Reachability of Chen-Fliess series: A Gradient Descent Approach*

Ivan Perez Avellaneda and Luis A. Duffaut Espinosa

Electrical and Biomedical Engineering

University of Vermont

*This material is based upon work supported by NASA under cooperative agreement VT-80NSSC20M0213.

The University of Vermont

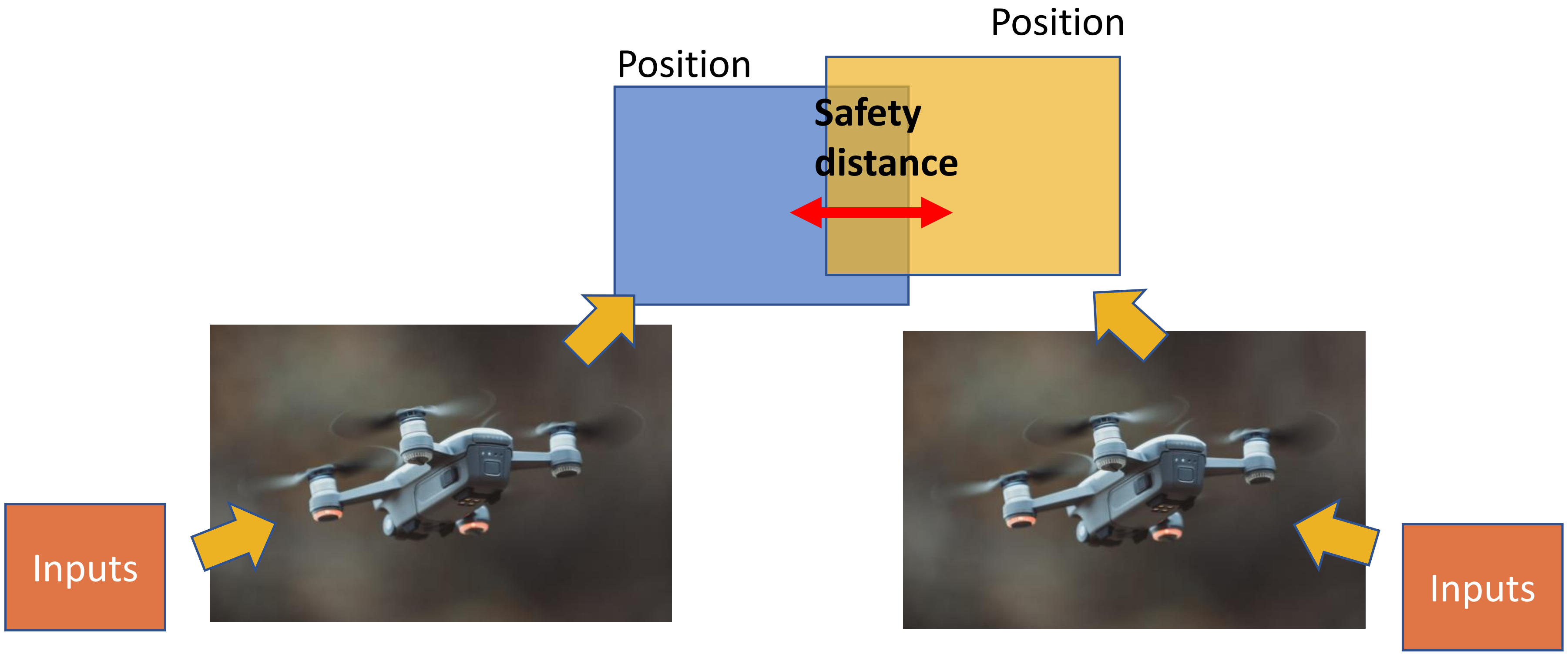
Outline

1. Motivation and Problem Statement
2. Preliminaries
    2.1. Chen-Fliess Series
    2.2. Mixed-Monotonicity
    2.3. Data-Driven Mixed-Monotonicity
3. Gradient Descent Method
4. Illustrative Simulations
5. Conclusion

1. Motivation and Problem Statement

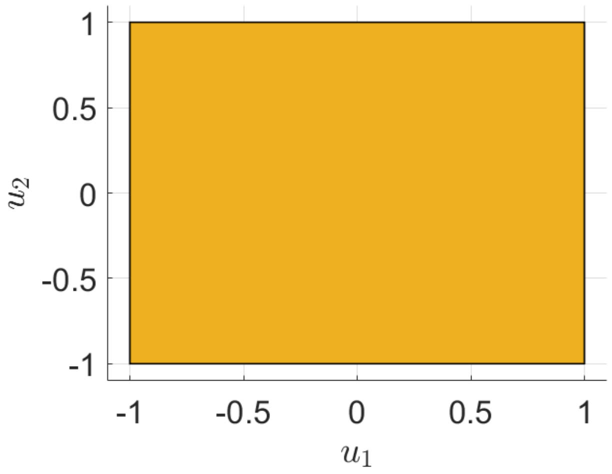
Definition 1 (Reachable set): The set of outputs of the system as a response to a set of inputs and a set of initial conditions.

