# Nonlinear System Reachable Set Computation: Learning Approach with Chen-Fliess Series

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# Work

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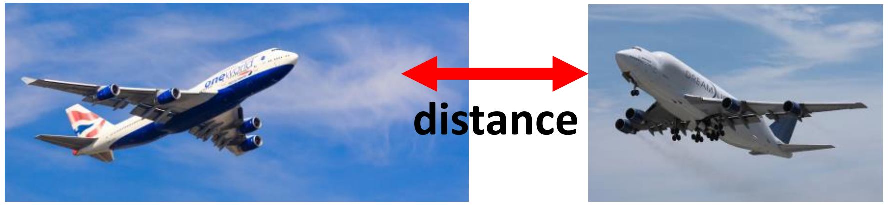
1. **I. Perez Avellaneda** and L. A. Duffaut Espinosa, “On Mixed-Monotonicity of Chen-Fliess series,” in IEEE 26th International Conference on System Theory, Control and Computing (ICSTCC), 2022, to appear.
2. **I. Perez Avellaneda** and L. A. Duffaut Espinosa, “Reachability of Chen-Fliess series: A Gradient Descent Approach” in $58^{th}$ Allerton Conference on Communication, Control and Computing, 2022, to appear.

# Reachable sets

**Definition (Reachable set):** The set of outputs of the system obtained from initial input and state sets.

**Importance:** air-traffic control



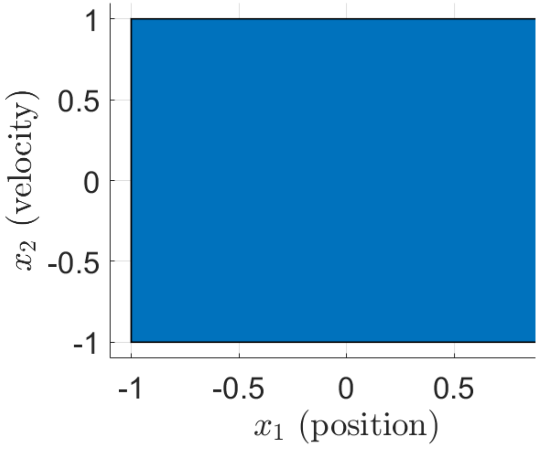

**Example:** Linear control system

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

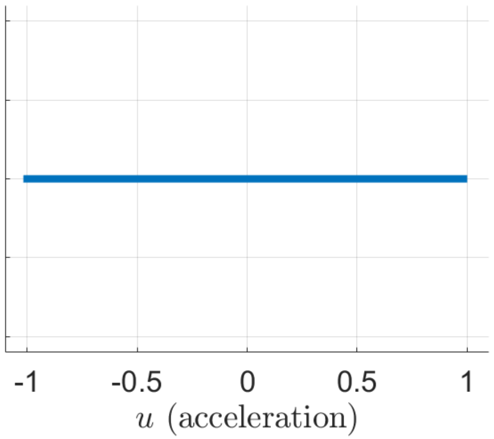
$$y = x,$$

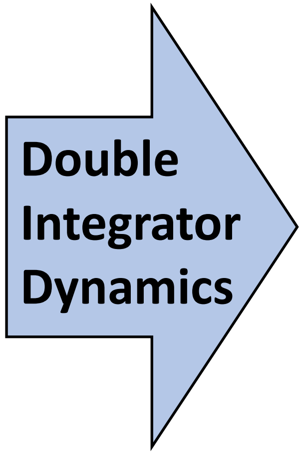
$$x(0) \in [-1,1] \times [-1,1], u \in [-1,1].$$

