Introduction

Model predictive control allows us to augment two features on the classical control formulation: constraints on states and inputs, and an optimal criteria to choose the latter. The incorporation of the dynamics of the system in an optimization problem makes this possible.

Example

Consider the dynamics

$$
\begin{aligned}
x(t+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \\
x_0 &= \begin{bmatrix} -4.5 \\ 2 \end{bmatrix}
\end{aligned}
$$

together with the following optimization problem:

$$
\begin{aligned}
\min_{u_0,u_1,u_2,x_{1,2},x_3} \quad & x_3' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_3 + \sum_{k=0}^{2} x_k' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k + u_k' 10 u_k \\
\text{s.t.} & -0.5 \leq u(k) \leq 0.5, k = 0, \cdots, 3 \\
& \begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}, k = 0, \cdots, 3 \\
& x_3 \in \mathcal{X}_f = \mathbb{R}^2
\end{aligned}
$$

We replace the dynamics by the following equality constraint

$$
\begin{bmatrix} B & 0 & 0 & A & -I & 0 & 0 \\ 0 & B & 0 & 0 & A & -I & 0 \\ 0 & 0 & B & 0 & 0 & A & -I \end{bmatrix} \tilde{z} = \tilde{0}
$$

where A is the matrix of the system, B is the control vector, $\tilde{0}$ is the zero vector and

$$\tilde{z} = (u(0), u(1), u(2), x(0), x(1), x(2), x(3))'.$$

This adds an equality constraint to the optimization problem. At this initial stage we obtain the optimal states and controls

$$\tilde{z}^* = (u^*(0), u^*(1), u^*(2), x^*(0), x^*(1), x^*(2), x^*(3))'$$

for the next stage we set $x^*(1)$ as initial condition and forget about the other outcomes of the optimization. Now the new unknown states and controls are indexed as follows $x(2), x(3), x(4), u(1), u(2), u(3)$. So the problem is now

$$\begin{aligned}
\min_{u_1,u_2,u_3,x_{2,3},x_4} \quad & x_4' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_4 + \sum_{k=1}^{3} x_k' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k + u_k' 10 u_k \\
\text{s.t.} - 0.5 \leq u(k) \leq 0.5, k = 1, \cdots, 4 \\
\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}, k = 1, \cdots, 4 \\
x_4 \in \mathcal{X}_f = \mathbb{R}^2
\end{aligned}$$

with the dynamics starting at x_1

$$\begin{aligned}
x(t+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \\
x_1 &= \begin{bmatrix} -2.5 \\ 1.5989 \end{bmatrix}
\end{aligned}$$

again we solve this problem and obtain $x^*(2)$ and continue this process with it as initial condition of the third stage and so on. We build a sequence of states by keeping track of the first optimal state and control at each stage. By doing this we obtain the following plot:

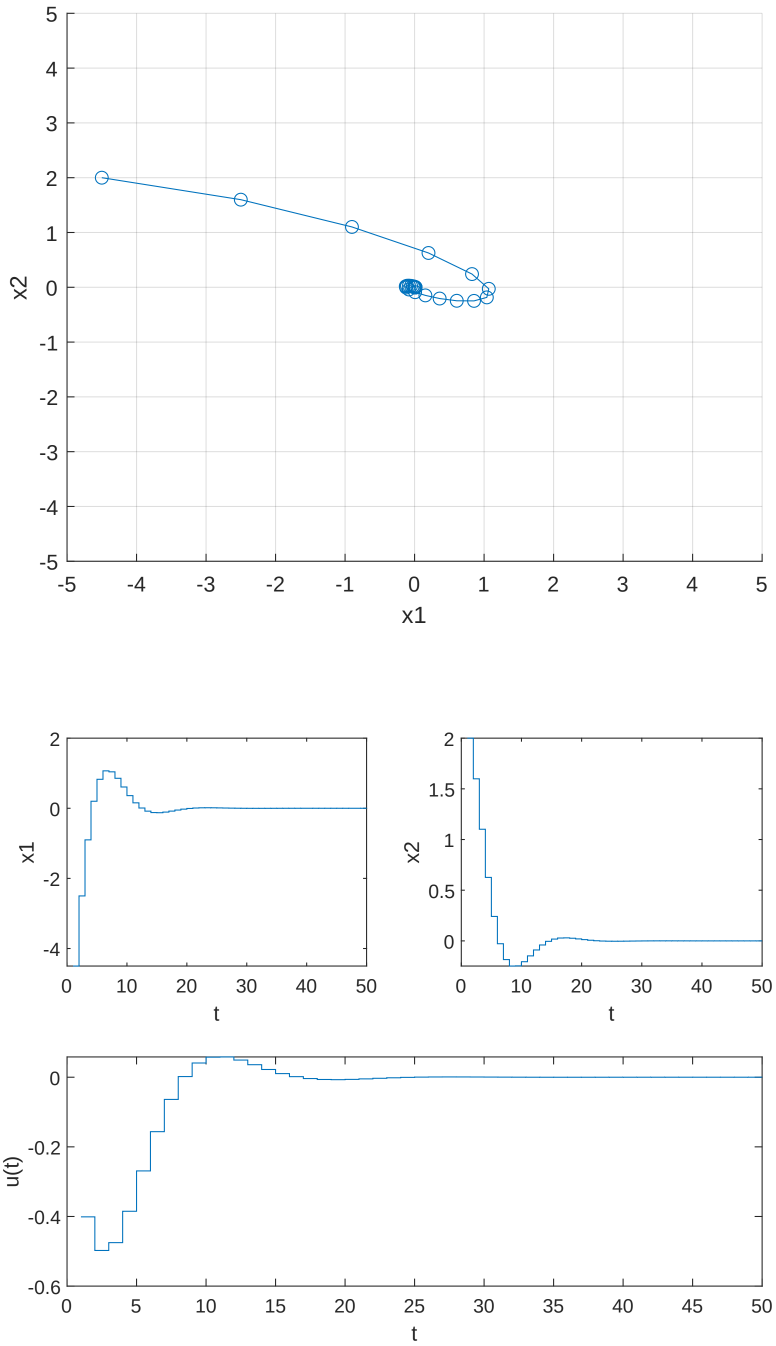
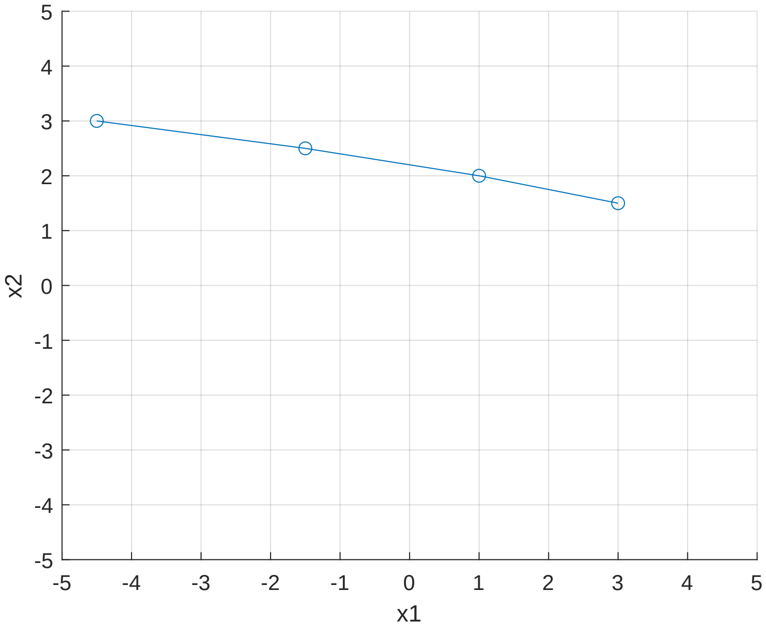


Figure 1: Above, trajectory solution for the initial value $(-4.5, 2)$. Below, its state coordinates and input.

If we start with $x_{0|0} = (-4.5, 3)'$ at the first stage we step on the problem of infeasibility. The optimization problem at stage 4 is non-solvable, because, as we are going see later, the feasible set of the optimization problem as well as the objective function depend on the initial state of each stage, therefore **it is possible that at a particular stage the feasible set be empty**.



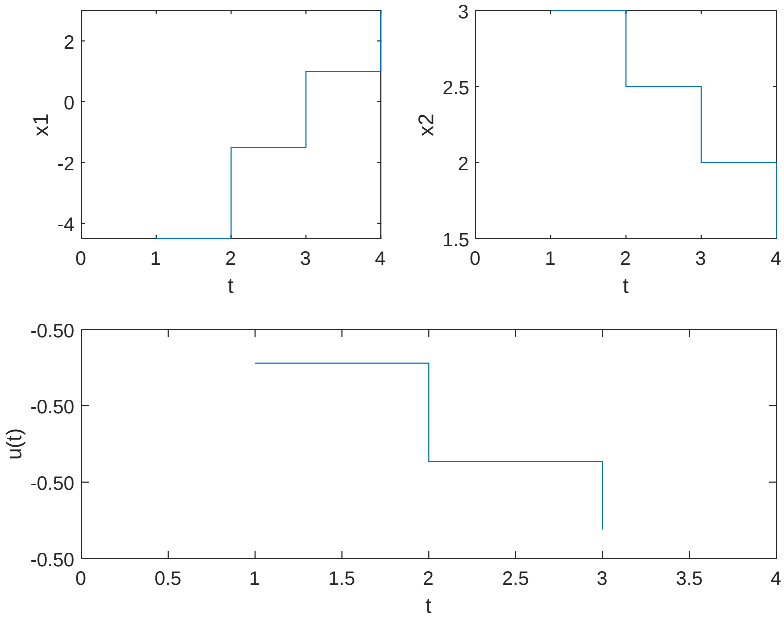


Figure 2: Above, trajectory solution for the initial value $(-4.5, 3)$ that becomes unfeasible at stage 3. Below, its state coordinates and input.

This problem setting is called "Receding Horizon Control". To make notation more understandable, the state k computed by the dynamics at the stage t is denoted $x_{t+k|t}$. In general the cost function of the optimization is the following:

$$J_{t\to t+N}(x_{t|t}, U_{t\to t+N}) := p(x_{t+N|t}) + \sum_{k=0}^{N-1} q(x_{t+k|t}, u_{t+k|t})$$

The cost function is also indexed by the subscript $\cdot|t$ in the same manner as the state. As depicted before, at each stage we use the N following states provided by the dynamic of the system and the N following inputs. This is, we start with the state $x_{t|t}$ and compute the N next states according to the behavior of the system $x_{t+k+1|t} = g(x_{t+k|t}, u_{t+k|t})$. By recursion, this defines a function of the sequence of the N inputs $U_{t\to t+N}$ and the initial condition $x_{t|t}$. A summary of the definitions is the following:

- N: time horizon
- $x_{t|t} = x(t)$: initial state at stage t

- $x_{t+k+1|t} = g(x_{t+k|t}, u_{t+k|t})$: system model.

- $x_{t+k|t}$: state vector obtained by applying the system model, the initial condition $x_{t|t}$ and the input sequence $u_{t|t}, \cdots, u_{t+k-1|t}$.

- $U_{t\to t+N} := (u'_{t|t}, \cdots, u'_{t+N-1|t})' \in \mathbb{R}^s$: vector of future inputs and $s = mN$

- $q(x_{t+k|t}, u_{t+k|t})$: stage cost

- $p(x_{t+N|t})$: terminal cost

the stage and terminal cost are positive real functions. For the linear case, which is the one we are going to treat, the matrices are positive definite:

$$p(x) > 0, \forall x \neq 0, p(0) = 0$$
$$q(x, u) > 0, \forall x \neq 0, u \neq 0, q(0, 0) = 0$$

The constrained finite time optimal control (CFTOC) setting is the following optimization problem defined for each stage:

$$\begin{aligned} J^*_{t\to t+N}(x_{t|t}) &= \min_{U_{t\to t+N}} J_{t\to t+N}(x_{t|t}, U_{t\to t+N}) \\ s.t\ x_{t+k+1|t} &= g(x_{t+k|t}, u_{t+k|t}), k = 0, \cdots, N-1 \\ h(x_{t+k|t}, u_{t+k|t}) &\leq 0, k = 0, \cdots, N-1 \\ x_{t+N|t} &\in \mathcal{X}_f \\ x_{t|t} &= x(t) \end{aligned}$$

We specify the following:

- $\mathcal{X}_f \subset \mathbb{R}^n$:terminal region that we want the system of states to reach at the end of the horizon. The mechanism works as follows, at the stage t we solve the optimization problem with the N inputs, after we obtain the optimal inputs, we use the element of $U^*_{t\to t+k}$ which is $u^*_{t|t}$ to compute $x_{t+1|t}$. This last element becomes the initial value of the next stage. This is $x_{t+1|t} = x_{t|t+1}$, and so on the procedure continues.

