of Sines : $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$

Low of Cosines : $c^2 = a^2 + b^2 - 2ab\cos(C)$



Norm of a Vector

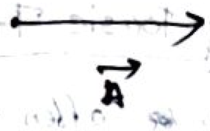
Defination

The norm of a vector $\vec{V}$ in $\mathbb{R}^n$, denoted as $\|\vec{V}\|$, is defined as the square root of the squares of its Components, which represent its magnitude or length.

$$\|\vec{V}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 \cdots v_n^2}$$

Ex: $\vec{V} = \begin{array}{cc} \begin{bmatrix} 3 \\ 4 \end{bmatrix} & \begin{array}{l} x \to v_1 \\ y \to v_2 \end{array} \end{array}$

Only one vector



, $\|\vec{V}\| = \sqrt{3^2 + 4^2} = 5$

Ed:

Euclidean Distance Between Two point

A and B in $\mathbb{R}^n$ is the norm of the vector Connecting A to B, representing the "Straight-line" distance between them;

$$d(A, B) = \sqrt{(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2}$$

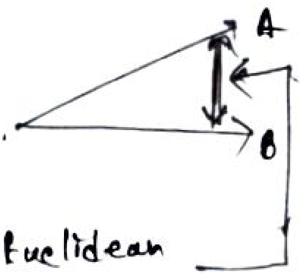
Ex: Distance between A(1, 2) and B(4, 6) is

$$d(A, B) = \sqrt{(4-1)^2 + (6-2)^2}$$

$$= \sqrt{9 + 16}$$

$$= \sqrt{25} = 5$$

Two vector



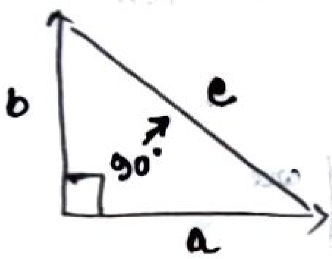
Euclidean Distance

The Pythagorean Theorem

In a right-Angled triangle the square of the length of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the lengths of the other two sides.

$$c^2 = a^2 + b^2$$

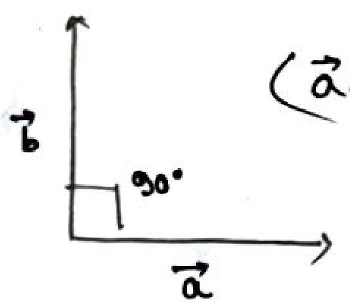
The theorem is a fundamental principle for defining norms and distances in Euclidean Spaces.



Orthogonality Defined

Two vector $\vec{a}, \vec{b}$ in R^n are Orthogonal to each other if their dot product is zero. $\vec{a} \cdot \vec{b} = 0$

- Orthogonality ~~De implement~~ implies the vectors from a right angle with each other in $\mathbb{R}^2$ and $\mathbb{R}^3$.



$(\vec{a} \cdot \vec{b} = 0 = 90°$

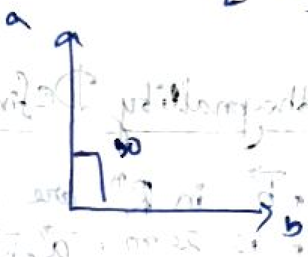
- The concept is visually represented by vectors $\vec{a}$ and $\vec{b}$ being perpendicular in the 2D Coordinate system.

Application of Orthogonality

It plays a crucial role in various aspects of linear algebra.

- It's fundamental in defining vector spaces, subspaces and in solving system of linear equations.
- Orthogonal vectors are used in finding the shortest distance from a point to a plane understanding, projections, and in the Gram-Schmidt process for orthogonalization.

Ex:- the vectors $\vec{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ are orthogonal since



$$2(-3) + 3(2) = 0$$

Fundamentals of Linear Algebra - Vector

Scalar:

A scalar is a single numerical value, often representing magnitude or quantity.

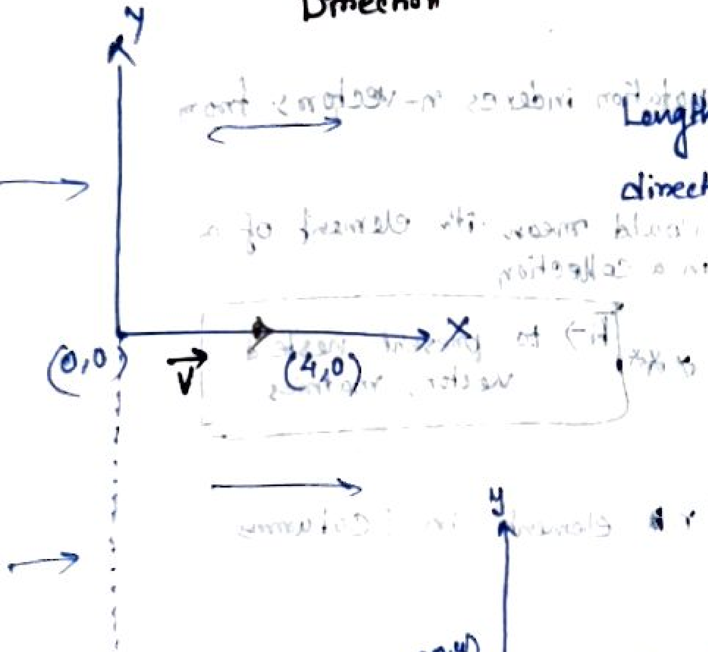
S = 22 (°) ⟶ 22° / 18° / 42° / 323°

Vector:

A vector is an ~~array~~ ordered array of numbers, which can represent both magnitude and direction in space.

ex:- A bird flies $\boxed{\text{south}}$ (10km/h) ⟶ Magnitude

Direction

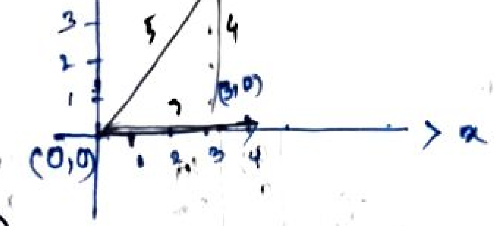


Length = 4 = magnitude

direction ⟶

$\vec{V}$ (0,0) ⟶ (4,0)

$\vec{W}$ (0,0) ⟶ (3,4)

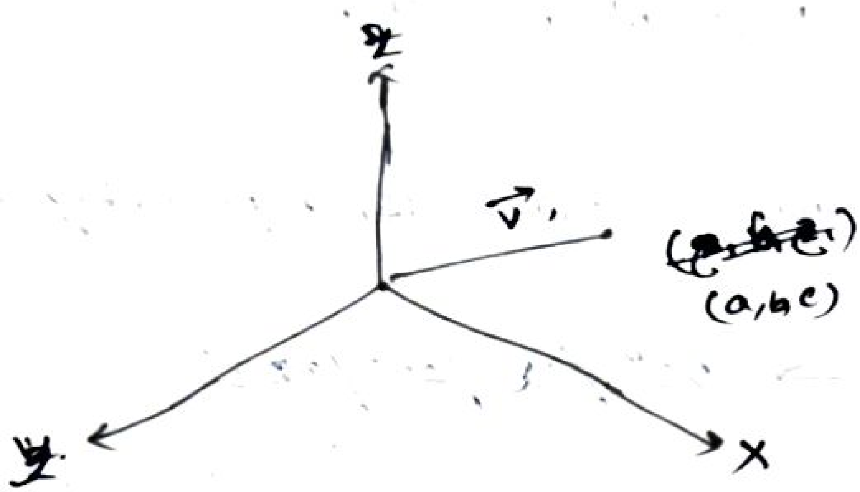


For representing 2D use [ ]

