On Mixed-Monotonicity of Chen-Fliess series*

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Outline

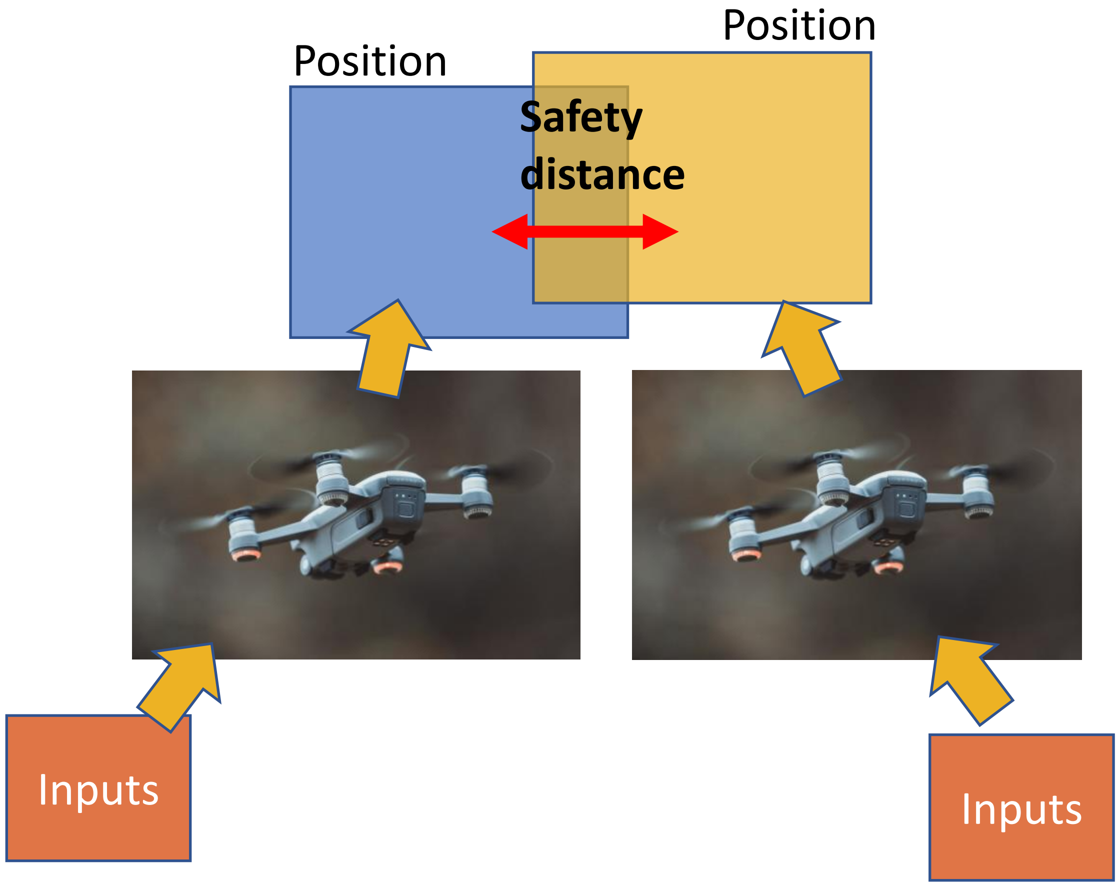
1. Motivation and Problem Statement

Definition 1 (Reachable set):

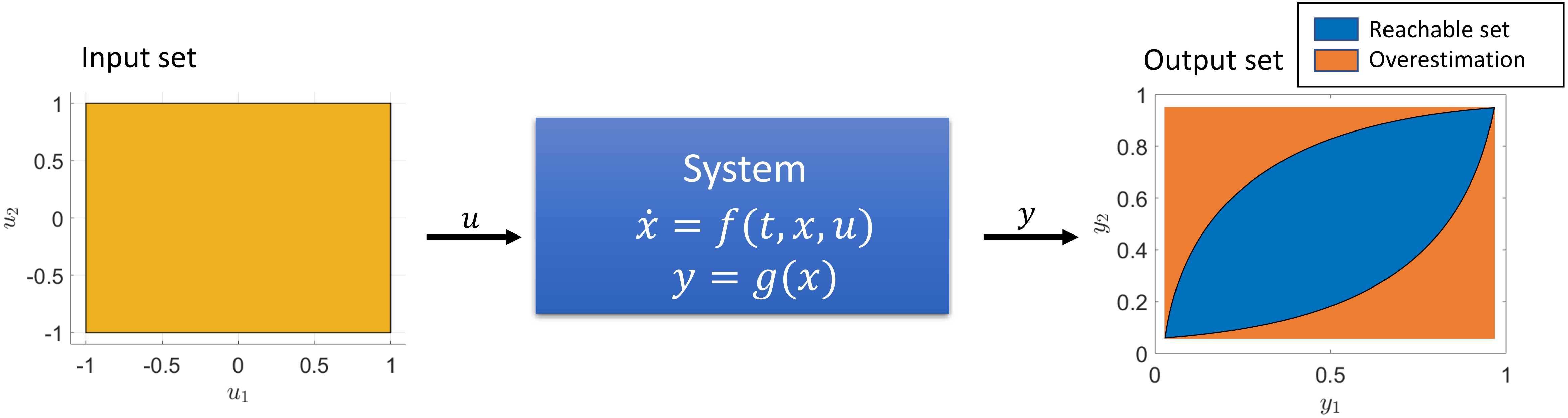
The set of outputs of the system as a response to a set of inputs and a set of initial conditions.