Initial States

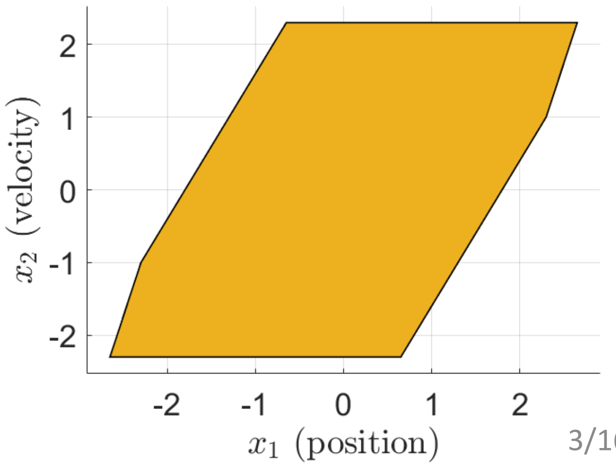



Input Set

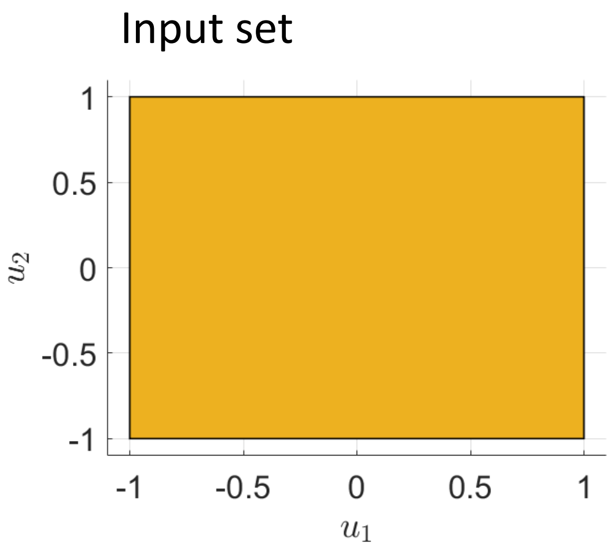


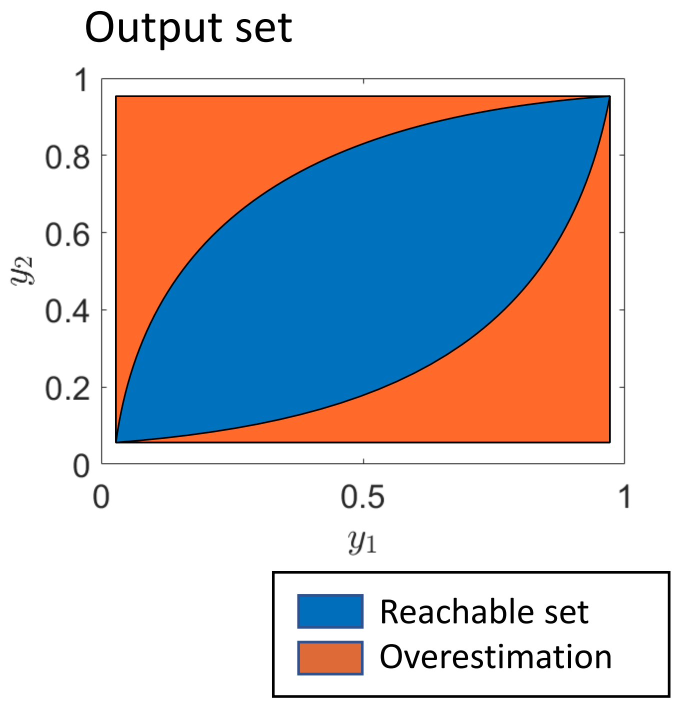

Reachable Set $t = 0.5\ s$



# Problem: tight overestimation of reachable sets of unknown systems

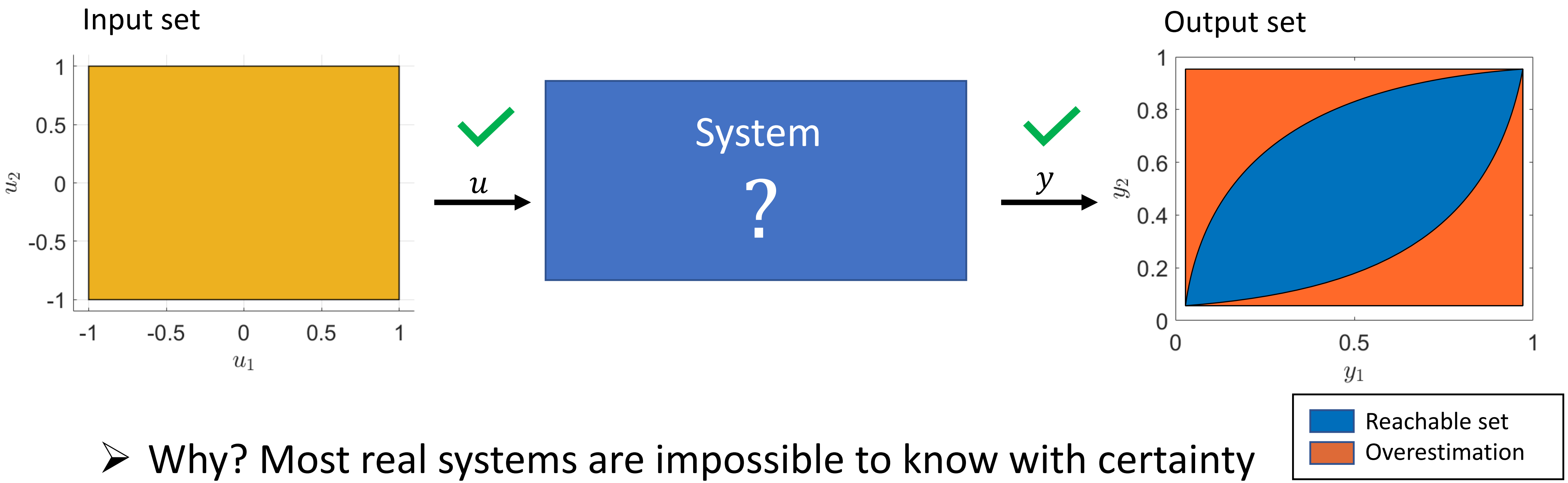



$u \rightarrow$ System $\dot{x} = f(t, x, u)$, $y = g(x)$ $\rightarrow y$




- Why? It only needs two points to compute.
- Approaches: mixed-monotonicity, Hamilton-Jacobi reachability, neural networks

# Problem: tight overestimation of reachable sets of unknown systems




- Why? Most real systems are impossible to know with certainty
- What to do? Use a data-driven approach
- Data-driven approaches: neural networks, Chen-Fliess series

# Chen-Fliess Series

**Definition (Chen-Fliess series):**

$$F_c[u](t) := \sum_{\eta \in X^*} \underbrace{(c,\eta)}_{\text{Vector in } \mathbb{R}^l} E_{x_i \nu}[u](t)$$

