Output Reachability of Chen-Fliess series: A Newton-Raphson Approach*

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1. Motivation and Problem Statement

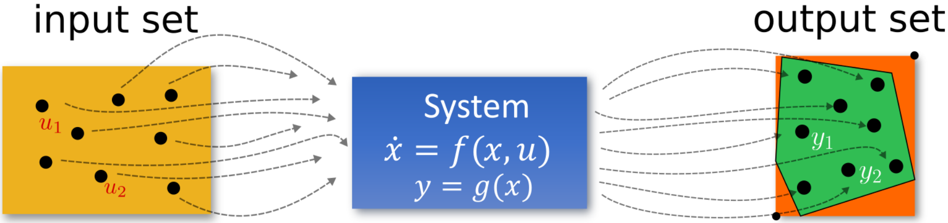

Figure 1: Reachable Set and Minimum Bounding Box (MBB).

Definition 1

The reachable set of a system subject to a set of inputs $\mathcal{U}$ and a set of initial conditions $\mathcal{X}_0$ is defined as

$$\text{Reach}(\mathcal{X}_0, \mathcal{U})(t) = \left\{ \phi(t, u, x_0) \in \mathbb{R}^n : \text{for some } u : [0, t] \to \mathcal{U}, x_0 \in \mathcal{X}_0 \right\} \tag{1}$$

- **Goal:** compute the reachable set not by simulating each possible trajectory one by one.
- **Methodologies:** set-based methods, mixed-monotonicity, Hamilton-Jacobi reachability, Koopman operators, neural networks.

1. Motivation and Problem Statement

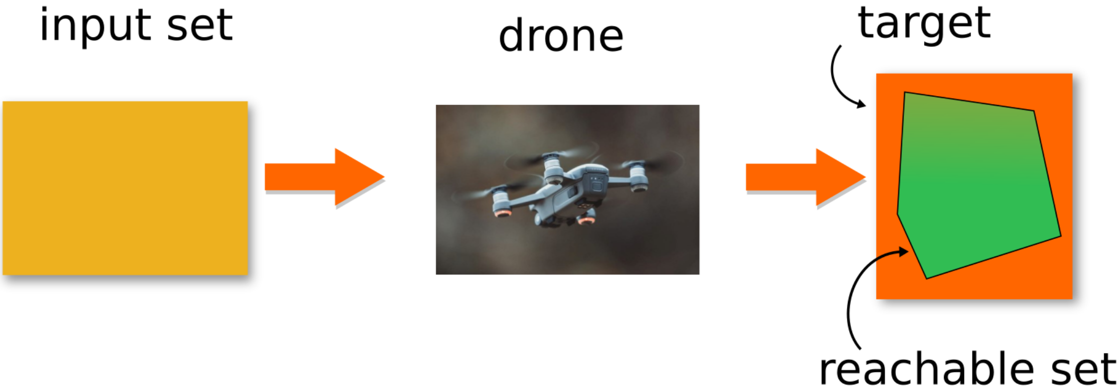

Figure 2: Goal reaching.

2.1 Preliminaries: Chen-Fliess series

A *word* is defined as the catenation $\eta = x_{i_1} \cdots x_{i_k}$ of letters from $X = \{x_0, x_1, \ldots, x_m\}$.
A *power series* c is defined as a function $c : X^* \to \mathbb{R}^\ell$ and represented as $c = \sum_{\eta \in X^*} (c, \eta)\eta$.

Definition 2 (Iterated integrals)

The map $E_\eta : L_1^m[0, T] \to C[0, t]$ for $u(t) = (u_1(t), \cdots, u_m(t))$ where $u_i(t) \in L_1[0, T]$ is defined iteratively for $\eta = \phi$ as $E_\phi[u](t) = 1$ and for $\eta = x_i\xi$ as

$$E_{x_i\xi}[u](t) = \int_0^t u_i(\tau) E_\xi[u](\tau) d\tau$$

Definition 3 (Chen-Fliess series)

The following operator is known as Chen-Fliess series of the power series $c = \sum_{\eta \in X^*} (c, \eta)\eta$

$$F_c[u](t) = \sum_{\eta \in X^*} \underbrace{(c, \eta)}_{\text{vector in } \mathbb{R}^\ell} E_\eta[u](t)$$

word

word

iterated integral

space of all words

2.1 Preliminaries: Chen-Fliess series

Theorem 1 (State-Space realization (Fliess 1983))