- $\mathcal{X}_{0\to N} \subset \mathbb{R}^n$: the set of initial conditions $x(0)$ for which there exists an input vector $U_{0\to N}$ so that the inputs $u_0, \cdots, u_{N-1}$ and the states $x_0, \cdots, x_N$ satisfy the model $x_{k+1} = g(x_k, u_k)$ and the constraints $h(x_k, u_k) \leq 0$ and that the state x_n lies in the terminal set $\mathcal{X}_f$. In the same manner we define $\mathcal{X}_{j\to N}$ as the set of states x_j at time j which can be steered into $\mathcal{X}_f$ at time N.

- $J^*_{0\to N}(x_0)$: value function

In general the problem may not have a minimum but an infimun. We will assume that there exists a minimum. This is the case when the set of feasible input vectors $U_{0\to N}$ (defined by h and $\mathcal{X}_f$) is compact and when the functions g, p and q are continuous. Also, there might be several input vectors $U^*_{0\to N}$ which yield the minimum ($J^*_{0\to N} = J_{0\to N}(x_0, U^*_{0\to N})$). In this case we will define one of them as the minimizer $U^*_{0\to N}$.

Feasible set

Now we are going to analyze the set of constraints that form the feasible set of the optimization problem. For this, as we set before, we assume the following CFTOC:

$$\begin{aligned}\min_{U_{t\to t+N|t}} \ & p(x_{t+N|t}) + \sum_{k=0}^{N-1} q(x_{t+k|t}, u_{t+k|t}) \\ s.t \ & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t}, k = 0, \cdots, N-1 \\ & x_{t+k|t} \in \mathcal{X}, u_{t+k|t} \in \mathcal{U}, k = 0, \cdots, N-1 \\ & x_{t+N|t} \in \mathcal{X}_f \\ & x_{t|t} = x(t),\end{aligned}$$

making use of the dynamics, we write the states in terms of the initial point at the stage and the inputs

$$x_{t+k|t} = A^k x_{t|t} + \sum_{i=0}^{k-1} A^{k-1-i} B u_{t+i|t}, \ k = 1, \cdots, N,$$

the constraints $x_{t+k|t} \in \mathcal{X}$ imply that $u_{t|t}$ belongs to a subset of $\mathcal{U}$ which is the intersection of $\mathcal{U}$ with N different sets obtained from $\mathcal{X}_f$, $u_{t+1|t}$ belongs to a subset of $\mathcal{U}$ which is the intersection with $N-1$ sets, continuing in the same manner, $u_{t+N-1|t}$ belongs to a subset of $\mathcal{U}$ which is the intersection with 1 sets.

Let us formalize this idea. For this we define the following sets

$$\begin{aligned}\mathcal{U}_j =& \{(u_{t|t}, u_{t+1|t}, \cdots, u_{t+j|t}, u_{j+1}, \cdots, u_{N-1}) \in \mathcal{U}^{N-1} | A^{j+1} x_{t|t} + \sum_{i=0}^{j} A^{j-i} B u_{t+i|t} \in \mathcal{X}\}, \\ & \forall j \in \{0, \cdots, N-1\},\end{aligned}$$

then the region of feasible sequence of N inputs is expressed as

$$\mathcal{U} = \bigcap_{j=0}^{N-1} \mathcal{U}_j$$

If the sets $\mathcal{U}$, $\mathcal{X}$ are the n-dimensional hypercubes $I_u = I_1^1 \times \cdots \times I_n^1$, $I_x = I_1^2 \times \cdots \times I_n^2$, $I_f = I_1^3 \times \cdots \times I_n^3$ where $I_i^1 = [-b_u; b_u]$, $I_i^2 = [-b_x; b_x]$ and $I_i^3 = [-b_f; b_f]$ for $i \in \{0, \cdots, n\}$. We analyze three types of feasible sets

- Case 1: $\mathcal{X}_f = I_f$
  It is immediate by imposing the N state to be in the hypercube

$$\begin{bmatrix} -b_x - Ax_{t|t} \\ -b_x - A^2 x_{t|t} \\ -b_x - A^3 x_{t|t} \\ \vdots \\ -b_f - A^N x_{t|t} \end{bmatrix} \leq \begin{bmatrix} B & 0 & 0 & \cdots & 0 \\ AB & B & 0 & \cdots & 0 \\ A^2 B & AB & B & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \cdots & B \end{bmatrix}_{N \times N} \begin{bmatrix} u_{t|t} \\ u_{t+1|t} \\ u_{t+2|t} \\ \vdots \\ u_{t+N-1|t} \end{bmatrix}$$
$$\leq \begin{bmatrix} b_x - Ax_{t|t} \\ b_x - A^2 x_{t|t} \\ b_x - A^3 x_{t|t} \\ \vdots \\ b_f - A^N x_{t|t} \end{bmatrix}$$

  and the input constraints as

$$-\bar{b}_u \leq I_{N \times N} \overline{u} \leq \bar{b}_u$$

  where $\bar{b}_u = [b_u, \cdots, b_u]^T$, $\overline{u} = [u_{t|t}, \cdots, u_{t+N-1|t}]^T$ and $I_{N \times N}$ is the $N \times N$ identity matrix

- Case 2: $\mathcal{X}_f = \mathbb{R}^n$
  We can neglect the last equation from the previous case by making the hypercube big enough that does not affect the system of constraints.

This is

$$\begin{bmatrix} -b_x - Ax_{t|t} \\ -b_x - A^2x_{t|t} \\ -b_x - A^3x_{t|t} \\ \vdots \\ -b_x - A^{N-1}x_{t|t} \end{bmatrix} \leq \begin{bmatrix} B & 0 & \cdots & 0 & 0 \\ AB & B & \cdots & 0 & 0 \\ A^2B & AB & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots & \\ A^{N-2}B & A^{N-3}B & \cdots & B & 0 \end{bmatrix}_{N-1\times N} \begin{bmatrix} u_{t|t} \\ u_{t+1|t} \\ u_{t+2|t} \\ \vdots \\ u_{t+N-1|t} \end{bmatrix}$$
$$\leq \begin{bmatrix} b_x - Ax_{t|t} \\ b_x - A^2x_{t|t} \\ b_x - A^3x_{t|t} \\ \vdots \\ b_x - A^{N-1}x_{t|t} \end{bmatrix}$$

the input constraints are the same.

- Case 3: $\mathcal{X}_f = 0$
  By equating the terminal state to zero and solving for the last input of the stage we have $Bu_{t+N-1|t} = -A^Nx_{t|t} - \sum_{i=0}^{N-2} A^{N-1-i}Bu_{t+i|t}$, now we applied the input boundary to the latter and obtain $-Bb_u \leq -A^Nx_{t|t} - \sum_{i=0}^{N-2} A^{N-1-i}Bu_{t+i|t} \leq Bb_u$. The polytope yields

$$\begin{bmatrix} -b_x - Ax_{t|t} \\ -b_x - A^2x_{t|t} \\ -b_x - A^3x_{t|t} \\ \vdots \\ -b_x - A^{N-1}x_{t|t} \\ -Bb_u - A^Nx_{t|t} \end{bmatrix} \leq \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ A^2B & AB & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ A^{N-2}B & A^{N-3}B & \cdots & B \\ A^{N-1}B & A^{N-2}B & \cdots & AB \end{bmatrix}_{N\times N-1} \begin{bmatrix} u_{t|t} \\ u_{t+1|t} \\ u_{t+2|t} \\ \vdots \\ u_{t+N-2|t} \end{bmatrix}$$
$$\leq \begin{bmatrix} b_x - Ax_{t|t} \\ b_x - A^2x_{t|t} \\ b_x - A^3x_{t|t} \\ \vdots \\ b_x - A^{N-1}x_{t|t} \\ Bb_u - A^Nx_{t|t} \end{bmatrix}$$

and the input constraints as

$$-\overline{b}_u \leq I_{N-1\times N-1}\overline{u} \leq \overline{b}_u$$

where $\overline{u} = [u_{t|t}, \cdots, u_{t+N-2|t}]^T$

Cost function

Proposition 1. *Given the initial states at stages t and $t+s$ with $s \in \mathbb{N}$ and the sequences of inputs $U_{t\to t+N}$, $U_{t+s\to t+s+N}$ such that $x_{t|t} = x_{t+s|t+s}$, $U_{t\to t+N} = U_{t+s\to t+s+N}$, then*

$$J_{t\to t+N}(x_{t|t}, U_{t\to t+N}) = J_{t+s\to t+s+N}(x_{t|t}, U_{t\to t+N})$$

Proof.

$$\begin{aligned}
J_{t+s\to t+N+s}(x_{t|t}, U_{t\to t+N}) &= x'_{N+s|t+s}Px_{N+s|t+s} + \sum_{k=t+s}^{N+s-1} x'_{k|t+s}Qx_{k|t+s} + u'_{k|t+s}Ru_{k|t+s} \\
&= x'_{N+s|t+s}Px_{N+s|t+s} + \sum_{i=t}^{N-1} x'_{i+s|t+s}Qx_{i+s|t+s} + u'_{i+s|t+s}Ru_{i+s|t+s} \\
&= x'_{N|t}Px_{N|t} + \sum_{i=t}^{N-1} x'_{i|t}Qx_{i|t} + u'_{i|t}Ru_{i|t} \\
&= J_{t\to t+N}(x_{t|t}, U_{t\to t+N})
\end{aligned}$$

where the second equality comes from defining $i = k - s$ and the third equation is valid by the dynamics of the same initial state and the same controls. □

This means that fixing the initial state and choosing the same inputs we obtain the same objective function. As we have shown the feasible set depends only on the initial state at each stage. Then this proposition means that the RHC problem is time invariant. This is, if two solutions ever meet each other, they have the same trajectory from that point of time on.

We set the terms of objective function as $q(x_{t+k|t}, u_{t+k|t}) = x'_{t+k|t}Qx_{t+k|t} + u'_{t+k|t}Ru_{t+k|t}$ and $p(x_{t+N|t}) = x'_{t+N|t}Px_{t+N|t}$. We define the following vectors and matrices

$$\widetilde{Q} \triangleq \begin{bmatrix} Q & 0 & 0 & \cdots & 0 & 0 \\ 0 & Q & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & Q & 0 \\ 0 & 0 & 0 & \cdots & 0 & P \end{bmatrix}_{Nn\times Nn}, \quad \widetilde{X}_t \triangleq \begin{bmatrix} x_{t+1|t} \\ \vdots \\ x_{t+N|t} \end{bmatrix}, \quad \widetilde{U}_t \triangleq \begin{bmatrix} u_{t|t} \\ \vdots \\ u_{t+N-1|t} \end{bmatrix}$$

$$\widetilde{A} \triangleq \begin{bmatrix} A \\ \vdots \\ A^N \end{bmatrix}, \quad \widetilde{B} \triangleq \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ A^2B & AB & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}_{Nn\times Nn}, \quad \widetilde{R} \triangleq \begin{bmatrix} R & 0 & 0 & \cdots & 0 \\ 0 & R & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & R \end{bmatrix}_{Nn\times Nn}$$

we can also express $\widetilde{B}$ as

$$\widetilde{B} = \sum_{i=0}^{N-1} \underline{\operatorname{diag}}_i(1_{N\times N}) \otimes A^i B$$

where $\underline{\operatorname{diag}}_i(1_{N\times N})$ is the i-th subdiagonal of the matrix of dimension N with all entries equal to 1. Then we can write the objective function as

$$J_{t\to t+N}(x_{t|t}, U_{t\to t+N}) = x'_{t|t} Q x_{t|t} + \widetilde{X}'_t \widetilde{Q} \widetilde{X}_t + \widetilde{U}'_t \widetilde{R} \widetilde{U}_t$$