words

Space of all words

Iterative integral

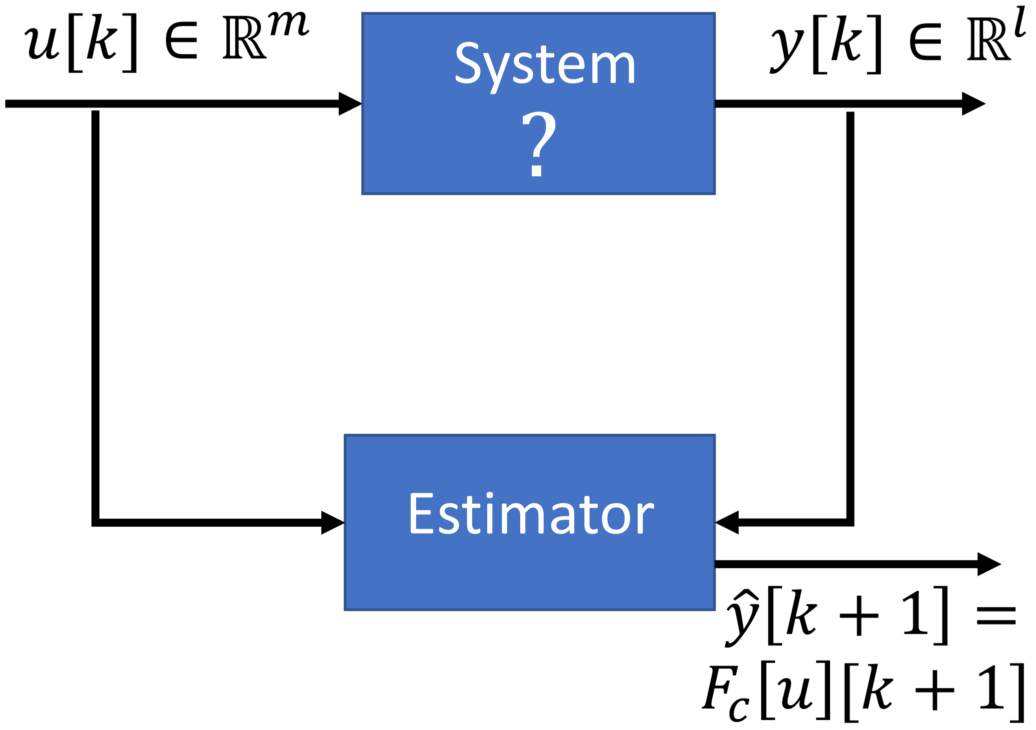
words

**Definition (Iterative integral):**

$$E_{x_i \nu}[u](t) := \int_0^t u_i(\tau) E_\nu[u](\tau) d\tau \,,$$

$$E_\phi[u](t) := 1$$ [Fliess, 1983]

**Remark:** we need to truncate to a word length to use them.




**Remark:** Chen-Fliess series does not estimate the system.

[Gray, Venkatesh, Duffaut Espinosa, 2019]

# Chen-Fliess Series

$u \rightarrow$ **System** $\rightarrow F_c[u](t)$

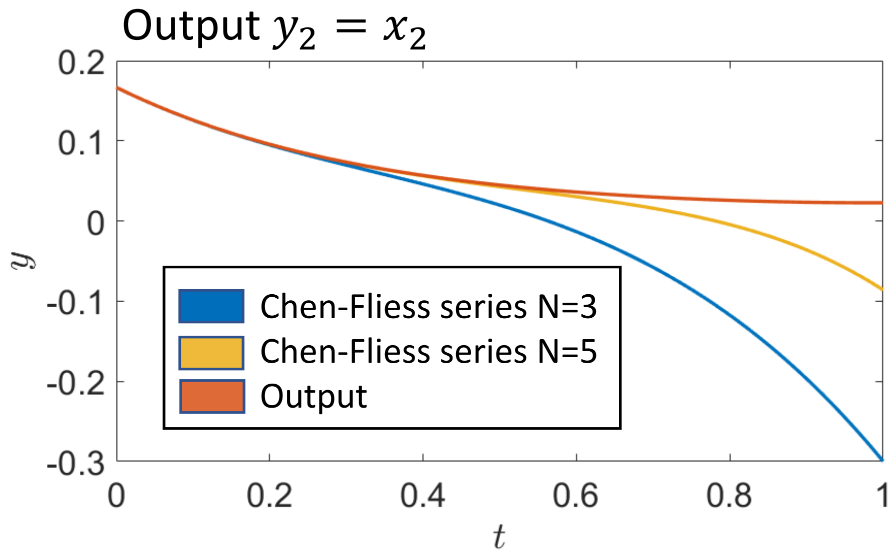
System

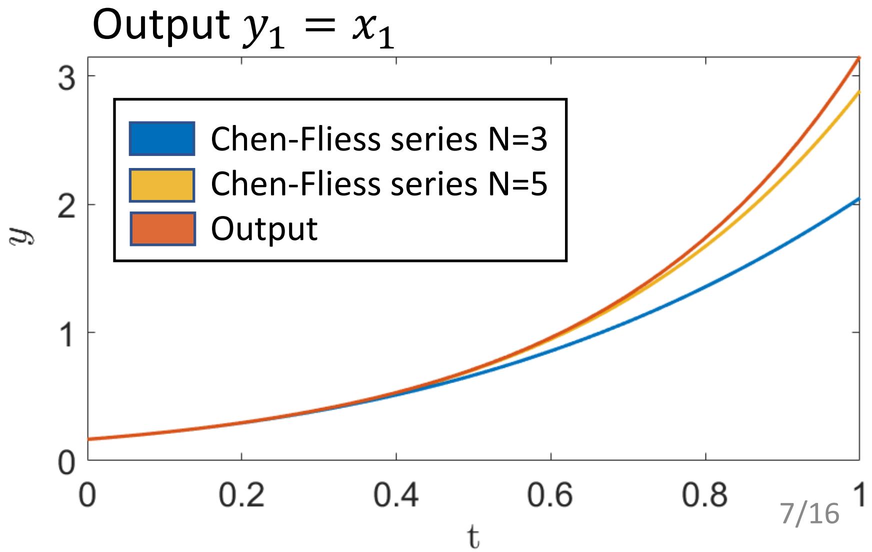
$$\dot{z} = g_0(z) + \sum_{i=1}^{m} g_i(z)u_i\,,$$
$$y_j = h_j(z), j = 1, \cdots, l,$$
$$z(0) = z_0$$