$\vec{V} = (4,0)$ $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$\vec{W} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\mathbb{R}^3$



$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Indexing in Vector

- Standard mathematical notation indexes n-vectors from $i = 1$, to $i = n$.
- Ambiguity in notation; a_i could mean ith element of a vector to the ith vector in a collection

$\boxed{\vec{A} \rightarrow \text{to present nested vector, matrices}}$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{bmatrix}$$

ith element in 1 columns

$$\vec{A} = \begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \\ \vdots \\ \vec{a_n} \end{bmatrix} \quad m \quad n \times m$$

$$\vec{a_3} = \begin{array}{c l} \begin{bmatrix} a_3 \\ a_3 \\ a_3 \\ \vdots \\ a_3 \end{bmatrix} & \begin{array}{l} \text{1st} \\ \text{2nd} \\ \text{3rd} \\ \vdots \\ \text{mth} \end{array} \end{array} \quad m \times 1$$

$a_1 = 1$

$a_2 = 2$

$a_3 = 3$

$\vdots$

$a_n = 10$

Module-2 Special Vectors and Operations

Zero Vector : $\vec{0}$: A vector with all element equal to zero.

Ex:-

$$\vec{0}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^3.$$

Unit Vector:- vector with a single element equal to ~~the~~ one and all other zero, denoted as e_i for ith unit vector in n dimensions.

Ex:-

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ in } \mathbb{R}^3$$

$\mathbb{R}^n$

$$e_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}$$

$i=5$

Sparsity in Vector

A sparse vector is characterized by having many of its ~~en~~ entries as zero. Its sparsity pattern indicates the positions of non zero entries.

Consider a vector $\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}$ in $\mathbb{R}^5$, Here $\vec{x}$ is sparse becouse majority of its element are zero.

Vector in Higer Dimensions.

we can visualized 2D, 3D linear Algebra allows us to work with vector in any number of Dimensions.

Application of Vectors - Word Count Vector

Ex :- Vectors can represent the frequency of words in document for example (25, 2, 0) indicates word counts in a 3-word dictionary.

$$\begin{array}{cl} \begin{bmatrix} 25 \\ 2 \\ 0 \end{bmatrix} & \begin{array}{l} 25 \times \text{word 1} \\ 2 \times \text{word 2} \\ 0 \times \text{word 3} \end{array} \end{array}$$

$$\begin{array}{cl} \begin{bmatrix} 3 \\ 2 \\ 10 \\ 0 \\ 4 \\ 2 \end{bmatrix} & \begin{array}{l} \text{I} \\ \text{reading} \\ \text{a} \longrightarrow \text{Stop} \\ \text{showers} \\ \text{books} \\ \text{Library} \end{array} \end{array}$$

Module 1: Foundations of Linear System and Matrices

Introduction to Linear System

Linear System from the ~~back bon~~ bedrock of linear Algebra, modeling vast array of problems. Thanks to ~~advance~~ ackencements in Computing we can solve even the most colossal system with remakable speed

A General Linear System

A linear ~~equn~~ System with m equn and n unknowns taken the from.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \quad \vdots \qquad \vdots \qquad\qquad \vdots \quad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Here, coefficients a_{ij} are known and x_j are the variables we want to determine.

Coefficient Labeling

The coefficient labeling a_{ij} in a linear system are labeled where the first index represents the row and and the second index dentoes the Column. This systematic Labeling is essential for structuring and Solving the linear equations

a_{ij} → row

→ column

Ex:-

The vectors $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent because there is noway to add these vectors together, with any Scalar multiples to equal the zero vector unless the scalar are zero.

Length of a Vector and Dot product

The length of a vector is deeply related to the dot product

- The dot product of a vector $\vec{v}$ with itself gives the square of the length of $\vec{v}$.

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$$

- In 2D space for $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. the dot product and length are related by:-

$$\vec{v} \cdot \vec{v} = x^2 + y^2 = \|\vec{v}\|^2$$

This relation is a restatement of the pythogorean theorem via dot product.

Linear Independence (4.48) (4.32)

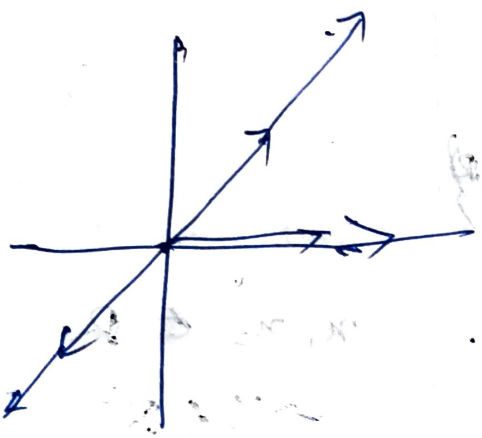
A set of ~~vet~~ vectors in linearly Independent if no vectors in the set can be written as Linear Combination of the Others. Otherwise, they are linearly dependent

Vectors $\vec{V_1}, \vec{V_2} \dots \vec{V_n}$ are linearly Independent if and only if the only solution to the equation $c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_n\vec{v_n} = \vec{0}$ is $c_1 = c_2 = \dots c_n = 0$.

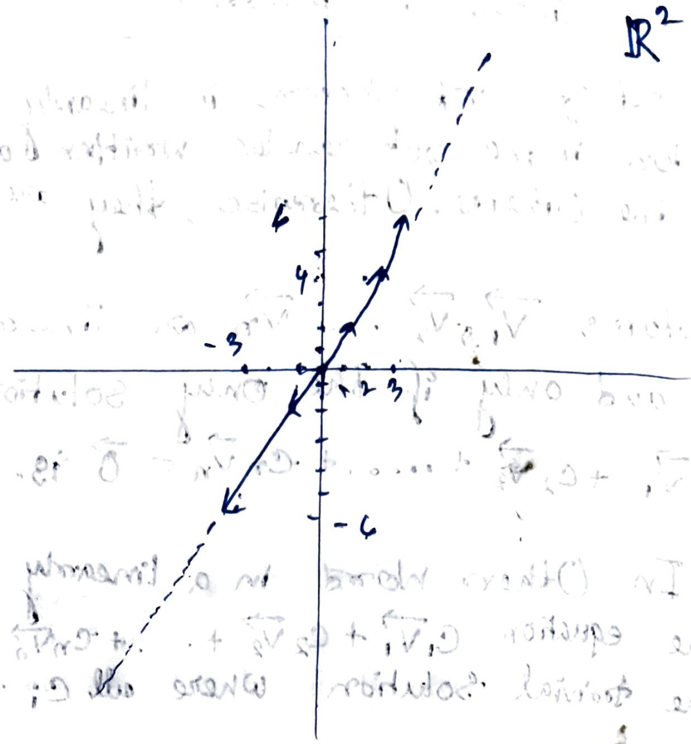
- In Others Word in a linearly Independent set, the equation $c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_n\vec{v_n} = \vec{0}$ has only the trivial Solution where all c_i are zero.

