An Interval Arithmetic Approach to Input-Output Reachability*

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Overview

1. Motivation and Problem Statement

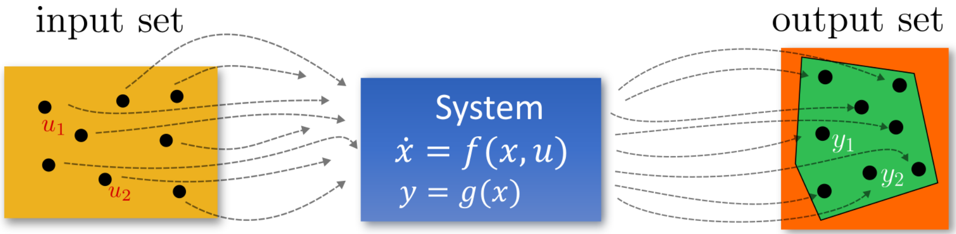


Figure 1: Reachable Set and Overestimation.

Definition 1 (Reachable Set)

Given the set of inputs $\mathcal{U}$ and a set of initial conditions $\mathcal{X}_0$, the reachable set at time t is

$$\text{Reach}(\mathcal{X}_0, \mathcal{U})(t) = \Big\{ \overbrace{\phi(t, u, x_0)}^{\text{trajectory}} \in \mathbb{R}^n : \text{for some } u : [0, t] \to \mathcal{U}, x_0 \in \mathcal{X}_0 \Big\} \quad (1)$$

- **Goal:** compute the reachable set not by simulating each possible trajectory one by one.
- **Methodologies:** set-based methods, mixed-monotonicity, Hamilton-Jacobi reachability, Koopman operators, neural networks.

1. Motivation and Problem Statement

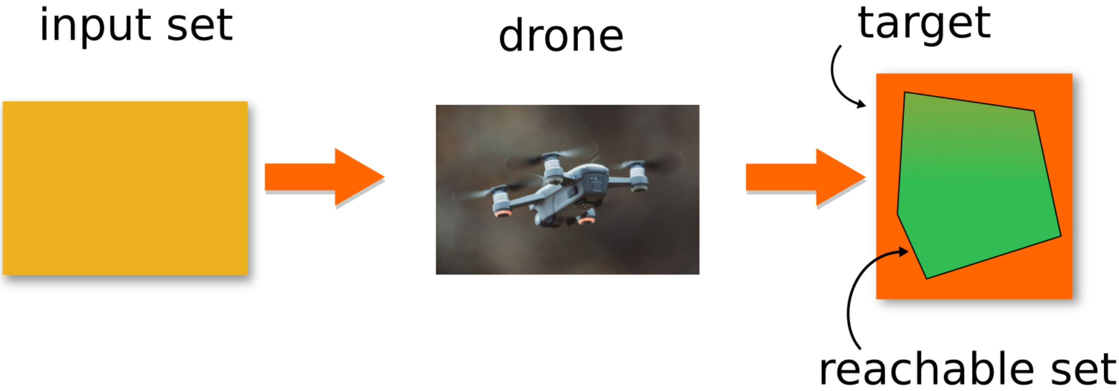


Figure 2: Target reaching.

2.1 Preliminaries: Chen-Fliess series

- An alphabet is a set of symbols (letters): $X = \{x_0, x_1, \ldots, x_m\}$
- A *word* is defined as the catenation $\eta = x_{i_1} \cdots x_{i_k}$ of letters from X.
- The set of words of any length: X^*

Example 1

$X = \{x_0, x_1\}$. We can form the words

$$\begin{aligned} \eta_1 &= x_0 x_1 \\ \eta_2 &= x_1 x_0 \\ \eta_3 &= x_1^2 = x_1 x_1 \end{aligned}$$

- A *power series* c is a function $c : X^* \to \mathbb{R}^\ell$ and represented as $c = \sum_{\eta \in X^*} (c, \eta)\eta$.