**Remark:** the coefficients are computed deterministically. [Fliess, 1983]

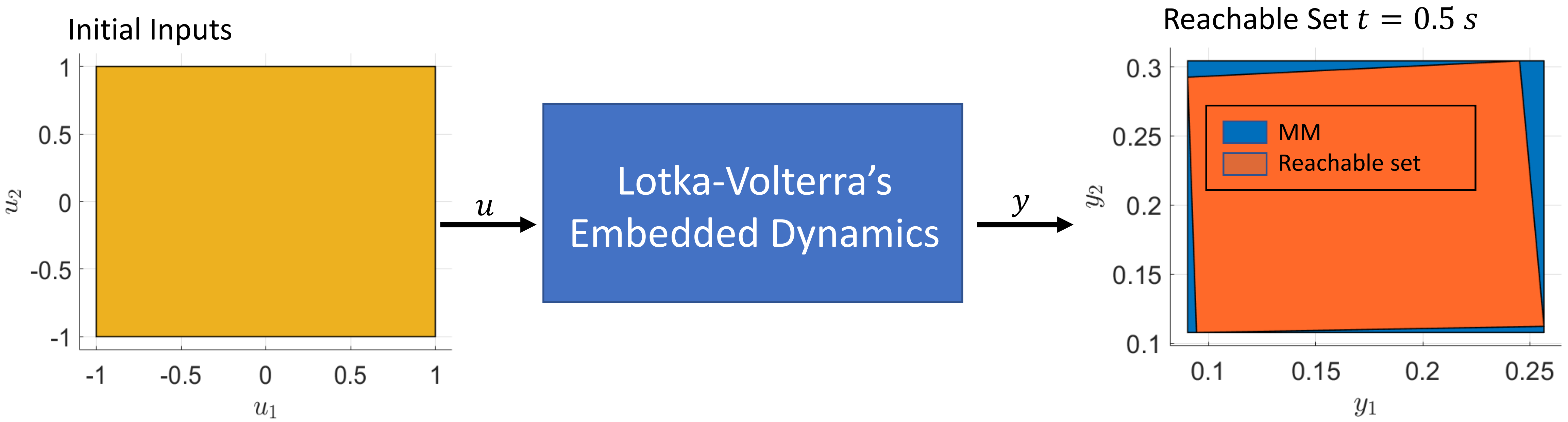
**Example:** Lotka-Volterra system

$$\dot{x}_1 = -x_1x_2 + x_1u_1,$$
$$\dot{x}_2 = x_1x_2 - x_2u_2,$$
$$y = x,$$
$$x(0) = (1/6,1/6),$$



# Mixed-Monotonicity (MM)




**Example:** Lotka-Volterra system

$\dot{x}_1 = -x_1 x_2 + x_1 u_1,$

$\dot{x}_2 = x_1 x_2 - x_2 u_2,$

$y = x,$

$x(0)$=(1/6,1/6)

$u_{1,2} \in [-1, 1]$

- The embedded dynamics generates two points of the overestimating hypercube: SW, NE.
- It is expressed in terms of a particular decomposition function.

[Abate, Dutreix, Coogan, 2020]

# Input-Output Mixed-Monotonicity (IOMM)

**Example:** Lotka-Volterra system

$$\dot{x}_1 = -x_1x_2 + x_1u_1,$$
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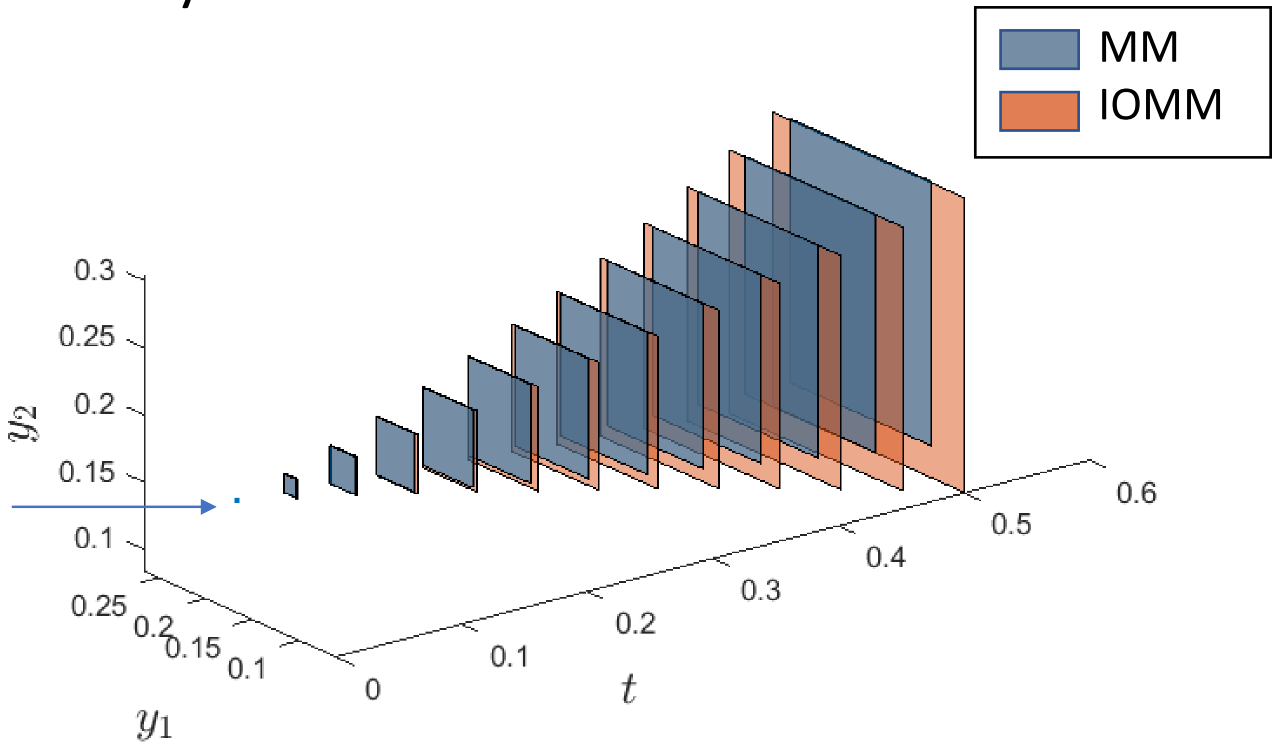
$$(u_1,u_2) \in [-1,1] \times [-1,1]$$