from the previous definitions we have

$$\widetilde{X}_t = \widetilde{A} x_{t|t} + \widetilde{B}\widetilde{U}_t$$

replacing this into the objective function we have

$$\begin{aligned}
&J_{t\to t+N}(x_{t|t}, U_{t\to t+N}) = x'_{t|t} Q x_{t|t} + (\widetilde{A} x_{t|t} + \widetilde{B}\widetilde{U}_t)'\widetilde{Q}(\widetilde{A} x_{t|t} + \widetilde{B}\widetilde{U}_t) + \widetilde{U}'_t \widetilde{R}\widetilde{U}_t, \\
&= x'_{t|t} Q x_{t|t} + (x'_{t|t}\widetilde{A}' + \widetilde{U}'_t\widetilde{B}')(\widetilde{Q}\widetilde{A} x_{t|t} + \widetilde{Q}\widetilde{B}\widetilde{U}_t) + \widetilde{U}'_t \widetilde{R}\widetilde{U}_t, \\
&= x'_{t|t} Q x_{t|t} + x'_{t|t}\widetilde{A}'\widetilde{Q}\widetilde{A} x_{t|t} + x'_{t|t}\widetilde{A}'\widetilde{Q}\widetilde{B}\widetilde{U}_t + \widetilde{U}'_t\widetilde{B}'\widetilde{Q}\widetilde{A} x_{t|t} + \widetilde{U}'_t\widetilde{B}'\widetilde{Q}\widetilde{B}\widetilde{U}_t + \widetilde{U}'_t \widetilde{R}\widetilde{U}_t, \\
&= x'_{t|t} Q x_{t|t} + x'_{t|t}\widetilde{A}'\widetilde{Q}\widetilde{A} x_{t|t} + 2x'_{t|t}\widetilde{A}'\widetilde{Q}\widetilde{B}\widetilde{U}_t + \widetilde{U}'_t \left(\widetilde{B}'\widetilde{Q}\widetilde{B} + \widetilde{R}\right)\widetilde{U}_t
\end{aligned}$$

so for example for the case $\mathcal{X}_f = \mathbb{R}^n$ the problem can be expressed as

$$J_{t\to t+N}(x_{t|t}, \widetilde{U}_t) = \min_{\widetilde{U}_t} 2x'_{t|t}\widetilde{A}'\widetilde{Q}\widetilde{B}\widetilde{U}_t + \widetilde{U}'_t \left(\widetilde{B}'\widetilde{Q}\widetilde{B} + \widetilde{R}\right)\widetilde{U}_t$$

$$\text{s.t} \quad \begin{bmatrix} I_{N\times N} \\ -I_{N\times N} \\ \widetilde{B} \\ -\widetilde{B} \end{bmatrix} \widetilde{U}_t \leq \begin{bmatrix} \bar{b}_u \\ \bar{b}_u \\ \bar{b}_x \\ \bar{b}_x \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\widetilde{A} \\ \widetilde{A} \end{bmatrix} x_{t|t}$$

where b_f is large enough that the corresponding row is negligible

$$\bar{b}_x = \begin{bmatrix} b_x \\ \vdots \\ b_x \\ b_f \end{bmatrix}_{Nn,1}$$

From the form of the feasible set we notice that it depends on $x_{t|t}$ so it may change for each iteration. For the example above we have

$$\widetilde{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \widetilde{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \\ 0 & 1 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}, \widetilde{B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \widetilde{R} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

Stability

Theorem 1. *Consider the setting of the MPC problem with the constraint on the terminal state equal to the origin. This is*

$$\begin{aligned} \min_{U_{t\to t+N|t}} & \sum_{k=0}^{N-1} x'_{t+k|t} Q x_{t+k|t} + u'_{t+k|t} R u_{t+k|t} \\ s.t\ & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t}, k = 0, \cdots, N-1 \\ & x_{t+k|t} \in \mathcal{X}, u_{t+k|t} \in \mathcal{U}, k = 0, \cdots, N-1 \\ & x_{t+N|t} = 0 \\ & x_{t|t} = x(t) \end{aligned}$$

where Q and R are positive definite matrices. If the problem is feasible at the first stage, then the state and the inputs converge to the origin

$$\lim_{t\to\infty} x(t) = 0$$
$$\lim_{t\to\infty} u(t) = 0$$

Proof. We solve the optimal problem at the first stage and take the initial optimal inputs

$$U^*_{0\to N} = (u^*_{0|0}, \cdots, u^*_{N-1|0})$$

with cost equal to $J^*_{0\to N}(x_{0|0}, U^*_{0\to N})$. Then we construct the following non-optimal sequence

$$U_{1\to N+1} = (u_{1|1}, \cdots, u_{N|1}) = (u^*_{1|0}, \cdots, u^*_{N-1|0}, 0)$$

with non-optimal cost equal to $J_{1\to N+1}$ which satisfies

$$J^*_{1\to N+1}(x_{1|1}, U^*_{1\to N+1}) \leq J_{1\to N+1}(x_{1|1}, U_{1\to N+1}) = J^*_{0\to N}(x_{0|0}, U^*_{0\to N}) - q(x_{0|0}, u_{0|0}) + 0$$

where $q(x, u) = x'Qx + u'Ru$. The system is time invariant so we have

$$J^*_{1\to N+1}(x_{1|1}, U^*_{1\to N+1}) = J^*_{0\to N}(x_{1|1}, U^*_{1\to N+1})$$

then we have the

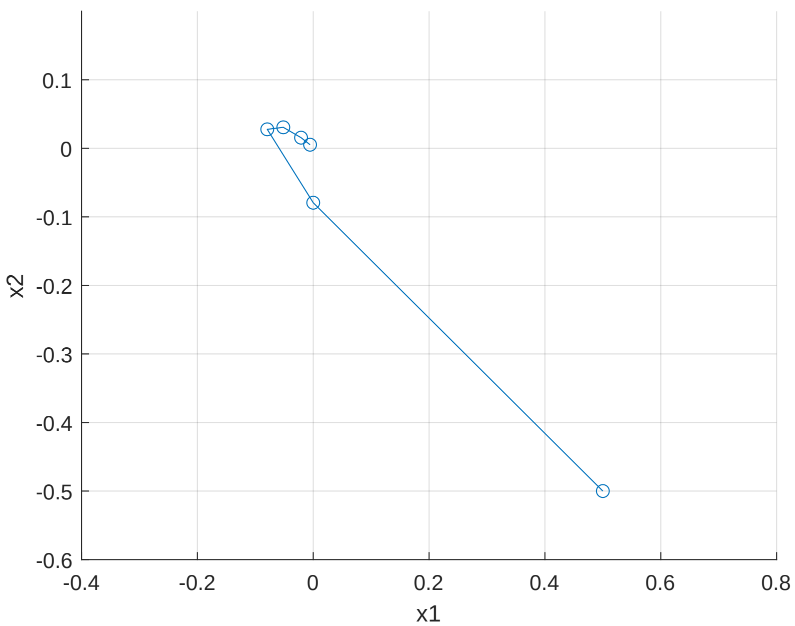
$$J^*_{0\to N}(x_{1|1}, U^*_{1\to N+1}) < J_{1\to N+1}(x_{1|1}, U_{1\to N+1})$$

using the same method, we construct the next term $J^*_{0\to N}(x_{2|2}, U^*_{2\to N+2})$ and we continue the process in the same way. Then we have the strictly decreasing sequence

$$\{J^*_{0\to N}(x_{0|0}, U^*_{0\to N}), J^*_{0\to N}(x_{1|1}, U^*_{1\to N+1}), J^*_{0\to N}(x_{2|2}, U^*_{2\to N+2}), \cdots\}.$$

As the cost is composed by positive definite matrices and the inputs are adding a zero each time, then the state tends to the origin. $\square$

Imposing the null terminal state condition to the example and setting $x_{0|0} = (0.5, -0.5)'$ we obtain the following outcomes



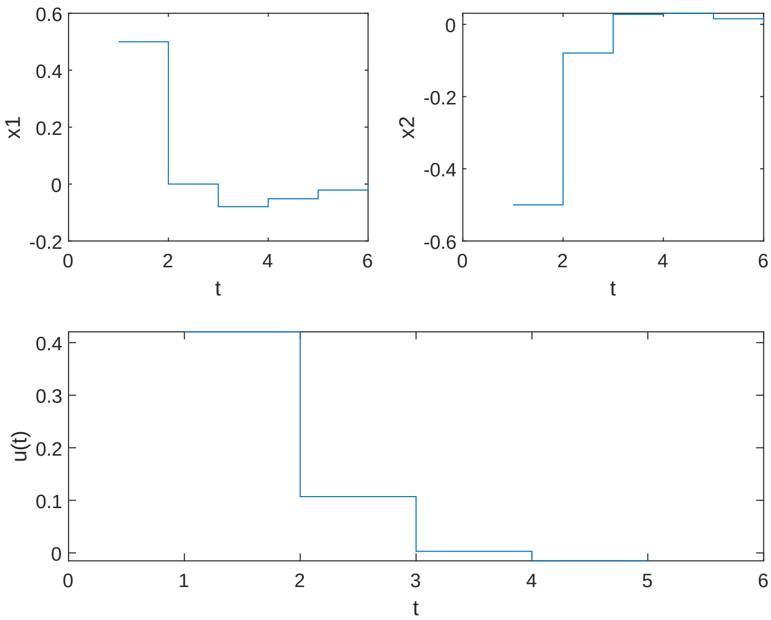


Figure 3: Above, trajectory solution for the initial value $(0.5, -0.5)$ of the problem with terminal constraint equal to zero. Below, its state coordinates and input.

Set invariance theory

Invariance sets are important in the discussion of stability and they are related to the studies of Lyapunov functions A basic idea is to find a positive invariant set included in the set of initial admissible states. This idea is extended to input output systems in this case we talk about control invariance or viability.

We define the following

- The preset of the set $\mathcal{S}$

$$\begin{aligned}\mathrm{Pre}(\mathcal{S}) &\triangleq \{x \in \mathbb{R}^n : f_a(x) \in \mathcal{S}\} \\ \mathrm{Pre}(\mathcal{S}) &\triangleq \{x \in \mathbb{R}^n : \exists u \in \mathcal{U} \text{ s.t. } f(x,u) \in \mathcal{S}\}\end{aligned}$$

  for autonomous and non-autonomous systems respectively.

- A set $\mathcal{O} \subset \mathcal{X}$ is **positive invariant** for the autonomous system subject to constraints if

$$x(0) \in \mathcal{O} \Rightarrow x(t) \in \mathcal{O}, \forall t \in \mathbb{N}_+$$

- The set $\mathcal{O}_\infty$ is called **maximal positive invariant** set if it contains the origin and every positive invariant set.
- A set $\mathcal{C} \subset \mathcal{X}$ is **control invariant** for the non-autonomous system subject to constraints if

$$x(t) \in \mathcal{C} \Rightarrow \exists u(t) \in \mathcal{U} \text{ such that } f(x(t), u(t)) \in \mathcal{C}, \forall t \in \mathbb{N}_+$$

- The set $\mathcal{C}_\infty$ is called **maximal control invariant** set if it contains the origin and every control invariant set.
- We denote the set of **feasible solutions** at stage i as $\mathcal{X}_i$, for $i = 0, \cdots, N$. This is,

$$\mathcal{X}_i = \{x \in \mathcal{X} : \exists u \in \mathcal{U}, Ax + Bu \in \mathcal{X}_{i+1}\}, \forall i \in \{0, \cdots, N-1\}$$
$$\mathcal{X}_N = \mathcal{X}_f.$$

considering the preset definition we obtain

$$\mathcal{X}_i = \text{ Pre}(\mathcal{X}_{i+1}) \cap \mathcal{X}$$

Theorem 2 (Geometric condition for invariance)**.** *The set $\mathcal{X}$ is control invariant iff $\mathcal{X} \subseteq Pre(\mathcal{X})$*

Proof. ($\Rightarrow$): $\mathcal{X}$ is control invariant, consider $x \in \mathcal{X}$, then by definition there exists a $u \in \mathcal{U}$ such that $Ax + Bu \in \mathcal{X}$, that is $x \in \text{Pre}(\mathcal{X})$.
($\Leftarrow$): if $\mathcal{X} \subseteq \text{Pre}(\mathcal{X})$, then for every element $x \in \mathcal{X}$, exists $u \in \mathcal{U}$ such that $Ax + uB \in \mathcal{X}$. □