Example 2

$$c_1 = \sum_{k=1}^{\infty} x_1^k, \qquad c_2 = \sum_{k=0}^{\infty} (k+1) \cdot x_0 x_1 x_2^k x_1$$

2.1 Preliminaries: Chen-Fliess series

Definition 2 (Iterated integrals)

Let $X = \{x_0, \cdots, x_m\}$, $\eta = x_i\xi \in X^*$, the input vector $u(t) = (u_1(t), \cdots, u_m(t)) \in L_1^m[0, T]$

$$E_\phi[u](t) := 1, \quad E_{x_i\xi}[u](t) := \int_0^t u_i(\tau) E_\xi[u](\tau) d\tau$$

Definition 3 (Chen-Fliess series)

Consider the power series $c = \sum_{\eta \in X^*} (c, \eta)\eta$, the Chen-Fliess series associated to c is

$$F_c[u](t) = \sum_{\eta \in X^*} \underbrace{(c, \eta)}_{\text{vector in } \mathbb{R}^\ell} E_\eta[u](t)$$

word

word

iterated integral

space of all words

2.1 Preliminaries: Chen-Fliess series

Theorem 1 (State-Space realization (Fliess 1983))

The Chen-Fliess series $F_c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ *represents the system*

$$\dot{z} = g_0(z) + \sum_{i=1}^{m} g_i(z)u_i, \;\; z(t_0) = z_0 \tag{2}$$

$$y = h(z) \tag{3}$$

if and only if

- $(c, \eta) = L_{g_\eta} h(z)|_{z_0}$ *where* $L_{g_\eta} h_j(z_0) := L_{g_{i_1}} \cdots L_{g_{i_k}} h_j(z)|_{z_0}$ *for* $\eta = x_{i_k} \cdots x_{i_1} \in X^*$ *and* $L_g f(z) = (\frac{\partial}{\partial z} f(z)) \cdot g(z)$. *Equivalently,*

$$L_{g_{i_1}} \cdots L_{g_{i_k}} h(z_0) = \frac{\partial}{\partial z}\left(\cdots\left(\frac{\partial}{\partial z}\left(\frac{\partial h(z)}{\partial z} \cdot g_{i_k}(z)\right) \cdot g_{i_{k-1}}(z)\right)\cdots\right) \cdot g_{i_1}(z)\bigg|_{z_0}. \tag{4}$$

- $c = \sum_{\eta \in X^*} (c, \eta)\eta$ *has finite Lie rank.*
- *There exist* $K, M \geq 0$ *such that* $|(c, \eta)| \leq K M^{|\eta|} |\eta|!$

2.2 Preliminaries: Mixed-Monotonicity

Consider the Lipschitz continuous function in each argument $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$. The continuous-time dynamical system is defined

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ x(0) &= x_0 \end{aligned}, \tag{5}$$

Definition 4 (Decomposition function)

Let $d : \mathcal{X} \times \mathcal{U} \times \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ locally Lipschitz continuous satisfying

i. $d(x, u, x, u) = f(x, u)$ for all $(x, u) \in \mathcal{X} \times \mathcal{U}$.

ii. $\frac{\partial d_i}{\partial x_j}(x, u, \hat{x}, \hat{u}) \geq 0$ for any $i \neq j$, and for any $(x, u, \hat{x}, \hat{u}) \in \mathcal{X} \times \mathcal{U} \times \mathcal{X} \times \mathcal{U}$.

iii. $\frac{\partial d_i}{\partial \hat{x}_j}(x, u, \hat{x}, \hat{u}) \leq 0$ for any i, j, and for any $(x, u, \hat{x}, \hat{u}) \in \mathcal{X} \times \mathcal{U} \times \mathcal{X} \times \mathcal{U}$.

iv. $\frac{\partial d_i}{\partial u_k}(x, u, \hat{x}, \hat{u}) \geq 0$, $\frac{\partial d_i}{\partial \hat{u}_k}(x, u, \hat{x}, \hat{u}) \leq 0$ for any i, k, and for any $(x, u, \hat{x}, \hat{u}) \in \mathcal{X} \times \mathcal{U} \times \mathcal{X} \times \mathcal{U}$.

d is a decomposition function of (5).