Applications:

1. Collision avoidance of quadcopters
2. Power systems safety
3. Robotics safety

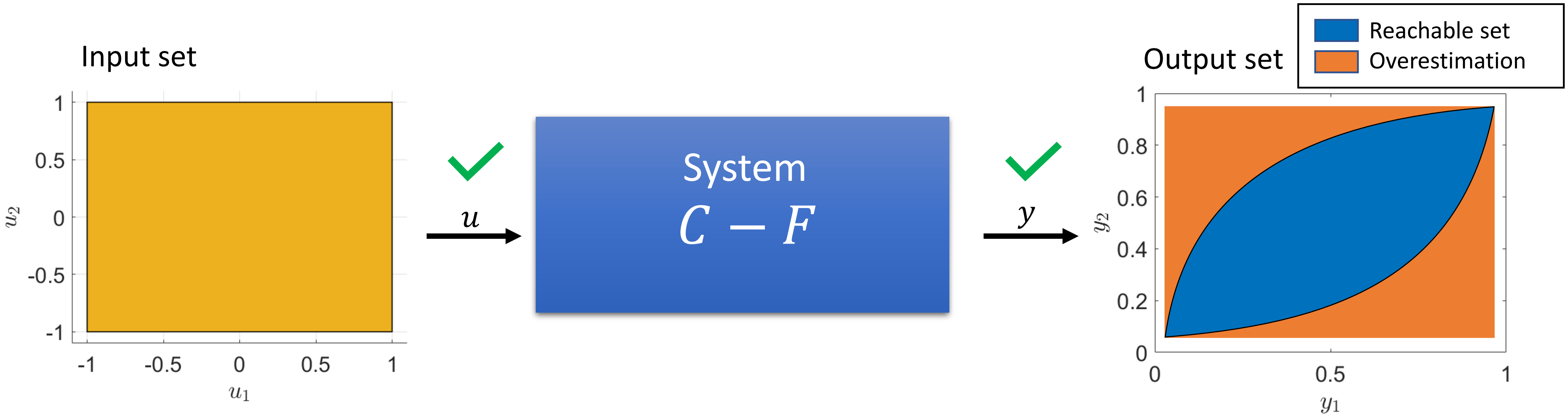


1. Motivation and Problem Statement



- Common methodologies: Set-based methods, Hamilton-Jacobi reachability, neural networks.
- Why a square? It only needs two points to compute.
- Approach: mixed-monotonicity.

1. Motivation and Problem Statement



- Why? Most real systems are impossible to know with certainty.
- What to do? Use a data-driven approach.
- Data-driven approaches: neural networks, Chen-Fliess series.
- Objective: extend MM to Chen-Fliess series and provide an overestimation.

2. Preliminaries
2.1 Chen-Fliess Series

Definition 2 (Chen-Fliess series):

$$y = F_c[u](t) := \sum_{\eta \in X^*} \underbrace{(c,\eta)}_{\text{Vector in } \mathbb{R}^l} E_\eta[u](t)$$

words

Space of all words

Iterative integral

words

Word: $x_{i_1}.x_{i_2}\cdots x_{i_n}$
Language: $X = \{x_1, \cdots, x_m\}$

Definition 3 (Iterative integral):