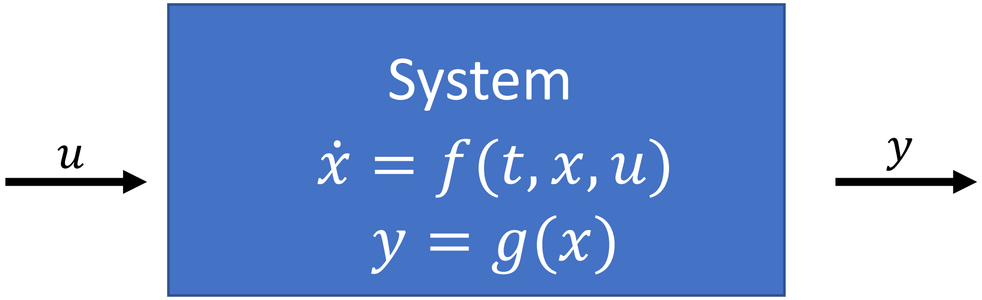
Application: Collision avoidance of quadcopters



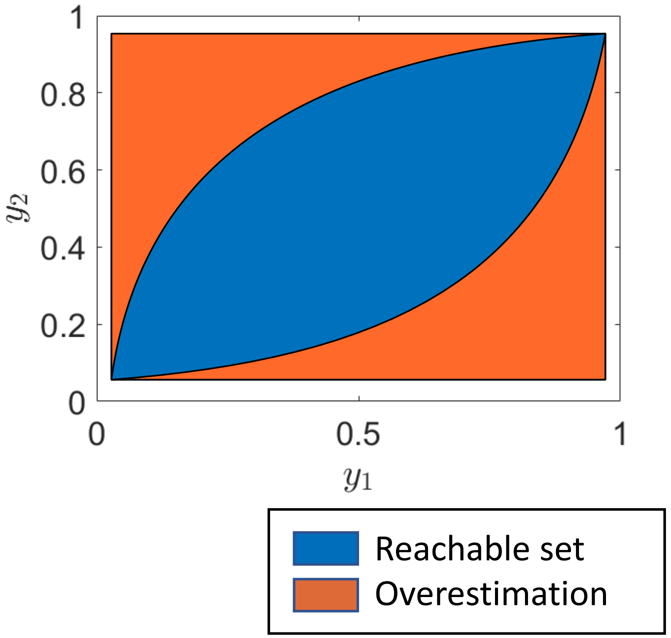
1. Motivation and Problem Statement

Input set



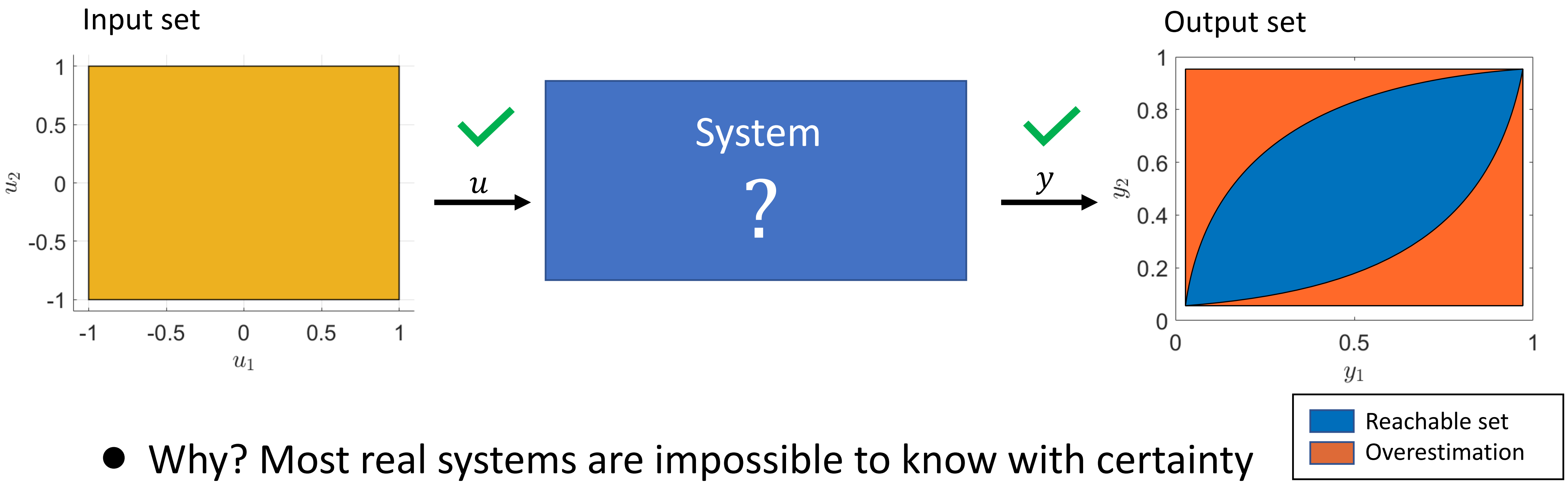


Output set



- Why? It only needs two points to compute.
- Approaches: mixed-monotonicity, Hamilton-Jacobi reachability, neural networks

1. Motivation and Problem Statement



- Why? Most real systems are impossible to know with certainty
- What to do? Use a data-driven approach
- Data-driven approaches: neural networks, Chen-Fliess series