2.2 Preliminaries: Mixed-Monotonicity

Theorem 2 (Coogan 2020)

The following is a **decomposition function** *of the system:*

$$d_i(x, u, \hat{x}, \hat{u}) = \begin{cases} \min\limits_{\substack{y \in [x, \hat{x}] \\ y_i = x_i \\ z \in [u, \hat{u}]}} f_i(y, z), \ x \le \hat{x}, \ u \le \hat{u}, \\ \max\limits_{\substack{y \in [\hat{x}, x] \\ y_i = x_i \\ z \in [\hat{u}, u]}} f_i(y, z), \ \hat{x} \le x, \ \hat{u} \le u. \end{cases} \tag{6}$$

Definition 5 (Coogan 2020)

Given the decomposition function d, the **embedding system** is

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = e(x, u, \hat{x}, \hat{u}) := \begin{bmatrix} d(x, u, \hat{x}, \hat{u}) \\ d(\hat{x}, \hat{u}, x, u) \end{bmatrix}, \tag{7}$$

where the input $(u, \hat{u}) \in \mathcal{U} \times \mathcal{U} \subset \mathbb{R}^{2m}$ and the state $(x, \hat{x}) \in \mathcal{X} \times \mathcal{X} \subset \mathbb{R}^{2n}$.

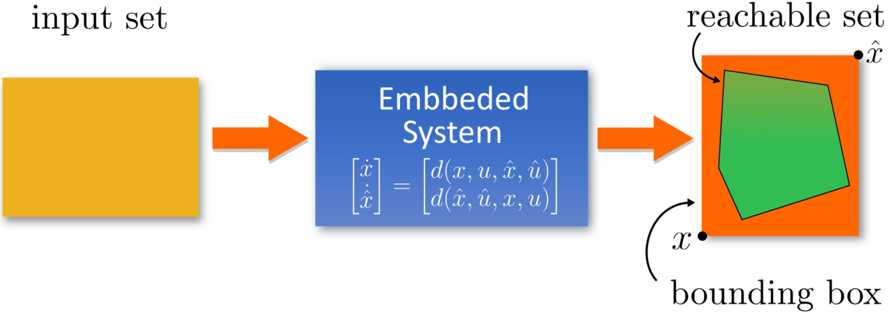
2.2 Preliminaries: Mixed-Monotonicity

- $\Phi^e(t, a, b)$: trajectory of the embedding system where $a, b \in \mathbb{R}^{2n}$.
- $[x,\ y] \subset \mathbb{R}^n$: hyperrectangle defined by the coordinatewise inequality $x \leq y$.
- $[\![a]\!] \subset \mathbb{R}^n$: hyperrectangle defined by $a \in \mathbb{R}^{2n}$ where $a = (x, y)$ and $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$.

Theorem 3 (Coogan 2020)

Consider the decomposition d associated with (5), $\mathcal{X}_0 = [\underline{x}, \overline{x}] \subset \mathcal{X}$ and $\mathcal{U} = [\underline{u},\ \overline{u}]$, then

$$\textit{Reach}([\underline{x}, \overline{x}], [\underline{u}, \overline{u}])(t) \subset [\![\Phi^e(t, (\underline{x}, \overline{x}), (\underline{u}, \overline{u}))]\!], \ \textit{for all } t \geq 0$$



2.3 Preliminaries: Other Approaches Using Chen-Fliess Series

1. Mixed-Monotonicity for Chen-Fliess series (**IPA** and LDE 2022a)
2. Gradient Descent using Chen-Fliess series (**IPA** and LDE 2022b)
3. Newton Optimization using Chen-Flies series (**IPA** and LDE 2023a)
4. Trust regions Optimization using Chen-Flies series (**IPA** and LDE 2023b)
5. Critical Points of Chen-Flies series (LDE and SG and **IPA** 2023c)
6. Backward and Inner Approximation using Chen-Flies series (**IPA** and LDE 2023d)