$\vec{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$

$\vec{a} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$



$\vec{a} = \begin{bmatrix}1\\2\end{bmatrix}$ Span$(\vec{a})$



$c \cdot \vec{a}$

$c = 1$ $\begin{bmatrix}1\\2\end{bmatrix}$

$c = 2 \rightarrow \begin{bmatrix}2\\4\end{bmatrix}$

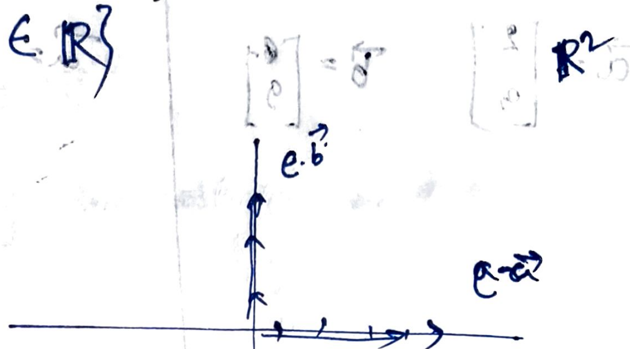
$c = 3 \rightarrow \begin{bmatrix}3\\6\end{bmatrix}$

$c = \cancel{4}\,-1 \rightarrow \begin{bmatrix}-1\\-2\end{bmatrix}$

$c = -3 \rightarrow \begin{bmatrix}-3\\-6\end{bmatrix}$

Span$\cdot(\vec{a}) = \{ c \cdot \vec{a} \mid c \in \mathbb{R} \}$

$\vec{a} = \begin{bmatrix}1\\0\end{bmatrix}$ $\vec{b} = \begin{bmatrix}0\\1\end{bmatrix}$



span$\cdot(\vec{a}, \vec{b}) = ?$

$(c_1 \cdot \vec{a} + c_2 \cdot \vec{b})$

Span$\cdot(\vec{a}, \vec{b} \cdot \mid \vec{a} \perp \vec{b}) = \{ c_1 \vec{a} \cdot c_2 \vec{b} \mid c_1, c_2 \mathbb{R} \}$

$\vec{V_1} = \begin{bmatrix}1\\2\end{bmatrix}$ $\vec{V_2} = \begin{bmatrix}3\\4\end{bmatrix}$ $x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$.

$x_1, x_2 \in \mathbb{R}$

$X \in \mathbb{R}^2$

$c_1, c_2 \in \mathbb{R}$

Span$(\vec{v_1}, \vec{v_2}) = \mathbb{R}^2$

$c_1 \vec{v_1} + c_2 \vec{v_2} \stackrel{?}{=} x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

$= \begin{bmatrix}1\\2\end{bmatrix}$

$= \begin{bmatrix}0\\4\end{bmatrix}$

Span of Vectors

The Span of a set of vectors is the set of all possible linear Combinations of those vectors. If ~~the~~ $V = \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$ is a set of vectors, then the span of V is written as Span (V) and includes any vector that can be expressed as $c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_k\vec{v_k}$ where c_i are scalars.

Example:

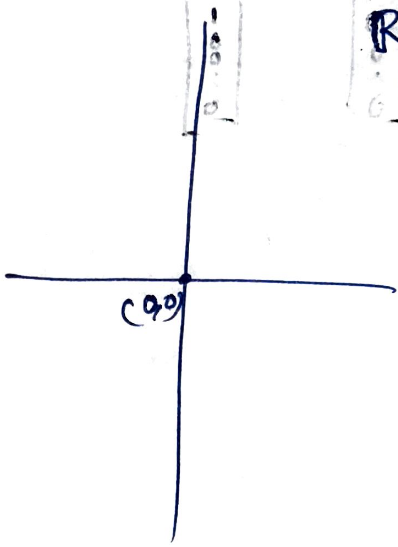
Consider two vectors in $\mathbb{R}^2$, $\vec{v_1} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix}3\\4\end{bmatrix}$ the span of $\{\vec{v_1}, \vec{v_2}\}$ is all of $\mathbb{R}^2$ because any vector in $\mathbb{R}^2$ can be expressed as a linear combination of $\vec{v_1}$ and $\vec{v_2}$

$\vec{a} = \vec{0} - \begin{bmatrix}0\\0\end{bmatrix}$

$c \cdot \vec{a} = \begin{bmatrix}0\times c\\0\times c\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$c=1 \Rightarrow \vec{a}\begin{bmatrix}0\times 1\\0\times 1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$c=n \Rightarrow \vec{a}\begin{bmatrix}0\times n\\0\times n\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$



$\text{span}(\vec{0}) = \vec{0}$

Example.

For $\vec{b} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$ it can be written as

$\mathbb{R}^3$ $\qquad \vec{b} = (-1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$ W

$\vec{b} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$ Can be present as linear Combination like this

$\vec{b} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \mathbb{R}^n$ $\qquad e_1 \; e_2 \; \cdots \; e_n \in \mathbb{R}^n$

$b_1, b_2, b_3 \cdots b_n \in \mathbb{R}^n$

$$\vec{b} = \beta_1 \cdot e_1 + \beta_2 \cdot e_2 + \cdots + \beta_n \cdot e_n$$

$$\vec{b} = b_1 \cdot e_1 + b_2 \cdot e_2 + \text{———} + b_n e_n$$

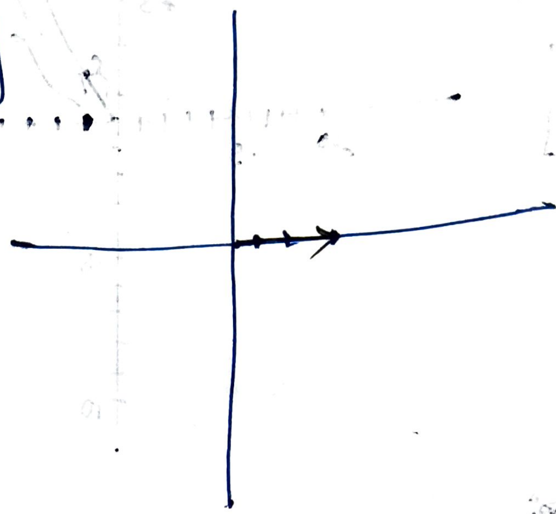
$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Ex

$\beta_1 = 3$, $\beta_2 = -2$

$\vec{a} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{b} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$(\beta_1 \cdot \vec{a} + \beta_2 \cdot \vec{b})$



$$\begin{bmatrix} 1 \times 3 \\ 2 \times 3 \end{bmatrix} + \begin{bmatrix} 0 \times -2 \\ 3 \times -2 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$\beta_1 \vec{a} + \beta_2 \vec{b} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\beta_1, \beta_2 \in \mathbb{R}$ (constant)

$x_1, x_2 \in \mathbb{R}$

$$\beta_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} \beta_1 \cdot 1 \\ \beta_2 \cdot 2 \end{bmatrix} + \begin{bmatrix} \beta_2 \cdot 0 \\ \beta_2 \cdot 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \beta_1 \\ 2\beta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3\beta_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \beta_1 \\ 2\beta_1 + 3\beta_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\beta_1 = x_1$