The Chen-Fliess series $F_c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ *represents the system*

$$\dot{z} = g_0(z) + \sum_{i=1}^{m} g_i(z) u_i, \;\; z(t_0) = z_0 \tag{2}$$

$$y = h(z) \tag{3}$$

if and only if $(c, \eta) = L_{g_\eta} h(z)|_{z_0}$ *and* $c = \sum_{\eta \in X^*} (c, \eta)\eta$ *has finite Lie rank.*
For $\eta = x_{i_k} \cdots x_{i_1} \in X^*$*,*

$$L_{g_\eta} h_j(z_0) := L_{g_{i_1}} \cdots L_{g_{i_k}} h_j(z)|_{z_0}, \tag{4}$$

and $L_g f(z) = (\frac{\partial}{\partial z} f(z)) \cdot g(z)$. *Equivalently,*

$$L_{g_{i_1}} \cdots L_{g_{i_k}} h(z_0) = \frac{\partial}{\partial z} \left(\cdots \left(\frac{\partial}{\partial z} \left(\frac{\partial h(z)}{\partial z} \cdot g_{i_k}(z) \right) \cdot g_{i_{k-1}}(z) \right) \cdots \right) \cdot g_{i_1}(z) \Bigg|_{z_0} \tag{5}$$

2.2 Preliminaries: first order derivatives of CFS

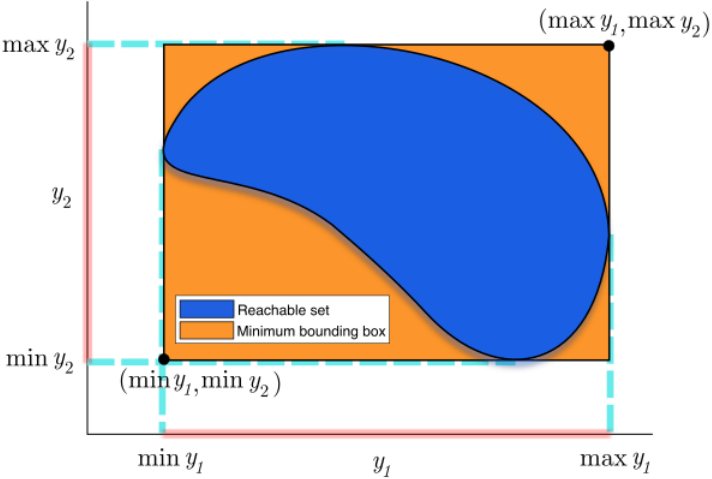

Figure 3: Output reachable set of a non-linear affine system and its MBB in terms of its maximum and minimum outputs.

2.2 Preliminaries: first order derivatives of CFS

Consider the alphabets $X = \{x_0, \cdots, x_m\}$ and $Y = \{y_1, \cdots, y_m\}$ and the set of all words in these two alphabets $Z^* = (X \cup Y)^*$.

Definition 4 (Extended iterated integral)

Given the alphabets X and Y associated with $u,\ v \in L_{\mathfrak{p}}^m[0,T]$, respectively. The iterated integral of $\eta \in Z^*$ for the input (u,v) is given by the mapping $\mathcal{E}_\eta : L_{\mathfrak{p}}^m[0,T] \times L_{\mathfrak{p}}^m[0,T] \to \mathcal{C}[0,T]$, where $\mathcal{E}_\emptyset[u,v](t) = 1$ and

$$\mathcal{E}_{z_i\eta}[u,v](t) := \begin{cases} \displaystyle\int_0^t u_i(\tau)\mathcal{E}_\eta[u,v](\tau)d\tau, & z_i \in X, \\ \displaystyle\int_0^t v_i(\tau)\mathcal{E}_\eta[u,v](\tau)d\tau, & z_i \in Y. \end{cases} \tag{6}$$

Definition 5 (Chen-Fliess series)

The following operator is the extended Chen-Fliess series of the power series $c = \sum_{\eta\in Z^*}(c,\eta)\eta$

$$\mathcal{F}_c[u,v](t) = \sum_{\eta\in Z^*}(c,\eta)\mathcal{E}_\eta[u,v](t) \tag{7}$$