- **IPA** = Ivan Perez Avellaneda
- LDE = Luis A. Duffaut Espinosa
- SG = W. Steven Gray

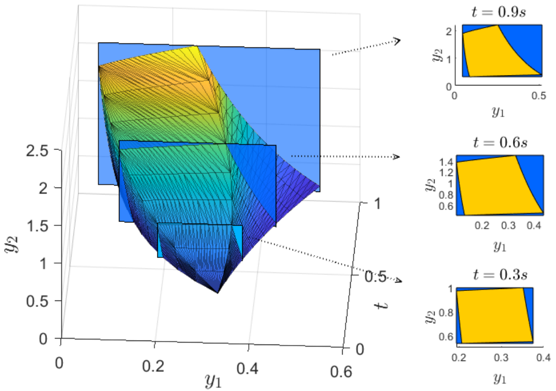


Figure 3: Output reachable set of a Lotka-Volterra system and its MBB in terms of its maximum and minimum outputs.

3.1 Main results: Interval Arithmetics

Goal: compute overestimation of the reachable set using IA and Chen-Fliess series.

Definition 6 (Interval Product)

Given $I_1 = [a_1,\ b_1] \subset \mathbb{R}$ and $I_2 = [a_2,\ b_2] \subset \mathbb{R}$, the product $I_1 \cdot I_2 := \left[\underline{I}, \overline{I}\right]$ where

$$\underline{I} = \min_{\substack{y_1 \in [a_1, b_1] \\ y_2 \in [a_2, b_2]}} y_1 y_2, \quad \overline{I} = \max_{\substack{y_1 \in [a_1, b_1] \\ y_2 \in [a_2, b_2]}} y_1 y_2.$$

Equivalently,

$$[a_1, b_1] \cdot [a_2, b_2] = [\min\{a_1a_2, a_1b_2, b_1a_2, b_1b_2\},\ \max\{a_1a_2, a_1b_2, b_1a_2, b_1b_2\}].$$

and

$$\underbrace{[a, b] \cdots [a, b]}_{n \text{ times}} = [a, b]^n.$$

Definition 7

Given a set $\mathcal{X} \subset \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, $\lambda \mathcal{X} := \{\lambda x \ : \ x \in \mathcal{X}\}$.

3.1 Main results: Interval Arithmetics

Example 3

- $[-2,1]\cdot[1,3] = [\min\{-2,-6,1,3\},\ \max\{-2,-6,1,3\}] = [-6,3]$
- $[-2,1]^2 = [-2,4]$

Example 4

$3[-2,1] = [-6,3]$.

Lemma 1

Consider $u : \mathbb{R} \to \mathbb{R}^m \in [\mathbf{a},\mathbf{b}]$ *(i.e.,* $u_i(t) \in [a,b]$*). For* $\eta = x_{i_1}\cdots x_{i_n} \in X^*$*, then*

$$\mathit{Reach}_\eta([\mathbf{a},\mathbf{b}])(t) \subset [a,b]^{|\eta|-|\eta|_{x_0}} \frac{t^{|\eta|}}{|\eta|!}, \forall\ t \in [0,T].$$

Example 5

Consider $F_c[u](t) = E_{x_1^2}[u](t)$ and $u \in [-2,1]$. We want to overaproximate $\mathrm{Reach}_c(\mathcal{U})(t)$

$$E_{x_1^2}[u](t) = \int_0^t u(\tau_1)\int_0^{\tau_1} u(\tau)d\tau d\tau_1$$

3.2 Main results: Interval Arithmetics and Iterated Integrals

Since $u \in [-2, 1]$

$$-2\tau_1 \leq \int_0^{\tau_1} u(\tau) d\tau \leq \tau_1,$$

it follows that

$$u(\tau_1) \int_0^{\tau_1} u(\tau) d\tau \in [-2, 1] \cdot [-2, 1]\tau_1$$

and

$$E_{x_1^2}[u](t) = \int_0^t u(\tau_1) \int_0^{\tau_1} u(\tau) d\tau d\tau_1 \in [-2, 1]^2 \frac{t^2}{2!}.$$

