The Discrete Mathematics Behind Casino Games

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I. Introduction

This document outlines the application of combinatorial mathematics in casino games. These calculations cover everything from casino advantage or "house edge" all the way down to the probability of outcomes occurring in different games. It is possible to calculate the probability of every outcome in every game. This document will look directly at the calculations of these probabilities and why they are so significant to the end goal of the owners of casinos, guaranteeing profit. Millions of gamblers play at these casinos every day, therefore, the casino, alternatively referred to as the "house", must ensure that the overall probability for which the gamblers lose must be greater than 50%. In various games, such as roulette, the house will grant itself as little as a 5% advantage of winning each spin to ensure that over the course of time, they will come out on top. Roulette and blackjack will be discussed in depth, regarding the probability of outcomes and where exactly the house is getting this key edge against its opponents.

II. House Edge

The casino would be silly to give themselves a 50/50 chance of losing to the player. That would mean that the casino winning all the time is due solely to luck. This is not the case however. The casinos don't beat the players because of luck, they beat the players because the odds are weighted in their favor. This advantage that they inevitably grant themselves is called the house edge. When referred to in numbers, it is the casino's average profit from a player's bet. As stated previously, in roulette, the house edge is approximately 5.26%, which means for every dollar that a player bets, the casino will average about 5 cents in winnings.

Of course, if a single dollar is bet just once, the winnings resultant of that bet will not be 95 cents—the return will either be the whole dollar or another whole dollar. The point of the house edge is to see what the average loss is. If all the roulette players in a casino collectively gamble $10 million on a Saturday night, the casino expects to pay back around $9,500,000 as winnings and keep around $500,000 as profit.

Since house edge is calculated with consideration to an infinite timeline, the more each player bets, the closer the player will get to that exact 95% return on their initial investment.

A. The Grind

Imagine a gambler sits down at a roulette table with $100, and bets $5 a spin. The gambler is betting about $150 per hour, even though they started with only $100. That's because some rounds are being played with resultant winnings and so on. After 13 hours of play (depending on if the player has lasted this long) bets will equal $1950. When calculating the 5% casino earnings on these bets totaling $1950, the casino will be walking away with $97.50 of the $100 initial betting money of the gambler. Leaving him/her with $2.50 if their original total.

This effect of the house slowly chipping away at the gambler's money is known as the grind. This grind effect can be minimized by playing games with a small house edge for short periods of time. This will ensure that some winnings at each table can be preserved, assuming that the player is in fact winning some money. While the house edge in roulette would continue to 5% every time, switching tables can affect the probability in ways that favor the player.

B. Calculating Average Loss for a Player

As seen above, while discussing the grind, there are several other factors that contribute to the house's overall edge against a player, and their total loss. In general, games with lower house edge will result in the player losing less money. A games house edge is only one factor contributing to the average loss of the player. Other factors weighing into average loss is average bet size, speed of play and total hours played. From these measurements, a formula can be derived to calculate average loss for the player:

*Bet/Round * House Edge * Rounds/Hour * Hours Played*

When considering the example provided in the grind section above, the formula after substitutions would be:

($5 bet per spin)(5% house edge)(~30 rounds/hr)(13 hours)=$97.50

While the actual loss of the player may vary due to this being probability and not concrete mathematics, the loss of the player should only vary in a slight standard deviation above and below this calculated amount.

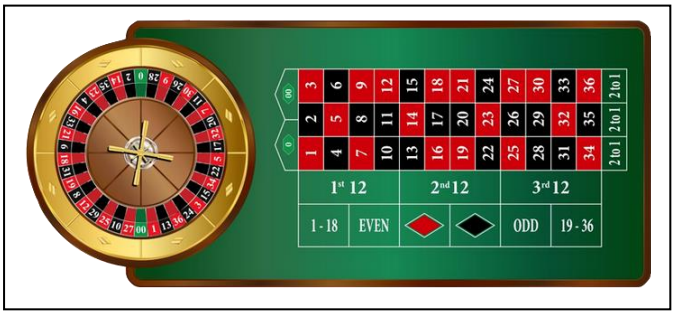
III. Roulette

Since, roulette was used to explain house edge and average loss at the start of this document, the calculations behind the house edge and various probabilities in the game of roulette will be discussed more in depth in this next section. There are two styles of the game Roulette, American and European. House edge varies drastically in both versions of the game.