Initial state




**Remark:** IOMM contains MM

[Perez Avellaneda & Duffaut Espinosa, 2022a]

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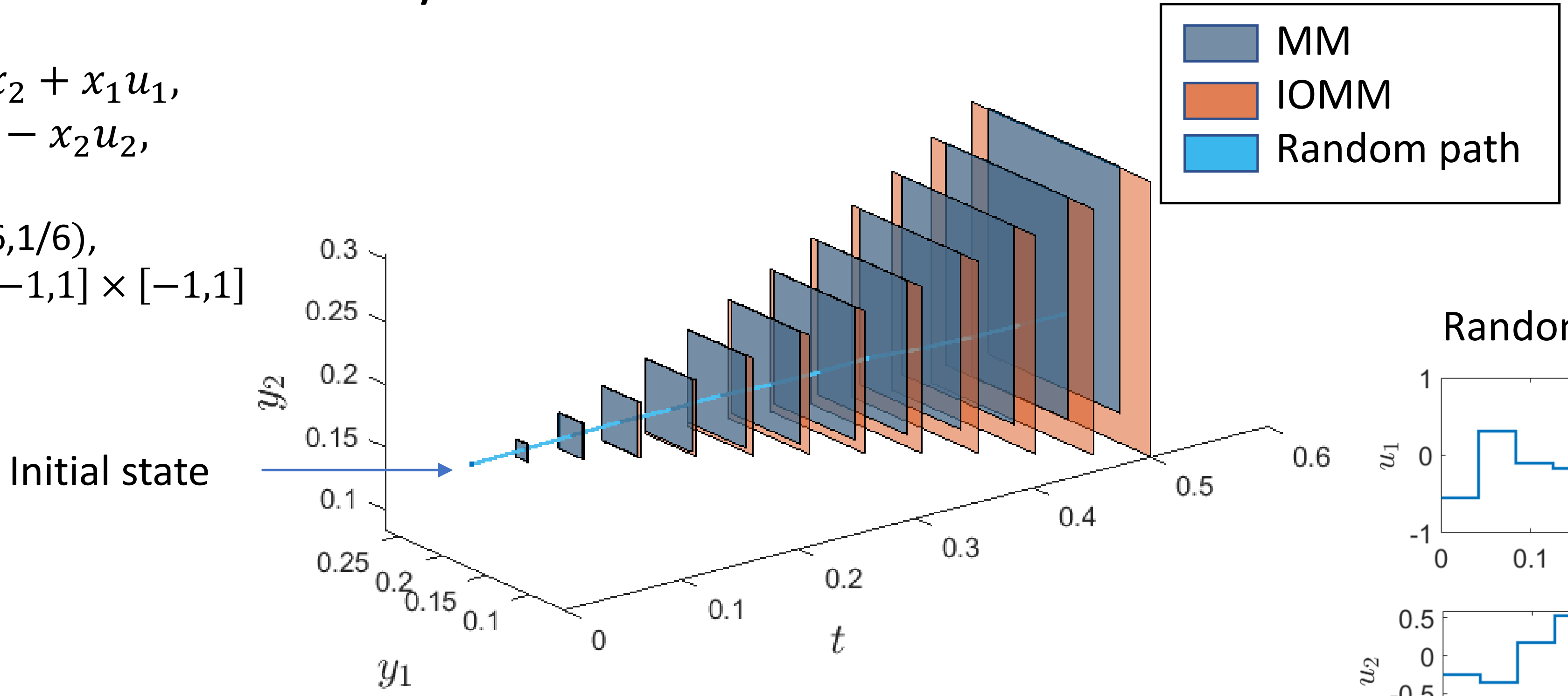
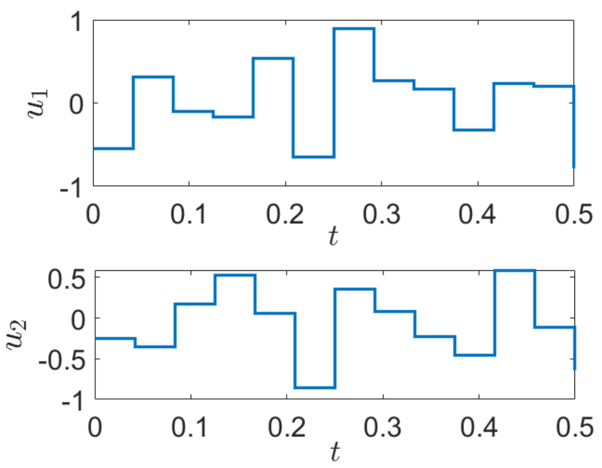