$$E_{x_i v}[u](t) := \int_0^t u_i(\tau) E_v[u](\tau)\, d\tau\,,$$

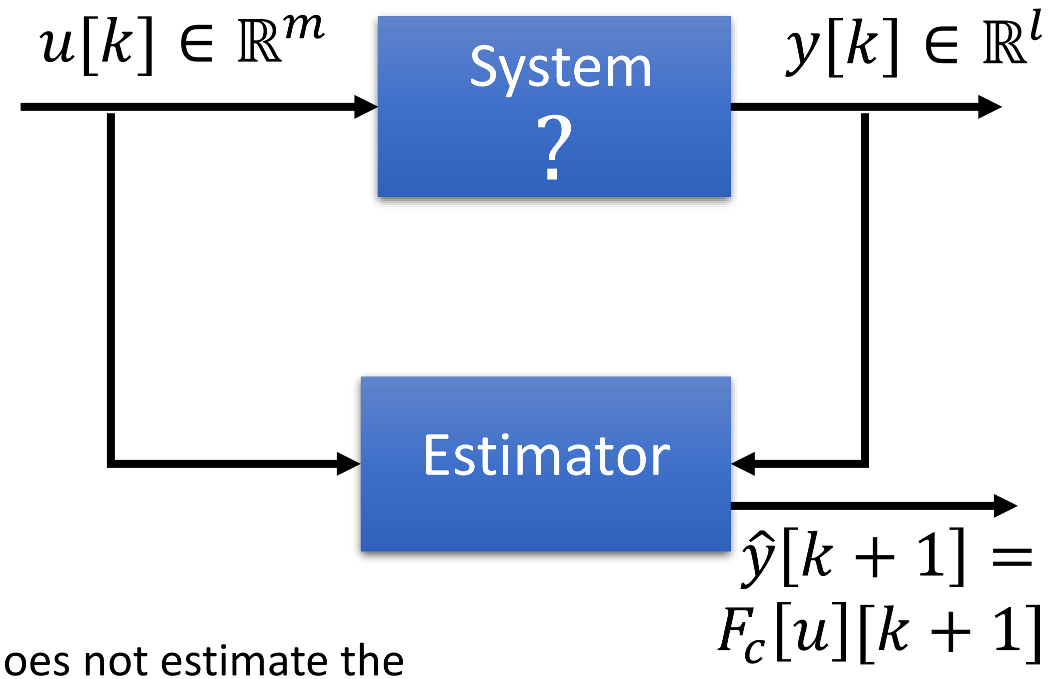
$$E_\phi[u](t) := 1$$

Remark: we need to truncate to a word length to use them

[Fliess, 1981]

2. Preliminaries
2.1 Chen-Fliess Series

Application:



Remark: Chen-Fliess series does not estimate the system, but it can be used as a framework to do so.

[Gray, Venkatesh, Duffaut Espinosa, 2019]

2. Preliminaries
2.1 Chen-Fliess Series

$u \longrightarrow$ System: $\dot{z} = g_0(z) + \sum_{i=1}^{m} g_i(z)u_i\,,\quad y_j = h_j(z), j = 1, \cdots, l,\quad z(0) = z_0 \longrightarrow F_c[u](t)$

Remark: the coefficients are computed deterministically in terms of Lie derivatives.

[Fliess, 1981]

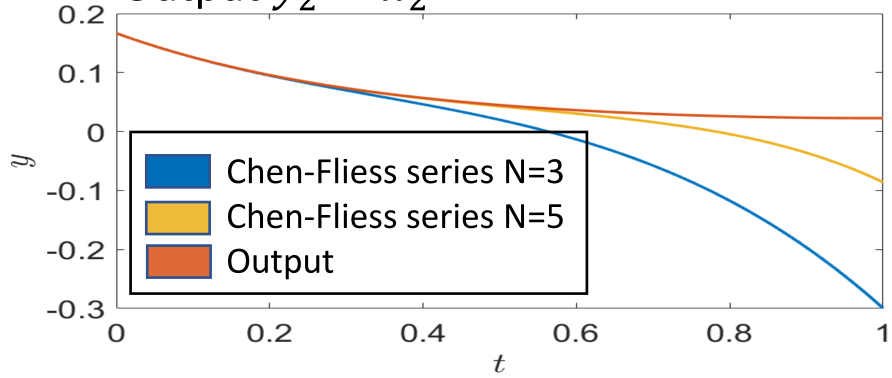
Example: Lotka-Volterra system

$$\dot{x}_1 = -x_1x_2 + x_1u_1,$$
$$\dot{x}_2 = x_1x_2 - x_2u_2,$$
$$y = x,$$
$$x(0) = (1/6,1/6).$$

Chen-Fliess series:

$$F_{c_2}[u] = 0.17 + 0.28E_{x_0}[u](t) - 0.17E_{x_2}[u](t) + \cdots$$

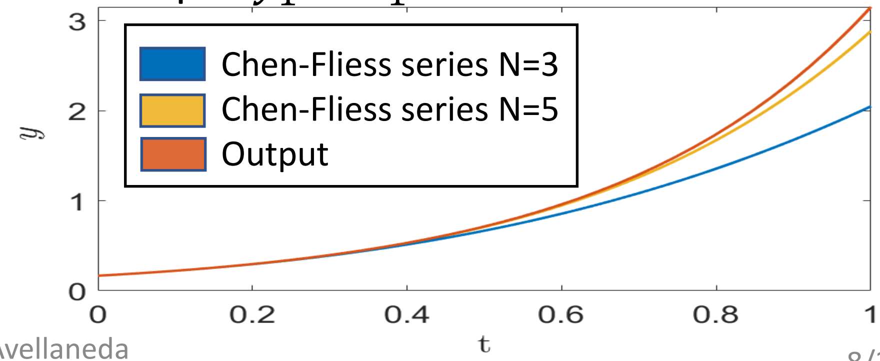
Output $y_2 = x_2$



Chen-Fliess series:

$$F_{c_1}[u] = 0.17 - 0.28E_{x_0}[u](t) + 0.17E_{x_1}[u](t) + \cdots$$

Output $y_1 = x_1$



2. Preliminaries
2.2 Mixed-Monotonicity (MM)

Definition 4 (Mixed-Monotone system):

