Optimization of Chen-Fliess series and Output Reachability of Nonlinear Systems

Ivan Perez Avellaneda and Luis Duffaut Espinosa

Department of Electrical and Biomedical Engineering, University of Vermont

Reachable Set

The reachable set of a system is the set of outputs as a response of the system to a set of inputs $\mathcal{U}$ and a set of initial conditions $\mathcal{X}_0$.

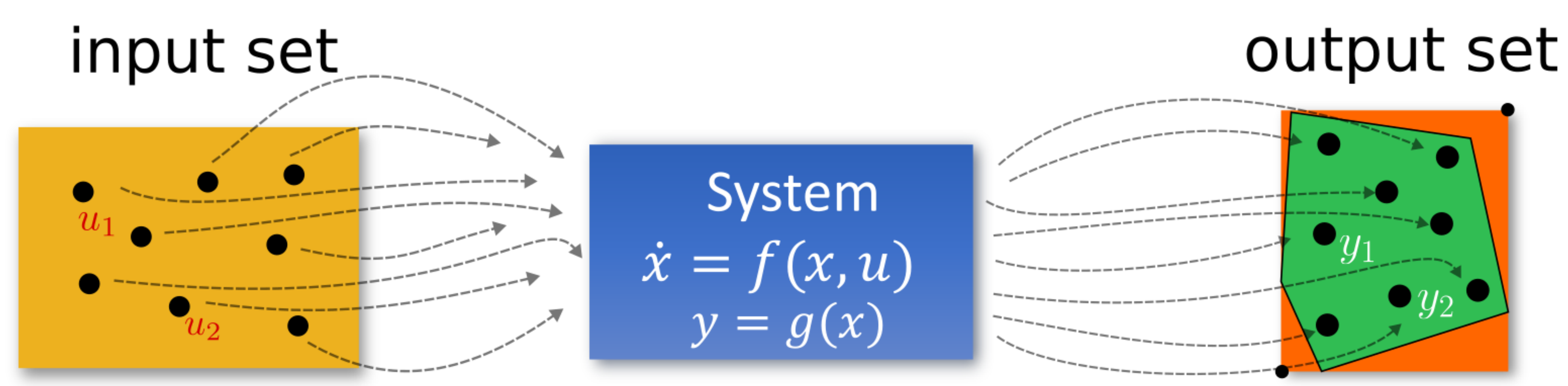


Figure 1. Reachable Set and Minimum Bounding Box (MBB).

Goal: compute the reachable set not by simulating each possible trajectory one by one.