2. Preliminaries
2.1 Chen-Fliess Series

Definition 2 (Chen-Fliess series):

$$F_c[u](t) := \sum_{\eta \in X^*} \underbrace{(c,\eta)}_{\text{Vector in } \mathbb{R}^l} E_{x_i \nu}[u](t)$$

words

Space of all words

Iterative integral

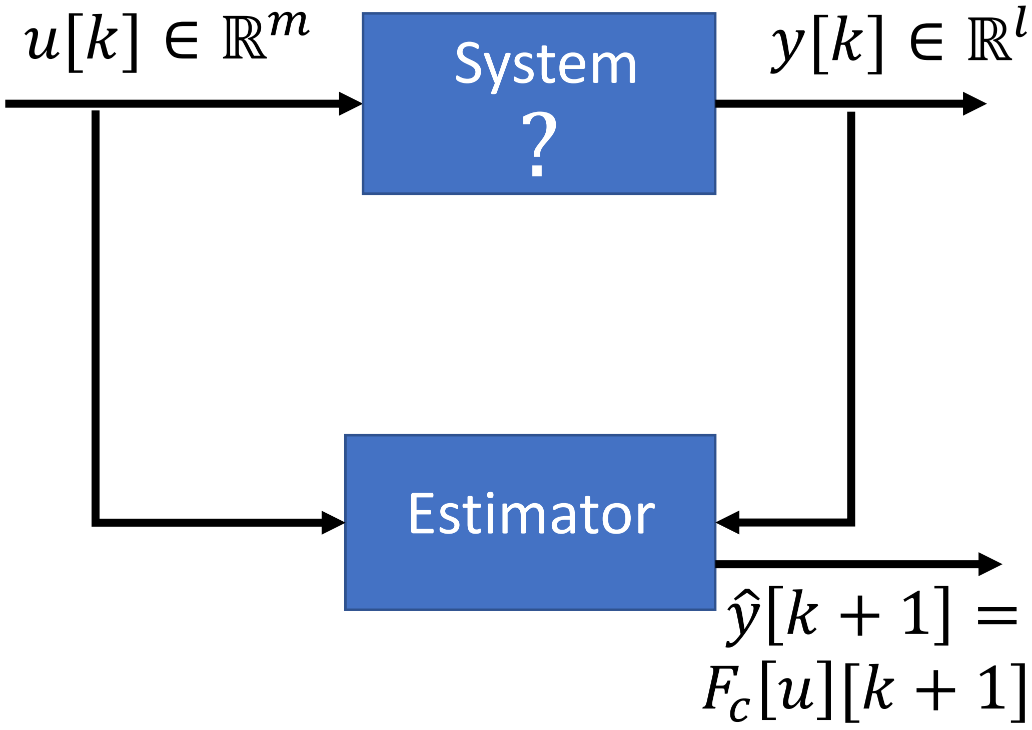
words

Definition 3 (Iterative integral):

$$E_{x_i \nu}[u](t) := \int_0^t u_i(\tau) E_\nu[u](\tau) d\tau \,,$$

$$E_\phi[u](t) := 1$$ [Fliess, 1981]

Remark: we need to truncate to a word length to use them.



Remark: Chen-Fliess series does not estimate the system.

[Gray, Venkatesh, Duffaut Espinosa, 2019]

2. Preliminaries
2.1 Chen-Fliess Series

u

System

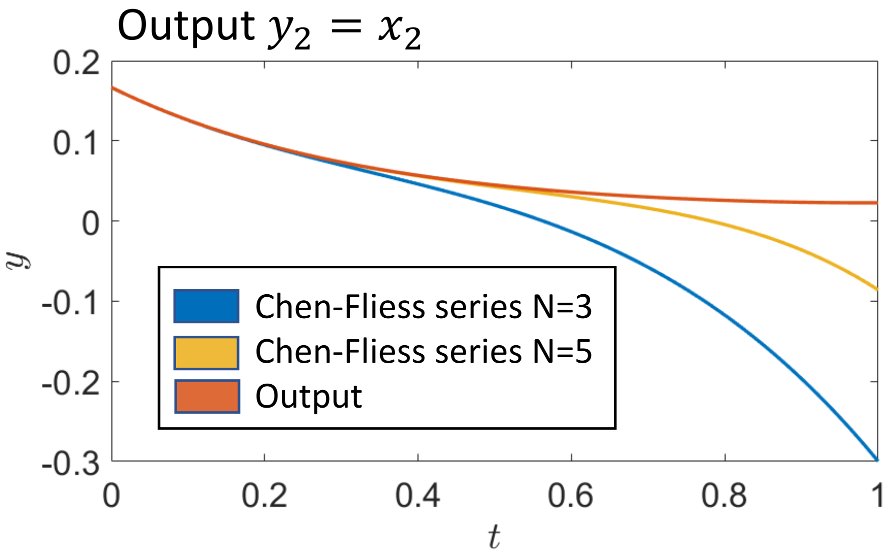
$$\dot{z} = g_0(z) + \sum_{i=1}^{m} g_i(z) u_i\,,$$
$$y_j = h_j(z), j = 1, \cdots, l,$$
$$z(0) = z_0$$