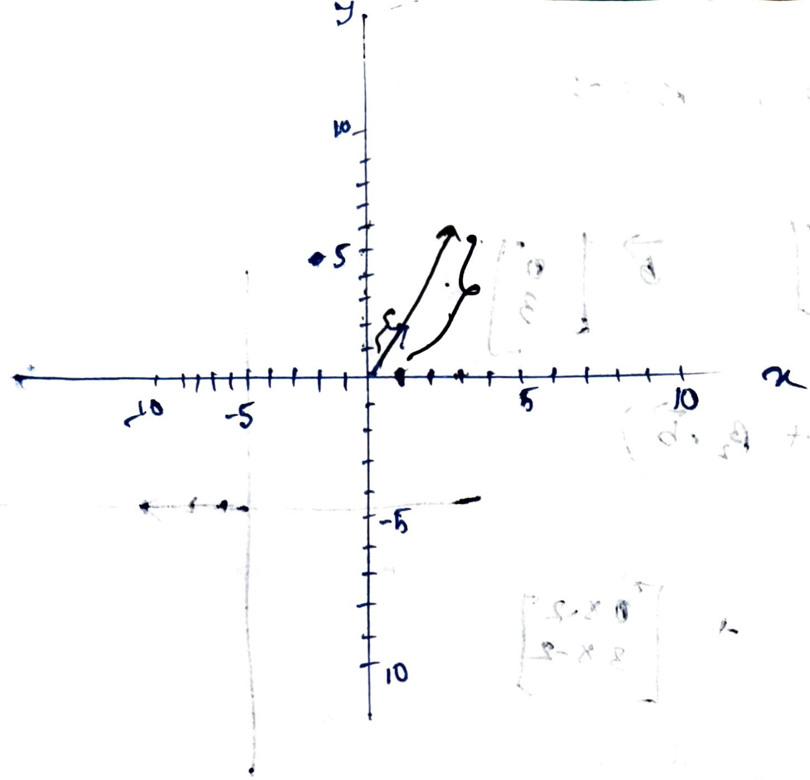
$2\beta_1 + 3\beta_2 = x_2$

$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$K = 3$

$K \cdot \vec{a} : \begin{bmatrix} 1 \times 3 \\ 2 \times 3 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 6 \end{bmatrix}$



Linear Combinations and Unit Vector (3:10:00)

A Linear Combination of vectors $a_1 \ldots\ldots a_m$ using scalars $\beta_1, \ldots\ldots \beta_m$ is the vector $\beta_1 a_1 + \ldots + \beta_m a_m$. The scalars are called the coefficients of the Linear Combination.

- Any vector $\vec{b}$ in n dimensions can be expressed as a Linear Combination of the standard unit vector $e_1 \ldots e_n$
- The coefficients in this combination are the entries of $\vec{b}$ itself.

$a_1 \cdot a_2 \ldots\ldots a_m \in \mathbb{R}^m$ (m = vectors)

$\beta_1 \cdot a_1 + \beta_2 a_2 + \beta_3 a_3 \ldots\ldots \beta_m + a_m$

$\beta_1, \beta_2, \ldots\ldots + \beta_m \in \mathbb{R}$

$\{\beta_1, \beta_2 \ldots \beta_m \in \mathbb{R}\}$ = "coefficient"

Application of Scalar - Vector Multiplication : Audio Scaling

- Scalar - Vector Multiplication in Audio processing can change the volume of an audio signal without altering its content.

- Example: Let $\vec{a}$ represent an audio signal. Multiplying $\vec{a}$ by a scalar β adjusts the ~~value~~ volume.

Example :

If $\beta = \frac{1}{2}$ or $\beta = -\frac{1}{2}$, then $\beta \vec{a}$ is perceived as the same audio signal but at a lower volume. the sign of β affects

$$\beta = \frac{1}{2} = 0.5 \; / \; \beta = -\frac{1}{2} = -0.5$$

$$\vec{a} = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}$$
R^3

$$\beta \cdot \vec{a} = \begin{bmatrix} 3 \times 0.5 \\ 6 \times 0.5 \\ 5 \times 0.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 3 \\ 2.5 \end{bmatrix}$$

$$\beta \cdot \vec{a} = \begin{bmatrix} 3 \times -0.5 \\ 6 \times -0.5 \\ 5 \times -0.5 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -3 \\ -2.5 \end{bmatrix}$$

Properties of Vector Addition

Vector Addition have several important properties:

- **Commatative:** $a+b = b+a$ for any vectors a, and b of the same size.

- **Associative:** $(a+b)+c = a+(b+c)$. we can write both as $a+b+c$.

- **Addition of Zero Vector:** $a+\vec{0} = \vec{0}+a = a$ adding the 0/ zero vector has no effect.

Subtracting a Vector from it self: $a - a = \vec{0}$. this yields the zero vector.

3

Scalar Multiplication

It involves multiplying each component of a vector by a Scalar value, effectively scaling the vector's magnitude.

$\mathbb{R}^n$ $\quad \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$ $\qquad \mathbb{R}$ $\quad c =$ $\qquad c.\vec{a} = \begin{bmatrix} c_1 \vec{a_1} \\ c_1 a_2 \\ \vdots \\ c a_n \end{bmatrix}_{n\times 1}$

Ex:- $\vec{C} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ by Scalar $k = -2$

$\vec{c}.k : \begin{bmatrix} 4\times -2 \\ -3 - 2 \end{bmatrix} : \begin{bmatrix} -8 \\ 6 \end{bmatrix}$

$\vec{C}, 0 = 0$ Always

Vector Operation:

Ex1:- Given Vectors $\vec{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$; then sum is

$$\vec{a} + \vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$\mathbb{R}^3$ ✓

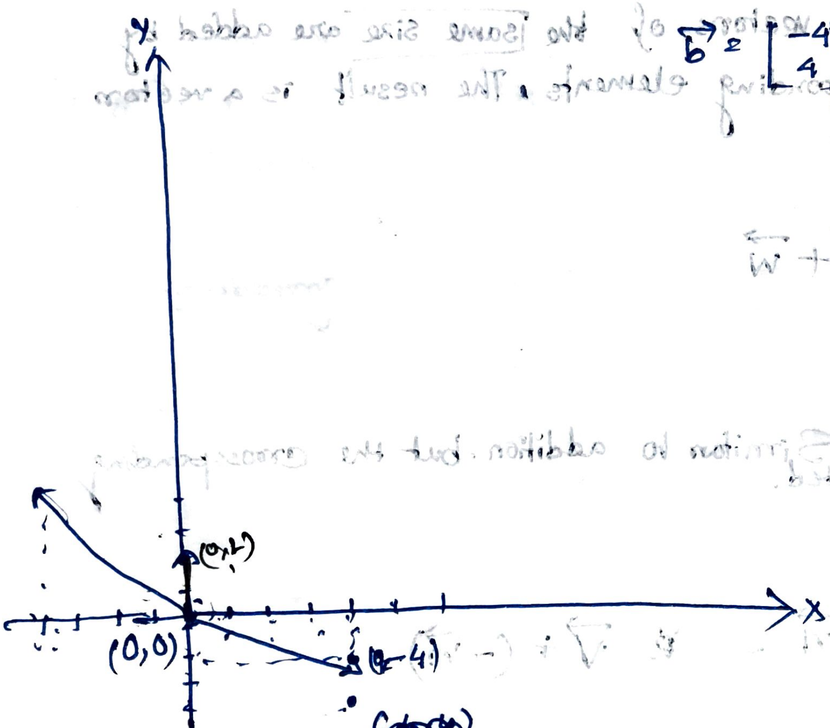
$$\begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}$$

$\mathbb{R}^2$

$$\begin{bmatrix} 1 \\ 9 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$



$$\vec{a} + \vec{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Application of Vectors → Representing Customer Purchase

Ex :- An n-vector p can record a Customer's purchases over time, with p_i being the quantity or Dollar value of item i purchased.

