The Problem of Stationary Functions of Chen-Fliess Series

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Overview

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System Theory

Definition (Control system in state-space form)

Consider the continuous functions $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^\ell$, $u(t) = (u_1(t), \cdots, u_m(t))$ with $u_i(t) \in L_p([0, T])$ for a $T > 0$ and the point $z_0 \in \mathbb{R}^n$. The following set of equations is called a **control system** in **state-space** representation.

$$\dot{z}(t) = f(z(t), u(t)) \tag{1}$$

$$y(t) = h(z(t)) \tag{2}$$

$$z(0) = z_0 \tag{3}$$

where (1) is called state dynamics, (2) is the output of the system and (3) is the initial condition of the system.

Example (Linear time-invariant system)

Consider the matrix $A \in \mathbb{R}^{n\times n}$, the vector $B \in \mathbb{R}^{n\times 1}$, the function $u(t) \in L_p([0, T])$ for a $T > 0$ and $z_0 \in \mathbb{R}^n$.

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) \\ y(t) &= z(t) \\ z(0) &= z_0. \end{aligned} \tag{4}$$

Solving the state dynamics

$$y(t) = \exp(At)z_0 + \int_0^t \exp(A(t-\tau))Bu(\tau)d\tau. \tag{5}$$

Given the initial condition z_0 and the matrix A, equation (5) provides an **input-output** representation of system (4). Expressing the exponential in its series form, we have

$$y(t) = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!} z_0 + \sum_{k=0}^{\infty} A^k B \int_0^t \frac{(t-\tau)^k}{k!} u(\tau)d\tau$$

this is equivalently written in terms of **iterated integrals** as

$$\begin{aligned} y(t) = &\sum_{k=0}^{\infty} A^k z_0 \int_0^t \int_0^\tau \cdots \int_0^{\tau_{k-1}} d\tau_k \cdots d\tau + \\ &+ \sum_{k=0}^{\infty} A^k B \int_0^t \int_0^t \int_0^\tau \cdots \int_0^{\tau_{k-1}} u(\tau_k) d\tau_k \cdots d\tau \end{aligned} \tag{6}$$

which is the Peano-Baker series.

Definition (Nonlinear affine control system)

Consider the continuous functions $g_i : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^\ell$, $u(t) = (u_1(t), \cdots, u_m(t))$ with $u_i(t) \in L_p([0, T])$ for a $T > 0$ and the point $z_0 \in \mathbb{R}^n$. The following set of equations is called a **nonlinear affine control system** in **state-space** representation.

$$\begin{aligned}
\dot{z}(t) &= g_0(z(t)) + \sum_{i=1}^{m} g_i(z(t))u_i(t) \\
y(t) &= h(z(t)) \\
z(0) &= z_0
\end{aligned} \tag{7}$$

Formal Language Theory

- Alphabet: $X = \{x_0, \cdots, x_m\}$.
- Word over X: the noncommutative concatenation sequence of letters $\eta = x_{i_1} \cdots x_{i_n}$ for some $n \in \mathbb{N}$.
- The length of a word: $\eta = x_{i_1} \cdots x_{i_n}$ is denoted $|\eta| = n$.
- The set of all words in X of length k is denoted X^k.
- The set of words of any length is denoted X^*.

Definition (Free monoid)

A free monoid is referred to the tuple $(X^*, \cdot, \phi)$ where $\cdot$ is the concatenation product $\cdot : X^* \times X^* \to X^*$, $\cdot(\eta, \xi) = \eta\xi$ and ϕ is the empty word that works as the identity $\phi\eta = \eta\phi = \eta$ for every $\eta \in X^*$.

Example

Consider the alphabet $X = \{x_0, x_1\}$. The words $x_0 x_1$ and $x_1 x_0$ are not the same. Also, we use the notation x_i^k to refer to the word $\underbrace{x_i \cdots x_i}_{k \text{ times}}$.

Definition (Formal power series)

Given an alphabet $X = \{x_0, \cdots, x_m\}$ a formal power series c is any function of the form

$$c : X^* \to \mathbb{R}^\ell$$

the image of a word $\eta \in X^*$ under c is denoted by (c, η) and is called coefficient of η in c. We can write c as the formal summation

$$c = \sum_{\eta \in X^*} (c, \eta)\eta.$$

The set of all formal power series is denoted $\mathbb{R}^\ell\langle\langle X\rangle\rangle$.