Computation of Pre($\mathcal{S}$)

We define the composition of affine mappings and polyhedra. Consider the polyhedron

$$\mathcal{P} \triangleq \{x \in \mathbb{R}^n : Hx \leq c\}$$

with $H \in \mathbb{R}^{n_p \times n}$ and an affine mapping $f(z)$

$$f : z \in \mathbb{R}^m \rightarrow Az + b, A \in \mathbb{R}^{m_A \times m}, b \in \mathbb{R}^{m_A}$$

for $m_A = n$, the composition of $\mathcal{P} \circ f$ is the following

$$\mathcal{P} \circ f \triangleq \{z \in \mathbb{R} : Hf(z) \leq c\}$$

for $m = n$, we define the composition of $f \circ \mathcal{P}$ as the following polyhedron

$$f \circ \mathcal{P} \triangleq \{y \in \mathbb{R}^{m_A} : y = Ax + b, \forall \in \mathbb{R}^n, Hx \leq c\}$$

This can be computed as

$$f \circ \mathcal{P} = \mathrm{conv}(F),\ F = \{AV_1 + b, \cdots, AV_k + b\}$$

where $V = \{V_1, \cdots, V_k\}$ and $\mathcal{P} = \mathrm{conv}(V)$. Consider the state and input constraints defined as

$$\begin{aligned} \mathcal{X} &= \{x \in \mathbb{R}^n : H_x x \leq b_x\}, \\ \mathcal{U} &= \{u \in \mathbb{R}^n : H_u x \leq b_u\}, \end{aligned}$$

and consider $\oplus$ the Minkowsky sum. This is

$$A \oplus B \triangleq \{x + y : x \in A, y \in B\}$$

$$A = \{y \in \mathbb{R}^n : A^y y \leq A^c\},\ B = \{y \in \mathbb{R}^n : B^z z \leq B^c\}$$

and it is computed as

$$\begin{aligned} A \oplus B \triangleq &\{x \in \mathbb{R}^n : x = y + z, A^y y \leq A^c, B^z z \leq B^c, y, z \in \mathbb{R}^n\} \\ &= \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^n, A^y y \leq A^c, B^z(x - y) \leq B^c, y, z \in \mathbb{R}^n\} \\ &= \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^n, \begin{bmatrix} 0 & A^y \\ B^z & -B^z \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} A^c \\ B^c \end{bmatrix}\} \\ &= \mathrm{Proy}_x \left(\{[x'y'] \in \mathbb{R}^{n+n} : \begin{bmatrix} 0 & A^y \\ B^z & -B^z \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} A^c \\ B^c \end{bmatrix}\} \right) \end{aligned}$$

with these we obtain a way to compute the pre-set of $\mathcal{X}$. Consider $\mathcal{X} = \{x \in \mathbb{R}^n : Hx \leq h\}$

- Autonomous

$$\mathrm{Pre}(\mathcal{X}) = \mathcal{X} \circ A.$$

Proof.

$$\begin{aligned}\mathrm{Pre}(\mathcal{X}) &\triangleq \{x \in \mathbb{R}^n : Ax \in \mathcal{X}\} \\ &= \{x \in \mathbb{R}^n : HAx \leq h\} \\ &\triangleq \mathcal{X} \circ A\end{aligned}$$

□

By a very intuitive corollary of the Caratheodory Theorem that claims that every polyhedra is the convex hull of their vertices we know that the preset (pre-image) is another polyhedron. The proof is easy by considering the pre-image of the vertices (works even if the matrix is singular).

- Non-autonomous

$$\mathrm{Pre}(\mathcal{X}) = (\mathcal{X} \oplus (-B \circ \mathcal{U})) \circ A$$

Proof.

$$\begin{aligned}\mathrm{Pre}(\mathcal{X}) &\triangleq \{x \in \mathbb{R}^n : \exists u \in \mathcal{U}, Ax + Bu \in \mathcal{X}\} \\ &= \{x \in \mathbb{R}^n : y = Ax + Bu,\ y \in \mathcal{X}, u \in \mathcal{U}\} \\ &= \{x \in \mathbb{R}^n : Ax = y + (-Bu),\ y \in \mathcal{X}, u \in \mathcal{U}\} \\ &= \{x \in \mathbb{R}^n : Ax \in \mathcal{C},\ \mathcal{C} = \mathcal{X} \oplus (-B \circ \mathcal{U})\} \\ &=\ \mathrm{Pre}(\mathcal{C}), \quad \mathcal{C} = \mathcal{X} \oplus (-B \circ \mathcal{U}) \\ &= \mathcal{C} \circ A, \quad \mathcal{C} = \mathcal{X} \oplus (-B \circ \mathcal{U}) \\ &= (\mathcal{X} \oplus (-B \circ \mathcal{U})) \circ A\end{aligned}$$

□

we characterize the set of inputs that steer the state to $\mathcal{X}$ in one step

$$\mathcal{U}_0 = ((A \circ \mathcal{X} \oplus (-\mathcal{X})) \circ B) \cap \mathcal{U}$$

Proof.

$$\begin{aligned}\mathcal{U}_0 &= \{u \in \mathcal{U} : \exists x \in \mathcal{X}, Ax + Bu \in \mathcal{X}\} \\ &= \{u \in \mathcal{U} : y = Ax + Bu,\ y \in \mathcal{X}, x \in \mathcal{X}\} \\ &= \{u \in \mathcal{U} : Bu = Ax - y,\ y \in \mathcal{X}, x \in \mathcal{X}\} \\ &= \{u \in \mathcal{U} : Bu = A \circ \mathcal{X} \oplus (-\mathcal{X})\} \\ &= \{u \in \mathcal{U} : Bu \in \mathcal{C}, \mathcal{C} = A \circ \mathcal{X} \oplus (-\mathcal{X})\} \\ &= \{u \in \mathcal{U} : \mathcal{C} \circ B, \mathcal{C} = A \circ \mathcal{X} \oplus (-\mathcal{X})\} \\ &= ((A \circ \mathcal{X} \oplus (-\mathcal{X})) \circ B) \cap \mathcal{U}\end{aligned}$$

$\square$

in general we have the set of feasible inputs at stage i

$$\mathcal{U}_i = ((A \circ \mathcal{X} \oplus (-\mathcal{X}_{i+1})) \circ B) \cap \mathcal{U}, \ \forall i \in \{0, \cdots, N-1\}$$

Computation of $\mathcal{O}_\infty$

Input f_a, $\mathcal{X}$
Output $\mathcal{O}_\infty$

1. Let $\Omega_0 \leftarrow \mathcal{X}$
2. Let $\Omega_{k+1} \leftarrow \text{Pre}(\Omega_k) \cap \Omega_k$
3. If $\Omega_{k+1} = \Omega_k$ then $\mathcal{O}_\infty \leftarrow \Omega_{k+1}$ else GOTO 2
4. END

Computation of $\mathcal{C}_\infty$

Input f, $\mathcal{X}, \mathcal{U}$
Output $\mathcal{C}_\infty$

1. Let $\Omega_0 \leftarrow \mathcal{X}$
2. Let $\Omega_{k+1} \leftarrow \text{Pre}(\Omega_k) \cap \Omega_k$
3. If $\Omega_{k+1} = \Omega_k$ then $\mathcal{C}_\infty \leftarrow \Omega_{k+1}$ else GOTO 2
4. END

Why is the output of this algorithm control invariant? because it satisfies the control invariance test.

Why is the output of this algorithm maximal? Suppose there is a control invariant polyhedron outside this set. We use the control invariance test to see that this polyhedron must be included in the first iteration, because for the first iteration we part from the biggest set we can handle which is the state constraint polyhedron $\mathcal{X}$. Now that this is included in the first iteration, by the same argument it must be in all the subsequent iterations.

Another algorithm involves computing N steps instead of 1:

$$\mathcal{X}_N = (\mathcal{X} \oplus \bigoplus_{i=0}^{N-1} (-A^i B \circ \mathcal{U})) \circ A^N$$

and contractions of it $\mathcal{X}_N^\alpha$. See Fiacchini, Alamir 2017.

Computation of $\mathcal{X}_i$

We express the constraints as

$$A_x x \leq b_x, \quad A_u u \leq b_u$$

and define the polyhedron

$$\begin{aligned}\mathcal{P}_i =&\{(U_i, x_i) \in \mathbb{R}^{m(N-i)+n} : G_i U_i - E_i x_i \leq W_i\} \\ =&\{(U_i, x_i) \in \mathbb{R}^{m(N-i)+n} : [G_i| - E_i](U_i, x_i)^T \leq W_i\}\end{aligned}$$

where G_i, E_i and W_i are defined as follows

$$G_i \triangleq \begin{bmatrix} A_u & 0 & \cdots & 0 \\ 0 & A_u & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & A_u \\ 0 & 0 & \cdots & 0 \\ A_x B & 0 & \cdots & 0 \\ A_x A B & A_x B & \cdots & 0 \\ A_x A^2 B & A_x A B & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ A_f A^{N-i-1} B & A_f A^{N-i-2} B & \cdots & A_f B \end{bmatrix}, \quad E_i \triangleq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^{N-i} \end{bmatrix}, \quad W_i \triangleq \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

Then $\mathcal{X}_i$ is the projection of the polyhedron $\mathcal{P}_i$ on the x_i space. This is

$$\mathcal{X}_i = \text{ Proy}_{\mathcal{X}} \mathcal{P}_i.$$

From the way $\mathcal{P}_i, \forall i \in \{0, \cdots, N\}$ is computed we observer that $\mathcal{X}_i, \forall i \in \{0, \cdots, N\}$ are independent of the stage, this is, they are the same for every stage.

For example for the horizon $N = 3$ we have that the polyhedron $\mathcal{P}_0$ is the set of points $(u_0, u_1, u_2, x_0) \in \mathbb{R}^{3m+n}$ that satisfy the following inequality

$$\begin{bmatrix} A_u & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_u \\ 0 & 0 & 0 \\ A_x B & 0 & 0 \\ A_x A B & A_x B & 0 \\ A_f A^2 B & A_f A B & A_f B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ -A_f A^3 \end{bmatrix} x_0 \leq \begin{bmatrix} b_u \\ b_u \\ b_u \\ b_x \\ b_x \\ b_x \\ b_f \end{bmatrix}$$

the polyhedron $\mathcal{P}_1$ is the set of points $(u_0, u_1, x_0) \in \mathbb{R}^{2m+n}$ that satisfy the following inequality

$$\begin{bmatrix} A_u & 0 \\ 0 & A_u \\ 0 & 0 \\ A_x B & 0 \\ A_f AB & A_f B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -A_x \\ -A_x A \\ -A_f A^2 \end{bmatrix} x_0 \leq \begin{bmatrix} b_u \\ b_u \\ b_x \\ b_x \\ b_f \end{bmatrix}$$

the polyhedron $\mathcal{P}_2$ is the set of points $(u_0, x_0) \in \mathbb{R}^{m+n}$ that satisfy the following inequality

$$\begin{bmatrix} A_u \\ 0 \\ A_f B \end{bmatrix} u_0 - \begin{bmatrix} 0 \\ -A_x \\ -A_f A \end{bmatrix} x_0 \leq \begin{bmatrix} b_u \\ b_x \\ b_f \end{bmatrix}$$

which is equivalent to

$$\left[\begin{array}{c|c} A_u & 0 \\ 0 & A_x \\ A_f B & A_f A \end{array}\right] \begin{bmatrix} u_0 \\ x_0 \end{bmatrix} \leq \begin{bmatrix} b_u \\ b_x \\ b_f \end{bmatrix}$$

For the example of the double integrator that we are working on with

$$A_u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A_x = A_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