A system $\dot{x} = f(x,u), x \in \mathcal{X} \subset \mathbb{R}^n, u \in \mathcal{U} \subset \mathbb{R}^m$ is mixed-monotone with respect to the decomposition function $d: \mathcal{X} \times \mathcal{U} \times \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ if

i. $d(x,u,x,u) = f(x,u)$,

ii. $\frac{\partial d_i}{\partial x_j}(x,u,\hat{x},\hat{u}) \geq 0$ for all $i \neq j$,

iii. $\frac{\partial d_i}{\partial \hat{x}_j}(x,u,\hat{x},\hat{u}) \leq 0$ for all i,j,

iv. $\frac{\partial d_i}{\partial u_k}(x,u,\hat{x},\hat{u}) \geq 0$ and $\frac{\partial d_i}{\partial \hat{u}_k}(x,u,\hat{x},\hat{u}) \leq 0$ for all i,k.

Definition 5 (Embedding system):

The embedding system of a mixed-monotone system with respect to the decomposition function $d: \mathcal{X} \times \mathcal{U} \times \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ is

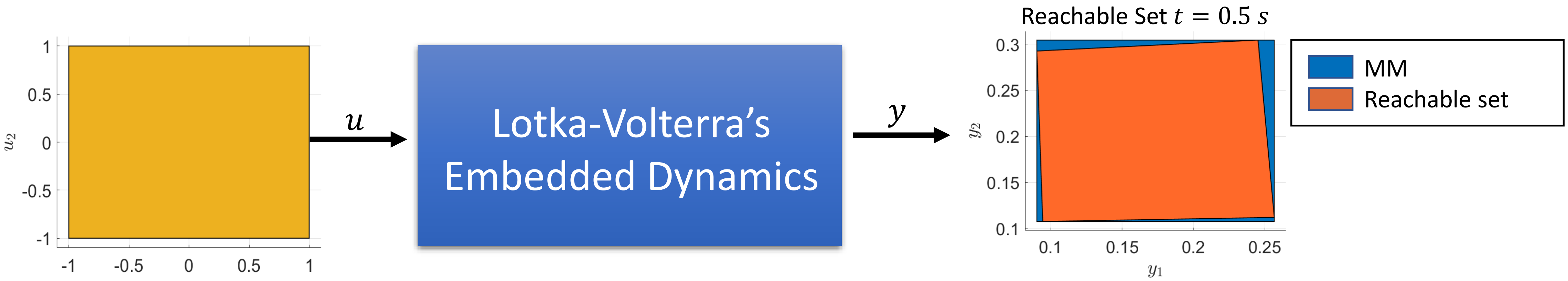
$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} d(x,u,\hat{x},\hat{u}) \\ d(\hat{x},\hat{u},x,u) \end{bmatrix}$$

[Abate, Dutreix, Coogan, 2020]

2. Preliminaries
2.2 Mixed-Monotonicity (MM)

Example: Lotka-Volterra system

$$\dot{x}_1 = -x_1 x_2 + x_1 u_1,$$
$$\dot{x}_2 = x_1 x_2 - x_2 u_2,$$
$$y = x,$$
$$x(0) = (1/6, 1/6),\ u_{1,2} \in [-1, 1].$$



[Abate, Dutreix, Coogan, 2020]

3. Input-Output Mixed-Monotonicity (IOMM)

Revisiting the objective:

- Extend mixed-monotonicity to Chen-Fliess series.
- Provide an overestimation of the reachable set of a Chen-Fliess series.