Example

Given the alphabet $X = \{x_0, x_1\}$, we can form the formal power series

$$c_1 = \sum_{k=0}^{\infty} x_0^k + x_0^k x_1.$$

A similar more general formal power series $c \in \mathbb{R}^n \langle\langle X \rangle\rangle$ is given by considering the matrix $A \in \mathbb{R}^{n \times n}$ and the vector $B \in \mathbb{R}^n$ and $z_0 \in \mathbb{R}^n$

$$c_2 = \sum_{k=0}^{\infty} (A^k z_0) x_0^k + (A^k B) x_0^k x_1. \tag{8}$$

Definition (Iterated integrals)

The map $E_\eta : L_1^m[0,T] \to C[0,t]$ for $u(t) = (u_1(t), \cdots, u_m(t))$ where $u_i(t) \in L_1[0,T]$ is defined iteratively as $E_\phi[u](t) = 1$ and

$$E_{x_i\xi}[u](t) = \int_0^t u_i(\tau) E_\xi[u](\tau) d\tau$$

Example

Consider the alphabet $X = \{x_0, x_1, x_2\}$ with each letter associated to $u_0(t) = 1, u_1(t) = \sin(t), u_2(t) = \cos(t)$ respectively, the iterated integral of the word $\eta = x_0 x_1^2$ is the following:

$$\begin{aligned} E_{x_0x_1^2}[u](t) &= \int_0^t u_0(\tau_1) \int_0^{\tau_1} u_1(\tau_2) \int_0^{\tau_2} u_1(\tau_3) d\tau_3 d\tau_2 d\tau_1 \\ &= \int_0^t \int_0^{\tau_1} \sin(\tau_2) \int_0^{\tau_2} \sin(\tau_3) d\tau_3 d\tau_2 d\tau_1 \end{aligned}$$

Definition (Chen-Fliess series)

Given any formal power series over X,

$$c = \sum_{\eta \in X^*} (c, \eta)\eta,$$

where each $(c, \eta) \in \mathbb{R}^\ell$, one can uniquely specify an input-output operator as

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta) E_\eta[u](t)$$

this operator is known as Chen-Fliess series.

Example

Consider the alphabet $X = \{x_0, x_1\}$ associated to the function $u(t) = (u_0(t), u_1(t))$ with $u_0(t) = 1$. The Chen-Fliess series of the formal power series in equation (8) represents the linear time-invariant system (4)

$$y(t) = F_{c_2}[u](t) = \sum_{k=0}^{\infty} (A^k z_0) E_{x_0^k}[u](t) + (A^k B) E_{x_0^k x_1}[u](t)$$

Theorem (Fliess 1983)

Consider the nonlinear affine system in (7), its output is represented by the following Chen-Fliess series

$$F_c[u](t) = \sum_{\eta \in X^*} (c,\eta) E_\eta[u](t)$$

with coefficients

$$(c, x_{i_1} \cdots x_{i_k}) = L_{g_{i_k}} \cdots L_{g_{i_1}} h(z_0)$$

where

$$L_{g_{i_k}} \cdots L_{g_{i_1}} h(z_0) = \frac{\partial}{\partial z}\left(\cdots \left(\frac{\partial}{\partial z}\left(\frac{\partial h(z)}{\partial z} \cdot g_{i_1}(z)\right) \cdot g_{i_2}(z)\right) \cdots \right) \cdot g_{i_k}(z) \Bigg|_{z_0}$$

and the power series $c = \sum_{\eta \in X^*} (c,\eta)\eta$ *has finite Lie rank.*

Definition (Substitution homomorphism)

Consider the alphabets $X = \{x_0, \cdots, x_m\}$ and $Y = \{y_0, \cdots, y_m\}$. Define the function $\sigma_X : (XUY)^* \to X$ such that $\sigma_X(y_i) = x_i$

$$\sigma_X(z_{i_1}\eta) = \begin{cases} z_{i_1}\sigma_X(\eta), \;\; z_{i_1} \in X \\ \sigma_X(z_{i_1})\sigma_X(\eta), z_{i_1} \in Y \end{cases}$$

Example

Consider the word $\xi = x_1x_2y_1x_0$, then $\sigma_X(\xi) = x_1x_2x_1x_0$. Consider $\xi = y_1y_1x_3y_2$, then $\sigma_X(\xi) = x_1x_1x_3x_2$.