A. *The Table*

Below is the layout of an American Roulette table. From observation, there are 38 cells where the ball can land on the wheel. There are 36 cells alternating black and red colors and 2 cells that are green. European Roulette varies, having only one green cell, which will change the house edge drastically (shown later).

AMERICAN ROULETTE TABLE:



On the right side of the wheel are the potential bets that can be placed. Inside bets are the single numbered cells within the main rectangle. Outside bets are the outer rectangles whose descriptions tell which groups of numbers being bet on (1st 12 includes the first 12 numbers, EVEN includes the even numbers, 2 to 1 includes all the numbers in the column above it, and so on). The payout in American Roulette is the same as European roulette despite the increased house edge, further increasing casino profit.

B. *Outside Bets*

Best odds to win on bigger individual bets comes from betting on the outside of the betting rectangle. Because of this, the payout is smaller to maintain house edge.

For example, if the odds of winning a black bet were to be calculated, consider the fact that there are 18 black numbers in total out of 38 total numbers. That means there is a 18/38 chance of winning the bet (47.47%). The same exact probability would hold true for red, even, odd, low or high bets as well, because the number of winning outcomes is the same. Because this probability is so close to being 50%, the payout that casinos give for a color win is 1:1.

When looking at the probability of a bet including 12 numbers (dozen & column bets), observe that there are 12 outcomes that win out of 38 in total. Similar to before, calculate the winning outcomes divided by the total outcomes (12/38) to get the 31.58% chance the player would have to win that bet. Since the probability of a player win is less common. The payout is bumped for this bet to 2:1.

C. *Inside Bets*

Inside bets are much harder to win, hence why casinos reward the players who win these bets handsomely in the payout. These bets are commonly for less money than the outside bets because the chance of winning is much lower. Inside bet payouts range from 5:1 all the way up to 35:1, but probabilities only range from 2%-16%.

If the player were to place a line bet, which includes 6 numbers where the chip is placed in-between two numbers on the bottom row, there would be 6 winning outcomes out of 38 in total. This in turn gives the player a 6/38 (15.79%) chance of winning. Since this probability is much lower than the outside bets seen previously, the payout is bumped up to 5:1. The one alternative to the line bet that is only seen in American Roulette is the basket bet. This bet is in the same layout as a line bet but it is all the way to the left side of the rectangle where the left column of the two-column set is the two variations of zero. Since this bet only contains 5 winning outcomes out of 38 in total, the probability is 5/38 (13.16%) with a bigger payout of 6:1.

Squares, streets and splits are all examples of inside bets that further decrease the probability of winning. Squares include four winning outcomes, therefore the probability to win is 4/38 (10.53%). Streets are like a line except it only includes one row of numbers. This means it has three winning outcomes, therefore the probability to win would be 3/38 (7.89%). Splits only contain two winning outcomes and are placed between two cells on the rectangle. This probability is 2/38 resulting in a 5.26% chance of winning for that bet.

The last and least likely bet to win is a straight-up. This bet is thrown on a single number and has the lowest probability of being one by the player. The probability of winning a straight-up bet is 1/38 (2.63%). Since this probability is so low, the casino rewards the player with a 35:1 payout.

D. *Calculating House Edge*

To calculate the house edge for American Roulette when placing chips on one of the 1:1 payout bets, calculate the probability of winning the bet and also the probability of losing the bet. From there, subtract the losing probability from the winning probability to obtain a negative fraction representing the expected return to the casino. [1]

$$\textit{Expected Return=(Winnings)(Winning Prob.)– (Loss)(Losing Prob.)}$$

$$\textit{Expected Return} = (1)(18/38) – (1)(20/38)$$

$$\textit{Expected Return} = -2/38$$

$$\textit{Expected Return} = -0.0526$$

Therefore, the house edge is about 5.26%

To calculate the house edge for the same type of bet in European Roulette, the only aspect that needs to be changed is the denominator for the probabilities. This is due to the fact that there is only one zero in comparison to the American Counterpart that has a zero and a double zero.

$$\textit{Expected Return= (Winnings)(Winning Prob.) – (Loss)(Losing Prob.)}$$

$$\textit{Expected Return} = (18/37) - (19/37)$$

$$\textit{Expected Return} = -1/37$$

$$\textit{Expected Return} = -0.027$$

Therefore, the house edge is about 2.7%

It can be observed that this calculation works for every type of bet as well. If a straight up bet were to be placed, the payout is 35:1. So if a dollar were placed on a number the house edge could be calculated as follows:

Expected Return=(Winnings)(Winning Prob.)-(Loss)(Losing Prob.)