Application

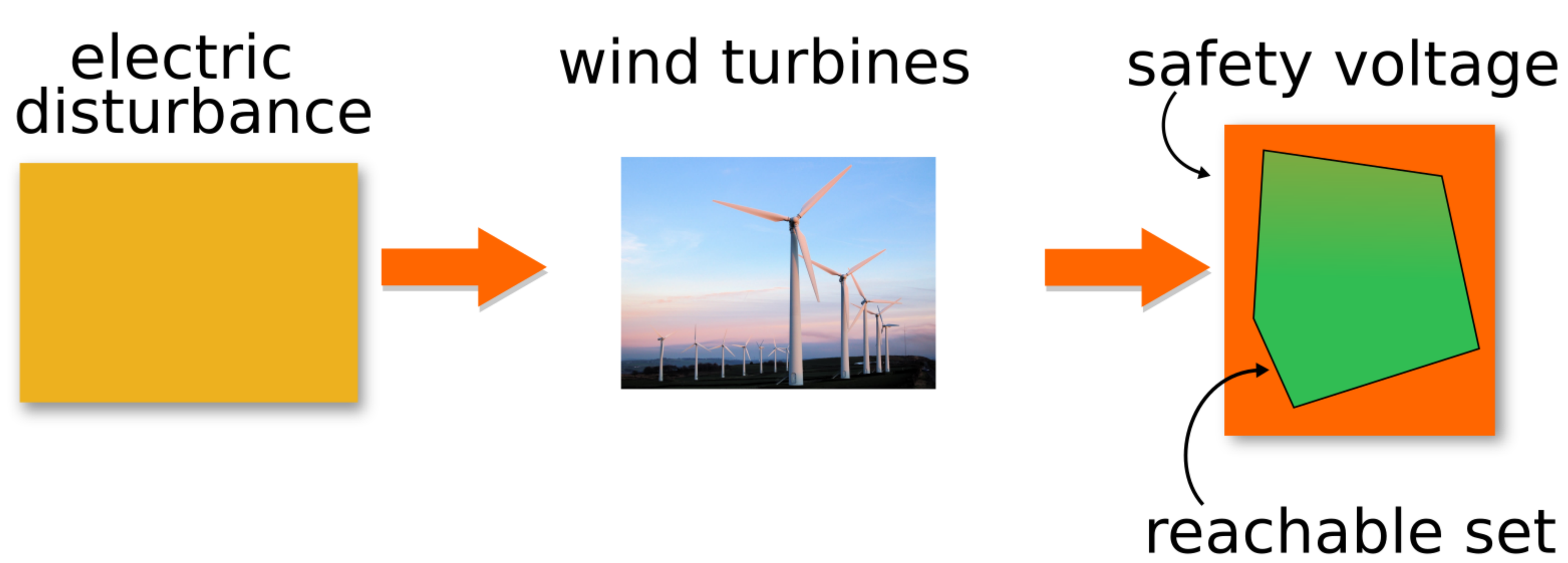


Figure 2. Safety of power systems.

Chen-Fliess Series (CFS)

CFS provide an input-output representation of an affine nonlinear system.

Iterated integral:

$$E_{x_i\xi}[u](t) = \int_0^t u_i(\tau) E_\xi[u](\tau)d\tau$$

Chen-Fliess series:

$$F_c[u](t) = \sum_{\eta \in X^*} (c,\eta) E_\eta[u](t)$$

State-Space realization: The CFS represents the system

$$\dot{z} = g_0(z) + \sum_{i=1}^m g_i(z)u_i, \; z(t_0) = z_0$$
$$y = h(z)$$

if and only if $(c,\eta) = L_{g_\eta} h(z)|_{z_0}$ and c has finite Lie rank.

Minimum Bounding Box

The minimum bounding box (MBB) of the reachable set is the smallest box containing the reachable set. The MBB is determined by two points which are the optima of the coordinates.