Definition (Shuffle set)

The following set is known as the shuffle set of two words $\eta, \xi \in X^*$:

$$\mathbb{S}_{\eta,\xi} = \{\nu \in X^* : \nu = \eta_1\xi_1\eta_2\xi_2 \cdots \eta_n\xi_n \in X^* \mid \\ \eta = \eta_{i_1}, \cdots, \eta_{i_n}, \xi = \xi_{i_1}, \cdots, \xi_{i_n}, \;\; n \geq 1\}$$

Example

Consider $\eta = x_1x_2$ and $\xi = x_3$, then

$$\mathbb{S}_{\eta,\xi} = \{x_1x_2x_3, x_1x_3x_2, x_3x_1x_2\}.$$

Consider $\eta = x_1x_2$ and $\xi = x_3x_4$, then

$$\mathbb{S}_{\eta,\xi} = \{x_1x_2x_3x_4, x_1x_3x_2x_4, x_3x_1x_2x_4, x_1x_3x_4x_2, x_3x_1x_4x_2, x_3x_4x_1x_2\}$$

Definition (Characteristic series)

The characteristic series of a language $L \subset X^*$ is the element in $\mathbb{R}\langle\langle X\rangle\rangle$ defined by $\mathrm{char}(L) = \sum_{\nu\in L} \nu$. Suppose, for example, $X = \{x_0, x_1\}$, then

$$\mathrm{char(X)} = x_0 + x_1$$

Derivatives in Banach spaces

Definition

Given $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ and the input functions $u \in L_1^m[0,t]$, the Chen-Fliess operator is Fréchet differentiable at u if and only if there exists $DF_c[u][.](t) : L_1^m[0,t] \to \mathbb{R}^l$ such that the following limit is satisfied:

$$\lim_{h \to 0} \frac{1}{||h||_p} \Big|\Big| F_c[u+h](t) - F_c[u](t) - DF_c[u][h](t) \Big|\Big|_q = 0.$$

Definition

Given $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ and the input functions $u,\ v \in L_{\mathfrak{p}}^m[0,t]$, the Chen-Fliess operator is Gâteaux differentiable at u in the direction of v if and only if there exists $\frac{\partial}{\partial v} F_c[u](t) \in \mathbb{R}^l$ such that the following limit is satisfied:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big(F_c[u+\varepsilon v](t) - F_c[u](t) - \frac{\partial}{\partial v} F_c[u](t)\varepsilon \Big) = 0.$$

Consider

$$F_c[u](t) = E_{x_1x_2x_3}[u](t)$$

then

$$F_c[u+\varepsilon v](t) = \int_0^t (u_1 + \varepsilon v_1)(\tau) \int_0^\tau (u_2 + \varepsilon v_2)(\tau_1) \int_0^{\tau_1} (u_3 + \varepsilon v_3)(\tau_2) d\tau_2 d\tau_1 d\tau$$
$$= \int_0^t u_1(\tau) \int_0^\tau u_2(\tau_1) \int_0^{\tau_1} u_3(\tau_2) d\tau_2 d\tau_1 d\tau + \varepsilon \int_0^t v_1(\tau) \int_0^\tau u_2(\tau_1) \int_0^{\tau_1} u_3(\tau_2) d\tau_2 d\tau_1 d\tau +$$
$$\varepsilon \int_0^t u_1(\tau) \int_0^\tau v_2(\tau_1) \int_0^{\tau_1} u_3(\tau_2) d\tau_2 d\tau_1 d\tau + \varepsilon \int_0^t u_1(\tau) \int_0^\tau u_2(\tau_1) \int_0^{\tau_1} v_3(\tau_2) d\tau_2 d\tau_1 d\tau +$$
$$\varepsilon^2 \int_0^t v_1(\tau) \int_0^\tau v_2(\tau_1) \int_0^{\tau_1} u_3(\tau_2) d\tau_2 d\tau_1 d\tau + \varepsilon^2 \int_0^t v_1(\tau) \int_0^\tau u_2(\tau_1) \int_0^{\tau_1} v_3(\tau_2) d\tau_2 d\tau_1 d\tau +$$
$$\varepsilon^2 \int_0^t u_1(\tau) \int_0^\tau v_2(\tau_1) \int_0^{\tau_1} v_3(\tau_2) d\tau_2 d\tau_1 d\tau + \varepsilon^3 \int_0^t v_1(\tau) \int_0^\tau v_2(\tau_1) \int_0^{\tau_1} v_3(\tau_2) d\tau_2 d\tau_1 d\tau +$$