2.2 Preliminaries: first order derivatives of Chen-Fliess series

Shuffle set of words: consider $\eta = x_1 x_2$ and $\xi = x_3 x_4$, then

$$\mathbb{S}_{\eta,\xi} = \{x_1x_2x_3x_4, x_1x_3x_2x_4, x_3x_1x_2x_4, x_1x_3x_4x_2, x_3x_1x_4x_2, x_3x_4x_1x_2\} \tag{8}$$

Theorem 2 (Gâteaux derivative)

let X and Y be alphabets associated with $u,\ v \in L_{\mathfrak{p}}^m[0,T]$, respectively

$$\frac{\partial}{\partial v} F_c[u](t) = \sum_{\eta \in X^*} \sum_{\xi \in \mathbb{S}_{\eta,Y}} (c, \sigma_X(\xi)) \mathcal{E}_\xi[u,v](t), \tag{9}$$

Define the elementary functions $e_i : [0,T] \to \mathbb{R}^m$, such that $e_1(t) = (1,0,\cdots,0)^\top, \ldots, e_m(t) = (0,0,\cdots,1)^\top$. The gradient is defined in terms of the Gâteaux derivative of $F_c[u](t)$ in the direction of e_i

$$\nabla F_c[u](t) = \left(\frac{\partial}{\partial u_1} F_c[u](t), \cdots, \frac{\partial}{\partial u_m} F_c[u](t) \right)^T. \tag{10}$$

(Perez Avellaneda and Duffaut Espinosa 2022)

3.1.1 Main results: differential languages

Goal: obtain a closed form of the first-order derivative of Chen-Fliess series.
Consider the alphabet $\delta X = \{\delta x_1, \cdots, \delta x_m\}$.

Definition 6 (Differential monoid)

The tuple $(Z, \cdot, \phi, \delta)$ where $Z = X \cup \delta X$, and $\cdot$ is concatenation operation, ϕ is the empty word, the substitution homomorphism $\sigma_X(\delta x_i) = x_i$, and the derivation function $\delta : Z \to Z$ such that $\delta(x_i) = \delta x_i$ for $x_i \neq x_0$, $\delta(\delta x_i) = 0$ for $x_i \neq x_0$ and $\delta(x_0) = \delta(\emptyset) = 0$. Let $\eta \in Z^*$ such that $|\eta|_X = n_1 \geq 1$ and $|\eta|_{\delta(X)} = n_2$ and consider the language

$$L_{\delta(\eta)} := \{\xi \in \mathbb{S}_{X^{n_1-1}, \delta X^{n_2+1}} \text{ s.t. } \sigma_X(\xi) = \sigma_X(\eta)\}. \tag{11}$$

The *derivative of* η is defined as

$$\delta(\eta) = \text{char}(L_{\delta(\eta)}) \in \mathbb{R}\langle Z \rangle. \tag{12}$$

When $n_1 = 0$, $L_{\delta(\eta)}$ is empty and $\delta(\eta) := 0$.

3.1.1 Main results: differential languages

Example 1

Let $\eta = x_0 x_{i_1} \in \mathbb{S}_{X^2, \delta X^0}$. Note that $\xi = x_0 \delta x_{i_1}$ is the only element in $\mathbb{S}_{X, \delta X}$ such that $\sigma_X(\xi) = \sigma_X(\eta)$. Then $\delta(x_0 x_{i_1}) = x_0 \delta x_{i_1}$. Hence, x_0 behaves as a constant with respect to δ.

Lemma 1

The derivative of $\eta \in X^n$ satisfies the following properties:

i) $\delta(\eta) = \sum_{j=1}^{n} x_{i_1} \cdots x_{i_{j-1}} \; \delta x_{i_j} \; x_{i_{j+1}} \cdots x_{i_n}$,

ii) $\delta^2(\eta) = 0$, *for* $|\eta| = 0$ *or* 1,

iii) $\delta(x_{i_1}^{n_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_{i_k}^{n_k}) = \sum_{l=1}^{k} x_{i_1}^{n_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_{i_l}^{n_l - 1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_{i_k}^{n_k} \sqcup\!\sqcup \delta x_{i_l}$

where $\{x_{i_1}, \ldots, x_{i_k}\} \subseteq X$.