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_i \\ \vdots \\ p_n \end{bmatrix} \rightarrow p_i = \text{value of item } i$$

$$\vec{p} = \begin{array}{c} 1 \rightarrow \\ \\ i \rightarrow \\ \\ n \rightarrow \end{array} \begin{bmatrix} \text{Math course} \\ \vdots \\ \text{NLP course} \\ \vdots \\ \text{DL course} \end{bmatrix} = \begin{array}{c} 1 \rightarrow \\ \\ i \rightarrow \\ \\ n \rightarrow \end{array} \begin{bmatrix} 1000 ₹ \\ \vdots \\ 3000 ₹ \\ \vdots \\ 6000 ₹ \end{bmatrix}$$

$p_i = 3000 ₹$

2.07

Vector Addition and Subtraction

Vector Addition :- Two vectors of the $\boxed{\text{same size}}$ are added by adding their corresponding elements. The result is a vector of the same size.

$$\vec{V} + \vec{W}$$

Vector Subtraction :- Similar to addition, but the crossponding elements are subtracted.

$$\vec{V} - \vec{W} = \vec{V} + (-\vec{W})$$

Homogeneous and Nonhomogeneous Systems.

A system is homogeneous if all constant terms b_i are zero; otherwise, its nonhomogeneous. Identifying this helps determine the nature of the solution set and the strategies for finding solutions.

$a+b=0 \Rightarrow$ homogeneous

$a+b\neq 0 \rightarrow$ nonhomogenous

Matrix

A matrix is a rectangular Array of real numbers arranged in Rows and Columns. For example. A is represented as,

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

→ row

↓ column.

$m\times n$ —

$n\rightarrow$ Column

$m\rightarrow$ row.

$m\times n$: row, column

Dimension $(A)=[m\times n]$

Ex:- 2×3 matrix

row, column

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$$

3×2.

$$A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2}$$

Matrix Structure : Rows, Columns and Dimensions

The rows of a matrix are the horizontal lines of entries while the columns are the vertical lines.

- The Dimensions of a matrix are given by the number of row x columns it has,
- An $m \times n$ matrix has m rows and n columns.

Type of Matrix

Identity Matrix

An Identity Matrix I_n is a square matrix with ones on the diagonal and zeros elsewhere.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3x3

Diagonal Matrix

An Diagonal Matrix is one kind of matrix where all off diagonal element are zero.

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \checkmark \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -7 \end{bmatrix} \checkmark$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -8 \end{bmatrix}$$

Ones Matrix :

An Ones matrix denoted by $1_{m\times n}$, it is a matrix in which all elements are one.

Ex:-

A $_{2\times3}$ ones matrix = $I_{2\times3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times3}$

Zero Matrix

A zero matrix, denoted by ~~Omxm~~ $O_{m\times n}$ is a matrix in which all element are zero.

Ex:- A 2X3 zero matrix

$$O_{2\times3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times3}$$

Core Matrix Operations

Sum :- The sum of two matrix A and B of the same dimentions is obtained by adding corresponding elements.

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

Difference :- The difference of two matrix A and B of the same dimensions is obtained by subtracting their corresponding elements.

$$(A-B)_{ij} = A_{ij} - B_{ij}$$

Ex. A+B

$$A \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \pm B \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} = A+B = \begin{bmatrix} 2 & 2 & 5 \\ 0 & 1 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$

A - B

$$\begin{bmatrix} 0 & -2 & -1 \\ 0 & 1 & 2 \\ -1 & -2 & -2 \end{bmatrix}$$

Scalar Multiplication of Matrix

Scalar Multiplication of a matrix A by a scalar α results in a new matrix where each entry of A is multiplied by α.

$\alpha = 3 \quad A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

$\alpha \cdot A = 3 \times \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 12 & 15 \end{bmatrix}$

Matrix Multipication

The product of an $m \times n$ matrix A and an $n \times p$ matrix B result in an $m \times p$ matrix C, where each entry c_{ij} is computed as the dot product of the ith row of A and the jth column of B.

$A^{m \times n}$ | $B^{n \times p}$ | * number of colums of A should be = to the number of Row of B

m rows, n columns | n rows, p columns

$A \times B = C^{ij}$

$A^{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad B^{n \times p} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & & b_{2p} \\ \vdots & \vdots & & \\ b_{n1} & b_{n2} & & b_{np} \end{bmatrix}$

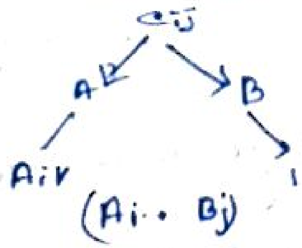
C_{mp}

$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \end{bmatrix}^T \begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \\ b_{nj} \end{bmatrix} \quad \vec{b_j}$

$\boxed{c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} b_{2j} + \cdots}$

Ex 6 **Matrix Multiplication Ex 1**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$



the product

$$AB = \begin{bmatrix} 1\times 2 + 2\times 1 & 1\times 0 + 2\times 2 \\ 3\times 2 + 4\times 1 & 3\times 0 + 4\times 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 2 \\ 4 & 6 \end{bmatrix}$$

The multiplication is

$$CD \begin{bmatrix} 3\times 0 + 5\times 4 & 3\times 2 + 5\times 6 \\ 7\times 0 + 9\times 4 & 7\times 2 + 9\times 6 \end{bmatrix} = \begin{bmatrix} 20 & 36 \\ 36 & 68 \end{bmatrix}$$

Module-4 Solving Linear System with matrix

Algebraic Laws ~~of~~ for Matrices

Just as with real numbers matrices follow certain algebraic Laws - for addition and multiplication (~~exp~~ except for Commutativity of multiplication):

1) Associative law.
2) Distributative law
3) Scalar Multiplication Law.
4) Commutative Law of Addition.

These laws ensure that matrix operation can be manipulated algebraically in predictable ways.

Commutative Law of for Matrix Addition

- Matrix addition is commutative.

$$A + B = B + A \qquad \boxed{Dim(A) = Dim(B)\ m \times n}$$

unlike matrix multiplication, we can add matrices in any order and the result will be same.

Associative Law of Matrices

The associative property holds for both matrix addition and multiplication.

For Matrix Addition

$$\boxed{(A+B)+C = A+(B+C)}$$

For Matrix Multiplication

$$\boxed{(AB)C = A(BC)}$$

This property allows us to add or multiply matrices without worrying about the grouping of terms.

Distributive Law of Matrix

Matrix addition and multiplication satisfy the distributive property

- Left Distribution

$$\boxed{A(B+C) = AB + AC}$$

- Right Distribution

$$\boxed{(A+B)C = AC + BC}$$

The distribution law lets us ~~explain~~ expand matrix multiplication over addition.

Associative law for Matrix Multiplication

$$\boxed{(AB)C = A(BC)}$$ ✓

Scalar Multiplication Law of Matrices

Given Matrices A and B.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \end{bmatrix}$$

and scalar $r = 2$

$$r(AB) = (rA)B = A(rB)$$

First, we find the product AB.

$$AB \begin{bmatrix} 1+(-2)+6 & 0+0+2 & 1-1+0 & 1-1+4 \\ 0+4+3 & 0+0+1 & 0+2+0 & 0+2+2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 2 & 0 & 4 \\ 7 & 1 & 2 & 4 \end{bmatrix}$$

$$r \cdot (AB) = \begin{bmatrix} 10 & 4 & 0 & 8 \\ 14 & 2 & 4 & 8 \end{bmatrix}$$

Module-2 : Determinants and their Properties

The determinant is a Scalar value that can be computed from the elements of square matrix and encodes certain properties of the matrix.

- Provides critical information about the matrix, such as whether it is invertible and the volume scaling for the linear transformation it represents

Calculating Determinants

For 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is given by

$\det(A) = ad - bc$

- The calculation of for larger matrix involves breaking them down into smaller matrices, this process is known as expansion by minors.