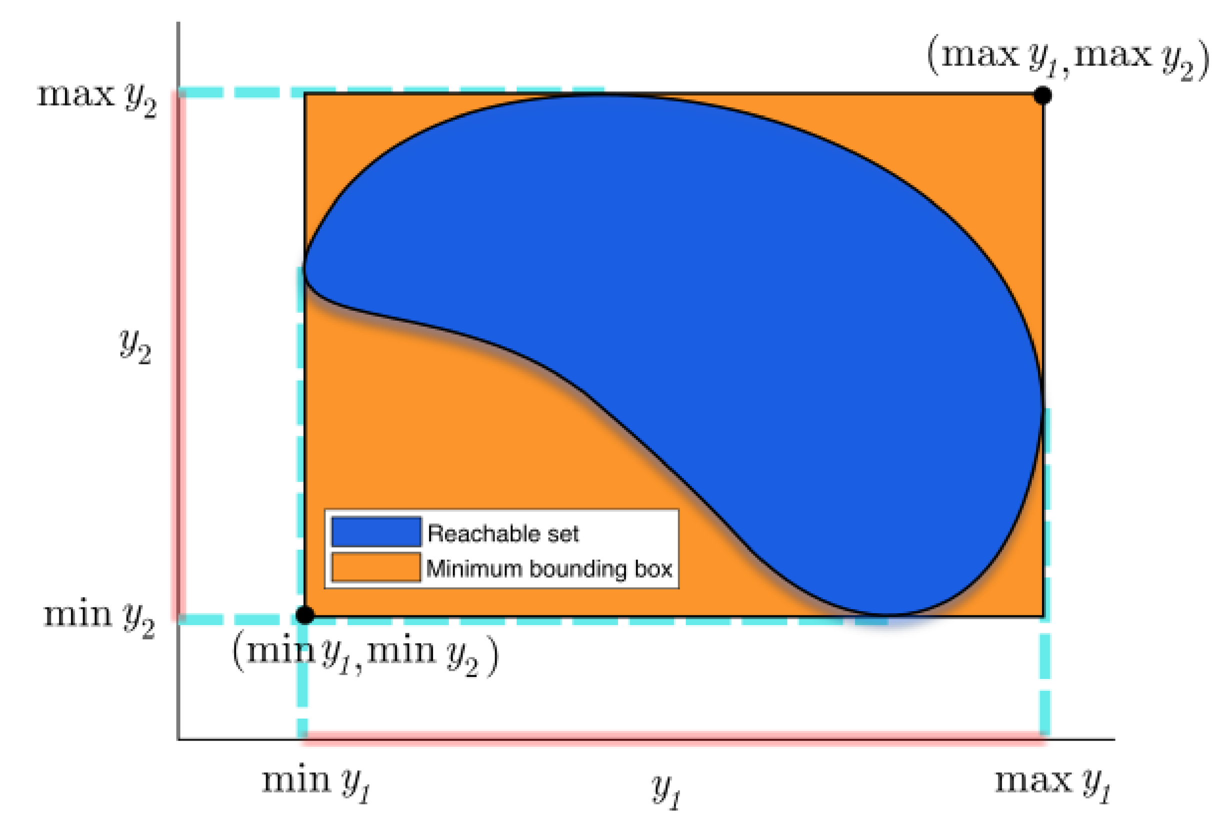


Figure 3. Output reachable set of a non-linear affine system and its MBB in terms of its maximum and minimum outputs.

Computing the MBB is equivalent to the optimization of CFS.

Chen-Fliess Series Calculus

The calculus and numerical optimization methods were extended for CFS.

- **First order derivative**: The Gâteaux derivative in the direction of $v \in L_\mathfrak{p}^m[0,t]$

$$\frac{\partial}{\partial v} F_c[u](t) = \sum_{\xi \in \mathbb{S}_{X^*,\delta X}} (c, \sigma_X(\xi)) \mathcal{E}_\xi[u,v](t)$$

Considering the directions $e_1(t) = (1,0,\cdots,0)^\top, \ldots, e_m(t) = (0,0,\cdots,1)^\top$, the gradient is given by

$$\nabla F_c[u](t) = \left(\frac{\partial}{\partial u_1} F_c[u](t), \cdots, \frac{\partial}{\partial u_m} F_c[u](t) \right)^T.$$

- **Second order derivative**: The 2nd order Gâteaux derivative in the direction of $v \in L_\mathfrak{p}^m[0,t]$

$$\frac{\partial^2}{\partial v^2} F_c[u](t) = 2! \sum_{\xi \in \mathbb{S}_{X^*,\delta X^2}} (c, \sigma_X(\xi)) \mathcal{E}_\xi[u,v](t)$$