Expected Return = (35)(1/38)-(1)(37/38)

Expected Return = 35/38 – 37/38 = 2/38

Expected Return = 0.0526

Therefore, the house edge is still 5.26%

E. Reccurrence of an Event in Roulette

There is no doubt that it is a great coincidence when the same number comes up again and again. The longest reliable series was registered at the hotel El San Chuan in Puerto Rico on June 9th, 1959. During the course of spins in American Roulette, the number ten occurred six times in a row. The probability of such (successive) events is determined by a multiplication of individual events. Therefore, the probability that the same number comes up six times in a row is:

(1/38) × (1/38) × (1/38) × (1/38) × (1/38) × (1/38)

$(1/38)^6 = 0.00000000033212259326167$

That is a very small number indeed, roughly three billionths only. If this probability was converted into true odds that would have to be offered by a casino the value is 3,010,936,384:1. The odds are reciprocal values of the probabilities. The higher they are, the lower the probabilities are.

There are no exceptions that the same color appeared more than 20 times in a row in practice. The record was registered in 1943, when the red color came up 32 times in a row! The probability of the repetition of the same color in American Roulette is much lower: $(18/38)^{32} = 0.00000000004127100756$ and the odds are 24,230,084,485:1. Thus this is even less likely than occurrence of a single number six times in a row. Again, it is clearly demonstrated what kind of importance (a negative one for players) has just one extra number in American Roulette.

IV. Blackjack

Blackjack is a casino game that relies on much more skill than luck. The goal of the player is to be closer to a sum of 21 (in cards) than the dealer without busting (exceeding 21). Both the player and the dealer are dealt two cards but one of the dealers' cards is face down, therefore the player can only see one. The player must then compare his/her cards to the one that the dealer is showing to determine whether to hit (accept another card from the dealer) or stand (stay at their current sum and risk the dealer beating it).

There are also other options for the player. Doubling down means to double the ante and receive only one card from the dealer. After doubling, the player can no longer accept any more cards from the dealer, meaning the player must be extremely confident in the card they are about to receive and their leverage over the dealer. Another option is to split. This option is only available if the player has been dealt two of the same cards. Therefore, if the player is dealt an 8 and an 8, they can elect to split the cards, essentially opening another ante of the same amount on the second 8. From there both hands will be in play and the goal remains the same, get both close to 21 and beat the dealer. Lastly, in some casinos the player can choose to surrender, giving up the cards while keeping their bet. This option is very rare because it substantially reduces the house edge.

A. When to Hit and When to Stand

For the player, the main concern is whether to hit or stand on a certain hand. The strategy that should be considered involves the possibilities of the downcard of the dealers and also the possibility for the player hand to bust.

Consider this scenario, if the dealer is showing a 6 and must stand on all sums of 17 (commonly the rule in most casinos), and the player is dealt a 13. Right away, a player not using the correct strategy would most likely hit and risk busting, because there are more cards that "equal" 10 than any other card. If the dealer is showing a 6 and has to stand on 17, the player could consider the same prediction. If the dealer were to flip their downcard and be revealing a 16, he/she would have to draw again, favoring a bust in probability because there are far more cards that are greater than or equal to 6 in the deck.