Ex:

Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\det(A) = (1\times4) - (3\times2) = 4 - 6 = -2$$

This determinant provides Critical insights into the properties of matrix A.

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 7 \end{pmatrix}$$

$\det(A) = -14 = -14$

Determinant of a 3x3 matrix

$$A = \begin{array}{cc} & \begin{array}{ccc} + & - & + \end{array} \\ \begin{array}{c} \\ - \\ + \end{array} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \end{array}$$

$$1 \cdot \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2 \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3 \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$-4 \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} - 5 \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} + 6 \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

$$+7 \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} - 8 \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} + 9 \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$1 \cdot (5 \cdot 9 - 8 \cdot 6) - 2(4 \cdot 9 - 7 \cdot 6) + 3(4 \cdot 8 - 7 \cdot 5)$$

$$1(45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= 1 \cdot (-3) - 2 \cdot (-6) + 3(-3)$$

$$= -3 + 12 - 9$$

$$= 0$$

Property of Determinant

- The determinant of an identity Matrix is 1.
- Swapping two rows (or Columns) of a matrix changes the Sign of its determinant.
- If a matrix has a row or column of zeros, its determinent is 0.
- The determinant of a product of matrices equals the product of their determinants.

Ex:- If I is an identity matrix of size $n \times n$ then $\det(I) = 1$

E for $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \det(I_2) = 1$

Swapping Row/Columns changes the sign

If a matrix A' is obtained by swapping two rows (or two columns) of matrix A, then $\det(A') = -\det(A)$

Ex:- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Swapping row

$A' = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ and $\det(A') = -\det(A)$

Matrix Row/Columns of Zero

If a matrix A has a row, or columns of zeros, then $\det(A) = 0$

Ex:- $A = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$, $\det(A) = 0$

Determinant of Product of Matrix

If A and B are square matrix than.

$\det(A \cdot B) = \det(A) \cdot \det(B)$ ✓

Ex : $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ True

$= 4$

$\det(A \cdot B) = 4$

$\det(A) \cdot \det(B) = 4$

Determinants in Geometry

Determinants have a Geometry interpretation:

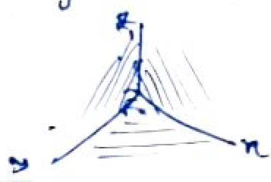
- The determinant of a 2×2 or 3×3 matrix represents the area or volume (respectively) of the parallelogram (or parallelepiped) formed by the column vectors of the matrix.

What are Determinants

Determinants offer a scalar value that summarizes the linear transformation described by a matrix. In geometric terms determinants help us understand.

- The area spanned by vectors in 2D space.
- the volume enclosed by vector in 3D space.

This scalar can also signify the Orientation and Scaling factor of the transformation.



Determinants in 2D: Area of a Parallelogram

Give two vector $\vec{a}$ and $\vec{b}$ in 2D, the determinant of the matrix formed by these vectors as columns given the area of the parallelogram spanned by these vectors.

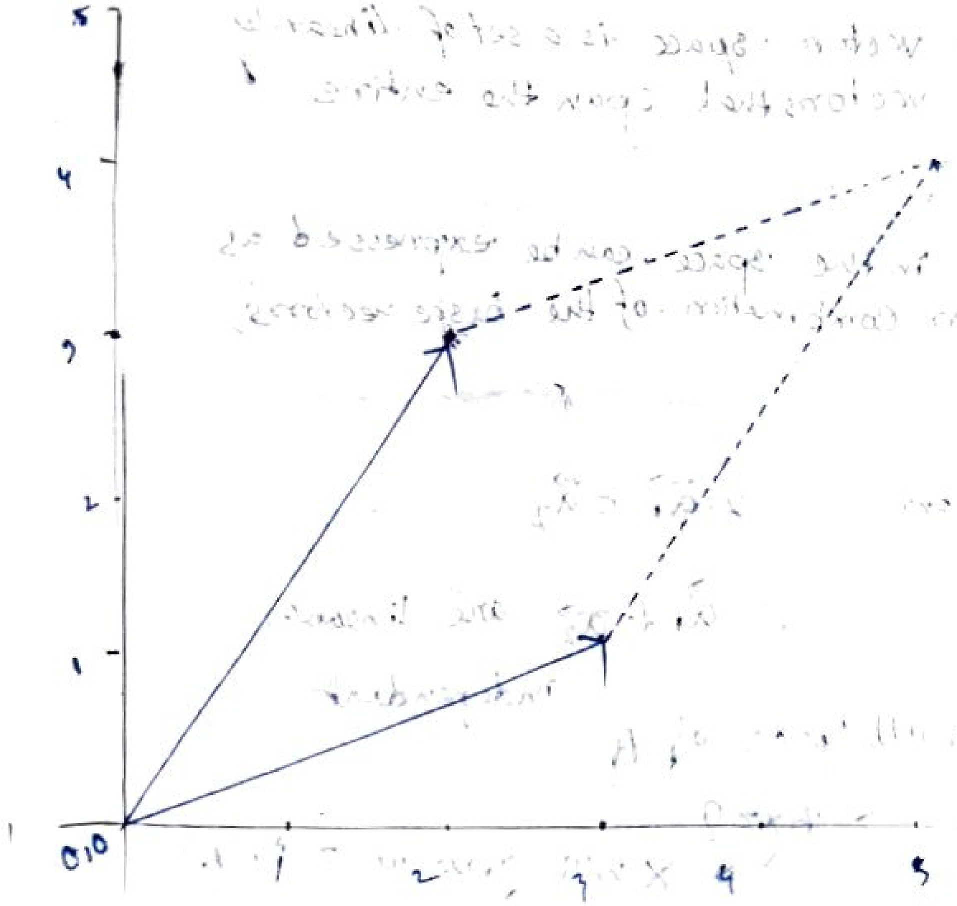
$$\text{Area} = \left| \det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \right) \right|$$

Ex $\vec{a} = [3, 2]$, $\vec{b} = [1, 4]$

$$\text{Area} = \left| \det\left(\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \right) \right| = 10 \text{ unit}^2$$

Visualizing the area of a Parallelogram

The plot visually demonstrates how the vectors $\vec{a}$ and $\vec{b}$ define a parallelogram whose area is given by the det determinant of the matrix Composed of these vector



Determinant play a Crucial Role in Understanding the geometric properties of spaces spanned by Vectors. they provide valuable insight into

- The Scaling effect of linear transformations.
- The ~~Sc~~ orientation and handedness of the Coordinate system.
- Practical applications in calculating area and volumes

Advance Linear Algebra

- Vector Space and projections

Basis : Defination

A basis of a vector space is a set of linearly independent vectors that span the entire vector space..