Note: one alphabet is not enough

Note 2: this is similar to $(x_1 + \varepsilon y_1)(x_2 + \varepsilon y_2)(x_3 + \varepsilon y_3)$

Definition

Consider the alphabets X and Y associated with $u,\ v \in L_{\mathfrak{p}}^m[0,T]$, respectively. The iterated integral of $\eta \in Z^*$ for the input $u \times v$ is given by the mapping $\mathcal{E}_\eta : L_{\mathfrak{p}}^m[0,T] \times L_{\mathfrak{p}}^m[0,T] \to \mathcal{C}[0,T]$, where $\mathcal{E}_\emptyset[u,v](t) = 1$ and

$$\mathcal{E}_{z_i\eta}[u,v](t) := \begin{cases} \displaystyle\int_0^t u_i(\tau)\mathcal{E}_\eta[u,v](\tau)d\tau, & z_i \in X, \\ \displaystyle\int_0^t v_i(\tau)\mathcal{E}_\eta[u,v](\tau)d\tau, & z_i \in Y. \end{cases} \tag{9}$$

The problem

Consider a series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, the compact set $\mathcal{U} \subset L_p^m[0, T]$, we want to solve the following problem.

$$\min_{u\in\mathcal{U}} F_c[u](t) \tag{10}$$

Find $u^*(t) \in L_p^m[0, T]$ such that

$$DF_c[u^*][h](t) = 0, \forall h \in L_p^m[0, T]$$

Consider the alphabet $\delta X = \{\delta x_1, \cdots, \delta x_m\}$.

Definition (Differential monoid)

The tuple $(Z, \odot, \phi, \delta)$ where $Z = X \cup \delta X$, $\mathcal{C}$ is concatenation operation, ϕ is the empty word and the derivation function $\delta : Z^* \to Z^*$ is defined for $\eta \in \mathbb{S}_{X^{n_1}, \delta X^{n_2}}$ for $n_1 \in \mathbb{N}^+$ and $n_2 \in \mathbb{N}$ as

$$\mathbb{D}_\eta := \{\xi \in \mathbb{S}_{X^{n_1 - 1}, \delta X^{n_2 + 1}} \mid \sigma_X(\xi) = \eta\}$$
$$\delta(\eta) := \mathrm{char}(\mathbb{D}_\eta)$$

and for $n_1 = 0$, $\delta(\eta) = \delta(\phi) = 0$.

Example

Let $\eta = x_0 x_{i_1} \in X^2$. Note that $\xi = x_0 \delta x_{i_1}$ is the only element in $\mathbb{S}_{X, \delta X}$ such that $\sigma_X(\xi) = \sigma_X(\eta)$. Then $\delta(x_0 x_{i_1}) = x_0 \delta x_{i_1}$. Hence, x_0 behaves as a constant with respect to δ.

Example

Let $\eta = x_1 x_2 \in X^2$, then $\mathbb{D}_\eta = \{\delta x_1 x_2, x_1 \delta x_2\}$ and

$$\delta(x_1 x_2) = \mathrm{char}(\mathbb{D}_\eta) = \delta x_1 x_2 + x_1 \delta x_2,$$

which matches Leibniz's derivative rule.

Lemma

The derivative of $\eta \in X^n$ satisfies the following properties:

1. $\delta(\eta) = \sum_{j=1}^{n} x_{i_1} \cdots x_{i_{j-1}} \delta x_{i_j}\ x_{i_{j+1}} \cdots x_{i_n}$
2. $\delta^2(\eta) = 0$, *for* $|\eta|_X = 0$ *or* 1