Therefore,

$$\text{Reach}_c(\mathcal{U})(t) \subset [-2, 1]^2 \frac{t^2}{2!}$$

Remark:

$$\min_{u\in\mathcal{U}} E_{\eta_1}[u](t) + \min_{u\in\mathcal{U}} E_{\eta_2}[u](t) \leq \min_{u\in\mathcal{U}} (E_{\eta_1}[u](t) + E_{\eta_2}[u](t))$$
$$\max_{u\in\mathcal{U}} (E_{\eta_1}[u](t) + E_{\eta_2}[u](t)) \leq \max_{u\in\mathcal{U}} E_{\eta_1}[u](t) + \max_{u\in\mathcal{U}} E_{\eta_2}[u](t).$$

3.3 Main results: Reachable Set Overestimation

Consider

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ x(0) &= x_0 \end{aligned} \tag{8}$$

$$y = h(x) \tag{9}$$

and $\phi(t, u, x_0)$ is the trajectory function.

Definition 8 (State-Space Reachable Set)

The RS of (8) with inputs in $\mathcal{U}$:

$$\text{Reach}(\mathcal{X}_0, \mathcal{U})(T) := \Big\{ \phi(T, u, x_0) \in \mathbb{R}^n : \text{for some } x_0 \in \mathcal{X}_0, u : [0, t] \to \mathcal{U} \Big\}.$$

Definition 9 (Input-Output Reachable Set)

The RS of the CFS $F_c[u](t)$ representing (8) and (9) with inputs in $\mathcal{U}$:

$$\text{Reach}_c(\mathcal{U})(T) := \Big\{ y = F_c[u](T) \in \mathbb{R}^\ell \ : \text{for some } u : [0, t] \to \mathcal{U} \Big\}.$$

3.3 Main results: Reachable Set Overestimation

Lemma 2

Given the interval $I = [a, b] \subset \mathbb{R}$, its n-th power I^n is given by

$$I^n = \begin{cases} [a^n, b^n], & a, b > 0, n \textit{ even} \\ [b^n, a^n], & a, b < 0, n \textit{ even} \\ [\min\{a^{n-1}b, ab^{n-1}\},\ \max\{|a|, |b|\}^n], & a < 0, b > 0, n \textit{ even} \\ [a^n, b^n], & ab > 0, n \textit{ odd} \\ [\min\{a^n, ab^{n-1}\},\ \max\{a^{n-1}b, b^n\}], & a < 0, b > 0,\ n \textit{ odd} \end{cases} \tag{10}$$

Theorem 4

Given $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ with $u(t) \in [\mathbf{a}, \mathbf{b}]$ for all $t > 0$. The RS of the CFS $F_c[u](t)$

$$\textit{Reach}_c([\mathbf{a}, \mathbf{b}])(t) \subset \left[F_{\underline{c}}[\mathbb{1}](t), F_{\overline{c}}[\mathbb{1}](t)\right],\ \forall t \in \mathbb{R},$$

where $(\underline{c}, \eta) = \min\left\{(c, \eta)[a, b]^{|\eta| - |\eta|_{x_0}}\right\},\ (\overline{c}, \eta) = \max\left\{(c, \eta)[a, b]^{|\eta| - |\eta|_{x_0}}\right\}$ *and* $\mathbb{1} = (1, \cdots, 1)^\top$ *for all* $t \in [0, T]$.