Example 2

Let $\eta = x_{i_1} x_{i_2} x_{i_3}$, then

$$\delta(x_{i_1} x_{i_2} x_{i_3}) = \delta x_{i_1} x_{i_2} x_{i_3} + x_{i_1} \delta x_{i_2} x_{i_3} + x_{i_1} x_{i_2} \delta x_{i_3} \tag{13}$$

$$\delta^2(x_{i_1} x_{i_2} x_{i_3}) = 2!(\delta x_{i_1} \delta x_{i_2} x_{i_3} + x_{i_1} \delta x_{i_2} \delta x_{i_3} + \delta x_{i_1} x_{i_2} \delta x_{i_3}) \tag{14}$$

$$\delta^3(x_{i_1} x_{i_2} x_{i_3}) = 3! \delta x_{i_1} \delta x_{i_2} \delta x_{i_3}. \tag{15}$$

3.1.1 Main results: differential languages

Consider the word $x_1\delta x_2 x_3\delta x_4 x_5 \in \mathbb{S}_{X^*,\delta X^2}$. Take the derivative three times

$$\delta^3(x_1\delta x_2 x_3\delta x_4 x_5) = 3!\delta x_1\delta x_2\delta x_3\delta x_4\delta x_5 \tag{16}$$

the word $\delta x_1 x_2\delta x_3 x_4 x_5 \in \mathbb{S}_{X^*,\delta X^2}$, which is different from $x_1\delta x_2 x_3\delta x_4 x_5$, also satisfies

$$\delta^3(\delta x_1 x_2\delta x_3 x_4 x_5) = 3!\delta x_1\delta x_2\delta x_3\delta x_4\delta x_5 \tag{17}$$

actually, there are $\binom{5}{2}$ different words ξ in $\mathbb{S}_{X^*,\delta X^2}$ such that $\sigma_X(\xi) = x_1x_2x_3x_4x_5$ and they all satisfy $\delta^3(\xi) = 3!\delta x_1\delta x_2\delta x_3\delta x_4\delta x_5$ therefore

$$\begin{aligned}&\frac{1}{3!}\delta^3(\delta x_1\delta x_2 x_3x_4x_5 + \delta x_1 x_2\delta x_3 x_4x_5 + \delta x_1 x_2x_3\delta x_4 x_5 + \delta x_1 x_2x_3x_4\delta x_5 + x_1\delta x_2\delta x_3 x_4x_5 +\\ &\quad x_1\delta x_2 x_3\delta x_4 x_5 + x_1\delta x_2 x_3x_4 x_5 + x_1x_2\delta x_3\delta x_4 x_5 + x_1x_2\delta x_3 x_4\delta x_5 + x_1x_2x_3\delta x_4\delta x_5)\\ &\quad = \binom{2+3}{2}\delta x_1\delta x_2\delta x_3\delta x_4\delta x_5\end{aligned} \tag{18}$$

For power series, they all have the same coefficient $(c, x_1x_2x_3x_4x_5)$, then this extends to power series by the linearity of the derivative.

3.1.1 Main results: differential languages

Lemma 2

The k-th derivative of char(X^*) satisfies

$$\delta^k(\text{char}(X^*)) = k!\ \text{char}(\mathbb{S}_{X^*,\delta X^k}). \tag{19}$$

Lemma 3

The k-th derivative of $c \in \mathbb{R}\langle\langle X\rangle\rangle$ satisfies

$$\delta^k(c) = k! \sum_{\xi \in \mathbb{S}_{X^*,\delta X^k}} (c, \sigma_X(\xi))\xi. \tag{20}$$

Lemma 4

Let $(Z, \odot, \emptyset, \delta)$ be a differential monoid. For $k, r \in \mathbb{N}$, it follows that

$$\frac{1}{k!}\delta^k\left(\text{char}\ \mathbb{S}_{X^*,\delta X^r}\right) = \binom{r+k}{r}\text{char}\left(\mathbb{S}_{X^*,\delta X^{r+k}}\right) \tag{21}$$

and, for $c \in \mathbb{R}\langle\langle X\rangle\rangle$, one has that

$$\sum_{\xi \in \mathbb{S}_{X^*,\delta X^r}} \frac{1}{k!}(c, \sigma_X(\xi))\delta^k(\xi) = \binom{r+k}{r} \sum_{\xi \in \mathbb{S}_{X^*,\delta X^{r+k}}} (c, \sigma_X(\xi))\xi. \tag{22}$$