Example

Let $\eta = x_{i_1} x_{i_2} x_{i_3}$, then

$$\begin{aligned}
\delta(x_{i_1} x_{i_2} x_{i_3}) &= \delta x_{i_1} x_{i_2} x_{i_3} + x_{i_1} \delta x_{i_2} x_{i_3} + x_{i_1} x_{i_2} \delta x_{i_3} \\
\delta^2(x_{i_1} x_{i_2} x_{i_3}) &= 2!(\delta x_{i_1} \delta x_{i_2} x_{i_3} + x_{i_1} \delta x_{i_2} \delta x_{i_3} + \delta x_{i_1} x_{i_2} \delta x_{i_3}) \\
\delta^3(x_{i_1} x_{i_2} x_{i_3}) &= 3! \delta x_{i_1} \delta x_{i_2} \delta x_{i_3}.
\end{aligned}$$

$$
\begin{tikzcd}
 & & 0 & & \\
\phi \arrow[urr, "\delta"] & \mathbb{S}_{X^0,\delta X^1} \arrow[ur, "\delta"] & \mathbb{S}_{X^0,\delta X^2} \arrow[u, "\delta"] & \mathbb{S}_{X^0,\delta X^3} \arrow[ul, "\delta"'] & \cdots \arrow[ull, "\delta"'] \\
\mathbb{S}_{X^1,\delta X^0} \arrow[ur, "\delta"] & \mathbb{S}_{X^1,\delta X^1} \arrow[ur, "\delta"] & \mathbb{S}_{X^1,\delta X^2} \arrow[ur, "\delta"] & \mathbb{S}_{X^1,\delta X^3} & \cdots \\
\mathbb{S}_{X^2,\delta X^0} \arrow[ur, "\delta"] & \mathbb{S}_{X^2,\delta X^1} \arrow[ur, "\delta"] & \mathbb{S}_{X^2,\delta X^2} \arrow[ur, "\delta"] & \mathbb{S}_{X^2,\delta X^3} & \cdots \\
\mathbb{S}_{X^3,\delta X^0} \arrow[ur, "\delta"] & \mathbb{S}_{X^3,\delta X^1} \arrow[ur, "\delta"] & \mathbb{S}_{X^3,\delta X^2} \arrow[ur, "\delta"] & \mathbb{S}_{X^3,\delta X^3} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$\delta^k(\eta) = k! \; \mathrm{char}(\mathbb{D}_\eta^k), \tag{11}$$

Lemma

The k-th derivative of $char(X^*)$ satisfies

$$\delta^k(char(X^*)) = k! char(\mathbb{S}_{X^*,\delta X^k}). \tag{12}$$

Lemma

The k-th derivative of $c \in \mathbb{R}\langle\langle X \rangle\rangle$ satisfies

$$\delta^k(c) = k! \sum_{\xi \in \mathbb{S}_{X^*,\delta X^k}} (c, \sigma_X(\xi))\xi \tag{13}$$

Additionally, the linearity of δ and (12) allow to write

$$\delta^k(c) = \sum_{\eta \in X^*} (c, \eta) \delta^k(\eta).$$

Lemma

Let $(Z, \odot, \emptyset, \delta)$ *be a differential monoid. For* $k, r \in \mathbb{N}$*, it follows that*

$$\frac{1}{k!}\delta^k\left(char\, \mathbb{S}_{X^*,\delta X^r}\right) = \binom{r+k}{r} char\left(\mathbb{S}_{X^*,\delta X^{r+k}}\right) \tag{14}$$

and, for $c \in \mathbb{R}\langle\langle X \rangle\rangle$*, one has that*

$$\sum_{\xi \in \mathbb{S}_{X^*,\delta X^r}} \frac{1}{k!}(c, \sigma_X(\xi))\delta^k(\xi) = \binom{r+k}{r} \sum_{\xi \in \mathbb{S}_{X^*,\delta X^{r+k}}} (c, \sigma_X(\xi))\xi.$$

Lemma

Given $c \in \mathbb{R}\langle\langle X \rangle\rangle$, the CFS of the sum of u and v is written as

$$F_c[u+v](t) = \sum_{k=0}^{\infty} \sum_{\eta \in X^*} \frac{1}{k!}(c,\eta) E_{\delta^k(\eta)}[u,v](t)$$

Notice that if the exponential of the derivative of $c \in \mathbb{R}^{\ell}\langle\langle X \rangle\rangle$ is defined as

$$e^{\delta(c)} = \sum_{k=0}^{\infty} \frac{1}{k!} \delta^k(c)$$

then Chen-Fliess series of the sum of two inputs $u, v \in L_{\mathfrak{p}}^m[t_0, t_1]$ is expressed as

$$F_c[u+v](t) = e^{\delta(c)}.$$

Theorem

Given $c \in \mathbb{R}\langle\langle X \rangle\rangle$, the Gâteaux derivative of $F_c[u](t)$ in the direction of v is

$$\frac{\partial}{\partial v} F_c[u](t) = \sum_{\xi \in \mathbb{S}_{X^*, \delta X}} (c, \sigma_X(\xi)) \mathcal{E}_\xi[u, v](t).$$