1

Define a partial order

$$u \preccurlyeq \hat{u}$$

2

Express the Chen-Fliess series as a difference of two positive series

$$F_c[u](t) = \mathcal{F}_{c^+}[u](t) - \mathcal{F}_{c^-}[u](t)$$

3

Obtain a decomposition function by lifting to a higher domain

$$d[u, \hat{u}] := \mathcal{F}_{c^+}[u](t) - \mathcal{F}_{c^-}[\hat{u}](t)$$

4

Use mixed-monotonicity

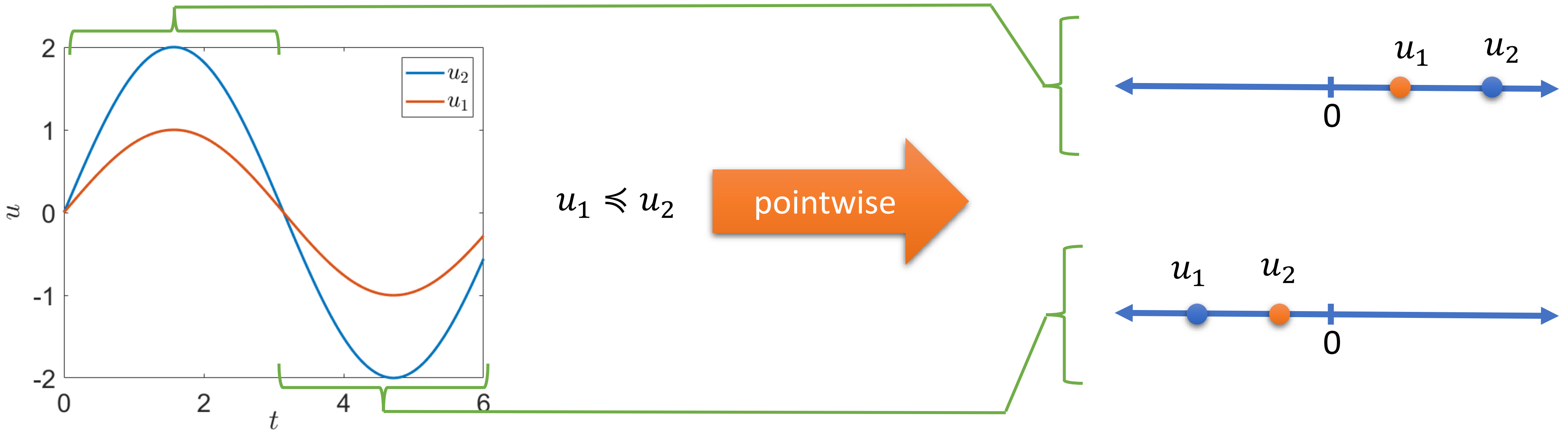
$$F_c[u](t) \in [d[\hat{u}, u](t), d[u, \hat{u}](t)]$$

3. Input-Output Mixed-Monotonicity (IOMM)

Definition 6 (Partial Order):

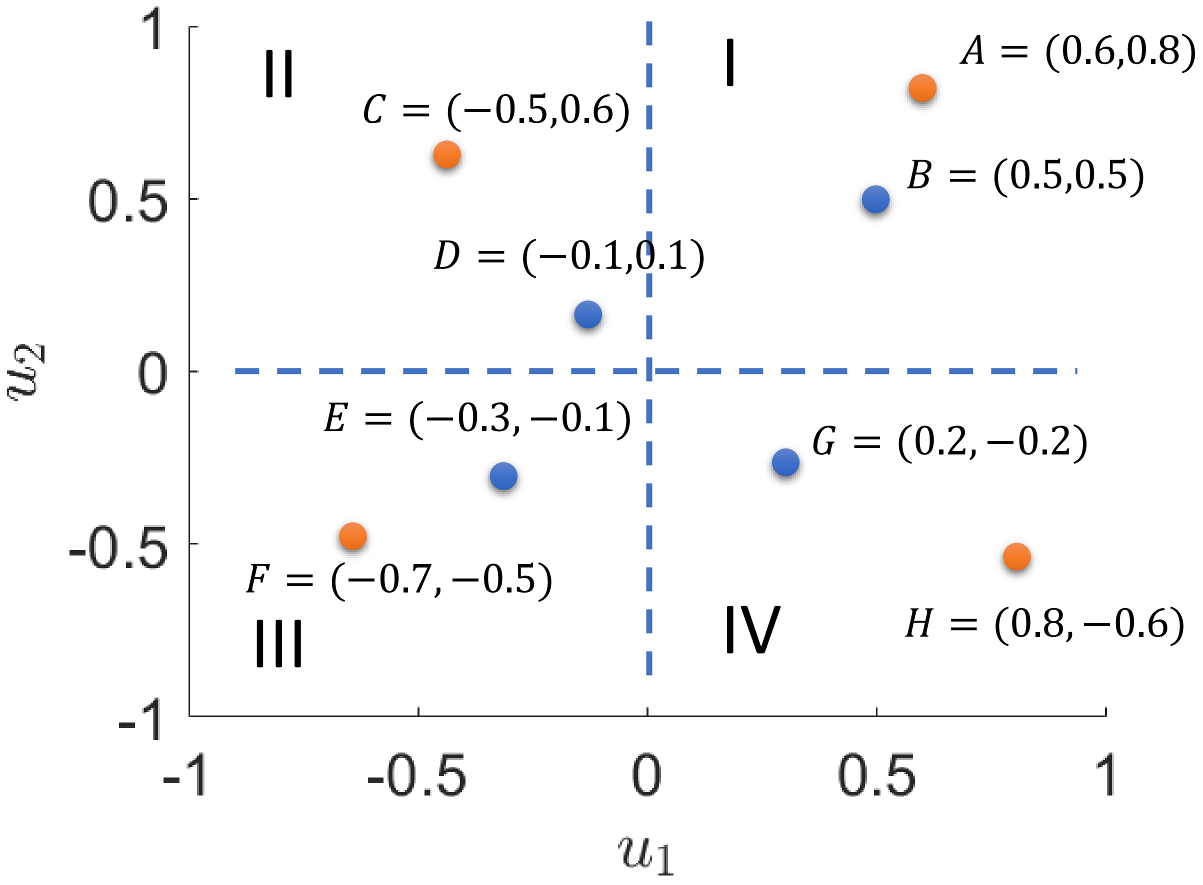
Consider the input functions $u_1: \mathbb{R} \to \mathbb{R}^m$ and $u_2: \mathbb{R} \to \mathbb{R}^m$. These functions are ordered $u_1(t) \preccurlyeq u_2(t)$ if and only if $u_1^+(t) \leq u_2^+(t)$ and $u_1^-(t) \leq u_2^-(t)$, where $\leq$ is the standard order of real numbers (componentwise) and $u^+(t) = \max\{u(t), 0\}$, $u^-(t) = -\min\{u(t), 0\}$.