4. Simulations

Example 6

Consider the system

$$\dot{x} = xu, \;\; y = x, \;\; x_0 = 1 \tag{11}$$

with $u \in [1, 2.8]$. The Chen-Fliess series is given by

$$F_c[u](t) = 1 + \sum_{k=1}^{\infty} E_{x_1^k}[u](t).$$

The embedding system (Mixed-Monotonicity):

$$\dot{x} = xu, \; \dot{\hat{x}} = \hat{x}\hat{u}, (x_0, \hat{x}_0) = (1, 1)$$

Using interval arithmetics:

$$(\bar{c}, \eta) = 2.8^k \text{ and } (\underline{c}, \eta) = 1, |\eta| = k$$

4. Simulations

$$\text{Reach}_c([1, 2.8])(t) \subset [F_{\underline{c}}[\mathbb{1}](t), F_{\overline{c}}[\mathbb{1}](t)] = \left[1 + \sum_{k=1}^{\infty} \frac{t^k}{k!}\ ,\ 1 + \sum_{k=1}^{\infty} 2.8^k \frac{t^k}{k!}\right].$$

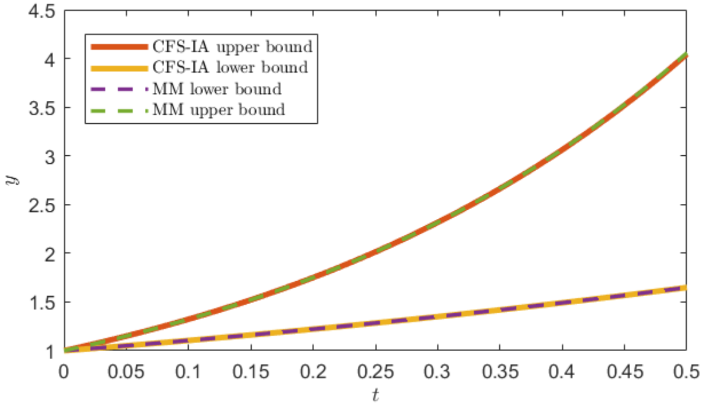


Figure 4: Reachable set overestimation using the mixed-monotonicity vs CFS with interval arithmetics.

4. Simulations

Example 7

Consider the Lotka-Volterra system

$$\dot{x}_1 = -x_1 x_2 + x_1 u_1, \quad \dot{x}_2 = x_1 x_2 - x_2 u_2, \quad y = x_1, \quad x_0 = (1/6, 1/6)^\top$$

The embedding system (Mixed-Monotonicity):

$$\begin{aligned}
\dot{x}_1 &= -x_1 \hat{x}_2 + x_1 u_1 \\
\dot{x}_2 &= x_2 x_1 - x_2 \hat{u}_2 \\
\dot{\hat{x}}_1 &= -\hat{x}_1 x_2 + \hat{x}_1 \hat{u}_1 \\
\dot{\hat{x}}_2 &= \hat{x}_2 \hat{x}_1 - \hat{x}_2 u_2
\end{aligned}$$

with $(x_{1,0}, x_{2,0}, \hat{x}_{1,0}, \hat{x}_{2,0}) = (1/6, 1/6, 1/6, 1/6)$

4. Simulation

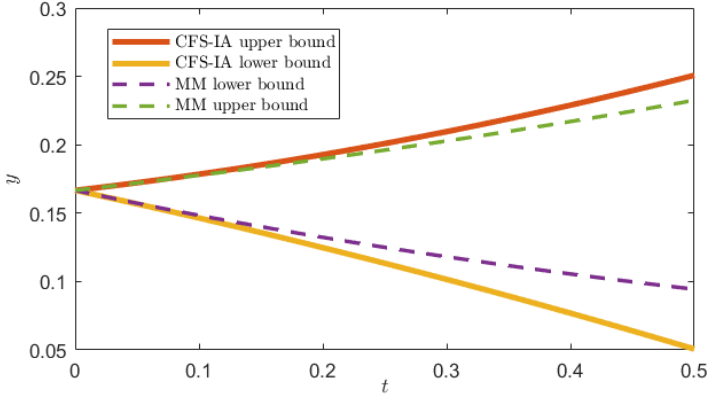


Figure 5: Reachable set overestimation using the mixed-monotonicity vs CFS with interval arithmetics.

5. Conclusions

1. **Closed form:** for overestimating reachable sets.
2. **Faster computation:** compared with non-convex optimization approaches.
3. **Good accuracy:** short periods of time.

Questions?
https://iperezav.github.io