Theorem

Consider the differential monoid $(Z, \odot, \emptyset, \delta)$, the derivation δ and the Gâteaux derivative $\frac{\partial}{\partial v}$ obey the following commutative diagram

$$\begin{tikzcd}
\mathbb{R}^\ell\langle\langle X \rangle\rangle \arrow[r, "F_{[\cdot]}[u](t)"] \arrow[d, "\delta"'] & L_{\mathfrak{q}}[t_0, t_1] \arrow[d, "\frac{\partial}{\partial v}"] \\
\mathbb{R}^\ell\langle\langle Z \rangle\rangle \arrow[r, "F_{[\cdot]}[u](t)"] & L_{\mathfrak{q}}[t_0, t_1]
\end{tikzcd}$$

This means

$$\frac{\partial}{\partial v} F_c[u](t) = F_{\delta(c)}[u](t) = \sum_{\eta \in X^*} (c, \eta) E_{\delta(\eta)}[u](t)$$

$$\vdots$$

$$\frac{\partial^k}{\partial v^k} F_c[u](t) = F_{\delta^k(c)}[u](t) = \sum_{\eta \in X^*} (c, \eta) E_{\delta^k(\eta)}[u](t)$$

or equivalently

$$\frac{\partial}{\partial v} F_c[u](t) = \sum_{\xi \in \mathbb{S}_{X^*, \delta X}} (c, \sigma_X(\xi)) \mathcal{E}_\xi[u, v](t)$$

$$\vdots$$

$$\frac{\partial^k}{\partial v^k} F_c[u](t) = k! \sum_{\xi \in \mathbb{S}_{X^*, \delta X^k}} (c, \sigma_X(\xi)) \mathcal{E}_\xi[u, v](t)$$

Partial derivative

Example

Consider $\eta = x_0 x_1 x_2 x_1$ and compute $\delta_{x_1}(\eta)$, then $\delta_{x_1}(x_0 x_1 x_2 x_1) = x_0 \delta x_1 x_2 x_1 + x_0 x_1 x_2 \delta x_1$. Similarly, $\delta_{x_2}(\eta) = x_0 x_1 \delta x_2 x_1$. $\square$

Lemma

Consider $c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$, the Gâteaux derivative in the i-th canonical direction satisfies

$$\frac{\partial}{\partial u_i} F_c[u](t) = F_{\delta_{x_i}(c)}[u](t).$$

Define the elementary functions $e_i : [0, T] \to \mathbb{R}^m$, such that $e_1(t) = (1, 0, \cdots, 0)^\top, \ldots, e_m(t) = (0, 0, \cdots, 1)^\top$. Thus, the Gâteaux derivative in the u_i direction is

$$\frac{\partial}{\partial u_i} F_c[u](t) = \sum_{\eta \in X^*} \sum_{\xi \in \mathbb{S}_{\eta, \delta x_i}} (c, \sigma_X(\xi)) \mathcal{E}_\xi[u, e_i](t).$$

$\nabla F_c : L_{\mathfrak{p}}^m[t_0, t_1] \to L_{\mathfrak{p}}^m[t_0, t_1]$ such that

$$\nabla F_c[u](t) = \left(\frac{\partial}{\partial u_1} F_c[u](t), \cdots, \frac{\partial}{\partial u_m} F_c[u](t) \right)^T. \tag{15}$$

Lemma

Consider the constant vector $v \in \mathbb{R}^m$, $u \in L_{\mathfrak{p}}^m[0, t]$ and the Chen-Fliess series $F_c[u](t)$, the Gateaux derivative and the gradient are related by

$$\frac{\partial}{\partial v} F_c[u](t) = v^T \nabla F_c[u](t).$$

Using the formula of the sum, we get

$$F_c[u+\varepsilon v](t) = F_c[u](t) + v^T \nabla F_c[u](t)\varepsilon + \sum_{k=2}^{\infty} \sum_{\eta \in X^*} \sum_{\xi \in \mathbb{S}_{\eta,\delta X^k}} (c, \sigma_X(\xi)) \mathcal{E}_\xi[u,v](t)\varepsilon^k. \quad (16)$$