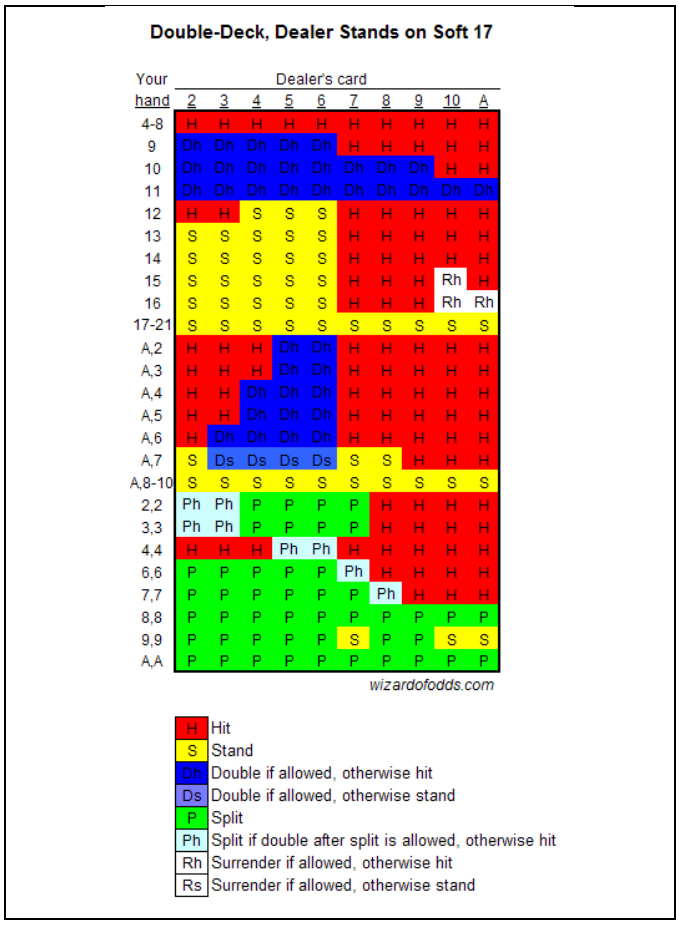
Consider the math behind this example. When playing with 2 decks, which is common in casino play, the player has 13 and the dealer is showing a 6. The player needs an 8 or below not to bust. Since there are 13 card rankings and the ace can be considered safe in this example, the probability that the player will not bust is 8/13. While this is true and seems safe to hit, consider the dealers probability of busting if they were to have a 10 as the downcard. They would have a 5/13 chance of not busting and complementarily, 8/13 chance of busting.

Now take into consideration that the dealers downcard could possibly not be a 10. There are 13 possibilities, 8 with the potential of making the dealers sum after the flip greater than 12. Once a sum is greater than 12 in blackjack, the probability of busting skyrockets, due to the fact there are so card with a value of 10. Therefore, if the dealer were to not have a 10 as his/her downcard, there is still more of a probability that the next card will put them above 12 which will favor the possibility of the dealer busting.

Below is a chart from wizardofodds.com using the strategy explained above with exact math from combinatorial analysis. Combinatorial analysis observes the distribution of all possible dealer end-states, given the dealer's upcard. First, calculate the probability of the player losing when standing. Then, calculate the probability of the player losing when taking a hit (busting). Then compare these two probabilities to determine whether to hit or stand. *If probability of loss when standing > probability of loss when hitting* then *hit* otherwise the player should *stand.* This obviously is a lot of math to prove while sitting at a

blackjack table, which is why players rely on these table to enhance their play. House edge is minimized by doing so, but can be further minimized with other strategies that consider the cards previously dealt and currently on the table as well.

HIT/STAND CHART:



[2]

B. *When to double, surrender, or split*

As seen above, there are also hands in which the probability of winning is so good for the player, that it is actually recommended to double down. Consider the example discussed earlier. The dealer is showing the 6 which already gives him/her the highest probability of busting. Therefore, if the player gets dealt a hand totaling 8, 9, 10 or 11 they can confidently double the ante and expect to best the dealer because the dealer will most likely either: bust if the player is dealt a low card on the hit, or the player will beat the dealer because the high card they were dealt makes their total closest to 21.

Hands can also be considered so bad for the player that the only reasonable action is to surrender. For example, flip the scenario in the dealer's favor. Imagine the dealer is showing a 10 and the player is sitting at a hard 16 to start. As discussed previously for a 16, the probability of busting is extremely high. Therefore, the player is inclined to believe the dealer has another 10 or something high as his/her second card. Leaving the player with no real option but to hit with a hope and a prayer. In casinos that allow surrender, this is the best option because the odds of winning are so low for the player.

C. *Calculating House Edge*

House edge in blackjack is one of the hardest to calculate because of the amount of varying conditions between games. Rules can vary in more than just one aspect. Shuffling, dealer stand rules, double down rules, number of decks, amount of people playing at the table, and other key factors can change the house edge in this game. Just amongst playing rules there are 12 common areas that vary across casinos.

1) *Table Conditions*

There are four distinct conditions of any blackjack game that directly affect the profit potential for card counters:

- The number of decks in play.
- Rules. Two dozen common rule variations.
- Players at table.
- Depth of deal.