Random inputs

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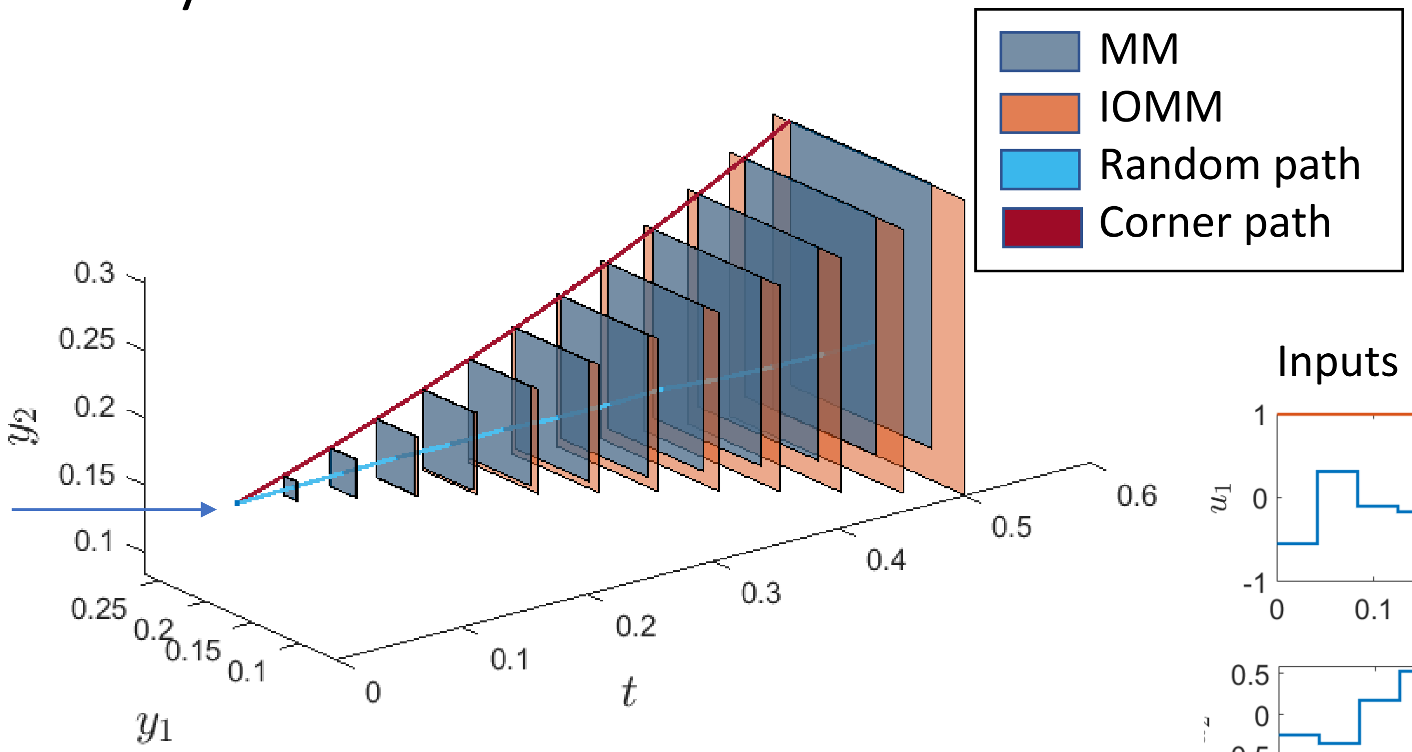
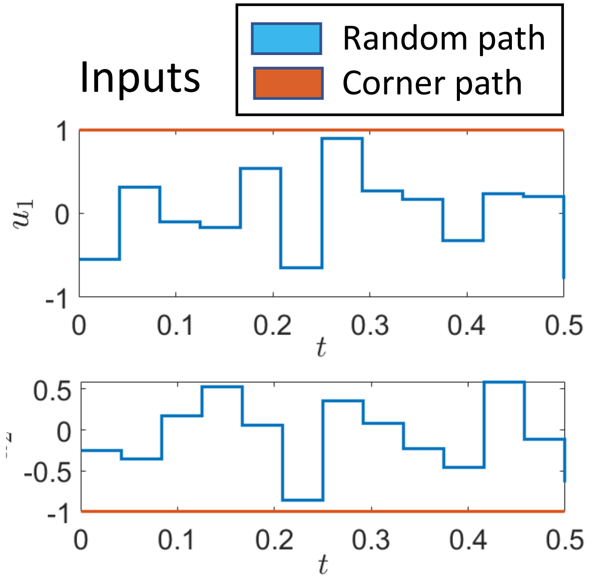



Inputs

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[Perez Avellaneda & Duffaut Espinosa, 2022a]

# Gradient Descent

**Problem:**

function

$$\min_{u \in \mathcal{U}} F_c[u](t)$$

Chen-Fliess series

**Theorem:** the Gâteaux derivative of a Chen-Fliess series in the direction of $v$ is the following:

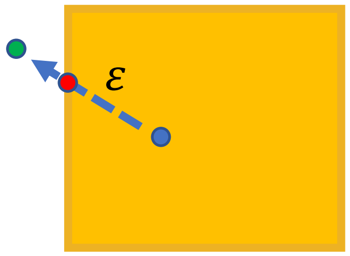
$$\frac{\partial}{\partial v} F_c[u](t) = \sum_{\eta \in X^*} \sum_{\xi \in \mathbb{S}_{X^*, Y^k}} (c, \sigma_X(\eta)) \mathcal{E}_\xi[u, v](t)$$