, $b_u = 0.5$, $b_x = 5$, $b_f = 1$, we obtain

$$\left[\begin{array}{c|cc} 1 & 0 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{array}\right] \begin{bmatrix} u_0 \\ x_{01} \\ x_{02} \end{bmatrix} \leq \left[\begin{array}{c} 0.5 \\ -0.5 \\ \hline 5 \\ 5 \\ 5 \\ 5 \\ \hline 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

which generates the following polynomial

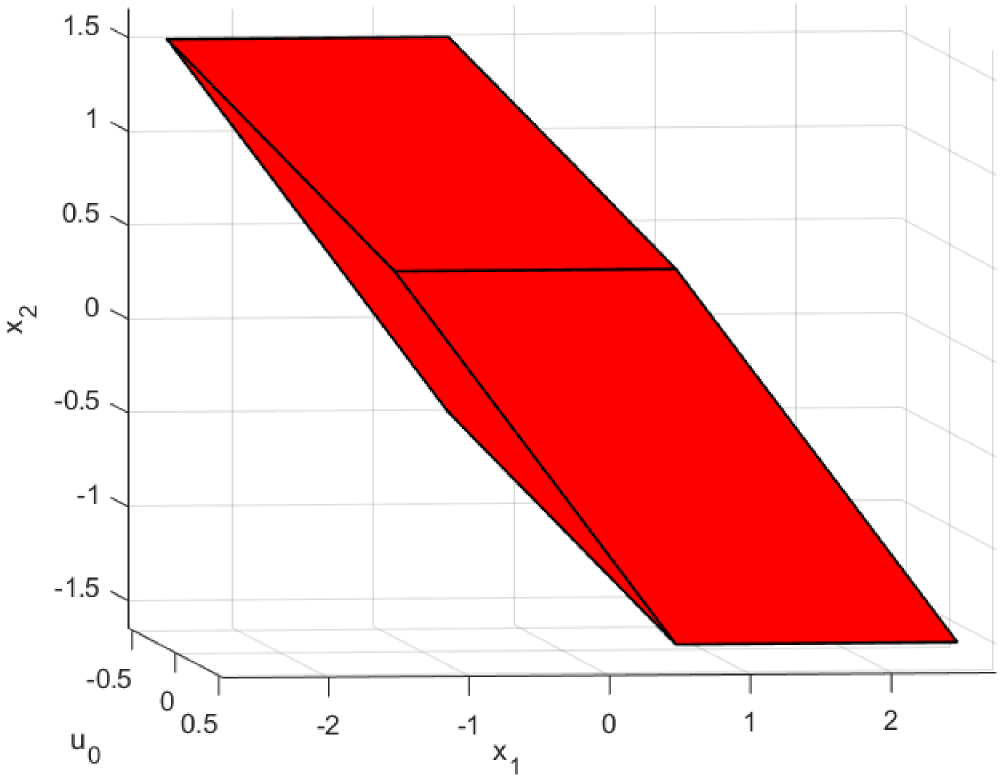


Figure 4: Polyhedron $\mathcal{P}_2$

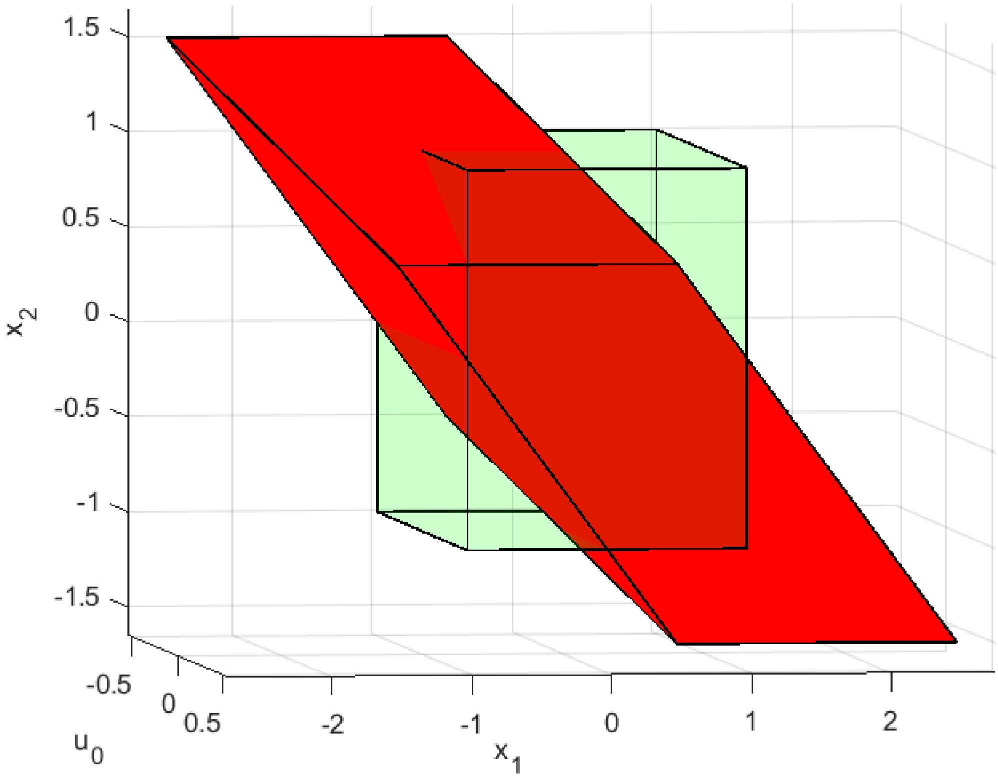


Figure 5: Feasible set $\mathcal{X}_f$ extended on the u_0 axis in green and the polyhedron $\mathcal{P}_2$ in red.

From these plots we infere that in general the feasible sets $\mathcal{X}_i$ do not need to follow a monotonic behavior. Also, notice that although $\mathcal{X}_i$ contains more intersections than $\mathcal{X}_{i+1}$, the polyhedron $\mathcal{P}_i$ increases in one its dimension compared to $\mathcal{P}_{i+1}$ so it is not the case, necessarily, that $\mathcal{X}_i$ is smaller than $\mathcal{X}_{i+1}$. It is clear that these feasible sets must be symmetric around the origin.

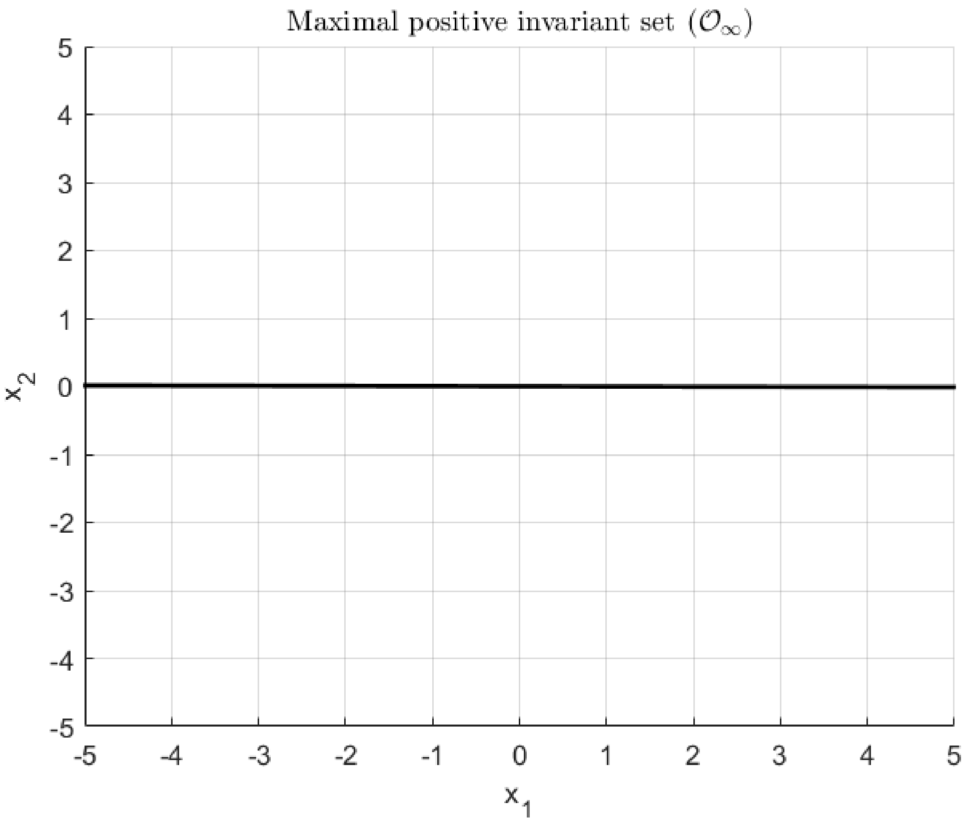
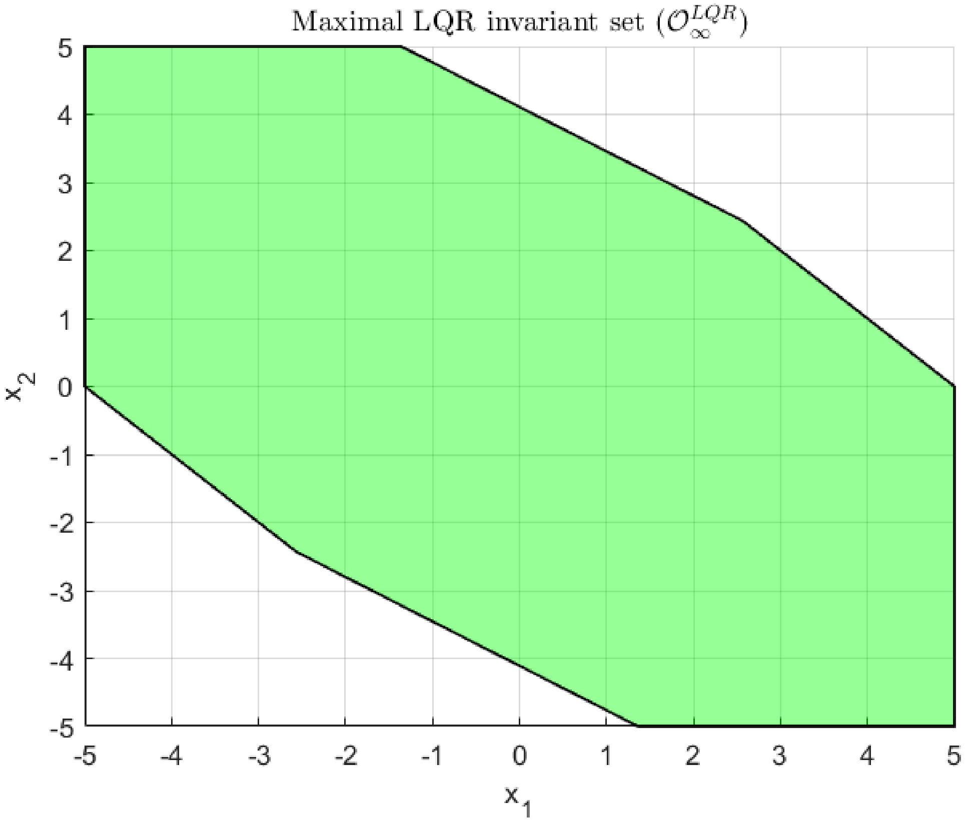
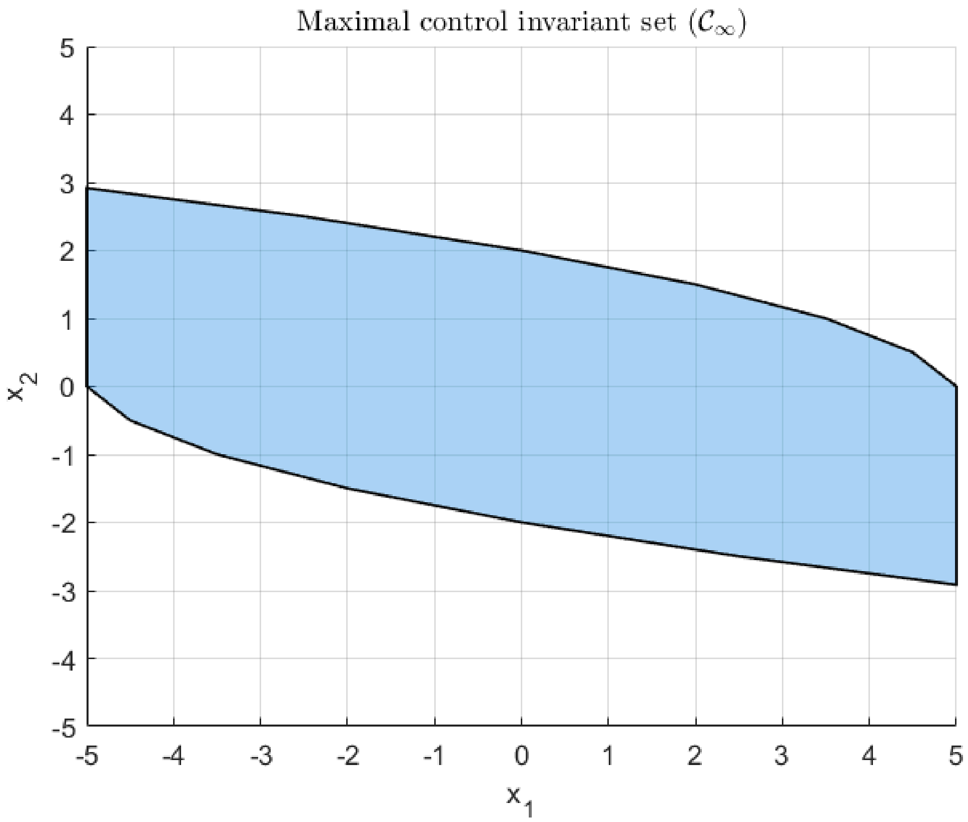