3.1.2 Main results: Chen-Fliess Series over Differential Languages

Lemma 5

Given $c \in \mathbb{R}\langle\langle X \rangle\rangle$, the Chen-Fliess series of the sum of u and v is written as

$$F_c[u+v](t) = \sum_{k=0}^{\infty} \sum_{\xi \in \delta^k(X^*)} \frac{1}{k!}(c, \sigma_X(\xi))\mathcal{E}_\xi[u,v](t). \tag{23}$$

Theorem 3

Given $c \in \mathbb{R}\langle\langle X \rangle\rangle$, the Gâteaux derivative of $F_c[u](t)$ in the direction of v is

$$\frac{\partial}{\partial v} F_c[u](t) = \mathcal{F}_{\delta(c)}[u,v](t) = \sum_{\xi \in \mathbb{S}_{X^*,\delta X}} (c, \sigma_X(\xi))\mathcal{E}_\xi[u,v](t). \tag{24}$$

3.2.1 Main results: second order derivatives of Chen-Flies series

Goal: obtain a closed form of the second-order derivative of Chen-Fliess series.

Definition 7

Let $Z_{x_i} := Z \setminus \{x_i\}$ be the alphabet where x_i has been removed from Z, $\eta \in Z^*$ such that $|\eta|_{Z_{x_i}} = n_1 \geq 1$ and $|\eta|_{\delta x_i} = n_2$ and consider the language

$$L_{\delta_{x_i}(\eta)} := \{\xi \in \mathbb{S}_{{Z_{x_i}}^{n_1-1}, \delta x_i^{n_2+1}} \text{ s.t. } \sigma_X(\xi) = \sigma_X(\eta)\}. \tag{25}$$

The *derivative of* η relative to x_i is $\delta_{x_i}(\eta) := \text{char}(L_{\delta_{x_i}(\eta)}) \in \mathbb{R}\langle Z\rangle$. When $|\eta|_{x_i} = 0$, $L_{\delta_{x_i}(\eta)}$ is empty and $\delta(\eta) := 0$.

Example 3

Consider $\eta = x_0x_1x_2x_1$ and compute $\delta_{x_1}(\eta)$. Since $L_{\delta_{x_1}(\eta)} = \{x_0\delta x_1x_2x_1, x_0x_1x_2\delta x_1\}$, then $\delta_{x_1}(x_0x_1^2) = x_0\delta x_1x_2x_1 + x_0x_1x_2\delta x_1$. Similarly, $\delta_{x_2}(\eta) = x_0x_1\delta x_2x_1$. □

Lemma 6

Consider $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$, the Gâteaux derivative in the i-th canonical direction satisfies

$$\frac{\partial}{\partial u_i}F_c[u](t) = F_{\delta_{x_i}(c)}[u](t). \tag{26}$$

3.2.1 Main results: second order derivatives of Chen-Flies series

Lemma 7

Let $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ *and* $u \in L_{\mathfrak{p}}^{m}[t_0, t_1]$, *then*

$$\frac{\partial^2}{\partial u_j \partial u_i} F_c[u](t) = \sum_{\xi \in \mathbb{S}_{X^*, \mathrm{supp}(\delta x_i \sqcup\!\sqcup\, \delta x_j)}} (c, \sigma_X(\xi)) \mathcal{E}_\xi[u, e_{i,j}](t) \tag{27}$$

where $e_{i,j}(t) = (0, \cdots, \underbrace{1}_{i-th}, 0, \cdots, \underbrace{1}_{j-th}, \cdots, 0)$.