Every vector in the space can be expressed as a unique linear combination of the basic vectors,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}_{2\times 2} \qquad \vec{a_1} \quad \vec{a_2}$$

$$2 \cdot \vec{a_1} = \vec{a_2}$$

$\therefore \vec{a_1}$ & $\vec{a_2}$ are linearly independent

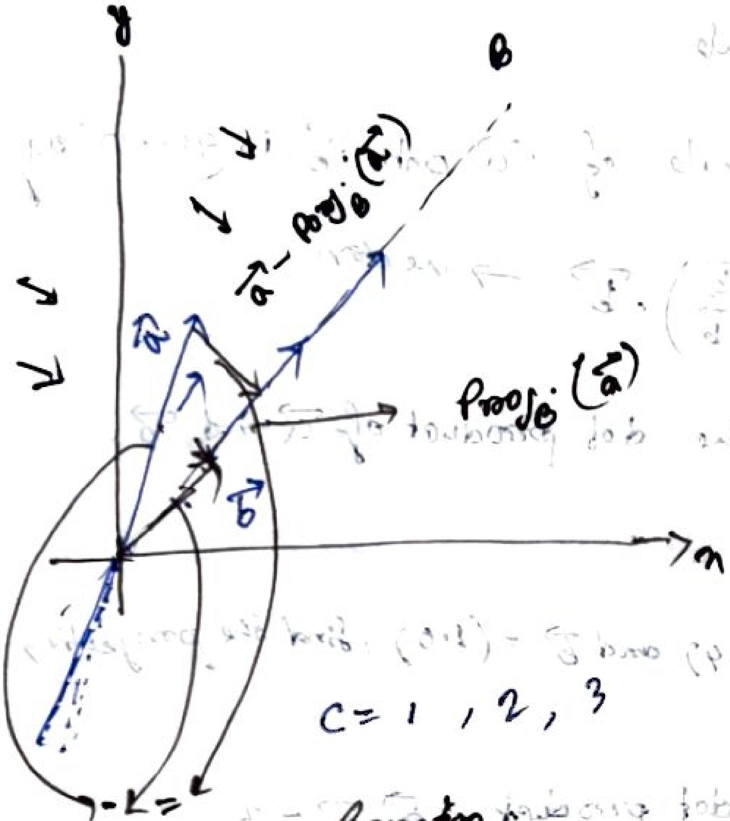
① Basis of null space of A

- $N(A) \longrightarrow Ax = 0$ $\to$ x will give us = N(A)
- N(A) = N (ARoblem)
- homogeneus case

Projection

The projection of a vector $\vec{a}$ onto another vector $\vec{b}$ is the orthogonal projection of $\vec{a}$ along $\vec{b}$.

It is denoted by $proj_{\vec{b}}(\vec{a})$ and represents the component of $\vec{a}$ in the direction of $\vec{b}$



$\mathbb{R}^2$

B is line in $\mathbb{R}^2$

$B = \{ c \cdot \vec{b} \mid c \in \mathbb{R} \}$

$Proj_B(\vec{a}) = ?$

$c = 1, 2, 3$

Projection

$Proj_B(\vec{a}) \longrightarrow$ what is c such that $c \cdot \vec{b} = Proj_B(\vec{a})$

$Proj_B(\vec{a}) \rightarrow$ same vector also on line B where $\vec{a} - Proj_B(\vec{a})$ is perpendicular / orthogonal to B

$\longrightarrow$ specification c.b

$(\vec{a} - Proj_B(\vec{a})) \cdot \vec{b} = 0$

$Proj_B(\vec{a}) = c \cdot \vec{b}$

$(\vec{a} - c \cdot \vec{b}) \cdot \vec{b} = 0$

$\vec{a}\vec{b} - c \cdot \vec{b} \cdot \vec{b} = 0$

$c \cdot \vec{b}\vec{b} = \vec{a}\vec{b}$

$c = \dfrac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}$

(2) $\text{proj}_{b}(a) = ?$

$$\text{Proj}_{B}(\vec{a}) = c\,\vec{b}$$
$$= \frac{\vec{a}\cdot\vec{b}}{(\vec{b}\cdot\vec{b})}\,\vec{b}$$

(*)

Projection formula

The projection formula of $\vec{a}$ onto $\vec{b}$ is given by

$$\text{Proj}_{\vec{b}}(\vec{a}) = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\cdot\vec{b} \rightarrow \text{vector}$$

where $\vec{a}\cdot\vec{b}$ is the dot product of $\vec{a}$ and $\vec{b}$.

Ex:

Given vectors $\vec{a} = (3,4)$ and $\vec{b} = (1,0)$, find the projection of $\vec{a}$ onto $\vec{b}$

- First compute the dot product $\vec{a}\cdot\vec{b} = 3$
- Compute $\vec{b}\cdot\vec{b} = 1$

then $\text{proj}_{\vec{b}}(\vec{a}) = \left(\frac{3}{1}\right)\cdot(1,0) = (3,0)$

the projection of $\vec{a}$ onto $\vec{b}$ is a vector of length 3 ~~and~~ in the direction of $\vec{b}$

Introduction to Vector projection.

· Given two vectors $\vec{a} = (4,3)$, $\vec{b} = (2,0)$ in 2-D space find the projection of $\vec{a}$ onto $\vec{b}$, denoted as $proj_{\vec{b}}(\vec{a})$.

$\Rightarrow$

$$\vec{a}.\vec{b} = \begin{bmatrix}4\\3\end{bmatrix}\begin{bmatrix}2 & 0\end{bmatrix} = 8$$

$$\vec{b}.\vec{b} = \begin{bmatrix}2\\0\end{bmatrix}\begin{bmatrix}2 & 0\end{bmatrix} = 4$$

Projection :

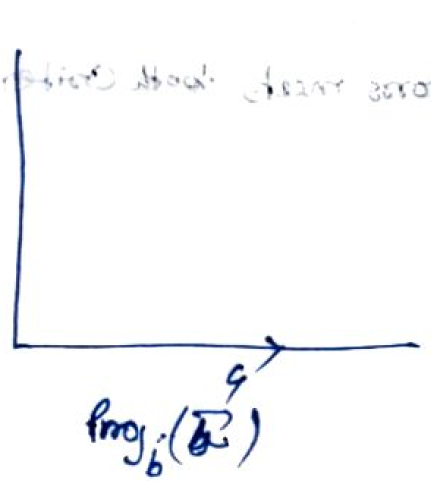
$$Proj_{\vec{b}}(\vec{a}) = \frac{8}{4} \cdot \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}2 \cdot 2\\2\end{bmatrix} = 2\begin{bmatrix}2\\0\end{bmatrix}$$

$$= \begin{bmatrix}4\\0\end{bmatrix}$$

The projection of $\vec{a}$ onto $\vec{b}$ is (4,0)

This means that $\vec{a}$'s component in the direction of $\vec{b}$ spans 4 units along the x-axis

The projection indicates that $\vec{a}$'s influence the direction of $\vec{b}$ is completely horizonal, with a magnitude of 4 units.



Orthonormal Bases:

An Orthonormal basis for a vector space is a basis where all vectors are orthogonal (perpendicular) to each other and each vector is of unit length

$\boxed{\text{Benifits}}$ → Simplify the calculation, including projections and transformations due to the properties of orthogonal vectors and normalization.

(i) $\vec{a} \cdot \vec{b} = 0$

(ii) $\|\hat{a}\| = 1$

$\|\hat{b}\| = 1$ $\quad \therefore |a| = |b| = 1$ ✓

Orthogonality vs Normalization

- Two vectors are orthogonal if their dot product is zero $\vec{v} \cdot \vec{w} = 0$
- A vector is normalized if it has a unit length, $\|\vec{v}\| = 1$
- An orthonormal set of vectors meets both criteria.

The Gram-Schmidt ~~Project~~ Process: Overview

The Gram-Schmidt Process is a method for orthogonalizing a set of vectors in an inner product space - turning them into an ~~orthogonal~~ Orthonormal set.

Given a set of linearly independent vectors, this process produces an orthonormal set spanning the same subspace.