**Theorem:** the gradient of a Chen-Fliess series is the following:

$$\nabla F_c[u](t) = \left( \frac{\partial}{\partial e_1} F_c[u](t), \ldots, \frac{\partial}{\partial e_m} F_c[u](t) \right)$$

**Algorithm 2** Gradient Descent

**Input:** $R$, $u_0$, $\varepsilon$, $\mathcal{U}$
**Output:** $F_c[u](t)$
*Initialization* : $u_0$
**for** $i = 1$ to $R$ **do**
$u_{i+1} = u_i - \varepsilon \nabla F_c[u_i](t)$,
$u_{i+1} \leftarrow \mathrm{sat}_{\mathcal{U}}(u_{i+1})$
**end for**
**return** $u_R$




[Perez Avellaneda & Duffaut Espinosa, 2022b]

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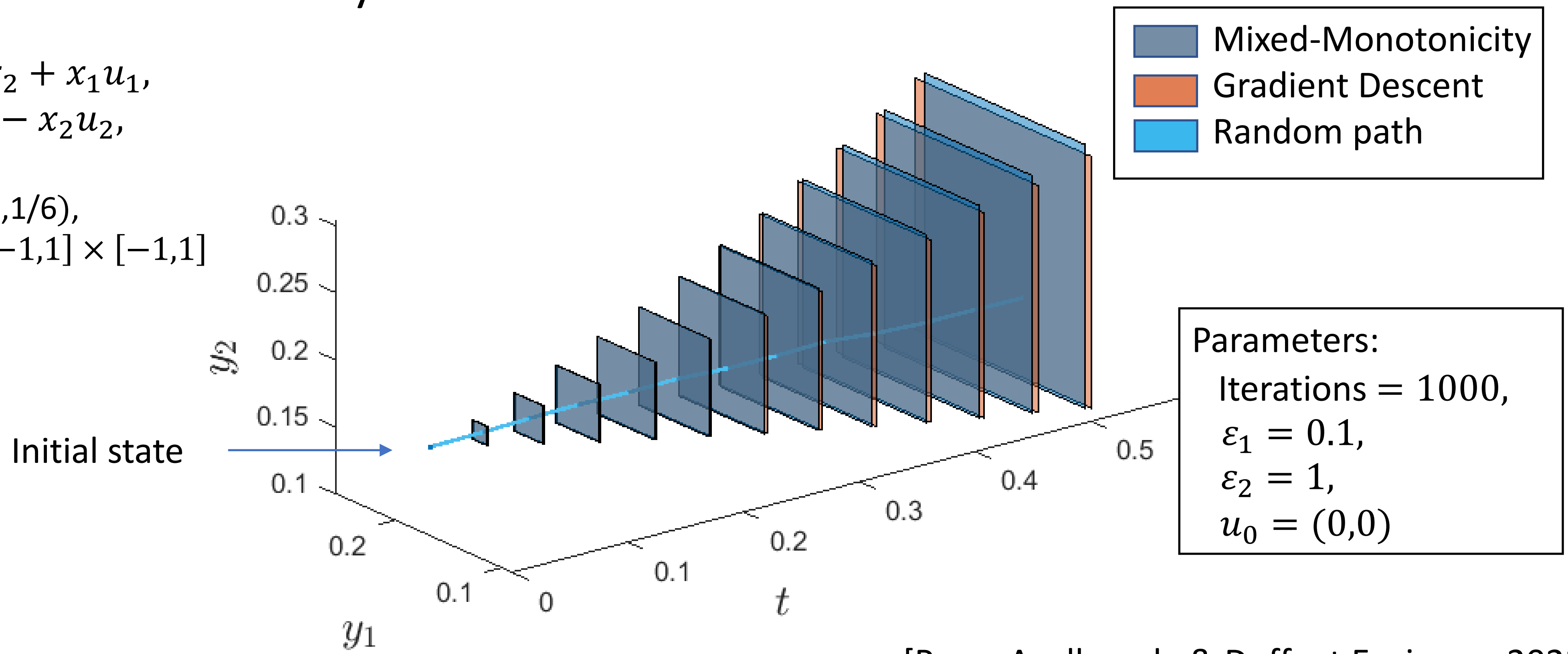
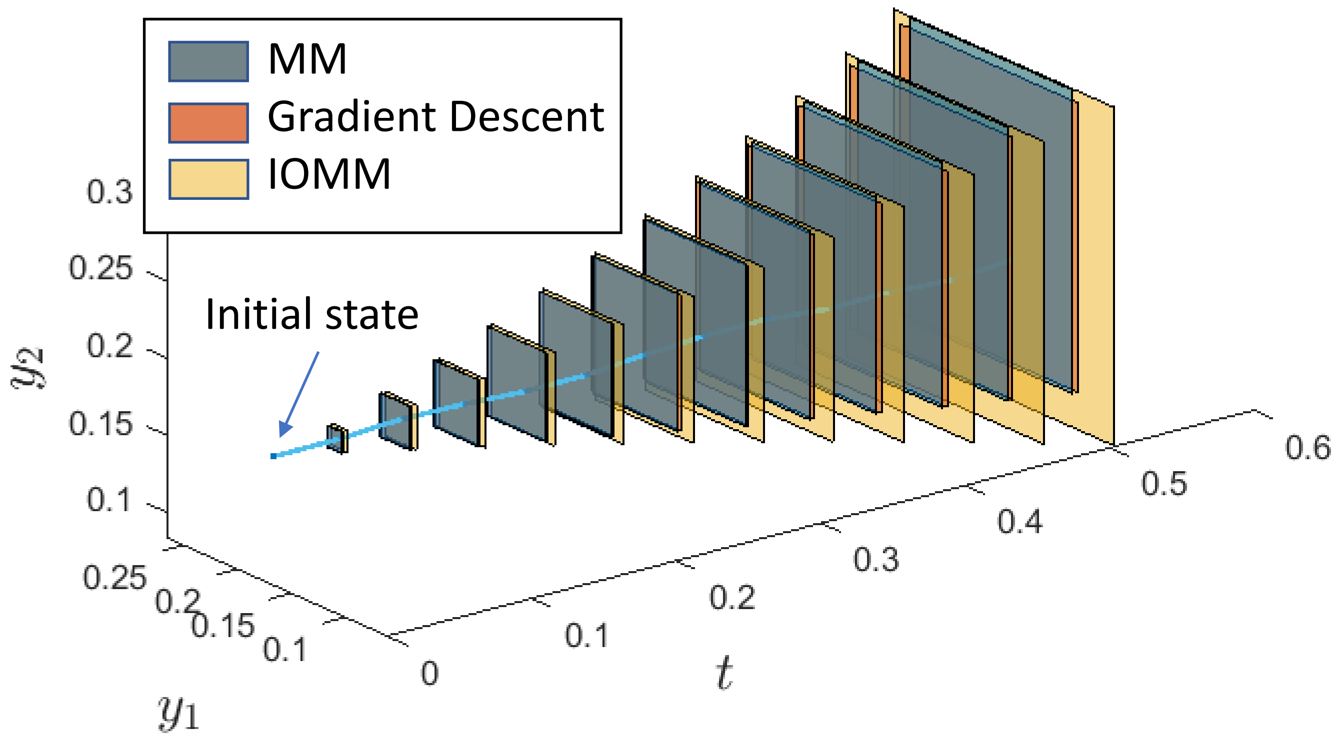