Example 1:



3. Input-Output Mixed-Monotonicity (IOMM)

Example 2:



I

$$B \preccurlyeq A$$
$$(0.5,0.5) = B^+ \leq A^+ = (0.6,0.8)$$
$$(0,0) = B^- \leq A^- = (0,0)$$

II

$$D \preccurlyeq C$$
$$(0,0.1) = D^+ \leq C^+ = (0,0.6)$$
$$(0.1,0) = D^- \leq C^- = (0.5,0)$$

III

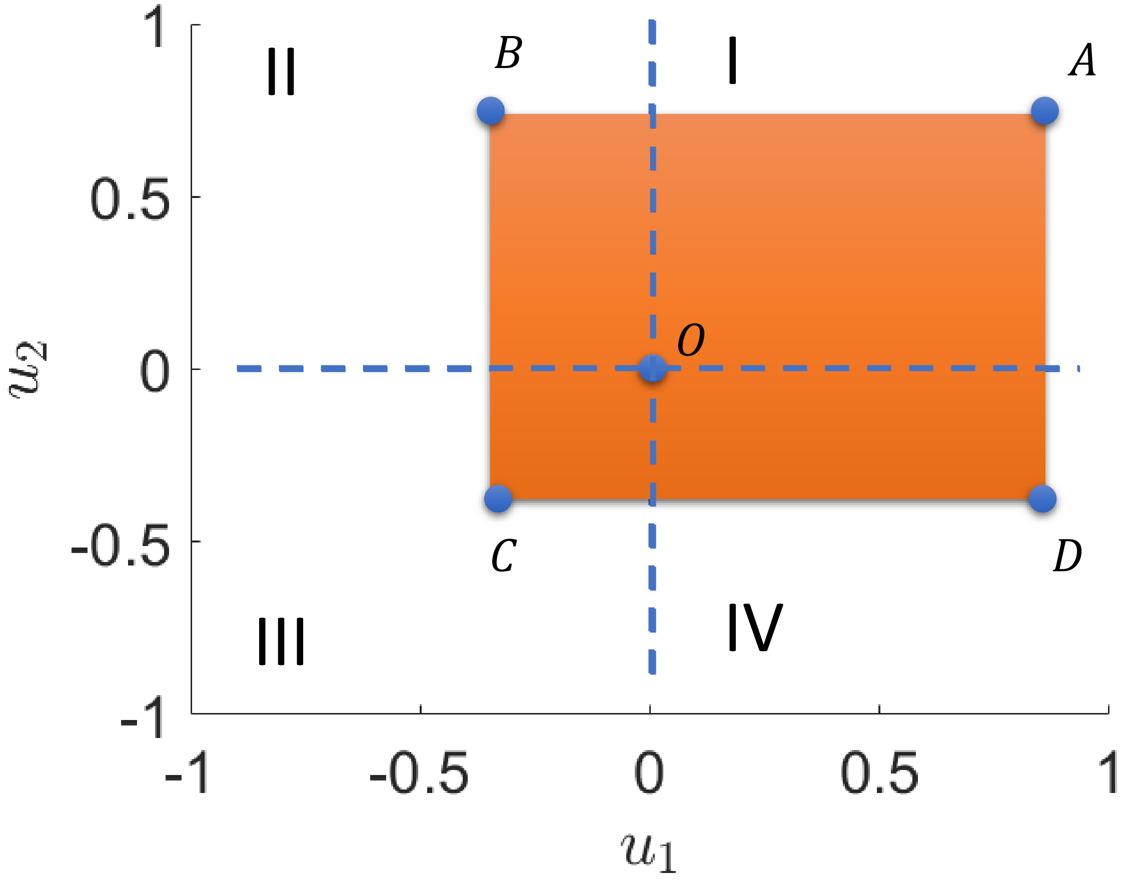
$$E \preccurlyeq F$$
$$(0,0) = E^+ \leq F^+ = (0,0)$$
$$(0.3,0.1) = E^- \leq F^- = (0.7,0.5)$$