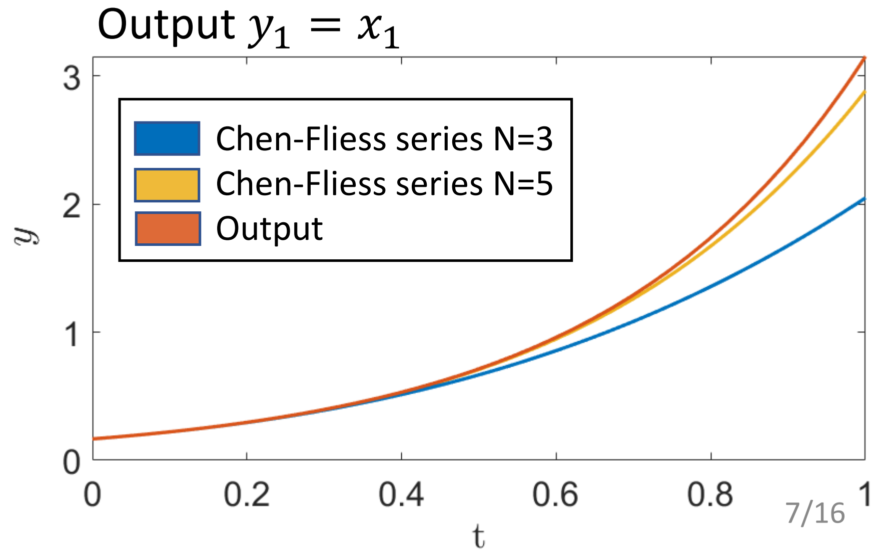
$F_c[u](t)$

Remark: the coefficients are computed deterministically. [Fliess, 1981]

Example: Lotka-Volterra system

$$\dot{x}_1 = -x_1 x_2 + x_1 u_1,$$
$$\dot{x}_2 = x_1 x_2 - x_2 u_2,$$
$$y = x,$$
$$x(0) = (1/6,1/6),$$





2. Preliminaries
2.2 Mixed-Monotonicity (MM)

Definition 4 (Mixed-Monotone system): A system is mixed-monotone with respect to the decomposition function $d: \mathcal{X} \times \mathcal{W} \times \mathcal{X} \times \mathcal{W} \rightarrow \mathbb{R}^n$ if

i. $d(x, w, x, w) = f(x, w)$,

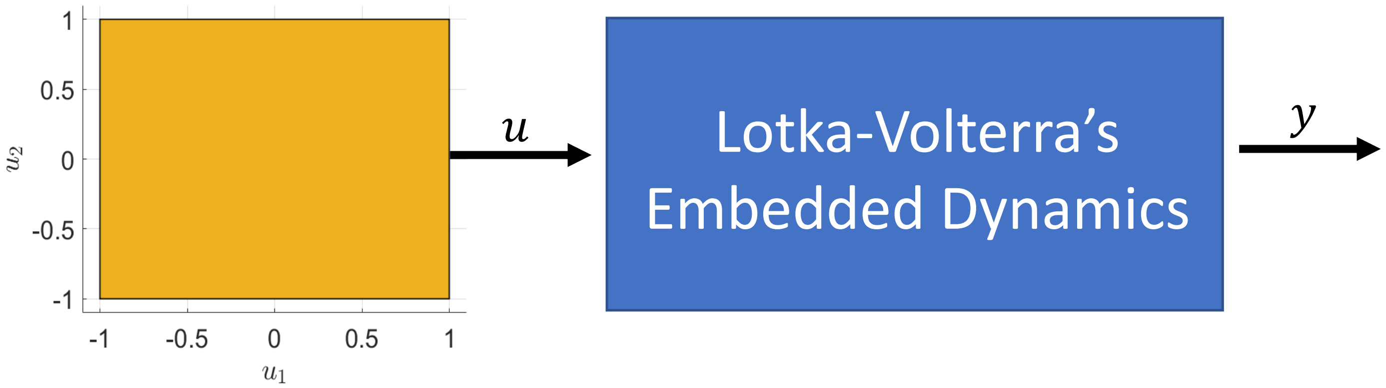
ii. $\frac{\partial d_i}{\partial x_j}(x, w, \hat{x}, \widehat{w}) \geq 0$ for all $i \neq j$,

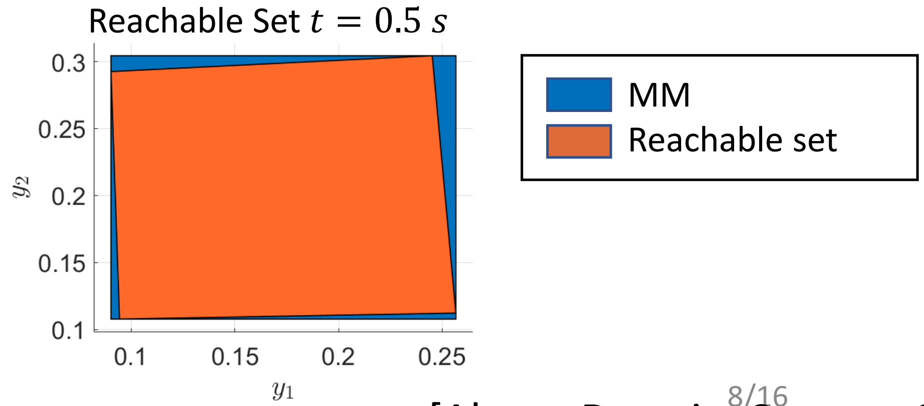
iii. $\frac{\partial d_i}{\partial \hat{x}_j}(x, w, \hat{x}, \widehat{w}) \leq 0$ for all i, j,

iv. $\frac{\partial d_i}{\partial w_k}(x, w, \hat{x}, \widehat{w}) \geq 0$ and $\frac{\partial d_i}{\partial \widehat{w}_k}(x, w, \hat{x}, \widehat{w}) \leq 0$ for all i, k