The Hessian of $F_c[u](t)$ is given by

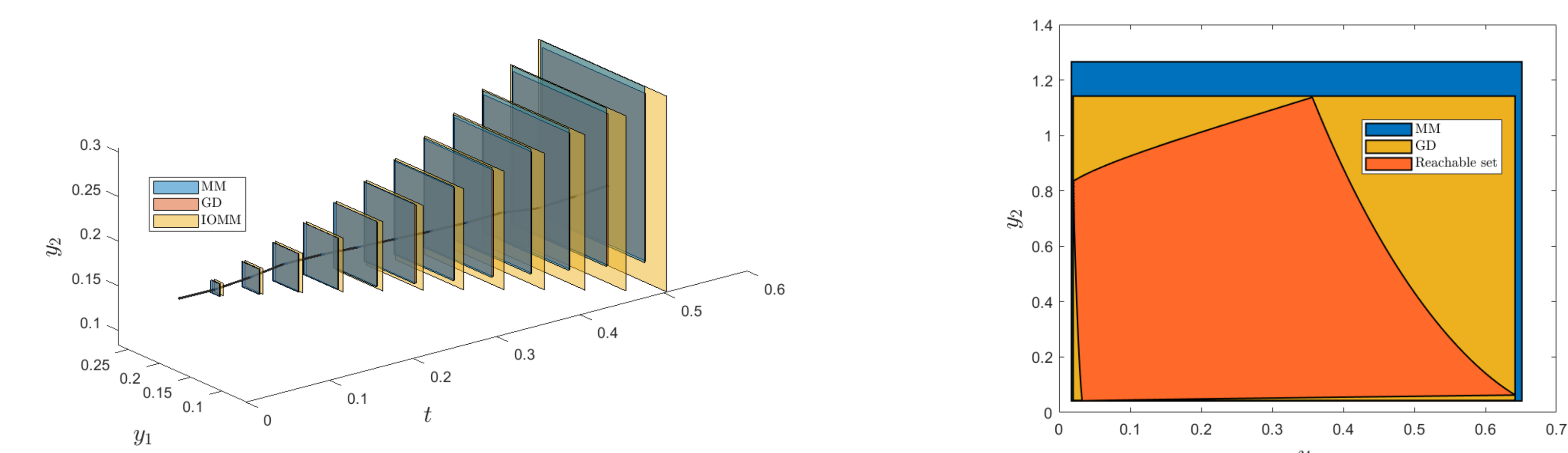
$$\nabla^2 F_c[u](t) = \begin{bmatrix} 2\frac{\partial^2}{\partial u_1^2} F_c[u](t) & \cdots & \frac{\partial^2}{\partial u_1 \partial u_m} F_c[u](t) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial u_m \partial u_1} F_c[u](t) & \cdots & 2\frac{\partial^2}{\partial u_m^2} F_c[u](t) \end{bmatrix}.$$

Gradient Descent

Consider the following MIMO Lotka-Volterra system given by

$$\dot{x}_1 = -x_1x_2 + x_1u_1,$$
$$\dot{x}_2 = x_1x_2 - x_2u_2,$$
$$y = x$$

with $x_0 = (1/6, 1/6)^\top$ and input set $\mathcal{U} = \{(u_1,u_2) : \; -1 \le u_1 \le 1, \; -1 \le u_2 \le 1\}$



(a) Estimation of the reachable sets up to $t = 0.5s$.

(b) Estimation of the reachable set for $t = 1.5s$

Figure 4. Reachable set computation using the gradient descent algorithm.

Newton Method

Consider the bilinear system

$$\dot{x} = xu,$$
$$y = x,$$
$$x(0) = 1$$

with input $-1 \le u \le 1$.
The Hessian of the CFS is

$$\nabla^2 F_c[u](t) = 2 \sum_{\xi \in \mathbb{S}_{X^*,x_0^2}} E_\xi[u](t).$$

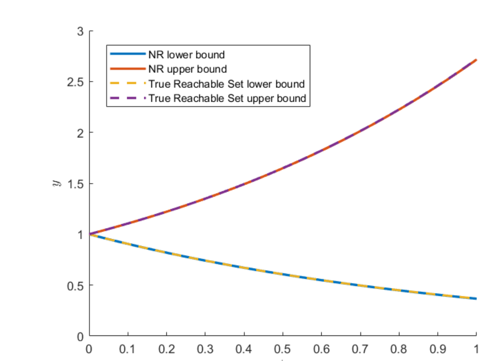


Figure 5. Reachable sets computation using the Newton optimization method.

References

[1] Ivan Perez Avellaneda and Luis A. Duffaut Espinosa.
On mixed-monotonicity of chen-fliess series.
In *2022 26th International Conference on System Theory, Control and Computing (ICSTCC)*, pages 98–103, 2022.

[2] Ivan Perez Avellaneda and Luis A. Duffaut Espinosa.
Output reachability of chen-fliess series: A newton-raphson approach.
In *2023 57th Annual Conference on Information Sciences and Systems (CISS)*, pages 1–6, 2023.

[3] Ivan Perez Avellaneda and Luis A. Duffaut Espinosa.
Input-output reachable set overestimation via chen-fliess series.
Transactions on Automatic Control, 2023 (under review).

[4] Ivan Perez Avellaneda and Luis A. Duffaut Espinosa.
Reachability of chen-fliess series: A gradient descent approach.
In *2022 58th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 1–7, 2022.