CS 412 - Homework 2

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Question 1.a)

pdf → $p(x|\alpha) = \alpha(1-x)^{\alpha-1}$ $\quad x \rightarrow$ observed flutterness, $\alpha > 0$

Likelihood Function $L(\alpha) = \prod_{i=1}^{n} p(x_i|\alpha) = \prod_{i=1}^{n} \alpha(1-x_i)^{\alpha-1}$

To find the MLE for α, we will take log (log-likelihood). Product turns into a sum, easier to differentiate

$$\ell(\alpha) = \log L(\alpha) = \sum_{i=1}^{n} \log[\alpha(1-x_i)^{\alpha-1}] = \underbrace{\sum_{i=1}^{n}\log(\alpha)}_{n\log(\alpha)} + (\alpha-1)\sum_{i=1}^{n}\log(1-x_i)$$

To maximize $\ell(\alpha)$, take derivative with respect to α, set it equal to 0 and solve for α.

$$\frac{d\ell(\alpha)}{d\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log(1-x_i) \qquad \frac{d\ell(\alpha)}{d\alpha} = 0 = \frac{n}{\alpha} + \sum_{i=1}^{n}\log(1-x_i) = \frac{n}{\alpha} = -\sum_{i=1}^{n}\log(1-x_i)$$

$$\frac{d(\log(\alpha))}{d\alpha} = \frac{1}{\alpha}$$

$$\boxed{\hat{\alpha} = \frac{n}{-\sum_{i=1}^{n}\log(1-x_i)}}$$

Question 1.b)

To get MAP estimate, we need likelihood function $p(X|\alpha)$ and prior distribution $p(\alpha)$. The posterior function $p(\alpha|X)$ is proportional to the product of these two and it contains all the knowledge about the unknown quantity α.

posterior $p(\alpha|X) \propto p(X|\alpha).p(\alpha)$

$$\propto \alpha^n \prod_{i=1}^{n}(1-x_i)^{\alpha-1} . \lambda\, \alpha^{\lambda-1} . e^{-\lambda\alpha}$$

take log of posterior → $\log p(\alpha|X) = (n+\lambda-1)\log\alpha + (\alpha-1)\sum_{i=1}^{n}\log(1-x_i) - \lambda\alpha$

take derivative → $\frac{d}{d\alpha}\log(p(\alpha|X)) = \frac{n+\lambda-1}{\alpha} + \sum_{i=1}^{n}\log(1-x_i) - \lambda$

Set derivative to 0 → $\frac{n+\lambda-1}{\alpha} + \sum_{i=1}^{n}\log(1-x_i) - \lambda = 0$

Solve for α → $\frac{n+\lambda-1}{\alpha} - \lambda = -\sum_{i=1}^{n}\log(1-x_i)$

(For the sake of simplicity, I will substitute C for $\sum_{i=1}^{n}\log(1-x_i)$)

$$\cancel{\alpha}\left(\frac{n+\lambda-1}{\cancel{\alpha}}\right) - \lambda\alpha = -\alpha C$$

$$n+\lambda-1-\lambda\alpha = -\alpha C$$

$$n+\lambda-1 = \lambda\alpha - \alpha C$$

$$n+\lambda-1 = \alpha(\lambda - C)$$

$$\boxed{\hat{\alpha}_{MAP} = \frac{n+\lambda-1}{\lambda-C}} \quad \text{where } C = \sum_{i=1}^{n}\log(1-x_i)$$

Note: If $\lambda = C$, then the solution would be undefined.