Definition 8

Let $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ and $u \in L_{\mathfrak{p}}^{m}[t_0, t_1]$. The Hessian of $F_c[u](t)$ is given by

$$\nabla^2 F_c[u](t) = \begin{bmatrix} 2\frac{\partial^2}{\partial u_1^2} F_c[u](t) & \cdots & \frac{\partial^2}{\partial u_1 \partial u_m} F_c[u](t) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial u_m \partial u_1} F_c[u](t) & \cdots & 2\frac{\partial^2}{\partial u_m^2} F_c[u](t) \end{bmatrix}. \tag{28}$$

Note: $\delta_1^2(x_1x_2x_1) = \delta_1(\delta_1x_1x_2x_1 + x_1x_2\delta_1x_1) = \delta_1x_1x_2\delta_1x_1 + \delta_1x_1x_2\delta_1x_1 = 2\delta_1x_1x_2\delta_1x_1$

3.2.1 Main results: second order derivatives of Chen-Flies series

Theorem 4

Let $c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ and $\epsilon > 0$. Then there exists $\varepsilon_0 \in (0, \varepsilon)$ such that

$$F_c[u + \varepsilon v] = F_c[u] + v^T \nabla F_c[u](t)\varepsilon + \frac{1}{\varepsilon_0} \int_0^{\varepsilon_0} \frac{1}{2} v^T \nabla^2 F_c[u + rv](t) v \varepsilon^2 dr. \tag{29}$$

Corollary 1

Let $c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ and $u^* \in L_{\mathfrak{p}}^m[t_0, t_1]$ such that $v^T \nabla F_c[u^*](t) = 0$. If there exists a neighborhood $\mathcal{B}$ of u^* in which

$$v^T \nabla^2 F_c[u^* + rv](t) v > 0, \forall r \in \mathbb{R} \text{ s.t. } u^* + rv \in \mathcal{B}, \tag{30}$$

then u^* is a local minimum in the direction v.

Example 4

Consider the bilinear system

$$\dot{x} = xu,\ y = x,\ x(0) = 1 \tag{31}$$

with input $u \in L_{\mathfrak{p}}[0, t]$. Its power series is $c = \sum_{n \geq 0} x_1^n \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$. The Hessian for the CFS of the output of the system is

$$\nabla^2 F_c[u](t) = 2 \sum_{\xi \in \mathbb{S}_{X^*, x_0^2}} E_\xi[u](t). \tag{32}$$

□

3.2.2 Main results: Newton-Raphson for Chen-Fliess series reachability

Algorithm 1 Newton-Raphson

Input: R, u_0, ε, v, $\mathcal{U}$
Output: $F_c[u](t)$
Initialization : u_0
1: **for** $i = 1$ to R **do**
2: $u_{i+1} = u_i - \frac{1}{\varepsilon}(\nabla^2 F_c[u_i](t))^{-1}\nabla F_c[u_i](t)^T$,
3: $u_{i+1} \leftarrow \text{sat}_{\mathcal{U}}(u_{i+1})$
4: **end for**
5: **return** u_R

Example 5

Consider the bilinear system

$$\dot{x} = xu,\ y = x,\ x(0) = 1 \tag{33}$$

□

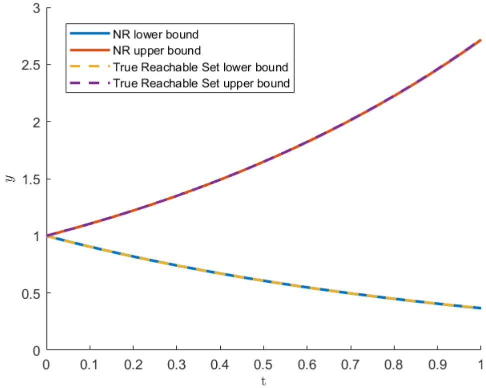

Figure 4: Estimation of reachable sets of (33) with initial state $x_0 = 1$ using Algorithm 1, $\varepsilon = 0.1$, $u_0 = 0$, $-1 \leq u(t) \leq 1$, and CFS truncation $N = 3$.

4. Conclusions

1. We have provided a derivative operation that acts on power series and, algebraically, it coincides with the derivatives of Chen-Fliess series from the analysis perspective.
2. This derivative on power series helps computing higher order derivatives of Chen-Fliess series.
3. We extended the Newton method of optimization for Chen-Fliess series.
4. The optimization by Newton provides the MBB.

Questions?

https://iperezav.github.io