Example: Lotka-Volterra system

$$\dot{x}_1 = -x_1 x_2 + x_1 u_1,$$
$$\dot{x}_2 = x_1 x_2 - x_2 u_2,$$
$$y = x,$$
$$x(0) = (1/6, 1/6)$$
$$u_{1,2} \in [-1, 1]$$





[Abate, Dutreix, Coogan, 2020]

2. Preliminaries
2.3 Input-Output Mixed-Monotonicity (IOMM)

- Alphabets: $X = \{x_1, \cdots, x_n\}$, $Y = \{y_1, \cdots, y_n\}$
- Word substitution: $\sigma_Y(x_i) = y_i$
- Extended Iterative Integral:

$$\mathcal{E}_{z_i\eta}[u,v](t) = \begin{cases} \int_0^t u_i(\tau)\mathcal{E}[u,v](\tau)d\tau, & z_i \in X \\ \int_0^t v_i(\tau)\mathcal{E}[u,v](\tau)d\tau, & z_i \in Y \end{cases}$$

Definition 5 (IOMM): A Chen-Fliess series $F_c[u](t)$ is input-output mixed-monotone if there exists a decomposition function $d[u,\hat{u}](t)$ such that

i. $d[u,u](t) = F_c[u](t)$,

ii. $d[u,\hat{u}](t)$ is non-decreasing in u,

iii. $d[u,\hat{u}](t)$ is non-increasing in $\hat{u}$,

Lemma 1: Every Chen-Fliess series is expressed as

$$F_c[u](t) = \mathcal{F}_{c^+}[u](t) - \mathcal{F}_{c^-}[u](t)$$

where

$$\mathcal{F}_{c^+}[u](t) = \sum_{k=0}^{\infty} \sum_{\eta \in X^*} \sum_{\xi \in \mathbb{S}_{\eta,Y^k}} (c^+,\xi)\mathcal{E}_\xi[u^+,u^-](t),$$

$$\mathcal{F}_{c^-}[u](t) = \sum_{k=0}^{\infty} \sum_{\eta \in X^*} \sum_{\xi \in \mathbb{S}_{\eta,Y^k}} (c^-,\xi)\mathcal{E}_\xi[u^+,u^-](t),$$

$$(c^+,\xi) = \max\{(-1)^k\big(c,\sigma_X(\xi)\big),0\}$$

$$(c^-,\xi) = -\min\{(-1)^k\big(c,\sigma_X(\xi)\big),0\}$$

Theorem 1: The following is decomposition function of the Chen-Fliess series

$$d[u,\hat{u}](t) \coloneqq \mathcal{F}_{c^+}[u](t) - \mathcal{F}_{c^-}[\hat{u}](t)$$

[Perez Avellaneda & Duffaut Espinosa, 2022a]