Figure 6: 400 iterations


Maximal LQR invariant set ($\mathcal{O}_\infty^{LQR}$)
5
4
3
2
1
0
-1
-2
-3
-4
-5
x_2
-5
-4
-3
-2
-1
0
1
2
3
4
5
x_1


Maximal control invariant set ($\mathcal{C}_\infty$)
x_1
x_2
5
4
3
2
1
0
-1
-2
-3
-4
-5

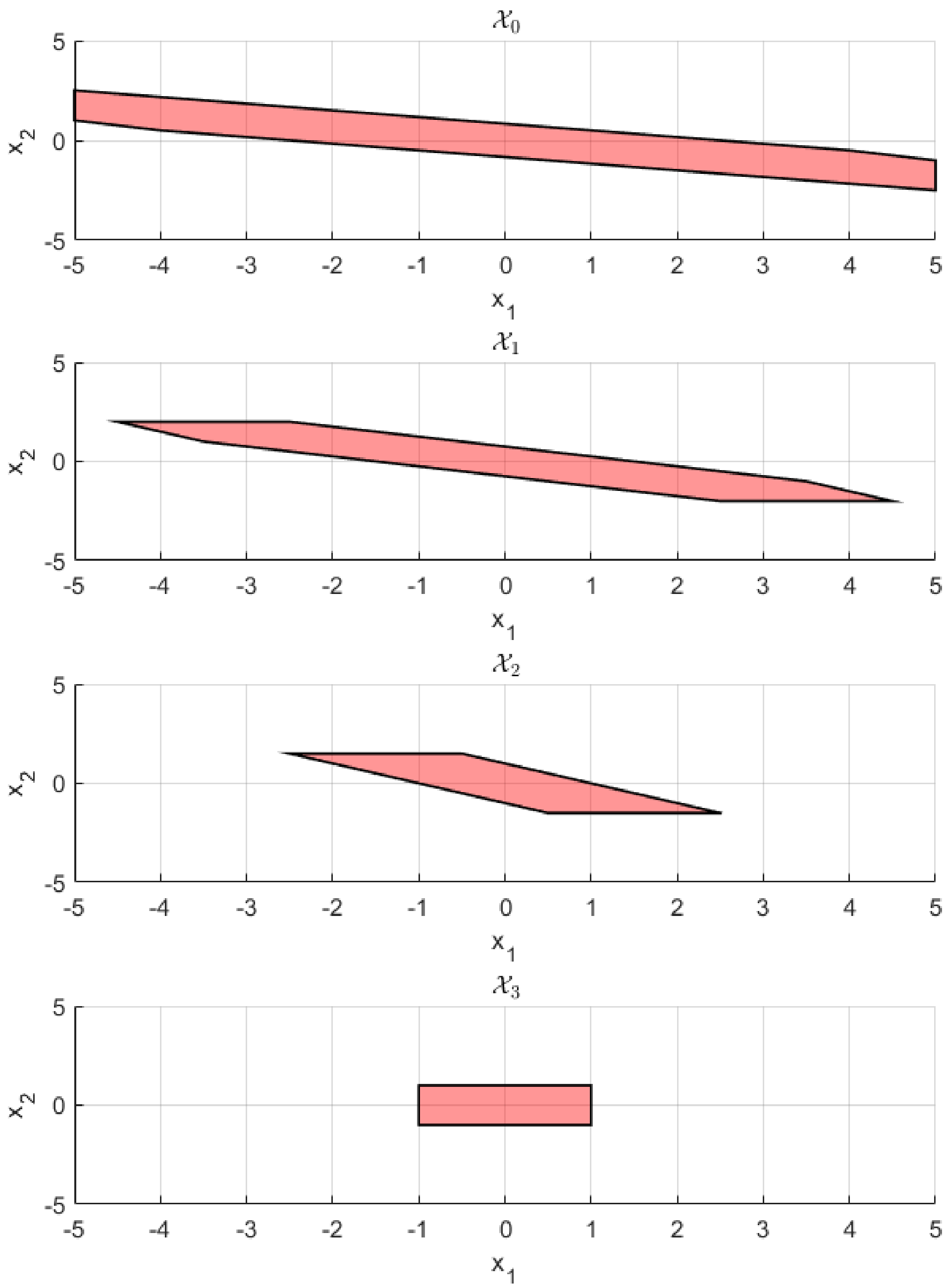


Figure 7: Feasible sets

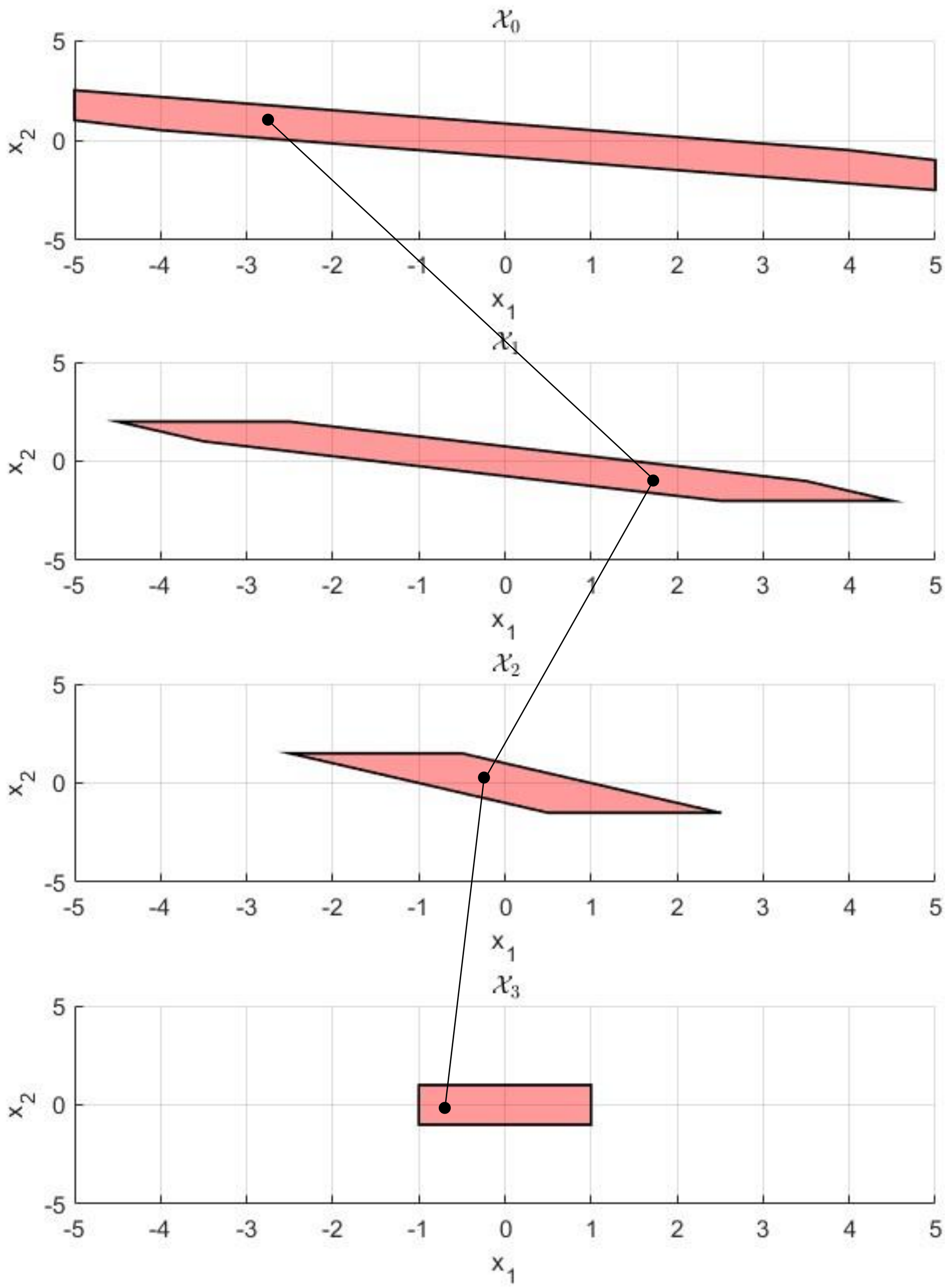


Figure 8: Example of a sequence $x^*_{0|0}, x^*_{1|0}, x^*_{2|0}, x^*_{3|0}$

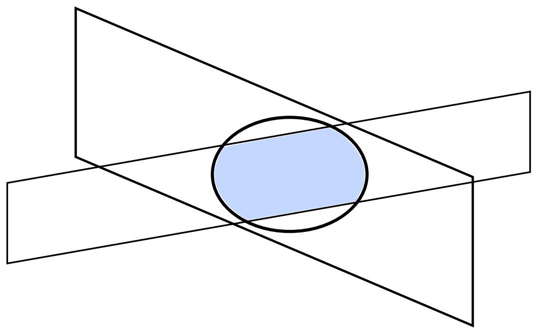


Figure 9: The shaded area is the intersection between $\mathcal{X}_1, \mathcal{X}_0$ and the image of $\mathcal{X}_0$ under the closed loop system represented by the circle.

Note that from the feasible sets figure, for this case, we observe that the feasible sets are increasing on the x_1 axis.

- **Determined index:** The smallest element $i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$ in the algorithm to compute $\mathcal{O}_\infty$ ($\mathcal{C}_\infty$), if the number exists we say that the set if finitely determined.

- **N-step Controllable Set** $\mathcal{K}_N(\mathcal{O})$: For a given target set $\mathcal{O} \subseteq \mathcal{X}$, the $N-$step controllable set $\mathcal{K}_N(\mathcal{O})$ of the system subject to constraints is defined recursively as:

$$\mathcal{K}_j(\mathcal{O}) \triangleq \mathrm{Pre}(\mathcal{K}_{j-1}(\mathcal{O})),\ \mathcal{K}_0(\mathcal{O}) = \mathcal{O},\ j \in \{1, \cdots, N\}$$

- **N-step (Maximal) Stabilizable Set:** For a given control invariant set $\mathcal{O} \subseteq \mathcal{X}$, the $N-$step (maximal) stabilizable set of the system subject to constraints is the $N-$step (maximal) controlable set $\mathcal{K}_N(\mathcal{O})$ ($\mathcal{K}_\infty(\mathcal{O})$):

The following two theorems ensure a monotonic behavior of the sequence of feasible sets.