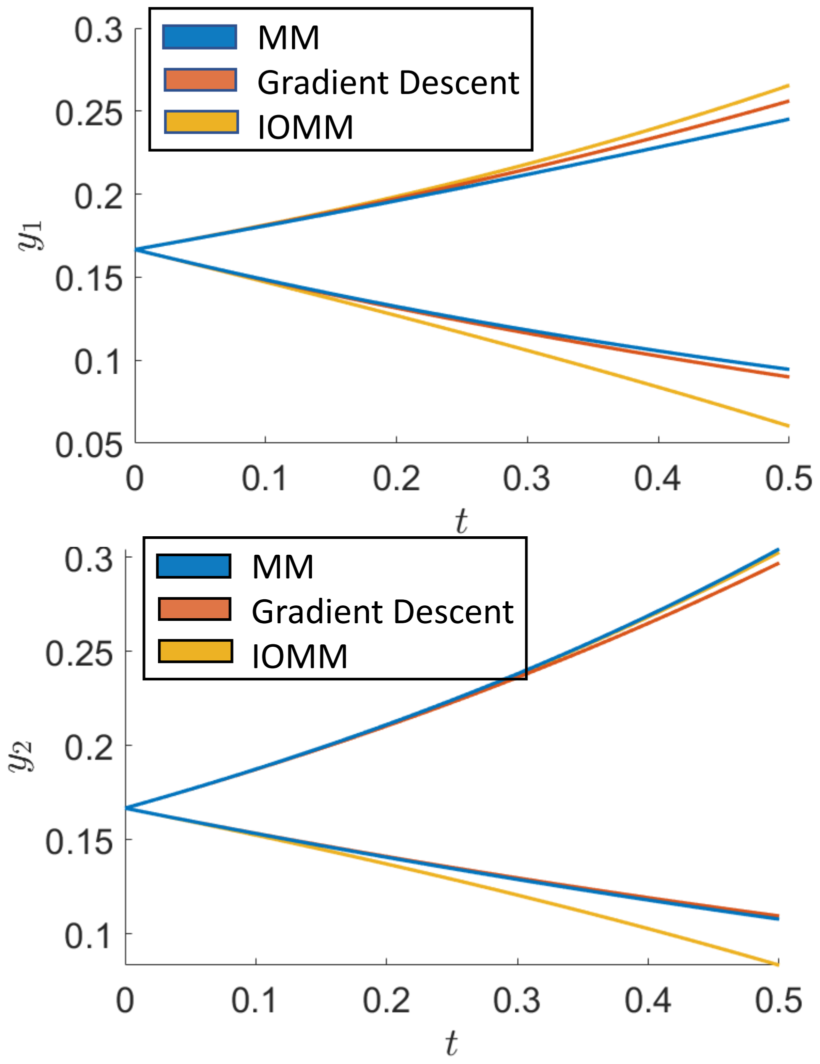

Parameters:
Iterations $= 1000$,
$\varepsilon_1 = 0.1$,
$\varepsilon_2 = 1$,
$u_0 = (0,0)$

[Perez Avellaneda & Duffaut Espinosa, 2022b]

**Remark:** the Gradient descent is close to MM in the convergence time range.

# Gradient Descent




**Remark:** the Gradient descent is closer to MM in the convergence time range.

[Perez Avellaneda & Duffaut Espinosa, 2022b]

# Conclusion and Future work

- Conclusion
  - The gradient descent approach estimates well the real reachable set of a system in the convergence time interval.
  - The gradient descent approach outperforms the IOMM approach.
  - The performance also depends on the truncation length of the words used in the Chen-Fliess series.

- Future work
  - Obtain a closed form of the inputs that zero the gradient of Chen-Fliess series.
  - Obtain the closed form of the truncation error of the gradient descent.
  - Obtain the reachable set of Chen-Fliess series with coefficients in a compact set.
  - Integrate the gradient descent approach with a Model Predictive Control setting to solve stability problems.

# Questions?

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