IV

$$G \preccurlyeq H$$
$$(0,0.2) = G^+ \leq H^+ = (0.8,0)$$
$$(0,0.2) = G^- \leq H^- = (0,0.6)$$

3. Input-Output Mixed-Monotonicity (IOMM)

Example 3:



$$\{u \in \mathbb{R}^2 : C_x \leq u_1 \leq A_x, D_y \leq u_2 \leq B_y\}$$
$$=$$
$$[O, A] \cup [O, B] \cup [O, C] \cup [O, D]$$

where

$$[O, A] \coloneqq \{u \in \mathbb{R}^2 : O \preccurlyeq u \preccurlyeq A\}$$

3. Input-Output Mixed-Monotonicity (IOMM)

Definition 7 (IOMM):

A Chen-Fliess series $F_c[u](t)$ is input-output mixed-monotone if there exists a decomposition function $d_{F_c}[u,\hat{u}](t)$ such that

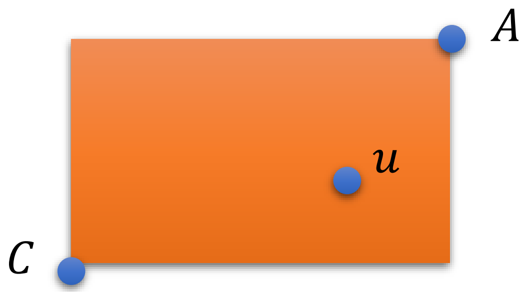
i. $d_{F_c}[u,u](t) = F_c[u](t)$,

ii. $d_{F_c}[u,\hat{u}](t)$ is non-decreasing in u,

iii. $d_{F_c}[u,\hat{u}](t)$ is non-increasing in $\hat{u}$.

Overestimation:

Consider the input function $u \in L_p^m[0,T]$ living in a square in the first orthant, this is C $\preccurlyeq u \preccurlyeq A$.



$$d_{F_c}[C,A](t) \leq F_c[u](t) = d_{F_c}[u,u](t) \leq d_{F_c}[A,C](t)$$

3. Input-Output Mixed-Monotonicity (IOMM)

Definition 8 (Extended iterative integral):

Consider the alphabets X and Y associated to the functions $u, v \in L_p^m[0, T]$, respectively

$$\mathcal{E}_{z_i \eta}[u, v](t) := \begin{cases} \int_0^t u_i(\tau)\mathcal{E}[u, v](\tau)d\tau, & z_i \in X \\ \int_0^t v_i(\tau)\mathcal{E}[u, v](\tau)d\tau, & z_i \in Y \end{cases}, \qquad \mathcal{E}_{\phi}[u, v](t) := 1$$

Lemma 1:

Consider the alphabets X and Y associated to the functions $u, v \in L_p^m[0, T]$, respectively

$$F[u + v](t) = \sum_{k=0}^{\infty} \sum_{\xi \in \mathbb{S}_{X^*, Y^k}} \left(c, \sigma_x(\xi)\right) \mathcal{E}_{\xi}[u, v](t)$$

Subtitution homomorphism from $Z = X \cup Y$ to X:

$$y_i \rightarrow x_i$$

3. Input-Output Mixed-Monotonicity (IOMM)

Lemma 2:

Consider the input function $u \in L_p^m[0,T]$ and its positive and negative parts u^+, u^- associated to the alphabets X and Y, respectively

$$F_c[u](t) = \mathcal{F}_{c^+}[u](t) - \mathcal{F}_{c^-}[u](t)$$

$$\mathcal{F}_{c^+}[u](t) = \sum_{k=0}^{\infty} \sum_{\eta \in X^*} \sum_{\xi \in \mathbb{S}_{\eta, Y^k}} \overbrace{(c^+, \xi)}^{\boxed{\max\{(-1)^k (c, \sigma_X(\xi)), 0\}}} \mathcal{E}_\xi[u^+, u^-](t),$$

$$\mathcal{F}_{c^-}[u](t) = \sum_{k=0}^{\infty} \sum_{\eta \in X^*} \sum_{\xi \in \mathbb{S}_{\eta, Y^k}} \overbrace{(c^-, \xi)}^{\boxed{-\min\{(-1)^k (c, \sigma_X(\xi)), 0\}}} \mathcal{E}_\xi[u^+, u^-](t),$$

Theorem 1:

The following is a decomposition function of the Chen-Fliess series $F_c[u](t)$

$$d_{F_c}[u, \hat{u}](t) := \mathcal{F}_{c^+}[u](t) - \mathcal{F}_{c^-}[\hat{u}](t)$$

3. Input-Output Mixed-Monotonicity (IOMM)

Theorem 2:

Consider the Chen-Fliess series $F_c[u](t)$ taking values in the hyper-rectangle $U = [u, \hat{u}] \subset K$. Then

$$Reach(F_c, U)(t) \subset [d_{F_c}[u, \hat{u}](t), d_{F_c}[\hat{u}, u](t)].$$

4. Illustrative Simulation

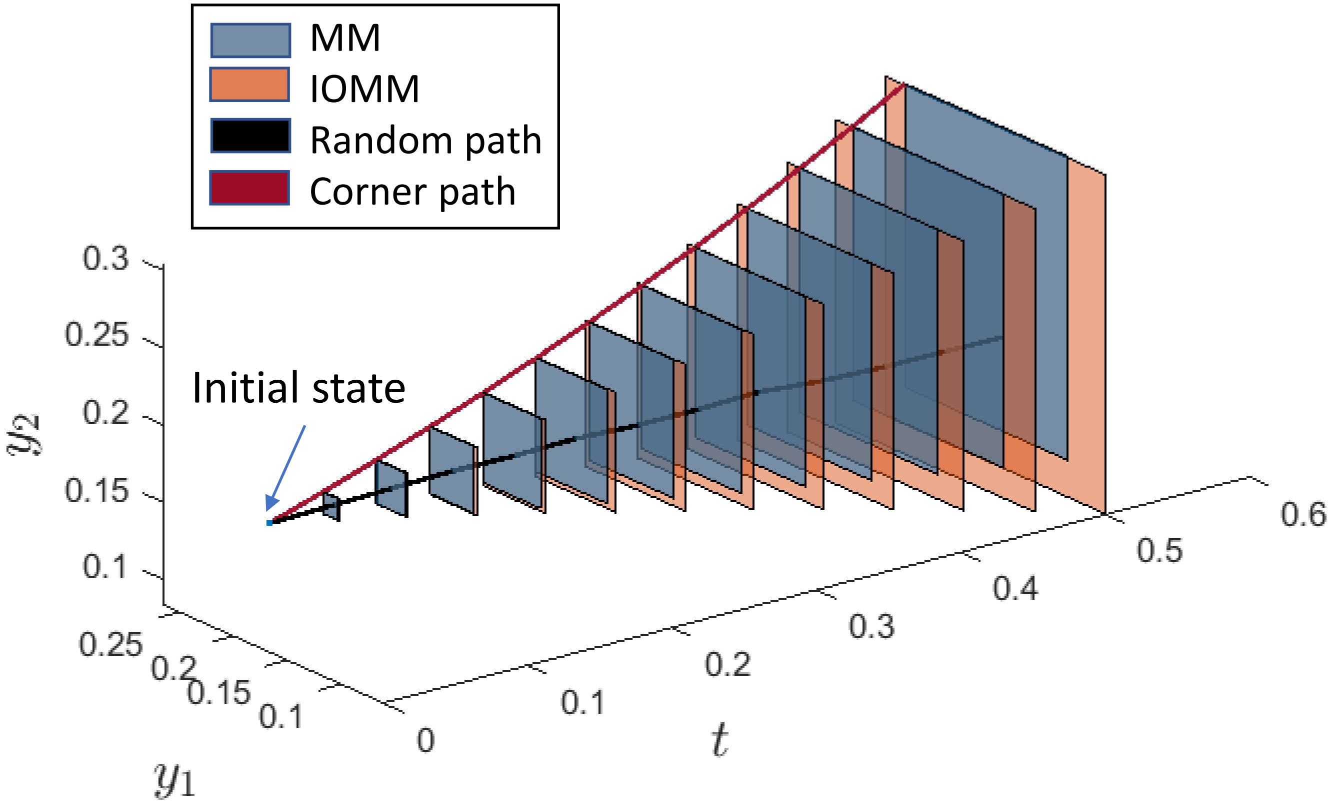
Example 4: Lotka-Volterra system

$$\dot{x}_1 = -x_1 x_2 + x_1 u_1,$$
$$\dot{x}_2 = x_1 x_2 - x_2 u_2,$$
$$y = x,$$
$$x(0) = (1/6, 1/6),$$
$$(u_1, u_2) \in [-1,1] \times [-1,1].$$

Positive and negative parts of the decomposition function:

$$\mathcal{F}_{c^+}[u](t) = 0.167 + 0.028\mathcal{E}_{x_0}[u^+, u^-](t) + 0.167\mathcal{E}_{y_2}[u^+, u^-](t) + \cdots$$

$$\mathcal{F}_{c^-}[u](t) = 0.167\mathcal{E}_{x_2}[u^+, u^-](t) + 0.028\mathcal{E}_{x_0 x_2}[u^+, u^-](t) + 0.028\mathcal{E}_{x_0 y_1}[u^+, u^-](t) + \cdots$$



Remark: IOMM always contains MM.

5. Conclusion

- We have provided a methodology to overestimate the reachable set of a system described as a Chen-Fliess series.
- The methodology provides a closed-form for the reachable set overestimation.
- The IOMM approach contains the sets that MM provides when the full Chen-Fliess series is considered.
- The accuracy of the method depends on the truncation of the Chen-Fliess series.
- Further work is needed to find the tightest overestimation.

Questions?

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