R^n

Process

Given vectors $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}$:

1) Start with $\vec{v_1} = \vec{a_1}$ and normalize it to get $\vec{e_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|}$

2) For each subsequent vector $\vec{a_k}$, subtract its projection on all previously computed orthogonal vectors.

$$\boxed{\vec{v_k} = \vec{a_k} - \sum_{i=1}^{k-1} \mathrm{Proj}_{\vec{e_i}}(\vec{a_k})}$$

3) Normalize $\vec{v_k}$ to get $\vec{e_k}$.

4) Repeat Steps 2-3 for all vectors

$\vec{}$

St-1 $\quad \vec{v_1} = \vec{a_1}$

normalize $\vec{v_1}$ to get $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}$

$$\frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{\vec{a_1}}{\|\vec{a_1}\|} = \vec{e_1}$$

St-2

$k=2$ $\quad \vec{v_k} = \vec{a_k} - \sum_{i=1}^{k-1} \mathrm{Proj}_{\vec{e_i}}(\vec{a_k})$

$$\vec{v_2} = \vec{a_2} - \sum_{i=1}^{1} \mathrm{Proj}_{e}(\vec{a_2}) = \vec{a_2} - \mathrm{Proj}_{\vec{e_1}}(\vec{a_2})$$

$K=3$

$$\vec{v_3} = \vec{a_3} - \sum_{j=1}^{2} \text{Proj}_{\vec{e_j}}(\vec{a_3}) = \vec{a_3} - \left(\text{Proj}_{\vec{e_1}}(\vec{a_3}) + \text{Proj}_{\vec{e_2}}(\vec{a_3})\right)$$

$j=1 \qquad j=2$

St3 $v_k \underset{k=2}{=}$ Normalized . $\vec{v_i}$

$$\vec{v_2} \longrightarrow \vec{e_2} \quad \left(\text{by } \frac{\vec{v_2}}{\|\vec{v_2}\|} = \vec{e_2}\right)$$

St4 → Repeate - Step -

Ex Lets apply the Gram-Schmidt Process to vectors.

$\vec{a_1} = (1,1,0)$

$\vec{a_2} = (1,0,1)$

Calculate $\vec{e_1}, \vec{v_2}$ then $\vec{e_2}$

Through the process, we will obtain an ~~orthog~~ orthonormal basis for the subspace spanned by $\vec{a_1}$ and $\vec{a_2}$

St-1

$\vec{v_1} = \vec{a_1}$ & normalize. $\vec{v_1} \to \vec{e_1}$

$\vec{v_1} = \vec{a_1} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix}$ & $\vec{e}_1 = \dfrac{\vec{v_1}}{\|\vec{v_1}\|} \Rightarrow \dfrac{\begin{bmatrix}1\\1\\0\end{bmatrix}}{\|\vec{v_1}\|}$

given $\|\vec{v_1}\| = \dfrac{\vec{v_1} \cdot \vec{v_1}}{\cancel{\ }} = [1\,1\,0] \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} = 2$

St-2 normalize $\vec{v_2} \to \vec{e_2}$

$k=2$ ($\vec{a_2}$), get $\vec{v_2}$ &

$\vec{v_2} = \vec{a_2} - \sum_{i=1}^{k-1=1} proj_{\vec{e_i}}(\vec{a_2}) = \vec{a_2} - proj_{\vec{e_1}}(\vec{a_2})$

$(i=1)$

$= \begin{bmatrix}1\\0\\1\end{bmatrix}$

$proj_{\vec{e_1}}(\vec{a_2}) = \dfrac{\vec{a_1}\vec{e_1}}{\vec{e_1}\cdot\vec{e_1}} \cdot \vec{e} = \dfrac{\begin{bmatrix}1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}1/2\\1/2\\0\end{bmatrix}}{\begin{bmatrix}1/2\\1/2\\0\end{bmatrix} \cdot \begin{bmatrix}1/2\\1/2\\0\end{bmatrix}} \cdot \begin{bmatrix}1/2\\1/2\\0\end{bmatrix}$

$= \left(\dfrac{1/2 + 1}{1/4 + 1/4}\right) \cdot \begin{bmatrix}1/2\\1/2\\0\end{bmatrix} = \dfrac{3/2}{1/2} \cdot \begin{bmatrix}1/2\\1/2\\0\end{bmatrix} = 3 \cdot \begin{bmatrix}1/2\\1/2\\0\end{bmatrix}$

$\begin{bmatrix}-1/4\\3/4\\4/4\end{bmatrix}$

$\begin{bmatrix}3/2\\3/2\\0\end{bmatrix}$ $e_2 = \dfrac{\vec{v_2}}{\|\vec{v_2}\|}$ $\dfrac{v_2}{v_2 v_2}$

$= 1/2 / 14/4$.

Application of Orthonormal Bases

- Simplification of complex vector operations.
- Basis for Fourier Series, Quantum Mechanics and signal processing
- Critical in numerical methods, machine learning algorithms, and data compression

Special Matrices.

Special Matrix have unique properties such as Symmetry (Symmetric matrices) all non-zero elements on the diagonal (diagonal matrices), or Orthogonality (orthogonal matrices)

Symmetric Matrix

A symmetric matrix is equal to its transpose. Here's an example.

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$A = A^T$$

Diagonal Matrix

A diagonal Matrix has non-zero element only on its diagonal.

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Orthogonal Matrix

An orthogonal Matrix is a square matrix whose columns and rows are orthogonal unit vectors (orthonormal vectors) and its transpose equal its inverse. Here's an example.

$$Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

Ex $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$Q^T Q = Q Q^T = I$$

Where I is the Identity matrix, Confirming that Q is Orthogonal.

Module-3 **Matrix Factorization**

Matrix Factorization techniques are essential for

- Simplifying Matrix operations and making Complex calculations more manageable.
- Solving Systems of Linear equn efficiently.
- Performing eigenvalue decomposition, singular value decomposition, and Other operations crucial in Machine learning and data analysis.
- Reducing computational Complexity which is especially important in high-Dimensional data processing.

These techniques underpin many algorithms in numerical Analysis. Optimization and beyond.

Matrix Factorization

It refers to decomposing a matrix into a product of two or more matrix, revealing its structure and simplifying further analysis.

Common type of matrix factorization include LU (Lower Upper), QR (Orthogonal Triangular) SVD (Singular Value Decomposition), and Eigen Decomposition.

QR Decomposition

- Decomposes a matrix into an orthogonal matrix (Q) and an upper triangular matrix (R)
- Solves linear least ~~sequence~~ squares problems and provides numerically stable solutions.
- Used extensively in signal processing and statistical analysis

LU Decomposition

- Decomposes a matrix into a lower triangular matrix (L) and an upper triangular matrix (U).
- Facilitates the solving of linear equations and matrix inversion.
- Common in engineering and physical sciences for systems of linear equations.

Singular Value Decomposition (SVD)

- Decomposes a matrix into three matrices. U (orthogonal) (diagonal) and V* (Conjugate transpose of an orthogonal matrix)
- Used in Data Compression, noise reduction, and principle Component Analysis (PCA)
- Provides insights into the structure and rank of the matrix

Eigen Decomposition

- Decomposes a matrix into eigenvalues and eigenvectors. revealing the matrix's fundamental properties.
- Critical for understanding linear transformations stability analysis, and systems of differential equations.
- Basis for many algorithms in numerical linear algebra. Including those used in AI for feature extraction and dimensionality reduction.

Choosing the Right Tool

choosi the choice among QR, LU, SVD and ED depends on the specific problem's requirements and data characteristics

QR & LU are preferred for solving linear systems, while SVD and ED offer deeper insights into the data's geometry and invaluable for ~~PCA~~ PCA and (ML) applications.