Several-Variable Calculus Midterm

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September 2023

CHAPTER 2

Equations:

- Dot product: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$
- Cross product: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = i \cdot det \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - j \cdot det \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + k \cdot det \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$
  $= (a_y \cdot b_z - a_z \cdot b_y, -a_x \cdot b_z + a_z \cdot b_x, a_x \cdot b_y - a_y \cdot b_x)$
- $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$
- $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos(\theta)$

- $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a} \times \|\mathbf{b}\| \times \sin(\theta)$ = the area of a parallelogram
- The parameterization of the **line** from A to B is $\alpha(t) = \mathbf{v} \cdot t + \mathrm{A}$
- The parameterization of the **curve** with radius r is $\alpha(t) = (r \cdot \cos(\theta), r \cdot \sin(\theta))$ for $0 \leq \theta \leq 2\pi$
- The parameterization of the **helix** is $\alpha(t) = (r \cdot \cos(\theta), r \cdot \sin(\theta), b \cdot \theta)$ where b is a scalar

Definitions:

- For a curve in $\mathbb{R}^n$, the **parameterization** has the form: $\alpha(t) = (x_1(t), x_2(t), ..., x_n(t))$ for $a \leq t \leq b$

- Position: $\alpha(t)$
- Velocity: $\mathbf{v}(t) = \alpha'(t)$
- Speed: $\|\mathbf{v}(t)\| = \|\alpha'(t)\|$
- Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \alpha''(t)$

- Arclength: $l = \|\alpha(t + \Delta \cdot t) - \alpha(t)\|$
- The **aclength of a curve** parameterized by $\alpha(t)$ from $a \leq t \leq b$ is $\int_a^b \|\alpha'(t)\| \cdot dt$
- Integral of a function with respect to arclength: $\int_\alpha f \cdot ds = \int_a^b f(\alpha(t)) \cdot \|\alpha'(t)\| \cdot dt$

- Tangent Vector: $\mathbf{T}(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}$
- Normal Vector: $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$
- Bionormal Vector: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$

- **Frenet Vectors** are a set of orthogonal vectors, ex. $(\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t))$
- **Frenet Vectors** form a **basis** in $\mathbb{R}^n$

- Curvature: The rate of turning $\kappa(\mathbf{t}) = \frac{\|\mathbf{T}'(t)\|}{\|\alpha'(t)\|}$
- Torsion: The rate of wobbling $\tau(t) = -\frac{c(t)}{\|\alpha'(t)\|}$

Identities:

- The vector $\mathbf{v}$ from A to B is equal to B-A
- $\mathbf{a} \perp \mathbf{b}$ if $\mathbf{a} \cdot \mathbf{b} = 0$
- $\mathbf{a} \times \mathbf{b}$ is always orthogonal to $\mathbf{a}$ and $\mathbf{b}$ if $\mathbf{a}$ and $\mathbf{b}$ are not multiples of each other
- $\|\mathbf{T}(t)\| = 1$
- $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are orthogonal to each other
- The binormal vector $\mathbf{B}(t)$ is orthogonal to $\mathbf{T}(t)$ and $\mathbf{N}(t)$
- $\mathbf{B}'(t) = c(t) \cdot \mathbf{N}(t)$ where $c(t)$ is a scalar

CHAPTER 3

Sketching Graphs:

- **Level sets** $f(\mathbf{x}) = c$, make output equal to a constant
- **Cross sections** $f(x_1, x_2, ..., x_n) = y$, make an input variable constant

Common Graphs:

- $x^2 + y^2 + z^2 = a^2$ is a sphere of radius a
- $x^2 + y^2 = a^2$ is a cylinder of radius a

Planes:

- For $\mathbf{v} = (x, y, z)$ and $\mathbf{p} = (p_1, p_2, p_3)$ on a plane, $\mathbf{v} - \mathbf{b}$ lies on the plane
- For $\mathbf{n} = (n_1, n_2, n_3)$ that is orthogonal to the plane, $\mathbf{n} \cdot (\mathbf{v} - \mathbf{p}) = 0$
  Therefore, $\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \mathbf{p}$

- If $\mathbf{n_1} \cdot \mathbf{n_2} = 0$, then planes P_1 and P_2 are perpendicular
  If $\mathbf{n_1}$ and $\mathbf{n_2}$ are multiples of each other, then planes P_1 and P_2 are parallel

Continuity:

- An **open set** is a set that does not contain its own boundary
- A function f is continuous at point c if $f\mathbf{x}$ approaches $f\mathbf{c}$ as $\mathbf{x}$ approaches $\mathbf{c}$
- If f and g are both continous, then $[f+g]$ $[k \cdot f]$ $[f \cdot g]$ $[\frac{f}{g}]$ and $[fog]$ are also continuous

Partial Derivatives:

- The **gradient** of f is $\nabla f(\mathbf{x}) = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$
- The **Jacobian matrix** or **derivative** is $Df(\mathbf{x} = [\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}])$
- The **first order affine approximation** of a function at point $\mathbf{a}$ is $l(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a}) \cdot [\mathbf{x} - \mathbf{a}]$

Differentiability:

- The function f is differentiable at point $\mathbf{a}$ if $\lim_{\mathbf{x} \to \mathbf{a}} \frac{f(\mathbf{x}) - l(\mathbf{x})}{\|\mathbf{x} - \mathbf{a}\|}$
- If f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$
- If some partial derivative of f does not exist at $\mathbf{a}$, then f is not differentiable at $\mathbf{a}$
- If all partial derivatives of f are continuous on its domain U, then f is differentiable on its domain