Second order derivation

Lemma

Let $c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ *and* $u \in L_{\mathfrak{p}}^m[t_0, t_1]$, *then*

$$\frac{\partial^2}{\partial u_j \partial u_i} F_c[u](t) = \sum_{\xi \in \mathbb{S}_{X^*,\{\delta x_i \delta x_j, \delta x_j \delta x_i\}}} (c, \sigma_X(\xi)) \mathcal{E}_\xi[u, e_{i,j}](t)$$

where $e_{i,j}(t) = (0, \cdots, \underbrace{1}_{i-th}, 0, \cdots, \underbrace{1}_{j-th}, \cdots, 0)$.

Notice that when the second derivatives exist, they satisfy Schwarz Theorem of the symmetry of second order differentiation

$$\frac{\partial^2}{\partial u_j \partial u_i} F_c[u](t) = \frac{\partial^2}{\partial u_i \partial u_j} F_c[u](t)$$

Definition

Let $c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ and $u \in L_{\mathfrak{p}}^m[t_0, t_1]$. The Hessian of $F_c[u](t)$ is given by

$$\nabla^2 F_c[u](t) = \begin{bmatrix} 2\frac{\partial^2}{\partial u_1^2} F_c[u](t) & \cdots & \frac{\partial^2}{\partial u_1 \partial u_m} F_c[u](t) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial u_m \partial u_1} F_c[u](t) & \cdots & 2\frac{\partial^2}{\partial u_m^2} F_c[u](t) \end{bmatrix}.$$

Lemma

For $c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$, the Gâteaux derivative and the Hessian are related as

$$\frac{\partial^2}{\partial v^2} F_c[u](t) = v^T \nabla^2 F_c[u](t) v.$$

Lemma

For $c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ and $\varepsilon > 0$, one has that

$$F_c[u+\varepsilon v](t) = F_c[u](t) + v^T \nabla F_c[u](t)\varepsilon \frac{1}{2} v^T \nabla^2 F_c[u](t) v \varepsilon^2 + \sum_{k=3}^{\infty} \sum_{\xi \in \mathbb{S}_{X^*,\delta(X)^k}} (c, \sigma_X(\xi)) \mathcal{E}_\xi[u,v](t)\varepsilon^k$$

Lemma

Let $r \in \mathbb{R}$, $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$, and X and Y be alphabets associated to $u \in L_{\mathfrak{p}}^m[t_0, t_1]$, $v \in \mathbb{R}^m$ respectively. It follows that

$$\frac{1}{2} v^T \nabla^2 F_c[u+rv](t) v = \sum_{k=2}^{\infty} \sum_{\xi \in \mathbb{S}_{X^*,\delta X^k}} \binom{k}{2} r^{k-2} (c, \sigma_X(\xi)) \mathcal{E}_\xi[u,v](t).$$

Sketch of the proof: Calculating the CFS of the expression in Lemma 28 for $r = 2$.

Theorem

Let $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ *and* $\varepsilon > 0$. *Then there exists* $\varepsilon_0 \in (0, \varepsilon)$ *such that*

$$F_c[u + \varepsilon v] = F_c[u] + v^T \nabla F_c[u + \varepsilon_0 v](t) \tag{17}$$

Theorem

Let $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ *and* $\varepsilon > 0$. *Then there exists* $\varepsilon_0 \in (0, \varepsilon)$ *such that*

$$\begin{aligned} F_c[u + \varepsilon v] &= F_c[u] + v^T \nabla F_c[u](t)\varepsilon \\ &\quad + \frac{1}{\varepsilon_0} \int_0^{\varepsilon_0} \frac{1}{2} v^T \nabla^2 F_c[u + rv](t) v \varepsilon^2 dr. \end{aligned} \tag{18}$$

Proof: Define the function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned}\phi(\gamma) = &\int_0^\gamma \int_0^\theta \frac{1}{2} v^T \nabla^2 F_c[u + rv](t) v \varepsilon^2 dr \\ &- (F_c[u + \varepsilon v](t) - F_c[u](t) - v^T \nabla F_c[u](t)\varepsilon)\frac{1}{2}\gamma^2.\end{aligned} \tag{19}$$