2. Preliminaries
2.3 Input-Output Mixed-Monotonicity (IOMM)

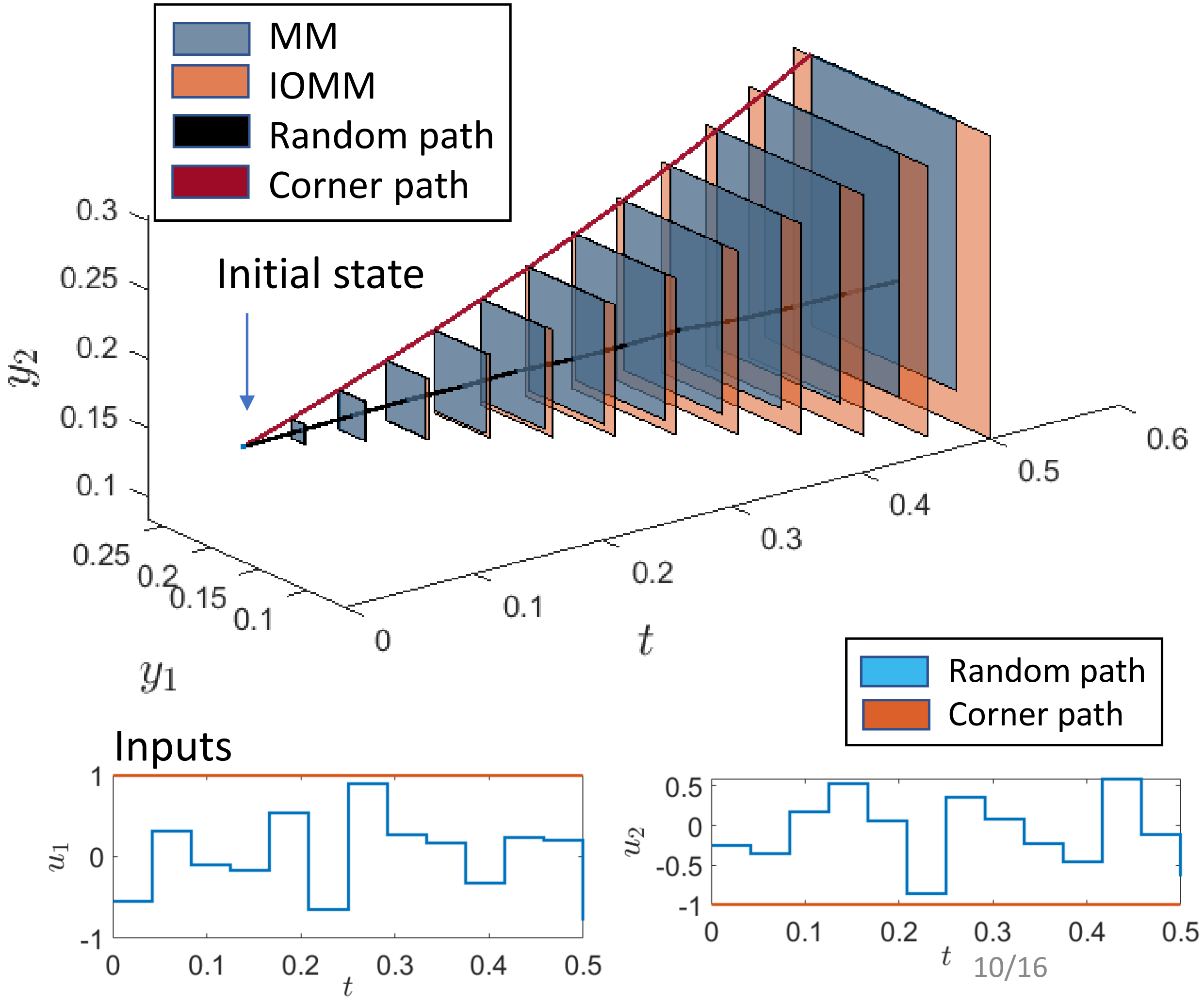
Theorem 2: Consider the Chen-Fliess series $F_c[u](t)$ taking values in the hyper-rectangle $U = [u, \hat{u}] \subset K$. Then

$$\mathrm{Reach}(F_c, U)(t) \subset [d[u, \hat{u}](t), d[\hat{u}, u](t)].$$

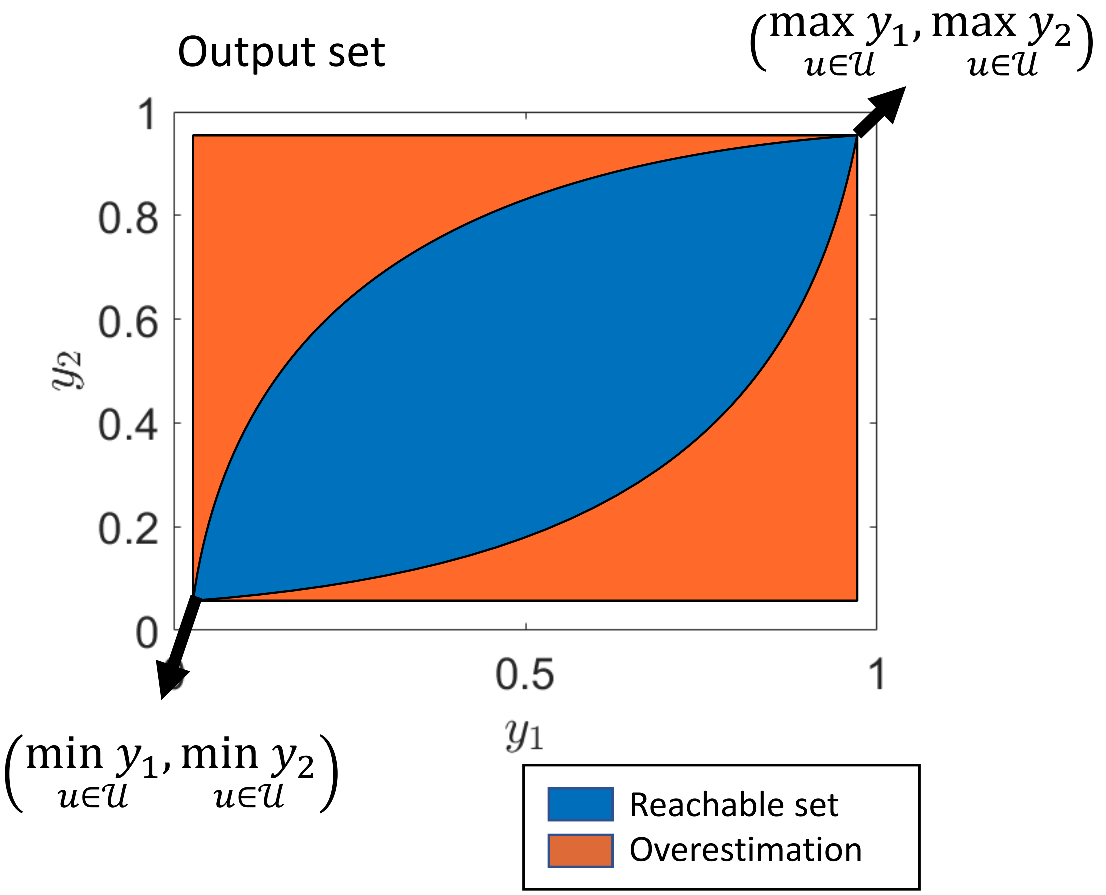
Example: Lotka-Volterra system

$$\dot{x}_1 = -x_1 x_2 + x_1 u_1,$$
$$\dot{x}_2 = x_1 x_2 - x_2 u_2,$$
$$y = x,$$
$$x(0) = (1/6, 1/6),$$
$$(u_1, u_2) \in [-1,1] \times [-1,1]$$