2) *How Number of Decks Effects House Edge*

Now let's consider the effect of the number of decks shuffled together. All other conditions being equal, single-deck games would be the most profitable. The more decks being used, the less profitable the game becomes. A single-deck game where blackjack pays 3:2, the player can double down on any two cards, and the dealer stands on soft 17, is very close to being a break-even proposition for a basic strategy player. With four or more decks in play without changing rules, the house gains about a ½ percent edge. This chart correlates the amount of decks with the advantage the house gains.

DECKS VS. CHANGE IN HOUSE EDGE:

# Decks	Change in House Edge
1	+0.02%
2	-0.31%
3	-0.43%
4	-0.48%
5	-0.52%
6	-0.54%
7	-0.55%
8	-0.57%

3) *How the Rules Affect House Edge*

The second condition that must be considered is the set of rules used. Some rules, notably those that offer the player more options, are favorable to the player, assuming the player applies the correct strategy. Such rules would be surrender, doubling after splitting allowed, re-splitting aces allowed, etc. Those rules that limit the player's options, such as doubling down on

10-11 only, or no re-splits, are disadvantageous to the player. [3]

Some rules neither limit nor offer options to the player, but alter the dealer's procedure. An example of one such rule would be dealer hits on soft 17 (Soft meaning there is an ace involved. The term soft refers to the fact that the ace can be 1 or 11, meaning there are two sums from the 1 combination). This is disadvantageous to the player.

In order to figure out the starting advantage, a benchmark game of blackjack must be established when rule addition and subtraction can directly affect the house edge. This benchmark will be considered Vegas Strip blackjack defined as follows:

- Dealer stands on soft 17.
- Player may double down on any 2 original cards.
- Player may not double down after splitting a pair.
- Player may split any pair.
- Player may re-split any pair except aces.
- Split aces receive only one card each.
- No surrender.
- Dealer either receives a hole card, or the player's original bet only is lost if the player doubles down or splits a pair and the dealer gets a blackjack.
- Insurance is allowed up to one-half the player's bet, and pays 2 to 1.
- Player blackjack is paid 3 to 2.

Now the effect of any other rule will be accounted for in determining the edge. These are the rule effects:

RULES AFFECT ON HOUSE EDGE:

Common Rules	1-Deck	2-Deck	Multi-Deck
Double on 10-11 only:	-0.26%	-0.21%	-0.18
Double on 9-10-11 only:	-0.13%	-0.11%	-0.09
Hits Soft 17:	-0.19%	-0.20%	-0.21
No Re-splits:	-0.02%	-0.03%	-0.04
Double After Splits:	+0.14%	+0.14%	+0.14
Re-split Aces:	+0.03%	+0.05%	+0.07
Draw to Split Aces:	+0.14%	+0.14%	+0.14
Late Surrender:	+0.02%	+0.05%	+0.08

These percentages are calculations from Peter Griffin's "Theory of Blackjack". [3] These calculations involve extensive combinatorial methods run across thousands of simulations. Calculating them by hand is near impossible. And yet, these advantages seem very small. But in the long run, it will give the casino the ability to profit over the players.

V. CONCLUSION

Although a player may use strategy to gain an advantage over other players in casino games, it is impossible to gain an advantage on the casino. This built in advantage is assured by every casino on the planet, otherwise they wouldn't be able to stay in business! The application of combinatorics and probability in these games allows the casino to find these desired areas and exploit them.

As discussed previously, roulette uses standard probability to determine the odds of winning and house edge. There is also the probability of recurring event formula to determine the best bet to be placed for the upcoming spin. This can help by determining the probability of a color occurring on a long streak of consecutive spins or the probability of a number occurring multiple times in a row.

Blackjack is based more on combinatorics, with house edge varying more with the application of different rules. Strategy can be applied by using combinatorial mathematics to determine the probability of the dealer or the player to bust when hitting for another card. Another strategy seen by players is card counting, which can give players with the ability a significant advantage over others (still doesn't completely eliminate house edge however). Blackjack has one of the lower house edges of casino games due to the skill and strategy that goes into it. This is why casinos alternate rules to make it more difficult and less advantageous for players. Without these varying rules a standard game of single deck Vegas blackjack with optimal strategy would be about even on the player-to-house advantage scale.

With the application of these discrete mathematics, casinos have provided a fun and addicting way of making or losing money. Players can win big some days and lose bigger on other days. The point is that the house advantage will never disappear. If a player's lucky, they may die with a profit at the casino, but due to these discrete mathematics, it is a fact that over the course of time, casinos will end up with the profit they guarantee themselves.

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