Theorem 3. *Consider the terminal constraint set $\mathcal{X}_f$ be equal to $\mathcal{X}$. Then,*

1. *The feasible set $\mathcal{X}_i$, $i = 0, \cdots, N-1$ is equal to the $(N-i)$-step controllable set:*

$$\mathcal{X}_i = \mathcal{K}_{N-i}(\mathcal{X})$$

2. *The feasible set $\mathcal{X}_i$, $i = 0, \cdots, N-1$ contains the maximal control invariant set:*

$$\mathcal{C}_\infty \subseteq \mathcal{X}_i$$

3. *The feasible set $\mathcal{X}_i$ is control invariant if and only if the maximal control invariant set if finitely determined and $N - i$ is equal to or greater than its determinedness index $\bar{N}$, i.e*

$$\mathcal{X}_i \subseteq \ Pre(\mathcal{X}_i) \Leftrightarrow \mathcal{C}_\infty = \mathcal{K}_{N-i}(\mathcal{X}), \forall i \leq N - \bar{N}$$

4. *$\mathcal{X}_i \subseteq \mathcal{X}_j$ if $i < j$ for $i = 0, \cdots, N-1$. The size of the feasible set $\mathcal{X}_i$ stops decreasing (with decreasing i) if and only if the maximal control invariant set is finitely determined and $N - i$ is larger than its determinedness index, i.e*

$$\mathcal{X}_i \subset \mathcal{X}_j \text{ if } N - \bar{N} < i < j < N$$

Furthermore,

$$\mathcal{X}_i = \mathcal{C}_\infty \text{ if } i \leq N - \bar{N}$$

Theorem 4. *Consider the terminal constraint set $\mathcal{X}_f$ be a control invariant subset of $\mathcal{X}$. Then,*

1. *The feasible set $\mathcal{X}_i$, $i = 0, \cdots, N-1$ is equal to the $(N-i)$-step controllable set:*

$$\mathcal{X}_i = \mathcal{K}_{N-i}(\mathcal{X})$$

2. *The feasible set $\mathcal{X}_i$, $i = 0, \cdots, N-1$ is control invariant and contained within the maximal control invariant set:*

$$\mathcal{X}_i \subseteq \mathcal{C}_\infty$$

3. *$\mathcal{X}_i \supseteq \mathcal{X}_j$ if $i < j$ for $i = 0, \cdots, N-1$. The size of the feasible set $\mathcal{X}_i$ stops increasing (with decreasing i) if and only if the maximal control invariant set is finitely determined and $N - i$ is larger than its determinedness index, i.e*

$$\mathcal{X}_i \supset \mathcal{X}_j \text{ if } N - \bar{N} < i < j < N$$

Furthermore,

$$\mathcal{X}_i = \mathcal{K}_\infty(\mathcal{X}_f) \text{ if } i \leq N - \bar{N}$$

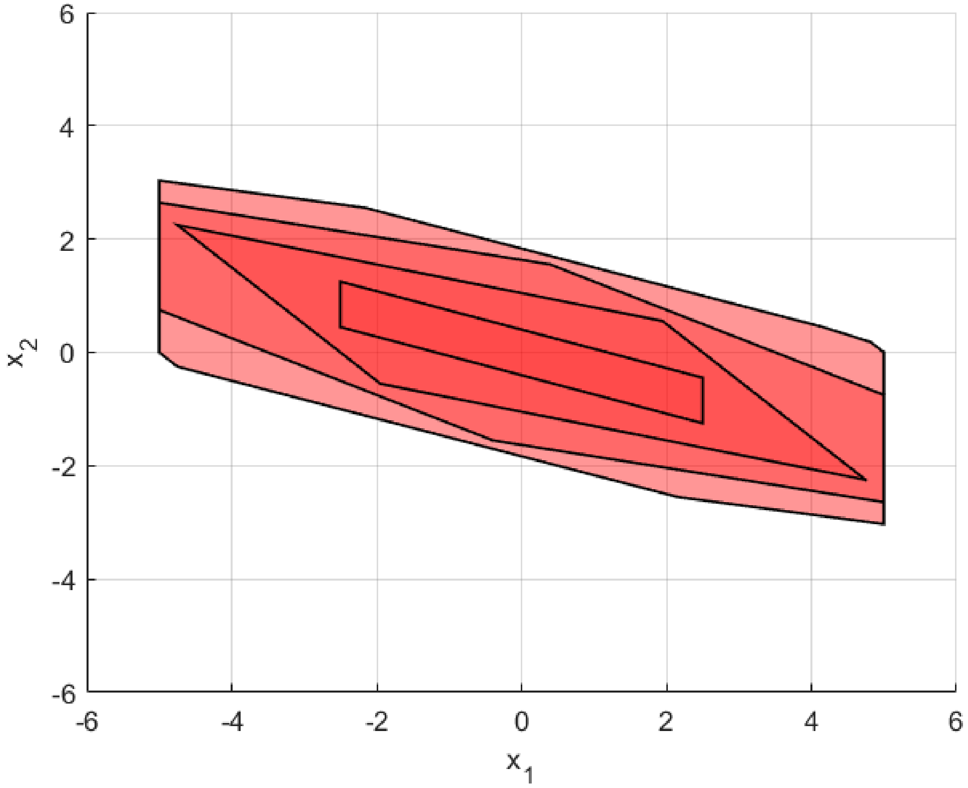


Figure 10: Feasible sets for the double integrator with $b_x = 5$, $b_u = 1$ and $H_f = [-0.32132 \ -0.94697; 0.32132 \ \ 0.94697; 1 \ \ 0; -1 \ \ 0], b_f = [0.3806; 0.3806; 2.5; 2.5]$. Notice the decreasing behavior of the sequence.

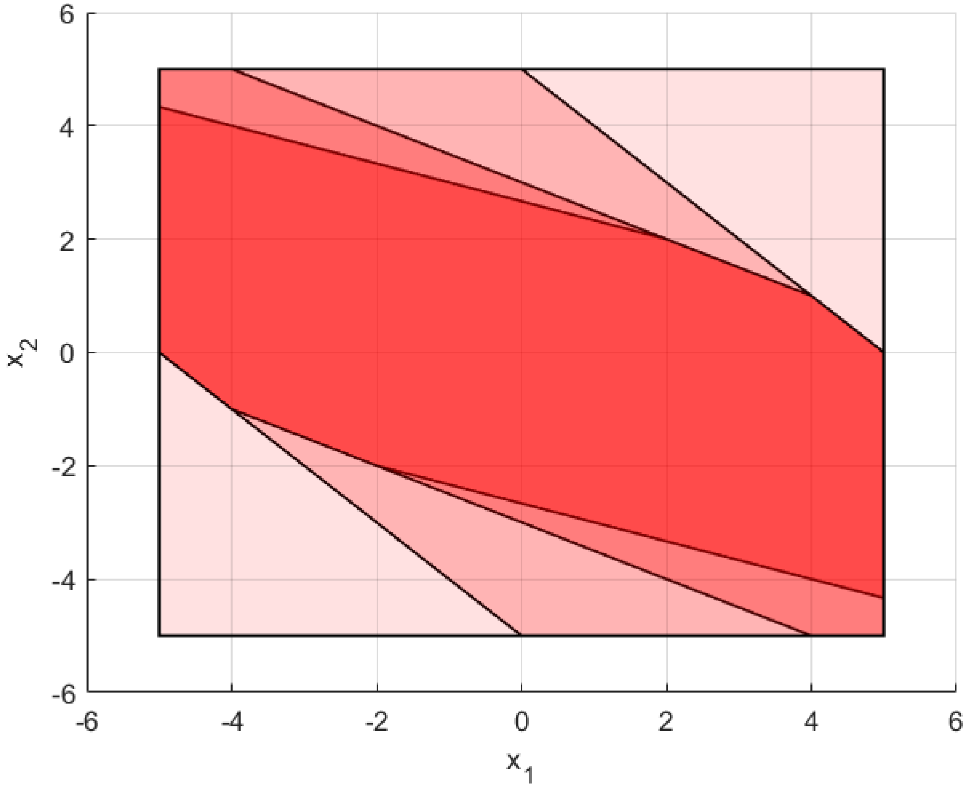


Figure 11: Feasible sets for the double integrator with $b_x = 5$, $b_u = 1$ and $b_f = 5$. Notice the increasing behavior of the sequence.

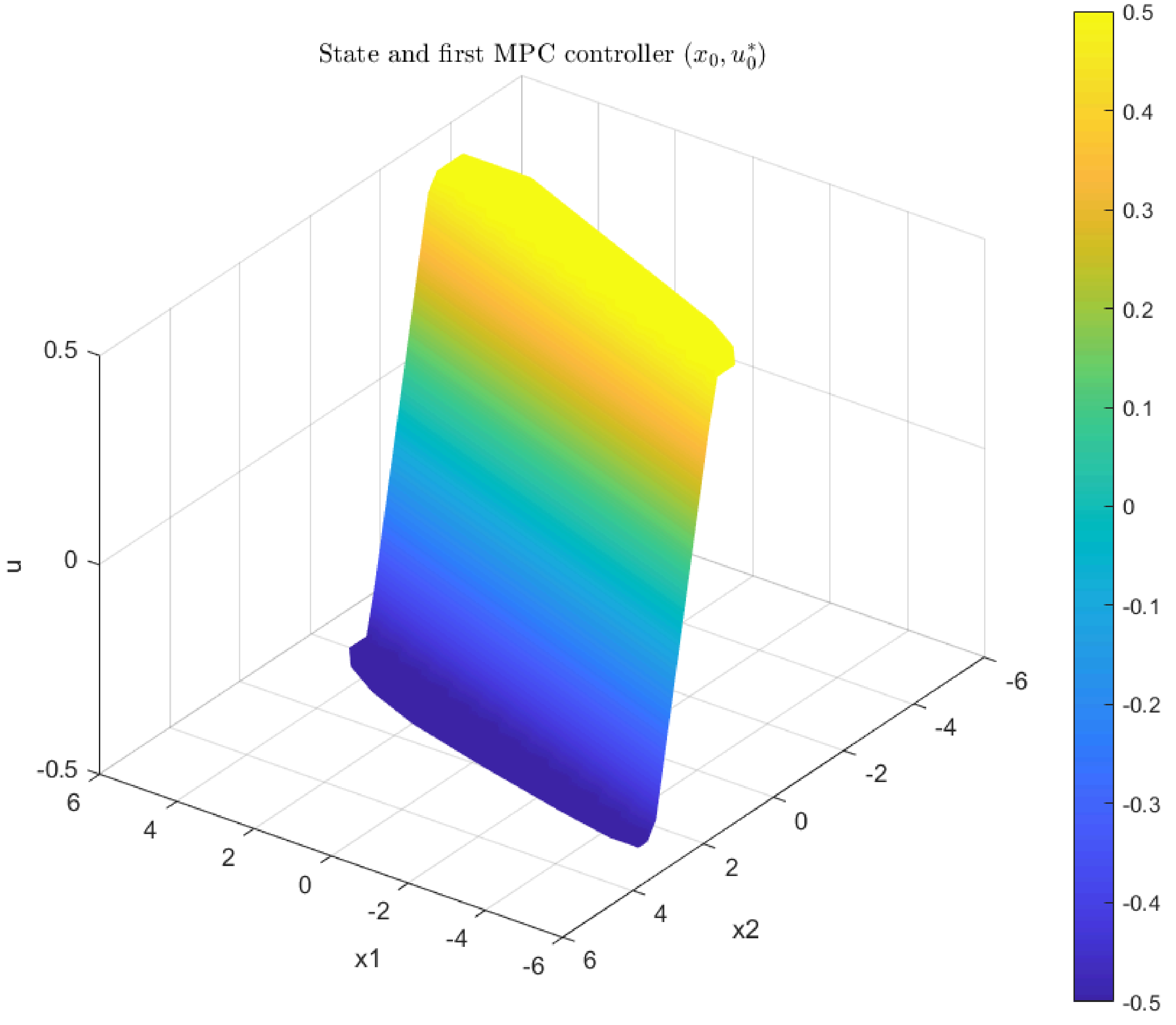
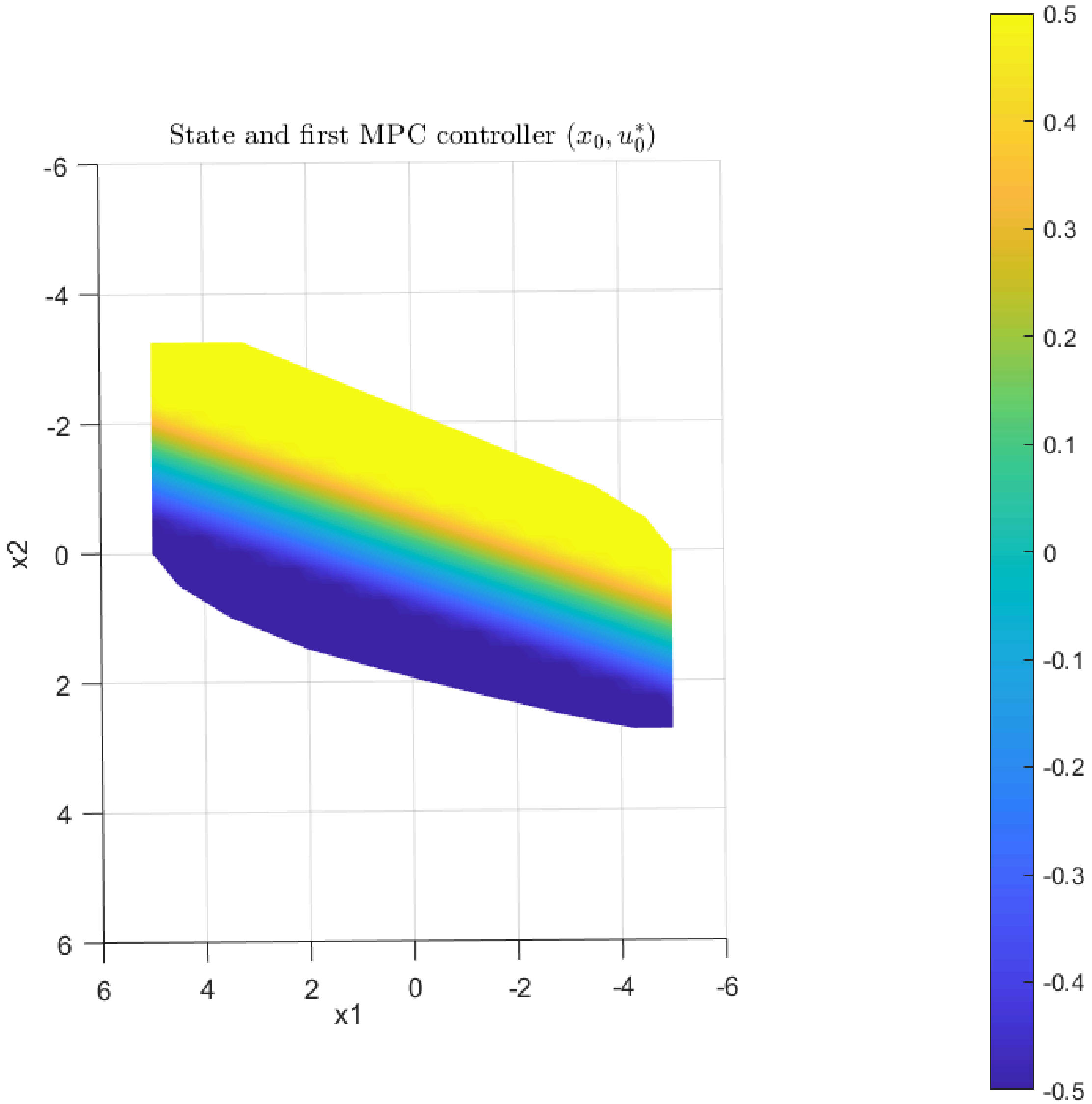