[Perez Avellaneda & Duffaut Espinosa, 2022a]



3. Gradient Descent Method



Problem:

$$\min_{u\in\mathcal{U}} F_c[u](t)$$

function

Chen-Fliess series

Definition 6 (Gateâux derivative): Given $c \in \mathbb{R}^l\langle\langle X\rangle\rangle$ and the input functions $u, v \in L_p^m[0,t]$, the Chen-Fliess series is Gâteaux differentiable at u the direction of v if and only if there exists $\frac{\partial}{\partial v}F_c[u](t) \in \mathbb{R}^l$ such that the following limit is satisfied:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(F_c[u+\varepsilon v](t) - F_c[u](t) - \frac{\partial}{\partial v}F_c[u](t)\varepsilon\right) = 0$$

3. Gradient Descent Method

Theorem 2: Let X and Y be alphabets associated to $u, v \in L_p^m[t_0, t_1]$, respectively. The Chen-Fliess series is Gâteaux differentiable in the direction of v if and only if

$$\lim_{\varepsilon \to 0} \sum_{k=2}^{\infty} \sum_{\eta \in X^*} \sum_{\xi \in \mathbb{S}_{\eta, Y^k}} \big(c, \sigma_X(\xi)\big) \mathcal{E}_\xi[u, v](t) \varepsilon^k = 0$$

And the Gâteaux derivative is expressed as

$$\frac{\partial}{\partial v} F_c[u](t) = \sum_{\eta \in X^*} \sum_{\xi \in \mathbb{S}_{X^*, Y^k}} \big(c, \sigma_{\mathrm{X}}(\eta)\big) \mathcal{E}_\xi[u, v](t)$$

Definition 7 (Gradient): the gradient of a Chen-Fliess series is defined as

$$\nabla F_c[u](t) = \left(\frac{\partial}{\partial e_1} F_c[u](t), \ldots, \frac{\partial}{\partial e_m} F_c[u](t) \right)$$

where $e_i(t) = (0, \cdots, \underbrace{1,}_{i\text{-th}} \cdots, 0)$

Example: Consider the system:

$$\dot{x} = Ax + Bu, y = Cx, x(0) = x_0$$

The output is expressed as

$$y(t) = C\exp(-At)x_0 + \int_0^t C\exp A(t - \tau) B u(\tau) d\tau$$

The Chen-Fliess series of the system is

$$F_c[u] = \sum_{k=0}^{\infty} CA^k x_0 \frac{t}{k!} + \sum_{k=0}^{\infty} CA^k B E_{x_0^k x_1}[u](t)$$

The Gâteaux derivative is the following

$$\nabla F_c[u](t) = \sum_{k=0}^{\infty} CA^k B \frac{t^{k+1}}{(k+1)!} = -\frac{dy}{du}$$

3. Gradient Descent Method

Theorem 3: Consider the constant vector $v \in \mathbb{R}^m$, the function $u \in L_p^m[0,t]$ and $c \in R_{LC}\langle\langle X \rangle\rangle$. Then there exists $\varepsilon_0 \in (0, \varepsilon)$ such that

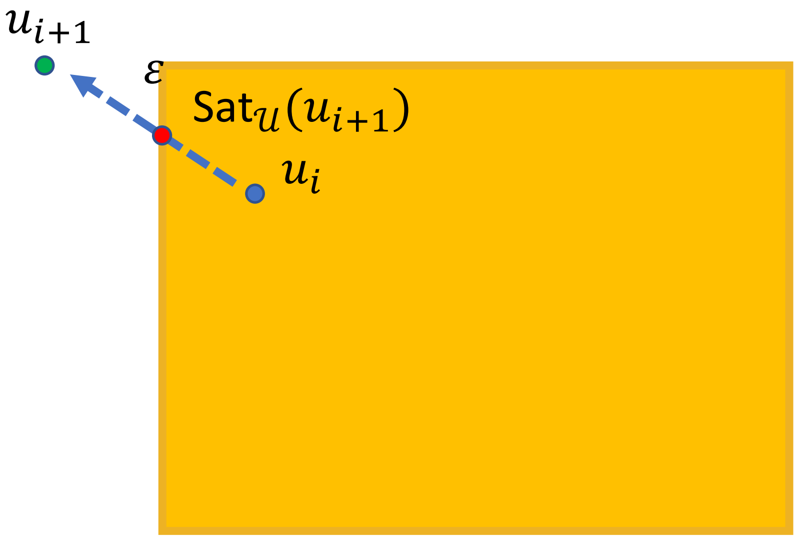
$$F_c[u + \varepsilon v] = F_c[u] + v^T \nabla F_c[u + \varepsilon_0 v](t)\varepsilon$$