Applying Lemmas 37 and 39 and by direct integration with respect to r, it follows that

$$\begin{aligned}\int_0^\gamma \int_0^\theta &\frac{1}{2} v^T \nabla^2 F_c[u + rv](t) v \varepsilon^2 dr = \\ &\frac{1}{4} v^T \nabla^2 F_c[u](t) v \varepsilon^2 \gamma^2 + \\ &+ \sum_{k=3}^{\infty} \sum_{\xi \in \mathbb{S}_{X^*, \delta X^k}} \frac{1}{2}(c, \sigma_X(\xi)) \mathcal{E}_\xi[u, v](t) \varepsilon^2 \gamma^k\end{aligned} \tag{20}$$

Using Lemma (38), the second term in the right hand side of (19) can also be written as

$$
\begin{aligned}
(F_c[u+\varepsilon v](t) - F_c[u](t) - v^T \nabla F_c[u](t)\varepsilon)\frac{1}{2}\gamma^2 = \\
\frac{1}{4} v^T \nabla^2 F_c[u](t) v \varepsilon^2 \gamma^2 + \\
+ \sum_{k=3}^{\infty} \sum_{\xi \in \mathbb{S}_{X^*,\delta X^k}} \frac{1}{2}(c, \sigma_X(\xi))\mathcal{E}_\xi[u,v](t)\varepsilon^k \gamma^2.
\end{aligned}
\tag{21}
$$

The fact that $\phi(\varepsilon) = 0$ follows from using (20), (21) and making $\gamma = \varepsilon$ in (19). Also, it is easy to see that $\phi(0) = 0$. Thus, by the continuity of $F_c[u]$, Rolle's Theorem guarantees the existence of $\varepsilon_0 \in (0, \varepsilon)$ such that the derivative of ϕ at ε_0 is zero. That is,

$$
\begin{aligned}
\phi'(\varepsilon_0) = & \int_0^{\varepsilon_0} \frac{1}{2} v^T \nabla^2 F_c[u + rv](t) v \varepsilon^2 dr \\
& - (F_c[u+\varepsilon v](t) - F_c[u](t) - v^T \nabla F_c[u](t)\varepsilon)\varepsilon_0 \\
& = 0.
\end{aligned}
$$

Back to the problem

Find $u^*(t) \in L_p^m[0, T]$ such that

$$DF_c[u^*][h](t) = 0, \forall h \in L_p^m[0, T]$$

Gradient descent algorithm:

$$u_{i+1} = u_i - \varepsilon \nabla F_c[u_i](t) \tag{22}$$

For example

$$u_2 = u_1 - \varepsilon \nabla F_c[u_0 - \varepsilon \nabla F_c[u_0](t)](t) \tag{23}$$

If $u(t) = u \in \mathbb{R}$ is a constant function, then $F_c^N[u](t)$ is a polynomial. For example,

$$\begin{aligned} F_c[u](t) &= E_{x_1}[u](t) + E_{x_1^2}[u](t) \\ &= \int_0^t u(\tau)d\tau + \int_0^t u(\tau) \int_0^{\tau_1} u(\tau_1)d\tau d\tau_1 \\ &= u \int_0^t d\tau + u^2 \int_0^t \int_0^{\tau_1} d\tau d\tau_1 \\ &= ut + u^2 \frac{t^2}{2} \end{aligned}$$

We can always provide a solution as in Galois theory by field extension.

An example of the more general case is the following:

$$F_c[u](t) = E_{x^2}[u](t) = \int_0^t u(\tau) \int_0^\tau u(\tau_1) d\tau_1 d\tau$$

the Fréchet derivative is

$$\begin{aligned} DF_c[u][h](t) &= \int_0^t u(\tau) \int_0^\tau h(\tau_1) d\tau_1 d\tau + \int_0^t h(\tau) \int_0^\tau u(\tau_1) d\tau_1 d\tau \\ &= \mathcal{E}_{x \sqcup\!\sqcup\, \delta x}[u, h](t) \\ &= E_x[u](t) E_{\delta x}[h](t) \end{aligned}$$

then $DF_c[u][h](t) = 0$, implies $E_x[u](t)E_{\delta x}[h](t) = 0$ and $E_x[u](t) = 0$. Solutions to this are $u^* = 0$ and symmetric functions on the interval $[0, t]$.
Remark: there is a factorization concept implicit in some sum of iterated integrals in two alphabets.

Thank you!

https://iperezav.github.io