Figure 12: State space and first optimal controller



Feasibility

A RHC problem is **persistenty feasible** if all points which are feasible at first stage develop in a feasible problem for the rest of the stages.

Proposition 2. $\mathcal{O}_\infty \subseteq \mathcal{X}_0$

Proof. The definition of $\mathcal{O}_\infty$ implies solving the optimization problem and for each element in this set, the sequence it generates comes from optimizing at each stage, then it is feasible at the initial stage. □

Lemma 1. *Let $\mathcal{O}_\infty$ be the maximal positive invariant set for the closed-loop system $x(k+1) = f_{cl}(x(k))$ with constraints. The RHC problem is persistently feasible if and only if $\mathcal{X}_0 = \mathcal{O}_\infty$*

Proof. Supose the RHC problem is persistently feasible and $x_0 \in \mathcal{X}_0$, this generates $x^*_{1|0}, \cdots, x^*_{N|0}$, as from the procedure we set $x_{0|1} = x^*_{1|0}$ which belongs to $\mathcal{X}_0$, because of the assumed persistent feasibility, indeed all $x^*_{t|0}$, $\forall t \in \mathbb{N}$ is in $\mathcal{X}_0$, then this set is positive invariant, therefore $\mathcal{X}_0 \subseteq \mathcal{O}_\infty$. From proposition 2 we have $\mathcal{O}_\infty \subseteq \mathcal{X}_0$, therefore $\mathcal{O}_\infty = \mathcal{X}_0$. Now assuming $\mathcal{O}_\infty = \mathcal{X}_0$, as we saw before the elements of $\mathcal{O}_\infty$ are persistently feasible, then $\mathcal{X}_0$ is persistently feasible. □

This result is important because $\mathcal{X}_0$ does not depend on P, Q, R, but $\mathcal{O}_\infty$ does then the approach to obtain persistent feasibility is to calibrate P, Q, R.

Lemma 2. *Consider the RHC problem with $\mathcal{X}_1$ control invariant, then the RHC problem is persistently feasible. Also $\mathcal{O}_\infty$ is independent of P, Q, R.*

Proof. Consider $x \in \mathcal{X}_1$. As this feasible set is control invariant we are able to find a control u that keeps x inside $\mathcal{X}_1$, call this element x_1 from there on, by the definition of the set we can generate the sequence of $N-2$ states $x_2, \cdots, x_N$ that satisfy all constraints. By definition x, call it x_0 now, is an element for which we can find a sequence of $N-1$ states $x_1, \cdots, x_N$ that satisfy all constraints, then $x_0 \in \mathcal{X}_0$, this means that $\mathcal{X}_1 \subseteq \mathcal{X}_0$. Now for every element of $\mathcal{X}_0$ we solve the optimization problem, then we keep the element $x_{0|1} = x^*_{1|0} \in \mathcal{X}_1 \subseteq \mathcal{X}_0$, this happens at every stage, then the set $\mathcal{X}_0$ is positive invariant and we have $\mathcal{O}_\infty = \mathcal{X}_0$. As before the problem is persistently feasible. □

Proposition 3. *If $\mathcal{X}_2$ is control invariant, then $\mathcal{X}_1$ is control invariant.*

Proof. Consider the element $x \in \mathcal{X}_2$, as the set is control invariant, we have that $x \in \mathcal{X}_1$, then $\mathcal{X}_2 \subseteq \mathcal{X}_1$. Then for each point $x \in \mathcal{X}_1$ we can find u that generates $x_2 \in \mathcal{X}_2$, but $\mathcal{X}_2 \subseteq \mathcal{X}_1$ so $x_2 \in \mathcal{X}_1$ and the set $\mathcal{X}_1$ is control invariant. □

in general we have

Corollary 1. *If $\mathcal{X}_j$ is control invariant, then $\mathcal{X}_i$ is control invariant for $i < j$.*

and this implies

Theorem 5. *If $\mathcal{X}_f$ is control invariant, then the RHC problem is persistently feasible.*

It is difficult to foresee a control invariant set just by giving the problem settings, but we can always obtain the maximal control invariant set and set it as the final state constraint.

In general

$$\mathcal{O}_\infty \neq \mathcal{C}_\infty$$

take an element in $\mathcal{C}_\infty$, it provides with a sequence of inputs that are able to make the dynamic stay inside $\mathcal{C}_\infty$, but this sequence is not the optimal one, so the optimal could take the dynamic out of $\mathcal{C}_\infty$ for the given initial state. But it is true that $\mathcal{O}_\infty \subseteq \mathcal{C}_\infty$.

Generalized Stability

With the set theoretical approach definition we can state

Theorem 6. *For the RHC setting, let us assume*

- *Q, R are symmetric and Q, R, P are positive definite.*
- *The set $\mathcal{X}, \mathcal{X}_f$ and $\mathcal{U}$ contain the origin in their interior and are closed.*
- *$\mathcal{X}_f$ is control invariant, $\mathcal{X}_f \subset \mathcal{X}$*
- *and the minimization problem*

$$\min_{v \in \mathcal{U}, Ax+Bv \in \mathcal{X}_f} -x'Px + x'Qx + v'Rv + (Ax+Bv)'P(Ax+Bv) \leq 0, \ \forall x \in \mathcal{X}_f$$

then the state of the close-loop system converges to the origin.

Readings

Blanchini1999:

- In general a controlled invariant set may not admit linear controllers
- Explains an example of a discrete dynamic with only state constraints (no input constraints), specifies a positive invariant ellipse inside the state rectangle constraints for the closed loop (control is linear). Then adds input constraints and the ellipse might not be positive invariant now, but claims that by slightly shrinking the ellipsoid we can derive a new invariant domain which is included in the new constraint set (property of positive invariance under scaling)

- Ellipsoids come from the Lyapunov functions (are ellipsoids)
- Nagumo theorem
- How a convex and compact invariant set containing the origin in its interior can shape a Lyapunov function.

Viability kernel= controlled invariant set
domain of attraction: initial states that converge to the equilibrium.
Receding horizon control: Automatic generation of high-speed solvers

Mattingley, Wang, Boyd 2011:

With RHC an optimization problem is solved at each time step (stage) to determine a plan of action over the fixed time horizon. The first input from this plan is applied to the system. At the next time step we repeat the planning process, solving a new optimization problem with the time horizon shifted one step forward.

In RHC, the designer specifies the objective and constraints as part of an optimization problem, whereas in a conventional design process, the designer adjusts controller gains and coefficients to indirectly handle constraints, often by trial and error.

- Form predictive model
- Optimize
- Execute

MPC algorithm:
Input: M, dynamics, x_0, $J(x,u) = \sum_{i=0}^{N} p(x_i) + q(u_i)$,
$\mathcal{C}(x,u) = \{(x,u) | [H_x^T H_u^T]^T [x^T u^T]^T \leq [b_x^T b_u^T]^T\}$
Output: $\{x_1^*, x_2^*, \cdots, x_M^*\}$

1. Let $x_t \leftarrow A^t x_0 + \sum_{i=0}^{t-1} A^{t-1-i} B u_i$, $\forall t \in \{1, \cdots, N\}$
2. Let $\{u_0^*, \cdots, u_{N-1}^*\} \leftarrow$ argmin $J(u, x_0)$ s.t $\mathcal{C}(u, x_0)$
3. Let $x_0 \leftarrow A x_0 + B u_0^*$
4. If iteration M, END, ELSE GOTO 1

the trajectory result of the MPC is not an optimal path.

Mirko Fiacchini, Mazen Alamir (2017) Computing control invariant sets is easy:

Finite determination of the algorithm, that is ensured for autonomous systems, cannot be assured in general in presence of control input. If, instead, the procedure is initialized with a non-decreasing sequence of control invariant sets are obtained that converges from the inside to the maximal control invariant set.

work examples

- $A = [1.2\ 1; 0\ 1.2]$, $B = [0.5; 0.3]$, $-2 \leq u \leq 2$, no constraints in the state, $\mathcal{X}$ = unitary box. Iterate 5 times
- Singular case $A = [1.2\ 1; 0\ 0]$, $B = [0.5; 0.3]$, $-2 \leq u \leq 2$, no constraints in the state, $\mathcal{X}$ = unitary box.
- $A = [1.2\ 1; 0\ 1.2]$, $B = [0.5; 0.3]$, $-2 \leq u \leq 2$, no constraints in the state, $\mathcal{X} = \{-10 \leq x_1 \leq 5, -1 \leq x_2 \leq 2\}$. Iterate 15 times

Borrelli, Bemporad, Morari 2011:

There is an algorithm to compute the horizon for which the problem is equivalent to the LQR see pag 189.

Definition 1. *A function $h(\theta) : \Theta \to \mathbb{R}^k$, where $\Theta \subset \mathbb{R}^s$ is piecewise affine (PWA) if there exists a strict partition $\mathbb{R}_1, \cdots, \mathbb{R}_N$ of Θ and $h(\theta) = H^i\theta + k^i$, $\forall\theta \in R_i$, $i = 1, \cdots, N$*

in PPWA the first P is for polyhedra and is the same as PWA, but the partition is a polyhedra partition. In PWQ the function is $h(\theta) = \theta' H^i \theta + k^i\theta + l^i$ and in PPWQ the partition is a polyhedra partition.

CRA: critical region of active constraints A. This relates the control u with the initial condition x_0.

Pag 103: computation of CRA for mp-QP and shows that the CRA are polyhedra.

pag 112: mp-QP algorithm: determines the partition of the feasible set into critical regions, and finds the expression of the functions $J^*(.)$ and $z^*(.)$ for each critical region. In principle, one could simply generate all the possible combinations of active sets. However, in many problems only a few active constraints sets generate full-dimensional critical regions inside the feasible region.

question: what are the requirements for the control invariant algorithm to be finite? for the uncontrollable system Am=eye(2) and Bm=[1;1] the Cinfty exists.