Algorithm 2 Gradient Descent

Input: R, u_0, ε, $\mathcal{U}$

Output: $F_c[u](t)$

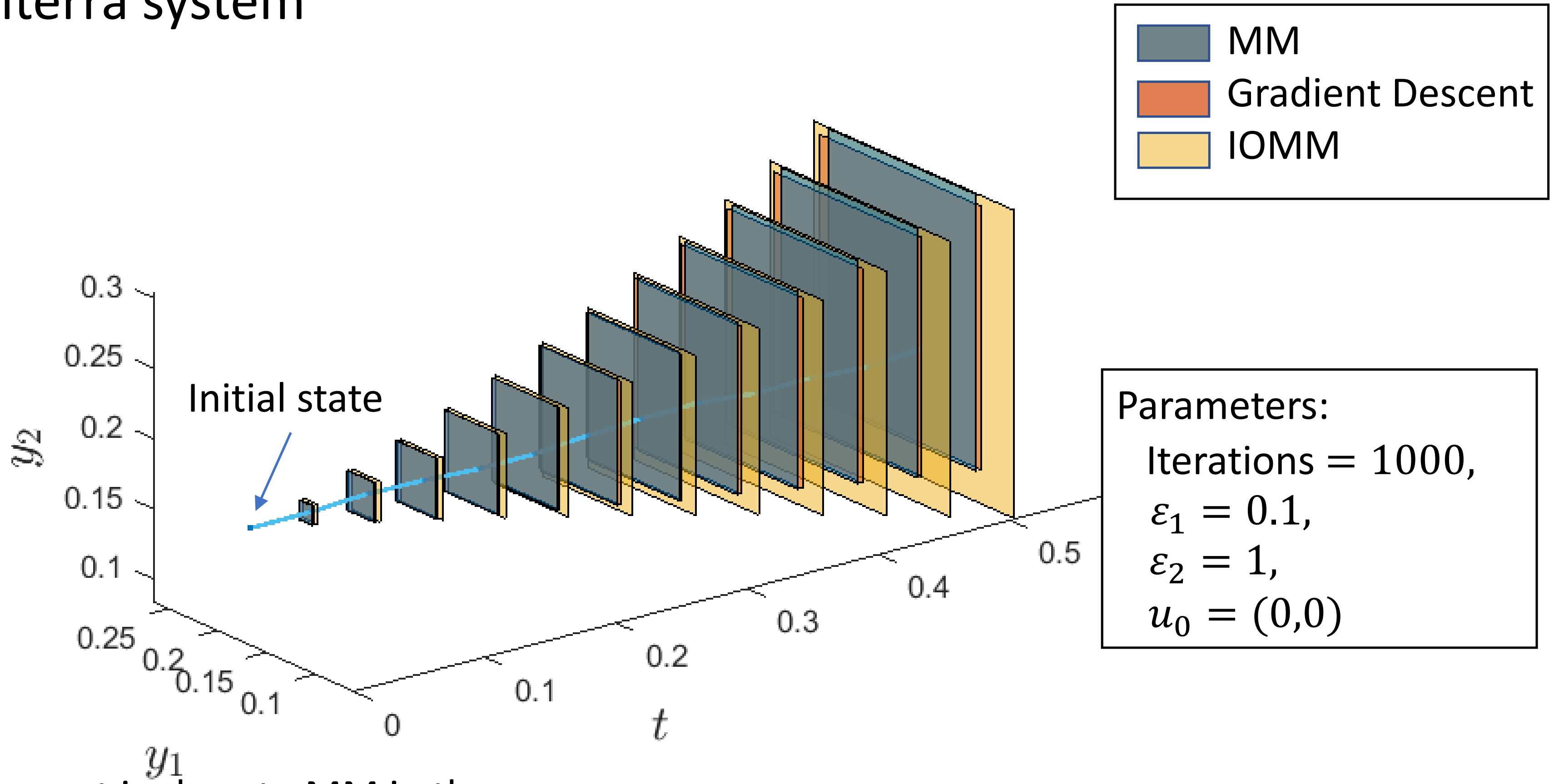
```
    Initialization : u_0
for i = 1 to R do
    u_{i+1} = u_i − ε∇F_c[u_i](t),
    u_{i+1} ← sat_U(u_{i+1})
end for
return u_R
```



4. Illustrative Simulation

Example: Lotka-Volterra system

$\dot{x}_1 = -x_1 x_2 + x_1 u_1,$
$\dot{x}_2 = x_1 x_2 - x_2 u_2,$
$y = x,$
$x(0) = (1/6,1/6),$
$(u_1, u_2) \in [-1,1] \times [-1,1]$



Parameters:
Iterations $= 1000,$
$\varepsilon_1 = 0.1,$
$\varepsilon_2 = 1,$
$u_0 = (0,0)$

Remark: the Gradient descent is close to MM in the convergence time range.

5. Conclusion

- The gradient descent approach estimates well the real reachable set of a system in the convergence time interval.
- The gradient descent approach outperforms the IOMM approach.
- The performance also depends on the truncation length of the words used in the Chen-Fliess series.

Questions?

Ivan.perez@uvm.edu